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Hyperfinite Construction of G-expectation*

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Abstract

We prove a lifting theorem, in the sense of Robinsonian nonstandard analysis, for the G-expectation. Herein, we use an existing discretization theorem for the G-expectation by T. Fadina and F. Herzberg (*Bielefeld University, Center for Mathematical Economics in its series Working Papers, 503, (2014)*).

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1 Introduction

The hyperfinite G-expectation is a nonstandard discrete analogue of G-expectation (in the sense of Robinsonian nonstandard analysis) which is infinitely close to the continuous time G-expectation. We develop the basic theory for the hyperfinite G-expectation. We prove a lifting theorem for the G-expectation. For the proof of the lifting theorem, we use an existing discretization theorem for the G-expectation from Fadina and Herzberg [8, Theorem 6]. Very roughly speaking, we extend the discrete time analogue of the G-expectation to a hyperfinite time analogue. Then, we use the characterization of convergence in nonstandard analysis to prove that the hyperfinite discrete-time analogue of the G-expectation is infinitely close to the (standard) G-expectation.

Nonstandard analysis makes consistent use of infinitesimals in mathematical analysis based on techniques from mathematical logic. This approach is very promising because it also allows, for instance, to study

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continuous-time stochastic processes as formally finite objects. Many authors have applied nonstandard analysis to problems in measure theory, probability theory and mathematical economics (see for example, Anderson and Raimondo [3] and the references therein or the contribution in Berg and Neves [4]), especially after Loeb [12] converted nonstandard measures (i.e. the images of standard measures under the nonstandard embedding *) into real-valued, countably additive measures, by means of the standard part operator and *Caratheodory*'s extension theorem. One of the main ideas behind these applications is the extension of the notion of a finite set known as *hyperfinite set* or more causally, a formally finite set. Very roughly speaking, hyperfinite sets are sets that can be formally enumerated with both standard and nonstandard natural numbers up to a (standard or nonstandard, i.e. unlimited) natural number.

Anderson [2], Hoover and Perkins [9], Keisler [10], Lindstrøm [11], a few to mention, used Loeb's [12] approach to develop basic nonstandard stochastic analysis and in particular, the nonstandard Itô calculus. Loeb [12] also presents the construction of a Poisson processes using nonstandard analysis. Anderson [2] showed that Brownian motion can be constructed from a hyperfinite number of coin tosses, and provides a detailed proof using a special case of Donsker's theorem. Anderson [2] also gave a nonstandard construction of stochastic integration with respect to his construction of Brownian motion. Keisler [10] uses Anderson's [2] result to obtain some results on stochastic differential equations. Lindstrøm [11] gave the hyperfinite construction (*lifting*) of L^2 standard martingales. Using nonstandard stochastic analysis, Perkins [15] proved a global characterization of (standard) Brownian local time. In this paper, we do not work on the Loeb space because the G-expectation and its corresponding G-Brownian motion are not based on a classical probability measure, but on a set of martingale laws.

Dolinsky et al. [7] and Fadina and Herzberg [8] showed the (standard) weak approximation of the G-expectation. Dolinsky et al. [7] introduced a notion of volatility uncertainty in discrete time and defined a discrete version of Peng's G-expectation. In the continuous-time limit, it turns out that the resulting sublinear expectation converges weakly to the G-expectation. To allow for the hyperfinite construction of G-expectation which require a discretization of the state space, in Fadina and Herzberg [8, Theorem 6] we refine the discretization by Dolinsky et al. [7] and obtain a discretization where the martingale laws are defined on a finite lattice rather than the whole set of reals.

The aim of this paper is to give an alternative, combinatorially inspired construction of the G-expectation based on the aforementioned Theorem 6. We hope that this result may eventually become useful for applications in financial economics (especially existence of equilibrium on continuous-time financial markets with volatility uncertainty) and provides additional intuition for *Peng*'s G-stochastic calculus. We begin the nonstandard treatment of the G-expectation by defining a notion of S-continuity, a standard part operator, and proving a corresponding lifting (and pushing down) theorem. Thereby, we show that our hyperfinite construction is the appropriate nonstandard analogue of the G-expectation. For details on nonstandard analysis, we refer the reader to Albeverio et al. [1], Cutland [5], Loeb and Wolff [13] and Stroyan and Luxemburg [16].

The rest of this paper is organised as follows: in Section 2, we introduce the G-expectation, the continuous-time setting of the sublinear expectation and the hyperfinite-time setting needed for our construction. In Section 3, we introduce the notion of S-continuity and also define the appropriate lifting notion needed for our construction. Finally, we prove that the hyperfinite G-expectation is infinitely close to the (standard) G-expectation.

2 Framework

The G-expectation $\xi \mapsto \mathcal{E}^G(\xi)$ is a sublinear function that takes random variables on the canonical space Ω to the real numbers. The symbol G is a function $G : \mathbb{R} \to \mathbb{R}$ of the form

$$G(\gamma) := \frac{1}{2} \sup_{c \in \mathbf{D}} c\gamma, \tag{1}$$

where $\mathbf{D} = [r_{\mathbf{D}}, R_{\mathbf{D}}]$ and $0 \leq r_{\mathbf{D}} \leq R_{\mathbf{D}} < \infty$. Let $\mathcal{P}^G$ be the set of probabilities on Ω such that for any $P \in \mathcal{P}^G$, B is a martingale with volatility $d\langle B \rangle_t / dt \in \mathbf{D}$ in $P \otimes dt$ a.e. Then, the dual view of the G-expectation via volatility uncertainty (cf. Denis et al. [6]) can be denoted as

$$\mathcal{E}^G(\xi) = \sup_{P \in \mathcal{P}^G} \mathbb{E}^P[\xi].$$

The canonical process B under the G-expectation $\mathcal{E}^G(\xi)$ is called *G-Brownian motion* (cf. Peng [14]).

2.1 Continuous-time construction of sublinear expectation

Let $\Omega = \{\omega \in \mathcal{C}([0,T]; \mathbb{R}) : \omega_0 = 0\}$ be the canonical space of continuous paths on $[0,T]$ endowed with the maximum norm $\|\omega\|_\infty = \sup_{0 \leq t \leq T} |\omega_t|$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}$. B is the canonical process defined by $B_t(\omega) = \omega_t$ and $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$ is the filtration generated by B. $\mathcal{P}_{\mathbf{D}}$ is the set of all martingale laws on Ω such that under any $P \in \mathcal{P}_{\mathbf{D}}$, the coordinate process B is a martingale with respect to $\mathcal{F}_t$ with volatility $d\langle B \rangle_t / dt$ taking values in $\mathbf{D}$, $P \otimes dt$ a.e., for $\mathbf{D} = [r_{\mathbf{D}}, R_{\mathbf{D}}]$ and $0 \leq r_{\mathbf{D}} \leq R_{\mathbf{D}} < \infty$.

$$\mathcal{P}_{\mathbf{D}} = \{P \text{ martingale law on } \Omega;\ d\langle B \rangle_t / dt \in \mathbf{D},\ P \otimes dt \text{ a.e.}\}.$$

Thus, the sublinear expectation is given by

$$\mathcal{E}_{\mathbf{D}}(\xi) = \sup_{P \in \mathcal{P}_{\mathbf{D}}} \mathbb{E}^P[\xi], \tag{2}$$

for any $\xi : \Omega \to \mathbb{R}$, ξ is $\mathcal{F}_T$-measurable and integrable for all $P \in \mathcal{P}_{\mathbf{D}}$. $\mathbb{E}^P$ denotes the expectation under P. It is important to note that the continuous-time sublinear expectation (2) can be considered as the classical G-expectation (for every $\xi \in \mathbb{L}^1_G$ where $\mathbb{L}^1_G$ is defined as the $\mathbb{E}[|\cdot|]-$norm completion of $\mathcal{C}_b(\Omega;\mathbb{R})$) provided (1) is satisfied (cf. Dolinsky et al. [7]).

2.2 Hyperfinite-time setting

Here we present the nonstandard version of the discrete-time setting of the sublinear expectation and the strong formulation of volatility uncertainty on the hyperfinite timeline. For the (standard) strong formulation of volatility uncertainty in the discrete-time see Fadina and Herzberg [8], and for the continuous-time see Dolinsky et al. [7] and Fadina and Herzberg [8].

Definition 2.1. *$^*\Omega$ is the *-image of Ω endowed with the *-extension of the maximum norm $^*\|\cdot\|_\infty$.*

$^*\mathbf{D} = {}^*[r_{\mathbf{D}}, R_{\mathbf{D}}]$ is the *-image of $\mathbf{D}$, and as such it is *internal.*
It is important to note that $st : {}^*\Omega \to \Omega$ is the standard part map, and $st(\omega)$ will be referred to as the *standard part* of ω, for every $\omega \in {}^*\Omega$. $^\circ z$ denotes the standard part of a hyperreal z.

Definition 2.2. *For every $\omega \in \Omega$, if there exists $\widetilde{\omega} \in {}^*\Omega$ such that $\|\widetilde{\omega} - {}^*\omega\|_\infty \simeq 0$, then $\widetilde{\omega}$ is a* nearstandard point *in $^*\Omega$. This will be denoted as $ns(\widetilde{\omega}) \in {}^*\Omega$.*

For all hypernatural N, let

$$\mathcal{L}_N = \left\{ \frac{K}{N\sqrt{N}}, \quad -N^2\sqrt{R_{\mathbf{D}}} \leq K \leq N^2\sqrt{R_{\mathbf{D}}}, \quad K \in {}^*\mathbb{Z} \right\} \tag{3}$$

and the hyperfinite timelime

$$\mathbb{T} = \left\{ 0, \frac{T}{N}, \cdots, -\frac{T}{N} + T, T \right\}. \tag{4}$$

We consider $\mathcal{L}_N^{\mathbb{T}}$ as the canonical space of paths on the hyperfinite timeline, and $X^N = (X_k^N)_{k=0}^N$ as the canonical process denoted by $X_k^N(\bar{\omega}) = \bar{\omega}_k$ for $\bar{\omega} \in \mathcal{L}_N^{\mathbb{T}}$. $\mathcal{F}^N$ is the internal filtration generated by X^N. The linear interpolation operator can be written as

$$\widetilde{} : \widehat{\cdot} \circ \iota^{-1} \to {}^*\Omega, \quad \text{for } \widetilde{\mathcal{L}_N^{\mathbb{T}}} \subseteq {}^*\Omega,$$

where

$$\widehat{\omega}(t) := (\lfloor Nt/T \rfloor + 1 - Nt/T)\omega_{\lfloor Nt/T \rfloor} + (Nt/T - \lfloor Nt/T \rfloor)\omega_{\lfloor Nt/T \rfloor + 1},$$

for $\omega \in \mathcal{L}_N^{N+1}$ and for all $t \in {}^*[0,T]$. $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y and $\iota : \mathbb{T} \to \{0, \cdots, N\}$ for $\iota : t \mapsto Nt/T$.

For the hyperfinite strong formulation of the volatility uncertainty, fix $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Consider $\left\{\pm\frac{1}{\sqrt{N}}\right\}^{\mathbb{T}}$, and let P_N be the uniform counting measure on $\left\{\pm\frac{1}{\sqrt{N}}\right\}^{\mathbb{T}}$. P_N can also be seen as a measure on $\mathcal{L}_N^{\mathbb{T}}$, concentrated on $\left\{\pm\frac{1}{\sqrt{N}}\right\}^{\mathbb{T}}$. Let $\Omega_N = \{\underline{\omega} = (\underline{\omega}_1, \cdots, \underline{\omega}_N); \underline{\omega}_i = \{\pm 1\}, i = 1, \cdots, N\}$, and let $\Xi_1, \cdots, \Xi_N$ be a *-independent sequence of $\{\pm 1\}$-valued random variables on Ω_N and the components of Ξ_k are orthonormal in $L^2(P_N)$. We denote the hyperfinite random walk by

$$\mathbb{X}_t = \frac{1}{\sqrt{N}} \sum_{l=1}^{Nt/T} \Xi_l \quad \text{for all } t \in \mathbb{T}.$$

The hyperfinite-time stochastic integral of some $F : \mathbb{T} \times \mathcal{L}_N^{\mathbb{T}} \to {}^*\mathbb{R}$ with respect to the hyperfinite random walk is given by

$$\sum_{s=0}^{t} F(s, \mathbb{X})\Delta\mathbb{X}_s : \Omega_N \to {}^*\mathbb{R}, \quad \underline{\omega} \in \Omega_N \mapsto \sum_{s=0}^{t} F(s, \mathbb{X}(\underline{\omega}))\Delta\mathbb{X}_s(\underline{\omega}).$$

Thus, the hyperfinite set of martingale laws can be defined by

$$\bar{\mathcal{Q}}_{\mathbf{D}'_N}^N = \left\{ \ P_N \circ (M^{F,\mathbb{X}})^{-1}; \ F : \mathbb{T} \times \mathcal{L}_N^{\mathbb{T}} \to \sqrt{\mathbf{D}'_N} \ \right\}$$

where

$$\mathbf{D}'_N = {}^*\mathbf{D} \cap \left(\frac{1}{N}{}^*\mathbb{N}\right)^2$$

and

$$M^{F,\mathbb{X}} = \left(\sum_{s=0}^{t} F(s, \mathbb{X})\Delta\mathbb{X}_s\right)_{t \in \mathbb{T}}.$$

Remark 2.3. *Up to scaling,* $\bar{\mathcal{Q}}_{\mathbf{D}'_N}^N = \mathcal{Q}_{\mathbf{D}'_n}^n$.

3 Results and proofs

Definition 3.1 (Uniform lifting of ξ)**.** *Let* $\Xi : \mathcal{L}_N^{\mathbb{T}} \to {}^*\mathbb{R}$ *be an internal function, and let* $\xi : \Omega \to \mathbb{R}$ *be a continuous function.* Ξ *is said to be a* uniform lifting *of* ξ *if and only if*

$$\forall \bar{\omega} \in \mathcal{L}_N^{\mathbb{T}} \Big(\widetilde{\bar{\omega}} \in ns({}^*\Omega) \Rightarrow {}^\circ\Xi(\bar{\omega}) = \xi(st(\widetilde{\bar{\omega}}))\Big),$$

where $st(\widetilde{\omega})$ is defined with respect to the topology of uniform convergence on Ω.

In order to construct the hyperfinite version of the G-expectation, we need to show that the *-image of ξ, $^*\xi$, with respect to $\widetilde{\omega} \in ns(^*\Omega)$, is the canonical lifting of ξ with respect to $st(\widetilde{\omega}) \in \Omega$. i.e., for every $\widetilde{\omega} \in ns(^*\Omega)$, $^\circ\left(^*\xi(\widetilde{\omega})\right) = \xi(st(\widetilde{\omega}))$. To do this, we need to show that $^*\xi$ is S-continuous in every nearstandard point $\widetilde{\omega}$.

It is easy to prove that there are two equivalent characterizations of S-continuity on $^*\Omega$.

Remark 3.2. *The following are equivalent for an internal function $\Phi : {}^*\Omega \to {}^*\mathbb{R}$:*

(1) $\forall \omega^{'} \in {}^*\Omega \left(^*\|\omega - \omega^{'}\|_\infty \simeq 0 \Rightarrow {}^*|\Phi(\omega) - \Phi(\omega^{'})| \simeq 0 \right).$

(2) $\forall \varepsilon \gg 0, \exists \delta \gg 0 : \forall \omega^{'} \in {}^*\Omega \left(^*\|\omega - \omega^{'}\|_\infty < \delta \Rightarrow {}^*|\Phi(\omega) - \Phi(\omega^{'})| < \varepsilon \right).$

(The case of Remark 3.2 where $\Omega = \mathbb{R}$ is well known and proved in Stroyan and Luxemburg [16, Theorem 5.1.1])

Definition 3.3. *Let $\Phi : {}^*\Omega \to {}^*\mathbb{R}$ be an internal function. We say Φ is S-*continuous *in $\omega \in {}^*\Omega$, if and only if it satisfies one of the two equivalent conditions of Remark 3.2.*

Proposition 3.4. *If $\xi : \Omega \to \mathbb{R}$ is a continuous function satisfying $|\xi(\omega)| \le a(1 + \|\omega\|_\infty)^b$, for $a, b > 0$, then, $\Xi = {}^*\xi \circ \widetilde{\cdot}$ is a uniform lifting of ξ.*

Proof. Fix $\omega \in \Omega$. By definition, ξ is continuous on Ω. i.e., for all $\omega \in \Omega$, and for every $\varepsilon \gg 0$, there is a $\delta \gg 0$, such that for every $\omega^{'} \in \Omega$, if

$$\|\omega - \omega^{'}\|_\infty < \delta, \text{ then } |\xi(\omega) - \xi(\omega^{'})| < \varepsilon. \tag{5}$$

By the Transfer Principle: For all $\omega \in \Omega$, and for every $\varepsilon \gg 0$, there is a $\delta \gg 0$, such that for every $\omega^{'} \in {}^*\Omega$, (5) becomes,

$$^*\|^*\omega - \omega^{'}\|_\infty < \delta, \text{ and } {}^*|^*\xi(^*\omega) - {}^*\xi(\omega^{'})| < \varepsilon. \tag{6}$$

So, $^*\xi$ is S-continuous in $^*\omega$ for all $\omega \in \Omega$. Applying the equivalent characterization of S-continuity, Remark 3.2, (6) can be written as

$$^*\|^*\omega - \omega^{'}\|_\infty \simeq 0, \text{ and } {}^*|^*\xi(^*\omega) - {}^*\xi(\omega^{'})| \simeq 0.$$

We assume $\widetilde{\omega}$ to be a nearstandard point. By Definition 2.2, this simply implies,

$$\forall \widetilde{\omega} \in ns(^*\Omega), \; \exists \omega \in \Omega : {}^*\|\widetilde{\omega} - {}^*\omega\|_\infty \simeq 0. \tag{7}$$

Thus, by S-continuity of ${}^*\xi$ in ${}^*\omega$,

$$^*|^*\xi(\widetilde{\widetilde{\omega}}) - {}^*\xi({}^*\omega)| \simeq 0.$$

Using the triangle inequality, if $\omega' \in {}^*\Omega$ with ${}^*\|\widetilde{\widetilde{\omega}} - \omega'\|_\infty \simeq 0$,

$$^*\|^*\omega - \omega'\|_\infty \leq {}^*\|^*\omega - \widetilde{\widetilde{\omega}}\|_\infty + {}^*\|\widetilde{\widetilde{\omega}} - \omega'\|_\infty \simeq 0$$

and therefore again by the S-continuity of ${}^*\xi$ in ${}^*\omega$,

$$^*|^*\xi({}^*\omega) - {}^*\xi(\omega')| \simeq 0.$$

And so,

$$^*|^*\xi(\widetilde{\widetilde{\omega}}) - {}^*\xi(\omega')| \leq {}^*|^*\xi(\widetilde{\widetilde{\omega}}) - {}^*\xi({}^*\omega)| + {}^*|^*\xi({}^*\omega) - {}^*\xi(\omega')| \simeq 0.$$

Thus, for all $\widetilde{\widetilde{\omega}} \in ns({}^*\Omega)$ and $\omega' \in {}^*\Omega$, if ${}^*\|\widetilde{\widetilde{\omega}} - \omega'\|_\infty \simeq 0$, then,

$$^*|^*\xi(\widetilde{\widetilde{\omega}}) - {}^*\xi(\omega')| \simeq 0.$$

Hence, ${}^*\xi$ is S-continuous in $\widetilde{\widetilde{\omega}}$. Equation (7) also implies

$$\widetilde{\widetilde{\omega}} \in m(\omega) \left(m(\omega) = \bigcap \{{}^*\mathcal{O}; \mathcal{O} \text{ is an open neighbourhood of } \omega\} \right)$$

such that ω is unique, and in this case $st(\widetilde{\widetilde{\omega}}) = \omega$.
Therefore,

$$^\circ\left({}^*\xi(\widetilde{\widetilde{\omega}})\right) = \xi(st(\widetilde{\widetilde{\omega}})).$$

□

Definition 3.5. *Let $\bar{\mathcal{E}} : {}^*\mathbb{R}^{\mathcal{L}_N^{\mathbb{T}}} \to {}^*\mathbb{R}$. We say that $\bar{\mathcal{E}}$ lifts $\mathcal{E}^G$ if and only if for every $\xi : \Omega \to \mathbb{R}$ that satisfies $|\xi(\omega)| \leq a(1 + \|\omega\|_\infty)^b$ for some $a, b > 0$,*

$$\bar{\mathcal{E}}({}^*\xi \circ \tilde{\cdot}) \simeq \mathcal{E}^G(\xi).$$

Theorem 3.6.

$$\max_{\bar{Q} \in \bar{\mathcal{Q}}^N_{\mathbf{D}'_N}} \mathbb{E}^{\bar{Q}}[\cdot] \textit{ lifts } \mathcal{E}^G(\xi). \tag{8}$$

Proof. From the standard approximation in Fadina and Herzberg [8, Theorem 6],

$$\max_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \to \mathcal{E}^G(\xi), \quad \text{as } n \to \infty. \tag{9}$$

For all $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, we know that (9) holds if and only if

$$\max_{Q \in {}^*\mathcal{Q}^N_{\mathbf{D}'_N}} \mathbb{E}^Q[{}^*\xi(\widehat{X}^N)] \simeq \mathcal{E}^G(\xi), \tag{10}$$

(see Albeverio et al. [1], Proposition 1.3.1). Now, we want to express (10) in term of $\bar{\mathcal{Q}}^N_{\mathbf{D'_N}}$. i.e., to show that

$$\max_{\bar{Q}\in\bar{\mathcal{Q}}^N_{\mathbf{D'}_N}} \mathbb{E}^{\bar{Q}}[{}^*\xi \circ \tilde{\cdot}] \simeq \mathcal{E}^G(\xi).$$

To do this, use

$$\mathbb{E}^Q[{}^*\xi \circ \hat{\cdot}] = \mathbb{E}^Q[{}^*\xi \circ \hat{\cdot} \circ \iota^{-1} \circ \iota]$$

and

$$\begin{aligned}
\mathbb{E}^Q[{}^*\xi \circ \hat{\cdot} \circ \iota^{-1} \circ \iota] &= \mathbb{E}^Q[{}^*\xi \circ \tilde{\cdot} \circ \iota] \\
&= \int_{{}^*\mathbb{R}^{N+1}} {}^*\xi \circ \tilde{\cdot} \circ \iota dQ, \quad \text{(transforming measure)} \\
&= \int_{{}^*\mathbb{R}^{\mathbb{T}}} {}^*\xi \circ \tilde{\cdot} d(Q \circ j), \\
&= \mathbb{E}^{Q\circ j}[{}^*\xi \circ \tilde{\cdot}]
\end{aligned}$$

for $j : {}^*\mathbb{R}^{\mathbb{T}} \to {}^*\mathbb{R}^{N+1}$, $(xt)_{t\in\mathbb{T}} \mapsto \left(\frac{xNt}{T}\right)_{t\in\mathbb{R}^{N+1}}$.
Thus,

$$\bar{\mathcal{Q}}^N_{\mathbf{D'}_N} = \{Q \circ j : Q \in {}^*\mathcal{Q}^N_{\mathbf{D'}_N}\}.$$

This implies,

$$\max_{\bar{Q}\in\bar{\mathcal{Q}}^N_{\mathbf{D'}_N}} \mathbb{E}^{\bar{Q}}[{}^*\xi \circ \tilde{\cdot}] = \max_{Q\in{}^*\mathcal{Q}^N_{\mathbf{D'}_N}} \mathbb{E}^Q[{}^*\xi \circ \hat{\cdot}].$$

□

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Appendix

Proof of Remark 3.2. Let Φ be an internal function such that condition (1) holds. To show that (1) $\Rightarrow$ (2), fix $\varepsilon \gg 0$. We shall show there exists a δ for this ε as in condition (2). Since Φ is internal, the set

$$I = \left\{ \delta \in {}^*\mathbb{R}_{>0} : \ \forall \omega' \in {}^*\Omega \ ({}^*\|\omega - \omega'\|_\infty < \delta \Rightarrow {}^*|\Phi(\omega) - \Phi(\omega')| < \varepsilon) \right\},$$

is internal by the Internal Definition Principle and also contains every positive infinitesimal. By Overspill (cf. Albeverio et al. [1, Proposition 1.27]) I must then contain some positive $\delta \in \mathbb{R}$.
Conversely, suppose condition (1) does not hold, that is, there exists some $\omega^{'} \in {}^*\Omega$ such that

$$^*\|\omega - \omega^{'}\|_\infty \simeq 0 \text{ and } ^*|\Phi(\omega) - \Phi(\omega^{'})| \text{ is not infinitesimal.}$$

If $\varepsilon = \min(1, ^*|\Phi(\omega) - \Phi(\omega^{'})|/2)$, we know that for each standard $\delta > 0$, there is a point $\omega^{'}$ within δ of ω at which $\Phi(\omega^{'})$ is farther than ε from $\Phi(\omega)$. This shows that condition (2) cannot hold either. $\square$