Center for
Mathematical Economics
Working Papers

536

February 2015

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http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

The El Farol Problem Revisited

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Abstract

The so-called El Farol problem describes a prototypical situation of interacting agents making binary choices to participate in a non-cooperative environment or to stay by themselves and choosing an outside option. In a much cited paper Arthur (1994) argues that persistent non-converging sequences of rates of participation with permanent forecasting errors occur due to the non-existence of a prediction model for agents to forecast the attendance appropriately to induce stable rational expectations solutions. From this he concludes the need for agents to use boundedly rational rules.

This note shows that in a large class of such models the failure of agents to find rational prediction rules which stabilize is not due to a non-existence of perfect rules, but rather to the failure of agents to identify the correct class of predictors from which the perfect ones can be chosen. What appears as a need to search for boundedly rational predictors originates from the non existence of *stable* confirming self-referential orbits induced by predictors selected from the wrong class.

Specifically, it is shown that, within a specified class of the model and due to a structural non-convexity (or discontinuity), symmetric Nash equilibria of the associated static game may fail to exist generically depending on the utility level of the outside option. If they exist, they may induce the least desired outcome while, generically, asymmetric equilibria are uniquely determined by a positive maximal rate of attendance.

The sequential setting turns the static game into a dynamic economic law of the Cobweb type for which there always exist nontrivial $\epsilon-$perfect predictors implementing $\epsilon-$perfect steady states as stable outcomes. If zero participation is a Nash equilibrium of the game there exists a unique perfect predictor implementing the trivial equilibrium as a stable steady state. In general, Nash equilibria of the one-shot game are among the $\epsilon-$perfect foresight steady states of the dynamic model.

If agents randomize over indifferent decisions the induced random Cobweb law together with recursive predictors becomes an iterated function system (IFS). There exist unbiased predictors with associated stable stationary solutions for appropriate randomizations supporting nonzero asymmetric equilibria which are not mixed Nash equilibria of the one-shot game. However, the least desired outcome remains as the unique stable stationary outcome for $\epsilon = 0$ if it is a Nash equilibrium of the static game.

Keywords: El Farol, participation games, repeated play, forecasting, rational expectations, Cobweb models

JEL classification : C7, C73, D83, D84

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1 Introduction

The so called El Farol problem stands for an interactive non-cooperative setting of N participants in a market or a game form who have to decide whether to stay by themselves (chose an outside option) or participate in a common activity whose benefits to each individual depend in a concave way on the number of active participants. There are numerous economic situations with the same prototypical setting and many examples, among them entry-exit decisions for firms in markets, activities in clubs, mechanisms for supporting a local public project, – or going to a popular bar in Santa Fe on Thursday night (see Arthur, 1994). He argues that within such a setting when participants dislike crowding (in a bar, at noontime for lunch in a restaurant, etc. and when there are limits to coordination) attendance rates will typically fluctuate due to consistent misanticipation of the number of attendees. He infers that a "rational" solution to the prediction problem does not exist since standard deductive reasoning will fail structurally to find or identify a self-confirming rational expectations solution. He writes (Arthur, 1994, p. 3):

> *"First, if there were an obvious model that all agents could use to forecast attendance and base their decisions on, then a deductive solution would be possible. But this is not the case here. Given the numbers attending in the recent past, a large number of expectational models might be reasonable and defensible. Thus, not knowing which model other agents might choose, a reference agent cannot chose his in a well-defined way. There is no deductively rational solution – no"correct" expectational model. From the agents' viewpoint, the problem is ill-defined and they are propelled into a world of induction.*
> *Second, and diabolically, any commonalty of expectations gets broken up: If all believe* few *will go, all will go. But this would invalidate that belief. Similarly, if all believe* most *will go, nobody will go, invalidating that belief. Expectations will be forced to differ."*

Applying such reasoning to an expectations mechanism in a repeated or recursive setting, Arthur shows by means of an example that it may induce complex or non-stationary sequences of plays. This is taken as an indication for the need to search for solutions under bounded rationality.

This note shows that, within a specified class of the problem and due to a structural non-convexity (or discontinuity), Nash equilibria of the associated static game may fail to exist generically. However, if they exist, they may induce the least desired outcome to all participants. The sequential setting becomes a dynamic economic law of the Cobweb type for which approximately perfect predictors always exist. They may induce the trivial equilibrium as a stable steady state as well as those with positive attendances. Moreover, a randomization by agents to break indifferences converts the random Cobweb law into an an iterated function system for which unbiased predictors exist which implement positive stationary rational expectations outcomes. What appears as a need to search for boundedly rational predictors originates from the non existence of *stable* confirming self-referential orbits induced by predictors selected from the wrong class.

2 The El Farol Game

Although Arthur does not give an explicit formal description of the model he has in mind, the formulation chosen here seems to be a generic representation for the scenario he describes which captures the essential features of interaction of the El Farol problem as a game of participation with crowding. The presentation assumes homogeneous identical agents. The implications of heterogeneity for the results are discussed in the final section.

Let $I = \{1, \ldots, i, \ldots, N\}$ denote the set of players/visitors to El Farol where $N \in \mathbb{N}$ is a relatively large number and let $n \in [0, 1]$ denote the proportion of players present at El Farol. Assuming identical (homogeneous and anonymous) preferences for each player i, denote by $u : [0, 1] \times \{0, 1\} \to \mathbb{R}$ the utility of a player i, defined by

$$u(n, x) = xv(n) + (1 - x)B \qquad 0 \le B \le 1 \tag{2.1}$$

where $v : [0, 1] \to [0, 1]$ is a concave function.

If v has an interior maximum, i.e. for some $0 < \bar{n} < 1$, $v(\bar{n}) \ge v(n)$, for all $n \in [0, 1]$ with $v(\bar{n}) > B \ge \max\{v(1/N), v(1)\}$, then individuals prefer medium crowded bars to under- or overcrowded ones, but they also dislike being alone in the bar. It will be assumed throughout that $B > v(1)$, which reflects the fact that players dislike maximum crowding. For concreteness, v could be any asymmetric unimodal map or the symmetric logistic map $v(n) = An\,(1 - n)$ for $0 \le A \le 4$ in which case u could also be written as

$$u(n, x) := \begin{cases} An\,(1 - n) & x = 1 \qquad 0 \le A \le 4 \\ B & x = 0 \qquad B > 0 \end{cases}.$$

Denote by $x := (x_1, \ldots, x_N)$ a list of choices for each player and define by

$$\nu(x) := \frac{\sum_{j=1}^{N} x_j}{N}$$

the proportion of players at El Farol induced by x. This yields a payoff function for each i given by $\pi_i(x) := u(\nu(x), x_i)$. Then, the pair $G := (I, \Pi)$ is a symmetric normal form game with payoff function $\Pi : \{0, 1\}^N \to \mathbb{R}^N$ given by

$$\Pi(x) := \prod \pi_i(x) = \prod_{i=1}^{N} u(\nu(x), x_i). \tag{2.2}$$

Clearly, $x = (1, \ldots, 1) =: \mathbf{1}$ and $x = (0, \ldots, 0) =: \mathbf{0}$ are the only candidates for a symmetric Nash equilibrium since partial attendance is never symmetric. $x = (1, \ldots, 1)$ is not a Nash equilibrium if $B > v(1)$, for any concave v. If $B > v(1/N)$, i.e. if being alone in the bar is less attractive than staying at home, $x = 0 = (0, \ldots, 0)$ is the unique symmetric Nash equilibrium. Conversely, if the condition $B < v(1/N)$ holds, G has no nontrivial symmetric equilibrium in pure strategies.

Existence of asymmetric Nash equilibria requires that the payoff for those participating must be at least as good as staying at home. Define the upper contour set of v with respect to

B as $v^+(B) := \{n \in (0,1) \,|\, v(n) \geq B\}$. Then, asymmetric equilibria exist only if $v^+(B) \cap \{1/N, \dots, (N-1)/N\}$ is nonempty. Under a generic condition there exists a unique maximal level of participation supporting the only asymmetric equilibria.

Lemma 1. *Let v be concave and assume that $1 \geq B > v(1)$ holds. Then:*

(1) $x = \mathbf{0}$ is the unique symmetric Nash equilibrium of G if and only if $B \geq v(1/N)$.

(2) Let $v^+(B) \cap \{1/N, \dots, (N-1)/N\} \neq \emptyset$.

1. *There exists a unique maximal group size (participation rate) $\hat{k} \in \{1, \dots, N-1\}$ and associated list $\hat{x} \in \{0,1\}^N$ with $\sum_1^N \hat{x}_j = \hat{k}$ such that $\mathbf{0} \neq \hat{x} \neq \mathbf{1}$ is an asymmetric Nash equilibrium. $\hat{k}$ is the only group size supporting asymmetric Nash equilibria if $v(\hat{k}/N) > B$.*
2. *If $v(\hat{k}/N) = B$ and $(\hat{k}-1)/N \in v^+(B)$, then $\hat{k}-1 = \sum_1^N x_j$ is an additional group size (participation rate) with associated asymmetric Nash equilibrium x. If $v((\hat{k}-1)/N) = v(\hat{k}/N) = B$, then all $k/N \in v^+(B)$ support asymmetric Nash Equilibria.*

Proof. For $i = 1, \dots, N$, let $\pi_i(x_1, \dots, x_N) \equiv \pi(x_i, x_{-i})$, where $x_{-i} := (x_j)_{j \neq i}$. Then,

$$\pi_i(0, \mathbf{0}_{-i}) \geq \pi_i(1, \mathbf{0}_{-i}) \quad \text{if and only if} \quad B \geq v(1/N).$$

Thus, $x_i = 0$ is a best response to $x_{-i} = \mathbf{0}_{-i}$ for every $i = 1, \dots, N$ so that $\mathbf{0}$ is a Nash equilibrium. Conversely, $\pi_i(1, \mathbf{1}_{-i}) = v(1) < B = \pi_i(0, \mathbf{1}_{-i})$ implies that $x = \mathbf{1}$ cannot be a Nash equilibrium.

For any $x \in \{0,1\}^N$, define $S(x) := \{j \in N \,|\, x_j = 1\}$. Then, $\mathbf{0} \neq x \neq \mathbf{1}$ is a Nash equilibrium if and only if:

$$v\left(\frac{1 + \sum_1^N x_j}{N}\right) = \pi_i(1, x_{-i}) \leq \pi_i(0, x_{-i}) = B \qquad \text{for} \quad i \notin S(x)$$

and

$$B = \pi_i(0, x_{-i}) \leq \pi(1, x_{-i}) = v\left(\frac{\sum_1^N x_j}{N}\right) \qquad \text{for} \quad i \in S(x).$$

Therefore, for any Nash equilibrium x one has

$$v\left(\frac{|S(x)|}{N}\right) = v\left(\frac{\sum_1^N x_j}{N}\right) \geq B \geq v\left(\frac{1 + \sum_1^N x_j}{N}\right) = v\left(\frac{1 + |S(x)|}{N}\right), \tag{2.3}$$

i.e. v must be decreasing below B at $S(x)$ for any Nash equilibrium.

Let $\hat{k} = \max\{k \in (1, \dots, (N-1)) \,|\, k/N \in v^+(B)\}$. Chose $\hat{x}$ such that $\hat{k} = |S(\hat{x})|$. Then, by construction, $v((\hat{k}+1)/N) < B$. Therefore, $\hat{x}$ is a Nash equilibrium. If $v(\hat{k}/N) > B$, concavity of v implies that $v((k+1)/N) > B$ for all $k \in \{1, \dots, \hat{k}-1\}$ such that $k/N \in v^+(B)$. Therefore, there exists no $k < \hat{k}$ which supports Nash equilibria.

If $v(\hat{k}/N) = B$, then $\hat{k}-1$ supports a Nash equilibrium if $v((\hat{k}-1)/N) \geq v(\hat{k}/N) = B$. However, if $v((\hat{k}-1)/N) = B$, then concavity of v implies that $v(k/N) = B$ for all $k \in v^+(B)$ with $k < k - 1$. $\square$

If x is an asymmetric Nash equilibrium, the payoff is weakly decreasing in the number of the participants implying that the size of the participating group has to be maximal on the set of outcomes which dominate B. This implies that there cannot exist asymmetric Nash equilibria with higher participation. For games with strict concavity this also excludes asymmetric equilibria with lower rates of participation except when $\hat{k}$ is on the boundary of the better set $v^+(B)$, in which case there are at most two levels of participation which support Nash equilibria.

More than two levels supporting Nash equilibria occur if and only if the function v is equal to B at three levels. Then, concavity implies that v attains its maximum at B on an interval. Thus, for strictly concave functions v and generic levels B, asymmetric equilibria have a unique size of participation whose equilibria are defined by redistributing participation.

In summary, with sufficient disutility of crowding relative to the outside option, symmetric Nash equilibria with positive attendance do not exist, while $x = 0$ becomes the unique symmetric equilibrium when B is large. Equilibria with positive attendance are non-symmetric which have a unique size generically.

3 Going to El Farol Repeatedly

Assume that all players decide independently and repeatedly to go to El Farol without prior communication or collusion, but *after* making a forecast/prediction of the relative number of players to be present causing a sequential (or informational) delay between the forecast and the actual decision of each agent which is assumed to be utility maximizing. Formally, let $0 \leq n^e \leq 1$ denote i's forecast. This induces his best response defining an attendance map of each player as

$$\xi(n^e) := \arg \max_{x \in \{0,1\}} u(n^e, x) \tag{3.1}$$

if he assumes that n^e visitors (including himself) will come. As a consequence the individual attendance decision becomes a mapping from the space of forecasts. This mapping is of the Cobweb type, - a terminology adapted from dynamic price theory - indicating that the outcome depends on the forecast only and is independent of the previous outcome (see also footnote 1).

Lemma 2.
Let v be continuous and concave such that there exists $n \in [0, 1]$ with $v(n) > B > v(1)$.

For $B < v(0)$, the best response correspondence ξ has the form

$$\xi(n^e) := \begin{cases} 1 & 0 \leq n^e < \bar{n}^e \\ \{0,1\} & n^e = \bar{n}^e \\ 0 & \bar{n}^e < n^e \leq 1 \end{cases} \tag{3.2}$$

where $\bar{n}^e$ satisfies $v(\bar{n}^e) = B$.

If $B > v(0)$, there exist two positive numbers $0 < n^e_1 < n^e_2 < 1$ defined by

$$v(n^e_1) = v(n^e_2) = B \tag{3.3}$$

such that

$$(3.4) \qquad \xi(n^e) := \begin{cases} 0 & 0 \le n^e < n_1^e \\ \{0,1\} & n^e = n_1^e \\ 1 & n_1^e < n^e < n_2^e \\ \{0,1\} & n^e = n_2^e \\ 0 & n_2^e < n^e \le 1 \end{cases}$$

The correspondence $\xi : [0,1] \to \{0,1\}$ has a closed graph (is upper hemi-continuous).

The best response map is a continuous function except at the positive critical levels $\bar{n}^e$ or $0 < n_1^e < n_2^e < 1$ respectively, which are determined by the preferences. At the critical values ξ consists of the two values zero and one implying that the correspondence is not convex valued at these two points.

3.1 The Dynamics of Expectations and Attendance

Given the sequential interpretation that forecasts are made before the decisions to attend are taken, it is useful to keep the attendance mechanism and the forecasting rules as two *conceptually separate* mappings. However, in order to examine whether stable equilibrium configurations exist and whether they can be reached it is necessary to analyze the dynamics of expectations and of attendances as a *joint* process arising from the interaction of these two mappings in order to understand the causes and failures of stable configurations. Let us first define the two essential characteristics of steady states of such processes.

Definition 1. *A pair of $(x, n^e) \in \{0,1\}^N \times [0,1]$ is called an equilibrium under best response and perfect foresight if*

$$(3.5) \qquad n^e = \nu(x) \quad \text{and} \quad x_i \in \xi(n^e) \quad \text{for every} \quad i \in I.$$

Equivalently, $x = (x_1, \ldots, x_n)$ is an equilibrium if

$$(3.6) \qquad x_i \in \xi(\nu(x)) \quad \text{for every} \quad i \in I.$$

Observe that such an equilibrium is a state with self-confirming expectations and best response to expectations which occur only at the critical levels $\bar{n}^e$, $\bar{n}_i, i = 1, 2$ respectively. Thus, generically, the induced participation is not a Nash equilibrium of the El Farol game $G = (I, \Pi)$ (see Lemma 1).

Given the best response behavior of all agents for any forecast n^e, the induced next state of attendance n_t is given by a recursive *attendance* mapping induced by the simultaneous best response of all agents as follows.

Definition 2. *Let $\Xi(n^e) := (\xi(n^e), \ldots, \xi(n^e))$ denote the $N-$fold product of the best response correspondence ξ. There exists a correspondence $F : [0,1] \to [0,1]$ describing the outcome/attendance map written as $F := \nu \circ \Xi$ defined by*

$$(3.7) \qquad F(n^e) := \begin{cases} 1 & 0 \le n^e < \bar{n}^e \\ \frac{1}{N}\{0,1,2,\ldots,N\} & n^e = \bar{n}^e \\ 0 & \bar{n}^e < n^e \le 1 \end{cases}$$

for $B < v(0)$ and by

$$F(n^e) := \begin{cases} 0 & 0 \leq n^e < n_1^e \\ \frac{1}{N}\{0,1,2,\ldots,N\} & n^e = n_1^e \\ 1 & n_1^e < n^e < n_2^e \\ \frac{1}{N}\{0,1,2,\ldots,N\} & n^e = n_2^e \\ 0 & n_2^e < n^e \leq 1 \end{cases} \tag{3.8}$$

for $B > v(0)$.

The set valued mapping F describes next periods attendance $n_t \in F(n^e_{t-1,t})$ under best response behavior of the agents for given expectations $n^e_{t-1,t}$. It is the recursive rule describing the one-step ahead determination of the next state inducing a dynamic process. Note that F is upper-hemicontinuous. It is a continuous function for all $n^e \neq \bar{n}^e$ or $n^e \neq n_1^e, n_2^e$ respectively, but it is not convex valued at these critical levels.

In models of temporary equilibria in markets mappings like F determine the next equilibrium price as a function of the current state and expectations. Such maps (when they are functions!) are called *economic laws*. They have been termed *Laws of the Cobweb type* if they are independent of actual or previous states, a property which corresponds to the essential feature of the common Cobweb models of dynamic price theory[1]. Keeping in line with that terminology (see Böhm & Wenzelburger, 1999), let us call the set valued mapping F the *El Farol Law* and observe that it is of the Cobweb type, since neither previous states nor previous expectations enter as an argument.

Using the law F to rewrite Definition 1, one finds that (x, n^e) is a best response perfect foresight equilibrium if $n^e \in F(\nu(x))$ and $n^e = \nu(x)$. As a consequence n is called a perfect foresight state if it is a fixed point of the El Farol Law F. The following proposition characterizes the structure of the set of equilibria which is essentially a consequence of the restriction of agents' decisions to binary choices.

Proposition 1.
Let v be continuous and concave such that there exists $n \in [0,1]$ with $v(n) > B > v(1)$.

1. *For any function v, the set of $\{B \in \mathbb{R} | B > v(1)\}$ for which there exists a nonzero best response perfect foresight equilibrium has Lebesgue measure zero.*
2. *Let $\tilde{F}$ denote the smallest pointwise convex valued correspondence containing the mapping F. Then, $\bar{n}^e \in \tilde{F}(\bar{n}^e)$ for $B < v(0)$ and $n_1^e \in \tilde{F}(n_1^e)$ and $n_2^e \in \tilde{F}(n_2^e)$ for $B > v(0)$ are the only fixed points in the two cases respectively.*

The result is immediate since the set of agents is finite and the critical levels cannot be fractions of N for general v and B. In other words, generically, zero may be the only perfect foresight equilibrium for F and there are no positive perfect foresight equilibria for F, while the positive critical levels of partial attendance are generic nontrivial fixed points of the convexified Cobweb law. This indicates that the nonconvexity of ξ and of F are the reasons for the generic failure of existence of equilibria with positive partial attendance.

[1]See Ezekiel (1938); Nerlove (1958); Waugh (1964); Pashigian (1987)

3.2 Predictors and Dynamics

Forecasting mechanisms or rules determining the prediction in any period are called *predictors*. These are given as mappings from past data to forecasts which are used to update the forecast at each step in time. They are called stationary (or recursive) if they are defined on the same finite dimensional space of observations and if they do not depend on time.

Let $T \geq 1$ denote the length of a finite history of observations and denote by $(x_{-T}, n_{-T}, n^e_{-T}) := (x_\tau, n_\tau, n^e_\tau)_{\tau=-T}^{-1} \in \{0,1\}^{NT} \times [0,1]^{2T}$ a specific list of observed data in an arbitrary period where $n^e_\tau \equiv n^e_{\tau-1,\tau}$ denotes the forecast made in period $\tau - 1$ for τ.

Definition 3. *A mapping* $\psi : \{0,1\}^{NT} \times [0,1]^{2T} \to [0,1]$,
$(x_{-T}, n_{-T}, n^e_{-T}) \mapsto \psi\left(x_{-T}, n_{-T}, n^e_{-T}\right) = n^e$ *is called a predictor.*

Definition 4. *Given the El Farol Law* F *and a predictor* ψ*, the pair of mappings* $(\psi, F \circ \psi)$ *defines the one-step movement of a set valued dynamical system on* $\{0,1\}^{NT} \times [0,1]^{2T}$*.*
An orbit of the system is a sequence $\{(x_t, n_t, n^e_t)\}_{t=0}^\infty$ *such that*

$$
\begin{aligned}
n^e_{t+1} &= \psi(x_{t-T}, n_{t-T}, n^e_{t-T}) \\
n_{t+1} &\in F(\psi(x_{t-T}, n_{t-T}, n^e_{t-T}))
\end{aligned}
\tag{3.9}
$$

with $x_t \in \Xi(n^e_t) \quad$ *and* $\quad n_t = \nu(x_t)$*.*

It is evident from the definition of the system $(\psi, F \circ \psi)$ that without the specification of a predictor no recursive dynamics is defined and that the chosen predictor drives/determines the dynamics of expectations and states in an essential way. In other words, a predictor becomes a structural input into an autonomous dynamical system (a control function of the system). The concept of a perfect predictor is introduced in two steps (see Böhm & Wenzelburger, 1999).

Definition 5. *For* $\epsilon \geq 0$*, an orbit* $\{(x_t, n_t, n^e_t)\}_{t=0}^\infty$ *of* $(\psi, F \circ \psi)$ *is said to be* ϵ*-perfect if*

$$
|n^e_{t+1} - n_{t+1}| \leq \epsilon \quad \text{for all} \quad t \geq 0. \tag{3.10}
$$

It is said to have the perfect foresight property if $\epsilon = 0$*, or equivalently if*

$$
n_{t+1} = n^e_{t+1} \tag{3.11}
$$

holds for all t *with* $n^e_{t+1} = \psi(x_t, n_t, n^e_t)$*,* $n_t = \nu(x_t)$*, and* $x_t \in \Xi(n^e_t)$*.*

Definition 6. *A predictor* ψ^* *is called* $\epsilon-$*perfect for* $\epsilon \geq 0$ *if it induces an* $\epsilon-$*perfect foresight orbit. It is called perfect for* $\epsilon = 0$*. Therefore,* ψ^* *is a perfect predictor if*

$$
n_{t+1} = n^e_{t+1} = \psi^*(x_t, n_t, n^e_t) \quad \text{for all} \quad t \geq 0, \tag{3.12}
$$

or equivalently if

$$
\psi^*(x_t, n_t, n^e_t) \in F(\psi^*(x_t, n_t, n^e_t)) \tag{3.13}
$$

holds for all t *and* $n^e_{t+1} = \psi(x_t, n_t, n^e_t)$*,* $x_t \in \Xi(n^e_t)$ *and* $n_t = \nu(x_t)$*.*

Under the assumptions of Proposition 1 one obtains the following result.

Proposition 2.
Assume $B > v(0)$ and let zero be the unique fixed point of the El Farol Law F as given in Definition 2.

1. *The constant predictor $\psi^*(x_{-T}, n_{-T}, n^e_{-T}) \equiv 0$ is the unique globally perfect predictor for F.*
2. *The state $x = n = n^e = 0$ is the unique globally stable steady state of the dynamical system $(\psi^*, F \circ \psi^*)$.*
3. *There exist $\epsilon_i > 0$, $i = 1, 2$, such that the constant predictors $\psi_i(x_{-T}, n_{-T}, n^e_{-T}) \equiv n^e_i$ are ϵ_i-perfect for $i = 1, 2$ respectively, i.e. there exist $(x_i, n_i) = (x_i, \nu(x_i))$ such that*
$$|n_i - n^e_i| = |\nu(x_i) - n^e_i| \leq \epsilon_i, \qquad i = 1, 2$$
4. *(x_i, n_i, n^e_i) are globally stable for $(\psi_i, F \circ \psi_i)$, $i = 1, 2$ respectively.*

Proof. For $i = 1, 2$, choose $k_i \in \{0, \ldots, N\}$ and $x_i \in \{0, 1\}^N$ such that $n^e_i \in [k_i/N, (k_i+1)/N]$ and $k_i = \sum_{j=1}^{N} x^j_i$. Then, $(x_i, n_i) = (x_i, \nu(x_i))$ is an approximately perfect foresight steady state for $\epsilon_i = 1/N$ induced by the constant predictor $\psi_i(x_{-T}, n_{-T}, n^e_{-T}) \equiv n^e_i$. Therefore, all four properties follow from Lemma 4 in Böhm & Wenzelburger (1999). □

The corresponding result for the case $B < v(0)$ follows as a corollary.

Corollary Let $B < v(0)$.

1. There exists no continuous globally perfect predictor for F.
2. There exists $\epsilon > 0$ such that the constant predictor $\psi^*(x_{-T}, n_{-T}, n^e_{-T}) \equiv \bar{n}^e$ is $\epsilon-$perfect, i.e. there exist $(x, n) = (x, \nu(x))$ such that $|n - \bar{n}^e| = |\nu(x) - \bar{n}^e| \leq \epsilon$,
3. $(x, n, \bar{n}^e)$ is globally stable for $(\psi^*, F \circ \psi^*)$.

One finds that for $B > v(0)$ the El Farol Law always has a unique globally stable solution which is obtainable under full rationality and noncooperative behavior of all agents provided they agree on choosing the unique perfect predictor – or that an outside analyst suggests that this is the only one for which consistency of expectations can be obtained *and* all agents follow the suggestion. Since it is the unique equilibrium with the perfect foresight property it can be found by rational deduction either by the outside analyst or by each individual agent. In such a case, convergence under perfect foresight is obtained in one step. No cycles occur and there is no room or need for considerations of bounded rationality. This holds for the perfect foresight case with zero attendance only, clearly not the case that Arthur (1994) had in mind originally. If there is no positive equilibrium with perfect foresight the perfect predictor for zero attendance (in the space of recursive forecasting rules for Cobweb laws) must be unique. This implies also that no other continuous recursive predictor could induce a positive *limiting self-confirming orbit.*

Statements 3 and 4 of the Proposition and the Corollary show that there exist approximately globally perfect predictors which support equilibria with positive asymmetric attendance close to the respective critical values under uniform prediction. The predictors must be constant value predictors independent of any history of attendance and past expectations in order to be globally $\epsilon-$perfect. The proof shows that the associated rates of attendances at steady states are defined by a unique group size. In conjunction with Lemma 1 one finds that a steady state with the largest group size implements a Nash equilibrium of the static game if $B > v(0)$ and $k_2/N < n_2^e$ (or if $B < v(0)$ and $\bar{k}/N < \bar{n}^e$) hold. In other words, not all non-zero steady states under best response behavior and self-confirming expectations induce Nash equilibria.

Thus, depending on the level of the reservation value B there exist distinct approximately perfect predictors with corresponding self-confirming positive levels of equilibrium attendance obtainable under deductive reasoning. Since the levels of attendance are not unique they always induce a coordination problem to implement the associated asymmetric attendances while the predictor is unique and globally approximately perfect. In other words, the lack of a deductive solution to find a self-confirming predictor (for positive attendance) as argued by Arthur has to be attributed to the generic non-existence of positive perfect foresight equilibria in the specific Cobweb environment when zero prediction errors are imposed. There is no such lack under approximately perfect predictions since deductive reasoning identifies the constant predictors and leads to unique ones (for $B < v(0)$) and to the corresponding statements in the proposition and the corollary.

The *multiplicity* of approximately perfect predictors (for $B > v(0)$) and not their non-existence imply an additional equilibrium selection issue which is beyond the individual non-cooperative rational of the game or of the recursive description of the model. It may seem that Arthur's reasoning, motivated by his experimental example, points at the induced coordination issue arising from multiple steady state attendances as an argument for inductive reasoning to justify the need for bounded rationality for expectations formation. However, the conceptual separation of the forecasting issue from the attendance mapping, reveals that there is no need to consider boundedly rational predictors in the El Farol model if *constant* and *approximately perfect* predictors are included in the set of allowable forecasting rules. Nevertheless, the coordination of attendances under uniform approximately perfect predictions remains a challenging problem to be solved under dynamic repetitions.

3.3 A Stochastic El Farol Law and Rational Expectations

Extending the above model to a situation with random attendance (or mixed plays of best responses), let the forecast n^e be interpreted as the subjective prediction of each agent for the *mean attendance* next period. Then, using (3.4) define for every agent $i \in N$ a pair of best response functions $\xi_{max} : [0,1] \to \{0,1\}$ and $\xi_{min} : [0,1] \to \{0,1\}$ as

$$(3.14) \qquad \xi_{max}(n^e) := \arg\max\{x | x \in \xi(n^e)\} \qquad \text{and} \qquad \xi_{min}(n^e) := \arg\min\{x | x \in \xi(n^e)\}$$

which satisfy $\xi_{max}(n^e) = \xi_{min}(n^e)$ for $n^e \neq n_1^e, n_2^e$ when $B > v(0)$ and $n^e = \bar{n}^e$ when $B < v(0)$, respectively. Let $P := (p_1, \dots, p_N)$, $0 < p_i < 1$ denote a probability distribution on $\{0,1\}^N$, where p_i is the probability according to which i randomizes over the pair of functions $(\xi_{min}(n^e), \xi_{max}(n^e))$. For a given realization $w \in \{0,1\}^N$ define the random choice of i as

$$(3.15) \qquad \xi(w_i, n^e) := w_i \xi_{max}(n^e) + (1 - w_i)\xi_{min}(n^e).$$

As a consequence, one obtains a random family of finitely many mappings (functions!) $F(\cdot, n^e) : [0,1] \to [0,1]$ given by

$$(3.16) \qquad F(w_1, \dots, w_n, n^e) := \frac{1}{N} \sum \xi_i(w_i, n^e)$$

describing the one step random change of the attendance as a stochastic difference equation

$$n_t = F(w_t, n_t^e). \tag{3.17}$$

Thus, $F(w, \cdot) : [0, 1] \to [0, 1]$ is a random El Farol Law which is again of the Cobweb type.

In order to define and investigate the dynamics of randomized sequential play under rational expectations and its stability we proceed as in Böhm & Wenzelburger (2002). Let $n^e = \psi(x_{-T}, n_{-T}, n^e_{-T})$ denote a mean value predictor for F. Then,

$$\begin{aligned} n^e &= \psi(x_{-T}, n_{-T}, n^e_{-T}) \\ n &= F(w, \psi(x_{-T}, n_{-T}, n^e_{-T})) \end{aligned} \tag{3.18}$$

defines a pair of finitely many difference equations describing the random one step evolution of expectations and attendance with randomized attendance at the critical levels, but which are identical deterministic functions at all other values of n^e. Therefore, given the discrete probability measure P on $\{0, 1\}^N$, the list $\{(\psi, F(w, \psi); P\}$ becomes an iterated function system IFS (see Barnsley, 1988; Arnold & Crauel, 1992). The IFS induces random orbits $\{(x_\tau, n^e_\tau, n_\tau)\}_0^\infty$ for any sample path $\omega := (\ldots, w_0, w_1, \ldots, w_\tau, \ldots) \in \Omega := \left\{\{0, 1\}^N\right\}^{\mathbb{N}}$ and initial states (x_0, n^e_0, n_0). To investigate the performance of a predictor consider the mean of the prediction error

$$\mathbb{E}\left\{n^e_t - n_t\right\} = n^e_t - \mathbb{E}\left\{F(w_t, n^e_t))\right\} \tag{3.19}$$

at any time t.

Definition 7. *A mean value predictor ψ^* is called unbiased if its induced mean error is zero, i.e. if*

$$\mathbb{E}_P\left\{F(w_t, \psi^*(x_-, n_-, n^e_-))\right\} = \psi^*(x_-, n_-, n^e_-). \tag{3.20}$$

Lemma 3. *Let F denote the random El Farol Law defined in* (3.16) *and denote by*

$$(EF)(n^e, P) := \mathbb{E}_P\left\{F(\cdot, n^e)\right\} \tag{3.21}$$

its expected value function with respect to P. $\psi^\star$ is an unbiased predictor for F, if ψ^* predicts a fixed point of the mean El Farol Law (EF).

The lemma follows directly from the fact that the mean law is of the Cobweb type. Therefore, unbiased predictors must be constant value predictors predicting fixed points of the mean law. By construction the mean law is discontinuous at the respective critical levels for the two cases, i.e. for $B > v(0)$ one has

$$(EF)(n^e, P) := \begin{cases} 0 & 0 \le n^e < n^e_1 \\ \frac{1}{N}\sum_1^N p_j & n^e = n^e_1 \\ 1 & n^e_1 < n^e < n^e_2 \\ \frac{1}{N}\sum_1^N p_j & n^e = n^e_2 \\ 0 & n^e_2 < n^e \le 1 \end{cases} \tag{3.22}$$

while for $B < v(0)$ one finds

$$(EF)(n^e, P) := \begin{cases} 1 & 0 \le n^e < \bar{n}^e \\ \frac{1}{N}\sum_1^N p_j & n^e = \bar{n}^e \\ 0 & \bar{n}^e < n^e <\le 1. \end{cases} \tag{3.23}$$

By construction the mean law takes on the value zero or one everywhere except at the respective positive critical levels. Therefore, for any (v, B, P) with $B < v(0)$, $\bar{n}^e \neq (EF)(\bar{n}^e, P)$ and (EF) has no fixed point unless $\sum p_j = N\bar{n}^e$. Thus, there is no unbiased predictor in most cases. Similarly, for any (v, B, P) with $B > v(0)$, $n_1^e \neq (EP)(n_1^e, P)$ and $n_2^e \neq (EP)(n_2^e, P)$, making zero the unique fixed point of the mean law generically. Thus, one has the following result.

Theorem 1. *Let v satisfy the conditions of Proposition 1 and denote by F a random law as defined in* (3.16) *for (v, B) with P given.*

(i) *If $B > v(0)$:*

1. *$\psi^* \equiv 0$ is the unique unbiased predictor if and only if $\sum_1^N p_j \neq Nn_i^e$, $i = 1, 2$.*
2. *Zero is the unique asymptotically stable solution of $\{(\psi^*, F(w, \psi^*)); P\}$.*
3. *There exist two unbiased predictors with constant positive value $\psi_i \equiv n_i^e$, i=1,2, if and only if $\sum_j p_i^j = Nn_i^e$, $P_i := (p_i^1, \dots, p_i^j, \dots, p_i^N)$, $i = 1, 2$.*
4. *All random orbits $\{n_i^e, n_t\}_{t=0}^\infty$ of the IFS $\{(\psi_i, F(w, \psi_i); P_i\}$ are stationary solutions with sample mean n_i^e and rational expectations $\mathbb{E}_{P_i}(n_t) = n_i^e$, $i = 1, 2$.*

(ii) *If $B < v(0)$:*

1. *The constant mean value predictor $\bar{n}^e \equiv \bar{\psi}$ is the unique unbiased predictor for (v, B, P) if and only if $\sum \bar{p}_j = N\bar{n}^e$.*
2. *All random orbits $\{(\bar{n}^e, n_t\}_{t=0}^\infty$ of the iterated function system $\left\{(\bar{\psi}, F(w, \bar{\psi})); \bar{P}\right\}$ are stationary solutions with sample mean $\bar{n}^e$ and rational expectations $\mathbb{E}_{\bar{P}}(n_t) = \bar{n}^e$.*

Theorem 1 extends the results from Proposition 2 to the case were agents jointly randomize their choices, independently across time, but not necessarily independently across agents. When the outside option dominates the situation of 'lonely participation' $(v(0) < B)$, time independent randomizations induce the unique but least desirable attendance as a stable outcome of the dynamics under rational expectations in all cases and with any randomization. The two positive critical levels n_1^e and n_2^e are fixed points of the mean law under a set of different restricted randomizations which do not correspond to mixed Nash equilibria of the one shot game. These include, however, the specific symmetric randomization with $p_j = n_i^e$ for all $j = 1, \dots, N$; $i = 1, 2$ which provide an intuitive explanation for symmetric stationary behavior.

When $v(0) > B$, zero attendance is no longer a limiting rational expectations outcome, so that the unbiased predictor $\bar{\psi}$ with associated probabilities $\sum \bar{p}_j = N\bar{n}^e$ induces a unique stationary unbiased solution $\{n_t\}_{t=0}^\infty$ with $\mathbb{E}_{\bar{P}}\{n_\tau\} = \bar{n}^e$ for all $\tau = 1, \dots,$.

4 Summary and Conclusions

What has become known in the literature as the "El Farol problem" can be described by a class of one shot *games of participation with crowding.* Their equilibrium properties are determined by a structural discontinuity originating from the binary decision which players face when choosing the maximizers of two concave functions. For all symmetric concave preferences it is the level of the outside option which determines whether the game has no

Nash equilibrium or whether the unique Nash equilibrium is the one with zero attendance, which is a least desired outcome for all agents. The crowding effect prevents the existence of Nash equilibria with all agents participating. Moreover, for generic concave functions and outside option levels, asymmetric equilibria have a unique size of participation, implying that different equilibria are characterized by the redistribution of a uniquely determined constant number of players.

Under a recursive and strictly non-cooperative setting of repeated decision making the El Farol problem induces a recursive economic law of the Cobweb type whose steady states under perfect foresight are typically not Nash equilibria of the original one-shot game. When the outside option is significant there exists a unique perfect predictor generically in the space of preferences. In this case, the dynamics under perfect foresight are globally converging to the boundary state with zero attendance. However, for all levels of the outside option, asymmetric equilibria with partial positive attendance rates can be obtained as stable steady states under approximately perfect predictors including Nash equilibria, where the prediction error can be kept below a maximum level of $1/N$. In exceptional (nongeneric) cases these exhibit perfect foresight with $\epsilon = 0$.

Since the El Farol Law is of the Cobweb type, $\epsilon-$perfect predictors must be constant functions independent of any history, a logic which is inherent in Cobweb laws. This feature can be used by each participating agent (or by an outside observer) to identify perfect predictors as those which minimize the forecasting error at any time. In other words, in contrast to the reasoning put forward by Arthur (1994), the occurrence of persistent non-converging sequences of rates of attendance with permanent forecasting errors in such models is not due to a non-existence of perfect rules, but rather to the failure of agents to identify and to use them. Nevertheless, when the $\epsilon-$perfect predictors are not unique or coexist with the zero perfect predictor a coordination problem or equilibrium selection issue arises as to which of the predictors and associated asymmetric attendance rates should be implemented. To solve the coordination or the selection question requires cooperative reasoning or other mechanisms which are beyond the rational of non-cooperative and strict individualistic behavior used in the definition of equilibrium.

The results under deterministic behavior were extended to situations where agents randomize over their binary choices to break indifferences. In this case, the stochastic economic law becomes a family of random maps which induce an iterated function system (IFS) when agents use stationary (recursive) forecasting. Their stationary equilibria under rational expectations do not implement mixed equilibria of the original game. Due to the fact that the random El Farol Law is again of the Cobweb type, there exists a unique unbiased predictor implying rational expectations along orbits of the induced iterated function system. As in the deterministic case, depending on the level of the outside option, all orbits converge either to the (deterministic) steady state with zero attendance or a stationary solution with positive partial attendance. With non-uniqueness of the unbiased predictors, there remains the same coordination or equilibrium selection question as in the deterministic case.

It seems that none of the above results loose their general validity when agents' preferences and outside options are heterogeneous. It is well known and part of common economic reasoning in markets that heterogeneous expectations are the source of nontrivial orbits which cannot be self-confirming for all agents under heterogeneity of forecasts. Thus, one may argue that stable attendance configurations of an El Farol Law with heterogeneity would exist and could best be described by imposing a positive ϵ on the measure of rational expectations, allowing possibly a large enough class of $\epsilon-$unbiased predictors which may differ across agents. Otherwise, for $\epsilon = 0$, there may indeed be no predictors inducing self-confirming and stabilizing orbits of positive attendances or participations for many interesting economic choice problems with outside options.

These findings confirm the two main structural features of rational expectations dynamics for models of the Cobweb type. First, perfect foresight or rational expectations dynamics require a *constant* predictor - an almost trivial choice in the space of predictors. This fact is counterintuitive to common reasoning used by agents and analysts trying to search and identify perfect predictors. Unfortunately, this is based on the false presumption that past states determine the next state, while the Cobweb law indicates that the last forecast determines the next state. What may look as a need to search for boundedly rational predictors originates primarily from the non existence of *stable* confirming self-referential orbits for the predictors used. The reason is the choice of a predictor within the wrong class for which there are no self-confirming rational expectations orbits. This is not a failure of an application of deductive reasoning, but rather a failure to identify the correct class of predictors. If there are several predictors in the correct class an additional equilibrium selection problem arises.

Second, the stability of the dynamics with rational expectations is trivially fulfilled for constant predictors. Thus, the principle reason for cyclical and non-converging dynamics with consistent and permanent forecast errors in models of the Cobweb type lies in the fact that agents (and the analysts of such models!) chose predictors from the wrong class. Therefore, the failure of observable stationary solutions or perfect foresight orbits in Cobweb models with adaptive, recursive, or learning predictors arises because of the misconception of agents and analysts of the underlying dynamics, a belief which seems to be deeply rooted in the minds of many economists (see for example the discussion and reasoning in Nerlove, 1958; Samuelson, 1976; Pashigian, 1987, and others). Thus, cycles in Cobweb models using non-constant stationary predictors are self-inflicted and their orbits cannot induce rational expectations along orbits.

The fundamental interaction between an economic law and the predictor indicates that economies with expectations feed back in their intertemporal structure are closed loop systems in contrast to physical systems. Thus, predictors become controls in economic systems where forecasts influence future outcomes, something unheard of in meteorology! The rationality of forecasts and the stability of the system are only two of many possible criteria according to which predictors are and may be chosen.

Finally, there remains the question of how to find or identify the perfect or unbiased predictor from finite time series data along an orbit. Note that under recursive predictors the data of any orbit is always contained in a subset of the graph of the predictor and the law itself, which is a time invariant object accessible to empirical estimation at least in principle. Thus, orbits generated with given predictors reveal mostly information about the predictor used and *little* of an economic law of the Cobweb type or its associated error function. Therefore, a learning procedure has to be designed in such a way that it improves information about the contours/level sets of the error function and thus of the economic law. Often such procedures are time dependent so that the dynamical system becomes nonautonomous and the sets on which the dynamics of states and expectations occur may become complex or are no longer forward invariant, making successful learning possibly a difficult computational issue.

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