Three Essays on the Economics of Patents

Inaugural-Dissertation zur Erlangung des Grades eines Doktors der Wirtschaftswissenschaften (Dr. rer. pol.) durch die Fakultät für Wirtschaftswissenschaften der Universität Bielefeld

vorgelegt von
Martin Harry Vargas Barrenechea

Bielefeld, July 6, 2015

Three Essays on the Economics of Patents

Inaugural-Dissertation zur Erlangung des Grades eines Doktors der Wirtschaftswissenschaften (Dr. rer. pol.) durch die Fakultät für Wirtschaftswissenschaften der Universität Bielefeld

vorgelegt von
Martin Harry Vargas Barrenechea

Bielefeld, July 6, 2015

Erstreferent: Prof. Dr. Herbert Dawid
Zweitreferent: Prof. i.R. Dr. Walter Trockel

Gedruckt auf alterungsbestndigem Papier nach DIN-ISO 9706

To my beloved family.

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to Prof. Dr. Herbert Dawid, my research supervisor, for his patient guidance, enthusiastic encouragement (out of the duty) and useful critiques of this research, I am one of most profited by his intelligence and imagination.

I would also like to extend my thanks to Prof. i.R. Dr. Walter Trockel for his guidance not only related to the career as economist and human being also for his continuous support and encouragement.

I owe a debt of gratitude to David Encaoua, Jeyhun Mammadov and John Sykes, whose comments helped to build this text.

Lastly I will like to thank the DFG for the financial support during my stay at the EBIM program in Bielefeld and Paris.

CONTENTS

1

INTRODUCTION

It is difficult nowadays to have a clear and definitive opinion of some topic. Each response relies on the shadow of the assumptions and circumstances. In the case of patents there is no difference respect to this point . The patent system in some situations is a hero and in other situations is a villain.

The area of patents is a very big research area as the innovation area is a big one, recently however some topics became hotter than others, one of this hot points is the explosion of patent litigation in the lasts decades. Those facts have pointed out the importance of the factors that drive the patent system as a system that incentive innovation and as legal framework that protects invention. Some of these factors are the patent strength and the complexity of the products consumed nowadays.

This new general technological state in developed economies have increased the complexity of the measurement of the positive and negative effects of paten systems. In special in the markets for manufactured products and in the markets for technology. So, the old school of innovation received new spices as the frequency of terms as probabilistic rights and patent fragmentation became common terms in the more often legal disputes for property rights (Farrell and Shapiro, 2008).

In this new scenario I consider this thesis as a small tour along the turbulent and complex universe of effects produced by the patent system, so chapters 2 and 3 are dedicated to the study of Patent litigation and chapter 4 evaluates the effects of patents in a dynamic scenario by using simulations.

The chapter 2 deals with the assessment of different damage rules (lost profits and unjust enrichment) and the impact of them in welfare and innovation. In this chapter we compare both damage rules in a very simple context with a linear demand for an homogeneous product, and a patented cost reduction innovation. We allow such patent be considered not just as an ironclad right also as a probabilistic right.

In chapter 3 we study in more detail the process of patent litigation incorporating the analysis of settlement as an additional option of agreement. So we use an earlier model by Aoki and Hu (2003) that deals with the effects of the legal system on the incentives for innovation and imitation. This model was developed for a product innovation and we extend it to deal with process innovations. A common knowledge reasonable royalty rate is taken as compensation for damages instead of the lost profits rule used in Aoki and Hu (2003). In the same way that the case of a product innovation, we analyse the impact of the legal system on innovation and welfare when the innovation to be considered is a cost reduction innovation.

Finally, the chapter 4 explores industrial dynamics and the effects of patents on a simple artificial industry, for this purpose we use the technology-performance space developed in Silverberg and Verspagen (2007) together with the basic schumpeterian dynamics of the Nelson and Winter models.

2

LICENSING PROBABILISTIC PATENTS AND LIABILITY RULES

In this chapter, a dynamic game is used to compare licensing of a cost reduction innovation under lost profits (LP) and unjust enrichment (UE), both damage rules used by courts in the calculation of damages when a patent has been infringed.

The novelty property right which belongs to a firm (patent holder) has a positive probability to be declared invalid in a court. The market is composed of two indistinguishable firms that compete in quantities (Cournot).

Licensing by using royalty rates is preferred to that of fixed fees. However, it is pragmatic to use basic licensing for non-drastic innovations and absolute licensing for radical innovations.

LP is better (in almost all cases) than UE for the industry and society. However in most cases consumers are better off under UE and in LP the patent holder benefits more.

2.1 INTRODUCTION

One of the most important mechanisms made for compensating and awarding innovation is the *patent system*. In this system there is an authority (i.e. the EPO in Europe) that gives rights of property on pieces of knowledge to an agent. This rights are known as *patents*.

Once a patent is granted, the patent holder has the exclusivity right to exploit the commercial potential of an innovation through a monopoly, by licensing to others or under other kind of contracts (i.e. cross licensing).

Economists have been interested in the incentives for licensing for example, when the inventor gives license to a firm due to an inability to exploit the commercial potential of the innovation (when its not a incumbent firm). However, licensing between competitors is not a uncomplicated decision, i.e in a Bertrand competition license the the monopoly profit is split, however it is expected that licensing in markets involve high levels of differentiation

Other topics that have received immense interest for economists is the contractual mechanism of licensing. Licensing contracts can be summarized in the groups : i) licensing just by a royalty rate; ii) just by a fixed fee and iii) a combination of both. The common approach has been game theory, in this the patent holder and one or several players are involved in a game of three stages:

1. at the first stage of the game, the patent holder decides how much to ask for the licenses and how many licenses he/she will offer;
2. at the second stage potential licensees decide whether to purchase the license or to use the backstop technology [1];
3. finally in the last stage, firms compete in the market[2].

The early literature has assumed that patents are indisputable property rights- *ironclad property rights*. Several authors have compared fixed fees against royalty rates under different conditions: insider/ outsider inventor and duopolistic/ monopolistic/ perfect competence. Sen and Tauman (2007) summarize and extend the early models. They consider a contract where royalties and fixed fees can be included together, the innovation is a cost reduction one (drastic or non-drastic[3]) and a outsider/incumbent inventor. They conclude that:

1. there is full diffusion of the innovation;
2. consumers are better off, firms are worse off and welfare is improved.
3. the optimal license contract includes a positive royalty rate for non-drastic innovations.
4. outsider innovator license by a fixed fee just if the market is a monopoly and the innovation is drastic.

In an ideal world it should be expected that primarily the ideas that increase the well being of the society should be patented. But the procedure for to select which ideas are valuable or which are not valuable it is not perfect and creates some collateral effects, as result patents could be declared invalid in court procedures.

Lemley and Shapiro (2005) and others have pointed out that half of all litigated patents are found to be invalid, some of them with considerable commercial importance. In consequence of the high quantity of litigations, nowadays economists have changed the concept of patent from the "right to exclude others" to "the right to sue others for infringement".

In practice, when potential users decide to infringe a patent, the patent holder could enforce the property rights by using the legal

1 The best technology available without the use of the innovation

2 see Kamien and Tauman (2002) and Sen and Tauman (2007) for a survey about licensing games under ironclad rights

3 In the case of a drastic innovation, the industry becomes a monopoly unless licensing is allowed.

system, in this arena the patent holder will try to prove *infringement* and the infringer(s) will try to invalid the patent. If the patent holder is successful in proving infringement, the court could authorize a compensation or damage payments and could order other actions in order to enforce the property rights. Commonly two liability rules are used to calculate damage payments: *Lost Profits (LP)* and *Unjust Enrichment (UE)* [4].

When the inventor is incumbent in the market, damages could be calculated in different ways, the most common way to do it is using the LP rule or the UE rule, both rules are based in a profile scenario. the *"no infringement"* scenario. Nevertheless it is more difficult when the inventor is outside the market because there is not a well defined profile scenario, because the value of innovations is unknown and could be estimated by several methods.

By comparing with the base scenario, LP compensate the share of profit lost by the patent holder caused by the infringement and UE transfers the competitor's profit excess to the patent holder. The impacts of this damages have been studied in different contexts as vertical relationship (outside inventor) and horizontal competition (incumbent inventor).

In the case of vertical relationship Schankerman and Scotchmer (2001) have analyzed how liability rules protect patents, they conclude that UE protects the patent holder better than LP in the case of research tools, however in the case of cost reduction innovations these results are reverse.

Anton and Yao (2007) explore the impacts of the LP rule on competence and innovation, assuming a linear demand scheme and a non-drastic innovation, they conclude that infringement is a dominant situation even under the use of different liability rules. In the other fold Choi (2009) compares different liability rules assuming a drastic innovation and a more general demand function. He concludes that LP benefits more to the patent holder.

The objective of this paper is to fill the gap left by the recent literature related to the licensing and liability rules. Firstly, I will try to compare LP against UE. Secondly I will compare fixed fees against royalty rates and its relation with damage rules

The starting point of my research is the contribution developed by Wang (1998), where he develops a duopoly model to study licensing under ironclad patents under a Cournot scenario. In this model royalty rate scheme is compared against fixed fee licensing for drastic and non-drastic innovations. Under this base model I add the development made by Anton and Yao and Choi (AYC) to include probabilistic patents in a take or leave it ex-post[5] licensing situation.

4 see Heath et al. (2002) for a complete comparison of damage rules between countries.
5 ex-post innovation or ex-ante trial.

In a difference of AYC I use a linear demand with homogeneous firms and homogeneous costs, this specification allow me to study drastic and non-drastic innovations. Also I compare the royalty rate scheme against the fixed fee scheme assuming probabilistic patents.

Surprisingly, my results show that licensing is impossible under UE and just significant innovations are licensed under the LP rule, for another side it is shown that licensing using a royalty rate is better than a fixed fee scheme from the point of view of the patent holder. Finally, comparison analysis shows that LP protects the patentee for large innovations better, and for small ones the patentee is better protected by UE.

This article is organized as follows. In section 2 assumptions are established and it is included a description of the licensing game. In the sections 3, 4 and 5 the game is solved. In section 6 a comparative analysis between LP and UE is executed. In section 7 the conclusions and important remarks of this work are analyzed. Proofs of the propositions are shown in the text and lengthy proofs are treated in an appendix.

2.2 THE GAME

The game is a non cooperative game that involves two players: patent holder (firm 1) and a competitor (firm 2), they produce the same good under fixed marginal costs c, and without loss of generality it is assumed that $c = 0$.

Let $p = 1 - q_1 - q_2$ be the inverse linear demand function that both face, where the subindex 1 is for the patent holder and the subindex 2 is for the competitor.

The *firm* 1 has patented a cost reduction innovation that reduces the marginal cost by γ, where $0 < \gamma$.

Let $\pi_i^s(q_i, q_j) = (1 + \gamma - q_1 - q_2)q_i$ be the profit associated with the use of the new cost reduction innovation by the firm i and let $\pi_i^i(q_i, q_j) = (1 - q_1 - q_2)q_i$ be the profit associated with the use of the old technology.

At the very beginning of the game the patent holder decides whether to license ($\mathcal{L}'$) or not ($\mathcal{N}'$). If decides licensing, he offers a fixed fee (F) or a royalty rate(r), and the offer is made it in the way take it or leave it, so if the offer is rejected there is not a new offer.

When the patentee decides to license the innovation by asking by a royalty rate (r) or a fixed fee (F), after the offer is made it by the patentee, the competitor chooses one the following three alternatives: 1) accept the offer of the patent holder ($\mathcal{L}$); 2) uses the backstop technology ($\mathcal{N}$) and 3) Infringe the patent ($\mathcal{I}$). If there is not a license offer by the patentee, the competitor just chooses between to use the backstop technology $\mathcal{N}'$ or to infringe the patent ($\mathcal{I}'$) (see Figure 1 below).

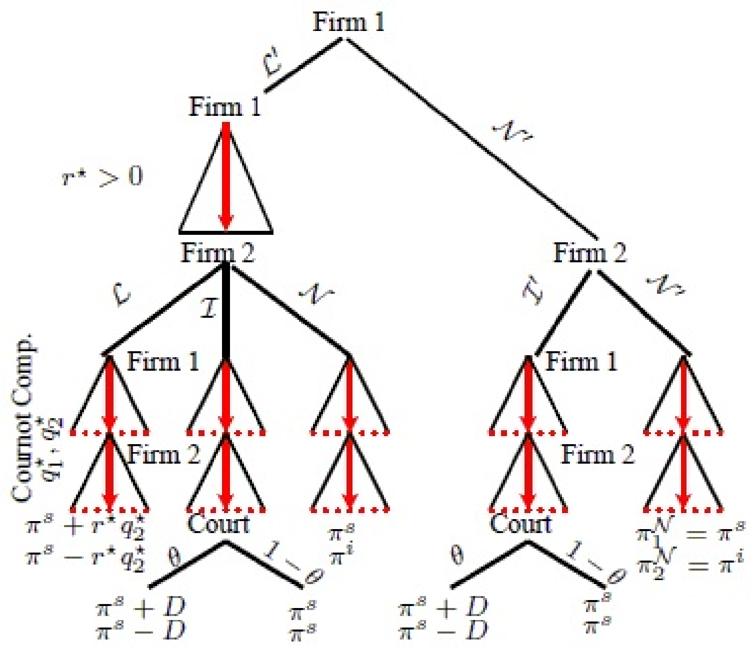

Figure 1.: Game with a royalty rate licensing scheme

Later firms decide the quantities to be offered in the market as solution of a Cournot game. In the cases that the competitor infringes the patent the patent holder reacts by starting a process in a court, with the objective to enforce its property rights. The result of the trial is unknown, but there is a common knowledge probability $\theta \in (0,1)$ that the patent will be declared valid after the trial, this parameter also reflects the *strength of the patent*.

Once the patent holder proves in court the existence of infringement, the court calculate damage payments, for calculations LP and UE are considered as estimative of a fair payment. The method that is going to be used by the court for calculating damages is common knowledge before trial.

Payoffs are characterized through the actions of the competitor, for example if the patent holder chooses not to license $\mathcal{N}'$ and the competitor chooses to use the backstop technology $\mathcal{N}$. Firms will be on the same situation if the patent holder chooses to license $\mathcal{L}'$ and asks a royalty rate r and the competitor decides to use the backstop technology $\mathcal{N}'$. Notice that in both situation players should choose the same quantities, so the payoffs in the terminal stories are the following:

1. $\pi_1^{\mathcal{N}} = \pi_1^s(q_1, q_2)$ and $\pi_2^{\mathcal{N}} = \pi_2^i(q_1, q_2)$, when competitor plays $\mathcal{N}$.

2. $\pi_1^{\mathcal{L}} = \pi_1^s(q_1, q_2) + L(q_2)$ and $\pi_2^{\mathcal{L}} = \pi_2^s(q_1, q_2) - L(q_2)$, when competitor plays $\mathcal{L}$, notice that L is the license's revenue (originated by a fixed fee or a royalty rate).

3. $\pi_1^{\mathcal{I}} = \pi_1^s(q_1, q_2) + \theta D(q_1, q_2)$ and $\pi_2^{\mathcal{I}} = \pi_2^s(q_1, q_2) - \theta D(q_1, q_2)$, happens when competitor plays $\mathcal{I}$, where D is the damage payment.

The solution criterion for the game described above that we are going to use is the Sub-Game Perfect Nash Equilibrium (SPNE), such solution is developed in the next three sections.

2.3 EQUILIBRIUM OUTCOMES

2.3.1 *Competition Stage*

This section is devoted to calculating the equilibrium payoffs under different scenarios as a solution of the Cournot problem.

Given a defined rule for the calculations of damages (LP or UE), a level of technology chosen by the incumbent firm ($\mathcal{N}$, $\mathcal{I}$, $\mathcal{L}$) and a licensing policy defined by the patent holder (to offer or not a license to the competitor using a fixed fee or a royalty rate), both firms compete by choosing quantities.

When the competitor decides to use the backstop technology ($\mathcal{N}$), the *Nash Equilibrium (NE)* is granted as

$$(q_1^{\mathcal{N}}, q_2^{\mathcal{N}}) = \begin{cases} \left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}, 0\right) & \text{if } 1 \leq \gamma \end{cases} \tag{1}$$

As was noted by Arrow (1962) big innovations could permit to the patent holder to reduce the price till levels below the competitive prices, this allows the patent holder to dominate the market. These kind of innovations are called *drastic*. In this particular setup an innovation is *non-drastic* if $0 \leq \gamma < 1$ and is defined *drastic* if $\gamma \geq 1$, and then equilibrium payoffs in this case are:

$$\begin{aligned} \pi_1^{\mathcal{N}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \\ \pi_2^{\mathcal{N}} &= \begin{cases} \left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ 0 & \text{if } 1 \leq \gamma \end{cases} \end{aligned} \tag{2}$$

A more complex situation emerges when the competitor infringes the patent ($\mathcal{I}$), once infringement is identified the patent holder will try to enforce the property rights by suing the incumbent firm. When the patent holder is successful in court (gains the trial), it is assumed here that the court will calculate a damage, payment based in the LP or UE rule, so profits are:

$$\begin{aligned} \pi_1^{\mathcal{I}} &= (1 - q_1 - q_2 + \gamma)q_1 + \theta D(q_1, q_2) \\ \pi_2^{\mathcal{I}} &= (1 - q_1 - q_2 + \gamma)q_2 - \theta D(q_1, q_2) \end{aligned} \tag{3}$$

Notice that the first term of the r.h.s. in the eq.(3), $(1 - q_i - q_j + \gamma)q_i$ is the profit gained by the sales and the second term represents the damage payments $\theta D(q_1, q_2)$.

Basically UE and LP both need a comparison scenario of *"no infringement"*, here $\pi_1^{\mathcal{N}}$ is used as the comparison value when LP is the liability rule used by the court, and then under LP, we have that the damage payments are:[6].

$$D^{LP} = \max\left\{\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1, 0\right\} \tag{4}$$

When the court uses UE as liability rule, the damage (D^{UE}) is the excess of profit of the competitor respect to $\pi_2^{\mathcal{N}}$, then

$$D^{UE} = \max\left\{(1 - q_1 - q_2 + \gamma)q_2 - \pi_2^{\mathcal{N}}, 0\right\} \tag{5}$$

The NE when damages are calculated by using the LP rule, and when the incumbent firm decides to infringe the patent deserves a special treatment. Given the structure of the damage rules quantities affect the level of damages, so then, the expected damage affects the equilibrium quantities.

Lemma 1. *The Cournot solution when competitor infringes and court uses LP rule for calculate damages is,*

$$(q_1^{\mathcal{I},\mathcal{LP}}, q_2^{\mathcal{I},\mathcal{LP}}) = \begin{cases} \left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right) & \text{if } \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right) & \text{if } \gamma \geq \frac{\theta}{3-2\theta} \end{cases} \tag{6}$$

, it produces

$$\begin{aligned} \pi_1^{\mathcal{I},\mathcal{LP}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma < \frac{\theta}{3-2\theta} \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \\ \pi_2^{\mathcal{I},\mathcal{LP}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \end{aligned} \tag{7}$$

6 Interested readers could see Anton and Yao (2007) for a more detailed analysis for non drastic innovation and a more general linear demand.

When $q_2^{\mathcal{I},\mathcal{LP}} = q_2^{\mathcal{N}}$ eq. (6), the patent holder gets the same profit as in the situation of no infringement. However, , the competitor has an advantageous situation because he enjoys a lower costs and can produce the same quantity that could have been produced under no infringement. Anton and Yao (2007) calls this equilibrium *Passive Infringement*, because the damage payment does not reflect the effects of the infringement.

When $\gamma > \frac{\theta}{3-2\theta}$ damage payments calculated with the lost profit rule are positive in equilibrium, then in equilibrium a *Active Infringement* is present, such situation means that the infringer does not care about to maintain the profit of the patent holder at the not infringement's profit level.

Lemma 2. *The Cournot solution when competitor infringes and court uses UE as liability rule is,*

$$(q_1^{\mathcal{IUE}}, q_2^{\mathcal{IUE}}) = \left((1-\theta)\frac{1+\gamma}{3-\theta}, \frac{1+\gamma}{3-\theta}\right) \tag{8}$$

Results in the lemmas 1 and 2 cannot be considered trivial, because the best replies that produces the NEs are non-smooth in both cases. Proofs of these lemmas are considered in the appendix[7].

By using the lemma 2,

$$\begin{aligned} \pi_1^{\mathcal{IUE}} &= \begin{cases} \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \\ \pi_1^{\mathcal{IUE}} &= \begin{cases} (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \end{aligned} \tag{9}$$

When the competitor accepts the offer ($\mathcal{L}$) against a given fixed fee (F), the following NE is obtained:

$$\left(q_1^{\mathcal{L},\mathcal{F}}, q_2^{\mathcal{L},\mathcal{F}}\right) = \left(\frac{1+\gamma}{3}, \frac{1+\gamma}{3}\right) \tag{10}$$

, and when the competitor accepts the offer ($\mathcal{L}$) against a given royalty rate (r), the following NE is obtained:

$$\left(q_1^{\mathcal{L},\mathcal{R}}, q_2^{\mathcal{L},\mathcal{R}}\right) = \left(\frac{1+\gamma+r}{3}, \frac{1+\gamma-2r}{3}\right) \tag{11}$$

7 Anton and Yao (2007) prove the lemma 1 and claimed that the lemma 2 is true, in the appendix I offer the proof for the lemma 2 and an alternative proof for the lemma 1.

These results produce the following equilibrium payoffs: for the fixed fee case

$$\begin{aligned} \pi_1^{\mathcal{L},\mathcal{F}} &= \left(\tfrac{1+\gamma}{3}\right)^2 + F \\ \pi_2^{\mathcal{L},\mathcal{F}} &= \left(\tfrac{1+\gamma}{3}\right)^2 - F \end{aligned} \tag{12}$$

; and

$$\begin{aligned} \pi_1^{\mathcal{L},\mathcal{R}} &= \left(\tfrac{1+\gamma+r}{3}\right)^2 + r\tfrac{1+\gamma-2r}{3} \\ \pi_2^{\mathcal{L},\mathcal{R}} &= \left(\tfrac{1+\gamma-2r}{3}\right)^2 \end{aligned} \tag{13}$$

for the royalty rate case.

2.3.2 *Competitor's Technology Stage*

By assuming the type of contract offered is known (a royalty rate or a fixed fee). To solve the game its necessary to analyze the behavior of the competitor in respect to the technology choice, where the alternatives are:

1. not infringe the patent $\mathcal{N}$ (use the backstop technology);
2. infringe the patent $\mathcal{I}$ (use the new technology without a permission of the patent holder);
3. accept to pay for the use of the new technology if a license is offered $\mathcal{L}$.

Lemma 3. *If the courts calculates damages using the* LP *rule or the* UE $\pi_2^{\mathcal{I}} \geq \pi_2^{\mathcal{N}}$.

Lemma 3 says that the competitor always prefers to infringe instead of using the backstop technology independently of the liability rule[8]. Then, it is necessary to just compare the competitor's payoff under licensing $\pi_2^{\mathcal{L}}$ against the payoff under infringement $\pi_2^{\mathcal{I}}$.

Let $\underline{F}$ be a fixed fee $F \geq 0$ such that $\pi_2^{\mathcal{L},\mathcal{F}} - \pi_2^{\mathcal{I}} = 0$, then

$$\underline{F} = \left(\frac{1+\gamma}{3}\right)^2 - \pi_2^{\mathcal{I}} \tag{14}$$

, notice that if $\underline{F}$ is negative there is no positive fixed fee that makes the licensing option as good as infringe for the competitor. In the appendix is proved that

Lemma 4. $\underline{F}^{\mathcal{LP}} \geq 0$ *but* $\underline{F}^{\mathcal{UE}} \geq 0$ *just if* $\gamma \geq \delta_1$*, where*

$$\delta_1 = \frac{12 - 5\theta + \theta^2 - 2\sqrt{(3-\theta)^2(3+\theta)}}{6 - 7\theta + \theta^2}$$

8 This result is also true under other liability rules as *lost royalties* and for more general specifications, see AYC.

For the case of licensing under a royalty rate, let $\underline{r}$ be a royalty rate $r \geq 0$ that makes $\pi_2^{\mathcal{L},\mathcal{R}} - \pi_2^{\mathcal{I}} = 0$, then if exists

$$\underline{r} = \frac{1+\gamma-3\sqrt{\pi_2^{\mathcal{I}}}}{2} \tag{15}$$

From eq. 12 and 13:

1. $\partial(\pi_2^{\mathcal{L},\mathcal{F}} - \pi_2^{\mathcal{I}})/\partial F < 0$;
2. $\partial(\pi_2^{\mathcal{L},\mathcal{R}} - \pi_2^{\mathcal{I}})/\partial r < 0$
3. $\pi^{\mathcal{L},\mathcal{F}} = \pi^{\mathcal{L},\mathcal{R}} > 0$ if $F = r = 0$
4. $\pi^{\mathcal{L},\mathcal{F}}$ and $\pi^{\mathcal{L},\mathcal{R}}$ are bounded.

then so, it is possible to create a one to one function between$\underline{r}$ and $\underline{F}$, in consequence

Lemma 5. $\underline{F} > 0$ *iff* $\underline{r} > 0$

by using the lemma 4 and 5, it is establish that

Lemma 6. *In the LP case always exist a positive fixed fee $\underline{F}$ (or royalty rate $\underline{r}$) such that $\pi_2^{\mathcal{L},\mathcal{F}} \geq \pi_2^{\mathcal{I},\mathcal{LP}}$ (or $\pi_2^{\mathcal{L},\mathcal{R}} \geq \pi_2^{\mathcal{I},\mathcal{LP}}$). However in the UE case the last statement is true just for $\gamma > \delta_1$.*

2.3.3 *Licensing Stage*

In a take it or leave it bargaining, the patent holder will ask for the fixed fee that makes the competitor indifferent between take the license or to infringe.

From eq. (12) it is observable that the patent holder will choose the greater F that allows the competitor to enjoy the same profit whilst under infringement, so then

$$F^\star = \underline{F}$$

In the case of the royalty rate from the eq. 13 it is easy to see that the patent holder gets the maximum level of fees when $r = \frac{1+\gamma}{2}$ and $\pi_2^{\mathcal{L},\mathcal{R}} = 0$, because $\underline{r} \leq \frac{1+\gamma}{2}$, so then, the patent holder will ask

$$r^\star = \underline{r}$$

as a royalty rate in exchange of a license, summarizing

Lemma 7. *When a licensing contract is offered the patent holder will ask for $F^\star = \underline{F}$ ($r^\star = \underline{r}$) when a fixed fee (royalty rate) is asked against the license.*

By using the definition of $\underline{r}$ (eq. (15)) in the payoff function $\pi_1^{\mathcal{L},\mathcal{R}}$ from eq.(13), the patent holder's payoff is

$$\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\pi_2^{\mathcal{I}} \tag{16}$$

From eq.((14)), $\pi_2^{\mathcal{I}} = \left(\frac{1+\gamma}{3}\right)^2 - F^\star$, by replacing $\pi_2^{\mathcal{I}}$ in the last equation produces

$$\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left[\left(\frac{1+\gamma}{3}\right)^2 - F^\star\right] \tag{17}$$

and by using eq. (12), produces

$$\pi_1^{\mathcal{L},\mathcal{R}} - \pi_1^{\mathcal{L},\mathcal{F}} = F^\star/4 \geq 0$$

, summarizing.

Proposition 1. *The patent holder will prefer to license using a royalty rate scheme instead or a fixed fee scheme.*

When the patent holder does not offer a license, the competitor infringes the patent, so then the patent holder has to compare $\pi_1^{\mathcal{L}}$ against $\pi_1^{\mathcal{I}}$ in order to offer or not a license. Then by comparing these payoffs we get that

Proposition 2. *The patent holder will never license under UE. However under LP a royalty rate's license is offered if* $\gamma > \delta_2$, *where*

$$\delta_2 = \frac{\theta(3-2\theta) + 3\sqrt{(3-\theta)^2(2-\theta)}}{18 - 15\theta + 4\theta^2}$$

.

In the last lemma δ_2 is near to 1, meaning that just big innovations are licensed when courts used the LP rule. This result also coincides with the result by Choi (2009) when he concludes that the royalty rate under UE is lower than the one under LP in a general demand case.

Here, in fact when UE is used by the court, there is not royalty rate in equilibrium, because the royalty rate is not so big to make the payoff under licensing big enough as the expected payoff under infringement, for the patent holder.

2.4 LP VS UE

In this section a comparison between LP and UE is made by using the results of previous sections.

By summarizing,

Lemma 8. *When the LP rule is used there are three possible scenarios:*

1. *passive infringement* $\gamma \leq \theta/(3-2\theta)$;
2. *active Infringement* $\gamma > \theta/(3-2\theta)$;
3. *licensing by a royalty rate* $\gamma > \delta_2$, *where* $0 \leq \theta/(3-2\theta) \leq \delta_2 \leq 1$.

However, when UE is used there is no licensing and the competitor infringes the patent in equilibrium.

Then so, the equilibrium payoffs under both regimes are

$$\left(\pi_i^{\mathcal{LP}}, \pi_i^{\mathcal{UE}}\right) = \begin{cases} \left(\pi_i^{\mathcal{I},\mathcal{LP}}, \pi_i^{\mathcal{I},\mathcal{UE}}\right) & 0 \leq \gamma < \delta_2 \\ \left(\pi_i^{\mathcal{L},\mathcal{R},\mathcal{LP}}, \pi_i^{\mathcal{I},\mathcal{UE}}\right) & \gamma \geq \delta_2 \end{cases}$$

where $i = 1, 2$, in consequence by using eq. 16 and 7 it is easy to obtain the equilibrium payoffs under the LP regime

$$\pi_1^{\mathcal{LP}} = \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma \leq \theta/(3-2\theta) \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \theta/(3-2\theta) \leq \gamma < \delta_2 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2\right) & \text{if } \delta_2 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2\right) & \text{if } 1 \leq \gamma \end{cases}$$

$$\pi_2^{\mathcal{LP}} = \begin{cases} \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma \leq \theta/(3-2\theta) \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \theta/(3-2\theta) < \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{18}$$

Because in the case of UE infringement is always present (see Lemma 3), then from eq. (9) the payoffs are

$$\pi_1^{\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases}$$

$$\pi_2^{\mathcal{UE}} = \begin{cases} (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{19}$$

and by comparing these equilibrium payoffs under LP against UE, is established that

Proposition 3. *The patent holder and the industry (competitor) are better off (is worse off) under LP (UE) for drastic and almost all the non-drastic innovations.*

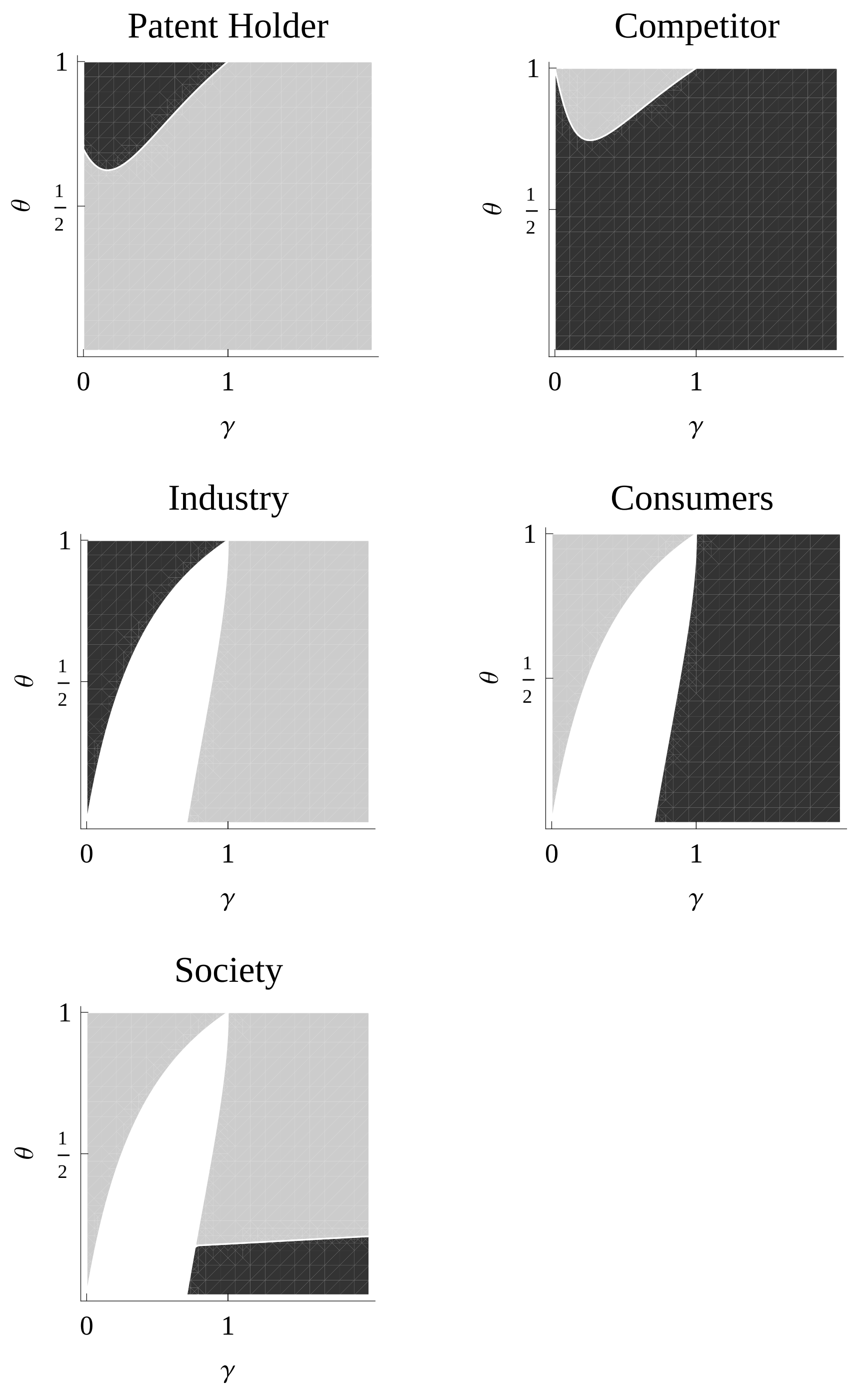

Figure 2.: LP against UE: $LP \succ UE$ in gray; $LP \prec UE$ in black; and $LP \approx UE$ in white

As is shown in the Figure 2 the patent holder is better off under LP (gray area) a mirror situation happens with the competitor that is better off under UE (black area), both situations are often observed except in a small area.

Schankerman and Scotchmer (2005) said that under LP the competitor is apprehensive about the losses of the patent holder, and in respect, its output is chosen endogenously to compensate the damages of the patent holder, so LP turns in a collusive mechanism of profit transfer.

When the effect on the industry is compared, industry is better off under UE in the area on passive infringement (black area), the reason comes from the fact that under passive infringement there are not transfers from the competitor to the patent holder under LP. This reduces the possibility to reach a collusive profit for the industry.

However, when active infringement is present both rules produce the same industry profit. This result coincides with the one found by Choi (2009) for a general quantity competition. Finally, when the innovation is drastic and LP is used there is licensing. This mechanism seems to be better than the UE damages infringement mechanism for sharing the surplus of the innovation. In a consequence a inverse situation is going to be observed by the consumers who are loosing surplus facing a higher price.(see lemma 4)

When a patent race is considered, the patent holder's equilibrium payoff is the reward of the winner and the competitor's payoff is the reward of the looser in the patent race, then as consequence of the proposition 3,

Proposition 4. *LP incentives more R&D than UE for drastic and almost all non-drastic innovations.*

In the case of the consumers, as consequence that the demand is linear, the consumer surplus is $(q_1 + q_2)^2/2 = Q^2/2$. Now, when the LP rule is used by using eq 6, 11 and lemma 8

$$Q^{\mathcal{LP}} = \begin{cases} \frac{2+\gamma}{3} & \text{if } 0 < \gamma < \theta/(3-2\theta) \\ (2-\theta)\frac{1+\gamma}{3-\theta} & \text{if } \theta/(3-2\theta) \leq \gamma < \delta_2 \\ \frac{2(1+\gamma)-r}{3} & \text{if } \gamma \geq \delta_2 \end{cases}$$

, now by using eq (7)

$$Q^{\mathcal{LP}} = \begin{cases} \frac{2+\gamma}{3} & \text{if } 0 < \gamma < \theta/(3-2\theta) \\ (2-\theta)\frac{1+\gamma}{3-\theta} & \text{if } \theta/(3-2\theta) \leq \gamma < \delta_2 \\ \frac{(1+\gamma)}{3} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2} & \text{if } \delta_2 \leq \gamma < 1 \\ \frac{(1+\gamma)}{3} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2} & \text{if } \gamma \geq 1 \end{cases} \tag{20}$$

For UE from eq (8)

$$Q^{\mathcal{UE}} = (2-\theta)\frac{1+\gamma}{3-\theta} \tag{21}$$

, so then, by direct comparison between $Q^{\mathcal{LP}}$ versus $Q^{\mathcal{UE}}$, it is established that

Proposition 5. *The consumers are better off under UE for drastic innovations, but under non-drastic innovations LP is at least as good as UE.*

Let $SW = Q^2/2 + \sum \pi_i$ be the social welfare, then by using eq (17),(18),(19) and (20), comparisons shows that

Proposition 6. *Under non-drastic innovations LP is at least as good as UE, for the society, but for drastic innovations society is better off under the LP (UE) for strong (weak) patents.*

When innovations are drastic and LP is used, there is licensing against a royalty rate. However, when patents are fragile (small θ), patentees have less power of bargain with , so then, patentees receive a comparative diminutive royalty rate, which in turn produces a fall in revenues of the patent holder and the industry (see Figure 2).

2.5 CONCLUSIONS

Throughout this chapter LP rule is compared against UE rule. As consequence of propositions 3 and 5 it is concluded that LP should be preferred. Firstly, because it has a positive impact on R&D and secondly it has a positive net effect on the society, even being considered a collusive mechanism.

Damages are a proven importance in determining the licensing terms and critical for the existence of a licensing contract (fixed fee or royalty rate). The results show that theres no licensing under UE, nevertheless, under LP just considerable innovations are licensing.

One implicit assumption of this model is the timing, this model lives and ends during the litigation time, by making so that injunctions have not been important, then even the results showed here predict no licensing, it could be possible when injunctions are considered[9].

Some questions remain unsolved, the first one is related to the objectives of damage rules in deterring infringement. The question in this direction is: Which new damage rule could deter infringement?, also an axiomatic point of view could be useful in such way that it could be important to know which ideal properties must be present in a damage rule.

9 see Farrell and Shapiro (2008) and Encaoua and Lefouili (2009) for the study of licensing when just injunctions are considered.

One of the problems with non-drastic innovations and lost profits are the non-smoothness of the payoff function, more work should be done trying to characterize some smooth approximation to this functions in a similar way as the approximation made it by Boone (2001) for characterizing intensity of competition.

Finally it seems important the study of damage rules with more than two firms as competitors or as innovators, Encaoua and Lefouili (2009) and Farrell and Shapiro (2008) study weak patents without consider a comparative approach between the different types of damage rules, such scenario provides many open questions about the collusive behavior of firms under different damage rules.

3

PROCESS INNOVATIONS, PATENT LITIGATION AND TIME EFFECTS

In this work we extend the model developed in (Aoki and Hu, 2003) in order to cover cost reduction innovations, instead of product innovations originally developed on that article. The results show that smaller innovations are more licensable. Regarding the time factors, infringers like faster innovation. Bigger innovations and longer imitation periods, under some suitable situations, litigation time could support innovation and discourage infringement. However the patent life has ambiguous effects and may promote infringement.

3.1 INTRODUCTION

Patents are, by their own right, an interesting topic of study, in part because there is no clear conclusion about the balance between the positive effects (promote innovations) and the negative ones (market power for example) of having a patent system. However, there are other points to take into account since nowadays, some of them relate to the complexity of patent rights. Such complexities are derived from the actual development of science and the efficiency of the legal system to determine whether an invention is in fact a nontrivial improvement of knowledge.

The other dimension is related with the complexity of the actual technology, i.e. a cell phone needs more than one hundred different patented technologies. So in many cases a developer of a product faces several patent holders in order to develop a final product, and several of those technologies could complement one of the others (fragmented patents).

The term probabilistic patents has its origins on the possibility that a patent can be declared invalid in a court. This happens because the control of the patent office is not absolute and sometimes that institution endorses patents to innovations that do not fulfill the requirements to be patented (most commonly inventive step). Even firms dealing with market competition could hold these weak patents (patent with high probability to be declared invalid under litigation)and license its competitors taking together with the other firms the market prize of the patent (Farrell and Shapiro, 2008).

There are several studies about patent litigation; the first ones study their relationship with settlements by using bargaining models. Inside these models they try several variations of consequences on the information of probabilistic patents. (Bebchuk, 1984; Meurer, 1989), in these earlier models the failure to reach agreements are mere consequence of information failures.

Another group of important studies come from Aoki and Hu (1999) that analyze the effect of the legal system on licensing and litigation, they characterize the legal system by the strength of patents and legal costs. They conclude that a legal system that induces a monopoly power incentive research, also they found that longer litigation is better for innovator and imitators (Aoki and Hu, 2003); those results were found for the case of product innovations.

Nevertheless, some analysts think that in the last step, the strength of a patent comes from a position of the responsible court that in some cases could be pro patent protection (Bessen and Meurer, 2005). This proposition has important consequences on markets and social welfare in the actual system of incentives to innovation and in all sectors and systems related with innovations. Many of these arguments and results of several years of empirical and theoretical research summarized in Bessen and Meurer (2008), show the importance of the study of the actual patent litigation policies and its deficiencies in some cases.

Consequently, there is a gap related to patent litigation and its relationship with innovation, licensing and settlement when the innovation is a process innovation. When an innovation is a process innovation there are several more difficulties, because in the case of product innovations markets with two competitors are duopolies with the same technology or just a monopoly. However in the case of process innovations, the competitors stay in the market even if they do not exploit the innovation and have an inferior technology.

In a market with two firms, firm 1 (patentee) and firm 2 (competitor, potential infringer or licensee) this work attempts to explore the effects of time factors in specific litigation and imitation times on the licensing, litigation and settlement of process innovations. In this way we use the base model by Aoki and Hu (2003). However it is necessary to work on some specific duopoly and simple games, in order to explain what is going to happen in those scenarios that differ from those that product innovations produce. In the section 3.2 we solve some static duopoly games that help out to describe a model that models licensing, imitation, litigation and settlement of a cost reduction innovation in a one shot Cournot game. Afterwards in the section 3.3 we describe the main model, that is basically a game in extensive form. This model is solved by backward induction throughout the subsections 3.4.1 to 3.4.4. After then we use the results found in the previous sections and compare these results in the sense of social welfare (consumer surplus added to the profits of the firms). Finally we finish with concise conclusions.

3.2 PARTICULAR EQUILIBRIA

It is important before considering our main model, to study some simple models which are later going to be used in order to extend the model by Aoki and Hu (2003) and to include process innovations instead of product innovations. So in this section we solve three simple duopoly models with linear demands, the setup being as follows:

- There are two firms: firm 1 and firm 2
- firm 1 and firm 2, both produce a homogeneous good and face an inverse linear demand given by:

$$p = 1 - q_1 - q_2 \tag{22}$$

  - both firms compete in this markets choosing quantities (Cournout).
- Without loss of generality, we assume that the firms initially produce under a constant marginal cost of $c_i = 0$;
- The firm 1 obtains a cost reduction innovation of size $\epsilon \in (0, 1)$, in this way its marginal cost is now $c_1 = -\epsilon$
- Such cost reduction innovation is patented.
- In this way the profit function for the firm i is:

$$\pi_i(q_i, q_j) = (1 - q_i - q_j - c_i)q_i$$

  - where q_j represents the offered quantity by the other firm.

3.2.1 *Duopoly under same technology*

At this case both firms produce goods with the same technology, and it is important to characterize a situation where an infringer uses the patented technology without any consequence. By solving the game, the equilibrium quantities are:

$$q_1^a = q_2^a = \frac{1+\epsilon}{3} \tag{23}$$

and the equilibrium profits are

$$\pi_1^a = \left(\frac{1+\epsilon}{3}\right)^2 \qquad \pi_2^a = \left(\frac{1+\epsilon}{3}\right)^2 \tag{24}$$

3.2.2 *Duopoly under different technologies*

In this case there is no licensing and firms just compete as Cournot with different costs, so firm 1 enjoys the innovation and firm 2 produces with the old technology. We are going to use this case when the potential infringer decides not to infringe and produce under the inferior technology, so by solving the game we have:

$$q_1^b = \frac{1+2\epsilon}{3} \qquad\qquad q_2^b = \frac{1-\epsilon}{3} \tag{25}$$

and the the equilibrium profits under this setup are easily calculated as

$$\pi_1^b = \left(\frac{1+2\epsilon}{3}\right)^2 \qquad\qquad \pi_2^b = \left(\frac{1-\epsilon}{3}\right)^2 \tag{26}$$

3.2.3 *Duopoly under the same technology and an expected reasonable royalty rate*

We should also consider the cases where there is infringement and a suit from the patentee, so the quantities chosen by the firms in the Cournot competition are made under the shadow of damage payments and injunctions. In this case both firms produce under the same technology, but in the case of firm 2, it infringes the patented technology and as consequence, firm 1 sues the other firm. If the patent is declared valid and infringed, then the infringer (firm 2) has to pay a reasonable royalty rate τ by each unit sold; the probability that the innovation will be declared valid is common knowledge $\theta \in (0,1)$

So in this case the payoff functions are:

$$\pi_1(q_1, q_2) = (1 - q_1 - q_2 + \epsilon)q_1 + \theta\tau q_2 \tag{27}$$

$$\pi_2(q_1, q_2) = (1 - q_1 - q_2 + \epsilon - \theta\tau)q_2 \tag{28}$$

where $\theta\tau q_2$ is the expected rent for the firm 1,so in this case the equilibrium quantities are:

$$q_1^c = \frac{1+\epsilon+\tau\theta}{3} \qquad\qquad q_2^c = \frac{1+\epsilon-2\tau\theta}{3} \tag{29}$$

and the equilibrium profits are

$$\pi_1^c = \left(\frac{1+\epsilon+\tau\theta}{3}\right)^2 + \theta\tau\frac{1+\epsilon-2\tau\theta}{3} \tag{30}$$

$$\pi_2^c = \left(\frac{1+\epsilon-2\tau\theta}{3}\right)^2 \tag{31}$$

We assume that $\tau \in (0,1)$ and in some cases we also are going to assume that $\tau \leq \epsilon$

3.3 MODEL

The main model is based in the model of (Aoki and Hu, 2003); that model studies the effect of time factors on the licensing of a product innovation. Here we will use the same structure in order to evaluate the impact of time factors on the imitation and litigation when the innovation is a process innovation[1].

The temporal setting and the description of the game is as follows:

1. At the very beginning firm 1 gets a cost reduction innovation with a patent life of γ periods, A license is offered to firm 2 in the form of a "take or leave it" offer. This offer is a fixed fee F, firm 2 then has two options: accept the license an produce with the same technology or reject it. If firm 2 accepts the offer the game ends and both firms produce under the same technology, for γ periods .

2. If there is no licensing firm 2 has to decide whether to imitate the technology or just stay with the old technology. If firm 2 decides not to imitate, then the game ends with a duopoly where the firms produce under different technologies, for γ periods.

3. If firm 2 decided to imitate, imitation needs an investment of h and takes α periods. when the imitation is complete, firm 1 can litigate in order to stop infringement or just leave the things as they are. if the firm 1 decides not to litigate the game ends, with both firms producing under the same technology. So then during the period of imitation firms produce under different technologies and after the $\gamma - \alpha$ remaining periods, they produce with the same technology.

4. If litigation is chosen by firm 1, the trial is going to end β periods after (where $\beta \leq \gamma - \alpha$). At the very beginning of the legal process, firm 1 can offer a settlement by a fixed fee K, this offer again is a "take or leave it" offer. If the firm 2 accepts, both firms end producing under the same technology for the remain $\gamma - \alpha$ periods.

5. If there is no settlement, both firms continue producing but under the shadow of a royalty rate τ that should be paid if the patent is declared valid and infringed. The probability that this happens is a common knowledge value $\theta \in (0,1)$. Also, if the patent is declared valid and infringed the infringer firm 2

1 In a difference of the model by Aoki and Hu (2003), our model is discrete, also we do not consider the effect of the temporal discount factor without loss of generality.

stops using the technology till the end of the life of patent, and produces under the old technology. Consequence of this legal process each firm pays a litigation cost by ℓ_1 and ℓ_2.

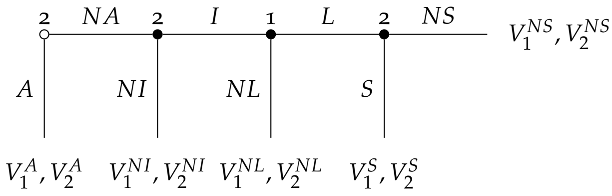

Figure 3.: Game tree of the process

The figure 3 shows the structure of the game, unfortunately the real game tree should be a lot bigger the one that is shown, even so we will call the figure 3 as tree of the game, because this graph gives a good reference of the over all game.

In order to define the values for all cases, we are going to use our basic results obtained in the last section,and we characterize the values of payoffs for firm i under scenario k as V_i^k.

Now we proceed to characterize the payoffs, using the equilibrium profits of section 3.2, utilizing a finite repeated game[2]

1. In the case that firm 2 accepts the license, it pays a fixed fee of F, and both firms produce under the same technology as a Duopoly during the patent life γ. We have the following payoffs:
$$V_1^A = \gamma\pi_1^a + F \tag{32}$$
$$V_2^A = \gamma\pi_2^a - F \tag{33}$$

2. In the case where there is no licensing and where firm 2 decides not to imitate the innovation, we see that firm 1 produces with the new technology and firm 2 produces with the old technology for the γ periods. In consequence payoffs are:
$$V_1^{NI} = \gamma\pi_1^b \tag{34}$$
$$V_2^{NI} = \gamma\pi_2^b \tag{35}$$

3. If firm 1 decides to imitate and makes it at a cost of h, after which if firm 1 (patent holder) decides not to litigate, both firms act as a duopoly with the same costs for the last $\gamma - \alpha$ periods, and in the first α periods (time required to imitate) firm 1 has lower costs than firm 2, so we have the following payoffs:
$$V_1^{NL} = \alpha\pi_1^b + (\gamma - \alpha)\pi_1^a \tag{36}$$
$$V_2^{NL} = \alpha\pi_2^b + (\gamma - \alpha)\pi_2^a - h \tag{37}$$

2 In a finite repeated game the equilibrium in each period is the Nash solution of the one shot game, so we just multiply the equilibrium profits on section 3.2 in order to calculate the payoffs

4. In the case where firm 1 decides to litigate, firm 1 makes a take it or leave it offer by K. If a settlement is achieved firm 2 produces under the same costs as firm 1 for the remaining periods, and pays K to the firm 1, then:

$$V_1^S = \alpha\pi_1^b + (\gamma - \alpha)\pi_1^a + K \tag{38}$$
$$V_2^S = \alpha\pi_2^b + (\gamma - \alpha)\pi_2^a - h - K \tag{39}$$

5. The most complex case emerges when there is no settlement after infringement and after both firms have been unable to reach a licensing accord. So after α periods, both firms produce under the shadow of expected cost and expected benefits of τq_2 for β periods. With the probability of θ that firm 1 will win, it is going to produce with lower costs than firm 2 for the remaining $\gamma - \alpha - \beta$ periods; with a probability of $(1-\theta)$ both will produce with the same costs for the remaining $\gamma - \alpha - \beta$ periods and in this case, both firms pay litigation costs by ℓ_i each period. So:

$$V_1^{NS} = \alpha\pi_1^b + \beta\pi_1^c + \theta(\gamma - \alpha - \beta)\pi_1^b + (1-\theta)(\gamma - \alpha - \beta)\pi_1^a - \beta\ell_1 \tag{40}$$
$$V_2^{NS} = \alpha\pi_2^b + \beta\pi_2^c + \theta(\gamma - \alpha - \beta)\pi_2^b + (1-\theta)(\gamma - \alpha - \beta)\pi_2^a - \beta\ell_2 - h \tag{41}$$

3.4 EQUILIBRIUM OF THE MODEL

In this section we solve the model by backward induction. First we determinate the equilibrium solution for the fixed fee optimal for settlement, and then with these results, we can explore the optimal choices of litigation, imitation and licensing recursively.

3.4.1 *Settlement*

As previously described firm 1 makes a "take-it-or-leave-it" option of a fixed fee settlement. If we assume that this fixed fee exists that value should be enough to compensate the payoff that firms should receive in the case of no settlement[3].

3 $\star$ symbol is used to denote the equilibrium solution.

The settlement conditions depends of the payoff of firm 2, by assuming a take it or leave it offer, $K^\star$ should be such that makes $V_2^{S\star} = V_2^{NS\star}$, so:

$$\begin{aligned} K^\star &= -\beta\pi_2^c - \theta(\gamma-\alpha-\beta)\pi_2^b - (1-\theta)(\gamma-\alpha-\beta)\pi_2^a + \beta\ell_2 + (\gamma-\alpha)\pi_2^a \\ &= (\theta(\gamma-\alpha-\beta)+\beta)\pi_2^a - \theta(\gamma-\alpha-\beta)\pi_2^b - \beta\pi_2^c + \beta\ell_2 \\ &= \theta(\gamma-\alpha-\beta)(\pi_2^a-\pi_2^b) + \beta(\pi_2^a-\pi_2^c) + \beta\ell_2 \geq 0 \end{aligned} \tag{42}$$

Now because $\pi_2^a \geq \pi_2^b$ and $\pi_2^a \geq \pi_2^c$ (the better situation happens when firm 2 uses the innovation without paying any fee or royalty), so $K^\star \geq 0$, it implies that the settlement condition depends only on the payoff of firm 1. In this case we should note that the equilibrium payoffs should hold :

$$V_1^{S\star} \geq V_1^{NS\star}$$

- by developing this condition

$$\begin{aligned} \alpha\pi_1^b + (\gamma-\alpha)\pi_1^a + K^\star &\geq \alpha\pi_1^b + \beta\pi_1^c + \theta(\gamma-\alpha-\beta)\pi_1^b + \\ &\quad (1-\theta)(\gamma-\alpha-\beta)\pi_1^a - \beta\ell_1 \\ K^\star &\geq \beta(\pi_1^c-\pi_1^a) + \theta(\gamma-\alpha-\beta)(\pi_1^b-\pi_1^a) - \beta\ell_1 \end{aligned}$$

by using the value of $K^\star$

$$\begin{aligned} &\theta(\gamma-\alpha-\beta)(\pi_2^a-\pi_2^b) + \\ &\quad \beta(\pi_2^a-\pi_2^c) + \beta\ell_2 \geq \beta(\pi_1^c-\pi_1^a) + \theta(\gamma-\alpha-\beta)(\pi_1^b-\pi_1^a) - \beta\ell_1 \\ &\quad \beta(\ell_1+\ell_2) \geq \beta(\pi_1^c+\pi_2^c-\pi_1^a-\pi_2^a) + \\ &\qquad \theta(\gamma-\alpha-\beta)(\pi_1^b+\pi_2^b-\pi_1^a-\pi_2^a) \end{aligned}$$

- in consequence, the settlement condition is:

$$\sum_i \ell_i \geq \left(\sum_i \pi_i^c - \sum_i \pi_i^a\right) + \theta\left(\frac{\gamma-\alpha}{\beta} - 1\right)\left(\sum_i \pi_i^b - \sum_i \pi_i^a\right) \tag{43}$$

Now we have:

$$\sum_i \pi_i^c - \sum_i \pi_i^a = \frac{\theta\tau}{9}(1+\epsilon-\theta\tau) \geq 0 \tag{44}$$

which is positive.

About the other term we have that

$$\sum_i \pi_i^b - \sum_i \pi_i^a = -\frac{\epsilon}{9}(2-3\epsilon) \tag{45}$$

when the effect of the innovation is lower than $2/3$, the term is negative and is going to be positive where $\epsilon > 2/3$.

With such facts it is easy to get to know the effects of some variables as the patent life and others on the suitability of litigation using the eq. 43 , just some few comments are necessary for the case of θ and τ and in the other ones the result is direct.

Because

$$\frac{\partial \sum \pi_i^c}{\partial \theta} = \frac{\tau}{9}(1 + \epsilon - 2\theta\tau) \geq 0 \tag{46}$$

bigger patent strength makes less suitable settlement just for bigger innovations because the term $\sum_i \pi_i^b - \sum_i \pi_i^a$ in the equation 43 is just positive for $\epsilon > 2/3$.

About the royalty rate we have that

$$\frac{\partial \sum \pi_i^c}{\partial \tau} = \frac{\theta}{9}(1 + \epsilon - 2\theta\tau) \geq 0 \tag{47}$$

so more royalty rate is less suitable to have a settlement. These facts are summarized in the following proposition.

Proposition 7. *It is more suitable to have settlement when:*

1. *litigation costs are higher,*
2. *patent life γ is longer when $\epsilon < 2/3$ and shorter when $\epsilon > 2/3$*
3. *imitation time α is shorter when $\epsilon < 2/3$ and longer when $\epsilon > 2/3$*
4. *litigation time β is shorter when $\epsilon < 2/3$ and longer when $\epsilon > 2/3$*
5. *the patent strength θ is lower if $\epsilon > 2/3$*
6. *the reasonable rate τ is lower*

It is important to see the results of proposition 7, and the effects that change in relation to the size of the innovation. This is because when an innovation is substantial, the cumulated profit of both firms producing under the same technology is lower than the rent when they produce under different technologies. There is a scenario to have a settlement accord in order that the patentee has more bargaining power (because it makes the offer in the take it or leave scheme), the patentee will ask for a bigger share in order to settle, and these effects are amplified by the patent life and decreased by the imitation time and the litigation time. For the case of smaller innovations ($\epsilon < 2/3$), the effects are reversed.

Now when there is a settlement, firm 2 pays $K^\star$ to firm 1, this value is positive as we saw before, the value of this is:

$$K^\star = \theta(\gamma - \alpha - \beta)(\pi_2^a - \pi_2^b) + \beta(\pi_2^a - \pi_2^c) + \beta\ell_2 \tag{48}$$

By doing easy calculations we get:

$$\frac{\partial K^\star}{\partial \alpha} = -\theta(\pi_2^a - \pi_2^b) \leq 0$$

Because $(\pi_2^a - \pi_2^b) > 0$,this means that if the time for imitation is longer the settlement should be lower. This happens as a consequence that the firm has less time to explore the benefits of innovation along bigger profits, also

$$\frac{\partial K^\star}{\partial \gamma} = \theta(\pi_2^a - \pi_2^b) \geq 0$$

by the oppossing reason, so bigger patent lifetime makes the settlement fee bigger. Now, about the effect of litigation time it comes

$$\begin{aligned}\frac{\partial K^\star}{\partial \beta} &= -\theta(\pi_2^a - \pi_2^b) + (\pi_2^a - \pi_2^c) + \ell_2 \\ &= -\frac{4}{9}\theta\left(\epsilon - \tau - \epsilon\tau + \theta\tau^2\right) + \ell_2\end{aligned}$$

So by trying to characterize an expected royalty rate (or desired), it should be fair to assume that $\tau = \epsilon$, it ends as:

$$\frac{\partial K^\star}{\partial \beta} = \frac{4}{9}\epsilon^2(1-\theta)\theta + \ell_2 \geq 0 \quad \text{if } \tau = \epsilon \tag{49}$$

So when the reasonable royalty rate is fair [4] litigation time has a positive effect on the settlement fee.

As expected

$$\frac{\partial K^\star}{\partial \ell_2} = \beta \geq 0 \tag{50}$$

, so bigger litigation costs increases the settlement fee.The effect of the strength of the patent θ on $K^\star$ is

$$\frac{\partial K^\star}{\partial \theta} = (\gamma - \alpha - \beta)(\pi_2^a - \pi_2^b) - \beta\frac{\partial \pi_2^c}{\partial \theta} \geq 0 \tag{51}$$

which is positive because $\frac{\partial \pi_2^c}{\partial \theta} \leq 0$, in that way innovations with more strength will be settled with greater settlement fees. Finally about the effect of the reasonable royalty rate we have that

$$\frac{\partial K^\star}{\partial \tau} = -\beta\frac{\partial \pi_2^c}{\partial \tau} \geq 0 \tag{52}$$

this result comes because $\frac{\partial \pi_2^c}{\partial \tau} \leq 0$.

In consequence we get the following proposition

Proposition 8. *$\frac{\partial K^\star}{\partial \alpha} \leq 0$; $\frac{\partial K^\star}{\partial \gamma} \geq 0$; $\frac{\partial K^\star}{\partial \beta} \geq 0$ when $\tau = \epsilon$; $\frac{\partial K^\star}{\partial \ell_2} \geq 0$; $\frac{\partial K^\star}{\partial \theta} \geq 0$; $\frac{\partial K^\star}{\partial \tau} \geq 0$*

This proposition reflects the fact that time factors amplify the impacts of rents difference under the different technologies, that are more or less in our setup captured by the patentee, in relation to the legal system (θ, τ, ℓ) these variables improve the bargaining power of the patentee and in consequence affect the settlement fee.

4 see Farrell and Shapiro (2008) for a discussion of the ratio between ϵ and τ, given that in some cases it could be possible that $\tau > \epsilon$

3.4.2 *Litigation*

By solving the game, we back another stage (see figure 3)to the choice of firm 1 to litigate or not to litigate. At this point we have to consider the two possible scenarios (Settlement and No Settlement). There is no Litigation if $V_1^{NL^*} < V_1^{S^*}$ when settlement takes place in the next stage, or $V_1^{NL^*} < V_1^{NS^*}$ when there is no settlement in the next stage. We consider the payoffs of firm 2 equal in equilibrium under both situations, it is understood the optimal fee is positive (see eq.(48)).

If there is settlement we have:

$$V_1^{S^*} - V_1^{NL^*} = K^\star > 0$$

So we have that

$$V_1^{S^*} > V_1^{NL^*} \tag{53}$$

Now if there is no settlement in the next stage, it happens because:

$$V_1^{NS^*} > V_1^{S^*}$$

By using the eq. 53 we have

$$V_1^{NS^*} > V_1^{NL^*} \tag{54}$$

This make us conclude that :

Proposition 9. *Litigation is always optimal for firm 1*

This result is quite important because it shows that independently the innovation's size and the patent strength is always optimal for the patentee to litigate when infringement happens.

3.4.3 *Imitation*

Imitation is going to take place if $V_2^{NI\star} < V_2^{S\star}$. This eventually happens independently of if there is or not settlement. This is because the settlement fixed fee is such that $V_2^{S\star} = V_2^{NS\star}$, so we can obtain the imitation condition as $V_2^{NI\star} < V_2^{S\star}$.

By developing this condition, we obtain

$$\begin{aligned} \alpha\pi_2^b + (\gamma - \alpha)\pi_2^a - h - K^\star > &\quad \gamma\pi_2^b \\ G = (\gamma - \alpha)(\pi_2^a - \pi_2^b) - h - K^\star > &\quad 0 \end{aligned}$$

It means that the imitation condition is $G > 0$. By deriving the time variables we have that:

$$\frac{\partial G}{\partial \alpha} = -(\pi_2^a - \pi_2^b) - \frac{\partial K^\star}{\partial \alpha} = -(1-\theta)(\pi_2^a - \pi_2^b) < 0 \tag{55}$$

$$\frac{\partial G}{\partial \gamma} = (\pi_2^a - \pi_2^b) - \frac{\partial K^\star}{\partial \gamma} = (1-\theta)(\pi_2^a - \pi_2^b) > 0 \tag{56}$$

$$\frac{\partial G}{\partial \beta} = -\frac{\partial K^\star}{\partial \beta} < 0 \quad \text{if } \tau = \epsilon \tag{57}$$

Essentially these effects represent the incentives to imitations because longer patent life, short imitation time and lower litigation time make the infringement premium greater for the potential infringer (firm 2), by summarizing:

Proposition 10. *There is more suitability to have imitation, when:*

1. *The imitation time α is lower;*
2. *The patent life γ is bigger;*
3. *The litigation time β is lower when $\tau = \epsilon$*

Also we can take derivatives in respect of the variables that represent the legal system, so:

$$\frac{\partial G}{\partial \theta} = -\frac{\partial K^\star}{\partial \theta} \leq 0 \tag{58}$$
$$\frac{\partial G}{\partial \tau} = -\frac{\partial K^\star}{\partial \tau} \leq 0 \tag{59}$$
$$\frac{\partial G}{\partial \ell_2} = -\frac{\partial K^\star}{\partial \ell_2} \leq 0 \tag{60}$$

If the legal variables have some direction to support the patentee system (higher patent strength, higher royalty rates, and high cost of litigation for the infringer). They reduce the feasibility of infringement, because these directly affect the settlement payment, making it greater and the premium of infringement is made lower, so:

Proposition 11. *There is less suitability to have imitation, when:*

1. *The patent strength θ is higher;*
2. *The reasonable royalty rate τ is higher;*
3. *The litigation cost for the firm 2 ℓ_2 is higher.*

One aspect that is quite interesting, is to explore at this stage the impact of time effects on the payoffs of patentee and potential infringer.

By starting with the infringer we proceed to calculate the derivatives $V_2^{S^*}$ [5] with respect to the time variables α, β and γ. It is observed in figure 3 given that litigation is active, there are two potential scenarios: Settlement and No Settlement. It was discussed before if the equilibrium payoffs for firm 2 are equal, we can have the impacts of time variables on the incentives to infringe just deriving $V_2^{S^*}$ respect to the time variables, so:

$$\frac{\partial V_2^{S^*}}{\partial \alpha} = -(\pi_2^a - \pi_2^b) - \frac{\partial K^\star}{\partial \alpha} = -(1-\theta)(\pi_2^a - \pi_2^b) < 0 \tag{61}$$

5 We obtain this equilibrium payoff by replacing the equilibrium values of $K^\star$ in equation 38, we act is similar way for the other equilibrium payoffs.

$$\frac{\partial V_2^{S*}}{\partial \beta} = -\frac{\partial K^*}{\partial \beta} \leq 0 \text{ if } \tau = \epsilon \tag{62}$$

$$\frac{\partial V_2^{S*}}{\partial \gamma} = \pi_2^a - \frac{\partial K^\star}{\partial \gamma} = \pi_2^a - \theta(\pi_2^a - \pi_2^b) = (1-\theta)\pi_2^a + \theta\pi_2^b > 0 \tag{63}$$

Because the impact of the imitation time is negative it is optimal for the infringer to imitate as fast as possible. One counterintuitive result is the impact of the patent life, the greater the incentive to infringe, because it is possible to get greater rents throughout settlement accord and enjoy the innovation together with the patentee, for a bigger period of time. However the impact of the litigation time is negative when $\tau = \epsilon$ (fair compensation), because longer periods of litigation reduce the premium of infringement along the litigation costs that eventually make the settlement fee greater, as is summarized in the following proposition.

Proposition 12. $\frac{\partial V_2^{S*}}{\partial \alpha} < 0$, $\frac{\partial V_2^{S*}}{\partial \beta} < 0$ *if* $\tau = \epsilon$, $\frac{\partial V_2^{S*}}{\partial \gamma} > 0$

By working in the same way as before, only this time for the patentee, we have to compare the two payoffs under both scenarios: 1) Settlement and 2) No Settlement.

Then we start deriving the payoff of the patentee in respect of the imitation time under the first scenario, as

$$\frac{\partial V_1^{S*}}{\partial \alpha} = \pi_1^b - \pi_1^a + \frac{\partial K^\star}{\partial \alpha} = \pi_1^b - \pi_1^a - \theta(\pi_2^a - \pi_2^b) = \frac{1}{9}\epsilon(2 + 3\epsilon - 4\theta)$$

This term in most of the cases is positive, and in particular it is positive if $\epsilon > 2/3$ (bigger innovations). However, it could be negative if the innovation is small enough at least $\epsilon < 2/3$ and with very high patent strength $\theta > \frac{2+\epsilon}{4} > 1/2$, so the patentee with bigger innovations benefit from a longer imitation.

$$\frac{\partial V_1^{S*}}{\partial \beta} = \frac{\partial K^\star}{\partial \beta} > 0 \text{ if } \tau = \epsilon \tag{64}$$

Litigation time has a direct consequential effect of increasing the settlement fee that is positive under the suitable assumption that $\tau = \epsilon$, meaning that longer periods of litigation benefit the patentee, when a fair royalty rate is applied. Finally making the derivatives of the payoff of patentee respect the patent life we get that:

$$\frac{\partial V_1^{S*}}{\partial \gamma} = \pi_1^a + \frac{\partial K^\star}{\partial \gamma} = \pi_1^a + \theta(\pi_2^a - \pi_2^b) > 0 \tag{65}$$

So this equation shows that in the scenario of a settlement that longer patent life benefits the patentee, and summarizing the results, we get the following proposition

Proposition 13. $\dfrac{\partial V_1^{S*}}{\partial \alpha} > 0$ *if* $\epsilon > 2/3$, $\dfrac{\partial V_1^{S*}}{\partial \beta} > 0$ *if* $\tau = \epsilon$, $\dfrac{\partial V_1^{S*}}{\partial \gamma} > 0$

By working under the last scenario "No Settlement" and making the calculations of the derivatives, we get that:

$$\frac{\partial V_1^{NS*}}{\partial \alpha} = (1-\theta)(\pi_1^b - \pi_1^a) > 0 \tag{66}$$

$$\frac{\partial V_1^{NS*}}{\partial \gamma} = \theta\pi_1^b + (1-\theta)\pi_1^a > 0 \tag{67}$$

This means that imitation and longer patent life improves the payoff of the patentee. By calculating the derivative in respect to the litigation time, we get:

$$\frac{\partial V_1^{NS*}}{\partial \beta} = \pi_1^c - \theta\pi_1^b + (1-\theta)\pi_1^a - \ell_1$$

The first term is

$$\pi_1^c - \theta\pi_1^b + (1-\theta)\pi_1^a = \frac{1}{9}\left(3\epsilon\left(1-\epsilon\theta^2\right) + (2+\epsilon)(1-\theta) + 2\epsilon^2\left(1-\theta^2\right)\right) > 0$$

which is positive, then:

$$\text{if } \tau = \epsilon \begin{cases} \dfrac{\partial V_1^{NS*}}{\partial \beta} > 0, & \text{if } \ell_1 < \pi_1^c - \theta\pi_1^b + (1-\theta)\pi_1^a; \\ \dfrac{\partial V_1^{NS*}}{\partial \beta} < 0, & \text{if } \ell_1 > \pi_1^c - \theta\pi_1^b + (1-\theta)\pi_1^a). \end{cases} \tag{68}$$

We conclude that longer litigation benefits the patentee under a fair royalty rate if the litigation costs for the patentee are smaller and reverting the effect if they are bigger or when there is no settlement.

Proposition 14. $\dfrac{\partial V_1^{NS*}}{\partial \alpha} > 0$, $\dfrac{\partial V_1^{NS*}}{\partial \beta} > 0$ *if* $\ell_1 < \pi_1^c - \theta\pi_1^b + (1-\theta)\pi_1^a$, $\dfrac{\partial V_1^{NS*}}{\partial \gamma} > 0$

We can collate these results as they impact on the incentives to innovate or to infringement, in the following table:

	Imitation time	Litigation time	Patent Life
Infringement	$-$	$-$ if $\tau = \epsilon$	$+$
Innovation	$+$ if $\epsilon > 2/3$	$+$ if $\tau = \epsilon$, and smaller ℓ_1	$+$

Table 1.: Impacts of time variables on infringement and innovation.

We can conclude that the effect of patent life have ambiguous effects as it supports innovation (making the patentee payoff better off in all situations) but also supports infringement (because longer patent life make the incentives to infringe greater). The imitation time as we have seen before, infringers like fast innovation; in the case of innovators with greater innovation they will prefer slower imitation. Time of litigations seems to show that it is worse for the infringer in all cases, and in some cases supports innovation on the probability that litigation costs for the patentee are diminished.

3.4.4 *Licensing*

In the study of licensing there are 3 possible scenarios:

1. No imitation
2. Imitation and no settlement
3. Imitation and settlement

But as we saw in equilibrium, the payoff of firm 2 is the same whether there is settlement or not. For this reason we have two possible fixed fee values, and three different scenarios for the patentee.

Licensing under not imitation

If we assume the case that the backward solution is consistent with no imitation, it is certain firm 1 is offered a license fee $F^{NI\star}$ that make $V_2^{NI\star} = V_2^{A\star}$ so we have that:

$$\begin{aligned} V_2^A = \gamma\pi_2^a - F^{NI\star} &= \gamma\pi_2^b = V_2^{NI} \\ F^{NI\star} &= \gamma(\pi_2^a - \pi_2^b) \\ F^{NI\star} &= \frac{4}{9}\epsilon \geq 0 \end{aligned} \tag{69}$$

Now we should see if licensing under this condition is feasible for the firm 1,

$$\begin{aligned} &V_1^A - V_1^{NI} = \\ =&\gamma\pi_1^a + F^{NI\star} - \gamma\pi_1^b = \\ =&\gamma(\pi_1^a - \pi_1^b) + \gamma(\pi_2^a - \pi_2^b) = \\ =&\gamma\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) = \frac{\gamma}{9}(2 - 3\epsilon)\epsilon \end{aligned} \tag{70}$$

When the imitation does not hold , the licensing condition depends solely upon the size of the cost reduction innovation,

Proposition 15. *When no imitate is optimal for the firm 2, there is a positive optimal fixed fee $F^{NI\star}$ if there is licensing, and the licensing condition is $\epsilon \leq 2/3$, meaning that big innovations are never licensed and small innovations are always licensed. Also it is more suitable to have licensing when the patent life is longer.*

Licensing under imitation and no settlement

Certainly firm 1 is offered a license fee $F^{I\star}$ that makes $V_2^{A\star} = V_2^{S\star} = V_2^{NS\star}$ so we can get this value by developing this equation as:

$$\begin{aligned}
&V_2^A = \gamma\pi_2^a - F^{I\star} = \alpha\pi_2^b + (\gamma-\alpha)\pi_2^a - h - K^\star = V_2^S \\
&F^{I\star} = \alpha(\pi_2^a - \pi_2^b) + h + K^\star \\
&F^{I\star} = \alpha(\pi_2^a - \pi_2^b) + h + \theta(\gamma-\alpha-\beta)(\pi_2^a - \pi_2^b) + \beta(\pi_2^a - \pi_2^c) + \beta\ell_2 \geq 0
\end{aligned} \tag{71}$$

Now we explore firm 1's payoffs under licensing V_1^A and no licensing, so the difference is:

$$\begin{aligned}
H =& V_1^A - V_1^{NS} = \\
=& \gamma\pi_1^a + F^{I\star} - (\alpha\pi_1^b + \beta\pi_1^c + \theta(\gamma-\alpha-\beta)\pi_1^b + (1-\theta)(\gamma-\alpha-\beta)\pi_1^a - \beta\ell_1) \\
=& F^{I\star} + \alpha(\pi_1^a - \pi_1^b) + \theta(\gamma-\alpha-\beta)(\pi_1^a - \pi_1^b) + \beta(\pi_1^a - \pi_1^c) + \beta\ell_1 = \\
& \text{by using the value of } F^{I\star} \text{ from eq.71} \\
=& \alpha\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) + \theta(\gamma-\alpha-\beta)\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) \\
& + \beta\left(\sum_i \pi_i^a - \sum_i \pi_i^c\right) + \sum_i \beta\ell_i
\end{aligned} \tag{72}$$

Unfortunately this expression is not always positive because the term $\left(\sum_i \pi_i^a - \sum_i \pi_i^c\right)$ is negative when $\tau = \epsilon$, the other terms being positive, so when H is positive there is licensing and in other cases there is no licensing.

If we derive H in order to the time variables we get that:

$$\begin{aligned}
\frac{\partial H}{\partial \alpha} &= (1-\theta)\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) = \frac{(1-\theta)}{9}(2-3\epsilon)\epsilon \\
\frac{\partial H}{\partial \gamma} &= \theta\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) = \frac{\theta}{9}(2-3\epsilon)\epsilon
\end{aligned}$$

So it means in both cases that the impacts are going to depend on the size of the innovation ϵ, so the impact of greater imitation time and longer patent life makes a license more suitable in small innovations and the effect is contrary in a significant innovation.

Litigation time we have that

$$\begin{aligned}
\frac{\partial H}{\partial \beta} =& -\theta\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) + \left(\sum_i \pi_i^a - \sum_i \pi_i^c\right) + \sum_i \ell_i \\
& -\frac{\theta}{9}(2-3\epsilon)\epsilon + \left(\sum_i \pi_i^a - \sum_i \pi_i^c\right) + \sum_i \ell_i
\end{aligned} \tag{73}$$

Now because we have that

$$\sum_i \pi_i^a - \sum_i \pi_i^c = \frac{1}{9}\theta\tau(-1-\epsilon+\theta\tau) < 0$$

we can say that $\frac{\partial H}{\partial \beta} > 0$ if and only if

$$\sum_i \ell_i > \theta\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) - \left(\sum_i \pi_i^a - \sum_i \pi_i^c\right)$$

Now in the case of θ we have that

$$\begin{aligned} \frac{\partial H}{\partial \theta} =& (\gamma-\alpha-\beta)\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) - \beta\sum_i \frac{\partial \pi_i^c}{\partial \theta} \\ & (\gamma-\alpha-\beta)\frac{1}{9}(2-3\epsilon)\epsilon - \beta\frac{1}{9}\tau(1+\epsilon-2\theta\tau) \end{aligned} \tag{74}$$

which is negative for bigger innovations, and finally calculating the derivative respect τ

$$\begin{aligned} \frac{\partial H}{\partial \tau} =& -\beta\sum_i \frac{\partial \pi_i^c}{\partial \tau} \\ & -\frac{\beta}{9}\theta(1+\epsilon-2\theta\tau) < 0 \end{aligned} \tag{75}$$

Proposition 16. *When there is imitation and no settlement as consistent choices, it is more suitable to have an accord of licensing , when there are:*

1. *Longer (Smaller) imitation periods and patent life for small (big) innovations*
2. *Longer litigation periods if litigation costs are high enough*
3. *Lower patent strength and reasonable royalty rates.*

Licensing under imitation and settlement

In this case we consider again the following fixed fee

$$F^{I\star} = \alpha(\pi_2^a - \pi_2^b) + h + K^\star$$

Using the definitions of the payoffs of the patentee we compare both payoffs in order to get the licensing conditions in this case, so:

$$\begin{aligned} & V_1^A - V_1^S = \\ =& \gamma\pi_1^a + F^{I\star} - \alpha\pi_1^b - (\gamma-\alpha)\pi_1^a - K^\star = \\ =& \alpha(\pi_1^a + \pi_2^a - \pi_1^b - \pi_2^b) + h = \\ =& \alpha\left(\sum_i \pi_i^a - \sum_i \pi_i^b\right) + h = \frac{\alpha}{9}(2-3\epsilon)\epsilon + h \end{aligned} \tag{76}$$

Proposition 17. *In the scenario where imitation and settlement are optimal choices, smaller innovations are licensed and it is more suitable to have licensing when the imitation time is longer and the cost of imitation are higher.*

What we see in the last three propositions is:

1. Small innovations are more suitable to be licensed
2. Longer patent life has a limited effect to promote licensing, eventually just in the case of no imitation or in some cases of small innovations.
3. Improve the patent strength and reasonable royalty rates (penalties for infringement)in some cases are against licensing, because improves the situation of the patentee when licensing contracts are bargained.

3.5 WELFARE ANALYSIS

3.5.1 *Welfare Analysis of simple games*

The welfare indicator we are going to use is the sum of the consumer surplus plus the sum of profits of the firms, and because the models are linear, the general form of social welfare is

$$SW = \sum_i \pi_i + \frac{\left(\sum_i q_i\right)^2}{2}$$

So in simple cases by using simple substitution of the formulas in the section 3.2 we have that, the social welfare for the cases: a) Duopoly under same technology; b) Duopoly under different technologies; c) Duopoly under same technology but under the shadow of reasonable royalty rates:

$$SW^a = \frac{4}{9}(1+\epsilon)^2 \tag{77}$$

$$SW^b = \frac{1}{18}\left(8+8\epsilon+11\epsilon^2\right) \tag{78}$$

$$SW^c = \frac{1}{18}(2+2\epsilon-\theta\tau)(4+4\epsilon+\theta\tau) \tag{79}$$

Now it should be interesting to compare different situations so we have that:

$$SW^a - SW^b = \frac{1}{18}(8-3\epsilon)\epsilon \tag{80}$$

$$SW^a - SW^c = \frac{1}{18}\theta\tau(2+2\epsilon+\theta\tau) \tag{81}$$

$$SW^c - SW^b = \frac{1}{18}\left(8\epsilon-3\epsilon^2-2\theta\tau-2\epsilon\theta\tau-\theta^2\tau^2\right) \tag{82}$$

The first two expressions are positive, but the last one needs an extra condition. Using the condition $\tau = \epsilon$, the three conditions are positive, so we have:

Proposition 18. *$SW^a > SW^b$, $SW^a > SW^c$ and $SW^b > SW^c$ if $\tau = \epsilon$*

3.5.2 *Welfare Analysis*

We are going to have several scenarios, listed below:

1. Ex ante licensing A, in this case

$$SW^A = \gamma SW^a \tag{83}$$

2. No ex ante licensing and no imitation NI, so

$$SW^{NI} = \gamma SW^b \tag{84}$$

3. No ex ante licensing, imitation, and no settlement NS, so

$$\begin{aligned} SW^{NS} =& \alpha SW^b + \beta SW^c + \theta(\gamma - \alpha - \beta)SW^b + \\ & (1-\theta)(\gamma - \alpha - \beta)SW^a - \beta \sum_i \ell_i - h \end{aligned} \tag{85}$$

4. No ex ante licensing, imitation, and settlement S, so

$$SW^S = \alpha SW^b + (\gamma - \alpha)SW^a - h \tag{86}$$

By fast comparisons of $SW^A \geq SW^{NI}$, $SW^A \geq SW^{NS}$, $SW^A \geq SW^S$, it means that the better situations happens when there is ex ante licensing,

Proposition 19. *The best situation from the point of view of social welfare is licensing.*

3.6 CONCLUSIONS

Related with the litigation costs of patents, Bessen and Meurer (2005) point out that innovations with small rewards at risk (less than one million of USD) make the median estimate of a half million in total litigation costs and for median rewards at risk (1-25 million) the estimated median of legal costs is 2 million, then these results are compatible with the possibility of firms settling in cases of weak and small patents that are the cause of the greater legal costs. This fact is consistent with the opinion and data of some authors that the greater share of patent disputes settle (see PWC (2014)) , comes as a consequence of the higher legal costs and is consistent with the results of the model.

Time factors are relevant in the sense that they amplify the effects of other variables, such as reasonable royalty rates and patent strength, but in some cases, time factors have direct effects, as can be shown in the table 1. Infringers will minimize the imitation time and innovators with significant innovations prefer longer imitation times. Also under suitable longer litigation times have desirable results in order to promote innovation and discourage infringement. However the patent life has an ambiguous effect because it supports innovation as infringement.

One relevant point is that much of the analysis shows that the legal system variables i.e. legal costs, patent strength and royalty rates have important interference on the bargaining power of patentees. So it is important that the patent strength that represents the probability that the patentee wins in a patent dispute represents the real meaning of innovation as a discovery, and its inventor has to receive an incentive in such a way that continues research.

Some of the results have shown the importance of the size of innovation in order to measure the magnitude of several policies. So we have to consider there should be different economic policies for different combinations of patent size and patent strength.

4

SCHUMPETERIAN COMPETITION, TECHNOLOGY SPACE AND PATENTS

Here we have a good produced by N_f firms in the market where the quantities supplied by firms determine the market price and the research policies are fixed. We use the model of Silverberg and Verspagen (2007) as structure for a technology-performance space, a space with N_c columns on the horizontal axis and without a upper bond to the vertical axis (performance). In this scenario we evaluate several combinations of patent breadth and patent life on productivity and social welfare.

4.1 INTRODUCTION

One of the most cited book in the economic literature is the one written by Nelson and Winter (1982), built on several decades of research studies of both authors based on the ideas originally developed by Schumpeter (see Schumpeter (1942) and Schumpeter (1961)). The result of such effort were several simple and enlightened evolutionary models of economies and industries.

The model that we are going to call NW82 developed in Nelson and Winter (1982, ch. 12), is the most simple of the family, and contains the basic structure of the modern models in evolutionary economics, however this model got some criticism oriented to the entry, exit and behavior of the firms inside the model. So, extensions of the model were incorporated in a new model called by us as NW84 (see Winter (1984)), it extends the NW82 model allowing adaptive behavior in the research policies, allowing also the possibility of entry and exit of firms from the industry.

The robust structure of the NW84 model was easily modified to explore the effect of intellectual property protection (see Winter (1993)). This was the first evolutionary model used in the study of patents on an evolutionary scenario. The conclusions of Winter in this last paper was that the patent system does not have positive effects on social welfare.

However, patents are not merely fixed policies and they could be strategically used against other firms, by giving considerable advantage to some firms by blocking the development of other firms. Therefore, it could be in this sense that firms invest in patents in order to obtain monopolistic benefits that will be eventually used by the firm for producing more innovation in order to perpetuate the control on a market, in consequence a virtuous cycle of innovation and investment it is created by highly concentrated industries. Vallée and Yıldızoglu (2007) explored this point in an evolutionary model, where firms were able to learn, and where firms were able to use its financial resources strategically in : i) savings; ii) investments; iii) patenting portfolios and iv) dividends distribution, with free entry and exit of firms, imitation of other firms and innovation. They show that stronger patent systems show negative results on social welfare and innovation when compared with milder patent systems.

Nevertheless behind the simplest way to generate innovation in the NW models there is a black box, eventually the innovation generation process is expected to be dominated by a Poisson distribution with parameters depending on the size of the R&D expenses. Also the line seems to be the space where this innovations live on. In consequence, there is no explicit structure about the cumulativeness of the innovation process. It is a quite interesting agenda to discuss and eventually include this innovation process in the models of the Nelson and Winter's type.

Recently Verspagen and his coauthors have make several attempts in order to made a definitive theory of how innovations are discovered and used by firms and industries inside a technology-performance space (defined as the cartesian product of a technological space and the performance of innovations, where technological space is conceived as the set of different technologies or techniques, i.e chemistry processes and mechanical processes could be different elements of a technological space), In their search they use historical data in order to observe empirical regularities as: 1) that there is a pattern in the appearance of radical innovations; 2) innovations are not following a time-homogeneous Poisson processes and 3) innovations are clustered (Silverberg, 2002). Based on these empirical observations and theoretical developments about technology (as the natural trajectories theory (Nelson and Winter, 1977)) they propose a model based on the theory of percolation, and they present a technology-performance space as the space where inventions are discovered. Such space in its simpler form is a half-plane where the bottom line represents different technologies sorted by technological proximity, and in the other dimension of this plane it is representing by the performance of the innovation. In this scenario a particular innovation is a cell, so innovations are arranged by technological type and performance, by using percolation each cell was categorized as excluded or not by nature. Using such model They were able to replicate some empirical regularities related to the innovation (Silverberg and Verspagen, 2005). The approach called NIP (Nothing is impossible at a price) was also explored in Silverberg and Verspagen (2007). In such approach all cells can be discovered but the difficulty for this discovery is heterogenous. Such models are in practice successful in order to replicate empirical regularities and incorporate general theories about innovation.

Based on the model of Silverberg and Verspagen (2007), Goldschlag (2014) has developed an Agent Based Model, taking the cells in the technology-performance space as product innovations and using firms as agents, in such scenario firms obtain temporal monopolistic power and during this time the firms use its resources to have more exploration throughout R&D on the technology-performance space. Also it incorporated intellectual protection on the model. He concludes that a patent system is not good in all cases and there is a considerable area of parameters where having a patent system makes a positive difference.

The model developed for Silverberg and Verspagen (2007) is a quite flexible and understandable model that show the mechanics behind the discovery of innovations. It is clear that this model can be easily exploited in the line of:

1. Industry dynamics, by using agent based Modeling (ABM), because the technological space presented by Silverberg and Verspagen (2007) is a reasonable model that describes the process of innovation's discovery;

2. Patents, because intellectual property rights are highly interesting in such setups. Since in such scenarios is possible not only to experiment with the patent life (length of a patent), also to evaluate different breadth (size) policies for patents.

We think there is a bigger number of applications in the approach of Verspagen et. al. maybe the most obvious and intuitive one, it is to relate this approach with the classical NW's structure. So we start describing the model in section 4.2, in this section we discuss the technology-performance space developed by Silverberg and Verspagen (2007), and the other elements that make the model able to simulate also Schumpeteriam competition. In section 4.3 we describe the simulation protocol and we summarize the parameters used in the simulation and we describe the results of the simulations throughout some tests and graphs. Finally in section 4.4 we conclude and discuss future research.

It is crucial to point out that this work could be considered complement to the one developed independently in Goldschlag (2014), differing from his model, here firms move based on the principles of local optimization and experimentation. Also there is highly iteration between firms, not just in the space of innovations also in the market of an homogeneous product, from such activities are generated resources for R&D in the technology-performance space.

4.2 MODEL

In this section , we develop a theoretical model that relates the technology-performance space construction made by Silverberg, Vespagen and co-authors (see Silverberg and Verspagen (2007)) and the called schumpeterian competition based on the Nelson and Winter (NW) models (see Nelson and Winter (1982)) .

4.2.1 *Technology-performance space*

The technology-performance space developed in Silverberg and Verspagen (2007) in its simplest version considers a square grid that wraps horizontally and is unbounded in the north side. In this two dimensional world the horizontal axis represents the different technological categories (technology space) and the vertical axis measures the performance of a particular invention on the productivity (other option could be a cost reduction). It is assumed that the elements of the technology-performance space are sorted by technological connectedness.

Each one of the cells a_{ij} (located on the ith row and jth column) on the technology-performance space, can be in one of the following states:

1. 0 not discovered yet (red)
2. 1 discovered but not viable (yellow)
3. 2 discovered and viable (green)

At the very beginning each innovation starts being not discovered, once is discovered it could be viable just if this innovation is close enough to operational innovations (discovered and viable). The figure 4 shows a particular realization of a technology-performance space, where the basement of the lattice is discovered and viable (green), there is a vast area of red cells (not discovered yet) and in the frontier of both there are innovations that are discovered but are not operational (viable).

The initial state of the technology-performance space, should at least contain a cell that is discovered and viable, discoveries are made by firms as a result of investments in R&D. Each innovation (cell) has a resistance value α_{ij}, it reflects the level of difficulty of being discovered, every R&D effort erodes the resistance of the cell, when this resistance is consumed the innovation turns discovered and could be viable or not.

The resistance value of the innovation located at ith row and jth column is α_{ij}, it is generated from a log normal distribution such that the mean of the realizations is μ^r and the standard error is σ^r.

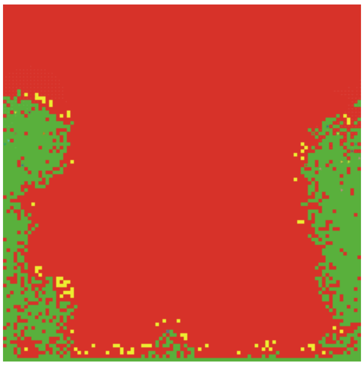

Figure 4.: A technology-performance space

Once each α_{ij} is generated, the dynamics of the resistance is given for the following equation

$$\alpha_{ij,t+1} = \alpha_{ij,t} - b\omega$$

, where $\omega \sim U(0,1)$ and b is the R&D effort by the firms.

In addition, each one of this cells has a level of productivity (or cost reduction efficiency) sw, this number is generated by a combination of the height $ycor$ of the cell and a realization of a normal distribution in the following way

$$sw = A_0 + \zeta h \tag{87}$$

, where $h \sim N(ycor, 1)$, A_0 is the initial productivity without any innovation and $\zeta \in (0,1)$ is a fixed parameter going to be used for calibration.

All cells start in the state 0 (undiscovered) except the ones that correspond to the base, then start at state 2 (discovered and viable). Once the resistance of a cell has been completely consumed, the cell turns to state 1 (discovered and not viable), such innovation could be also be able to turn state 2 (discovered and viable) if the cell is neighbor to a cell on state 2, the neighborhood considered for such evaluation is a Moore neighborhood of size 1 (see Figure 5).

4.2.2 *Short-run behavior*

The behavior in the short run is completely based in the model developed in Nelson and Winter (1982, chap. 12). So there is a fixed number of firms N_f, each one of them produce a quantity q with the following production function:

$$q = AK \tag{88}$$

where A is a productivity factor and K is the capital of the firm, based on the last equation, the profit is :

$$\Pi = P(A - c)K \tag{89}$$

where c is the variable cost of production as a rate of the current capital, and P is the current market price.

The market price is determinate by the inverse demand function as follows

$$P = \begin{cases} 2, & \text{if } Q \leq D/2; \\ D/Q, & \text{if } Q > D/2. \end{cases} \tag{90}$$

, where $Q = \sum q_i$ is the aggregate quantity of the good.

4.2.3 *Innovation and patents*

In this model each period the firms make R&D activities by using a percentage of the liquid profit, for the activities of exploration in the technology-performance space, so the budget destined for this activities is given by

$$B = \tau \Pi \tag{91}$$

the firm then uses this budget to explore each one of the cells inside a Moore neighborhood with the same intensity b for each one (see eq.(92)), the size of the search neighborhood is r (see figure 5), eventually bigger values of r not necessarily mean better situation in terms of exploration of the technology-performance space, because there is less resources for each cell compared with the situation of a lower search radius, however the firm could be benefit for bigger search radius, once the other firms have weaken the resistance of the surrounding cells.

$$b = \frac{B}{(2r+1)^2} \tag{92}$$

, when a site has been completely weakened the last firm that explore the cell claims for the right of the cell, it happens just if the cell is free and undiscovered, also claims its breadth (the neighborhood of size ϕ of undiscovered cells). This right is the patent and its exclusion to others have a duration of v periods (patent life).

> The required inventive step governs which innovations are protectable, and the breadth governs how different another product must be to avoid infringement — Scotchmer (2004, pp. 84)

About the duration of a patent, its definition and use is well known and very intuitive, however the breadth is a concept that needs a more detailed explanation. We instead of the product definition pointed up by Scotchmer (see above), we assume that the breadth of a innovation is immediately translated to a neighborhood in the technology-performance space.

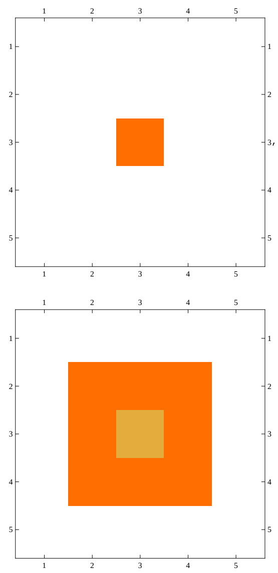

Figure 5.: Moore neighborhoods of size 0 and 1

Hence v denotes how many periods the protection is valid and the patent breadth ϕ tells how much extends the right on a neighborhood in the technology-performance space. The cells that belongs to the patent breadth of an innovation have to be undiscovered cells (cells on state 0). The undiscovered cells inside the breath of a patent that are property of another firm can be discovered, but cannot be exploited during the life of the patent which the breadth belongs, unless the patent related belongs to the same firm.

4.2.4 *Movement of the firm in the technology-performance space*

The place where the firm is determines the current level of productivity that the firms enjoys, once the firms have completed the exploration process, each one of them can move to another cell, if there is a movement such process is called innovation when its movement is to a new discovered cell and is called imitation if the movement is to a cell discovered for another firm, there is no movement to a cell protected by property rights of other firm.

It is important also to point out that the firm will just move to a cell inside its vision of exploration (the neighborhood with radius r). The cell also has to be in the state 2 (discovered and viable). Lastly, the firm cannot move to sites protected by property rights owned by other different firms.

The achievable set of firm is Ω (the set of cells where the firm can move at the current period). Once Ω is determined (this not empty set because at least such set can contents cell where the firm is), the firm will have two kind of movements to an element of Ω:

1. Move 1: the firm moves to the site with the greater

$$\beta = \frac{sw}{1 + 0.001 * d}$$

, where d is the distance to the actual point where the firm is, in this way there is a cost included in the movements, further movements make β lower. The movement "Move 1" reflects the wish of the firm to move to the most productive cell weighted by distance.

2. Move 2: the firm moves randomly to a random cell inside the set of achievable sites Ω, this movement exists in order to prevent that firms stayed in a local maxima.

The choice of the movement is made by chance, so the probability of the movement 2 is $\eta \in (0, 1)$.

Once the firm has moved inside the technology-performance space, it takes the productivity sw of the cell where the firm is at its new productivity level A.

4.2.5 *investment and capital update*

About the dynamics of the capital, it follows almost explicitly the specification in Nelson and Winter (1982), so defined the markup as

$$\rho = \frac{P_t A_t}{c} \tag{93}$$

and the market share as

$$s = \frac{q_t}{Q_t} \tag{94}$$

We can define the ratio of desired investment related to capital as $1.03 - \dfrac{2-s}{\rho(2-2s)}$ [1] and the financial restriction is $(1-\tau)\pi$ so the investment rate by unit of capital is

$$I(\rho, s, \pi, 0.03) = \max\left(0, \min\left(1.03 - \frac{2-s}{\rho(2-2s)}, (1-\tau)\pi\right)\right) \tag{95}$$

the term 0.03 is the depreciation of capital and $\pi = \dfrac{\Pi}{K}$. Then with such investment the dynamics of firm's capital is

$$K_{t+1} = (I(\rho, s, \pi, 0.03) + 0.97)K_t \tag{96}$$

1 see Jonard and Yildizoğlu (1998) for a detailed example, about how to calculate conjectures in the case of a duopoly

4.3 SIMULATIONS

4.3.1 *Simulation Protocol*

We implement the model described in the last section in Netlogo (see figure 6), and we carry on about of 20 runs of 1.000 periods each one, and after each run we save the results only for the last period. some relevant statics that we are interested on are :

1. IHH: Inverse Herfindhal Index that represents
2. max A: maximum productivity of the firms
3. P: Current price
4. $CS = 47(\ln 2 - \ln P)$: Consumer surplus
5. $FS = \sum \pi$: Firms' surplus
6. $SW = CS + FS$: Social Welfare

The values for parameters used in the simulations are summarized in the Table 2.

Mean resistance	$\mu^r = 0.05$	Neighborhood radius	$r = [5, 30]$
s.d. resistance	$\sigma^r = 0.12$	Patent Life	$v = [0, 10, 20]$
Initial capital	$K_0 = 0.16$	Patent Breadth	$\phi = [0, 5, 10]$
Initial productivity	$A_0 = 0.16$	Probability of movement 2	$\eta = 0.3$
Number of columns	$N_c = 100$	cost of production	$c = 0.16$
Number of firms	$N_f = 32$	Demand parameter	$D = 47$
% for exploration	$\tau = 0.2$	parameter in sw	$\zeta = 0.002$

Table 2.: Simulation parameters

Because we should consider 2 cases for r, 3 cases for v, and 3 cases ϕ, we have 18 combinations, by running each combinations 20 times gives us a database of 360 observations, our choice of the parameters where made it in order to keep as near as possible to the parameters used in Nelson and Winter (1982, chap. 12).

The figure 6 shows a particular run of the model, the results shown are in general similar as the ones obtained in the Nelson and Winter model, so the market gradually ends highly concentrated, the price decreases on time and productivity increases.

The pseudocode[2] of a run is as follows:

1. `setup`: the world is generated; each cell is provided with attributes as `resistance, sw, state`; `N_f` firms are created, initialized and located in the world.

2 The complete program is in the Appendix B

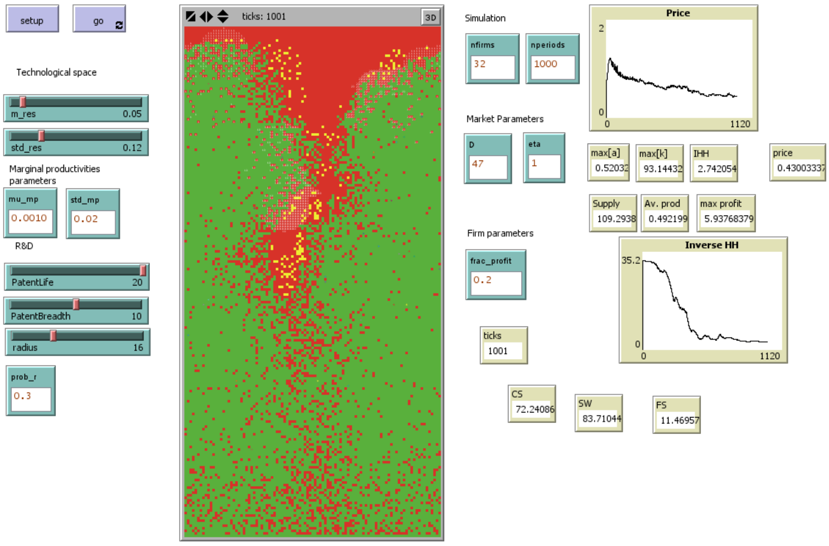

Figure 6.: Results from `TechSpaShumpComp.nlogo`

2. `update-space`: update the states of the technology-performance space, by an analysis of resistance of cells (state 0 to state 1) and an analysis of the composition of each cell's neighborhood (state 1 to state 2)
3. `market`: calculate individual supplies for firms `q = A * K`; calculate the aggregate supply `qtot =` $\sum q$ and calculate the market clearing price `price = D / qtot`; calculate attributes for the firm as `profit, share, rho` and update the capital `k`
4. `R&D`: for each firm establishes a budget for R&D activities `budget`; exerting exploration by eroding the resistance of the neighboring cells; each firm chooses randomly to execute `move1` or `move2`.
5. for each cell the value of the patent life `protected = protected -1` and patent breadth is generated if should be the case (give a label to a cell with a patent life)
6. `statistics`: relevant statistics are calculated as `max_a, CS, FS, SoW`
7. Procedures 2 till 6 are repeated 1000 times

4.3.2 *Simulation Results*

Our methodology here is exploratory, we make an descriptive analysis using box plots based on the results of the last period of each run. [3]

3 In case of ambiguity, some tests for comparisons should be used. Nonetheless, in our analysis we did not confront with such situation.

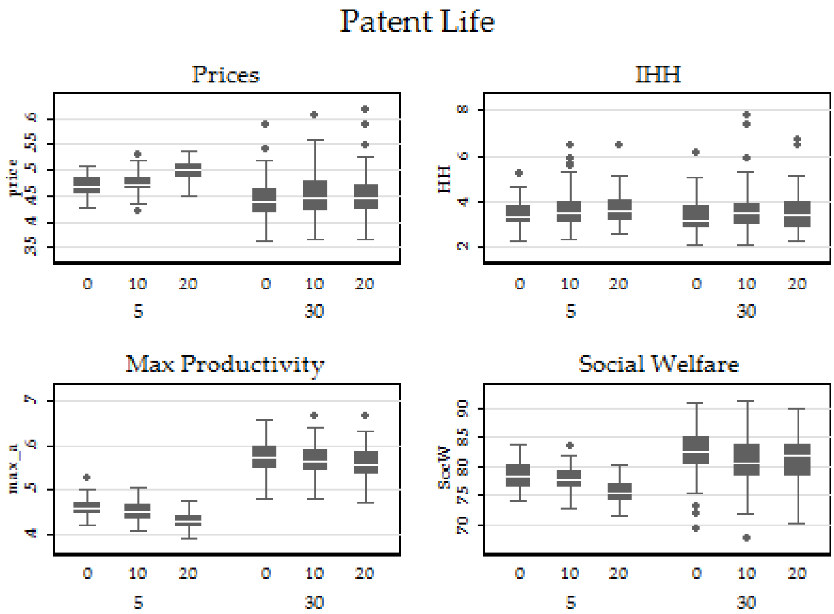

Figure 7.: Effects of patent life [0, 10, 20] under different radius of search [5, 30]

We start analyzing the effects of the patent life. So, the figure 7 shows the distributions of different variables under different combinations of duration for the patent (zero, ten or twenty periods) and different radius of search of firms in the technology-performance space (5 or 30). In a first sight it is evident that when $r = 30$ the impact of duration of patents is almost null, however results change dramatically when the radius of search is $r = 5$, in such case increments in the duration of patents just have negative effects; at first prices become bigger, social welfare also decreases as duration increases. Lastly even the productivity is affected negatively for longer patent duration

Now when we analyze the consequences of different patent breadth we cannot say nothing definitive, because the box plots do not show a clear tendency, it is more under the results of the figure 8 we do not see any impact, and this results are eventually the same under different radius of search.

It should be interesting from the point of view of the social welfare to see if there is some optimal combination of patent policies, by observing the results in the figure 9, we cannot see clearly that there is an optimal policy. However there is a worst scenario where patent life is longer and breadth is bigger. This scenario results in the poorer results in social welfare, also it seems reasonable that one of the best policies from the point of view of social welfare do not have a patent system.

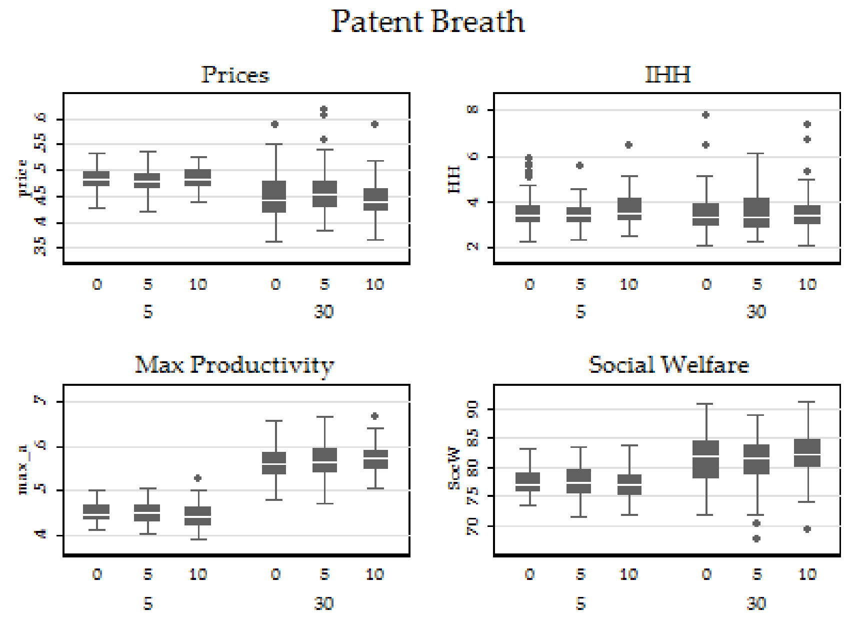

Figure 8.: Effects of patent breadth [0, 5, 10] under different radius of search [5, 30].

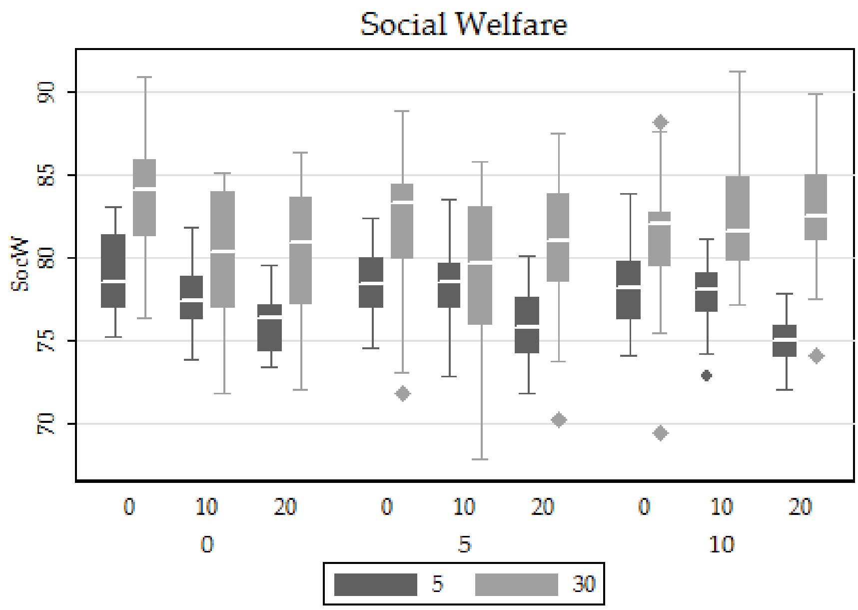

Figure 9.: Distribution of social welfare of combinations of patent policies (patent life [0, 10, 20] and patent breadth [0, 5, 10]) under different search radius [5, 30]

Finally, repetition of the same exercise but this time observing the distribution of the maximum level of productivity, shows a poor per-

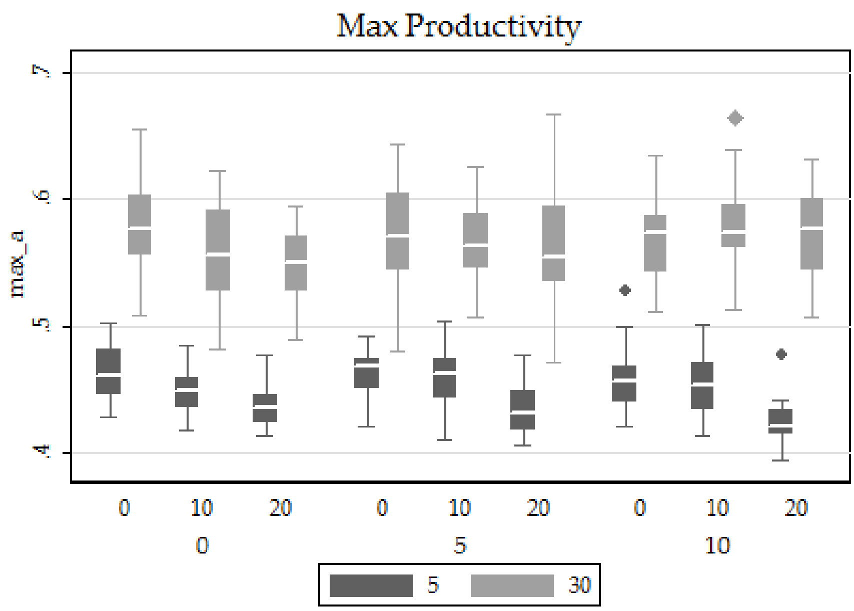

Figure 10.: Distribution of the maximum productivity with combinations of patent policies (patent life [0, 10, 20] and patent breadth [0, 5, 10]) under different search radius [5, 30]

formance of the patent system in special when the radius of search is smaller ($r = 5$).

After we see all the boxplots, we can conclude that there is a better performance in terms of social welfare and productivity when the radius of search is bigger. So in this case a good economic policy should be to not have a patent system and promote the research and development in divergent areas, not just concentrate research on a specific area (which should increase the research radius in the technology-performance space).

4.4 CONCLUSIONS

We develop a simple model that encompass several important points from schumpeterian dynamics and from the technology-performance space developed Silverberg and Verspagen (2007), in order to analyze several patent policies related to the patent life and patent breadth.

The main objective of a patent system is to promote innovation, however our findings show us that the patent system effects are bad for the social welfare and for the productivity (innovation). When there is smaller search radius in the technology space the results are even sharper against the patent system, because longer patent duration (patent life) and bigger patent breadth, difficult innovative search when the search radius is small.

We were not able to find an optimal combination of patent life and patent breadth that benefits society, contrarily we find that the worst situation is a society with a stronger patent system. As we saw in the introduction such results are not new, it more seems to be almost standard than many researchers express themselves against a stronger patent system.

There are shortcomings in our simple model, and so more work is necessary in particular in the development of the entrance and exit of firms that should be crucial in some highly dynamic scenarios, also one point that is lost in our analysis here is the learning process of firms in order to explore the technology-performance space. these points should be studied with more deepness.

There are also several options to explore, these related to the complexity of products (products are made with several different parts). One area that have been partially explored, is the complexity developed by the NK approach (see Chang (2009)). Also, Dosi et al. (2008) explore several aspects of product quality allied with evolutionary economic concepts, eventually the model developed here could follow similar lines. Our model could be extended to the case of more goods (potentially substitutes as cell phones and cameras). So in this case the innovations discovered in the technology-performance space should have different impact in the quality and costs of different substitute goods.

BIBLIOGRAPHY

J.J. Anton and D.A. Yao. Finding Lost Profits: An Equilibrium Analysis of Patent Infringement Damages. *Journal of Law, Economics, and Organization*, 23(1):186, 2007.

Reiko Aoki and Jin-Li Hu. Licensing vs. litigation: the effect of the legal system on incentives to innovate. *Journal of Economics & Management Strategy*, 8(1):133–160, 1999.

Reiko Aoki and Jin-Li Hu. Time factors of patent litigation and licensing. *Journal of Institutional and Theoretical Economics JITE*, 159(2): 280–301, 2003.

K.J. Arrow. Economic Welfare and the Allocation of Resources for Invention, 1962.

Lucian Arye Bebchuk. Litigation and settlement under imperfect information. *The RAND Journal of Economics*, 15(3):404–415, 1984.

James Bessen and Michael J Meurer. Lessons for patent policy from empirical research on patent litigation. *Lewis & Clark L. Rev.*, 9:1, 2005.

James Bessen and Michael James Meurer. *Patent failure: How judges, bureaucrats, and lawyers put innovators at risk*. Princeton University Press, 2008.

J. Boone. Intensity of competition and the incentive to innovate. *International Journal of Industrial Organization*, 19(5):705–726, 2001.

Myong-Hun Chang. Industry dynamics with knowledge-based competition: a computational study of entry and exit patterns. *Journal of Economic Interaction and Coordination*, 4(1):73–114, 2009.

J.P. Choi. Alternative damage rules and probabilistic intellectual property rights: Unjust enrichment, lost profits, and reasonable royalty remedies. *Information Economics and Policy*, 21(2):145–157, 2009.

Giovanni Dosi, Luigi Marengo, Corrado Pasquali, and Marco Valente. Patents, appropriability and competition in complex products industries: an evolutionary model. *Preliminary draft, LEM Working Paper*, 2008.

D. Encaoua and Y. Lefouili. Licensing Weak Patents. *The Journal of Industrial Economics*, 57(3):492–525, 2009.

Joseph Farrell and Carl Shapiro. How strong are weak patents? *The American Economic Review*, 98(4):1347–69, 2008.

Nathan Goldschlag. Percolating patents: Balancing the effects of patents on innovation. *Unpublished*, 2014.

C. Heath, J. Henkel, and M. Reitzig. Who Really Profits from Patent Infringements? Innovation Incentives and Disincentives from Patent Indemnification Rules. Technical report, 2002.

Nicolas Jonard and M Yildizoğlu. Technological diversity in an evolutionary industry model with localized learning and network externalities. *Structural Change and Economic Dynamics*, 9(1):35–53, 1998.

M. I. Kamien and Y. Tauman. Patent Licensing: The Inside Story. *The Manchester School*, 70(1):7–15, 2002.

M.A. Lemley and C. Shapiro. Probabilistic Patents. *Journal of Economic Perspectives*, 19(2):75, 2005.

Michael J Meurer. The settlement of patent litigation. *The RAND Journal of Economics*, 20(1):77–91, 1989.

R.R. Nelson and S.G. Winter. in search of useful theory of innovation. *Research Policy*, 6:36–76, 1977.

R.R. Nelson and S.G. Winter. *An evolutionary theory of economic change*. Belknap Press, 1982.

PWC. 2014 patent litigation study: As case volume leaps, damages continue general decline. Technical report, Price Waterhouse Coopers, 2014.

M. Schankerman and S. Scotchmer. Damages and Injunctions in Protecting Intellectual Property. *The RAND Journal of Economics*, 32(1): 199–220, 2001.

M. Schankerman and S. Scotchmer. Still Looking for Lost Profits: The Case of Horizontal Competition. *Department of Economics, UCB*, (E05-344), 2005.

Joseph Alois Schumpeter. *Socialism, capitalism and democracy*. Harper and Brothers, 1942.

Joseph Alois Schumpeter. *The theory of economic development: An inquiry into profits, capital, credit, interest, and the business cycle*, volume 55. Transaction Books, 1961.

S. Scotchmer. *Innovation and Incentives*. MIT Press, 2004.

D. Sen and Y. Tauman. General licensing schemes for a cost-reducing innovation. *Games and Economic Behavior*, 59(1):163–186, 2007.

G. Silverberg. The discrete charm of the bourgeoisie: quantum and continuous perspectives on innovation and growth. *Research Policy*, 31:1275–1289, 2002.

G. Silverberg and B. Verspagen. A percolation model of innovation in complex technology spaces. *Journal of Economic Dynamics and Control*, 29(1-2):225–244, 2005.

Gerald Silverberg and Bart Verspagen. Self-organization of r&d search in complex technology spaces. *Journal of economic interaction and coordination*, 2(2):211–229, 2007.

Thomas Vallée and Murat Yıldızoglu. Social and technological efficiency of patent systems. In *Innovation, Industrial Dynamics and Structural Transformation*, pages 407–424. Springer, 2007.

X.H. Wang. Fee versus royalty licensing in a Cournot duopoly model. *Economics Letters*, 60(1):55–62, 1998.

S.G. Winter. Schumpeterian competition in alternative technological regimes. *Journal of Economic Behavior & Organization*, 5(3-4):287–320, 1984.

S.G. Winter. Patents and welfare in an evolutionary model. *Industrial and Corporate Change*, 2(1):211, 1993.

A

ADDITIONAL PROOFS FOR CHAPTER 2

Proof Lemma 1. The best response function for the patent holder $\phi_1(q_2)$ is the same whether $D^{LP} > 0$ or $D^{LP} = 0$, assuming that $q_2 \in [0, 1+\gamma]$

$$\phi_1(q_2) = \frac{1 - q_2 + \gamma}{2} \tag{97}$$

Let,

$$\begin{aligned} x(q_1, q_2) &= (1 - q_1 - q_2 + \gamma)q_2 - \theta \max\left\{\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1, 0\right\} \\ &= x_1(q_1, q_2) - \theta \max\{x_2(q_1, q_2), 0\} \end{aligned} \tag{98}$$

be the competitor's payoff function. If the innovation is drastic $\pi_1^{\mathcal{N}}$ is the monopoly profit, so then $\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1 \geq 0$ for any $q_1, q_2 \geq 0$, then $D^{LP} > 0$. Assuming a internal solution

$$\phi_2(q_1) = \frac{1 + \gamma - (1+\theta)q_1}{2} \quad \text{if } \gamma \geq 1 \tag{99}$$

When the innovation is non drastic, for a given $q_1 \in [0, 1+\gamma)$. $x(q_1, q_2)$ reach maximum at $\tilde{q_2}$. $0 < \tilde{q_2} < \hat{q_2} = \frac{1+\gamma-q_1}{2}$ where $\hat{q_2}$ is the maximum of $x_1(q_1, q_2)$. Then $\partial x_1(q_1, q_2)/\partial q_2 > 0$ for $q_2 \in [0, \frac{1+\gamma-q_1}{2})$.

The best response depends on the sign of $x_2(q_1, q_2)$, this sign could be positive, negative or zero. $\partial x_2(q_1, 0)/\partial q_2 = q_1 > 0$ for any q_2, means that if $x_2(q_1, 0)$ is positive $x_2(q_1, q_2)$ is also positive, then just it is necessary to see what happens with $x_2(q_1, q_2)$ when $x_2(q_1, 0) < 0$. Let

$$\begin{aligned} q_1^{a,b} &= \frac{(1+\gamma) \pm \sqrt{(1+\gamma)^2 - 4\pi_1^{\mathcal{N}}}}{2} \\ &= \frac{(1+\gamma)}{2} \pm \sqrt{\left(\frac{1+\gamma}{2}\right)^2 - \left(\frac{1+2\gamma}{3}\right)^2} \end{aligned}$$

such that $x_2(q_1^a, 0) = x_2(q_1^a, 0) = 0$, where the upper index a refers to the inferior value and b to the superior one.

Notice that for a given q_1, $x_2(q_1, 0)$ reach minimum at $q_1^c = \frac{(1+\gamma)}{2}$, for $\gamma < 1$ it holds that

$$0 < q_1^a < q_1^{\mathcal{N}} < q_1^c < q_1^b < 1 + \gamma \tag{100}$$

Then by the convexity of $x_2(q_1, 0)$, $x_2(q_1, 0) > 0$ for $q_1 \in (0, q_1^a) \cup (q_1^b, 1+\gamma)$, and $x_2(q_1, 0) \leq 0$ when $q_1 \in [q_1^a, q_1^b]$.

Now when $q_1 \in [q_1^a, q_1^b]$, $x_2(q_1, 0) \leq 0$, then let q_2^a be such that $x_2(q_1, q_2^a) = 0$ for $q_1 \in [q_1^a, q_1^b]$, then

$$\begin{aligned} q_2^a &= (1+\gamma) - q_1 - \pi_1^{\mathcal{N}}/q_1 \\ &\geq (1+\gamma) - q_1^b - \pi_1^{\mathcal{N}}/q_1^a = 0 \end{aligned} \tag{101}$$

, in consequence $0 \leq q_2^a < 1 + \gamma$. Now by evaluating the directional derivative in the direction $(0,1)$ of x at (q_1, q_2^a), $\partial^+ x/\partial q_2(q_1, q_2^a) = 1 + \gamma - 2q_2^a - q_1 - \theta q_1$,

$$\phi_2(q_1) = \begin{cases} q_2^a & \text{if } q_1 \in A \\ \frac{1+\gamma-(1+\theta)q_1}{2} & \text{if } q_1 \in B \end{cases} \tag{102}$$

where
$A = \left\{(\gamma, q_1) : q_1 \in [q_1^a, q_1^b], \gamma < 1, 1 + \gamma - 2q_2^a - q_1 - \theta q_1 \leq 0\right\}$
and
$B = \left\{(\gamma, q_1) : q_1 \in [q_1^a, q_1^b], \gamma < 1, 1 + \gamma - 2q_2^a - q_1 - \theta q_1 > 0\right\}$.

In the cases when given a q_1, $x_2(q_1, 0) > 0$ or $q_1 \in B$, the equilibrium has a positive damage and the the Nash Equilibrium (NE) by eq. (22), (23) and (27) is $\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$. However, when $q_1 \in A$ $x_2 \leq 0$, by using eq(26) with $\phi_1(q_2^a)$ instead of q_1, $q_2^a = q_2^{\mathcal{N}}$, so then by (22) $\phi_1(q_2^{\mathcal{N}}) = q_1^{\mathcal{N}}$. By using the restriction $1 + \gamma - 2q_2^a - q_1 - \theta q_1 > 0$ with the equilibrium quantities, this condition becomes in $\gamma < \theta/(3 - 2\theta)$, then when it holds by using eq. (27) the NE is $(q_1^{\mathcal{N}}, q_2^{\mathcal{N}}) = \left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right)$. So then, when $\gamma > \theta/(3-2\theta)$, $x_2 \geq 0$, and the Nash equilibrium is $\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$ $\square$

Proof Lemma 2. There is a symmetry respect to last proof, this time $\phi_2(q_1)$ is the same whether $D^{UE} > 0$ or $D^{UE} = 0$, when $q_1 \in [0, 1+\gamma]$ is

$$\phi_2(q_1) = \frac{1 - q_1 + \gamma}{2}$$

and 0 if $q_1 > 1 + \gamma$.

Let,

$$\begin{aligned} y(q_1, q_2) &= (1 - q_1 - q_2 + \gamma)q_1 + \theta \max\left\{(1 - q_1 - q_2 + \gamma)q_2 - \pi_2^{\mathcal{N}}, 0\right\} \\ &= y_1(q_1, q_2) + \theta \max\{y_2(q_1, q_2), 0\} \end{aligned}$$

be the payoff of the patent holder. If the innovation is drastic $\pi_2^{\mathcal{N}} = 0$, then

$$\phi_1(q_2) = \frac{1 + \gamma - (1+\theta)q_2}{2} \quad \text{if } \gamma \geq 1 \text{ and } q_2 \in [0, 1+\gamma)$$

When the innovation is non drastic, for a given $q_2 \in [0, 1+\gamma)$, $y(q_1, q_2)$ reach maximum at $\tilde{q_1}$, where $0 < \tilde{q_1} < \hat{q_1} = (1 + \gamma - q_2)/2$ and $\hat{q_1}$ is the maximum of $y_1(q_1, q_2)$. Then $\partial y_1(q_1, q_2)/\partial q_1 > 0$ for $q_1 \in [0, (1+\gamma-q_2)/2)$ and $\partial y_2(q_1, q_2)/\partial q_1 = -q_2$ for any q_1.

The best response depends on the sign of $y_2(0, q_2)$, this sign could be positive, negative or zero. There are two values of q_2 that make $y_2(0, q_2) = 0$,

$$\begin{aligned} q_2^{a,b} &= \frac{(1+\gamma) \pm \sqrt{(1+\gamma)^2 - 4\pi_2^{\mathcal{N}}}}{2} \\ &= \frac{(1+\gamma)}{2} \pm \sqrt{\left(\frac{1+\gamma}{2}\right)^2 - \left(\frac{1-\gamma}{3}\right)^2} \end{aligned}$$

where the upper index a refers to the inferior value and b to the superior one. $y_2(0, q_2)$ reach maximum at $q_2^c = \frac{(1+\gamma)}{2}$, in a consequence

$$0 < q_2^a < q_2^c < q_2^b < 1 + \gamma$$

.

By concavity of $y_2(0, q_2)$, $y_2(0, q_2) < 0$ for $q_1 \in (0, q_2^a) \cup (q_2^b, 1+\gamma)$ and $y_2(0, q_2) \geq 0$ when $q_2 \in [q_2^a, q_2^b]$, then

$$\phi_1(q_2) = \frac{1 + \gamma - q_2}{2} \quad \text{if } q_2 \in (0, q_2^a) \cup (q_2^b, 1+\gamma)$$

.

If it assumed that $q_2 \in (0, q_2^a) \cup (q_2^b, 1+\gamma)$ then the best response of patent holder is $\frac{1+\gamma-q_2}{2}$, then the equilibrium is $q_1 = q_2 = (1+\gamma)/3$, but $(1+\gamma)/3 \in [q_2^a, q_2^b]$ then this is not a NE.

By assuming that $q_2 \in [q_2^a, q_2^b]$ there is

$$\begin{aligned} q_1^a &= (1+\gamma) - q_2 - \pi_2^{\mathcal{N}}/q_2 \\ &\geq (1+\gamma) - q_2^b - \pi_1^{\mathcal{N}}/q_2^a = 0 \end{aligned}$$

that makes $y_2(q_1^a, q_2) = 0$, where $0 \leq q_2^a < 1 + \gamma$. By analyzing the derivative on the left (or in direction $(-1, 0)$) at q_1^a ,$\partial^- y/\partial q_1(q_1^a, q_2) = -(1 + \gamma - 2q_1^a - q_2 - \theta q_2)$, it is bobtained that

$$\phi_2(q_1) = \begin{cases} q_1^a & \text{if } \partial^- y/\partial q_1(q_1^a, q_2) \leq 0 \\ \frac{1+\gamma-(1+\theta)q_1}{2} & \text{if } \partial^- y/\partial q_1(q_1^a, q_2) > 0 \end{cases} \tag{103}$$

If is assumed that $\partial^- y/\partial q_1(q_1^a, q_2) > 0$ the NE is $\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$ and it holds that $\partial^- y/\partial q_1 \left(q_1^a, (1-\theta)\frac{1+\gamma}{3-\theta}\right) > 0$.

Finally, if $\partial^- y/\partial q_1(q_1^a, q_2) \leq 0$ in equilibrium, $q_2^a(\phi_1) = (1+5\gamma)/3$, so then $\phi_2((1+5\gamma)/3) = q_2^{\mathcal{N}}$ and the condition $\partial^- y/\partial q_1(q_1^a, q_2) \leq 0$ becomes in $-(1-2\gamma)/3 + (1+\theta)(1-\gamma)/3 \leq 0$, but the first term is always positive. then the unique NE is $\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$ $\square$

Proof Lemma 3. When $\gamma \leq \theta/(3-2\theta)$, $\pi_2^{\mathcal{I},\mathcal{LP}} = \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) > \left(\frac{1-\gamma}{3}\right)^2 = \pi_2^{\mathcal{N}}$. When $\theta/(3-2\theta) \leq \gamma < 1$, $G(\gamma,\theta) = \pi_2^{\mathcal{I},\mathcal{LP}} - \pi_2^{\mathcal{N}}$, then $G = \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 - \left(\frac{1-\gamma}{3}\right)^2$, now notice that $G_{11} = \left(\frac{1}{3-\theta}\right)^2 - \frac{4\theta+1}{9}$, because at $\theta = 0$ $G_{11} = 0$ and because $dG_{11}/d\theta = 2(3-\theta)^{-3} - 4/9 < (2)^{-2} - 4/9 < 0$, $G_{11} < 0$ for $\theta \in (0,1)$, then G is concave in γ for $\theta \in (0,1)$.

$G(1,\theta) = \left(\frac{2}{3-\theta}\right)^2 - \theta$, moreover $G_2(1,\theta) = 8(3-\theta)^{-3} - 1 < 0$ for $\theta \in (0,1)$, $G(1,0) = \left(\frac{2}{3}\right)^2$ and $G(1,1) = 0$ then by continuity $G(1,\theta) > 0$ for $\theta \in (0,1)$.

$$G(\theta/(3-2\theta),\theta) = \left(\frac{1}{3-2\theta}\right)^2 - \theta\left(\frac{1}{3-2\theta}\right)^2 - \left(\frac{1-\theta}{3-2\theta}\right)^2 = \frac{\theta(1-\theta)}{(3-2\theta)^2} > 0$$

because G is concave in γ and $G(\theta/(3-2\theta),\theta)$, $G(1,\theta) > 0$, $G > 0$ for $\gamma > \theta/(3-2\theta)$ and $\theta \in (0,1)$. Also, when $\gamma > 1$, $\pi_2^{\mathcal{I},\mathcal{LP}} \geq \pi_2^{\mathcal{N}} = 0$.

For the UE case, if $\gamma < 1$ $\pi_2^{\mathcal{I},\mathcal{UE}} = (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 > \left(\frac{1-\gamma}{3}\right)^2 = \pi_2^{\mathcal{N}}$ and in the case $\gamma > 1$, $\pi_2^{\mathcal{I},\mathcal{UE}} \geq \pi_2^{\mathcal{N}} = 0$

□

Proof Lemma 4. By using (14),

$$\underline{F}^{\mathcal{LP}} = \begin{cases} \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma \leq \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} < \gamma < 1 \\ \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{104}$$

and after some algebra

$$\underline{F}^{\mathcal{LP}} = \begin{cases} \frac{1}{9}\gamma(1+3\gamma) > 0 & \text{if } 0 < \gamma \leq \frac{\theta}{3-2\theta} \\ \frac{\theta\left(3-5\theta+\theta^2+\gamma^2\left(30-23\theta+4\theta^2\right)+\gamma\left(24-22\theta+4\theta^2\right)\right)}{9(3-\theta)^2} & \text{if } \frac{\theta}{3-2\theta} < \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(57-50\theta+9\theta^2\right)}{36(3-\theta)^2} > 0 & \text{if } 1 \leq \gamma \end{cases}$$

it is straightforward to see that the first and third term are positive, in the case of the second term. Notice that $\partial\underline{F}^{\mathcal{LP}}/\partial\gamma = \frac{\theta\left(24-22\theta+4\theta^2+2\gamma\left(30-23\theta+4\theta^2\right)\right)}{9(3-\theta)^2} > 0$, $\underline{F}^{\mathcal{LP}}$ at $\gamma = \theta/(3-2\theta)$ is $\frac{\theta(3+\theta)}{9(3-2\theta)^2} > 0$, then the second term is also positive.

Now in the case of UE,

$$\underline{F}^{\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{3}\right)^2 - (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{3}\right)^2 - (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{105}$$

and after some algebra

$$\underline{F}^{\mathcal{UE}} = \begin{cases} -\frac{\theta\left(6-7\theta+\theta^2+\gamma^2\left(6-7\theta+\theta^2\right)-2\gamma\left(12-5\theta+\theta^2\right)\right)}{9(3-\theta)^2} & \text{if } 0 < \gamma < 1 \\ \frac{(1+\gamma)^2\theta(3+\theta)}{9(3-\theta)^2} > 0 & \text{if } 1 \leq \gamma \end{cases} \tag{106}$$

the first term is not always positive, taking the roots of the polynomial, it is observable that the expression is greater than zero when $\gamma > \frac{12-5\theta+\theta^2-2\sqrt{27-9\theta-3\theta^2+\theta^3}}{6-7\theta+\theta^2}$. Finally, the last case is positive.

□

Proof Proposition 2. from (16) $\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\pi_2^{\mathcal{I}}$

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} = \begin{cases} \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\frac{1+2\gamma}{3-\theta}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 \leq \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2\right) & \text{if } \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2\right) & \text{if } \gamma \geq 1 \end{cases}$$

then after some algebra

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} - \pi_1^{\mathcal{I},\mathcal{LP}} =$$
$$= \begin{cases} -\frac{(1-\gamma)(5\theta+\gamma(9+7\theta))}{36(3-\theta)^2} < 0 & \text{if } 0 \le \gamma < \frac{\theta}{3-2\theta} \\ \frac{\theta\left(-9+3\theta+\theta^2+2\gamma\theta(-3+2\theta)+\gamma^2\left(18-15\theta+4\theta^2\right)\right)}{36(3-\theta)^2} & \text{if } \frac{\theta}{3-2\theta} \le \gamma < 1 \\ \frac{(1+\gamma)^2(1-\theta)^2\theta}{16(3-\theta)^2} > 0 & \text{if } \gamma \ge 1 \end{cases}$$

it is observable that the first term is negative and the third one is positive. Let $H(\theta,\gamma) = \pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} - \pi_1^{\mathcal{I},\mathcal{LP}}$, where $\frac{\theta}{3-2\theta} \le \gamma < 1$, so then, $H(\theta, \frac{\theta}{3-2\theta}) = -\frac{(1-\theta)\theta}{4(3-2\theta)^2} < 0$, $H(\theta,1) = \frac{(1-\theta)^2\theta}{4(3-\theta)^2} > 0$ and $\partial H(\theta,\gamma)/\partial\gamma = \frac{\theta\left(-\theta(3-2\theta)+\gamma\left(18-15\theta+4\theta^2\right)\right)}{18(3-\theta)^2} > \frac{\theta^2}{54-54\theta+12\theta^2} > 0$, then $H(\theta,\gamma) \ge 0$ if $\gamma > \gamma^s$, where $\gamma^s = \frac{\theta(3-2\theta)+3\sqrt{(3-\theta)^2(2-\theta)}}{18-15\theta+4\theta^2}$ is a root of $H(\theta,\gamma) = 0$.

For the case of UE by preceding as in the LP case,

$$\pi_1^{\mathcal{L},\mathcal{R}\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left((1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2\right) & \text{if } 0 \le \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left((1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2\right) & \text{if } \gamma \ge 1 \end{cases}$$

then after some algebra and using the definition of $\pi_1^{\mathcal{I}\mathcal{UE}}$

$$\pi_1^{\mathcal{L},\mathcal{R}\mathcal{UE}} - \pi_1^{\mathcal{I}\mathcal{UE}} = \begin{cases} -\frac{\theta\left(18-15\theta+\theta^2-2\gamma\theta(3+\theta)+\gamma^2\left(18-15\theta+\theta^2\right)\right)}{36(3-\theta)^2} & \text{if } 0 \le \gamma < 1 \\ -\frac{(1+\gamma)^2(1-\theta)\theta}{4(3-\theta)^2} < 0 & \text{if } \gamma \ge 1 \end{cases}$$

The second term is negative, In the case of the first term, let $H(\theta,\gamma) = 18 - 15\theta + \theta^2 - 2\gamma\theta(3+\theta) + \gamma^2\left(18 - 15\theta + \theta^2\right)$, where $\gamma \in (0,1)$, so then, $H(0,\gamma) = 18\left(1+\gamma^2\right)$, $H(1,\gamma) = 4(-1+\gamma)^2$ and $\partial H(\theta,\gamma)/\partial\theta = -15 + 2\theta + \gamma^2(-15+2\theta) - 2\gamma(3+2\theta) < 0$. Then $H(\theta,\gamma) > 0$, $\pi_1^{\mathcal{L},\mathcal{R}\mathcal{UE}} - \pi_1^{\mathcal{I}\mathcal{UE}} = -\frac{H(\theta,\gamma)}{36(3-\theta)^2} < 0$.

□

Proof Proposition 3. After some algebra,

$$\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} = \begin{cases} \frac{\theta\left(3-5\theta+\theta^2\right)-2\gamma\left(-9+21\theta-8\theta^2+\theta^3\right)+\gamma^2\left(27-15\theta-2\theta^2+\theta^3\right)}{9(3-\theta)^2} & \text{if } 0 \le \gamma < \frac{\theta}{3-2\theta} \\ \frac{\theta\left(9-12\theta+2\gamma(-6+\theta)\theta+2\theta^2+\gamma^2\left(36-30\theta+5\theta^2\right)\right)}{9(3-\theta)^2} & \text{if } \frac{\theta}{3-2\theta} \le \gamma < \delta_2 \\ \frac{\theta\left(2\gamma\theta(-9+2\theta)+3\left(3-5\theta+\theta^2\right)+\gamma^2\left(54-45\theta+8\theta^2\right)\right)}{12(3-\theta)^2} & \text{if } \delta_2 \le \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(21-26\theta+5\theta^2\right)}{16(3-\theta)^2} & \text{if } \gamma \ge 1 \end{cases}$$

Let be $H(\theta,\gamma) = \theta\left(3-5\theta+\theta^2\right) - 2\gamma\left(-9+21\theta-8\theta^2+\theta^3\right) + \gamma^2\left(27-15\theta-2\theta^2+\theta^3\right)$, where $\delta_2 \le \gamma < 1$. $H(\theta,\gamma) = 0$ has two roots $\gamma^{a,b} = \frac{-9+21\theta-8\theta^2+\theta^3 \pm 3\sqrt{9-51\theta+85\theta^2-50\theta^3+12\theta^4-\theta^5}}{27-15\theta-2\theta^2+\theta^3}$. At Figure 3 are drawn both roots, It is observable that the behavior of the roots is not continuous because there are some θ that make the $9-51\theta+85\theta^2-50\theta^3+12\theta^4-\theta^5$ negative (see dashed line), then there are no roots for this area. The graph is shaded when $0 \le \gamma < \frac{\theta}{3-2\theta}$, $\gamma^{a,b}$ divide this area in two parts, at the first one $H(\theta,\gamma) > 0$ and in the second one $H(\theta,\gamma) < 0$, explicitly $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} \ge 0$ when $\gamma \le \gamma^a$ or when $\gamma \le \gamma^b$ and $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} < 0$ in another case.

Let $H(\theta,\gamma) = 2\gamma\theta(-9+2\theta) + 3\left(3-5\theta+\theta^2\right) + \gamma^2\left(54-45\theta+8\theta^2\right)$, where $\frac{\theta}{3-2\theta} \le \gamma < 1$, because $H(\theta, \frac{\theta}{3-2\theta}) = \frac{9\left(3-4\theta+\theta^2\right)^2}{(3-2\theta)^2} > 0$, $H(\theta,1) = 9\left(5-6\theta+\theta^2\right) > 0$ and $\partial H(\theta,\gamma)/\partial\gamma = 2\left((-6+\theta)\theta + \gamma\left(36-30\theta+5\theta^2\right)\right) > \frac{6\theta\left(6-5\theta+\theta^2\right)}{3-2\theta} > 0$. Then $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} > 0$ for $\frac{\theta}{3-2\theta} \le \gamma < \delta_2$

Let $H(\theta,\gamma) = 9 - 12\theta + 2\gamma(-6+\theta)\theta + 2\theta^2 + \gamma^2\left(36-30\theta+5\theta^2\right)$, where $\frac{\theta}{3-2\theta} \le \gamma < 1$, because $H(\theta, \frac{\theta}{3-2\theta}) = \frac{3(-3+\theta)^2\left(3-7\theta+4\theta^2\right)}{(3-2\theta)^2} > 0$, $H(\theta,1) = 63 - 78\theta + 15\theta^2 > 0$ and $\partial H(\theta,\gamma)/\partial\gamma = 2\left(\theta(-9+2\theta) + \gamma\left(54-45\theta+8\theta^2\right)\right) > 2\left(\theta(-9+2\theta) + \frac{\theta}{3-2*\theta}\left(54-45\theta+8\theta^2\right)\right) > \frac{2\theta\left(27-21\theta+4\theta^2\right)}{3-2\theta} > 0$. Then $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} > 0$ for $\delta_2 \le \gamma < 1$. By direct observation $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} > 0$ for $\gamma > 1$

In the case of the competitor, after some algebra

$$\pi_2^{\mathcal{LP}} - \pi_2^{\mathcal{UE}} = \begin{cases} -\frac{\theta\left(6-7\theta+\theta^2\right)+\gamma\left(9-30\theta+11\theta^2-2\theta^3\right)+\gamma^2\left(27-12\theta-4\theta^2+\theta^3\right)}{9(-3+\theta)^2} & 0 \le \gamma < \frac{\theta}{3-2*\theta} \\ -\frac{\theta\left(9-12\theta+2\gamma(-6+\theta)\theta+2\theta^2+\gamma^2\left(36-30\theta+5\theta^2\right)\right)}{9(-3+\theta)^2} & \frac{\theta}{3-2*\theta} \le \gamma < 1 \\ -\frac{(1+\gamma)^2(-6+\theta)\theta^2}{9(-3+\theta)^2} & \gamma \ge 1 \end{cases}$$

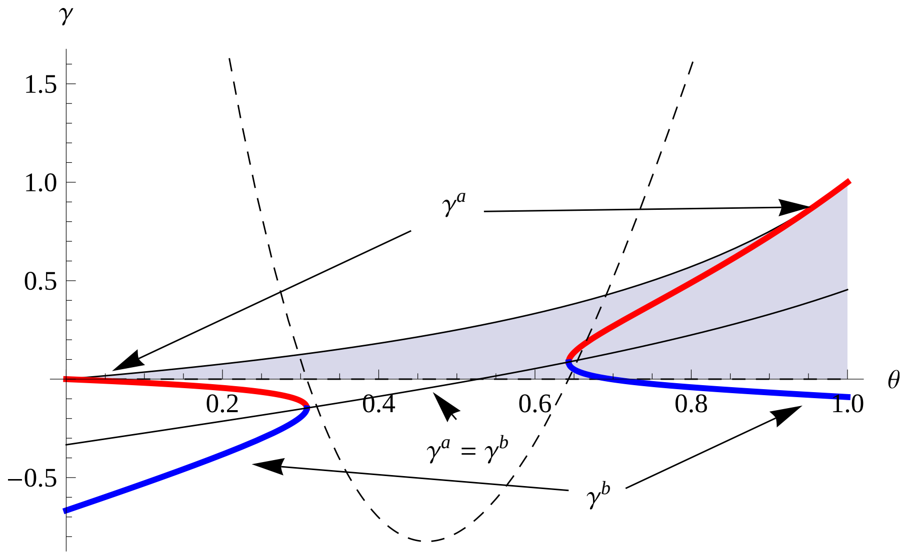

Figure 11.: Roots of $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} = 0$ for $0 \leq \gamma < \frac{\theta}{3-2\theta}$

in the last case is easy to see that the expression is positive, in the second case both roots are under $\theta/(3-2\theta)$ and the expression is negative at $\theta = \gamma = 1/2$, finally in the first case both roots are inside the region of interest then after some analysis is straightforward to see that $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} > 0$ if $\theta > \frac{-9+30\theta-11\theta^2+2\theta^3-3\sqrt{9-132\theta+238\theta^2-116\theta^3+17\theta^4}}{2\left(27-12\theta-4\theta^2+\theta^3\right)}$.

In the case of the industry

$$\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} = \begin{cases} \frac{\theta(-3+2\theta)+\gamma^2\theta(-3+2\theta)+\gamma\left(9-12\theta+5\theta^2\right)}{9(-3+\theta)^2} & 0 \leq \gamma < \frac{\theta}{3-2*\theta} \\ 0 & \frac{\theta}{3-2*\theta} \leq \gamma < \delta_2 \\ \frac{\theta\left(-9+3\theta+\theta^2+2\gamma\theta(-3+2\theta)+\gamma^2\left(18-15\theta+4\theta^2\right)\right)}{36(-3+\theta)^2} & \delta_2 \leq \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(189-138\theta+29\theta^2\right)}{144(-3+\theta)^2} & \gamma \geq 1 \end{cases}$$

In the first case both roots are outside the region of study and at $\theta = 1/2, \gamma = 1/10$ the expression is negative, then $\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} < 0$ if $0 \leq \gamma < \frac{\theta}{3-2*\theta}$, the third case follows by notice that one of the roots is δ_2 (the other one is negative) and at $\theta = 1/10, \gamma = 9/10$ the expression is positive, then $\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} > 0$ if $\frac{\theta}{3-2*\theta} \leq \gamma < \delta_2$, and the last case follows directly. □

Proof Proposition 4. after some algebra

$$Q^{\mathcal{LP}} - Q^{\mathcal{UE}} = \begin{cases} \frac{\theta-\gamma(3-2\theta)}{3(3-\theta)} & 0 \leq \gamma < \frac{\theta}{3-2*\theta} \\ 0 & \frac{\theta}{3-2*\theta} \leq \gamma < \delta_2 \\ -\frac{(1+\gamma)(-3+2\theta)}{3(-3+\theta)} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta * \left(\frac{1+2*\gamma}{3}\right)^2} & \delta_2 \leq \gamma < 1 \\ -\frac{(1+\gamma)(-3+2\theta)}{3(-3+\theta)} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta * \left(\frac{1+\gamma}{2}\right)^2} & \gamma \geq 1 \end{cases}$$

in the case when $0 \leq \gamma < \frac{\theta}{3-2*\theta}$, it is observable that $\partial(Q^{\mathcal{LP}} - Q^{\mathcal{UE}})/\partial\gamma < 0$ and at $\gamma = \frac{\theta}{3-2*\theta}$ $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} = 0$, then $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} > 0$ for $0 \leq \gamma < \frac{\theta}{3-2*\theta}$. In the third case, because both roots of the polynomial are below δ_2 then the term is positive or negative, because at $\theta = 1/10, \gamma = 9/10$ the expression is negative, by noticing that if the third term is negative this implies that the fourth it is also negative, then $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} > 0$ for $\gamma \geq \delta_2$. □

B

NETLOGO'S PROGRAM USED IN CHAPTER 4

```
globals[
  ;D demand parameter
  par ;; parameters for the logN distr.
  current-row ;; list of current row patches
  qtot ;; total supply
  HH ;; Inverse Herfindahl Index
  price ;; price of market
  ave_a
  max_a
  min_a
  max_profit
  ave_profit
  CS
  FS
  SocW
  ]
breed[firms firm]

firms-own[
  A ;; productivity
  K ;; capital
  q ;; quantity produced
  profit ;; profit
  rho ;; markup
  share ;; market share

  ]
patches-own[
  res ;; resistance
  w ;; marginal productivity
  state ;; initially all are in state 0 undiscovered
  sw ;; productivity of the site
  protected ;; periods left for patent protection
  ]
;; function that returns parameters
;; in LogNormal distribution
;; for make the mean and standard distribution
;; as asked
to-report Mulog [mu sigma]
  let beta  ln (1 + (sigma ^ 2) / (mu ^ 2))
  let M (ln mu) - (beta / 2)
  let S sqrt beta
  report (list M S)
end

to setup
```

```
  ca ;; clear all
  reset-ticks
  ;; give resistance, colors and productivities
  ;; to each site
  give-characteristics
  ;; create firms
  initialize-firms
end

to go
  update-space
  market
  RD
  ifelse random-float 1 < (1 - prob_r) [move1][move2]
  ask patches[
    if protected > 0 [set protected protected - 1]
    if protected = 0 [set plabel ""]
    ]
  statistics
  tick
  if ticks > nperiods [stop]
end

to give-characteristics
  ;; get parameters for the lognormal distribution
  set par Mulog m_res std_res
  ask patches[
    set res exp(random-normal (item 0 par) (item 1 par))
    set w ( random-normal mu_mp std_mp )
    set sw 0.16
    set state 0
    set pcolor red
    if pycor = 0 [
      set res 0
      set state 2
      set pcolor green
      ]
     ]
  set current-row patches with [pycor = min-pycor + 1]
  ask current-row [set pcolor white]
  while [any? current-row with [pcolor = white]]
  [
    ask current-row with [pcolor = white] [
      let ssw sw
      ask patches at-points [ [0 1]]
      [
```

```
        set sw ssw + w ;; adding productivities
        set pcolor white
          ]
    ]
      ;; advance to the next row
      ask current-row [set pcolor red]
      set current-row patch-set
      [patch-at 0 1] of current-row
  ]
  ;; going back to the bottom
  set current-row patches with [pycor = min-pycor]
  ask current-row [
    set state 2
    set pcolor green
     ]
  ask patches [set sw 0.16 + .002 * (random-normal pycor 1)]
end

to initialize-firms
  create-firms nfirms
  ask firms[
    setxy random-xcor min-pycor
    set a 0.16
    set k 12.89
    set plabel ""
    ]
end

to update-space
    ask patches[
    if (res <= 0) and (state = 0) [
      set state 1
      set pcolor yellow
      ]
    if (any? neighbors with [state = 2]) and (state = 1 )
    [
      set state 2
      set pcolor green
      ]
    ]
end

to RD
  ask firms[
    ;; budget for research
    let budget max list ( frac_profit * profit) 0
```

```
    let bc  budget / ((2 + radius) ^ 2)
    ;; id for the firm
    let tempid who
    ask patches in-radius radius [
      set res res -  bc * (random-float 1) ;; update resistance
      if (res <= 0) and (state = 0) and (plabel = "")[
        ask (
          patches in-radius patentbreadth with
          [plabel = "" and state = 0]) [
          set plabel tempid
          set protected patentlife
          ]
        ]
      ]
    ]
end

to market
  ask firms[
    set q A * K
    ]
  ;; calculate total supply
  set qtot sum [q] of firms
  ;; calculate HHI
  set HH 1 / ( sum[(q  / qtot) ^ 2] of firms)
  set price D / qtot ^ (eta)
  ask firms[
    set profit (a * price - 0.16) * k
    set share q / qtot
    set rho price * a / 0.16
    let di 1.03 - (2 - share) / (rho * (2 - 2 * share))
    set k
    k * (( max list (min list di (.8 * profit / k)) 0 ) + 0.97)
    ]
end


to move1
    ask firms [
      let tempid who
      let cap k
      let geto patches in-radius radius with
      [ state = 2 and ((plabel = tempid) or (plabel = ""))]
      move-to max-one-of geto
      [(price * sw )/ (1 + 0.01 * distance firm tempid)]
      set a  sw
```

```
  ]
end

to move2
    ask firms [
    let tempid who
    move-to one-of patches in-radius radius with
    [ state = 2 and ((plabel = tempid) or (plabel = ""))]
    set a  sw
      ]
end

to statistics
  let ktot sum [k] of firms
  set ave_a (sum [a * k] of firms) / ktot
  set max_a max [a] of firms
  set min_a min [a] of firms
  set max_profit max [profit] of firms
  set ave_profit (sum [profit * k] of firms) / ktot
  set CS  47 * (ln 2 - ln price)
  set FS sum [profit] of firms
  set SocW FS + CS
end
```