

# Order $g^2$ susceptibilities in the symmetric phase of the Standard Model

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**Abstract.** Susceptibilities of conserved charges such as baryon minus lepton number enter baryogenesis computations, since they provide the relationship between conserved charges and chemical potentials. Their next-to-leading order corrections are of order  $g$ , where  $g$  is a generic Standard Model coupling. They are due to soft Higgs boson exchange, and have been calculated recently, together with some order  $g^2$  corrections. Here we compute the complete  $g^2$  contributions. Close to the electroweak crossover the soft Higgs contribution is of order  $g^2$ , and is determined by the non-perturbative physics at the magnetic screening scale.

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## 1 Introduction

In the early Universe all charges which are violated at a rate smaller than the Hubble expansion rate can be considered conserved. For instance, in the minimal Standard Model (with zero neutrino masses) baryon number  $B$  and the  $n_f = 3$  flavor lepton numbers  $L_i$  are conserved below the electroweak scale, while at higher temperatures only the differences  $X_i \equiv B/n_f - L_i$  are conserved. All equilibrium properties are determined by the temperature  $T$  together with the values of *all* conserved charges  $Q_i$  or equivalently by the corresponding chemical potentials  $\mu_i$ . These properties are encoded in the grand canonical partition function

$$\exp(-\Omega/T) = \text{tr} \exp [(\mu_i Q_i - H)/T], \quad (1.1)$$

where  $H$  is the Hamiltonian.

It is rather plausible that initially the values of conserved charges were practically zero, for example if one assumes that the Universe underwent an early period of inflation. Since there is something rather than nothing, some processes must have created at least the charge that we know is non-vanishing at present, i.e., the baryon number, or baryon asymmetry of the Universe. Such a process, called baryogenesis, must proceed out of thermal equilibrium. For example, in leptogenesis [1] a non-vanishing value of some  $X_i$  is generated. Afterwards this quantity is conserved and its value determines the equilibrium properties, such as the expectation values of baryon number  $B$  or lepton number  $L$ .

The values of the charges and thus of the chemical potentials are usually small, so that the grand canonical potential is only needed to lowest non-trivial order, which is  $O(\mu^2)$ .<sup>1</sup>

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<sup>1</sup>We assume that the charges  $Q_i$  are odd under CPT. Then their expectation values vanish when  $\mu = 0$ , and  $\Omega$  contains no terms linear in  $\mu$ .

Then the  $\mu$ -dependence is fully determined by the second derivatives at zero  $\mu$ , the so-called susceptibilities

$$\chi_{ij} \equiv -\frac{1}{V} \left. \frac{\partial^2 \Omega}{\partial \mu_i \partial \mu_j} \right|_{\mu=0}. \quad (1.2)$$

One important use of the grand canonical potential is to determine the relation between  $B$  or  $L$  and the  $Q_i$ . Strictly speaking one cannot introduce a chemical potential for  $B + L$  in the symmetric phase where electroweak sphalerons rapidly violate  $B + L$ . Nevertheless, one can *formally* introduce a chemical potential for  $B + L$  as long as one computes only the expectation value of  $B + L$  and not higher moments. The reason is that for the resulting partition function

$$\exp(-\Omega'/T) = \text{tr} \exp \left\{ [\mu_{B+L}(B + L) + \mu_i Q_i - H]/T \right\} \quad (1.3)$$

one only needs the expansion to first order in  $\mu_{B+L}$ . Then, even though  $B + L$  does not commute with  $H$ , the operator ordering does not matter because of the trace. The expectation value can then be written as

$$\langle B + L \rangle = - \left. \frac{\partial \Omega'}{\partial \mu_{B+L}} \right|_{\mu_{B+L}=0}. \quad (1.4)$$

This relation can be used to determine  $B + L$  and thus  $B$  from the value of  $B - L$  before the electroweak crossover, neglecting possible effects of the non-equilibrium epoch when the electroweak sphaleron transitions are shut off.

Another use of the susceptibilities (1.2) has been pointed out recently [2] in the context of leptogenesis. There the asymmetry can be obtained from a set of kinetic equations. One coefficient in these equations quantifies the amount of washout of the asymmetry. It was found that at leading order in the right handed neutrino Yukawa couplings the washout rate can be factorized into a product of a spectral function which contains dynamical information, and the inverse of a matrix of susceptibilities. The spectral function has been computed at next-to-leading order which is  $O(g^2)$  in the Standard Model couplings  $g$ .<sup>2</sup> It turned out that deep in the symmetric phase the NLO corrections to the susceptibilities already start at order  $g$ . The  $O(g)$  contribution computed in [2] is an infrared effect caused by the exchange of a soft Higgs boson. Close to the electroweak crossover the effective thermal Higgs mass can become very small. If it becomes of the order of the magnetic screening scale, the perturbative expansion for the susceptibilities can be expected to break down.

In this paper we compute the complete  $O(g^2)$  corrections to the susceptibilities, thereby completing the  $O(g^2)$  result for the washout rate. We obtain contributions both from hard ( $\sim T$ ) and smaller momenta, which, depending on the value of the thermal Higgs mass, can be soft ( $\sim gT$ ) or even smaller ('ultrasoft'). We use dimensional reduction, a framework which allows us to systematically treat the contributions at the different scales and the required resummations.

Part of the  $O(g^2)$  susceptibilities have already been computed in [2]. Dimensional reduction in the presence of chemical potentials has been considered in [3], where the focus was on a electroweak phase transition. Therefore only those terms which depend on the Higgs field were computed.

<sup>2</sup>For our power counting we make no distinction between the different Standard Model couplings. In this respect we differ from [3].

This paper is organized as follows. In section 2 we recall the role of gauge charges and gauge fields in the presence of chemical potentials for global charges. Section 3 outlines our use of dimensional reduction. The hard Higgs contribution is obtained in section 4, and the dimensionally reduced theory is described in section 5. Depending on the value of the effective Higgs mass we obtain either soft (section 6) or both soft and ultrasoft contributions (section 7). Finally, in section 8 we illustrate our results by computing the relation of  $B$  and  $B - L$  near the electroweak crossover.

## 2 Chemical potentials and gauge charges

We write the partition function (1.1) as a path integral with imaginary time  $t = -i\tau$ ,

$$\exp(-\Omega/T) = \int \mathcal{D}\Phi \exp \left\{ \int_0^{1/T} d\tau \left[ \mu_i Q_i + \int d^3x \mathcal{L} \right] \right\}, \quad (2.1)$$

where  $\Phi$  stands for all fields in our theory with the Lagrangian  $\mathcal{L}$ . The temporal component of the gauge fields act as Lagrange multipliers which enforce Gauss' law. We work in a finite volume and take the volume to infinity in the end. Then, with spatial periodic boundary conditions, the total gauge charges vanish. These conditions are enforced by the constant modes of the temporal component of the gauge fields.

In the presence of chemical potentials for global charges the temporal components of the gauge fields can develop constant expectation values which act like chemical potentials for the corresponding gauge charges. We will only consider the symmetric phase of the electroweak theory, where only the weak hypercharge gauge field  $B_\mu$  can develop an expectation value.

It is convenient to perform the path integral (2.1) in two steps [4]. First one integrates over all fields except over the constant mode of  $B_0$  which we denote by  $\bar{B}_0$ . We denote the result of this integration by  $\exp(-\tilde{\Omega}/T)$ . In the presence of chemical potentials  $\tilde{\Omega}$  may contain terms linear in  $\bar{B}_0$ . The linear terms can arise when some of the global charges are correlated with the hypercharge. Then the integral over  $\bar{B}_0$

$$\exp(-\Omega/T) = \int d\bar{B}_0 \exp \left( -\tilde{\Omega}/T \right) \quad (2.2)$$

can lead to  $\mu$ -dependent contributions.

Here we are interested in small values of the conserved charges which corresponds to small values of the chemical potentials. Therefore we need to keep only those terms in  $\tilde{\Omega}$  which are at most quadratic in the chemical potentials. Then (2.2) can be evaluated in the saddle point approximation,

$$\exp(-\Omega/T) = \text{const} \times \exp \left[ -\tilde{\Omega}(\text{saddle point})/T \right]. \quad (2.3)$$

Here  $\tilde{\Omega}$  is evaluated at the saddle point

$$\frac{\partial \tilde{\Omega}}{\partial \bar{B}_0} = 0, \quad (2.4)$$

and the constant in (2.3) is independent of the chemical potentials. The relation (2.4) determines the expectation value of  $\bar{B}_0$  and is usually referred to as ‘equilibrium condition’. Note that it follows from the saddle point approximation to (2.2).

Our convention is such that the hypercharge gauge field enters the covariant time derivative for species  $\alpha$  with hypercharge  $y_\alpha$  as follows,

$$D_0 = \partial_t + iy_\alpha g_1 B_0 + \dots = i(\partial_\tau + y_\alpha g_1 B_0) + \dots, \quad (2.5)$$

where  $y_\varphi = 1/2$  for the Higgs field, and  $g_1$  is the weak hypercharge gauge coupling. Note that  $B_0$  is purely imaginary. The constant mode acts like a chemical potential  $\mu_\alpha = y_\alpha \mu_Y$  for each species  $\alpha$  with the ‘hypercharge chemical potential’

$$\mu_Y \equiv g_1 \bar{B}_0. \quad (2.6)$$

It is, like  $\bar{B}_0$ , purely imaginary.

### 3 Dimensional reduction

A useful tool for consistently treating the contributions from the different momentum scales at high temperature is dimensional reduction [5–8]. The constant gauge field modes (see section 2) can also be conveniently treated within this framework. Thus the computation of the grand canonical partition function is conveniently done as follows: in a first step one integrates out hard field modes with momenta of order  $T$ . This includes all fermion fields because in the imaginary time formalism their (Matsubara-) frequencies cannot vanish and are always of order  $T$ . The result is an effective action containing  $\tilde{\Omega}_{\text{hard}}$ , which aside from the zero modes is field independent, and an effective Lagrangian  $\mathcal{L}_{\text{soft}}$  for a 3-dimensional field theory, and momenta of order  $gT$  or less. In a second step one integrates over soft modes which are the zero frequency modes with spatial momenta of order  $gT$ . This yields  $\tilde{\Omega}_{\text{soft}}$  plus an effective Lagrangian for the ultrasoft ( $p \ll gT$ ) fields  $\mathcal{L}_{\text{ultrasoft}}$ . When the Higgs mass in  $\mathcal{L}_{\text{soft}}$  is small compared to  $gT$ , there are also important contributions from an ultrasoft spatial momentum scale smaller than  $gT$ , as will be discussed below. After these steps one obtains  $\tilde{\Omega}$  and from that  $\Omega$  using (2.3). In this way we obtain the grand canonical potential as a sum of three parts,

$$\tilde{\Omega} = \tilde{\Omega}_{\text{hard}} + \tilde{\Omega}_{\text{soft}} + \tilde{\Omega}_{\text{ultrasoft}}. \quad (3.1)$$

In principle it would be possible to treat the constant mode of the gauge fields as part of the 3-dimensional gauge field, without introducing the notion of a gauge charge chemical potential. Then the distinction between constant and non-constant gauge fields would have to be made only when integrating out the soft fields. In such an approach the mass term for  $B_0$  would not only contain the Debye mass for the soft field, but also a linear and a quadratic term in the constant mode. This point of view was taken in [4]. For a next-to-leading order calculation it is more convenient to distinguish the two as in [2], because the masses for the non-constant modes are only needed at order  $g^2 T^2$ , while  $g^2 T^2 \mu_\varphi^2 \sim g^4 T^2 \bar{B}_0^2$ . Furthermore, in this way we can easily read off the fermionic contributions to  $\Omega$  from [2].

### 4 Hard contributions

We compute  $\tilde{\Omega}_{\text{hard}}$  in the Standard Model in 4 dimensions. We need the terms of the Lagrangian which contain the Higgs field  $\varphi$ ,

$$\begin{aligned} \mathcal{L}_\varphi = & -\varphi^\dagger D^2 \varphi - m_0^2 \varphi^\dagger \varphi - \lambda (\varphi^\dagger \varphi)^2 \\ & - [(h_e)_{ab} \bar{l}_{a,L} \varphi e_{b,R} + (h_u)_{ab} \bar{q}_{a,L} \tilde{\varphi} u_{b,R} + (h_d)_{ab} \bar{q}_{a,L} \varphi d_{b,R} + \text{h.c.}]. \end{aligned} \quad (4.1)$$

We treat all particles as massless and perform a perturbative expansion in the parameters  $m_0^2$ ,  $\lambda$ ,  $h_i$ , and  $g_i$ , where  $g_2$  and  $g_3$  are the weak SU(2) and color SU(3) gauge couplings, respectively. We treat all couplings as being of order  $g$ , and  $m_0^2 \sim g^2 T^2$ . We use dimensional regularization by working in  $d = 3 - 2\varepsilon$  spatial dimensions. Then infrared divergences coming from massless propagators vanish automatically. The Higgs chemical potential (see (2.6)) introduces the following additional terms:

$$\delta\mathcal{L} = \mu_\varphi \left[ \varphi^\dagger (\partial_\tau \varphi) - (\partial_\tau \varphi^\dagger) \varphi \right] + \mu_\varphi^2 \varphi^\dagger \varphi + 2g_1 \mu_\varphi B_0 \varphi^\dagger \varphi + 2g_2 \mu_\varphi \varphi^\dagger A_0 \varphi. \quad (4.2)$$

Even though we only need an expansion up to order  $\mu_\varphi^2$ , we find it convenient to include the quadratic term in (4.2) in the Higgs propagator and later expand the loop integrals. Note that there are also  $\mu_\varphi$ -dependent vertices whose effects cannot be covered by a frequency shift in the propagator. We will see that the diagrams containing these vertices vanish at order  $\mu^2$  because the sum integral (4.5) is zero.

In the calculation for the hard contributions the following 1-loop sum-integrals  $\mathcal{J}_p \equiv T \sum_{p_0} \int_{\mathbf{p}}$  with  $\int_{\mathbf{p}} \equiv (2\pi)^{-d} \int d^d p$  appear:

$$J_0(\mu_\varphi) \equiv \mathcal{J}_p \ln(-p^2) = -\frac{\pi^2 T^4}{45} - \mu_\varphi^2 \frac{T^2}{6} + O(\mu_\varphi^4), \quad (4.3)$$

$$J_1(\mu_\varphi) \equiv \mathcal{J}_p \frac{1}{-p^2} = \frac{T^2}{12} - \frac{\mu_\varphi^2}{8\pi^2} + O(\mu_\varphi^4). \quad (4.4)$$

Here and below we denote  $p^2 = p_0^2 - \mathbf{p}^2$ , and  $p_0 = in2\pi T + \mu_\varphi$  with summation over all integer  $n$ . The only 2-loop sum-integral which cannot be reduced to products of 1-loop integrals is only needed at zero chemical potential, where it vanishes exactly,

$$J_2 \equiv \mathcal{J}_{p,q} \frac{1}{p^2 q^2 (p+q)^2} \Big|_{\mu=0} = 0. \quad (4.5)$$

This result has been found to order  $O(\varepsilon)$  in [9, 10], and to all orders in [11].

We then obtain the following contributions to  $-\tilde{\Omega}_{\text{hard}}/V$ : the leading order is given by the 1-loop diagram

$$\text{---} \circ \text{---} = -2J_0(\mu_\varphi) = 2 \left( \frac{\pi^2 T^4}{45} + \mu_\varphi^2 \frac{T^2}{6} + O(\mu_\varphi^4) \right). \quad (4.6)$$

There is also one 1-loop diagram with a Higgs mass insertion

$$\text{---} \bullet \text{---} \circ \text{---} = -2J_1(\mu_\varphi) = -2m_0^2 \left( \frac{T^2}{12} - \frac{\mu_\varphi^2}{8\pi^2} + O(\mu_\varphi^4) \right). \quad (4.7)$$

At 2 loops we have the Higgs self interaction,

$$\begin{aligned} \frac{1}{2} \text{---} \circ \text{---} \circ \text{---} &= -6\lambda J_1^2(\mu_\varphi) \\ &= -\lambda \frac{T^2}{2} \left( \frac{T^2}{12} - \frac{\mu_\varphi^2}{4\pi^2} \right) + O(\mu_\varphi^4). \end{aligned} \quad (4.8)$$



$$\begin{aligned}
& + 3 \left[ \frac{1}{4\pi^2} \text{tr}(h_u h_u^\dagger) \mu_\varphi^2 - \frac{3}{8\pi^2} \text{tr}(h_u^\dagger h_u \mu_q^2 + h_u h_u^\dagger \mu_u^2) \right] \\
& + 3 \left[ \frac{1}{4\pi^2} \text{tr}(h_d h_d^\dagger) \mu_\varphi^2 - \frac{3}{8\pi^2} \text{tr}(h_d^\dagger h_d \mu_q^2 + h_d h_d^\dagger \mu_d^2) \right] \\
& + \left[ \frac{1}{4\pi^2} \text{tr}(h_e h_e^\dagger) \mu_\varphi^2 - \frac{3}{8\pi^2} \text{tr}(h_e^\dagger h_e \mu_\ell^2 + h_e h_e^\dagger \mu_e^2) \right] + O(\mu^4) .
\end{aligned} \tag{4.12}$$

Here the  $h_i$ , are the Yukawa coupling matrices (see (4.1)). The chemical potential matrices are matrices in family space. They are determined by the zero mode  $\bar{B}_0$ , or hypercharge chemical potential, and by the chemical potentials in (1.1),

$$\mu_\alpha = y_\alpha \mu_Y + \sum_i \mu_i T_{i,\alpha} . \tag{4.13}$$

The matrices  $T_{i,\alpha}$  are the generators of the symmetry transformation corresponding to the charge  $Q_i$ , acting on fermion type  $\alpha$  with  $\alpha \in \{q, u, d, \ell, e\}$ . For example, the generator matrices of  $B-L$  are proportional to the unit matrix, with  $T_{B-L,q} = T_{B-L,u} = T_{B-L,d} = 1/3$  and  $T_{B-L,\ell} = T_{B-L,e} = -1$ .

## 5 The dimensionally reduced theory

Aside from the hard contribution  $\tilde{\Omega}_{\text{hard}}$  the hard modes also determine the effective Lagrangian for the bosonic modes with zero Matsubara frequency, and with soft or ultrasoft momenta. The derivation of an effective three-dimensional theory of the Standard Model has been done in [8] at zero  $\mu$ . At order  $g^2$  we need the following  $\mu$ -independent terms:<sup>4</sup>

$$\begin{aligned}
-\mathcal{L}_{\text{soft}, \mu_\varphi=0} &= \frac{1}{4} F_{ij} F_{ij} + \frac{1}{4} W_{ij} W_{ij} \\
&+ \varphi^\dagger \mathbf{D}^2 \varphi + m_3^2 \varphi^\dagger \varphi + \lambda_3 (\varphi^\dagger \varphi)^2 \\
&- \frac{1}{2} (\partial_i B_0)^2 - \frac{1}{2} m_{D,1}^2 B_0^2 - \frac{1}{2} (D_i A_0)^2 - \frac{1}{2} m_{D,2}^2 \text{Tr}(A_0^2) \\
&- h_1 \varphi^\dagger \varphi B_0^2 - h_2 \varphi^\dagger \varphi \text{Tr}(A_0^2) .
\end{aligned} \tag{5.1}$$

For the finite density effects we also need to include

$$-\delta \mathcal{L}_{\text{soft}} = -\mu_\varphi^2 \varphi^\dagger \varphi - \rho_1 \varphi^\dagger B_0 \varphi - \rho_2 \varphi^\dagger A_0 \varphi . \tag{5.2}$$

The quadratic scalar operators can be combined, yielding a  $\mu_\varphi$  dependent mass [6, 8]

$$\begin{aligned}
m_{3,\mu_\varphi}^2 &\equiv -\mu_\varphi^2 + m_3^2 \\
&= m_0^2 - \mu_\varphi^2 + T^2 \left( \frac{1}{2} \lambda + \frac{3}{16} g_2^2 + \frac{1}{16} g_1^2 + \frac{1}{4} h_t^2 \right) ,
\end{aligned} \tag{5.3}$$

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<sup>4</sup>The term  $\varphi^\dagger A_0 B_0 \varphi$  term does not contribute at  $O(g^2)$ .



where  $h_t$  is the (real) top Yukawa coupling. As discussed at the end of section 3, the Debye masses for  $A_0$ ,  $B_0$  are only needed at order  $g^2 T^2$  [8],

$$m_{D,1}^2 = \left( \frac{N_s}{6} + \frac{5n_f}{9} \right) g_1^2 T^2, \quad (5.4)$$

$$m_{D,2}^2 = \left( \frac{2}{3} + \frac{N_s}{6} + \frac{5n_f}{9} \right) g_2^2 T^2, \quad (5.5)$$

where  $N_s = 1$  is the number of Higgs doublets and  $n_f = 3$  is the number of families. The couplings are only needed at tree level,

$$g_{i,3}^2 = g_i^2 T \quad (i = 1, 2, 3), \quad \lambda_3 = \lambda T, \quad h_1 = g_1^2 y_\varphi^2 T, \quad h_2 = \frac{1}{4} g_2^2 T \quad (5.6)$$

and also the new parameters in  $\delta\mathcal{L}_{\text{soft}}$ ,

$$\rho_1 = 2\mu_\varphi y_\varphi g_1, \quad \rho_2 = 2\mu_\varphi g_2. \quad (5.7)$$

In our calculation for the soft contributions we encounter the standard 1-loop integrals

$$I_0(m) = \int_{\mathbf{k}} \ln(k^2 + m^2) = \frac{2m^d \Gamma(1 - \frac{d}{2})}{d (4\pi)^{d/2}} = -\frac{m^3}{6\pi} + O(\varepsilon), \quad (5.8)$$

$$I_1(m) = \int_{\mathbf{k}} \frac{1}{(k^2 + m^2)} = m^{d-2} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{d/2}} = -\frac{m}{4\pi} + O(\varepsilon). \quad (5.9)$$

In the case  $m = m_{3,\mu_\varphi}$  we expand in powers of  $\mu_\varphi^2$ ,

$$I_0(m_{3,\mu_\varphi}) = -\frac{m_3^3}{6\pi} + \frac{\mu_\varphi^2 m_3}{4\pi} + O(\mu_\varphi^4), \quad (5.10)$$

$$I_1(m_{3,\mu_\varphi}) = -\frac{m_3}{4\pi} + \frac{\mu_\varphi^2}{8\pi m_3} + O(\mu_\varphi^4). \quad (5.11)$$

The only 2-loop integral we need is [6, 9]

$$\begin{aligned} I(m_a, m_b, m_c) &= \int_{\mathbf{k}_1, \mathbf{k}_2} \frac{1}{(\mathbf{k}_1^2 + m_a^2)(\mathbf{k}_2^2 + m_b^2)[(\mathbf{k}_1 + \mathbf{k}_2)^2 + m_c^2]} \\ &= \frac{1}{16\pi^2} \left[ \frac{1}{4\varepsilon} + \ln \left( \frac{\bar{\mu}}{m_a + m_b + m_c} \right) + \frac{1}{2} \right] + O(\varepsilon). \end{aligned} \quad (5.12)$$

where  $\bar{\mu}$  is the  $\overline{\text{MS}}$  scale parameter. In the special case  $m_a = m_{3,\mu_\varphi}$ ,  $m_b = m \in \{0, m_{D,1}, m_{D,2}\}$  and  $m_c = m_{3,\mu_\varphi}$  it is useful to expand in  $\mu_\varphi^2$ ,

$$\begin{aligned} I(m_{3,\mu_\varphi}, m, m_{3,\mu_\varphi}) &= \frac{1}{16\pi^2} \left[ \frac{1}{4\varepsilon} + \ln \left( \frac{\bar{\mu}}{2m_{3,\mu_\varphi} + m} \right) + \frac{1}{2} \right] \\ &= \frac{1}{16\pi^2} \left[ \frac{1}{4\varepsilon} + \ln \left( \frac{\bar{\mu}}{2m_3 + m} \right) + \frac{1}{2} + \frac{\mu_\varphi^2}{m_3(2m_3 + m)} \right] + O(\mu_\varphi^4). \end{aligned} \quad (5.13)$$

## 6 Soft contributions for soft Higgs mass

In this section we consider temperatures high enough so that  $m_3^2$  is of order  $(gT)^2$  and positive. At lower temperatures, close to the electroweak crossover, the thermal mass squared can be almost canceled by the negative zero temperature  $m_0^2$ , making  $m_3^2$  smaller than  $O(g^2T^2)$ . This case will be discussed in section 7.

At 1 loop we have

$$\text{Diagram} = -2TI_0(m_{3,\mu_\varphi}) = 2T \left( \frac{m_3^3}{6\pi} - \frac{\mu_\varphi^2 m_3}{4\pi} + O(\mu_\varphi^4) \right). \quad (6.1)$$

At 2 loops the Higgs self-interaction gives

$$\begin{aligned} \frac{1}{2} \text{Diagram} &= -6\lambda T^2 [I_1(m_{3,\mu_\varphi})]^2 \\ &= -\frac{3\lambda T^2}{8\pi^2} (m_3^2 - \mu_\varphi^2) + O(\mu_\varphi^4). \end{aligned} \quad (6.2)$$

Note that the  $\mu_\varphi^2$ -term has the same parametric form as the one in (4.8). The sum of (6.2) and (4.8) yields the  $O(\lambda)$  correction, that has been computed in [2] by a Higgs mass resummation. The interaction between Higgs and the gauge fields gives

$$\begin{aligned} \frac{1}{2} \text{Diagram} &= \frac{T^2}{4} (g_1^2 + 3g_2^2) \int_{\mathbf{k}_1, \mathbf{k}_2} \frac{(2\mathbf{k}_1 + \mathbf{k}_2)^2}{(\mathbf{k}_1^2 + m_{3,\mu_\varphi}^2) \mathbf{k}_2^2 [(\mathbf{k}_1 + \mathbf{k}_2)^2 + m_{3,\mu_\varphi}^2]} \\ &= -\frac{T^2}{4} (g_1^2 + 3g_2^2) \left\{ [I_1(m_{3,\mu_\varphi})]^2 + 4m_{3,\mu_\varphi}^2 I(m_{3,\mu_\varphi}, 0, m_{3,\mu_\varphi}) \right\} \\ &= \frac{\mu_\varphi^2 T^2}{32\pi^2} (g_1^2 + 3g_2^2) \left[ \frac{1}{2\varepsilon} + \frac{1}{2} + 2 \ln \left( \frac{\bar{\mu}}{2m_3} \right) \right] + \dots \end{aligned} \quad (6.3)$$

$$\begin{aligned} \frac{1}{2} \text{Diagram} &= -\mu_\varphi^2 T^2 [g_1^2 I(m_{3,\mu_\varphi}, m_{D,1}, m_{3,\mu_\varphi}) + 3g_2^2 I(m_{3,\mu_\varphi}, m_{D,2}, m_{3,\mu_\varphi})] \\ &= -\frac{\mu_\varphi^2 T^2}{32\pi^2} \left\{ y_\varphi^2 g_1^2 \left[ \frac{1}{2\varepsilon} + 1 + 2 \ln \left( \frac{\bar{\mu}}{2m_3 + m_{D,1}} \right) \right] \right. \\ &\quad \left. + 3g_2^2 \left[ \frac{1}{2\varepsilon} + 1 + 2 \ln \left( \frac{\bar{\mu}}{2m_3 + m_{D,2}} \right) \right] \right\} + \dots, \end{aligned} \quad (6.4)$$

$$\begin{aligned} \frac{1}{2} \text{Diagram} &= -\frac{1}{2} g_1^2 T^2 I_1(m_{3,\mu_\varphi}) I_1(m_{D,1}) - \frac{3}{2} g_2^2 T^2 I_1(m_{3,\mu_\varphi}) I_1(m_{D,2}) \\ &= -\frac{T^2}{32\pi^2} (g_1^2 m_{3,\mu_\varphi} m_{D,1} + 3g_2^2 m_{3,\mu_\varphi} m_{D,2}) \\ &= \frac{\mu_\varphi^2 T^2}{32\pi^2} \frac{1}{2m_3} (g_1^2 m_{D,1} + 3g_2^2 m_{D,2}) + \dots \end{aligned} \quad (6.5)$$

where we omitted terms of orders other than  $\mu_\varphi^2$ . Adding up all contributions we obtain the finite result

$$-\frac{12}{VT^2} [\tilde{\Omega}(\mu) - \tilde{\Omega}(0)]_{\text{soft}} = 2\mu_\varphi^2 \left\{ -\frac{3m_3}{\pi T} + \frac{9\lambda}{4\pi^2} + \frac{3}{32\pi^2} [g_1^2 C_1 + 3g_2^2 C_2] \right\} + O(\mu_\varphi^4) \quad (6.6)$$

with

$$C_i \equiv \frac{m_{D,i}}{m_3} - 1 - 4 \ln \left( \frac{2m_3}{2m_3 + m_{D,i}} \right). \quad (6.7)$$

After integrating out the soft fields we are left with an effective theory for the ultrasoft ones. For soft  $m_3$  the ultrasoft sector contains only the spatial gauge fields. At the order we are considering the effective Lagrangian is independent of  $\mu_\varphi$ , so that this sector does not contribute to the susceptibilities, and  $\tilde{\Omega}_{\text{ultrasoft}} = 0$ .

## 7 Ultrasoft Higgs mass

When  $m_3^2$  in (5.1) becomes small, the perturbative expansion used in section 6 can break down, which can be seen in (6.5) where  $m_3$  appears in the denominator. This term is of the same order as the soft 1-loop Higgs contribution if  $|m_3^2| \lesssim g^2 T m_D \sim g^3 T^2$ . For such small  $m_3$  it is necessary to include the Higgs field in an effective theory for momenta  $\ll gT$ , which is obtained by integrating out the temporal components of the gauge fields.

First consider  $\tilde{\Omega}_{\text{soft}}$ . Since  $m_3 \ll gT$  we have to put  $m_3 = 0$  in the diagrams in section 6. Then the only non-vanishing contribution comes from the diagram (6.4) with  $m_3 \rightarrow 0$ . The other diagrams in section 6 vanish in dimensional regularization. Then  $\tilde{\Omega}_{\text{soft}}$  contains an infrared divergence which will cancel against an ultraviolet divergence in  $\tilde{\Omega}_{\text{ultrasoft}}$ , leaving an order  $g^2 \ln(1/g) T^2 \mu_\varphi^2$  contribution to  $\tilde{\Omega}$ .

The effective Lagrangian for the ultrasoft fields now reads

$$-\mathcal{L}_{\text{ultrasoft}} = \frac{1}{4} F_{ij} F_{ij} + \frac{1}{4} W_{ij} W_{ij} - \varphi^\dagger \mathbf{D}^2 \varphi + \bar{m}_3^2 \mu_\varphi \varphi^\dagger \varphi + \bar{\lambda}_3 \left( \varphi^\dagger \varphi \right)^2 \quad (7.1)$$

with the parameters [8]

$$\bar{m}_3^2 = m_3^2 - \frac{1}{4\pi} (3h_2 m_{D,2} + y_\varphi h_1 m_{D,1}) \quad (7.2)$$

$$\bar{\lambda}_3 = \lambda_3. \quad (7.3)$$

The negative  $O(g^3 T^2)$  contribution to  $\bar{m}_3^2$  results from integrating out the temporal components of the gauge fields. It leads to interesting effects depending on *how* soft  $\bar{m}_3$  is.

Here we have to distinguish several cases. Consider first  $\bar{m}_3^2 \sim g^3 T^2$  and positive. Then we are still in the symmetric phase. The loop expansion parameter is now  $g^{1/2}$ . The next-to-leading order (NLO) starts only at  $O(g^{3/2})$  coming from the 1-loop diagram (6.1), and the 2-loop diagrams (6.2) and (6.3) contribute at order  $g^2$ . Combining this with the soft contribution we find

$$-\frac{12}{VT^2} \left[ \tilde{\Omega}(\mu) - \tilde{\Omega}(0) \right]_{\text{soft+ultrasoft}} = 2\mu_\varphi^2 \left[ -\frac{3\bar{m}_3}{\pi T} + \frac{9\lambda}{4\pi^2} + \frac{3}{32\pi^2} (g_1^2 \bar{C}_1 + 3g_2^2 \bar{C}_2) \right]. \quad (7.4)$$

with

$$\bar{C}_i \equiv -1 - 4 \ln \left( \frac{2\bar{m}_3}{m_{D,i}} \right). \quad (7.5)$$

Note that in this expression we have parametrically  $\ln(m_{D,i}/\bar{m}_3) \sim \ln(1/g)$ .

There is another way to obtain (7.4). Since we are only interested in the  $O(\mu_\varphi^2)$  terms we can expand the path integral

$$\exp(-\tilde{\Omega}_{\text{ultrasoft}}/T) = \int \mathcal{D}\Phi_{\text{ultrasoft}} \exp \left\{ \int d^3x \mathcal{L}_{\text{ultrasoft}} \right\} \quad (7.6)$$

to second order in  $\mu_\varphi$ . In (7.1)  $\mu_\varphi$  only appears in the effective Higgs mass so that

$$\left[ \tilde{\Omega}(\mu) - \tilde{\Omega}(0) \right]_{\text{ultrasoft}} = -VT\mu_\varphi^2 \langle \varphi^\dagger \varphi \rangle + O(\mu_\varphi^4). \quad (7.7)$$

The expectation value of  $\varphi^\dagger \varphi$  has been extracted from the 2-loop effective potential [6, 8, 13],

$$\langle \varphi^\dagger \varphi \rangle_{2\text{-loop}} = -\frac{\bar{m}_3 T}{2\pi} + \frac{T^2}{16\pi^2} \left\{ 6\lambda + (g_1^2 + 3g_2^2) \left[ \frac{1}{4\varepsilon} + \ln \left( \frac{\bar{\mu}}{2\bar{m}_3} \right) + \frac{1}{4} \right] \right\}, \quad (7.8)$$

which again leads to (7.4).

However, (7.7) is also valid when  $\bar{m}_3$  becomes as small as the magnetic screening scale  $g^2 T$  of the electroweak theory. In this case the only momentum scale left is  $g^2 T$ . In a non-abelian gauge theory the physics at this scale is non-perturbative, and the loop expansion can no longer be applied, which is the so called Linde problem [12]. Nevertheless, the expansion in  $g$  (modulo logarithms) still exists, only the numerical coefficients in the series cannot be computed by summing diagrams.

Since the 3-dimensional fields have mass dimension 1/2, and since the only mass scale in the ultrasoft theory is  $g^2 T$ , we have  $\langle \varphi^\dagger \varphi \rangle \sim g^2 T$ . Thus the ultrasoft fields contribute to  $\tilde{\Omega}$  at order  $g^2$ . A reliable determination of  $\langle \varphi^\dagger \varphi \rangle$  can only be done by lattice simulation of the 3-dimensional gauge plus Higgs system. A recent lattice study with  $m_H = (125\text{--}126)$  GeV for a SU(2) + Higgs theory can be found in [16]. An older but more comprehensive study of the SU(2) theory can be found in [17] and a study including the U(1) gauge fields has been performed in [18]. Near the electroweak crossover  $\langle \varphi^\dagger \varphi \rangle$  turned out to be a rather smooth function of the temperature.

Finally, for negative  $\bar{m}_3^2$  the Higgs field develops an expectation value, which in presence of chemical potentials for global charges also leads to a non-zero expectation value of the temporal component of the SU(2)-gauge field [4]. We have not studied this case.

## 8 Relation between $B$ and $B - L$

To illustrate the use of our results for  $\tilde{\Omega}$  we compute the relation between the baryon number  $B$  and  $B - L$  in the symmetric phase, which was done in [4] at leading order. For this purpose we introduce chemical potentials for  $B - L$ , and formally (cf. the discussion in section 1) also for  $B + L$ . We use equation (2.6), and express all chemical potentials (4.13) appearing in  $\tilde{\Omega}$  in terms of  $\mu_{B-L}$ ,  $\mu_{B+L}$ , and  $\mu_Y$ . All chemical potential matrices are proportional to the unit matrix in family space and are given by

$$\begin{aligned} \mu_q &= \frac{\mu_Y}{6} + \frac{\mu_{B-L} + \mu_{B+L}}{3}, \\ \mu_u &= \frac{2\mu_Y}{3} + \frac{\mu_{B-L} + \mu_{B+L}}{3}, \\ \mu_d &= -\frac{\mu_Y}{3} + \frac{\mu_{B-L} + \mu_{B+L}}{3}, \end{aligned}$$

$$\begin{aligned}
 \mu_\ell &= -\frac{\mu_Y}{2} + \mu_{B+L} - \mu_{B-L}, \\
 \mu_e &= -\mu_Y + \mu_{B+L} - \mu_{B-L}, \\
 \mu_\varphi &= \frac{\mu_Y}{2}.
 \end{aligned}
 \tag{8.1}$$

Then we enforce the saddle point condition (2.4) in order to express  $\mu_Y$  in terms of  $\mu_{B-L}$  and  $\mu_{B+L}$ , which gives  $\Omega'$  as defined in (1.3). Then (1.4) yields a linear relation between  $\langle B+L \rangle$  and  $\mu_{B-L}$ . Thus the expectation values of all asymmetries are proportional to  $\mu_{B-L}$ , which determines the baryon number in terms of  $\langle B-L \rangle$ ,

$$\langle B \rangle = \kappa \langle B-L \rangle.
 \tag{8.2}$$

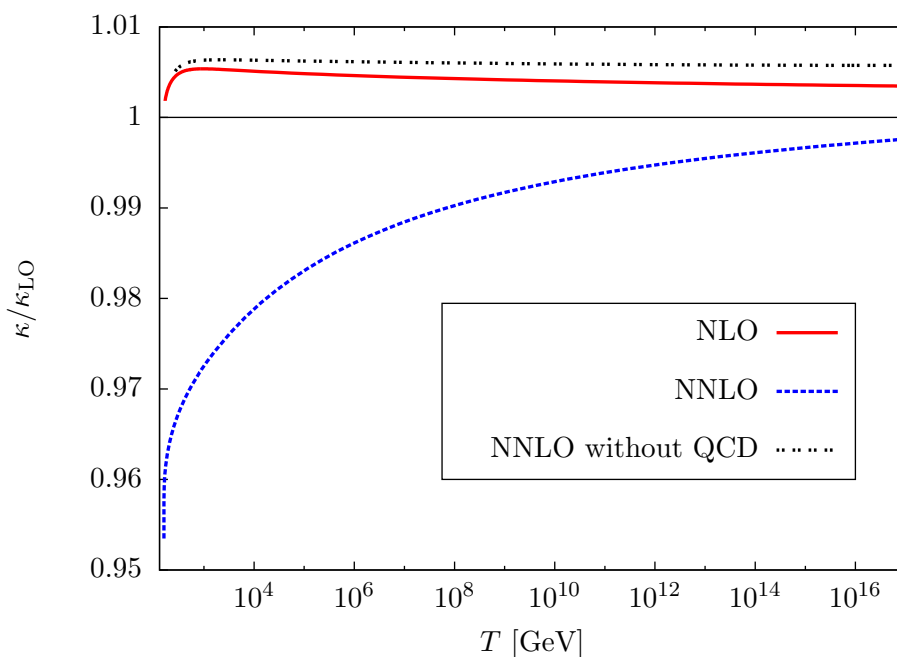
For  $m_3$  of order  $gT$  we obtain using (6.6)

$$\begin{aligned}
 \kappa &= \frac{4(2n_f + N_s)}{22n_f + 13N_s} + \frac{m_3}{\pi T} \frac{24n_f N_s}{(22n_f + 13N_s)^2} \\
 &+ \frac{g_1^2}{16\pi^2} \frac{236n_f^2 - (12C_1 - 212)n_f N_s + 75N_s^2}{(22n_f + 13N_s)^2} \\
 &+ \frac{g_2^2}{16\pi^2} \frac{9(12n_f^2 - 4(C_2 - 1)n_f N_s + 3N_s^2)}{(22n_f + 13N_s)^2} \\
 &- \frac{g_3^2}{16\pi^2} \frac{96(8n_f^2 + 11n_f N_s + 3N_s^2)}{(22n_f + 13N_s)^2} \\
 &+ \frac{h_t^2}{16\pi^2} \frac{6(6n_f^2 - 41n_f N_s - 18N_s^2)}{(22n_f + 13N_s)^2} \\
 &- \frac{\lambda}{16\pi^2} \frac{384n_f N_s}{(22n_f + 13N_s)^2} \\
 &- \frac{m_0^2}{(\pi T)^2} \frac{12n_f N_s}{(22n_f + 13N_s)^2},
 \end{aligned}
 \tag{8.3}$$

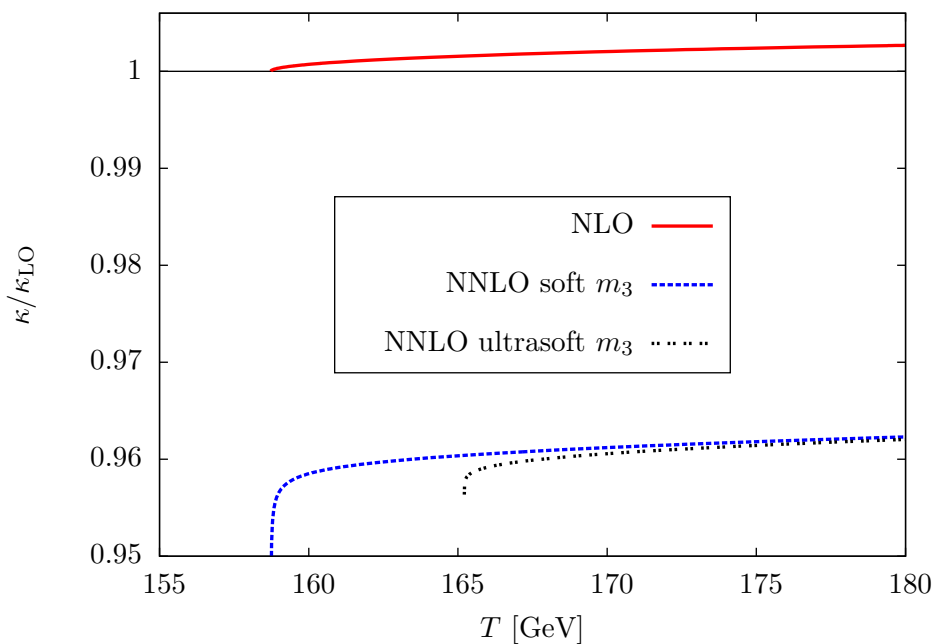
with the same definitions as in (5.5) and (6.7). When  $\bar{m}_3^2 \sim g^3 T^2$  the result for  $\kappa$  can be obtained from (8.3) by replacing  $m_3$  by  $\bar{m}_3$  and  $C_i$  by  $\bar{C}_i$  defined in (7.2) and (7.5).

The size of the corrections to  $\kappa$  are shown in figure 1 over a wide range of temperatures. The next-to-leading (NLO) corrections in Standard Model couplings are entirely due to the Higgs, and they are quite small. The next-to-next-to-leading order (NNLO) is significantly larger. This is caused by the relatively large QCD corrections. When the QCD corrections are left out, the remaining NNLO corrections are even smaller than the NLO, indicating that the perturbation series is well behaved. We also find that the NNLO Higgs correction has about the same size as the electroweak corrections coming from other chemical potentials.

Figure 2 shows a closer look at the most interesting region near the electroweak crossover at  $T \sim 160$  GeV. When  $m_3$  is treated as soft, the NNLO corrections diverge like  $1/m_3$  when  $m_3$  approaches zero. The perturbation series should be improved at small  $m_3$  by assuming  $\bar{m}_3 \sim g^{3/2} T$  and using (7.4). It then diverges logarithmically when  $\bar{m}_3$  vanishes. Clearly, the loop expansion breaks down here. However, since  $\langle \varphi^\dagger \varphi \rangle$  is rather smooth when computed non-perturbatively on the lattice, we expect that the result for  $\kappa$  using (7.7) with the non-perturbative  $\langle \varphi^\dagger \varphi \rangle$  [16–18] should be rather smooth as well. It should be given by a smooth extrapolation of the NNLO for ultrasoft  $m_3$  in figure 2 from higher to lower temperatures, without the sharp falloff.



**Figure 1.** Size of the radiative corrections to  $\kappa$  defined in (8.2) relative to the leading order result with  $m_H = 126$  GeV. The electroweak corrections are rather small, and the perturbation series is well behaved. The complete NNLO is dominated by the QCD corrections except at the highest temperatures.



**Figure 2.** The ratio of  $B$  and  $B - L$  at low temperatures with  $m_H = 126$  GeV. Shown are the LO, NLO and the NNLO result with soft and ultrasoft effective Higgs masses.

## 9 Conclusions

We have computed the  $O(g^2)$  Higgs contribution to the susceptibilities in the symmetric phase of the Standard Model, thus completing the  $O(g^2)$  calculation of [2]. Close to the electroweak crossover the loop expansion breaks down, and the infrared Higgs contributions are determined by the non-perturbative electroweak magnetic screening scale  $g^2 T$ . Nevertheless, the corrections are parametrically of order  $g^2$ . We have obtained a relation which can be used to determine its coefficient by a lattice simulation of the 3-dimensional gauge field plus Higgs theory. We have applied our result to compute the relation of  $B$  and  $B - L$ . The corrections are small in the regime where perturbation theory is valid. Our results indicate that this holds even when perturbation theory breaks down. We find that the QCD corrections dominate except at the highest temperatures, and that the corrections are below 5%.

For leptogenesis our result completes the  $O(g^2)$  computation of the washout rate [2]. Now two out of three rates<sup>5</sup> entering leptogenesis computations have been obtained at this order, the only missing piece being the  $CP$ -asymmetry.

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<sup>5</sup>The radiative corrections to the production rate are known both in the non-relativistic [13, 14] and relativistic regime [15].

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