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Network Design and Imperfect Defense

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The question how to optimally design an infrastructure network that may be subject to intelligent threats is of highest interest. We address this problem by considering a Designer-Adversary game of optimal network design for the case of imperfect node defense. In this two-stage game, first the Designer defends network connectivity by forming costly links and additionally protecting nodes. Then, the Adversary attacks a fixed number of nodes, aiming to disconnect the network. In contrast to the existing literature, defense is imperfect in the sense that defended nodes can still be destroyed with some fixed probability.
We completely characterize the solution of the game for attack budgets of one and two nodes, while for larger budget we present a partial characterization of the solution. To do so, we determine the minimum number of links necessary to construct a network with any degree of connectivity and any given number of essential nodes.

Keywords: Network Design, Network Defense, Designer-Adversary Games, Node Destruction

JEL-Classification: C69, C72, D85.

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1 Introduction

Infrastructure networks are a crucial part of the modern society. Airports, internet servers and distribution centers are only some examples. Given this evident importance, the question arises how one should design such networks to optimally defend them against threats like intelligent attacks or natural disasters.

We propose a model of network design with two players, borrowed from Dziubiński and Goyal (2013). The first player, called the Designer, forms costly bilateral links within a given set of nodes. She may also protect nodes against deletion, however protection is imperfect. The second player, called the Adversary, can then attack a fixed number of nodes, where unprotected nodes along with their respective links are deleted with certainty, protected nodes only with some given probability. The Designer intends to retain a connected residual network after the attack, while the Adversary has the opposing goal.

We aim to characterize equilibrium solutions of the resulting extensive-form zero-sum game. That is, we want to identify the set of possible equilibrium defended networks, all networks the Designer may construct contingent of the model's priors: the attack budget of the adversary, the costs of link formation and defense, and the probability of deletion of defended links.

It is important to understand that in the proposed model the Designer has two defense mechanisms at hand. First, she may decide to directly defend nodes against deletion. Second, she may increase the connectivity of the network to defend it against separation. Thus it is on the one hand possible to design a network of sufficiently high connectivity such that direct node defense is not necessary, on the other hand a minimally connected network with all nodes being protected, or even some intermediate solutions will turn out to be also possibly optimal. This tradeoff between two substitutable defense mechanisms is a key characteristic of the model.

In a first step we fully characterize the set of equilibrium defended networks for attack budgets of one or two nodes. In case the Adversary can attack one node we show that the possible equilibrium defended networks are the empty network, the centrally-protected star and the non-protected circle (Proposition 1). Protected nodes will be present in equilibrium only for high chances of defense and small network size (Corollary 1).
For an attack budget of two nodes the possible equilibrium defended networks are the empty network, the centrally-protected star, the fully protected circle and the Harary graph of order 3, as well as one or two networks with an intermediate number of defended nodes (Proposition 3).
As for a general attack budget of k_a nodes, we aim to limit the set of possible equilibrium defended networks by using the same strategies as before in order to identify all non-connected, 1-connected and maximally connected possible equilibrium defended networks (Lemma 4), as well as a set of k_a-connected networks that include all possible equilibria (Conjecture 2).

The paper also contributes to the literature of graph theory, generalizing an early and seminal result of Harary (1962), who gives the minimum number of links needed to construct a network with a given degree of connectedness.[1] Here, we determine the minimum number of links necessary to construct a network with a given degree of connectedness and a given number of essential nodes (Proposition 2, Proposition 4 and Conjecture 1). Essential nodes are those nodes whose deletion will result in a strictly less connected network.

Related literature

This paper is connected to two strands of literature. The first and obvious is the literature on network design, where various papers contributed in the last years. Closest to this work are Dziubiński and Goyal (2013), who solve the same model we look at here, but for perfect node defense. In Goyal and Vigier (2013) and Cerdeiro et al. (2015) contagion is added, i.e. all undefended nodes connected to an attacked node will also be deleted. In Goyal and Vigier (2010) the players decide on sizes of attack and defense, while the node destruction is decided with a Tullock contest. Among others, Hoyer and De Jaegher (2010) look at node and link deletion, but disregard defense in their model. Finally, in the case of decentralized defense (see e.g. Dziubinski and Goyal, 2013; Hoyer, 2012) any node is assumed to be a rational agent, aiming to protect itself against deletion.

The second, older strand of literature is the graph theoretic literature on connectedness, where Harary (1962) provided an early seminal contribution, identifying the minimum number of links necessary to construct a network of given degree of connectedness.

The rest of the paper is organized as follows. Section 2 presents the model, while Section 3 presents the easily accessible solution for an attack budget of one. In Section 4 the solution of the game for an attack budget of two is characterized, while in Section 5 the same ideas are used for a partial characterization of the solution in case of a general attack budget. Section 6 finally concludes.

2 The Model

We want to introduce imperfect defense into the network design model of Dziubiński and Goyal (2013). In order to do so consider the following. Let N be a given set of nodes with cardinality n. A link (edge) between two nodes $i, j \in N$ is denoted by ij, and we define the complete network (graph) by $g^N = \{ij \mid i, j \in N\}$, i.e. the network where any two nodes are linked. Finally, we can now define the set of all networks $\mathbb{G} = \{g \mid g \subseteq g^N\}$.

This model is defined as follows. There are two players, the Designer and the Adversary, playing a two-stage game. In the first stage the Designer chooses

[1] A network is said to be connected of degree k if it cannot be disconnected by deleting any $k-1$ nodes along with their respective links.

network g, where each link comes at a constant and exogenously given cost c_l. At the same time she also may defend nodes at a cost c_d. Denote the set of these nodes by $D \subseteq N$. In the second stage, the Adversary, having an exogenously given attack budget k_a, chooses which k_a nodes to attack, denoted by $A \subseteq N$. Unprotected nodes are destroyed with probability 1, while protected nodes are destroyed with probability $\pi \in (0, 1)$. Like this, for the case $\pi = 0$ we obtain exactly the model of Dziubiński and Goyal (2013).

In the paper at hand we will study the Connectivity Game, i.e. we study the case that the Designer aims to retain a connected residual network, while the Adversary tries to disconnect the network.[2] Defining X to be the set of all links adjacent to deleted nodes, we obtain an ex-post payoff to the Designer given as

$$u_D(g, D, A) = \begin{cases} 1 - c_l|g| - c_d|D| & \text{if } g - X \text{ connected,} \\ -c_l|g| - c_d|D| & \text{otherwise,} \end{cases}$$

where $g - X$ is the residual network after the attack. Moreover, defining $\Pi(g, D, A)$ as the probability that network g with defense D gets disconnected by attack A, the corresponding expected payoff is given by

$$\mathbb{E}u_D(g, D, A) = 1 - \Pi(g, D, A) - c_l|g| - c_d|D|.$$

As we want the Adversary to have the opposite goal of the Designer, we simply define his ex-post utility as

$$u_A(g, D, A) = -u_D(g, D, A).$$

Clearly, the Connectivity Game sets rather extreme incentives for the players. In terms of the Designer he gets a constant payoff of 1 whenever the network is connected and 0 otherwise. Some different utility functions are reasonable and existent in the network design literature, a prominent example being a utility function defined as the sum over all components of functions convex in component sizes (see e.g. Dziubiński and Goyal, 2013). However, retaining connectivity is still a relevant and important goal in many examples and also constitutes the starting point of the corresponding graph-theoretic literature in the 1960s and 1970s. Moreover, we will see that also in this simple version of the model we will be able to obtain some interesting findings.

The two different defense strategies the Designer has at hand now become apparent. On the one hand, directly defending nodes against deletion at a cost c_d is an obvious strategy, while this defense is supposed to be imperfect in our model. On the other hand, adding more costly links to the network in order to increase the connectivity constitutes a second strategy of defense, as e.g. a circle network cannot be disconnected by deletion of only one node, independently of node defense. While these two defense strategies clearly behave as substitutes

[2] A network g is said to be connected if there exists a path between any two nodes, i.e. for any $i, j \in N$ there exist nodes $\{i, k_1, k_2, ..., k_{p-1}, k_p, j\}$, such that any two adjacent nodes are linked in g.

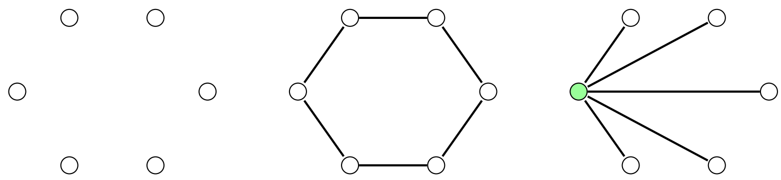


Figure 1: Possible equilibrium networks for $n = 6$ nodes and attack budget $k_a = 1$. Green colored node is protected.

in the model, we will see in the following that it may still be optimal to combine both.

The combination of the two defense mechanisms in an optimal defense strategy represents a central difference to the case of perfect defense, which is found in Dziubiński and Goyal (2013) and constitutes the limit case of $\pi = 0$ in the model at hand. It is easy to see that for perfect defense, regardless of the attack budget k_a of the Adversary the optimal strategy for the Designer is one out of the following three networks: the empty network for high costs, a star network with protected center, or an unprotected $(k_a + 1)$-connected network with minimal number of links (see Figure 1 for $k_a = 2$).[3] Thus, in this limit case the Designer chooses only one out of the two defense mechanisms, either a minimally connected but protected network or an unprotected but highly connected one.

We will now analyze the model first for attack budgets $k_a = 1$ and $k_a = 2$, and finally for a general attack budget $k_a \geq 3$.

3 Attack Budget 1

We want to start by the analysis of the game in case of an attack budget $k_a = 1$. While we will see that in this case the possible equilibria do not differ from the results in the perfect defense framework, we will see that being able to choose between the two available defense strategies (direct defense vs. increased connectivity), directly defending nodes is part of the equilibrium solution for the Designer only for low costs and high success probability of node defense.

As a starting point, the following result shows that if the Adversary can attack at most one node, then for any number of nodes $n \geq 3$ the equilibrium defended networks are as in Figure 1.

Proposition 1.
Let the attack budget of the Adversary be 1*. For any number of nodes* $n \geq 3$*, the set of possible equilibrium defended networks is given by the undefended empty*

[3]These networks, known as Harary graphs, were identified by Harary (1962) and are known to have $\lceil (k_a + 1)n/2 \rceil$ links.

network $g^{\emptyset}$, the undefended circle g^c and the centrally-protected star (henceforth CP-star) g^s.

- *The circle is the equilibrium defended network if $c_l \leq 1/n$ and $c_l \leq c_d + \pi$.*
- *The empty network is the equilibrium defended network if $c_l \geq 1/n$ and $c_d \geq 1 - \pi - (n-1)c_l$.*
- *The CP-star is the equilibrium defended network if $c_d \leq 1 - \pi - (n-1)c_l$ and $c_d \leq c_l - \pi$.*

The proof is rather immediate, understanding that the g^c is the Harary graph of order 2, and the star is the tree with the lowest number of essential nodes.

Proof. 1. There cannot be any equilibrium network $\tilde{g}$ with or without defense with $m > n$ links, as

$$u_D(\tilde{g}, D, A) \leq 1 - mc_l < 1 - nc_l = u_D(g^C, \emptyset, A) \quad \forall D, A \in N.$$

2. By Harary (1962) we know that a 2-connected network needs to have at least n links. Moreover, the circle g^c is the unique 2-connected network with n links: In any 2-connected network with n links each node has exactly 2 links. Thus deleting one node results by definition in a 1-connected residual network with $n-1$ nodes and $n-2$ links (a tree), and exactly two leafs, what necessarily constitutes a line. Now, the only possibility to get a 2-connected network out of a line by adding one node and 2 links is constructing a circle.

3. Any 1-connected network trivially has at least $n-1$ links. Further, it has to be protected in order to constitute a payoff higher than 0 (the payoff of the empty network). It is clear that in any such tree the leafs are non-essential nodes. The CP-Star is the only tree with a unique essential node.

4. No non-connected network can generate higher payoff than the empty network (same revenue, lower costs).

5. The cost combinations given in Proposition 1 are immediately result from the comparison of the expected utility to the Designer yielded by the three networks $g^{\emptyset}, g^c$ and g^s. □

The equilibria for different costs are depicted in Figure 2, for $\pi \in \{0,\ 1/(4n),\ 1/(2n),\ 3/(4n),\ 1/n\}$. Why it is enough to consider a maximum π of $1/n$ is shown in the following Lemma.

Lemma 1.
For $\pi > \frac{1}{n}$, the CP-star cannot be an equilibrium defended network.

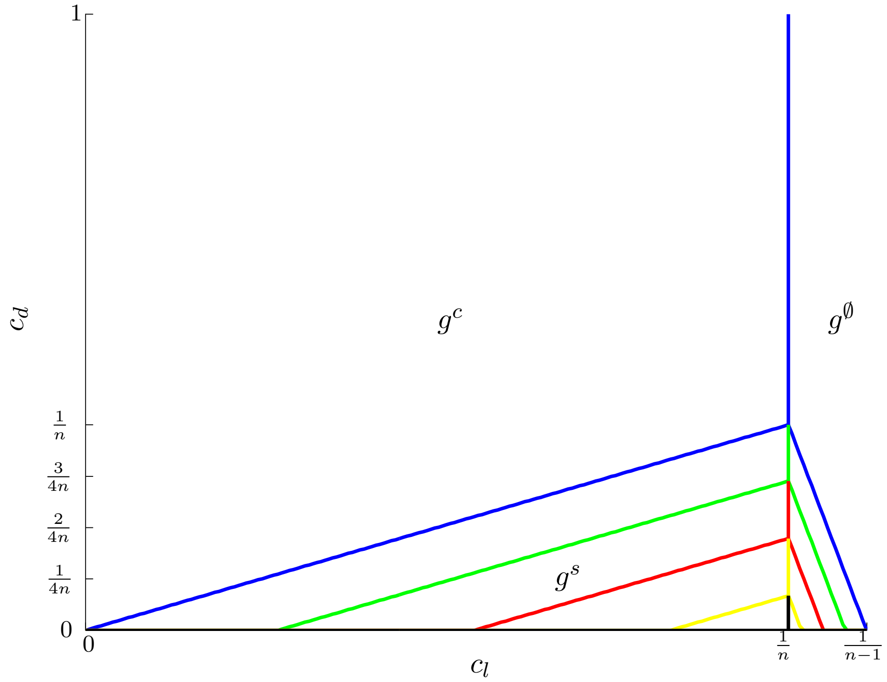


Figure 2: Equilibrium defended networks for $\pi \in [0, 1/n]$. As the success probability of attack π increases, the triangle region, where the CP-star is the optimal defended network, shrinks. For $\pi > 1/n$ (black lines), the CP-star cannot be an equilibrium any more.

Proof. The utility of the Designer from choosing the circle or the CP-star is

$$u_D(g^c, \emptyset, A) = 1 - nc_l,$$
$$u_D(g^s, \{1\}, \{1\}) = 1 - \pi - (n-1)c_l - c_d,$$

such that she would prefer the CP-star over the circle whenever

$$\pi + c_d \leq c_l. \tag{3.1}$$

If now $\pi > \frac{1}{n}$, inequality (3.1) yields $c_l > \frac{1}{n}$ for any positive defense costs c_d, and consequently

$$u_D(g^c, \emptyset, A) = 1 - nc_l < 0 = u_D(g^\emptyset, \emptyset, A),$$
$$u_D(g^s, \{1\}, \{1\}) = 1 - \pi - (n-1)c_l - c_d < 0 = u_D(g^\emptyset, \emptyset, A),$$

such that the Designer will prefer to choose the unprotected empty network. □

Notice that this result does not depend on the relative sizes of payoff and costs. The threshold $\pi = 1/n$ stays the same if the payoff would increase linearly in the number of nodes, i.e. if we considered a payoff in case of successful defense

$$\tilde{u}_D(g, D, A) = n - c_l|g| - c_d|D|,$$

Lemma 1 has also a direct consequence for large networks. The following corollary can also be understood as a limit analysis for $n \to \infty$.

Corollary 1.
For any positive success probability π and large enough network size n, a defended network cannot be an equilibrium.

We see that especially for large networks, as the Designer has the choice between the two defense mechanisms of direct defense and high network connectivity, the possibility of unsuccessful defense in most cases lets her decide in favor of higher connectivity of the network.

In the following section we will see that the picture slightly changes if the attack budget of the Adversary increases. Already for an attack budget $k_a = 2$ the Designer suddenly also has "intermediate" choices, i.e. it may be optimal for her to choose a network including a degree of connectivity larger than 1 and at the same time defending some nodes.

4 Attack Budget 2

We want to characterize the possible equilibrium defended networks in case of an attack budget $k_a = 2$ to the Adversary. We will find that the networks we found in the previous section are still part of the solution. In fact, in Section 5 we will define a set of possible equilibrium defended networks recursively.
What notably complicates the following analysis as compared to the previous section is that the Designer from now on has not only the possibility to choose between defending the network by either strategically defending essential nodes or increasing the network connectivity, but combining these two measures by constructing 2-connected networks with a strict subset of nodes being essential and thus defended.

Before turning to the results we want to further elaborate on this central point by presenting an easily accessible example.

Example 1. The set of possible equilibrium defended networks for $n = 8$ nodes and 2 units of attack are the empty network $g^{\emptyset}$, the CP-star g^s, the fully defended circle g^c and the Harary graph of order 3 $g^{h,3}$ (the wheel),together with a bipartite network of groups 3 and 5, where the smaller group is protected. All these networks are depicted in Figure 3.[4]

Instead of presenting all lengthy calculations to determine cost regions for each

[4]Dziubiński and Goyal (2013) present a similar example, however for a total number of nodes $n = 6$. The authors happen to miss the fact that for such a low number of nodes the maximal bipartite network with group sizes 2 and 4, the smaller group being fully defended, cannot be payoff-better than both the circle (Figure 3b) and the Harary graph of order 3 (Figure 3d).

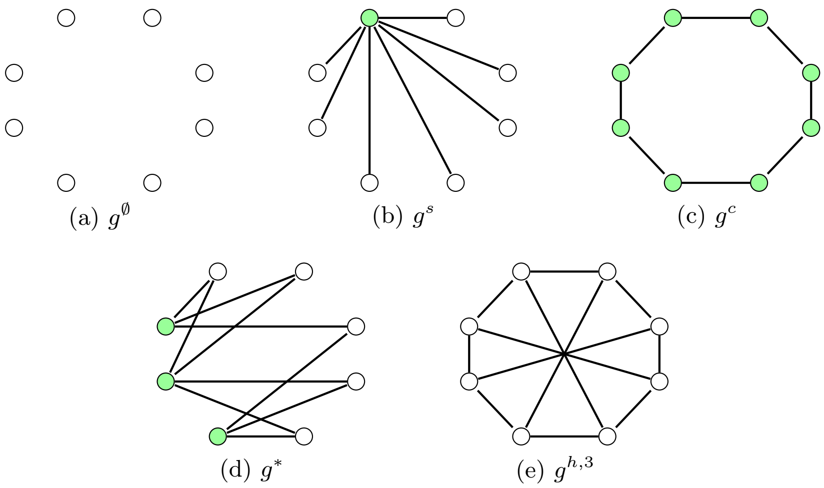


Figure 3: Possible equilibrium networks for $n = 8$ nodes and attack budget $k_a = 2$. Green colored nodes are protected.

network to be the equilibrium solution, in this example we show the solutions graphically in Figure 4 for various values of π.[5] The figures show that indeed there exist cost ranges and values of π for each of the 5 networks to be the equilibrium solution of the network design game. While g^* is only a possible solution for low but positive success probability of attack π and low costs, obviously all networks with defended nodes vanish as solutions for large π.

Let us now start the analysis of the game in case of $k_a = 2$ by collecting some first intuitive and easily provable facts.

Lemma 2.
Let the attack budget be $k_a = 2$. The following statements hold true.

- *The only possible non-connected equilibrium defended network is the undefended empty network $g^{\emptyset}$.*
- *The only possible 1-connected equilibrium defended network is the CP-star g^s.*
- *The only possible 3-connected equilibrium defended network is the undefended Harary graph of order 3, $g^{h,3}$.*[6]

All of these statements are the same or equivalent to those in the previous section (Proposition 1), such that a proof can be omitted here.

So far, we have found nothing qualitatively different from the previous case,

[5] The corresponding calculations are of course available from the author.
[6] It should be clear that this result does not only include all wheels but all 3-connected networks with $\lceil 3n/2 \rceil$.

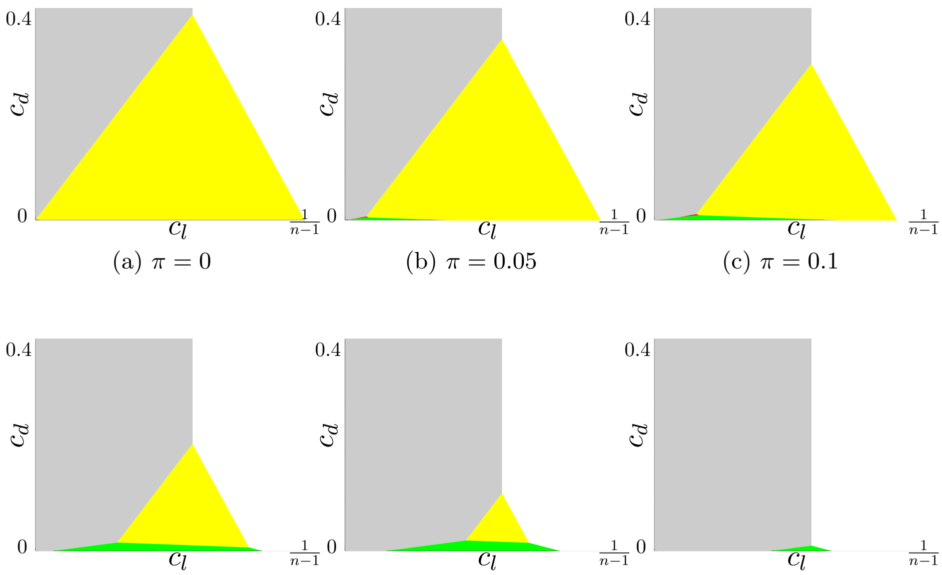


Figure 4: Equilibrium defended networks for all values of c_l and c_d, for different values of π. Grey area g^h, white area $g^\emptyset$, yellow area g^s, green area g^c, red area g^*.

as the networks that are equilibria in the boundary case of perfect defense are necessarily part of the solution here. However, in the following we will show that in the class of 2-connected networks there are more candidates to be found.

To provide some intuition why we should expect more possible equilibrium defended networks in the class of 2-connected networks, notice that the expected payoff of the CP-star is $1 - \pi$, while of the Harary graph of order 3 it is 1, net of costs. Any 2-connected network will yield a payoff of $1 - \pi^2$, net of costs. Clearly it is

$$1 \geq 1 - \pi^2 \geq 1 - \pi \qquad \forall \pi \in [0, 1],$$

such that we can see that whenever we can find a network 2-connected network with less links than the Harary graph, there will be a cost level such that for low π this network may be payoff-better in expected terms. On the other side, the network may make use of more links and more defense than the CP-star, as it yields a higher probability of success.

In order to find these 2-connected networks, we first need to identify optimal combinations of links and essential nodes. As link formation and units of defense are costly, the Designer needs to make sure that for any number of defended nodes, she uses the smallest possible number of links to create the network. In the following, we will call a network minimal if there does not exist another network with the same number of essential nodes and degree of connectedness and strictly less links.[7]

[7]Note that this definition is not the same as the non-existence of a non-essential link. It is

We will first present a lemma that essentially tells us that we have to distinguish not 2 groups of nodes (essential and non-essential), but 3 groups of nodes. The reason is that an essential node that is connected to non-essential nodes needs to have more links than one that is not. We will also collect some properties of nodes in each group.

Lemma 3.
In minimal 2*-connected networks with* p *essential nodes*

- *the* $q = n - p$ *non-essential nodes have at least* 2 *links, both to essential nodes.*
- *the* $p_q \leq p$ *essential nodes that are connected to non-essential nodes have at least 3 links.*
- *the remaining* $p - p_q$ *essential nodes have at least 2 links.*

The proof is the special case $k_a = 2$ of Lemma 5 in the following section.
Notice that we have already established two lower bounds for the number of links. First, any node has at least two links, yielding a lower bound of n links in the network. Second, the first bullet point of Lemma 3 yields a lower bound of $2q$ links. While for small q this lower bound may be even lower than the other one, we will in the subsequent proposition that it will be tight for large enough q.

The following proposition will now give the minimum number of links necessary to construct a 2-connected network with given number of essential nodes. The main idea will be to find the optimal number of nodes p_q of essential nodes to have links to non-essential nodes.

Proposition 2.
Let $n \geq 4$. *The minimum number of links in a 2-connected network with* $2 \leq q \leq n - 2$ *non-essential nodes is*

$$2q + [p - p_q + 1]\,\mathbb{1}_{\{p>p_q\}} + \mathbb{1}_{\{p=p_q, 3p>2q\}}, \tag{4.1}$$

where $N(q) \subseteq P$ *is the minimal set of neighbors such that*

$$p_q = \min\left\{p, \max\left\{2, \left\lfloor \frac{2(q+1)}{3} \right\rfloor\right\}\right\}. \tag{4.2}$$

In the proof, we first show that (4.1) is a lower bound for such networks by combining the ideas of Lemma 3, and then show that networks with these specifications can indeed be constructed.

easy to construct a network where all links are essential but there exists a different network with the same degree of connectedness and the same number of essential nodes, but less links.

Proof. Denote by P the set of essential nodes and Q the set of non-essential nodes. Let finally $P_q \subseteq P$ denote those nodes in P that are connected to nodes in Q.

To establish 4.1 as a lower bound for the number of links needed to construct a 2-connected network with q non-essential nodes, we observe that this is a special case of the more general Proposition 4, for $k_a = 2$. Then it is easy to understand that (5.1) collapses to (4.1), for $q > 1$.
To understand that (5.2) in case of $k_a = 2$ collapses to (4.2) observe first that

$$\begin{aligned}
&\left\lceil \frac{k_a(q+1) - 1 - \mathbb{1}_{\{k_a[p-(k_a q-3)/(k_a+1)]\ \text{even}\}}}{k_a+1} \right\rceil \\
\stackrel{k_a=2}{=} &\left\lceil \frac{2q+1-\mathbb{1}_{\{2[p-(2q-3)/3]\ \text{even}\}}}{3} \right\rceil \\
= &\left\lceil \frac{2q+1-\mathbb{1}_{\{q/3\in\mathbb{N}\}}}{3} \right\rceil \\
= &\left\lceil \frac{2q+\mathbb{1}_{\{q/3\notin\mathbb{N}\}}}{3} \right\rceil,
\end{aligned}$$

and then

$$\left\lceil \frac{2q+\mathbb{1}_{\{q/3\notin\mathbb{N}\}}}{3} \right\rceil = \left\lfloor \frac{2(q+1)}{3} \right\rfloor.$$

This already establishes the lower bound.

It is left to show that for each n, q a 2-connected network with links as in (4.1) exists. In order to understand that observe that for $p > p_q$ the number of nodes in p_q is $\lfloor 2(q+1)/3 \rfloor$, and thus the Harary graph of order 3 for p_q nodes has $q+1$ links.
Connecting each two nodes in P_q via at least one node in Q or a line of all nodes in $P \backslash P_q$. If $P_q = P$ and $3p > 2q$ then one direct link must be added. Figures 5a - 5c show examples of such minimal networks.
Finally, it is clear that this construction is only valid for $P_q \geq 4$. However, for smaller P_q the construction is straightforward, as is shown in Figures 5d - 5f. □

In Figure 6 the minimum number of links given by Proposition 2 are shown for all possible numbers of essential nodes, for the cases of $n = 15$ and $n = 17$ nodes in the network. What is left is to determine those networks within this set of candidates who may indeed be equilibrium solutions of the defense game, for specific combinations of costs c_l and c_d, as well as attack probability π. The following proposition will identify these networks by making use of the linearity of costs. The idea of the proof can also be seen in Figure 6. By linearity of costs, and as all 2-connected networks yield the same payoff of $1 - \pi^2$, net of costs, the possible equilibrium defended networks are those lying on the lower left side of the convex hull of all points given by Proposition 2 in the cost space.[8]

[8]Please be sure to understand that characterizations of minimal networks here leave room

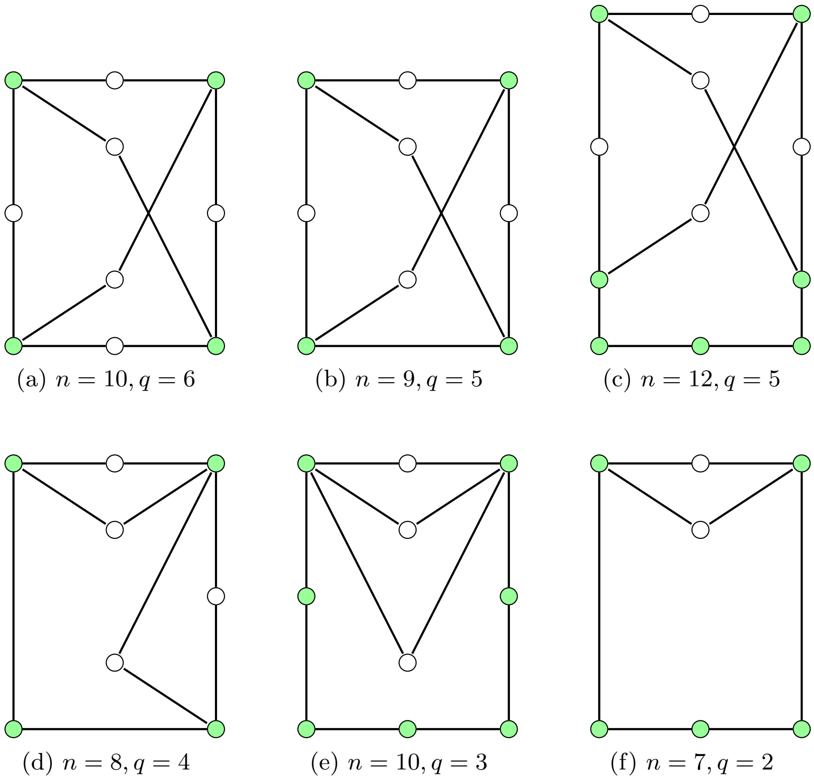


Figure 5: Minimal 2-connected networks for different n, q. Green colored nodes are essential.

Proposition 3.
Let $n \geq 7$. The set of possible equilibrium defended networks is given by

$$\Lambda(n) = \begin{cases} \{g^{\emptyset}, g^{h,3}, g^{s}, g^{c}, g^{*}\} & \text{if } n \equiv 0, 3, 4 \bmod 5 \\ \{g^{\emptyset}, g^{h,3}, g^{s}, g^{c}, g^{*}, \tilde{g}\} & \text{if } n \equiv 1, 2 \bmod 5, \end{cases}$$

where $g^{\emptyset}$ the undefended empty network, $g^{h,3}$ the undefended Harary graph of order 3, g^{s} the CP-star, g^{c} the completely defended circle, g^{*} the minimal 2-connected network for $q^{*} = \lceil (3n-2)/5 \rceil$, and $\tilde{g}$ a network such that $p = p^{*} + 2$ and $L(\tilde{g}) = L(g^{*}) - 1$.

Proof. The proof is structured in five parts.

1. For any $p < p^{*}$, the minimal network cannot be payoff-better than both g^{*} and $g^{h,3}$.
Observe first that q^{*} is the minimal number of essential nodes such that $p^{*}_{q} = p^{*}$,

not only for permutations of the set of nodes but also for all networks that have the same degree of connectedness and contain the same number of links as well as of essential and non-essential nodes. There are, for example, many ways to construct Harary graphs of higher orders.

as otherwise by Equation (4.2) it would need to hold that

$$
\begin{aligned}
& p_q^* = \left\lfloor \frac{2(\lceil \frac{3n-2}{5} \rceil + 1)}{3} \right\rfloor < n - \left\lceil \frac{3n-2}{5} \right\rceil \\
\iff \quad & n > \left\lfloor \frac{2(\lceil \frac{3n-2}{5} \rceil)}{3} + \frac{2}{3} + \left\lceil \frac{3n-2}{5} \right\rceil \right\rfloor \\
\iff \quad & n > \left\lfloor \frac{5}{3} \left\lceil \frac{3n-2}{5} \right\rceil + \frac{2}{3} \right\rfloor > \left\lfloor \frac{5}{3} \left(\frac{3}{5} n - \frac{2}{5} \right) + \frac{2}{3} \right\rfloor = n,
\end{aligned}
$$

what constitutes a contradiction. Equivalently, for $q = q^* - 1$

$$
\begin{aligned}
& p_q = \left\lfloor \frac{2(\lceil \frac{3n-2}{5} \rceil - 1 + 1)}{3} \right\rfloor = n - \left\lceil \frac{3n-2}{5} \right\rceil - 1 = p \\
\iff \quad & n = \left\lfloor \frac{5}{3} \lceil \frac{3n-2}{5} \rceil - 1 \right\rfloor \\
\iff \quad & n \leq \left\lfloor \frac{5}{3} (\frac{3n-2}{5} + 1) - 1 \right\rfloor = \left\lfloor n - \frac{2}{5} \right\rfloor = n - 1,
\end{aligned}
$$

what is again a contradiction.
Thus, in this network the number of links is $2q^* + \mathbb{1}_{\{3n>5q\}}$ by Equation (4.1), and as $q^* = \lceil (3n-2)/5 \rceil$ it is

$$
3n > 5q^* \qquad \iff \qquad n \equiv 2, 4 \text{mod } 5.
$$

Moreover, we know that for any g such that $q > q^*$ it is $|g| = 2q$, by Proposition 2, as $3n < 5(q^* + 1) \leq 5q$. For this network g to be an equilibrium solution, it needs to hold that

$$
|g| c_l + p c_d \leq |g^*| c_l + p^* c_d \tag{4.3}
$$

$$
|g| c_l + p c_d \leq \left\lceil \frac{3n}{2} \right\rceil c_l + 0 c_d, \tag{4.4}
$$

in order to be payoff-better than both g^* and the Harary graph of order 3. We show that these two equations cannot both be satisfied for g such that $q = q^* + 1$ and therefore $|g| = 2(q^* + 1)$. For all $\bar{q} > q^* + 1$ the proof is then a direct consequence. Equation (4.3) yields

$$
\begin{aligned}
& 2(q^* + 1) c_l + (p^* - 1) c_d \leq (2q^* + 1) c_l + p^* c_d \\
\iff \quad & c_l \leq c_d,
\end{aligned} \tag{4.5}
$$

where we used that $|g^*| \leq 2q^* + 1$. On the other hand, Equation (4.4) yields

$$
\begin{aligned}
& 2(q^* + 1) c_l + (p^* - 1) c_d \leq \left\lceil \frac{3n}{2} \right\rceil c_l \\
\iff \quad & \left(\left\lceil \frac{3n}{2} \right\rceil - 2 \left\lceil \frac{3n-2}{5} \right\rceil - 2 \right) c_l \geq \left(n - \left\lceil \frac{3n-2}{5} \right\rceil - 1 \right) c_d.
\end{aligned} \tag{4.6}
$$

Now, for both (4.5) and (4.6) to be satisfied at the same time it needs to hold that

$$\begin{aligned} &\left(\left\lceil\frac{3n}{2}\right\rceil - 2\left\lceil\frac{3n-2}{5}\right\rceil - 2\right) \geq \left(n - \left\lceil\frac{3n-2}{5}\right\rceil - 1\right) \\ \iff \quad &\left\lceil\frac{3n}{2}\right\rceil - n \geq \left\lceil\frac{3n-2}{5}\right\rceil + 1, \end{aligned}$$

while

$$\begin{aligned} &\frac{1}{2}n + 1 \geq \underbrace{\left\lceil\frac{3n}{2}\right\rceil}_{\leq \frac{3n}{2}+1} - n \geq \underbrace{\left\lceil\frac{3n-2}{5}\right\rceil}_{\geq \frac{3n-2}{5}} + 1 \geq \frac{3n-2}{5} + 1 \\ \iff \quad &\frac{1}{2}n \geq \frac{3n-2}{5} \\ \iff \quad &n \leq 4, \end{aligned}$$

what is a contradiction to the assumption that $n \geq 7$.

2. For $n > p > p^*$, the only possible networks satisfy $n - p = 3r + 2$ for $r \geq 0$.
For $n > p > p^*$, we know that $p_q < p$ and thus the number of links in a corresponding network g is

$$2q + [p - p_q + 1]\,,$$

for p_q defined as in (4.2). Comparing two minimal networks g and $\tilde{g}$ with q and $q + 1$ non-essential nodes, it is

$$\begin{aligned} &2(q+1) + [p - 1 - p_{q+1} + 1] - 2q - [p - p_q + 1] \\ =&1 - (p_{q+1} - p_q), \end{aligned}$$

such that whenever $p_{q+1} = p_q + 1$ then g cannot be an equilibrium defended network because for $\tilde{g}$ corresponding to $q+1$ the network has the same number of links while having one essential node less.
Observe now that for $p_q < p$ we have that $p_q = \lfloor 2(q+1)/3 \rfloor$ and thus we know that the only possible equilibrium defended networks satisfy $q = 3r + 2$ for $r \geq 0$.

3. For any of these, only the maximum (i.e. the one with maximal r) is possible. Denote this by g^r.
Take two networks g^{r-1} and g^r, where $0 \leq r - 1 < r$ and $q^{r-1} = 2 + 3(r-1)$, $q^r = 2 + 3r$. For g^{r-1} to be an equilibrium defended network for some cost level it needs to be payoff-better than both g^r and the completely defended circle.

$$\begin{aligned} &(n - 2 - 3(r-1))c_d + (n + r)c_l \leq (n - 2 - 3r)c_d + (n + r + 1)c_l \\ &(n - 2 - 3(r-1))c_d + (n + r)c_l \leq nc_d + nc_l \end{aligned}$$

what translates to

$$\begin{aligned} 3c_d &\leq c_l \\ (2 + 3(r-1))c_d &\geq rc_l, \end{aligned}$$

what cannot be both satisfied at the same time.

4. For $n \equiv 3, 4 mod\ 5$, $n - p^* = 3(r+1) + 2$, thus g^r cannot be an equilibrium defended network by the arguments of step 3.
Let $n \equiv 3, 4 \bmod 5$, then

$$q^* = \left\lceil \frac{3n-2}{5} \right\rceil = \left\lceil \frac{3n}{5} \right\rceil.$$

For $n = 8$, it is $q^* = 5 = 3 \cdot 1 + 2$, and for $n = 8 + 5r, r \geq 1$ it is thus

$$q^* = \left\lceil \frac{3n}{5} \right\rceil = \left\lceil \frac{3 \cdot 8 + 15r}{5} \right\rceil = 3(r+1) + 2.$$

For $n = 9 + 5r$ the same argument yields the result.

both5. For $n \equiv 0 mod\ 5$, $n - p^* = 3r$, while $|g^*| = |g^r|$, thus g^r cannot be an equilibrium defended network, as g^* will be cheaper for all positive costs c_d.
Let $n \equiv 0 \bmod 5$, then

$$q^* = \left\lceil \frac{3n-2}{5} \right\rceil = \frac{3n}{5},$$

and thus for $n = 5r, r \geq 1$ it is $q^* = 3r$.
Now, it is by (4.1)

$$|L(g^*)| = 2q^*,$$

as $3(5r - 3r) = 2 \cdot 3r$, and

$$|L(g^r)| = 2(q^* - 1) + 2 = 2q^*,$$

as in g^r it is $p_q = p - 1$.

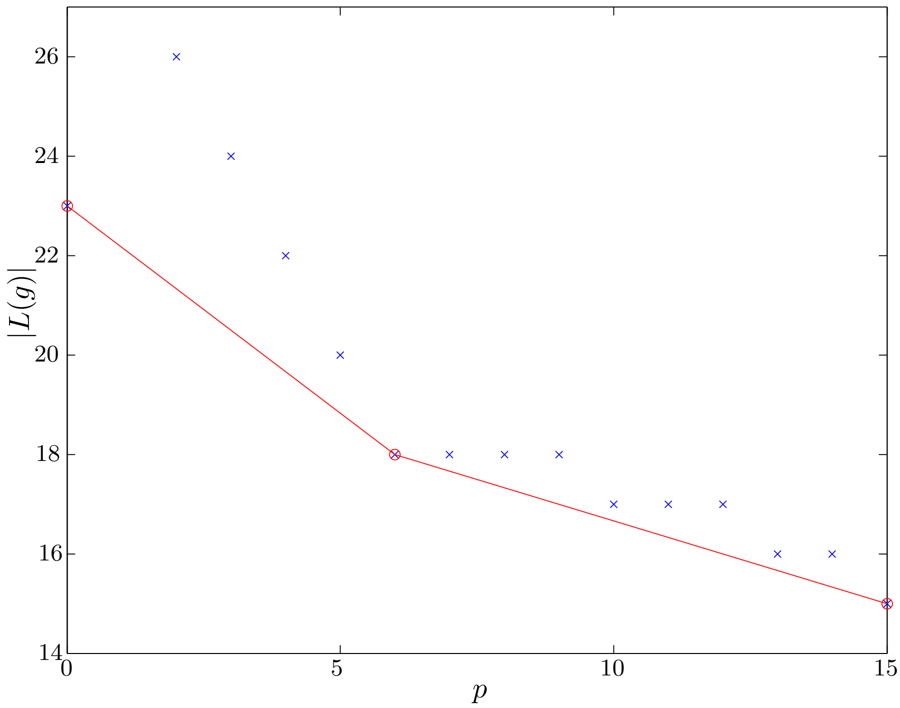
□

In Figure 6, for all numbers of essential nodes, the minimum number of necessary links to construct a 2-connected network is given, and the possible equilibrium defended networks as given in Proposition 3 are identified. This shows that indeed all of the networks given in Proposition 3 are possible equilibrium solutions of the defense game.

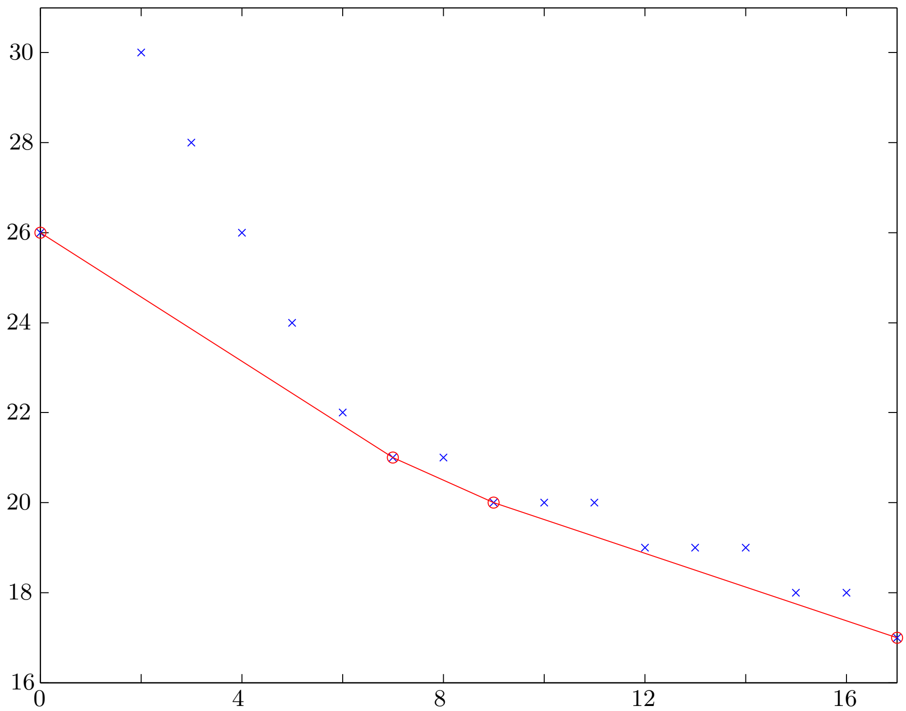
Note finally that in the definition of the game in Section 2 we did not allow the Adversary to attack the same node twice instead of attacking two different nodes. However, in case of $k_a = 2$ we can now see that this would not change the results qualitatively, as the only possibility would be to attack the CP-star center node twice. However, as

$$(1 - \pi)^2 < 1 - \pi^2 \iff \pi < 1,$$

the set $\Lambda(n)$ of possible equilibrium defended networks would stay the same.

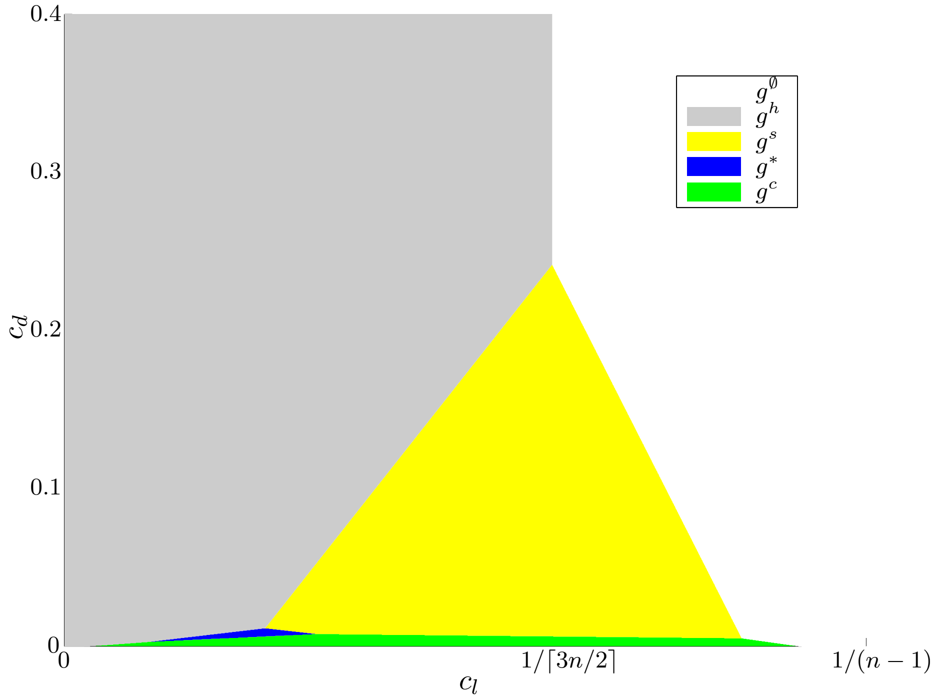


(a) $n = 15$

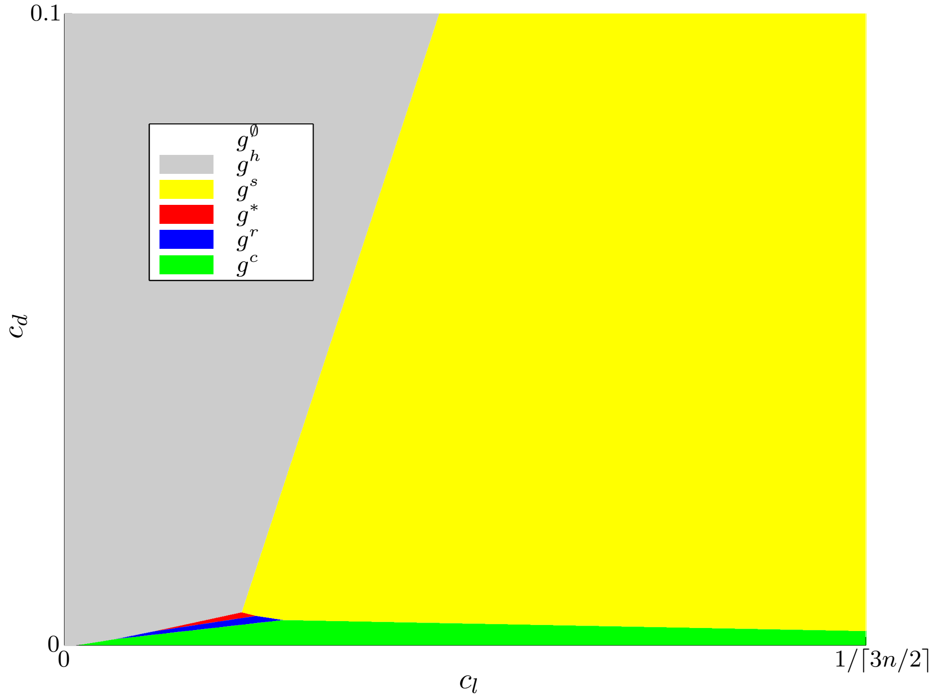


(b) $n = 17$

Figure 6: Possible 2-connected equilibrium defended networks for $n = 15, n = 17$.



(a) $n = 15, \pi = 0.15$



(b) $n = 17, \pi = 0.08$

Figure 7: Equilibrium defended networks for $n = 15, n = 17$, for a given defense probability π and all cost combinations.

Limit behavior

In Section 3 we saw that the only defended network, the CP-star, was not part of the solution set if the number of nodes n was big enough compared to the destruction probability π. We now argue that this result does not hold anymore already in the present case of $k_a = 2$. The main reason for this is that the difference in links between the star and the unprotected Harary graph is now growing in n. Precisely, we have

$$\begin{aligned} u_D(g^s, \{c\}, \{c\}) &= 1 - \pi - (n-1)c_l - c_d, \\ u_D(g^{h,3}, \emptyset, A) &= 1 - \lceil \tfrac{3n}{2} \rceil c_l, \end{aligned}$$

and thus

$$\begin{aligned} & u_D(g^{h,3}, \emptyset, A) \geq u_D(g^s, c, c) \\ \iff \quad & \underbrace{\left[\lceil \tfrac{3n}{2} \rceil - (n-1)\right]}_{\overset{n\to\infty}{\longrightarrow} \frac{n}{2}} c_l \leq \pi + c_d, \end{aligned}$$

such that we see that here in the limit of $n \to \infty$ only for zero link costs the Harary graph of order 3 can be payoff-better than the CP-star.

Again, we can instead of u_D alternatively consider a utility function where the payoff grows with the number of nodes in the network. As in the previous Section, consider a utility function $\tilde{u}_D$, where in case of a connected residual network the payoff is

$$\tilde{u}_D(g, D, A) = n - c_l|g| - c_d|D|,$$

and consider again the payoff of g^s and $g^{h,3}$. One gets

$$\begin{aligned} & \tilde{u}_D(g^{h,3}, \emptyset, A) \geq \tilde{u}_D(g^s, \{c\}, \{c\}) \\ \iff \quad & \left[\lceil \tfrac{3n}{2} \rceil - (n-1)\right] c_l \leq n\pi + c_d. \end{aligned}$$

This yields that in the limit $n \to \infty$ the Harary graph of order 3 is payoff-better than the CP-star if and only if

$$\frac{1}{2} c_l \leq \pi,$$

such that again the CP-star is a possible solution of the defense game for any network size.

Having fully characterized the set of possible equilibrium defended networks for an attack budget of $k_a = 2$, we now have an idea how to approach the problem for a general attack budget. However, while the ideas will stay the same, there will arise some problems forcing us to only partially characterize the solution in the general setup.

5 Attack Budget k_a

We now want to accomplish the same analysis of the previous section for a general attack budget k_a. The idea will be the same as before, such that we need to characterize the minimum number of links needed to construct a k_a-connected network with a given number of p essential links. Observe however that we now have to consider a lot more networks, as for any degree of connectedness $1 \leq k < k_a$ there may still be networks that are part of the solution. This fact makes it impossible for us to completely characterize the set of possible equilibrium defended networks $\Lambda(n)$. However, we will show that some obvious candidates are part of the solution and we will furthermore define a set of networks that we can show to include all possible k_a-connected equilibrium defended networks.

We start again with a lemma that comprises some first and easily derivable results, where we see that the obvious candidates, that is the empty network, the CP-star and the Harary graph of $k_a + 1$ are again part of the solution.

Lemma 4.
Let the attack budget be k_a. The following statements hold true.

- *The only possible non-connected equilibrium defended network is the undefended empty network $g^{\emptyset}$.*
- *The only possible 1-connected equilibrium defended network is the CP-star g^s.*
- *The only possible $(k_a + 1)$-connected equilibrium defended network is the undefended Harary graph of order $k_a + 1$, g^{h,k_a+1}.*

Again we omit a proof, as it is structurally equivalent to the proof of Proposition 1 in Section 3.

Instead, we turn to characterize more possible equilibria of the defense game. Intuitively it should be clear that in the general case we get a more diverse set of possible equilibrium defended networks. While it is easy to characterize the networks that are non-connected, 1-connected or $(k_a + 1)$ connected (as we saw in the previous Lemma), we now also have to think about all networks of connectivity $2, 3, ..., k_a$, and each time with any number of essential nodes.

To assess this problem we aim to partially characterize the set $\Lambda(n, k_a)$ of possible equilibrium defended networks by recursively identifying a class of networks $\Gamma(n, k_a)$ that we can show to include $\Lambda(n, k_a)$. The idea will be the same as in the previous section: We identify the minimal k_a-connected networks for any number of essential nodes and subsequently identify those that may be a solution to the game by exploiting the linearity of costs.

The following lemma, equivalently to Lemma 3 assesses the role of non-essential nodes in a k_a-connected network.

Lemma 5.
In minimal k_a-connected networks with p essential nodes, the $q = n - p$ non-essential nodes have at least k_a links, all of them to essential nodes. Thus, any such network has at least $k_a q$ links.

Proof. Remember that it is clear that any node in a k_a-connected network has at least k_a links, as the deletion of all neighbors leaves a node isolated and thus disconnects a network.
Let now $\bar{g}$ be a network such that $P = \{1, .., p\}$ are the essential nodes. Suppose further that $\bar{g}$ contains a link between two nodes $i, j \notin P$. Then it is clear that $\bar{g} - ij$ is again k_a-connected with the same set of separators. To see that note that by Menger's Theorem there are at least k_a node-disjoint paths between any two nodes in $\bar{g}$, and as $(N - P)$ are non-separating there are also at least k_a node-disjoint paths between any two nodes in $N \setminus \{i\}$ for any $i \in (N - P)$. Then it is already clear that all of these paths remain existing in $\bar{g} - ij$, leaving the connectivity and the set of non-separators unchanged.

Suppose now there are no links between non-separating nodes. Knowing that any node in a k_a-connected network has at least k_a links we know that there are $k_a q$ nodes from non-separating to separating nodes, what yields the result. □

We now characterize all minimal k_a-connected equilibrium defended networks. Remember first that the minimal k_a-connected network has $\lceil k_a n/2 \rceil$ links. We denote this network by g^{h,k_a}, the Harary graph of order k_a. It is clear that in this network all nodes are essential, such that the possible equilibrium defended network is the fully defended Harary graph.

Similar to the case of $k_a = 2$ in Section 4, we can now deduce the number of links necessary to construct a k_a-connected network with exactly p essential nodes, i.e. p nodes contained in separating k_a-cuts. Remember that by Lemma 5 the $q = n - p$ non-essential nodes all have exactly k_a links, all of them to essential nodes, yielding a lower bound of $k_a q$ links. Moreover, we know that any neighbor of a non-essential node has to have at least $k_a + 1$ links. Thus the idea is, again similar to Section 4, to determine the optimal number p_q of essential nodes connected to non-essential nodes.
A remaining issue is the construction of the networks we find. The idea in the previous section was to establish the number of links as a lower bound and subsequently provide a construction algorithm for a network of this degree of connectedness, as well as numbers of essential nodes and links. In this general case we will also provide a construction algorithm for a network and argue that it is a valid candidate. However, a proof for this conjecture is not provided. Notice therefore that the following proposition only provides a lower bound on the number of links, while afterwards we will add the conjecture and the construction algorithm for the networks.

Proposition 4.
Let n be big enough, then a lower bound for the minimum number of links in a k_a-connected network with $1 \leq q \leq n - k_a$ non-essential nodes is given by

$$G(p_q) = \begin{cases} k_a(q+1) + \left\lceil \frac{\max\{0,(k_a+1)p_q - k_a(q+1)\} + k_a(p-p_q-1)}{2} \right\rceil & \text{if } p_q < p, \\ k_a q + \left\lceil \frac{\max\{0,(k_a+1)p_q - k_a q\}}{2} \right\rceil & \text{if } p_q = p, \end{cases} \tag{5.1}$$

where $p_q = |P_q|$ and $P_q \subseteq P$ is the set of neighbors of non-essential nodes, such that

$$p_q = \min\left\{p, \max\left\{k_a, \left\lceil \frac{k_a(q+1) - 1 - \mathbb{1}_{\{k_a[p-(k_a q-3)/(k_a+1)] \ \text{even}\}}}{k_a+1} \right\rceil \right\}\right\}. \tag{5.2}$$

Proof. 1. $G(p_q)$ as defined in (5.1) is a lower bound for a k_a-connected network of q non-essential nodes with p_q neighbors.
By Lemma 5, it is clear that non-essential neighbors have exactly k_a links. Further, there need to be at least k_a links between the p_q neighbors of non-essential nodes and the remaining $p-p_q$ essential nodes, yielding $k_a q + k_a \mathbb{1}_{\{p>p_q\}}$ links. Moreover, the p_q neighbors of non-essential nodes need to have at least $k_a + 1$ links, while the remaining $p - p_q$ nodes need to have at least k_a links, what together yields (5.1).

2. To determine the minimum number of links, we need to find the p_q that minimizes (5.1). It is enough to find the minimum p_q such that $G(p_q) \leq G(p_q + 1)$ and to show that for this p_q it is also $G(p_q) \leq G(p_q - 1)$.

Define $F(p_q)$ to be equal to $G(p_q) - k_a q$ when disregarding the ceiling function, i.e.

$$F(p_q) = \begin{cases} k + \frac{\max\{0,(k_a+1)p_q - k_a(q+1)\} + k_a(p-p_q-1)}{2} & \text{if } p_q < p, \\ \frac{\max\{0,(k_a+1)p_q - k_a q\}}{2} & \text{if } p_q = p, \end{cases}$$

then for $G(p_q) \leq G(p_q+1)$, it is either $F(p_q) \leq F(p_q+1)$ or $F(p_q) = F(p_q+1) + \frac{1}{2}$ and

$$\max\{0, (k_a+1)p_q - k_a q - k_a \mathbb{1}_{\{p>p_q\}}\} + k_a(p - p_q - 1)\mathbb{1}_{\{p>p_q\}} \tag{5.3}$$

even.

Consider the first of these two cases. We need to distinguish between $p_q < p - 1$ and $p_q = p - 1$.

- $p_q < p - 1$. Then

$$\begin{aligned} & F(p_q) \leq F(p_q+1) \\ \iff \quad & \max\{0, (k_a+1)p_q - k_a(q+1)\} + k_a(p - p_q - 1) \\ & \quad \leq \max\{0, (k_a+1)(p_q+1) - k_a(q+1)\} + k_a(p - p_q - 2) \\ \iff \quad & \max\{0, (k_a+1)p_q - k_a(q+1)\} + k_a \leq \max\{0, (k_a+1)p_q - k_a q + 1\} \\ \iff \quad & p_q \geq \frac{k_a(q+1) - 1}{k_a+1}. \end{aligned}$$

- $p_q = p - 1$. Then

$$\begin{aligned} & F(p_q) \leq F(p_q+1) \\ \iff \quad & k_a + \max\{0, (k_a+1)p_q - k_a(q+1)\}/2 \\ & \quad \leq \max\{0, (k_a+1)(p_q+1) - k_a q\}/2 \\ \iff \quad & 2k_a + \max\{0, (k_a+1)p_q - k_a(q+1)\} \\ & \quad \leq \max\{0, (k_a+1)p_q - k_a q + k_a + 1\}/2 \\ \iff \quad & p_q \geq \frac{k_a(q+1)-1}{k_a+1}. \end{aligned}$$

We observe that the two boundaries coincide.

Consider instead the second case, such that $F(p_q) = F(p_q+1) + \frac{1}{2}$ and (5.3) even. We again distinguish $p_q < p-1$ and $p_q = p-1$

- $p_q < p - 1$. Then

$$\begin{aligned} & F(p_q) - F(p_q+1) = \tfrac{1}{2} \\ \iff \quad & [\max\{0, (k_a+1)p_q - k_a(q+1)\} + k_a(p - p_q - 1)]/2 \\ & \quad - [\max\{0, (k_a+1)p_q - k_a q + 1\} - k_a(p - p_q - 2)]/2 = \tfrac{1}{2} \\ \iff \quad & \max\{0, (k_a+1)p_q - k_a(q+1)\} + k_a \\ & \quad - \max\{0, (k_a+1)p_q - k_a q + 1\} = 1 \\ \iff \quad & \max\{0, (k_a+1)p_q - k_a q + 1\} \\ & \quad - \max\{0, (k_a+1)p_q - k_a(q+1)\} = k_a - 1 \\ \iff \quad & p_q = \frac{k_a(q+1)-2}{k_a+1}. \end{aligned}$$

- $p_q = p - 1$. Then

$$\begin{aligned} & F(p_q) - F(p_q+1) = \tfrac{1}{2} \\ \iff \quad & k_a + \big[\max\{0, (k_a+1)p_q - k_a(q+1)\} \\ & \quad - \max\{0, (k_a+1)p_q - k_a q + k_a + 1\}\big]/2 = \tfrac{1}{2} \\ \iff \quad & \max\{0, (k_a+1)p_q - k_a q + k_a + 1\} \\ & \quad - \max\{0, (k_a+1)p_q - k_a(q+1)\} = 2k_a - 1 \\ \iff \quad & p_q = \frac{k_a(q+1)-2}{k_a+1}. \end{aligned}$$

Notice again that the two boundaries coincide. Finally, we plug $\bar{p}_q = \frac{k_a(q+1)-2}{k_a+1}$

into (5.3) and notice that the maximum functions are zero, such that

$$\begin{aligned}
&\max\{0,(k_a+1)\bar{p}_q-k_aq-k_a\mathbb{1}_{\{p>\bar{p}_q\}}\}+k_a(p-\bar{p}_q-1)\mathbb{1}_{\{p>\bar{p}_q\}}\\
=&k_a\left(p-\frac{k_a(q+1)-2}{k_a+1}-1\right)\\
=&k_a\left(p-\frac{k_aq-3}{k_a+1}\right),
\end{aligned} \tag{5.4}$$

such that we know that the condition simplifies to (5.4) being even.

It is left to show that for the optimal p_q as defined in (5.2) it is also $G(p_q) \leq G(p_q-1)$. We will consider the sufficient criterion $F(p_q) \leq F(p_q-1)$ and show that this is always satisfied for p_q given by (5.2). Observe that we only need to consider $p_q < p$ here, as otherwise the above the above would yield $G(p-1) > G(p)$ and we would directly know that $p_q = p$ is optimal. Thus, for $p_q < p$ it is

$$\begin{aligned}
&F(p_q) \leq F(p_q-1)\\
\iff\quad &\max\{0,(k_a+1)p_q-k_a(q+1)\}+k_a(p-p_q-1)\\
&\leq \max\{0,(k_a+1)p_q-k_a(q+2)-1\}+k_a(p-p_q)\\
\iff\quad &\max\{0,(k_a+1)p_q-k_a(q+1)\}\\
&\quad-\max\{0,(k_a+1)p_q-k_a(q+2)-1\}\leq k_a\\
\iff\quad &p_q\leq\frac{k_a(q+2)}{k_a+1},
\end{aligned}$$

and it is clear that for all $k_a \geq 1$

$$\frac{k_a(q+1)-2}{k_a+1}\leq\frac{k_a(q+2)}{k_a+1}.$$

Altogether, we have now shown that (5.1) and (5.2) yield a lower bound for a k_a-connected network with p critical links. □

In the following it is argued that the lower bound given in Proposition 4 might be tight, meaning that there indeed exist networks of any given degree of connectedness and number of essential nodes, that have exactly the number of links given in (5.1) and (5.2) of Proposition 4. We formally give this statement in the following conjecture and will subsequently present a corresponding construction algorithm.

Conjecture 1.
$G(p_q)$ as given by (5.1) and (5.2) is the minimum number of links in a k_a-connected network with $1 \leq q \leq n-k_a$ non-essential nodes.

The idea is to construct such a k_a-connected network with exactly $G(p_q)$ links. Proposition 4 then guarantees that there cannot be a network with the given characteristics and less links.

Consider the following construction. Define the set of non-essential nodes as $Q = \{n_1, ..., n_q\}$, their neighbors $P_q = \{c_1, ..., c_{p_q}\}$ and all other essential nodes $P \backslash P_q$ as $\{c_{p_q+1}, ..., c_p\}$. Consider now the following construction. Let $p_q < q$ and $p_q = p$. Then for $i = 1, ..., p$, n_i is connected to $c_i, ..., c_{i+k_a-1}$ (mod p), while for $i = p+1, ..., q$, n_i is connected to $c_{i-p}, c_{i-p+\lceil p/k_a \rceil}, ..., c_{i-p+(k_a-1)\lceil p/k_a \rceil}$ (mod p). If $q - p < \lceil \frac{p}{k_a} \rceil$, then connect the smallest c_i with k_a links to $c_{i+\lceil p/k_a \rceil}$ (mod p), until all essential nodes have $k_a + 1$ neighbors.

Let $p_q < q$ and $p_q < p$. Again, for $i = 1, ..., p_q$, n_i is connected to $c_i, ..., c_{i+k_a-1}$ (mod p_q), while for $i = p_q + 1, ..., q$, n_i is connected to

$$c_{i-p_q}, c_{i-p_q+\lceil p_q/k_a \rceil}, ..., c_{i-p_q+(k_a-1)\lceil p_q/k_a \rceil} \pmod{p_q}.$$

Now, the nodes $c_{p_q+1}, ..., c_p$ form a Harary graph of order k_a in the following way. Consider the nodes $c_{q-p_q+1}, c_{q-p_q+1+\lceil p_q/k_a \rceil}, ..., c_{q-p_q+1+(k_a-1)\lceil p_q/k_a \rceil}$ (mod p_q) as being one node $\tilde{c}$. Then $c_{p_q+1}, ..., c_p$ and $\tilde{c}$ form a Harary graph of order k_a, where each node in $\tilde{c}$ gets one connection. Finally, if one of the nodes in $c_{p_q+1}, ..., c_p$ and $\tilde{c}$ has $k_a + 1$ links and there exists a node in $c_1, ..., c_{p_q}$ having only k_a links, w.l.o.g. the node with $k_a + 1$ links will be $\tilde{c}$, where the node with smallest index in $c_1, ..., c_{p_q}$ having only k_a links is added to $\tilde{c}$.
Finally, if still nodes in $c_1, ..., c_{p_q}$ have only k_a links, then connect the smallest c_i with k_a links to $c_{i+\lceil p/k_a \rceil}$ (mod p_q), until all essential nodes have $k_a + 1$ neighbors.
In this construction, if $c_{p_q+1}, ..., c_p$ are less than k_a nodes, the formation of a Harary graph is not possible. This is the case as

$$\begin{aligned} & \left\lceil \frac{k_a(p - p_q + 1)}{2} \right\rceil > \frac{(p - p_q + 1)(p - p_q)}{2} \\ \iff \quad & \frac{k_a(p - p_q + 1)}{2} > \frac{(p - p_q + 1)(p - p_q)}{2} \;\vee\; \left[k_a = p - 1 \;\wedge\; k_a p \text{ odd}\right], \end{aligned}$$

where the latter case is a contradiction, while the first case yields the above condition. In this case $c_{p_q+1}, ..., c_p$ form a completely connected subgraph of and all of them are additionally connected to $k_a - (p - p_q - 1)$ nodes from $c_{q-p_q+1}, c_{q-p_q+1+\lceil p_q/k_a \rceil}, ..., c_{q-p_q+1+(k_a-1)\lceil p_q/k_a \rceil}$ (mod p_q) as well as those nodes in $c_1, ..., c_{p_q}$ having only k_a links.

Finally, let $p_q \geq q$. Observe that for $k_a \geq 3$ this can only be the case if $q \leq k_a$, as from (5.2) it follows that

$$\begin{aligned} & p_q = k_a \\ \iff \quad & \left\lceil \frac{k_a(q+1) - 1 - \mathbb{1}_{\{k_a[p-(k_a q-3)/(k_a+1)] \text{ even}\}}}{k_a + 1} \right\rceil \leq k_a, \end{aligned}$$

and for $q = k_a + 1$ it is

$$\begin{aligned}
&\left\lceil \frac{k_a(k_a+2) - 1 - \mathbb{1}_{\{k_a[p-(k_a q-3)/(k_a+1)] \text{ even}\}}}{k_a+1} \right\rceil \\
&= \left\lceil k_a + \frac{k_a - 1 - \mathbb{1}_{\{k_a[p-(k_a q-3)/(k_a+1)] \text{ even}\}}}{k_a+1} \right\rceil \\
&\overset{k_a \geq 2}{=} k_a + 1.
\end{aligned}$$

Having understood this, we deduce that $q \leq p_q = k_a$, as any non-essential node has to be connected to k_a distinct essential nodes.
The construction in this case works as follows. All nodes in Q are connected to all nodes in P_q, such that the k_a nodes in P_q each have q links. Notice that these nodes need at least one more link each.
Now, consider as before all nodes in P_q as one artificial node $\tilde{c}$. Then all nodes in $P \backslash P_q$ together with $\tilde{c}$ form a Harary graph of order k_a, where the k_a links of $\tilde{c}$ are divided such that every node get one of the links. Finally, there might be links missing in P_q. If every node misses one link, then add links in pairs. If they lack more links, then add a Harary graph of this degree for all nodes in P_q.

Argumentation for Conjecture 1. Consider first a network constructed as above for the case where $p_q = q$. Consider furthermore $k_a = 3$. For such an example consider Figure 8. In the following we will distinguish non-essential nodes Q between "outside" nodes $n_1, ..., n_{p_q}$, and "inside" nodes $n_t, t > p_q$. This terminology is motivated by the above construction, see Figures 8 and 9.

We need to show that there exist three node-distinct paths between any two nodes in the network, while not using one arbitrary non-essential node.
This can however be heavily simplified. We will instead show that there exist 1) three different paths between any two nodes in P_q, 2) node-distinct paths from any 3 nodes in P_q to any other 3 nodes in P_q, and 3) node-distinct paths from any node in P_q to any 3 nodes in P_q, while we are always free not to use one arbitrary node in Q.

1) Take any two nodes in P_q, call them c_s and c_t. W.l.o.g. let $s < t$ and call $c_{s+1}, ..., c_{t-1}$ the right-hand side and $c_{t+1}, .., c_{s-1}$ the right-hand side. Observe that 3 links leave c_s to the outside, call them the left, middle and right link. The middle link can leave to the right (node c_{s+1}) or to the left (c_{s-1}). Analogously, 3 links leave c_t to the outside. Distinguish now the following cases.

- c_s and c_t are directly connected through an inside node. Then there are additionally two paths left and one path right on the outside, or vice versa.

- There is an inside connection from c_s to some node in P_q on the left-hand side, and an inside connection from c_t to some node on the right-hand side. Leave c_s with two paths to the right, one of which will hit the node connected to c_t via the inside. The other reaches c_t via the right link.

Leave c_s to the left and via the inside connection. Both paths will hit c_t from the left-hand side, one via the left, one via the middle link.

- There is an inside connection from c_s to some node in P_q on the left-hand side, and an inside connection from c_t to some node on the left-hand side. Leave c_s with two paths to the right-hand side, hitting c_t through the right and the middle link. Leave c_s to the left to encounter the node connected to c_t via the inside. Leave c_s via the inside, and continue on the left-hand side to hit c_t via the left link.

In any case, there are 4 node-distinct paths, such that 3 paths remain whenever one chooses to omit some non-essential node.

2) Take any node c_s and any group $(c_{t_1}, c_{t_2}, c_{t_3})$, all within P_q. Again, one can find node-distinct paths from c_s to the three nodes $(c_{t_1}, c_{t_2}, c_{t_3})$, while it is possible not to use one arbitrary non-essential node. If $t_1, t_2, t_3 \leq p_q$, then the proof is as in case (1) above, noticing that the three nodes lie on the circle of essential nodes connected to non-essential nodes, such that there is a left, center and right node.
If on the other hand for some t_i it is $t_i > p_q$, then exchange c_{t_i} for either the node c_u such that $u \leq p_q$ and c_{t_i} and c_u are connected, or if this node is also part of $(c_{t_1}, c_{t_2}, c_{t_3})$, any other node $\tilde{c}_u$ that is not part of $(c_{t_1}, c_{t_2}, c_{t_3})$ and connected to a node c_r such that $r > p_q$. Like this we again reduced the problem to be similar to case (1).

3) Take any two groups $(c_{s_1}, c_{s_2}, c_{s_3})$ and $(c_{t_1}, c_{t_2}, c_{t_3})$ of nodes within P_q. Again, the same arguments can be used to show that there exist node-distinct paths from c_{s_i} to c_{t_i}, $i = 1, 2, 3$, with the possibility not to use any non-essential node. The most difficult case is if $(c_{s_1}, c_{s_2}, c_{s_3})$ and $(c_{t_1}, c_{t_2}, c_{t_3})$ lie on two different sides of the circle $c_1, .., c_{p_q}$ (if some $s_1, s_2, s_3, t_1, t_2, t_3 > p_q$, the argument works as in case (2) above). As in case (1) there are two node-distinct paths to the left and two to the right on the outside, while one path may start via the inside. Thus indeed all arguments work as in case (1).

Due to the symmetry of the construction the extension of the above to higher degrees of connectedness is straightforward. □

Finally, we want to use the result of Conjecture 1 to characterize the possible equilibrium defended networks $\Lambda(n, k_a)$ in case of an arbitrary attack budget k_a. To this end we will recursively define a set of network that we can show to include $\Lambda(n, k_a)$. These networks will be those with minimum number of essential nodes for any number of links more than the Harary graph g^{h,k_a} of order k_a, and less than g^*, which is again the network with the minimum number of non-essential nodes such that $p = p_q$. Formally, the conjecture is the following.[9]

[9] Notice that despite the inclusion of a proof the following result remains a conjecture, as this proof relies on the validity of Conjecture 1.

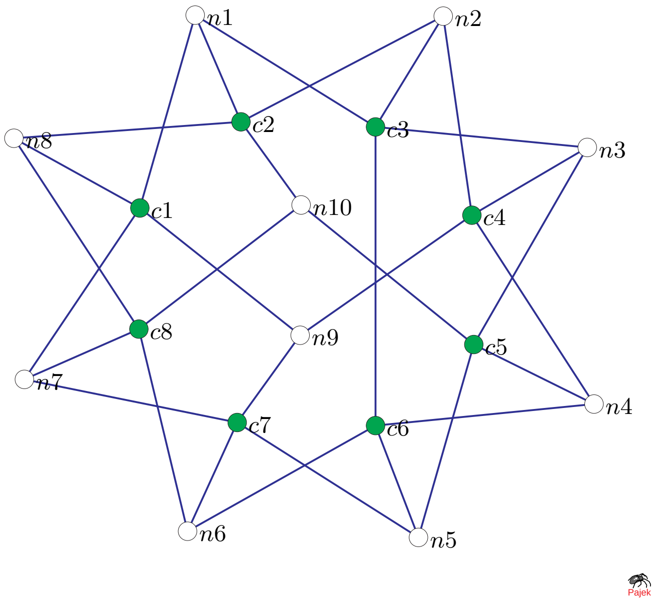


Figure 8: A 3-connected network of 18 nodes, where 8 are essential and 10 are non-essential, while $p_q = p$. There are 4 distinct paths between any two essential nodes.

Conjecture 2.
The set of possible k_a-connected equilibrium defended networks is a subset of

$$\Gamma(n, k_a) = \{g_1^{min}, g_2^{min}, ..., g^*\},$$

where g^* is the minimal network with minimum number of non-essential nodes such that $p = p_q$, while g_l^{min} for $1 \leq l < |g^*| - \lceil \frac{nk_a}{2} \rceil$ is the network with $\lceil \frac{nk_a}{2} \rceil + l$ links and the minimum possible number of essential nodes.

To understand the idea of Conjecture 2, consider Figures 10a and 10a. The networks that are possible equilibria of the game are those with optimal combinations of essential nodes and number of links on the flatter right part of the graphs. However, one can see that not all of these are necessarily in $\Lambda(n, k_a)$.

Proof. It is clear that no network with more essential nodes than g^* other than the g_l^{min} for $1 \leq l < |g^*| - \lceil \frac{nk_a}{2} \rceil$ can be equilibrium defended networks, as there always exist one g_l^{min} with the same number of links and less essential nodes.

We need to prove that no network with less essential nodes than g^* can be an equilibrium defended network. Note first that for these networks it is always

$$k_a q \geq (k_a + 1)p, \tag{5.5}$$

as p^* is the largest to satisfy $p_q^* = p^*$. Let us therefore consider the largest p to satisfy (5.5), that is let

$$p = \left\lfloor \frac{kn}{2k+1} \right\rfloor.$$

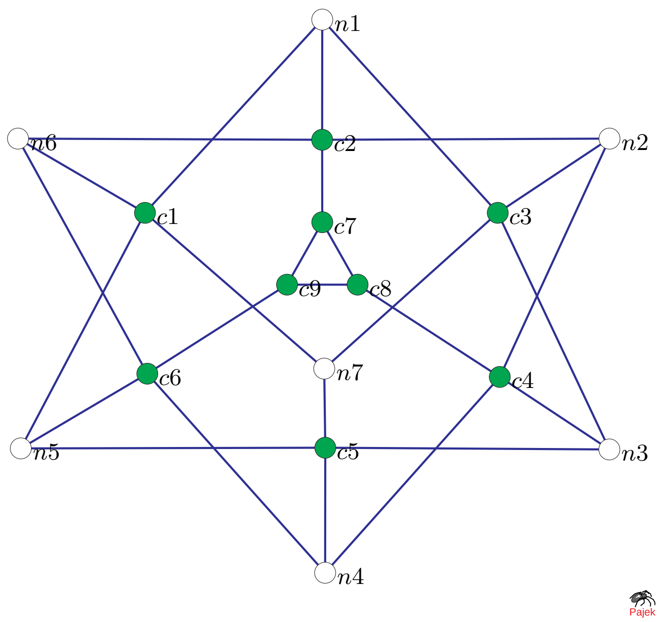


Figure 9: A 3-connected network of 16 nodes, where 9 are essential and 7 are non-essential, while $p_q = 6$.

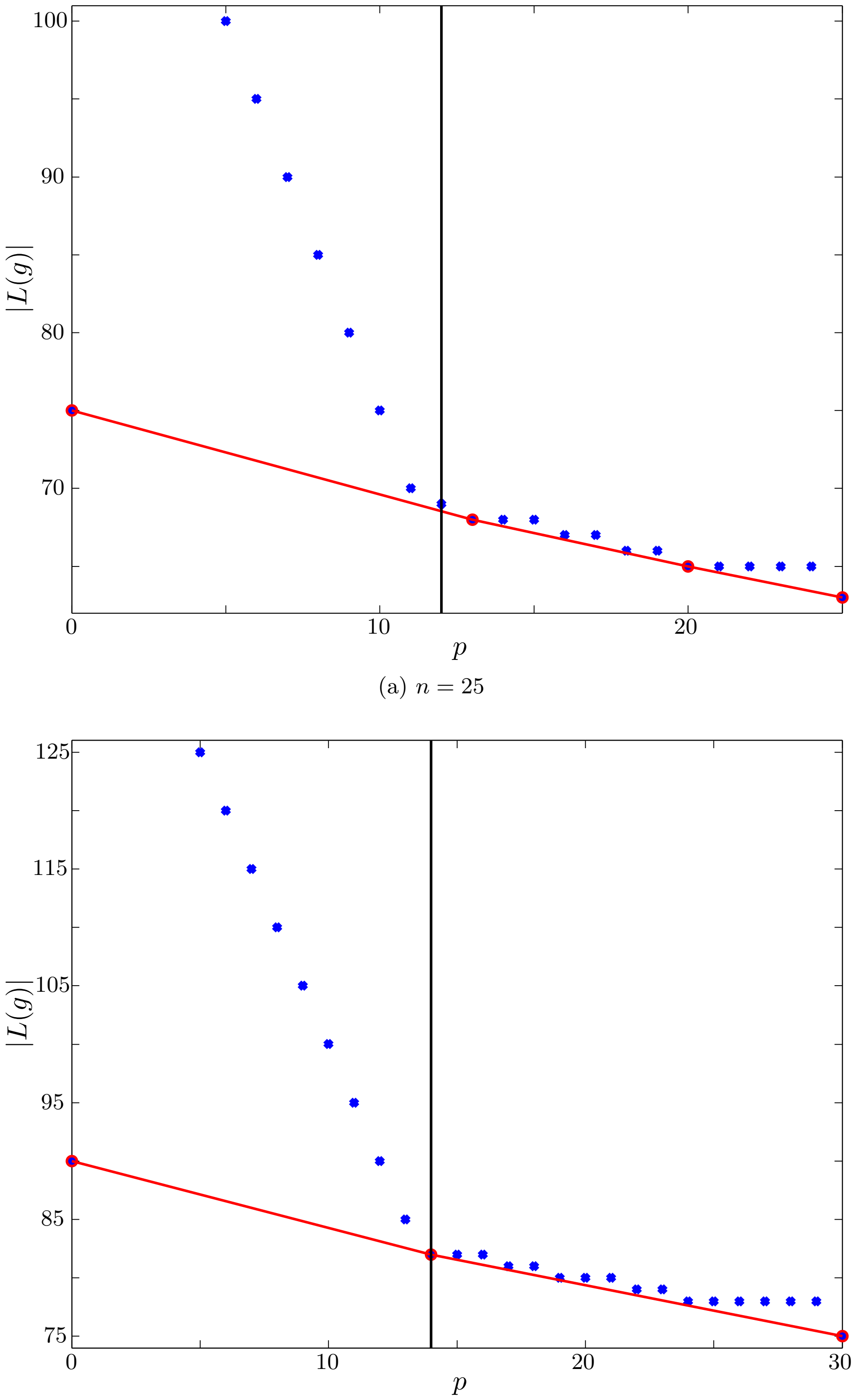
The idea will be to calculate the slope of a line between the two points referring to the network g with p essential nodes and the Harary graph of order $k_a + 1$ (consider e.g. Figure 10a). We know that g has $k_a(n - p)$ links, while g^{h,k_a+1} has $\lceil (k_a + 1)n/2 \rceil$ links. This yields a slope of

$$\begin{aligned}
& \frac{k_a(n - \lfloor \frac{k_a n}{2k_a+1} \rfloor) - \lceil \frac{(k_a+1)n}{2} \rceil}{\lfloor \frac{k_a n}{2k_a+1} \rfloor} \\
\geq \quad & \frac{k_a n - \frac{k_a^2 n}{2k_a+1} - \frac{(k_a+1)n}{2} - 1}{\frac{k_a n}{2k_a+1}} \\
= \quad & \frac{(4k_a + 2)k_a n - 2k_a^2 n - (2k_a + 1)(k_a + 1)n - (4k_a + 2)}{2k_a n} \\
= \quad & -\frac{k_a n + n + 4k_a + 2}{2k_a n}.
\end{aligned}$$

Further, we know that in order to add one non-essential node at the expense of one essential node if $\tilde{p} \leq p$ one needs to add exactly k_a links, yielding a slope of $-k_a$. On the other hand, the above slope is even larger than -1, as

$$\begin{aligned}
& \frac{k_a n + n + 4k_a + 2}{2k_a n} \leq 1 \\
\iff \quad & k_a n + n + 4k_a + 2 \leq 2k_a n \\
\iff \quad & n(k_a - 1) \geq 4k_a + 2 \\
\iff \quad & n \geq \frac{4k_a + 2}{k_a - 1} > 4,
\end{aligned}$$

what yields the result. □



(a) $n = 25$

(b) $n = 30$

Figure 10: Minimal 5-connected networks for $n = 25, n = 30$. Possible equilibria in red circles. The vertical black line marks the largest number of essential nodes such that $p_q = p$

6 Conclusion

We proposed a model of network design with imperfect defense as an extension of Dziubiński and Goyal (2013). The Designer chooses a network for a given number of nodes and can additionally choose to protect nodes against deletion, where protection is imperfect. Subsequently, the Adversary attacks a fixed number of nodes. In the model at hand, the Designer strives for retaining a connected residual network, while the Adversary tries to disconnect the network.

As a first step we succeeded to fully characterize the set of possible equilibria of the game, i.e. the possible equilibrium defended networks, for an attack budget of one or two nodes.
In case of a budget of one, this set consists of the undefended empty network, the centrally-protected star and the unprotected circle. We further showed that the centrally-protected star as the only network using node defense can only be an equilibrium solution for a low number of nodes and a low probability of destruction of defended nodes.
For an attack budget of two nodes we showed that more solutions are possible for the game. The set of possible equilibria is constituted by the unprotected empty network, the centrally-protected star, the fully protected circle, the unprotected wheel, and one or two intermediately defended 2-connected networks, depending on the total number of nodes.

In case of a general attack budget of k_a nodes the set of possible equilibrium defended networks was partially characterized. Given a number of nodes and an attack budget, we defined a group of networks that we demonstrated to include all possible k_a-connected equilibria.

As the main technical contribution we extended the seminal result of Harary (1962), who showed that the k-connected network of n nodes with minimal number of links has $\lceil kn/2 \rceil$ links. Similarly, we identified the minimum number of links for a k-connected network with p essential nodes.

Our results suggest that for the problem of optimal network design not only the cost structure is an important variable for the decision, but also the extent of the threat. If the connectivity of a (large) network shall be secured against the threat of a single node-deletion, when high enough connectivity is the best choice. If, in turn, a threat of several simultaneous attacks is given, then facing imperfect defense it may be optimal to mix the two mechanisms of direct node defense and a high degree of connectivity.

Regarding future research, it would be most interesting yet technically challenging to analyze the game for different utility functions. The here proposed connectivity game undoubtedly is a valid starting point for the analysis and already yielded interesting insights, yet one could think of other, presumably more realistic utility functions, e.g. a utility function being additively separable in components of the residual network and convex and increasing in component

sizes (see e.g. Dziubiński and Goyal (2013)).
Finally, the literature on network design proposes different rules of attack, such as contagion of attack to connected and unprotected nodes, or even link attack. Regarding these on the basis of imperfect defense would add to the understanding of the network design game.

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