Finiteness Properties of the Braided Thompson's Groups and the Brin-Thompson Groups

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Abstract

A group G is of type F_n if there is a $K(G,1)$ complex that has finite n-skeleton. It is of type F_∞, if it is of type F_n for all $n \in \mathbb{N}$. Here the property F_1 is equivalent to G being finitely generated and the property F_2 equivalent to being finitely presented. An interesting question in the study of these finiteness properties is how they change, if the group under consideration is changed. One family of examples to consider, when attacking such a question, are Thompson's groups, in particular F and V. It is well known, that both groups are of type F_∞ and there are quite a few generalizations of Thompson's groups in the literature. The question to consider here is whether these generalizations inherit the property of being of type F_∞.

In this thesis we will give an introduction to the classical Thompson's groups F and V and discuss generalizations of them. In particular we will study the higher-dimensional Brin-Thompson groups sV for $s \in \mathbb{N}$ and the braided Thompson's groups V_{br} and F_{br}. We will prove that both generalizations inherit the property of being of type F_∞.

The proof of the Main Theorem requires the analysis of certain simplicial complexes. One family of complexes that we need to consider are generalizations of matching complexes of a graph to arcs on surfaces, that we introduce in this thesis. We will also give bounds on their connectivity properties for certain underlying graphs.

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Introduction

Since their introduction in 1965 Richard Thompson's groups F, T and V have enticed a lot of research. For example T and V were among the first examples of groups known to be infinite, finitely presented and simple. The group F is infinite, torsion-free and of infinite geometric dimension. The later meaning, that a classifying space for F, i.e. a CW-complex with fundamental group F and trivial higher homotopy groups, has to have cells in arbitrary dimension. But surprisingly there is such a classifying space having only finitely many cells in each dimension. In the language of topological finiteness properties, as introduced by Wall ([Wal65, Wal66]), the group F is of type F_∞ but not of type F. Here the fact that F is torsion-free is important as it is easy to give examples of groups of infinite geometric dimension, for example every non-trivial finite group has this property. Due to this result finiteness properties of Thompson's and Thompson-like groups have been of interest.

The fact that the classical Thompson's groups F, T and V are finitely presented, equivalently are of type F_2, was already shown by Thompson himself. By now it is also well-known that all three of them are of type F_∞, by work of Brown. We will reprove this for F and V in this thesis.

Over the years quite a few generalizations of the classical Thompson's groups appeared in the literature. The first were the so-called Higman-Thompson groups $G_{n,r}$ defined by Higman in [Hig74], allowing for n-ary splits and r roots in the tree-model for V. Here we have $G_{2,1} = V$. These groups were later generalized by Brown ([Bro87]) to a family of groups $F_{n,r} \leq T_{n,r} \leq G_{n,r}$. He also showed them to be of type F_∞. In this thesis, we are mainly concerned with two other generalizations of the classical Thompson's groups.

First we will be concerned with the *braided Thompson's groups*. In [Bri07] and [Deh06], Brin and Dehornoy independently introduced a braided version of V, which we will denote V_{br}. This group contains a copy of F as well as an copy of the braid group B_n for every n. It was shown to be finitely presented by Brin ([Bri06]). Later Brady, Burillo, Cleary and Stein ([BBCS08]) introduced another braided Thompson group, which we denote F_{br}. It contains copies of the pure braid groups PB_n in a similar way to how V_{br} contains B_n. They also proved that F_{br} is finitely presented.

The second kind of generalized Thompson's groups that we will consider in detail are the *higher dimensional Thompson groups* or *Brin-Thompson groups* sV, for $s \in \mathbb{N}$. They were introduced by Brin ([Bri04]) and shown to be finitely presented by Hennig and Matucci ([HM12]). They are higher dimensional analogues of V, thought of as a group of homeomorphisms of the Cantor set, and in the case $s = 1$ we have $1V = V$. For $s = 2, 3$, Kochloukova, Martínez-Pérez and Nucinkis ([KMPN13]) proved that sV is of type F_∞.

We will show:

Main Theorem. *The braided Thompson's groups V_{br} and F_{br}, as well as the Brin-Thompson groups sV, for $s \in \mathbb{N}$, are of type* F_∞.

The proof is geometric and based on the articles [BFM+14, FMWZ13]. The starting point is the key observation that each Thompson's group that we consider acts naturally on a complex associated to a poset, and that there are in each case invariant subcomplexes which we term "Stein spaces", that are easier to understand locally. This was first done by Stein in the case of Thompson's group F, cf. [Ste92]. The Stein spaces will prove to be sufficiently nice to use "Brown's Criterion" ([Bro87]), the standard tool in determining the finiteness properties of a group. The simplicial complexes that will arise in the necessary analysis of the local structures of these complexes, are closely related to "matching complexes" of graphs in the classical cases and the case of sV, respectively of arcs on surfaces in the braided case. The later complexes consist of arc systems on a surface in which the nodes of a graph are embedded such that the arc systems yield a matching of the graph. They might be of independent interest and we will give bounds on their connectivity and show them to be highly connected.

The Main Theorem can not only be understood as continuing the program to determine finiteness properties of Thompson-like groups and therefore increasing our understanding of these structures, but also as part of a general attempt to understand the property of being of type F_n. As it seems mysterious what we learn about a group if we know it is of type F_{12} but not of type F_{34}, it is interesting to know how the finiteness properties of a group change, if at all, if the group changes. The generalizations of the classical Thompson's groups that we consider here, are closely related to their classical relatives, so the Main Theorem shows that, at least for Thompson's groups, the process of "braiding" the group or "raising" the dimension does not change the finiteness properties. Following this train of thought, we should mention that, as V_{br} can be thought of as "sticking" braid groups in a Thompson-like structure, Witzel and Zaremsky identified further examples of groups for which this is possible in [WZ14]. These groups exhibit what they call a "cloning system", and they also determine the finiteness length of such Thompson-like groups, continuing the program of analyzing how finiteness properties may change when the group is changed. Our Main Theorem for the groups V_{br} and F_{br} can also be deduced from their work once the local properties of the relevant spaces are understood. We also refer to work of Thumann ([Thu14]), who uses "Operad groups" in order to unify a lot of the proofs for Thompson-like groups being of type F_∞ in the literature.

This thesis is organized as follows. In Section 1 the notion of finiteness properties of a group is introduced, as well as some technical facts, that are used to determine such properties. In Section 2, we define matching complexes of graphs and on surfaces and calculate their connectivity properties. These complexes are used in the proof of the Main Theorem but are also of independent interest. Thompson's groups and their generalizations are discussed in Section 3 and Section 4 is concerned with the classical results on the finiteness properties of the groups F and V. These proofs are included here in order to introduce the ideas used in the proof of the Main Theorem, that will finally be carried out in Section 5 for the braided Thompson's groups, and for the Brin-Thompson groups in Section 6.

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Statement on my contributions to the main results

As the original results of this thesis are already contained in the articles *The braided Thompson's groups are of type* F_∞ by Kai-Uwe Bux, Martin Fluch, Stefan Witzel, Matthew Zaremsky and myself, and *The Brin–Thompson groups* sV *are of type* F_∞ by Martin Fluch, Stefan Witzel, Matthew Zaremsky and myself, there is a big overlap with these articles. So it seems appropriate to clarify my own contributions to the results.

In the case of the braided Thompson's groups, one of the main problems is to come up with the right model for descending links arising in the Stein space. This is where “matching complexes on surfaces” had to be introduced and their connectivity properties determined. I observed that fibers of the projection from the surface matching complex to the classical matching complex are not well-behaved and therefore a transfer of connectivity properties from the later complex to the surface matching complex by Quillen-type theorems can not succeed. Moreover I wrote up most of Section 1 on the introduction of the groups and “spraiges” of the published article and overcame most of the technical problems there.

As for the second paper on the Brin–Thompson groups sV, it turned out that classical matching complexes only work as models for subcomplexes of the descending links, namely the “very elementary” part. I came up with notion of “very elementary” used in the proof, which allows the analysis of descending links in the Stein space to be done in two steps. Namely by first using Quillen-type arguments to deduce the connectivity properties of the very elementary part of the descending link from those of a matching complex of the complete graph, and only then deduce the connectivity properties of the full descending links using Morse-theoretic tools. Additionally I observed the importance of considering an oriented version of the graph sK_n as underlying graph for the model of the “very elementary” descending link.

1 Basic Definitions and Properties

In this first section we will formally introduce the notion of finiteness properties of a group, as they are the main object of this thesis. We follow this up by giving an overview of the technical tools needed to prove our Main Theorem in Subsection 1.2 and Subsection 1.3.

Proofs of most facts are only sketched or omitted completely. Where this is the case, we made an afford to give sufficient references.

1.1 Finiteness Properties

In this subsection we will define and collect the basic properties of the (topological) finiteness properties F_n of a group. The properties F_n were introduced by C.T.C. Wall in [Wal65, Wal66]. A good reference for definitions and basic properties is [Geo08]. Our exposition follows closely [Wit14], Section 1.3.

Throughout this section an *n-cell* will be a topological space homeomorphic to the closed unit ball D^n in $\mathbb{R}^n$, considered as a topological space. The *boundary* of an n-cell is the subspace that is identified via the above homeomorphism with the unit sphere $S^{n-1} \subset D^n$. In particular $S^{-1} = \emptyset$.

Recall that a *CW-complex* X is a topological space that is obtained from the empty set by gluing in cells of increasing dimension along their boundary. We refer to [Hat02] for a rigorous definition. The *n-skeleton*, denoted $X^{(n)}$, of X is then the union of all of its cells up to dimension n. More generally a *subcomplex* of X is a union of some cells of X. Furthermore we require that a group G acting on a CW-complex X preserves the cell structure of X, meaning that G acts by cell-permuting homeomorphisms of X. We will call X in this case a *G-CW-complex*.

Recall further that a topological space X is *n-connected* if it is non-empty and $\pi_i(X)$ is trivial for $0 \leq i \leq n$. In particular (-1)-connected just requires X to be non-empty. We say that X is *n-aspherical* if it satisfies the same conditions, except possibly for $i = 1$. A CW-complex X is said to be *n-spherical* if it is $(n-1)$-connected and n-dimensional. It is *properly n-spherical* if it is n-spherical and not contractible, i.e. not n-connected.

We remark that a non-empty CW-complex is connected as a topological space if and only if it is path-connected, i.e. 0-connected (cf. [Geo08], Proposition 1.2.21). For future reference, recall further that every compact subset of a CW-complex X meets only finitely many cells ([Geo08], Proposition 1.2.6).

Definition 1.1.1. A connected CW-complex X is called a *classifying space* for a group G or a $K(G,1)$-space, if its fundamental group $\pi_1(X)$ is isomorphic to G and all its higher homotopy groups are trivial.

One reformulation of the latter condition is to require that the universal cover $\widetilde{X}$ of X is contractible. It is a well-known fact that classifying spaces exist for every group (see for example the proof of Proposition 1.1.3) and are unique up to homotopy equivalence (cf. [Geo08], Corollary 7.1.7). Clearly the universal cover $\widetilde{X}$ of each classifying space X of a group G is in particular a G-CW-complex, since we can identify G with the fundamental group of X.

Even though $K(G,1)$-spaces exist for any group G, there does not need to be one with nice properties. One property one might want classifying spaces to have is encoded in the definition of the topological finiteness properties that we will study.

Definition 1.1.2. A group G is of *type* F_n if there exists a $K(G,1)$-space with finite n-skeleton (i.e. finitely many cells in dimensions up to n), or equivalently such that $X^{(n)}$ is compact. If G is of type F_n for all $n \in \mathbb{N}$, it is of *type* F_∞. If there exists a finite $K(G,1)$-space, then G is of *type* F.

The following Proposition might motivate the interest in finiteness properties.

Proposition 1.1.3. *Every group G is of type* F_0*. G is finitely generated if and only if it is of type* F_1*, and it is finitely presented if and only if it is of type* F_2*.*

Proof. For any group G we can build a $K(G,1)$-space in the following way. Let $G = \langle S \mid R \rangle$ be a presentation of G. Start with a single 0-cell and attach to it a 1-cell for each generator $s \in S$ of G. At this point pick an orientation for each 1-cell and glue in a 2-cell for each element $r \in R$ along its boundary in the way prescribed by r, which is a word in $S \cup S^{-1}$. By now the fundamental group of the space is G. The space we just built is sometimes called the *presentation complex* or *Cayley-2-complex* (cf. [Geo08], Example 1.2.17). Finally kill all the higher homotopy groups by gluing in cells from dimension 3 on. Note that this does not change the fundamental group, since it only depends on the 2-skeleton. In any case we have built a $K(G,1)$-space for G, that has a single 0-cell, whence the first assertion. It has finite 1-skeleton if S was finite and finite 2-skeleton if R was finite.

Conversely suppose we are given a $K(G,1)$-space. Consider its 1-skeleton which is a graph. Hence we can find a spanning tree T and collapsing that to a point is a homotopy equivalence ([Spa66], Corollary 3.2.5). So we obtain a $K(G,1)$-space with a single 0-cell, this shows the first assertion. The 2-skeleton of the resulting space also serves as a presentation complex for G (after choosing an orientation on the 1-cells) similarly to the above construction. We get one generator of G for each 1-cell, and one relation for each 2-cell by reading of the word in the generators given by the boundary of the 2-cell. As G is the fundamental group of the space under consideration this indeed yields a presentation for G. Clearly if the 1-skeleton was finite to begin with, then we obtain a finite set of generators. If the 2-skeleton was finite, G is finitely presented. □

Example 1.1.4.

i) The free group on n generators is of type F. It is the fundamental group of a wedge of n circles, which is a classifying space since it is 1-dimensional.

ii) Clearly every group of type F is of type F_∞.

iii) Every non-trivial finite group is of type F_∞ but not of type F. The last fact is due to non-trivial finite groups having torsion elements. (See [Geo08], Corollary 7.2.5, Proposition 7.2.12).

iv) Having torsion is not the only reason for a group of type F_∞ to be not of type F. The most important example here is Thompson's group F. It is torsion-free (as stated in Subsection 3.1), of type F_∞ (Subsection 4.1) but not of type F ([Geo08], Proposition 9.2.6).

v) For every n, there are groups that are of type F_{n-1} but not of type F_n.

One of the first families of examples is due to Bieri ([Bie76]). Let L^n denote the direct product of n free groups on two generators and K_n be the kernel of the map $L^n \to \mathbb{Z}$ that maps each generator to 1. Bieri showed that K_n is of type F_{n-1} but not of type F_n.

Here is one important feature of the properties F_n:

Lemma 1.1.5 ([Geo08], Corollary 7.2.4)**.** *For every n, if G is a group and H is a subgroup of finite index, then H is of type F_n if and only if G is of type F_n.* □

The definition of the properties F_n that we have given is not easy to work with. Here are some equivalent reformulations that are a little more applicable (a proof can be found in [Wit14], Lemma 1.17):

Lemma 1.1.6. *Let G be a group and $n \geq 2$. The following are equivalent:*

i) G is of type F_n.

ii) There is a contractible free G-CW-complex X_2 that has finite n-skeleton modulo the action of G.

iii) There is a finite, $(n-1)$-aspherical CW-complex X_3 with fundamental group G.

iv) There is a $(n-1)$-connected, free G-CW-complex X_4 that is finite modulo the action of G.

□

Now it is often the case that given a group G that one is interested in, one knows the "right" space to act on, i.e. one has a contractible G-CW-complex X. But the canonical action of G on X is seldom free. It is also not clear from the definitions how to show that a given group is not of type F_n. Since these are well known problems, there is a standard tool to deal with them, namely a criterion given by Ken Brown. We will state Brown's Criterion first in full generality and then in a special case, that we will use later on. But first we need additional notation (cf. [Bro87]).

Let X be a G-CW-complex. By a *G-invariant filtration* $(X_\alpha)_{\alpha\in I}$ of X, where I is some directed set, we mean a family of G-invariant subsets of X, such that $X = \bigcup X_\alpha$ and $X_\alpha \subseteq X_\beta$ whenever $\alpha \leq \beta$.

A *directed system* of groups is a family of groups $(G_\alpha)_{\alpha\in I}$, again I some directed set, together with morphisms $f_\alpha^\beta\colon G_\alpha \to G_\beta$ whenever $\alpha \leq \beta$, such that $f_\beta^\gamma \circ f_\alpha^\beta = f_\alpha^\gamma$ whenever $\alpha \leq \beta \leq \gamma$. Such a directed system is called *essentially trivial* if for every α there exists a $\beta \geq \alpha$ such that the morphism f_α^β is trivial.

Since the homotopy groups π_i are functorial, we see that each G-invariant filtration (X_α) induces a directed system of homotopy groups $(\pi_i(X_\alpha))$. We will now state Brown's Criterion:

Proposition 1.1.7 ([Bro87], Theorem 2.2, Theorem 3.2)**.** *Let G be a group that acts on an $(n-1)$-connected CW-complex X. For $0 \leq k \leq n$, suppose that the stabilizer of every k-cell is of type F_{n-k}. Let $(X_\alpha)_{\alpha \in I}$ be a filtration in G-invariant subcomplexes that are compact modulo the action of G. Then G is of type F_n if and only if the directed system $(\pi_i(X_\alpha))_{\alpha \in I}$ is essentially trivial for $0 \leq i \leq n$.* □

Brown's original proof is algebraic in nature and uses a relation of the topological finiteness properties F_n, introduced here, to the homological finiteness properties FP_n, which we will not define here. A topological proof, by inductively building up a $K(G,1)$-space within the homotopy type of X, is sketched in [Geo08].

Since we are mainly concerned with the property F_∞, we will use the following weaker version of Brown's Criterion.

Proposition 1.1.8 ([Bro87], Corollary 3.3)**.** *Let X be a contractible G-CW-complex and suppose that all cell stabilizers are of type F_∞. Let $(X_j)_{j \geq 1}$ be a filtration in G-invariant subcomplexes that are compact modulo the action of G. Suppose that the connectivity of the pair (X_{j+1}, X_j) tends to ∞ as j tends to ∞. Then G is of type F_∞.* □

Recall that a CW-pair (X, A) is n-connected if the inclusion $A \to X$ induces an isomorphism in π_i for $i < n$ and an epimorphism in π_n.

1.2 Discrete Morse Theory

As we have seen at the end of the last subsection, the property F_∞ is closely related to the connectivity properties of CW-pairs (X_{j+1}, X_j) where $X_j \subseteq X_{j+1}$. A standard tool in determining such connectivity properties is a discrete version of Morse theory, as introduced by Bestvina and Brady in [BB97]. We collect the main notations and results that we will use in this subsection.

Definition 1.2.1. Let Y be a piecewise Euclidean complex. A function

$$f\colon \mathrm{vt}(Y) \to \mathbb{R},$$

where $\mathrm{vt}(Y)$ denotes the set of 0-cells of Y, is called a *Morse function* if

(1) Each cell has a unique vertex of maximal f-value

(2) The image of f is discrete in $\mathbb{R}$.

We often call $f(y)$ the *height* of the vertex y.

If we are dealing with simplicial complexes, condition (1) of the definition amounts to saying that no two adjacent vertices have the same height. As a second remark, it is not unusual that a Morse function f has range not $\mathbb{R}$, but some $\mathbb{R}^n$, where the tuples are ordered lexicographically. Indeed, we will do this several times. But if the image of f in the first component is discrete and finite in all the others, this is not a problem since we actually just need the image of f to be order-equivalent to $\mathbb{Z}$ (cf. [Wit14], Section 1.8).

For $t \in \mathbb{Z}$ let $Y^{\leq t}$ be the full subcomplex of Y spanned by vertices of height at most t. Similarly define $Y^{<t}$ and let $Y^{=t}$ be the set of vertices at height t. This gives rise to a filtration $(Y^{\leq t})_{t \in \mathbb{Z}}$ of Y. For any vertex y, the *descending star* $\mathrm{st}{\downarrow}_f(y)$, with respect to f, is defined to be the subcomplex of cells σ, that contain y as their vertex of maximal height. The *descending link* $\mathrm{lk}{\downarrow}_f(y)$ then is the set of "local directions" at y pointing into $\mathrm{st}{\downarrow}_f(y)$. More details can be found in [BB97]. The following is a consequence of Corollary 5 of [BB97].

Lemma 1.2.2. *Let f be a Morse function on Y. Then the following holds:*

i) For each vertex y with $f(y) = t$ suppose that $\mathrm{lk}{\downarrow}(y)$ is $(k-1)$-connected, then the pair $(Y^{\leq t}, Y^{<t})$ is k-connected.

ii) For any vertex y with $f(y) \geq t$ suppose that $\mathrm{lk}{\downarrow}(y)$ is $(k-1)$-connected, then the pair $(Y, Y^{<t})$ is k-connected.

□

Comparing the first statement of the Lemma with our specialized version of Brown's Criterion (Proposition 1.1.8) immediately shows its value. We will use it throughout this thesis.

The second statement will be useful to determine the connectivity properties of certain models for our descending links later. We use it for example in Subsection 2.1 on matching complexes of graphs, to get an upper bound for the connectivity of the complexes. There it is used in the following way. If we can build up from a subspace X to a contractible space Y by gluing in vertices along, say, n-connected links, the Morse Lemma says that we never change the i^{th} homotopy group for $i \leq n$. And hence the space $X = Y^{<t}$, for some t, is at least n-connected, since for Y all homotopy groups are trivial as it is contractible.

1.3 Posets

In this subsection we collect terminology and results from the theory of partially ordered sets that we will need. See for the basics on posets [Tro95] and [Koz08] for the geometric realization.

Recall that a *partially ordered set* or *poset* is a tuple $(\mathcal{P}, \leq)$ consisting of a set $\mathcal{P}$ and a binary relation $\leq$, that is reflexive, antisymmetric and transitive, called a *partial order*. We will usually drop the binary relation in the notation, if it is clear from the context and only speak of the poset $\mathcal{P}$. If $x \leq y$ or $y \leq x$ we call x and y *comparable*. Otherwise x, y are *incomparable*. If $x \leq y$ are distinct we will write $x < y$. A *chain* in $\mathcal{P}$ is a subset $S \subseteq P$ that is totally ordered, i.e. each pair of distinct elements is comparable.

An element $x \in \mathcal{P}$ is called *minimal (resp. maximal)* if there is no element $y \in \mathcal{P}$ such that $y < x$ (resp. $y > x$). If $x \leq y$ for all $y \in \mathcal{P}$, we call it the *minimum* and denote it by $\mathbf{0}$. Analogously we have the *maximum* $\mathbf{1}$. Clearly the minimum, if it exists, is unique and the unique minimal element, but not any minimal element is the minimum. The poset $\mathcal{P}$ is *bounded* if it has both a minimum and a maximum.

For $x, y, z \in \mathcal{P}$, if $x \leq z$ and $y \leq z$, then z is an *upper bound* for x and y. The poset $\mathcal{P}$ is *directed*, if any pair of distinct elements has an upper bound. If the set of upper bounds of x, y (with the induced order) has a unique minimum z, we call z the *least upper bound* or *join* and denote $x \vee y := z$. Dually we have *lower bounds* and the *greatest lower bound* or *meet* $x \wedge y$. A poset $\mathcal{P}$ is a *lattice* if for any two distinct elements their join and meet exist. Clearly every lattice is a directed poset.

Observation 1.3.1. *If a lattice $\mathcal{P}$ has a minimal (maximal) element, then it is unique.* □

Here is the first Lemma we will need:

Lemma 1.3.2. *Let $\mathcal{P}$ be a graded poset with unique minimum* $\mathbf{0}$ *such that for any two elements of $\mathcal{P}$ their join exists. Then $\mathcal{P}$ is a lattice.*

Proof. We only need to show the existence of meets, i.e. greatest lower bounds. Let $x, y \in \mathcal{P}$. The minimum $\mathbf{0}$ clearly is a lower bound of x and y. Now suppose towards a contradiction, that z and z' are both maximal lower bounds (these exist as $\mathcal{P}$ is graded). Then x as well as y are common upper bounds for z and z'. So by definition their join $z \vee z'$, which exists, is a common lower bound of x and y. This contradicts the maximality of z and z'. □

We now turn to the geometric side of things. Recall that any poset $\mathcal{P}$ determines an abstract simplicial complex, consisting of a vertex for each element of $\mathcal{P}$ and a k-simplex for each chain $x_0 < x_1 < \cdots < x_k$ in $\mathcal{P}$, the face relation is given by inclusion of chains. By abuse of notation we will denote the simplicial complex also by $\mathcal{P}$. The *geometric realization* $|\mathcal{P}|$ is the CW-complex obtained by gluing together standard k-simplices in $\mathbb{R}^k$ for each k-simplex in $\mathcal{P}$, along subsimplices corresponding to the faces, i.e. subchains. Further denote by $\mathcal{P}_c$ the poset of chains in $\mathcal{P}$, where the order relation is given by inclusion and observe that $|\mathcal{P}_c|$ is the barycentric subdivision of $|\mathcal{P}|$. Hence we can identify simplices in $|\mathcal{P}|$ with elements of $\mathcal{P}_c$.

We need two lemmas concerning contractability of geometric realizations:

Lemma 1.3.3. *Suppose the poset $\mathcal{P}$ is directed. Then the geometric realization $|\mathcal{P}|$ is contractible.*

Proof. Let $S^k \to |\mathcal{P}|$ be a continuous map. Since S^k is compact, its image in $|\mathcal{P}|$ meets only finitely many cells. Call the finite subcomplex, that supports the image, K. Each cell of K corresponds to a finite chain in $\mathcal{P}$. As $\mathcal{P}$ is directed, there is a common upper bound $v \in \mathcal{P}$ for all the vertices of these chains. So the cone $v * K$ exists in $|\mathcal{P}|$. Hence we can collapse the image of S^k to a point and see that, for any given k, the homotopy group $\pi_k(|\mathcal{P}|)$ is trivial. By the Whitehead Theorem ([Hat02], Theorem 4.5) we conclude that $|\mathcal{P}|$ is contractible. □

The second Lemma is basically the statement of Section 1.5 of [Qui78].

Lemma 1.3.4. *Let $\mathcal{P}$ be a poset and $f\colon \mathcal{P} \to \mathcal{P}$ be a poset map, i.e. it respects the order. Suppose there exists a $x_0 \in \mathcal{P}$, such that we have $x \geq f(x) \leq x_0$, for all $x \in \mathcal{P}$. Then $|\mathcal{P}|$ is contractible.*

Proof. Note that the map $|f|$ induced by f is simplicial. By the first inequality each map $S^k \to |\mathcal{P}|$ is homotopic to a map $S^k \to |f(\mathcal{P})|$. But by the second inequality, the subposet $f(\mathcal{P}) \cup \{x_0\}$ is directed. Hence the claim follows by the previous Lemma. □

Another result of [Qui78] about connectivity properties of posets (more precisely their geometric realizations) that we will frequently refer to is Theorem 9.1. For easier reference we will restate it here using our notations.

For ease of notation we will say that a poset $\mathcal{P}$ has a topological property, if its geometric realization has that property. Also recall that we can identify the link $\mathrm{lk}(\sigma)$ of a simplex σ in a simplicial complex with the poset of cofaces of σ, so $\mathrm{lk}(\sigma) = \{\tau \in \mathcal{P}_c \mid \sigma < \tau\}$ for any simplex σ in $|\mathcal{P}|$. Further $|\mathcal{P}|$ and $|\mathcal{P}_c|$ are homotopy equivalent, as $|\mathcal{P}_c|$ is the barycentric subdivision of $|\mathcal{P}|$. With this notations Quillens Theorem states the following:

Proposition 1.3.5 ([Qui78], Theorem 9.1)**.** *Let $f : \mathcal{P} \to \mathcal{P}'$ be a map of posets. Suppose that $\mathcal{P}'$ is $(n-1)$-connected. If for each $\sigma \in \mathcal{P}'_c$ the link $\mathrm{lk}(\sigma)$ is $(n-k(\sigma)-2)$-connected and the fiber $f^{-1}(\sigma)$ is $(k(\sigma)-1)$-connected, then $\mathcal{P}$ is $(n-1)$-connected.* □

2 Matching Complexes

If we want to apply Brown's Criterion (Proposition 1.1.8) and the Morse theory as discussed in Subsection 1.2 in order to determine the finiteness properties of a group, we need to calculate the connectivity properties of descending links, i.e. of certain simplicial complexes. In the case of Thompson's groups, these descending links will be closely related to so called *matching complexes*. We introduce these complexes in this section and determine their connectivity properties.

In Subsection 2.1, we first introduce the well-known notion of a *matching complex of a graph*, and in Subsection 2.2 we generalize this notion to what we call a *matching complex on a surface*, that we will need when dealing with the braided Thompson's groups in Section 5.

We remark that the idea of "defect" introduced in Subsection 2.1 and the proof of Proposition 2.1.3 are given as in [BFM$^+$14]. Also the content of Subsection 2.2 is primarily the same as Section 3 in that article.

The content of this section might be of interest in its own right.

2.1 Matching Complexes of Graphs

Recall that a *graph* Γ is given as a collection of *nodes (or vertices)* $V(\Gamma)$ together with a set of *edges* $E(\Gamma)$ and a function $\mathtt{Ends}$ that assigns each edge $e \in E(\Gamma)$ an unordered pair $\{v, w\}$ of nodes of Γ. The nodes v, w are then called the *ends* of e. Note that we allow for *loops*, i.e. edges that connect a node to itself, $\mathtt{Ends}(e) = \{v, v\}$, and *multiple edges* between two given nodes, i.e. for $e \neq e'$ their ends may coincide. A graph Γ without loops and multiple edges will be called *simple*.

An edge e of a graph Γ is *oriented*, if we have the additional data that one of its ends, say v, is the *initial node* of e and e then points from v to w. Γ is *oriented* if all edges of Γ have an orientation. Aside from an orientation on the edges, a graph Γ can be equipped with different additional data. For example a *labeling* of the nodes (edges) is a function ℓ_V (ℓ_E) from the set of nodes (edges) to some set of labels $\mathcal{L}$.

Two families of graphs will appear frequently in the rest of this thesis. For $n \in \mathbb{N}$ we will denote by K_n the *complete graph on n nodes*. That is the graph with n distinct nodes, labeled 1 to n, and exactly one edge between each pair of distinct nodes. The *linear graph on n edges*, that is the graph with $n+1$ nodes, labeled v_0 to v_n, and exactly one edge e_i with $\mathtt{Ends}(e_i) = \{v_{i-1}, v_i\}$ for $1 \leq i \leq n$, will be denoted with L_n. See Figure 2.1. Note that when dealing with K_n, n denotes the number of nodes, but considering L_n, n denotes the number of edges. This is to ease future notation when we are dealing with the braided Thompson's groups in Section 5.

For given graphs Γ and Γ', we say that Γ' is a *subgraph* of Γ if $V(\Gamma') = V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$ such that $\mathtt{Ends}_\Gamma |_{E(\Gamma')} = \mathtt{Ends}_{\Gamma'}$. If Γ' is a subgraph of Γ we write $\Gamma' \leq \Gamma$ and if additionally $\Gamma' \neq \Gamma$ we call Γ' a *proper* subgraph.

Now consider the following complex $\mathcal{H}(\Gamma)$ for a given graph Γ. $\mathcal{H}(\Gamma)$ consists of a k-simplex for any subgraph of Γ with $(k+1)$-edges, the face relation is given by inclusion. Hence the 0-skeleton of $\mathcal{H}(\Gamma)$ consists of one vertex for each subgraph of Γ with exactly one edge. $\mathcal{H}(\Gamma)$ has a 1-simplex for each subgraph with 2 edges and so on. In particular there is exactly one ℓ-simplex, corresponding to Γ itself, if Γ

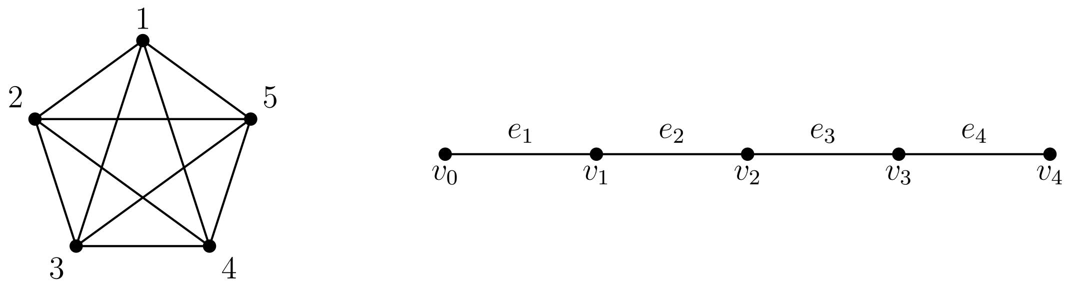

FIGURE 2.1: The graphs K_5 and L_4.

has $\ell + 1$ edges. Call $\mathcal{H}(\Gamma)$ the *subgraph space of* Γ. We obtain

Observation 2.1.1. *Let* Γ *be a graph and* $\ell+1$ *be the number of edges of* Γ*. Its subgraph space is a* ℓ*-simplex and hence contractible. The complex of proper subgraphs is homotopy equivalent to an* $(\ell-1)$*-sphere and therefore* $(\ell-2)$*-connected.*

Proof. If we discard the graph Γ itself and pass to the complex of proper subgraphs, we remove the barycenter of the ℓ-simplex. So the space of proper subgraphs of Γ is homotopy equivalent to the boundary of the ℓ-simplex. $\square$

We now introduce the complexes that we will use to model descending links in the following sections.

Definition 2.1.2. The *matching complex* $\mathcal{M}(\Gamma)$ of a graph Γ is the simplicial complex consisting of a k-simplex for each collection $\{e_0, \ldots, e_k\}$ of $k+1$ pairwise disjoint edges of Γ. The face relation is given by inclusion.

We remark here that each matching of Γ, i.e. each collection of pairwise disjoint edges, can be thought of as the subgraph of Γ consisting of these edges. Hence $\mathcal{M}(\Gamma)$ can be viewed as a subcomplex of the subgraphspace $\mathcal{H}(\Gamma)$, which is contractible. We will use this to analyze the connectivity properties of the matching complexes $\mathcal{M}(K_n)$.

The Matching Complexes for K_n and L_n

For $\ell \in \mathbb{N}$ let $\nu(\ell) := \lfloor \frac{\ell-2}{3} \rfloor$. We will first show that $\mathcal{M}(K_n)$ is $(\nu(n)-1)$-connected. By the discussions above, $\mathcal{M}(K_n)$ is embedded in the contractible space $\mathcal{H}(K_n)$. Consider a simplex Γ in $\mathcal{H}(K_n)$, i.e. a subgraph of K_n. We denote by $e(\Gamma)$ the number of edges of Γ and by $r(\Gamma)$ the number of non-isolated nodes of Γ. The *defect* of Γ will be the number $d(\Gamma) = 2e(\Gamma) - r(\Gamma)$. Note that a subgraph Γ of K_n is a matching if and only if $d(\Gamma) = 0$. In other words the defect of a subgraph measures the failure of being a simplex of $\mathcal{M}(K_n)$. Observe that $\mathcal{M}(K_n)$ already contains the 0-skeleton of $\mathcal{H}(K_n)$ and that a proper subgraph $\Gamma' < \Gamma$ can not have a higher defect than Γ.

Now consider the function $h(\Gamma) := (d(\Gamma), -e(\Gamma))$ on the vertex set of the barycentric subdivision $\mathcal{H}'(K_n)$ of $\mathcal{H}(K_n)$. We consider the values of h ordered lexicographically. Note that adjacent vertices of $\mathcal{H}'(K_n)$ have different e-values and hence

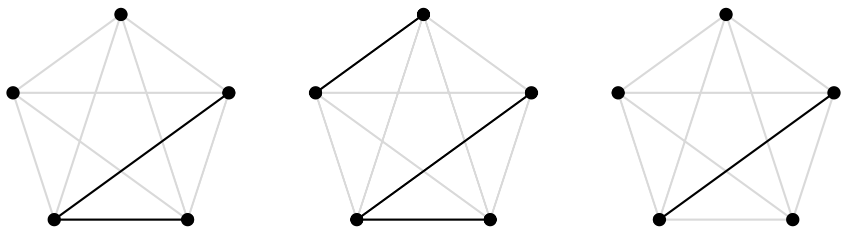

Figure 2.2: Three simplices in $\mathcal{H}(K_5)$. From left to right: a graph Γ with defect 1, a graph in the uplink of Γ and a graph in the downlink of Γ.

different h-values. So h is a height function in the sense of Subsection 1.2 and we adopt the appropriate notations there.

Fixing a vertex Γ in $\mathcal{H}'(K_n)$, we denote by $\mathrm{lk}{\downarrow}(\Gamma)$ its descending link with respect to h. There are two types of vertices in $\mathrm{lk}{\downarrow}(\Gamma)$. On the one hand there are graphs $\widetilde{\Gamma} > \Gamma$ with $h(\widetilde{\Gamma}) < h(\Gamma)$. This implies that $d(\widetilde{\Gamma}) = d(\Gamma)$. On the other hand we have graphs $\Gamma' < \Gamma$ and $h(\Gamma') < h(\Gamma)$. This is equivalent to $d(\Gamma') < d(\Gamma)$. We define the *uplink* (respectively the *downlink*) of Γ to be the full subcomplex of $\mathrm{lk}{\downarrow}(\Gamma)$ spanned by vertices of the first type (respectively second type). Any vertex of the downlink is a subgraph of any vertex in the uplink and hence $\mathrm{lk}{\downarrow}(\Gamma)$ is the join of the uplink and the downlink. Confer Figure 2.2 for an idea of defect, uplink and downlink.

We are now in a position to prove

Proposition 2.1.3. *The matching complex $\mathcal{M}(K_n)$ of the complete graph K_n is $(\nu(n) - 1)$-connected.*

Proof. As a base case note that $\mathcal{M}(K_n)$ is non-empty, hence (-1)-connected, for $n \geq 2$. Suppose that $n \geq 5$. By the fact that $\mathcal{H}(K_n)$ is contractible (Observation 2.1.1) and that each vertex of $\mathcal{H}(K_n)$ is already contained in $\mathcal{M}(K_n)$, we can build up from $\mathcal{M}(K_n)$ to $\mathcal{H}(K_n)$ by gluing in simplices in increasing h-order along their descending links. By the second part of the Morse Lemma 1.2.2, it suffices to prove that for any Γ with $e(\Gamma) \geq 2$ and $d(\Gamma) \geq 1$, the descending link $\mathrm{lk}{\downarrow}(\Gamma)$ is $(\nu(n) - 1)$-connected to conclude the proof.

First consider the downlink. A subgraph $\Gamma' < \Gamma$ fails to be in the downlink if and only if it has the same defect as Γ. This amounts to saying that each edge in $\Gamma \setminus \Gamma'$ is disjoint from every other edge in Γ. Denote by Γ_0 the unique subgraph of Γ consisting of precisely all such edges, if any exist. By Observation 2.1.1 the space of all proper subgraphs of Γ is a $(e(\Gamma) - 2)$-sphere. The complement of the downlink in this space is either empty or contractible with cone point Γ_0. Hence the downlink is either $(e(\Gamma) - 3)$-connected or contractible.

Now for the uplink. It consists of graphs $\widetilde{\Gamma}$ that are obtained from Γ by adding edges that are disjoint from all other edges of Γ and each other, since then and only then $d(\widetilde{\Gamma}) = d(\Gamma)$. So the uplink is again a matching complex of a complete graph, namely $\mathcal{M}(K_{n-r(\Gamma)})$ and by induction is $(\nu(n - r(\Gamma)) - 1)$-connected.

Hence the descending link $\mathrm{lk}{\downarrow}(\Gamma)$ is $(e(\Gamma) + \nu(n - r(\Gamma)) - 2)$-connected. Since we

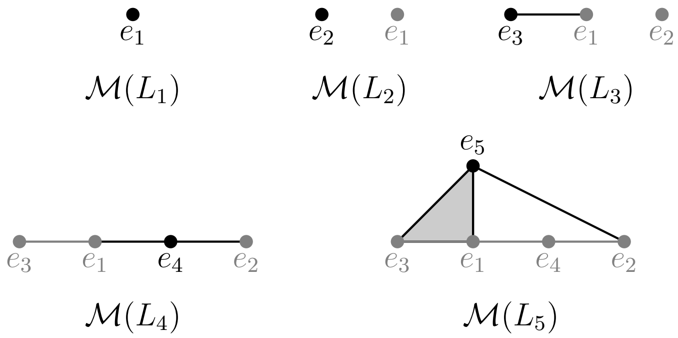

FIGURE 2.3: The matching complexes $\mathcal{M}(L_n)$ for $1 \leq n \leq 5$. The vertices are labeled by the single edge contained in the corresponding matching.

have assumed $d(\Gamma) \geq 1$ and $e(\Gamma) \geq 2$, we get

$$\begin{aligned} e(\Gamma) + \nu(n - r(\Gamma)) - 2 &= \nu(3e(\Gamma) + n - r(\Gamma) - 3) - 1 \\ &= \nu(n + d(\Gamma) + e(\Gamma) - 3) - 1 \\ &\geq \nu(n) - 1 \end{aligned}$$

and this concludes the proof. □

The rough method we just used can be applied to various situations. We will use it primarily in Section 6 and refer back to this simpler case.

We now turn to the family of linear graphs L_n. Recall that in this case n denotes the number of edges of the linear graph. It is readily checked, that the matching complex $\mathcal{M}(L_n)$ is non-empty if $n \geq 1$ and connected if $n \geq 4$. See Figure 2.3. Here we will even give the concrete homotopy type of $\mathcal{M}(L_n)$.

Proposition 2.1.4. *Let $n \geq 1$. Then $\mathcal{M}(L_n)$ is contractible if $n = 3k + 1$, it is homotopy equivalent to a k-sphere if $n = 3k + 2$ and to a $(k - 1)$-sphere if $n = 3k$.*

Proof. As base cases we have that $\mathcal{M}(L_0)$ is empty, $\mathcal{M}(L_1)$ is contractible, $\mathcal{M}(L_2)$ is a 0-sphere, as is $\mathcal{M}(L_3)$.

Now let $n \geq 4$. Clearly $L_{n-3} < L_{n-2} < L_{n-1} < L_n$ and so are the corresponding matching complexes. We describe $\mathcal{M}(L_n)$ in the following way. Consider the subcomplex Z of all matchings, that do not use the edge e_{n-1}. Clearly this contains the complex $\mathcal{M}(L_{n-2})$ and we get an additional simplex in Z for each simplex of $\mathcal{M}(L_{n-2})$ by extending the matching by the edge e_n. So Z is $\mathcal{M}(L_{n-2})$ coned of by the point e_n, hence Z is contractible and since it contains a copy of $\mathcal{M}(L_{n-2})$ it also contains $\mathcal{M}(L_{n-3})$. What we are missing of $\mathcal{M}(L_n)$ are the matchings using the edge e_{n-1}. For this we similarly consider the space Y, that is the cone over $\mathcal{M}(L_{n-3})$ with cone point e_{n-1}. We obtain $\mathcal{M}(L_n)$ now from gluing the contractible spaces Y and Z along their intersection $\mathcal{M}(L_{n-3})$.

If $n = 3k + 1$ then L_{n-3} is contractible by induction, hence so is $\mathcal{M}(L_n)$.

The other two cases follow from the Freudenthal suspension theorem (cf. [Hat02], Theorem 4.23). This gives us that if $\mathcal{M}(L_{n-3})$ is $(\ell - 1)$-connected, which we know by induction, then $\mathcal{M}(L_n)$ is ℓ-connected. The concrete homotopy type follows from induction and dimension arguments. □

We remark that both results, Proposition 2.1.3 and Proposition 2.1.4, are well-known.

A proof of Proposition 2.1.3 for example can be found in [Ath04, BLVv94]. We reproved it here to introduce the methods, which will play an important role later, e.g. Subsection 6.3.

The connectivity properties of matching complexes of linear graphs are, for example, contained in [Koz08], Chapter 11. Kozlov uses a version of discrete Morse theory due to Forman to determine the homotopy types of, so called, *independence complexes* $\mathrm{Ind}(L_n)$ of linear graphs. That is the simplicial complex given by sets of pairwise non-adjacent vertices, the face relation is given by inclusion. It is readily checked that $\mathrm{Ind}(L_n)$ is isomorphic to $\mathcal{M}(L_{n+1})$, as L_n is the adjacency graph of L_{n+1}.

2.2 Matching Complexes on Surfaces

We will now generalize the notion of a matching complex of a graph to arcs on surfaces. Recall from Subsection 2.1 the families of complete graphs K_n and linear graphs L_n. Recall that in the case of K_n, n denotes the number of nodes and in the case of L_n, n denotes the number of edges.

Throughout this subsection, let $\mathcal{S}$ be a closed surface with (possibly empty) boundary $\partial\mathcal{S}$. P denotes a finite set of points in $\mathcal{S} \setminus \partial\mathcal{S}$. The main reference for the spaces we will consider in this subsection is [Hat91]. Contrary to Hatcher we define an *arc* to be a simple path in $\mathcal{S} \setminus \partial\mathcal{S}$ that intersects P precisely at its endpoints, and whose endpoints are distinct. The difference being, that we do not allow "loops", i.e. we do not allow the endpoints of an arc to coincide. Also in [Hat91] the points in P may lie in the boundary of $\mathcal{S}$. We prohibit this also. In Section 5, we will only consider the case where $\mathcal{S}$ is a disc, so this is a good example to keep in mind. But for the proofs in this subsection we need the more general setup.

The Arc Complex

Let $\{\alpha_0, \dots, \alpha_k\}$ be a collection of arcs. If all the α_i are disjoint from each other, except possibly at their endpoints, and no two distinct arcs α_i and α_j are homotopic relative P, we call $\{\alpha_0, \dots, \alpha_k\}$ an *arc system*. Clearly the homotopy classes, relative P, of arc systems form the simplices of a simplical complex, where the face relation is given by passage to subsystems.

Definition 2.2.1. Let Γ be a simple graph with $|P|$ nodes and identify P with the set of nodes of Γ. We call an arc in $\mathcal{S}$ *compatible* with Γ, if its endpoints are connected by an edge in Γ. Let $\mathcal{HA}(\Gamma)$ be the simplicial complex with a k-simplex for each arcsystem $\{\alpha_0, \dots, \alpha_k\}$, such that all arcs α_i are compatible with Γ. We call $\mathcal{HA}(\Gamma)$ the *arc complex* on $(\mathcal{S}, P)$ corresponding to Γ.

We include a technical Lemma that will allow us to use actual arcs, rather than homotopy classes.

Lemma 2.2.2. *Given finitely many homotopy classes of arcs $[\alpha_0], \dots, [\alpha_k]$ there are representatives $\alpha_0, \dots, \alpha_k$ such that $|\alpha_i \cap \alpha_j|$ is minimal among all representatives of*

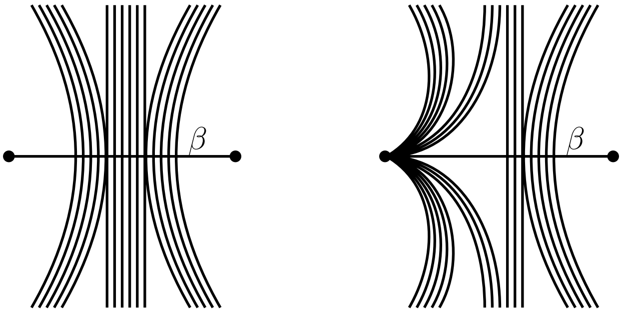

Figure 2.4: The Hatcher flow. Roughly speaking: Pick an arc β. For each simplex not in the star of β, look at the intersections with β and then continuously "push" the intersecting arcs away from β.

$[\alpha_i]$ and $[\alpha_j]$ for $0 \leq i < j \leq k$. In particular, any simplex of $\mathcal{HA}(\Gamma)$ is represented by arcs that are disjoint except possibly at their endpoints.

Proof. If $|P| \leq 2$ there is at most one arc and nothing to show. If $|P| \geq 3$ we consider the points in P as punctures. Then $\mathcal{S}$ has negative Euler characteristic so we may equip it with a hyperbolic metric. The following references are stated for closed curves but also apply to arcs, see [FM12], Section 1.2.7. For each homotopy class $[\alpha_i]$ we take α_i to be the geodesic within the class ([FM12], Proposition 1.3). Then any two of the arcs intersect minimally ([FM12], Corollary 1.9). $\square$

Proposition 2.2.3. *For any $n \geq 2$ the complex $\mathcal{HA}(K_n)$ is contractible.*

The proof here is essentially the same as the proof of the theorem in [Hat91], so we will not be overly precise. Indeed there is only one extra step, which we will point out when it comes.

Proof. Fix an arc β, i.e., a vertex in $\mathcal{HA}(K_n)$. We will retract $\mathcal{HA}(K_n)$ to the star of β. We use the "Hatcher flow" introduced in [Hat91]. Let $\sigma = \{\alpha_0, \dots, \alpha_k\}$ be a simplex in $\mathcal{HA}(K_n)$ and let p be a point in σ, expressed in terms of barycentric coordinates $p = \sum_{i=0}^{k} c_i\alpha_i$, with $c_i \geq 0$ and $\sum_{i=0}^{k} c_i = 1$. Interpret p geometrically by saying that each α_i is thickened to a "band" of thickness c_i. Wherever the bands cross β, pinch them into a single band of thickness θ. Now the Hatcher flow is as follows. At time $t \in [0, 1]$, push p to the point p_t obtained by leaving $(1-t)\theta$ worth of the band in place and pushing the remaining $t\theta$-thick part of the band all the way to one end of β; see Figure 2.4. The additional consideration we have to make is, if at any point we create a new arc whose endpoints coincide, discard this from p_t. This is allowed, since if none of the α_i are loops then there will always exist at least one non-loop arc used in p_t. One checks that this flow is continuous and respects the face relation, and at time $t = 1$ we have deformed $\mathcal{HA}(K_n)$ into the star of β, so we conclude that $\mathcal{HA}(K_n)$ is contractible. $\square$

As a remark, note that the above proof yields contractibility for more general $\mathcal{HA}(\Gamma)$; the only requirement is that there exists a node of Γ that shares an edge with every other node.

We now want to consider a subspace of $\mathcal{HA}(K_n)$ that is related to the matching complex of a complete graph, which we call the matching complex on a surface.

Matching Complexes on Surfaces

Recall from Subsection 2.1, that the matching complex $\mathcal{M}(\Gamma)$ of a graph Γ is given by collections of pairwise disjoint edges. We transfer that notion to the setting of surfaces and arcs:

Definition 2.2.4. Let $\mathcal{MA}(K_n)$ be the subcomplex of $\mathcal{HA}(K_n)$, whose simplices are given by arc systems whose arcs are pairwise disjoint including at their endpoints. For a subgraph Γ of K_n, let $\mathcal{MA}(\Gamma)$ be the preimage of $\mathcal{M}(\Gamma)$ under the map $\mathcal{MA}(K_n) \to \mathcal{M}(K_n)$ that sends an arc with endpoints labeled i and j to the edge of K_n with endpoints i and j. We call $\mathcal{MA}(\Gamma)$ the *matching complex* on $(\mathcal{S}, P)$ corresponding to Γ.

The rest of this section is dedicated to verifying the connectivity properties of the complexes $\mathcal{MA}(K_n)$ and $\mathcal{MA}(L_n)$.

Define for $n \in \mathbb{Z}$ the numbers $\nu(n) := \lfloor \frac{n-2}{3} \rfloor$ and $\eta(n) := \lfloor \frac{n-1}{4} \rfloor$ and note, that both tend to ∞ as n tends to ∞.

We remark here, that one could continue and use the proof of Proposition 2.1.3, the Morse theory used there, the map $\mathcal{MA}(K_n) \to \mathcal{M}(K_n)$ from above, and Proposition 2.2.3, to show that $\mathcal{MA}(K_n)$ is $(\nu(n)-1)$-connected. This was done in detail in [BFM+14], Theorem 3.8. But as was remarked there, this does not readily generalize to a proof for $\mathcal{MA}(L_n)$, hence we will not give the details here. Instead we will focus on the methods from [BFM+14] that can be used to prove connectivity properties of both $\mathcal{MA}(K_n)$ and $\mathcal{MA}(L_n)$. These methods are inspired and based on the proof of Proposition 5.2 in [Put12].

We first need a lemma that allows us to make certain assumptions about maps from spheres to $\mathcal{MA}(\Gamma)$. To state it we need to recall some definitions. By a *combinatorial k-sphere (respectively k-disk)* we mean a simplicial complex that can be subdivided to be isomorphic to a subdivision of the boundary of a $(k+1)$-simplex (respectively to a subdivision of a k-simplex). An m-dimensional *combinatorial manifold* is an m-dimensional simplicial complex in which the link of every simplex σ of dimension k is a combinatorial $(m-k-1)$-sphere. In an m-dimensional *combinatorial manifold with boundary* the link of a k-simplex σ is allowed to be homeomorphic to a combinatorial $(m-k-1)$-disk; its *boundary* consists of all the simplices whose link is indeed a disk.

A simplicial map is called *simplexwise injective* if its restriction to any simplex is injective.

Lemma 2.2.5. *Let Y be a compact m-dimensional combinatorial manifold. Let X be a simplicial complex and assume that the link of every k-simplex in X is $(m-2k-2)$-connected. Let $\psi\colon Y \to X$ be a simplicial map whose restriction to ∂Y is simplexwise injective. Then after possibly subdividing the simplicial structure of Y, ψ is homotopic relative ∂Y to a simplexwise injective map.*

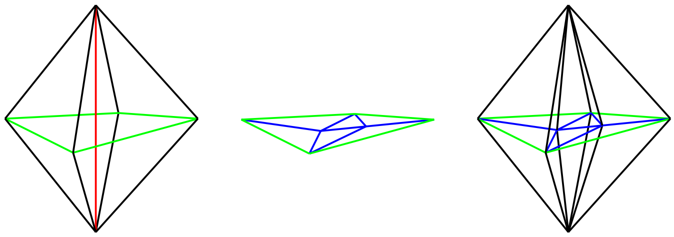

FIGURE 2.5: Illustration of the proof of Lemma 2.2.5. The red edge is the simplex σ, that is, both of its vertices are mapped to the same vertex under ψ. The green circle is the link of σ. The link of $\psi(\sigma)$ is simply connected by assumption, so ψ can be extended to a filling disk B (blue).

Compare the statement of the lemma to the statement of the claim in the proof of Proposition 5.2 in [Put12]. As a remark, the assumption that Y is compact is not necessary, but it makes the end of the proof simpler.

Proof. The proof is by induction on m and the statement is trivial for $m = 0$.

If ψ is not simplexwise injective, there exists a simplex whose vertices do not map to pairwise distinct points. In particular we can choose a simplex $\sigma \subseteq Y$ of maximal dimension $k > 0$ such that for every vertex x of σ there is another vertex y of σ with $\psi(x) = \psi(y)$. By assumption, σ is not contained in ∂Y. Maximality of the dimension of σ implies that the restriction of ψ to the $(m-k-1)$-sphere $\mathrm{lk}_Y(\sigma)$ is simplexwise injective. It also implies that $\psi(\mathrm{lk}_Y(\sigma)) \subseteq \mathrm{lk}_X(\psi(\sigma))$. Note further that $\psi(\sigma)$ has dimension at most $(k-1)/2$. Therefore its link in X is $(m-k-1)$-connected by assumption. Hence there is an $(m-k)$-disk B with $\partial B = \mathrm{lk}_Y(\sigma)$ and a map $\varphi\colon B \to \mathrm{lk}_X(\psi(\sigma))$ such that $\varphi|_{\partial B}$ coincides with $\psi|_{\mathrm{lk}_Y(\sigma)}$. Inductively applying the lemma, we may assume that φ is simplexwise injective.

We now replace Y by Y', the space obtained by replacing the closed star of σ by $B * \partial\sigma$. The map $\psi'\colon Y' \to X$ is the map that coincides with ψ outside the open star of σ, coincides with φ on B and is affine on simplices. It is clearly homotopic to ψ, since the image of B under φ is contained in $\mathrm{lk}_X(\psi(\sigma))$. Since the restriction of ψ' to B is simplexwise injective, the restriction to any k-simplex of $B * \partial\sigma$ is injective. Since Y is compact, by repeating this procedure finitely many times we eventually obtain a map that is simplexwise injective. $\square$

Our general procedure to analyze $\mathcal{MA}(\Gamma)$ for a graph Γ will use Morse theoretic ideas and notions from Subsection 1.2, as well as a variant of the "Hatcher flow" introduced in the proof of Proposition 2.2.3. Here is an overview of the strategy of proof:

Given a graph Γ. Pick an edge e of Γ, say with endpoints v and w. Identify the vertices of Γ with the distinguished points P in the surface $\mathcal{S}$. Note that the 0-skeleton of $\mathcal{MA}(\Gamma)$ consists of arc systems with just one arc. Define a map

$$q\colon \mathcal{MA}(\Gamma)^{(0)} \to \{0,1,2,3\}$$

by sending an arc α to 0 if it has neither v nor w as an endpoint, to 1 if it has v but not w, to 2 if it has w but not v, and to 3 if it has both. As two arcs are adjacent if

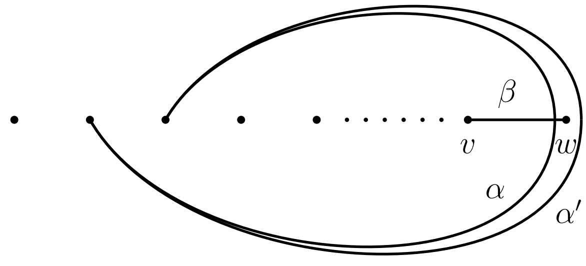

FIGURE 2.6: Pushing the arc α off of β to obtain the arc α'.

they are disjoint even on endpoints, adjacent arcs have different q-value. So q will serve as a Morse function. Observe that for any arc α, say with endpoints v_1 and v_2, its link $\mathrm{lk}(\alpha)$ in $\mathcal{MA}(\Gamma)$ consists of all arc systems $\{\alpha_0, \ldots, \alpha_k\}$ such that the α_i are all disjoint from α, even on endpoints, since then and only then $\{\alpha_0, \ldots, \alpha_k, \alpha\}$ constitutes an simplex in $\mathcal{MA}(\Gamma)$. Hence $\mathrm{lk}(\alpha)$ in $\mathcal{MA}(\Gamma)$ is isomorphic to $\mathcal{MA}(\Gamma')$, where Γ' is the graph obtained from Γ by removing the stars of v_1 and v_2. Note that the surface on which $\mathcal{MA}(\Gamma')$ is considered, is not $\mathcal{S}$, but rather $\mathcal{S}$ with a new boundary component obtained by "slicing" $\mathcal{S}$ along α. As Γ' has fewer vertices and edges, the complexes $\mathcal{MA}(\Gamma')$ will be highly connected by induction. Hence the idea is to build up from $\mathcal{MA}(\Gamma)^{q=0}$ to $\mathcal{MA}(\Gamma)$ by gluing in vertices along their relative links in increasing q-order. By the second part of the Morse Lemma 1.2.2, it follows that the pair $(\mathcal{MA}(\Gamma), \mathcal{MA}(\Gamma)^{q=0})$ is highly connected. But even though $\mathcal{MA}(\Gamma)^{q=0}$ is highly connected by induction, it is typically not as highly connected as we want it to be. So we need another argument. We want to prove that the inclusion $\iota\colon \mathcal{MA}(\Gamma)^{q=0} \to \mathcal{MA}(\Gamma)$ induces the trivial map in π_k up to the desired connectivity bound for $\mathcal{MA}(\Gamma)$. We will do this the following way. Fix an arc β with endpoints v and w, and let $\overline{\psi}\colon S^m \to \mathcal{MA}(\Gamma)^{q=0}$ be a simplicial map, where S^m denotes an m-sphere. We want to prove that $\psi = \iota \circ \overline{\psi}$ is homotopy equivalent to the constant map sending S^m to β, if m is not too large. This is where a variant of the Hatcher flow becomes useful. Look at arcs in the image of S^m crossing β and pick one closest to w, say α. Now "push" α over w and off of β, to the arc α'. See Figure 2.6. We can homotope ψ to a map ψ' using α' instead of α, assuming that the mutual link $\mathrm{lk}(\alpha) \cap \mathrm{lk}(\alpha')$ is sufficiently high connected. The last assertion can be engineered to be true, if we have enough control over the structure of Γ. This is where Lemma 2.2.5 becomes crucial.

We will carry this out first for subgraphs Γ of the linear graph L_n. Recall that L_n is the graph on $(n+1)$ nodes, having n edges connecting the vertices $i-1$ and i. Observe that in this setting, removing the star of two adjacent vertices results in removing at most 3 edges.

Theorem 2.2.6. *Let Γ_n be any subgraph of a linear graph, with Γ_n having n edges. Then $\mathcal{MA}(\Gamma_n)$ is $(\eta(n)-1)$-connected.*

Proof. We induct on n, with the base case being that $\mathcal{MA}(\Gamma_n)$ is non-empty for $n \geq 1$, which is clear. Now assume $n \geq 5$. We will freely apply Lemma 2.2.2 to represent simplices by systems of arcs. Choose an edge e in Γ_n with at least one endpoint of degree 1. Let v and w be the endpoints of e, say w has degree 1. Let q be the function defined above. For an arc α with $q(\alpha) = 1$, the descending link of α with respect to q is isomorphic to $\mathcal{MA}(\Gamma_{n'})$, where $\Gamma_{n'}$ is a subgraph of Γ_n with n' edges. Since every vertex has degree at most 2, $n' \geq n-3$, so by induction $\mathcal{MA}(\Gamma_{n'})$ is $(\eta(n)-2)$-connected. Similarly if $q(\alpha) = 3$ then the descending link of α is isomorphic to $\mathcal{MA}(\Gamma_{n'})$, now with $n' \geq n-2$, so again induction tells us that $\mathcal{MA}(\Gamma_{n'})$ is $(\eta(n)-2)$-connected. Note that $q(\alpha) = 2$ actually does not occur in the present situation (we defined q this way for the sake of consistency with the proof of Theorem 2.2.8 below).

The Morse Lemma 1.2.2 now implies that the pair $(\mathcal{MA}(\Gamma_n), \mathcal{MA}(\Gamma_n)^{q=0})$ is $(\eta(n)-1)$-connected, that is, the inclusion $\iota\colon \mathcal{MA}(\Gamma_n)^{q=0} \hookrightarrow \mathcal{MA}(\Gamma_n)$ induces an isomorphism in π_m for $m \leq \eta(n)-2$ and an epimorphism for $m = \eta(n)-1$. We could now invoke induction and use that $\mathcal{MA}(\Gamma_n)^{q=0}$ is $(\eta(n)-2)$-connected to conclude that $\mathcal{MA}(\Gamma_n)$ is $(\eta(n)-2)$-connected as well. However, since we even want $\mathcal{MA}(\Gamma_n)$ to be $(\eta(n)-1)$-connected, we need a different argument and we may as well apply this for all m. We want to show that $\pi_m(\mathcal{MA}(\Gamma_n)^{q=0} \hookrightarrow \mathcal{MA}(\Gamma_n))$ is trivial for $m < \eta(n)$. In other words, every sphere in $\mathcal{MA}(\Gamma_n)^{q=0}$ of dimension at most $(\eta(n)-1)$ can be collapsed in $\mathcal{MA}(\Gamma_n)$.

First we check a hypothesis on $\mathcal{MA}(\Gamma_n)$ that allows us to apply Lemma 2.2.5, namely that the link of a k-simplex should be $(m-2k-2)$-connected. A k-simplex σ is determined by $k+1$ disjoint arcs. Hence, the link of σ is isomorphic to $\mathcal{MA}(\Gamma_{n'})$ where n' is at least $n-(3k+3)$. By induction, this is $(\eta(n-3k-3)-1)$-connected. Moreover,

$$\begin{aligned}\eta(n-3k-3)-1 &= \left\lfloor \frac{n-3k-4}{4} \right\rfloor - 1 \\ &\geq \frac{n-3k-4}{4} - 2 \\ &\geq \eta(n)-2k-3 \geq m-2k-2\,.\end{aligned}$$

We conclude that the hypothesis of Lemma 2.2.5 is satisfied.

Let S^m be a combinatorial m-sphere. Let $\overline{\psi}\colon S^m \to \mathcal{MA}(\Gamma_n)^{q=0}$ be a simplicial map and let $\psi := \iota \circ \overline{\psi}$. It suffices by simplicial approximation ([Spa66], Theorem 3.4.8) to homotope ψ to a constant map. By Lemma 2.2.5 we may assume that ψ is simplexwise injective. Fix an arc β with endpoints v and w. We claim that ψ can be homotoped in $\mathcal{MA}(\Gamma_n)$ to land in the star of β, which will finish the proof, as $\operatorname{st}(\beta)$ is contractible. We will proceed in a similar way to the Hatcher flow used in the proof of Proposition 2.2.3. None of the arcs in the image of ψ use v or w as vertices, but among the finitely many such arcs, some might cross β. Pick the one, say α, intersecting β at a point closest along β to w, and let x be a vertex of S^m mapping to α. By simplexwise injectivity, none of the vertices in $\operatorname{lk}_{S^m}(x)$ map to α. Let α' be the arc with the same endpoints as α such that together α and α' bound a disk whose interior contains no boundary components, punctures or points of P other than w. See Figure 2.6 for an example. Note that there is no edge in $\mathcal{MA}(\Gamma_n)$ from α to α', so none of the vertices in $\operatorname{lk}_{S^m}(x)$ map to α'. Note also that $\psi(\operatorname{lk}_{S^m}(x)) \subseteq \operatorname{lk}\alpha'$ by choice of α.

Define a simplicial map $\psi' \colon S^m \to \mathcal{MA}(\Gamma_n)$ that sends the vertex x to α' and sends all other vertices y to $\psi(y)$. We claim that we can homotope ψ to ψ'. Once we do this, we will have reduced the number of crossings with β, and so continuing this procedure we will have homotoped our map so as to land in the star of β, finishing the proof.

The mutual link $\mathrm{lk}(\alpha) \cap \mathrm{lk}(\alpha')$ is isomorphic to $\mathcal{MA}(\Gamma_{n'})$, where $\Gamma_{n'}$ now is the graph obtained from Γ_n by removing e, and removing any edge sharing an endpoint with an endpoint of α. Here n' is the number of edges of the resulting graph. Since Γ_n is a subgraph of a linear graph, we have thrown out at most 4 edges, and so $n' \geq n-4$. Hence by induction $\mathrm{lk}(\alpha)\cap\mathrm{lk}(\alpha')$ is $(\eta(n)-2)$-connected, and in particular $(m-1)$-connected. Since $\mathrm{lk}_{S^m}(x)$ is an $(m-1)$-sphere, this tells us that there exists an m-disk B with $\partial B = \mathrm{lk}_{S^m}(x)$ and a simplicial map $\varphi \colon B \to \mathrm{lk}(\alpha) \cap \mathrm{lk}(\alpha')$ so that φ restricted to ∂B coincides with ψ restricted to $\mathrm{lk}_{S^m}(x)$. Since the image of B under φ is contained in $\mathrm{lk}(\alpha)$, we can homotope ψ, replacing $\psi|_{\mathrm{st}_{S^m}(x)}$ with φ. Since the image of B under φ is contained in $\mathrm{lk}(\alpha')$, we can similarly homotope ψ', replacing $\psi'|_{\mathrm{st}_{S^m}(x)}$ with φ. These both yield the same map, so we are finished. $\square$

Corollary 2.2.7. *$\mathcal{MA}(L_n)$ is $(\eta(n)-1)$-connected.* $\square$

As a remark, we expect that a better connectivity bound should be possible. Indeed, one can check that $\mathcal{MA}(L_n)$ is already connected for $n \geq 4$, and, by Proposition 2.1.4, $\mathcal{M}(L_n)$ is $(\nu(n)-1)$-connected, which for large n is stronger than being $(\eta(n)-1)$-connected. For now however, we will content ourselves with this bound.

Now that we have dealt with the family of linear graphs L_n (and subgraphs thereof), we turn to the complete graphs K_n. The methods used in the proof of Theorem 2.2.6 can also be used to show that $\mathcal{MA}(K_n)$ is $(\nu(n)-1)$-connected.

Theorem 2.2.8. *The complex $\mathcal{MA}(K_n)$ is $(\nu(n)-1)$-connected.*

Proof. The base case is that $\mathcal{MA}(K_n) \neq \emptyset$ for $n \geq 2$, which is clear. Let $n \geq 5$. Choose any edge e, with endpoints v and w. Let q be as above. For an arc α with $q(\alpha) = 1$, the descending link of α is isomorphic to $\mathcal{MA}(K_{n-3})$. If $q(\alpha) = 2$ or 3, the descending link is isomorphic to $\mathcal{MA}(K_{n-2})$. In any case, by induction all descending links are $(\nu(n)-2)$-connected. Hence we need only check that $\iota \colon \mathcal{MA}(K_n)^{q=0} \to \mathcal{MA}(K_n)$ induces the trivial map in π_m for $m < \nu(n)$.

First we check the hypothesis of Lemma 2.2.5. The link of a k-simplex is a copy of $\mathcal{MA}(K_{n-2k-2})$, which by induction is $(\nu(n-2k-2)-1)$-connected. We need this to be bounded below by $m-2k-2$. Indeed,

$$\nu(n-2k-2)-1 \geq \frac{n-2k-4}{3} - 2 \geq \nu(n) - 2k - 3 \geq m - 2k - 2\,.$$

Now we consider a simplicial map $\overline{\psi} \colon S^m \to \mathcal{MA}(K_n)^{q=0}$, with $\psi := \iota \circ \overline{\psi}$. We claim that we can homotope ψ to a constant map. By the same argument as in the proof of Theorem 2.2.6, the problem reduces to inspecting the mutual link $\mathrm{lk}(\alpha) \cap \mathrm{lk}(\alpha')$, where α and α' are again as in Figure 2.6. This mutual link is isomorphic to $\mathcal{MA}(K_{n-3})$, since compatible arcs may use any endpoints other than the endpoints of α, or the point w. Hence by induction $\mathrm{lk}(\alpha) \cap \mathrm{lk}(\alpha')$ is $(\nu(n)-2)$-connected, and by the same argument as in the proof of Theorem 2.2.6, we can eventually homotope ψ to land in the star of β, so we are done. $\square$

3 Thompson's Groups

After having collected the technical facts needed in the first two sections, we dedicate this section to Thompson's groups, as they are the groups we want to study.

We start with introducing the classical Thompson's groups F, T and V. The main reference for this is [CFP96]. Our focus in this is rather on giving the reader the right ideas to think about Thompson's groups as will be needed later, than on proofs. Also our introduction of the groups sV, V_{br} and F_{br} here will be less formal and focused on giving an intuition for those groups. Formal definitions will be given in the sections dealing with the finiteness properties of these generalizations.

3.1 Thompson's Group F

Let $[0,1]$ be the unit interval. A real number is called *dyadic* if it is of the form $k/2^\ell$, where $k \in \mathbb{Z}, \ell \in \mathbb{N}$. Consider the set F of piecewise linear homeomorphisms of $[0,1]$ to itself that are differentiable except at finitely many dyadic points and linear with slope a power of 2 on intervals where they are differentiable. In other words, for a homeomorphism $f \in F$, we have a sequence $0 = x_0 < x_1 < \cdots < x_n = 1$ of dyadic numbers at which f is not differentiable. On the intervals $[x_i, x_{i+1}]$ we have $f(x) = a_i x + b_i$, where a_i is a power of 2 and b_i is dyadic. It is easy to see that the homeomorphism f^{-1} is in F and that f induces a bijection on the set of dyadic numbers in $[0,1]$. The last statement implies that the set F is closed under composition and hence F is a group.

Definition 3.1.1. The set F together with composition is *Thompson's group F*.

Here are two important functions in F:

$$A(x) := \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1 & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases} \qquad B(x) := \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8} & \text{if } \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1 & \text{if } \frac{7}{6} \leq x \leq 1 \end{cases}$$

See Figure 3.1 for the graphs of A and B.

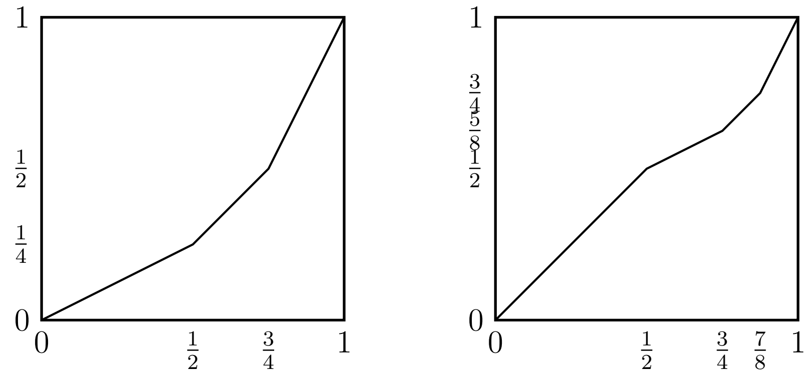

Figure 3.1: The functions A and B in F.

There is another way to think about elements of F. We follow the exposition in [Bel04] here. Consider the unit interval $[0,1]$. A *standard dyadic interval* in $[0,1]$ will be an interval of the form $\left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right]$, where $k, \ell \in \mathbb{N}$. A *dyadic subdivision* of $[0,1]$ is any decomposition of $[0,1]$ into standard dyadic intervals. Note that the pieces of a dyadic subdivision come in a natural order. Dyadic subdivisions are best pictured as a sequence of halvings of the unit interval. I.e. first cut the unit interval in half and then continue halving some of the resulting intervals until the dyadic subdivision is obtained. A *dyadic rearrangement* then is a piecewise linear homeomorphism f of $[0,1]$ induced by a pair of dyadic subdivisions $\mathcal{D}, \mathcal{C}$, where $\mathcal{D}$ and $\mathcal{C}$ have the same number of pieces. f then maps the i^{th} piece of $\mathcal{D}$ to the i^{th} piece of $\mathcal{C}$, where the pieces are ordered in the obvious way. By Theorem 1.1.2 of [Bel04] the group of dyadic rearrangements is isomorphic to F. Under this isomorphism the maps A and B from above correspond to the following dyadic rearrangements:

$$A\colon \qquad \left\{\left[0,\frac{1}{2}\right],\left[\frac{1}{2},\frac{3}{4}\right],\left[\frac{3}{4},1\right]\right\} \longmapsto \left\{\left[0,\frac{1}{4}\right],\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{1}{2},1\right]\right\}$$

$$B\colon \quad \left\{\left[0,\frac{1}{2}\right],\left[\frac{1}{2},\frac{3}{4}\right],\left[\frac{3}{4},\frac{7}{8}\right],\left[\frac{7}{8},1\right]\right\} \longmapsto \left\{\left[0,\frac{1}{2}\right],\left[\frac{1}{2},\frac{5}{8}\right],\left[\frac{5}{8},\frac{3}{4}\right],\left[\frac{3}{4},1\right]\right\}$$

This reinterpretation of F as the group of dyadic rearrangements leads to the well-known model of *paired tree diagrams* for elements of F. We need some notation. A *rooted binary tree* is a finite tree $\mathcal{T}$ where all vertices have degree 3, except the *leaves*, that have degree 1 and the *root*, that has degree 2, except if the root is a leaf. A *caret* is a subtree of a rooted binary tree $\mathcal{T}$ consisting of a non-leaf vertex and its two descendants, that are the leaves of the caret. A caret will be called *elementary* if its leaves are leaves of $\mathcal{T}$. Now each dyadic subdivision $\mathcal{D}$ gives rise to a rooted binary tree, where the root corresponds to the interval $[0,1]$ and the leaves to the standard dyadic intervals in $\mathcal{D}$. The non-leaf, non-root vertices correspond to intermediate steps in the subdivision. See Figure 3.2 for an example. Hence given any element $f \in F$, we can represent it by a pair of trees, one for the subdivision of the domain and one for the codomain. Such a representation is a *paired tree diagram* that we usually denote by $f = (\mathcal{T}_-, \mathcal{T}_+)$. A common convention is to draw the tree representing the codomain of f upside down and below the domain tree, such that the leaves match up. Doing this we also speak of *split-merge diagrams*, where we call each caret in the domain tree a *split* and each caret in the codomain tree a *merge*. This is closely related to the language of "strand diagrams", cf. [Bel04] and [BM14].

Clearly we obtain for each element of f a paired tree diagram. But those are not unique. For example all of the diagrams in Figure 3.3 represent the identity.

The ambiguity of the diagrams in Figure 3.3 is due to the fact, that the domain tree and the codomain tree have *opposing* elementary carets. This is saying, that in the paired tree diagram $(\mathcal{T}_-, \mathcal{T}_+)$ both trees have an elementary caret whose leaves have the same labels (recall that there is a natural labeling of the leaves by $1, \dots, n$ from left to right), or that we see a split directly followed up by a merge in the picture. A *reduction* of a paired tree diagram is the operation of removing opposing elementary carets in $\mathcal{T}_-$ and $\mathcal{T}_+$. This corresponds to eliminating unnecessary "cuts" in the respective dyadic subdivisions. A paired tree diagram is *reduced* if there are

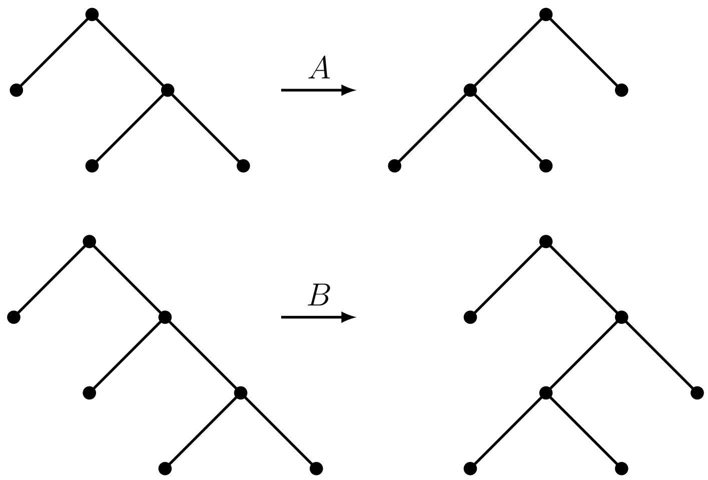

FIGURE 3.2: Paired tree diagrams for A and B.

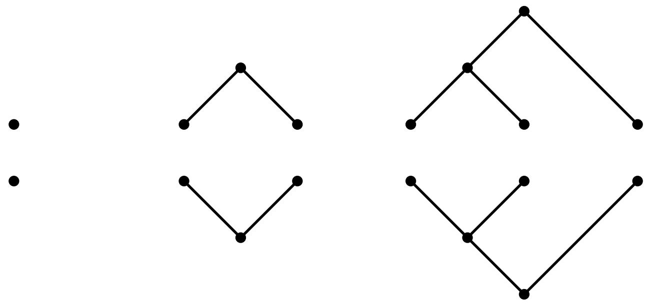

FIGURE 3.3: Three distinct paired tree diagrams, all of which represent the identity in F.

no opposing carets. In Figure 3.3 the leftmost diagram is reduced. The inverse operation of a reduction is an *expansion* of the diagram. The following fact is not hard to see:

Proposition 3.1.2 ([Bel04], Theorem 1.2.4)**.** *Each $f \in F$ has a unique reduced paired tree diagram.*

Using our model of reduced paired tree diagrams or split-merge diagrams it becomes particularly easy to determine the product of elements of $f, g \in F$. Let $(\mathcal{T}_-, \mathcal{T}_+)$ be the reduced paired tree diagram for f, respectively $(\mathcal{S}_-, \mathcal{S}_+)$ for g. By a sequence of expansions we can obtain diagrams $(\mathcal{T}'_-, \mathcal{T}'_+)$ and $(\mathcal{S}'_-, \mathcal{S}'_+)$ such that $\mathcal{T}'_+ = \mathcal{S}'_-$. Then $(\mathcal{T}'_-, \mathcal{S}'_+)$ is a diagram for the product fg. This is readily verified by reinterpreting the paired tree diagrams as piecewise linear maps. In the language of split-merge diagrams this procedure can be described by "stacking" the bottom of the diagram for f on top of the diagram for g. The resulting diagrams can be reduced to a split-merge diagram for fg. Additionally to the reduction we had before, namely a merge directly following a split is doing nothing, we also need the "inverse" reduction, i.e. we declare that a merge directly followed by a split is doing nothing. In this way we can reduce the stacked diagram fg to a split-merge diagram. See Figure 3.4 for the reduction moves and an example.

To summarize we have introduced three ways of thinking about elements of F.

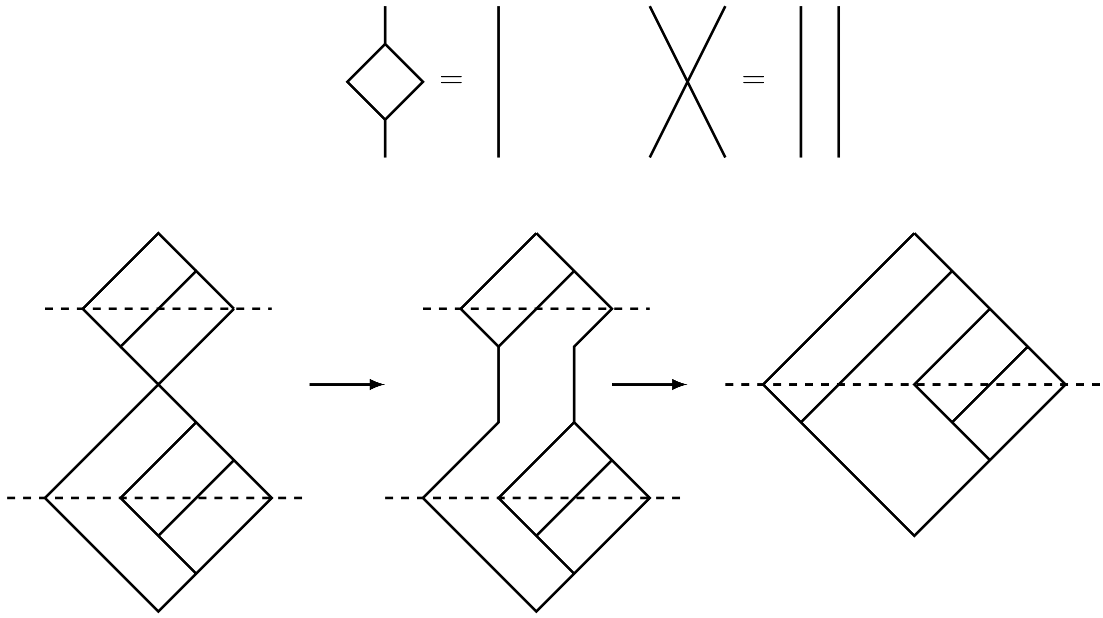

FIGURE 3.4: On top the two reduction moves on split-merge diagrams. Below the product AB and the reduction to a reduced split-merge diagram.

Firstly they can be viewed as piecewise linear maps from the unit interval to itself. Secondly we can represent them as pairs of trees $(\mathcal{T}_-, \mathcal{T}_+)$, that are unique up to reduction. Finally we have a notion of split-merge diagrams, again up to reduction. We will use all three models throughout this thesis.

To close this introductory section on F, we will collect and restate some well-known facts.

Firstly F is finitely presented. We give two standard presentation for F and refer to [CFP96] for proofs. Even though we will not use the explicit finite presentations, we restate them for completeness. Recall the elements A, B of F and define a family $\{X_i\}$ of elements of F by $X_0 := A$ and $X_n := A^{-(n-1)}BA^{n-1}$. See Figure 3.5. In particular we have $X_1 = B$.

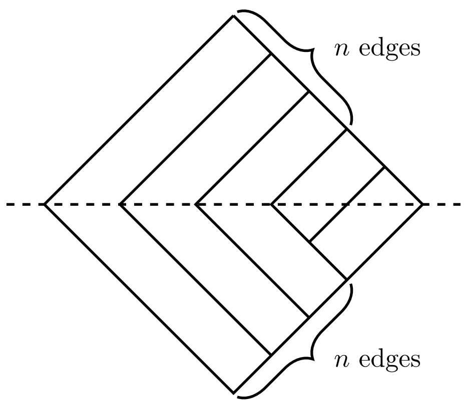

FIGURE 3.5: The reduced split-merge diagram for $X_n \in F$.

Then the following holds:

$$F = \langle X_0, X_1 \mid X_2X_1 = X_1X_3, X_3X_1 = X_1X_4 \rangle \ ,$$

where we read the X_i as words in A, B and their inverses.

This is the standard finite presentation. Sometimes it is more practical to work with the following presentation:

$$F = \langle X_0, X_1, \cdots \mid X_k^{-1}X_nX_k = X_{n+1} \text{ for } k < n \rangle$$

The second fact we want to state is that F is infinite and torsion free. From the point of view of piecewise linear homeomorphisms this is pretty clear. We sketch the argument given in [Bel04]. For each non-trivial element $f \in F$ there is a smallest point t_0 in $[0, 1]$ such that the right-derivative of f at t_0 is 2^m for $m \neq 0$. Then clearly the derivative of f^n is 2^{mn} at t_0 and hence all positive powers of f are distinct.

3.2 Thompson's Groups T and V

Having introduced F and different ways of thinking about it, we now turn our attention to the other classical Thompson's groups, namely T and V. Both of these were also introduced by Thompson and shown to be infinite, finitely presented and simple. They were the first known examples of such groups.

As for F, we can introduce T as a group of piecewise linear homeomorphism, but instead of the unit interval $[0, 1]$, we now consider the unit circle S^1, thought of as the unit interval with endpoints identified. T is defined to be the group of piecewise linear homeomorphisms of S^1 to itself that map images of dyadic numbers to dyadic numbers, are differentiable except at finitely many images of dyadic numbers and the derivatives are, where they are defined, powers of 2. We can also think of elements of T in terms of dyadic rearrangements and introduce unique reduced tree diagrams for them. The only difference to F is that an element $t \in T$ is allowed to cyclically permute (and affinely transform) the pieces. For example consider the map

$$C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{4} & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}$$

in T. Note that C is defined on $[0, 1]$ but for elements of T we consider $0 = 1$. The domain and codomain tree for C are the same, but the leaves are cyclically permuted: The standard dyadic interval $[3/4, 1]$ for example is mapped to $[1/2, 3/4]$. To keep track of this permutation, we number the leaves of the trees correspondingly, or draw arrows between the leaves in case of split-merge diagrams. See Figure 3.6 for the diagrams of C.

Clearly we have to be careful about reductions now. An opposing caret is no longer a pair of opposite drawn elementary carets, but rather a pair of elementary carets for which the leaves are identified, in the right order, by the permutation. (Otherwise the diagrams in Figure 3.6 would not be reduced.) We also write $(\mathcal{T}_-, \rho, \mathcal{T}_+)$ for the tree diagrams as before, where now ρ is a cyclic permutation. Everything we said for F, especially for multiplication, works the same way for T.

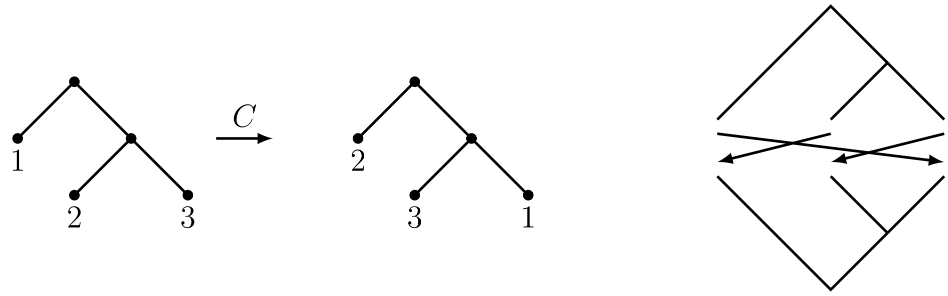

FIGURE 3.6: The paired tree diagram and a split-merge diagram for $C \in T$.

It is obvious that F is a subgroup of T, namely the subgroup of elements, whose reduced tree diagrams are of the form $(\mathcal{T}_-, \mathrm{id}, \mathcal{T}_+)$. In this sense we view the functions A and B from above as elements of T and obtain the following presentation (cf. [CFP96], Lemmas 5.2 and 5.3 and Theorem 5.8):

$$\begin{aligned} T = \langle A, B, C \,|& [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2], C^{-1}B(A^{-1}CB), \\ & ((A^{-1}CB)(A^{-1}BA))^{-1}B(A^{-2}CB^2), (CA)^{-1}(A^{-1}CB)^2, C^3 \rangle \end{aligned}$$

From the point of view of tree diagrams $(\mathcal{T}_-, \rho, \mathcal{T}_+)$ it seems unnecessary to restrict ourselves to cyclic permutations ρ. Indeed it is possible to take an arbitrary permutation ρ in the appropriate symmetric group S_n. Doing this leads to a description of the third classical Thompson's group V. Since V is playing a main role in the rest of this thesis we will introduce it in more detail than T.

Contrary to how we introduced F, we will define V in terms of paired tree diagrams and then give an alternate description of V in terms of bijections of the half open interval $[0, 1)$.

We extend our definition of *paired tree diagrams* from the last Subsection, to, instead of pairs of trees, be triples of the form $(\mathcal{T}_-, \rho, \mathcal{T}_+)$, where $\mathcal{T}_-$ and $\mathcal{T}_+$ are rooted binary trees with the same number of leaves n and $\rho \in S_n$. If $\rho = \mathrm{id}$ this yields a paired tree diagrams as defined for F. Additionally we label the leaves of $\mathcal{T}_-$ by $1, \dots, n$ from left to right. The labeling of the leaves of T_+ then depends on the permutation ρ. Namely the $\rho(i)^{\text{th}}$ leaf of T_+ is labeled i, for all i. There is an equivalence relation on the set of paired tree diagrams, given by *reduction* and *expansion*. By a reduction we mean the following: Suppose $\mathcal{T}_-$ and $\mathcal{T}_+$ have an elementary caret with left leaf labeled i and right leaf labeled $i + 1$. Such a pair is again called *opposite*. We obtain a new paired tree diagram $(\mathcal{T}'_-, \rho', \mathcal{T}'_+)$ by defining $\mathcal{T}'_\pm$ to be $\mathcal{T}_\pm$ with the opposing carets removed and letting $\rho' \in S_{n-1}$ be the permutation that maps the new leaf of $\mathcal{T}'_-$ to the new leaf of $\mathcal{T}'_+$ and behaves exactly like ρ on the rest of the leaves. We say that $(\mathcal{T}'_-, \rho', \mathcal{T}'_+)$ is obtained by *reducing* $(\mathcal{T}_-, \rho, \mathcal{T}_+)$. The inverse of a reduction is an *expansion*. A paired tree diagram is *reduced* if there is no reduction possible. Again it is not hard to see, that this constitutes an equivalence relation and each equivalence class has a unique reduced representative. See Figure 3.7 for an example.

We can define a binary operation on equivalence classes of paired tree diagrams as before when we introduced F. Let $(\mathcal{T}_-, \rho, \mathcal{T}_+)$ and $(\mathcal{S}_-, \xi, \mathcal{S}_+)$ be paired tree diagrams. By repeatedly applying expansions we can obtain equivalent tree diagrams

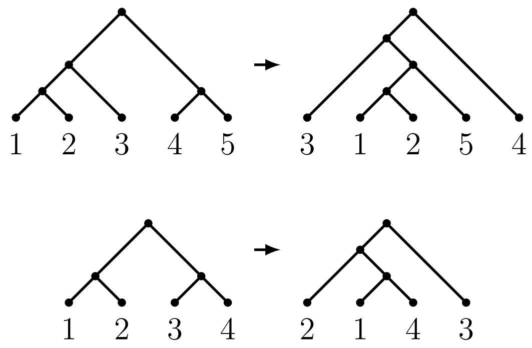

FIGURE 3.7: An example of a reduction of paired tree diagrams.

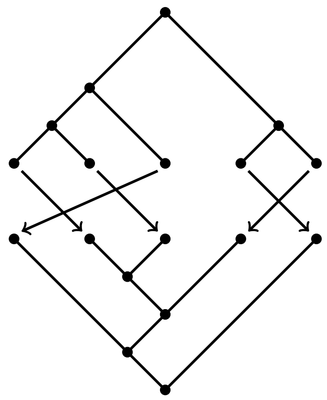

FIGURE 3.8: An element of V.

$(\mathcal{T}'_-, \rho', \mathcal{T}_+)$ and $(\mathcal{S}'_-, \xi', \mathcal{S}'_+)$ such that $\mathcal{T}'_+ = \mathcal{S}'_-$. We then define the *product* of the given equivalence classes of paired tree diagrams to be the equivalence class of $(\mathcal{T}'_-, \rho'\xi', \mathcal{S}'_+)$. Again one verifies that this is well-defined and is a group operation (cf. for example [CFP96]).

Definition 3.2.1. Thompson's group V is the group of equivalence classes of paired tree diagrams with the above defined multiplication.

Observation 3.2.2. *The way we defined V, it is clear from what has been said before, that $F \leq T \leq V$.* □

It will be convenient to draw paired tree diagrams the way we did for T. See Figure 3.8 for an example.

To get a notion of V as a set of functions, recall the correspondence between rooted binary trees and dyadic subdivisions of the unit interval $[0, 1]$ and also of $[0, 1)$. An element of V then corresponds to a right-continuous bijection of $[0, 1)$ in the following way: $\mathcal{T}_-$ gives a dyadic subdivision of the domain interval and $\mathcal{T}_+$ of the codomain interval and the i^{th} piece of the subdivision of the domain is mapped to the $\rho(i)^{\text{th}}$ piece of the codomain. These bijections have again the properties, that they map images of rational dyadic numbers to images of rational dyadic numbers, they are linear on intervals of differentiability and have derivative a power of 2.

Hence we can think of elements of V also, less formally, as sequences of halvings of the unit interval, one for the domain and for the codomain, and then identifying

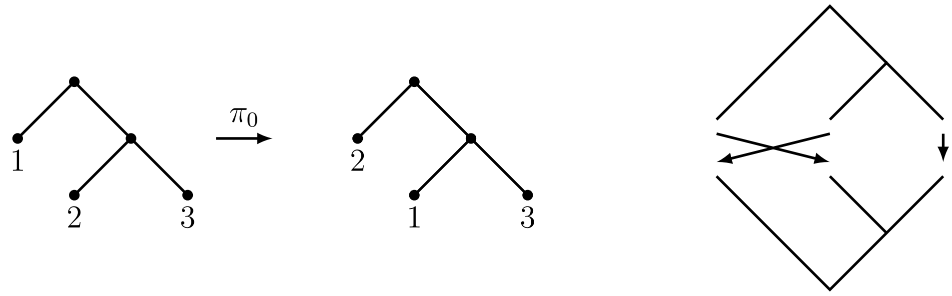

FIGURE 3.9: The element π_0 of V.

the resulting pieces by a permutation and while doing that affinely transforming the pieces, if necessary.

For completeness we again restate a finite presentation for V as given in [CFP96]. By Observation 3.2.2 the elements A, B and C defined before are in V. To introduce non-cyclic permutations, we consider additionally the element π_0 given by the diagram in Figure 3.9.

Recall that we defined $X_0 = A$ and $X_n = A^{-(n-1)}BA^{n-1}$ for $n \geq 1$. Additionally we let $C_n = A^{-(n-1)}CB^{n-1}$ for $n \geq 1$ and $\pi_1 = C_2^{-1}\pi_0 C_2$ and $\pi_n = A^{-(n+1)}\pi_1 A^{n-1}$. With this notation we have the following presentation of V by [CFP96], Lemma 6.1, Theorem 6.9:

$$\begin{aligned} V = \langle A, B, C, \pi_0 \mid & [AB^{-1}, X_2], [AB^{-1}, X_3], BC_2C_1^{-1}, BC_3(C_2X_2)^{-1}, C_2^2(C_1A)^{-1}, C_1^3, \\ & \pi_1^2, \pi_3\pi_1(\pi_1\pi_3)^{-1}, (\pi_2\pi_1)^3, \pi_1X_3(X_3\pi_1)^{-1}, B\pi_2\pi_1(\pi_1X_2)^{-1}, \\ & B\pi_3(\pi_2B)^{-1}, C_3\pi_2(\pi_1C_3)^{-1}, (\pi_1C_2)^3 \rangle \end{aligned}$$

Let us recall some facts on T and V. It is again clear that both are infinite, since they contain F as a subgroup. But they are no longer torsion-free. T contains, for example, a copy of each cyclic group and V a copy of each symmetric group. The following is one of the most prominent results on T and V already shown by Thompson in hand-written notes.

Theorem 3.2.3 ([CFP96], Theorem 5.8 and Theorem 6.9)**.** *T and V are simple.*

T and V were among the first examples of infinite, finitely presented simple groups.

3.3 The Groups sV, V_{br} and F_{br}

To close our introductory section on Thompson's groups, we want to mention the two generalizations of V, that we will consider in Sections 5 and 6 and give a very rough idea of what they are.

First we will deal with a braided version of V (and the corresponding version of F). For this we take the point of view of paired tree diagrams $(\mathcal{T}_-, \rho, \mathcal{T}_+)$ for elements of V. Now instead of "sticking" a permutation between the two trees, one can also use braids. By doing this we obtain the group V_{br}, introduced independently by Brin ([Bri07]) and Dehornoy ([Deh06]). Clearly one has to do some work to assure that the notions of reduction and multiplication still work. We will do this in detail in

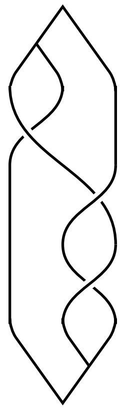

FIGURE 3.10: An element of V_{br}.

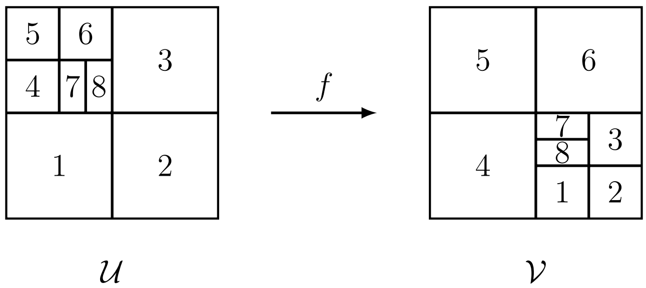

FIGURE 3.11: An element of $2V$.

Section 5. For now it suffices to have a picture in mind, see Figure 3.10. The group V_{br} was shown to be finitely presented (cf. [Bri06]) and contains F and a copy of the braid group B_n for each $n \in \mathbb{N}$. In 2008 Brady, Burillo, Cleary and Stein introduced the braided version of F, that we denote F_{br}, and showed it to be finitely presented ([BBCS08]). Again it can be easily understood from the point of view of paired tree diagrams, that $F_{br} \leq V_{br}$, namely the subgroup where the braid in between the trees is pure (i.e. induces the trivial permutation).

The second generalization of V we will deal with are the groups we denote as sV for $s \geq 1$. They were introduced by Brin in 2004,2005 ([Bri04, Bri05]) and are a higher dimensional analogue of V. For that reason they are usually termed *Brin-Thompson groups* or *higher dimensional Thompson groups*. Recall that elements of V can be thought of as maps from one, by a sequence of halvings sliced up, unit interval $[0,1]$ to another unit interval, that is cut into the same number of pieces. For the higher-dimensional groups, we no longer think of the unit interval but rather unit s-cubes $[0,1]^s$. The cube $[0,1]^s$ can be halved by hyperplanes in s different directions, as can any resulting piece of such an operation. Analogously to V, an element of sV can be described as a sequence of halvings of the domain and codomain and an identification of the resulting pieces by a permutation, maybe affinely deforming the pieces. In particular we have $V = 1V$ and Brin showed that all the groups sV are simple and finitely presented. See Figure 3.11 for an example of an element of $2V$. We will give a formal definition and further intuition for the groups sV in Section 6.

4 Finiteness Properties of the Classical Thompson's Groups

In this chapter we will reprove the well-known fact that the classical Thompson's groups F and V are of type F_∞. We chose to include these proofs since they give the classical case and our proof of the Main Theorem is inspired by them.

4.1 Thompson's Group F

We start with the group F. Recall from Subsection 3.1 that we can think of elements of F as paired tree diagrams $(\mathcal{T}_-, \mathcal{T}_+)$. We also spoke informally of split-merge diagrams, that arise if we draw the tree $\mathcal{T}_+$ upside down and below $\mathcal{T}_-$ such that the leaves of both match up. In order to use the tools introduced in Section 1 to prove that F is of type F_∞, we need to make this more precise.

Split-Merge Diagrams

We will again freely use the language of strand diagrams (cf. [Bel04, BM14]). Let $(\mathcal{T}_-, \mathcal{T}_+)$ be a paired tree diagram. Recall that we called a caret in $\mathcal{T}_-$ a *split* and a caret in $\mathcal{T}_+$ a *merge*. Then we defined a *split-merge diagram* to be the picture representing the paired tree diagram as in Figure 3.4. We extend these definitions from trees (i.e. finite rooted binary trees) to forests (i.e. unions of finitely many trees).

Definition 4.1.1. An (n,m)*-split-merge diagram* is a split-merge diagram that begins on n strands, the *heads*, and ends on m strands, the *feet*. Equivalently we can think of an (n,m)-split-merge diagram as a paired forest diagram $(\mathcal{F}_-, \mathcal{F}_+)$, where $\mathcal{F}_-$ has n roots and $\mathcal{F}_+$ has m roots and both have the same number of leaves. By an n*-split-merge diagram* we will mean a split-merge diagram with n heads and an arbitrary number of feet. A *split-merge diagram* is a (n,m)-split-merge diagram for some n, m. We denote by $\mathtt{S}$ the set of all split-merge diagrams. The symbols $\mathtt{S}_n$ and $\mathtt{S}_{n,m}$ are defined accordingly.

We also extend the notions of *reduction* and *expansion* to split-merge diagrams, i.e. the operations of removing or adding a pair of opposing elementary carets to the forests $\mathcal{F}_-$ and $\mathcal{F}_+$. Recall that opposing means that the left and right leaves are labeled the same way, in the case of F that is saying, that they match up. This again forms an equivalence relation and each equivalence class of split-merge diagrams has a unique reduced representative. This is not immediate, a proof is sketched in [BS08], Proposition 1. We will just call an equivalence class of a split-merge diagram a split-merge diagram. In particular

Observation 4.1.2. *The set of* $(1,1)$*-split-merge diagrams is in bijection to the elements of* F*.* $\square$

The multiplication defined for F, by stacking $(1,1)$-split-merge diagrams on top of each other, carries over to arbitrary split-merge diagrams. Except that we can not multiply arbitrary split-merge diagrams σ and τ, but only those, where the number

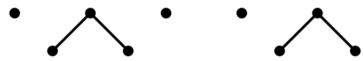

FIGURE 4.1: The elementary forest $\mathcal{F}^{(5)}_{\{2.5\}}$.

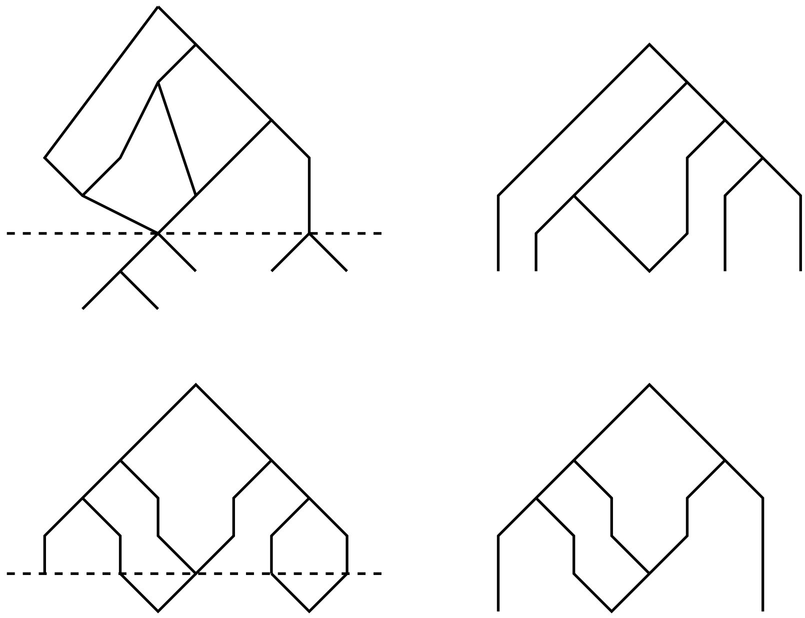

FIGURE 4.2: A splitting by some $\mathcal{F}$ and an elementary merging by $\mathcal{F}^{(3)}_{\{2,3\}}$.

of feet of σ equals the number of heads of τ. In other words $\sigma\tau$ is only defined for $\sigma \in \mathtt{S}_{n,m}$ and $\tau \in \mathtt{S}_{m,n'}$ for some n, m, n'. This yields a groupoid structure on $\mathtt{S}$. The inverse of a split-merge-diagram $\sigma = (\mathcal{F}_-, \mathcal{F}_+)$ is given by $(\mathcal{F}_+, \mathcal{F}_-) = \sigma^{-1}$.

There is an important family of forests that will become essential in defining the correct space for F to act on. For $n \in \mathbb{N}$ and $J \subset \{1, \ldots n\}$ let $\mathcal{F}^{(n)}_J$ be the forest with n roots and a single caret attached to the i^{th} root for each $i \in J$. Observe that these forests are characterized by the fact, that each caret is elementary. We also call such a forest *elementary*. See Figure 4.1 for an example.

The last notion we need is that of (elementary) splittings and (elementary) mergings. Fix an (n, m)-split-merge diagram σ. For any forest $\mathcal{F}$ with m roots and ℓ leaves, the *splitting of σ by $\mathcal{F}$* is the (n, ℓ)-split-merge diagram obtained by multiplying σ from the right with $(\mathcal{F}, 1_\ell)$, where we denote by 1_ℓ the trivial forest on ℓ roots for an arbitrary $\ell \in \mathbb{N}$. Analogously the *merging of σ by $\mathcal{F}'$* is the split-merge diagram obtained by right-multiplying with $(1_m, \mathcal{F}')$ for $\mathcal{F}'$ a forest with ℓ roots and m leaves. A splitting or merging is *elementary* if $\mathcal{F}$ is an elementary forest. See Figure 4.2 for examples. Note that a splitting or merging of $\sigma \in \mathtt{S}_n$ does not change the number of heads, so the resulting split-merge diagram is again an element of $\mathtt{S}_n$. We write $x \leq y$ for $x, y \in \mathtt{S}_n$ if y is obtained by a splitting of x, and $x \preceq y$ if the splitting is elementary. It is readily verified that the pair $(\mathtt{S}_n, \leq)$ is a poset.

A Space for F

From now on we will focus on the poset $(\mathtt{S}_1, \leq)$. That is the set of split-merge diagrams with one head and an arbitrary number of feet together with the relation of splitting.

Observation 4.1.3. *The poset* $(\mathtt{S}_1, \leq)$ *is directed.*

Proof. We need to show that any pair of distinct split-merge diagrams $x, y \in \mathtt{S}_1$ have an upper bound. Recall that split-merge diagrams are equivalence classes of paired forest diagrams. Let $(\mathcal{F}_-, \mathcal{F}_+)$ be a representative for x and $(\mathcal{G}_-, \mathcal{G}_+)$ for y. Here $\mathcal{F}_-$ and $\mathcal{G}_-$ are trees, since $x, y \in \mathtt{S}_1$. Let n be the number of roots of $\mathcal{F}_+$ and ℓ be the number of leaves. Consider the splitting of x by $\mathcal{F}_+$, that is the $(1, \ell)$-split-merge diagram represented by $(\mathcal{F}_-, 1_\ell)$, where 1_ℓ again denotes the trivial forest on ℓ roots. So we have $x \leq (\mathcal{F}_-, 1_\ell)$. Analogously we obtain $y \leq (\mathcal{G}_-, 1_m)$, where $\mathcal{G}_+$ is a forest with m leaves. Now since $\mathcal{F}_-$ and $\mathcal{G}_-$ are trees, there is a tree $\mathcal{T}$ having both as a subtree. Say $\mathcal{T}$ has k leaves. Clearly $(\mathcal{T}, 1_k)$ is a splitting of $(\mathcal{F}_-, 1_\ell)$ and also of $(\mathcal{G}_-, 1_m)$. Hence x and y have an upper bound. □

By Lemma 1.3.3 we conclude that the geometric realization $|\mathtt{S}_1|$ of $\mathtt{S}_1$ is contractible.

Note that there is a natural left action of F on the poset $\mathtt{S}_1$ given by multiplication of split-merge diagrams. An element $f \in F$ is a $(1,1)$-split-merge diagram, so for $x \in \mathtt{S}_1$ we have $fx = y$ where y is again a $(1, n)$-split-merge diagram and n is the number of feet of x. As the F-action preserves the number of feet, this extends to an simplicial action on $|\mathtt{S}_1|$.

Observation 4.1.4. *The action of* F *on* $|\mathtt{S}_1|$ *is free.*

Proof. It suffices to show that vertex stabilizers are trivial, since the action of F preserves the number of feet and adjacent vertices of $|\mathtt{S}_1|$ have a different number of feet. Let $f \in \mathrm{Stab}_F(x)$ be represented by the paired tree diagram $(\mathcal{T}_-, \mathcal{T}_+)$, where x is a vertex of $|\mathtt{S}_1|$. So x is represented by a $(1, n)$-split-merge diagram for some n. Say $x = (\mathcal{F}_-, \mathcal{F}_+)$, where $\mathcal{F}_-$ is a tree with ℓ leaves and $\mathcal{F}_+$ is a forest on n roots with ℓ leaves. We have $fx = x$, in terms of split-merge diagrams that is

$$(\mathcal{T}_-, \mathcal{T}_+)(\mathcal{F}_-, \mathcal{F}_+) = (\mathcal{F}_-, \mathcal{F}_+).$$

Using the groupoid structure on $\mathtt{S}$, the set of all split-merge diagrams, and the fact, that everything is defined up to reduction of the diagrams, we find the inverse of $(\mathcal{F}_-, \mathcal{F}_+)$ to be $(\mathcal{F}_+, \mathcal{F}_-)$. Multiplying from the right with this element yields that $(\mathcal{T}_-, \mathcal{T}_+)$ can be reduced to the trivial diagram. Hence f is trivial. □

Observation 4.1.5. *Let* x *be a* $(1, n)$*-split-merge diagram. The* F*-orbit of* x *contains a* $(1, n)$*-split-merge diagram of the form* $(\mathcal{T}, 1_n)$ *where* $\mathcal{T}$ *is a tree.*

Proof. Let $(\mathcal{F}_-, \mathcal{F}_+)$ be a representative of x. As $x \in \mathtt{S}_{1,n}$, $\mathcal{F}_-$ is a tree and $\mathcal{F}_+$ is a forest on n roots. Consider an arbitrary tree $\mathcal{T}$ with n leaves. There is an expansion $(\mathcal{T}', \mathcal{F}_+)$ of the $(1, n)$-split-merge diagram $(\mathcal{T}, 1_n)$, such that $(\mathcal{T}', \mathcal{F}_+)(\mathcal{F}_+, \mathcal{F}_-)$ is a $(1,1)$-split-merge diagram and hence represents an element $f \in F$. Now fx is represented by $(\mathcal{T}', \mathcal{F}_+)$, since $(\mathcal{F}_+, \mathcal{F}_-)(\mathcal{F}_-, \mathcal{F}_+) = 1_n$. Hence fx is represented by $(\mathcal{T}, 1_n)$. □

Consider the map $h\colon \mathtt{S}_1 \to \mathbb{N}$ that assigns to each split-merge diagram its number of feet. As $x \le y$ for $x, y \in \mathtt{S}_1$ implies $h(x) \le h(y)$, this is a Morse function in the sense of Subsection 1.2. We adopt the notations introduced there. For example, let $|\mathtt{S}_1|^{\le n}$ be the full subcomplex of $|\mathtt{S}_1|$ spanned by vertices x with $h(x) \le n$. This yields a filtration $(|\mathtt{S}_1|^{\le n})_n$ of $|\mathtt{S}_1|$.

Corollary 4.1.6. *$|\mathtt{S}_1|^{\le n}$ is finite modulo the action of F.*

Proof. Observation 4.1.5 shows that the 0-skeleton of $F \backslash |\mathtt{S}_1|^{\le n}$ is finite. Even stronger, there is just one vertex in the quotient for each $k \le n$. As $|\mathtt{S}_1|^{\le n}$ is locally finite, since there are only finitely many ways to split any given split-merge diagram into one with n feet, the claim follows. □

At this point we have verified all the assumption of Brown's Criterion (Proposition 1.1.8). If we could show that the connectivity of the pairs $(|\mathtt{S}_1|^{\le n+1}, |\mathtt{S}_1|^{\le n})$ tends to ∞ as n tends to ∞, we would be able to conclude that F is of type F_∞. This amounts to analyzing the descending links in $|\mathtt{S}_1|$ with respect to the height function h. To have an easier time doing that, we will not analyze the whole space $|\mathtt{S}_1|$, but a subcomplex that we term the "Stein space for F". In order to define it, recall that we introduced the relation $\preceq$ on $\mathtt{S}_1$. For $x, y \in \mathtt{S}_1$, we have $x \preceq y$ if y is obtained from x by an elementary splitting. Note that $\preceq$ is not transitive. But it is true, that if $x \preceq z$, then $x \preceq y \preceq z$ for each $x \le y \le z$. This enables us to define a simplex in $|\mathtt{S}_1|$, i.e. a chain $x_0 \le \cdots \le x_k$ to be *elementary* if $x_0 \preceq x_k$. By the above discussion each face of an elementary simplex is again elementary.

Definition 4.1.7. The *Stein space* X_F for F is the subcomplex of $|\mathtt{S}_1|$ consisting of the elementary simplices.

Clearly the action of F on X_F is still free and the quotient of $X_F^{\le n}$ modulo F finite. But we have to make sure that the space X_F is contractible. For this we take [Bro92], Section 4, as a guide, where Brown described the Stein space for V.

We use the standard notion of intervals in a poset. Hence the open interval (x, y) will denote the set $\{z \in \mathtt{S}_1 \mid x < z < y\}$. Closed and half open intervals are defined accordingly.

Let $x, y \in \mathtt{S}_1$ such that $x \le y$. We denote by y_0 the maximal element in $[x, y]$ such that $x \preceq y_0$. It is obtained from x by adding single carets to each foot of x that is split in y. See Figure 4.3 for an example. We will call y_0 also the *elementary core* of y.

Lemma 4.1.8. *Let $x, y \in \mathtt{S}_1$. Suppose $x < y$ and $x \not\preceq y$. Then $|(x, y)|$ is contractible.*

Proof. Firstly we have $x < y_0$ since $x < y$, and $y_0 < y$ since $x \not\preceq y$, hence $y_0 \in (x, y)$. Let $z \in (x, y)$. Clearly $x < z_0 \le z < y$. Hence $z_0 \in (x, y)$. Moreover we have $z_0 \le y_0$, since otherwise it would be impossible for y to be a splitting of z. Now the inequalities $z \ge z_0 \le y_0$ provide a contraction of $|(x, y)|$ by Lemma 1.3.4. □

Corollary 4.1.9. *X_F is contractible.*

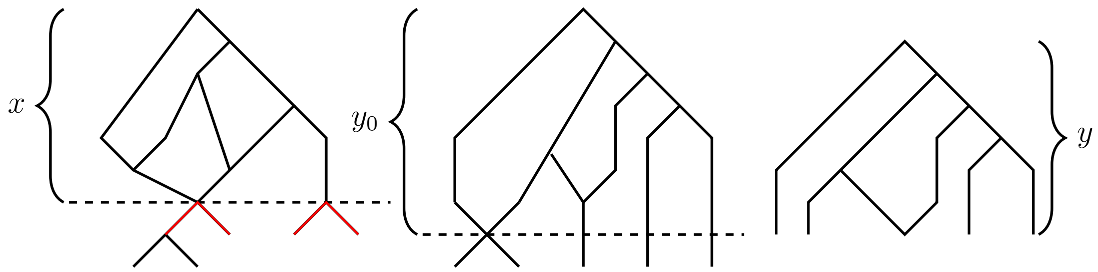

FIGURE 4.3: The splitting of x by $\mathcal{F}$ from Figure 4.2. The elementary core is highlighted on the right and drawn in the middle.

Proof. Since $|\mathtt{S}_1|$ is contractible by Observation 4.1.3 and Lemma 1.3.3, it suffices to show that we can build up from X_F to $|\mathtt{S}_1|$ without changing the homotopy type. We will do this by gluing the closed intervals $|[x, y]|$ for $x \not\prec y$ onto X_F in increasing order, where the order is given by the number $h(y) - h(x)$. This implies that when we glue in $|[x, y]|$, the space $|[x, y) \cup (x, y]|$ is already glued in. But this is the suspension of $|(x, y)|$ and hence contractible by Lemma 4.1.8. Clearly the space $|[x, y]|$ itself is contractible as a directed poset by Lemma 1.3.3. Hence we only ever attach contractible spaces along contractible subspaces and never change the homotopy type. This concludes the proof. $\square$

We are left with verifying that the connectivity of the pair $(X_F^{\leq n+1}, X_F^{\leq n})$ tends to ∞ as n tends to ∞. We will do this using part (i) of the Morse Lemma 1.2.2. So we have to analyze the connectivity of the descending links with respect to the Morse function h.

Descending Links

Recall that we have the Morse function $h\colon \operatorname{vt}(X_F) \to \mathbb{N}$, where $h(x) = n$, the number of feet of the $(1, n)$-split-merge diagram x. The descending link $\operatorname{lk}\!\downarrow(x)$ is the full subcomplex of X_F spanned by vertices adjacent to x and of smaller height. In other words a $(1, m)$-split-merge diagram y is a vertex of $\operatorname{lk}\!\downarrow(x)$ if and only if $h(y) = m < n = h(x)$ and $y \prec x$, or equivalently if y is obtained from x by an elementary merging. Such an elementary merging of x is given by a forest $F_J^{(m)}$ on m roots that has n leaves. We can best picture this by drawing a rectangle for the element x and the feet emerging on the bottom. Then any set of disjoint merges that we attach to the feet of x yields a vertex of the descending link. See Figure 4.4. Labeling the feet of x by $0, \ldots, n-1$, we can think of the feet as vertices of a linear graph L_{n-1}. Then the elementary merges of x are in one-to-one correspondence with the matchings of the graph L_{n-1}. And hence:

Observation 4.1.10. *Let $x \in X_F^{=n}$. The descending link $\operatorname{lk}\!\downarrow(x)$ with respect to h is isomorphic to the barycentric subdivision of the matching complex $\mathcal{M}(L_{n-1})$ of the linear graph L_{n-1}. Hence $\operatorname{lk}\!\downarrow(x)$ is at least $(\lfloor (n-1)/3 \rfloor - 2)$-connected.*

Proof. The construction of the isomorphism is described in the previous paragraph. The connectivity statement follows from Proposition 2.1.4. $\square$

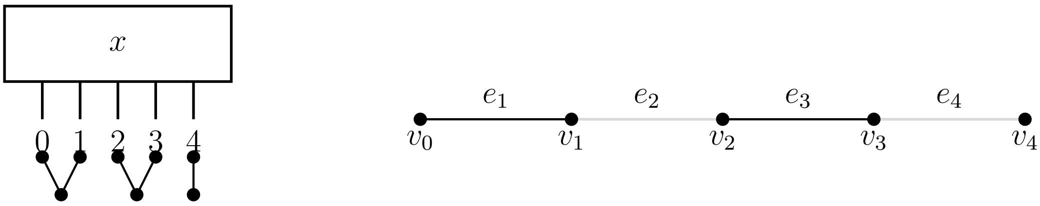

FIGURE 4.4: The correspondence between the descending link $\mathrm{lk}{\downarrow}(x)$ and $\mathcal{M}(L_{n-1})$.

We are now in the position to prove the Theorem of this subsection:

Theorem 4.1.11. *Thompson's group F is of type* F_∞.

Proof. We apply Brown's Criterion as stated in Proposition 1.1.8 to the action of F on X_F. By Corollary 4.1.9 X_F is contractible and by Observation 4.1.4 cell stabilizers are trivial, hence of type F_∞. The filtration $(X_F^{\leq n})$ is a filtration in cocompact subspaces by Corollary 4.1.6.

It follows from the first part of the Morse Lemma 1.2.2 and Observation 4.1.10 that the connectivity of the pair $(X_F^{\leq n+1}, X_F^{\leq n})$ tends to ∞ as n tends to ∞.

We conclude that Thompson's group F is of type F_∞. □

At this point we are done with the blueprint for the proofs to come in the rest of the thesis. We remark, that in the case of F we do not need to invoke Brown's Criterion to give a proof of Theorem 4.1.11. We give the details.

Alternate Proof of Theorem 4.1.11. The group F acts freely and cocompactly on $X_F^{\leq n}$ by Observation 4.1.4 and Corollary 4.1.6. As X_F is contractible by Corollary 4.1.9, it is m-connected for all m. By the second part of the Morse Lemma 1.2.2 and Observation 4.1.10, we have that $X_F^{\leq n}$ is $(\lfloor (n-1)/3 \rfloor - 1)$-connected. Hence by Lemma 1.1.6 the group F is of type $\mathrm{F}_{(\lfloor (n-1)/3 \rfloor)}$ and the space $F \backslash X_F^{\leq n}$ is a witness to that by definition.

The theorem follows, since $(\lfloor (n-1)/3 \rfloor)$ tends to ∞ as n tends to ∞. □

4.2 Thompson's Group V

We turn our attention now to the group V. Recall from Section 3 that V is the group of paired tree diagrams $(\mathcal{T}_-, \rho, \mathcal{T}_+)$, where we allow for the leaves to be permuted by an appropriate permutation ρ.

In complete analogy to the situation for F, we will first introduce the general class of *split-permute-merge diagrams*, then we define a *Stein space* X_V for V and use it to prove that V is of type F_∞. As everything is analogous to the situation before, we will not be overly verbose.

Split-Permute-Merge Diagrams

Recall that we can picture an element $(\mathcal{T}_-, \rho, \mathcal{T}_+)$ of V as in Figure 3.8.

Definition 4.2.1. An *(n,m)-split-permute-merge diagram* $(\mathcal{F}_-,\rho,\mathcal{F}_+)$ is a split-merge diagram $(\mathcal{F}_-,\mathcal{F}_+)$, together with a permutation $\rho\in S_\ell$, where ℓ is the number of leaves in $\mathcal{F}_-$ and $\mathcal{F}_+$. By an *n-split-permute-merge diagram* we will mean a split-permute-merge diagram with n heads and an arbitrary number of feet. A *split-permute-merge diagram* is a (n,m)-split-permute-merge diagram for some n,m. We denote by $\mathtt{S}$ the set of all split-permute-merge diagrams. The symbols $\mathtt{S}_n$ and $\mathtt{S}_{n,m}$ are defined accordingly.

We remark here that we use the same symbol to denote the sets of split-merge and split-permute-merge diagrams. This is justified by the fact that every split-merge diagram is a split-permute-merge diagram, where the permutation is the identity.

The equivalence relation induced by *reduction* and the multiplication we defined for V readily extend, with the obvious restriction for multiplication, to $\mathtt{S}$ as in the case of F. So we have:

Observation 4.2.2. *V is in one-to-one correspondence with the set of $(1,1)$-split-permute-merge diagrams.* □

The notions of *(elementary) splittings* and *(elementary) mergings* also extend to split-permute-merge diagrams.

We remark here that we could continue exactly as for F to prove that V is of type F_∞. We would only have to be more careful when proving, that the corresponding sublevel sets of the Stein space are cocompact for the action of V. But since the symmetric groups, that constitute the main difference between F and V, are finite, this is still true. But we will take another approach here, that is closer to the situation for the braided group V_{br} in Chapter 5. Instead of dealing with a bigger space, we will, in a sense, "put" the symmetric groups into the cell stabilizers. For that we introduce the notion of *dangling*.

Note that we can identify the symmetric group S_n with a subgroup of $\mathtt{S}_{n,n}$ by the map $\rho\mapsto(1_n,\rho,1_n)$, where 1_n again denotes the trivial forest on n roots. We obtain in particular for any $n,m\in\mathbb{N}$ an right action of the group S_m on $\mathtt{S}_{n,m}$ by permuting the feet.

Definition 4.2.3. For $\sigma\in\mathtt{S}_{n,m}$ denote by $[\sigma]$ the orbit of σ under the action of S_m and call $[\sigma]$ an *dangling (n,m)-split-permute-merge diagram.*

We denote by $\mathcal{P}_{n,m}$ the set of all dangling (n,m)-split-permute-merge diagrams. Again the symbols $\mathcal{P}_n$ and $\mathcal{P}$ are defined analogously. Note that S_1 is trivial, so we identify $\mathtt{S}_{n,1}$ with $\mathcal{P}_{n,1}$ and in particular V with $\mathcal{P}_{1,1}$.

Observation 4.2.4. *Let $\sigma\in\mathtt{S}_{n,m}$ and $\tau_1,\tau_2\in\mathtt{S}_{m,\ell}$. If $[\sigma\tau_1]=[\sigma\tau_2]$, then $[\tau_1]=[\tau_2]$.*

Proof. The assumption $[\sigma\tau_1]=[\sigma\tau_2]$ implies that there is a permutation $\xi\in S_\ell$ such that

$$\sigma\tau_1(1_\ell,\xi,1_\ell)=\sigma\tau_2.$$

σ is of the form $(\mathcal{F}_-,\rho,\mathcal{F}_+)$, then left multiplying by $(\mathcal{F}_+,\rho^{-1},\mathcal{F}_-)$ proves the claim. □

We have again a partial ordering on $\mathcal{P}$ that is induced by splitting. That is, for $x = [\sigma_x] \in \mathcal{P}$ and $y \in \mathcal{P}$, we have $x \leq y$ if there is a forest $\mathcal{F}$ with m leaves such that $y = [\sigma_x(\mathcal{F}, \mathrm{id}, 1_m)]$. To check that this is well-defined let σ'_x be another representative for x, i.e. $\sigma'_x = \sigma_x(1_n, \rho, 1_n)$, where n is the number of feet of σ_x and $\rho \in S_n$. Then we can rewrite the product $\sigma_x(1_n, \rho, 1_n)(\mathcal{F}, \mathrm{id}, 1_m)$ as $\sigma_x(\mathcal{F}, \mathrm{id}, 1_m)(1_m, \rho', 1_m)$ which also represents y, as y is a dangling split-permute-merge diagram. It is as easily seen that the notion of elementary splitting is invariant under dangling, so the setup transfers from the situation for F. Hence we can also define the relation $\preceq$. Again this is not transitive, but if $x \preceq y$ and $x \leq z \leq y$, then $x \preceq z \preceq y$.

A Space for V

We study the geometric realization $|\mathcal{P}_1|$ of $\mathcal{P}_1$ in analogy to the situation for F.

Lemma 4.2.5. *Let $x, y \in \mathcal{P}_1$. Then x and y have a least upper bound. If they have a lower bound then they have a greatest lower bound.*

Proof. Let $x = [\sigma], y = [\tau] \in \mathcal{P}_1$. We first need to show that there is a common upper bound for x and y. This is the same as in the proof of Observation 4.1.3.

Suppose now that there are two minimal upper bounds for x and y. Say z and w. Let σ be the dangling $(1, k)$-split-permute-merge diagram (T, ρ, F) and τ be the dangling $(1, \ell)$-split-permute-merge diagram (U, ξ, G). Say T has n leaves and U has m leaves. Then there exists a (k, ℓ)-split-permute merge diagram (H_-, π_1, H_+) such that $[\sigma(H_-, \pi_1, H_+)] = y$ and $[\sigma(H_-, \mathrm{id}, 1_p)] = z$. Here H_- has p leaves. Moreover there is another (k, ℓ)-split-permute-merge diagram (I_-, π_2, I_+) such that $[\sigma(I_-, \pi_2, I_+)] = y$ and $[\sigma(I_-, \mathrm{id}, 1_q)] = w$, where q is the number of leaves of I_-. In particular we have

$$[\sigma(H_-, \pi_1, H_+)] = [\sigma(I_-, \pi_2, I_+)]$$

By Observation 4.2.4 this tells us that $[(H_-, \pi_1, H_+)] = [(I_-, \pi_2, I_+)]$. Since z and w are minimal upper bounds the split-permute-merge diagrams (H_-, π_1, H_+) and (I_-, π_2, I_+) are reduced. But reduced representatives are unique, hence in particular $H_- = I_-$. So $z = w$. We conclude that x and y have a least upper bound.

Finally suppose x and y have maximal lower bounds z and w. Then, of course, x and y are upper bounds of z and w. Let v be the least upper bound of z and w. Then v is a lower bound of x and y and by maximality of z and w, we must have $z = v = w$. $\square$

Corollary 4.2.6. *The poset $(\mathcal{P}_1, \leq)$ is directed and hence $|\mathcal{P}_1|$ is contractible by Lemma 1.3.3.* $\square$

Definition 4.2.7. The *Stein space* X_V for V is the subcomplex of $|\mathcal{P}_1|$ consisting of elementary simplices.

As the notion of *elementary core* for $x \leq y$, $x, y \in \mathcal{P}_1$, carries over to the present setup, we can use the same proofs as in Lemma 4.1.8 and Corollary 4.1.9 to conclude:

Corollary 4.2.8. *X_V is contractible.* $\square$

Note that there is a well-defined simplicial action of V on $|\mathcal{P}_1|$. Since we have identified V with $\mathcal{P}_{1,1}$, we have for a vertex $x = [\sigma_x]$ of $|\mathcal{P}_1|$ and $g = [g] \in V$:

$$gx = [g\sigma_x].$$

Since this action preserves the relations $\leq$ and $\preceq$ it extends to the desired simplicial action.

There is a coarser cell structure on X_V. Recall that the closed interval $[x, y]$ is defined to be $\{z \mid x \leq z \leq y\}$ and that for $x \preceq y$ the interval is contained in X_V. So each vertex in $[x, y]$ is obtained from x by an elementary splitting. If we number the feet of σ, where $x = [\sigma]$, from left to right by $1, \ldots, n$ then there is an simplicial isomorphism from $|[x, y]|$ to the geometric realization of the power set of $\{1, \ldots, n\}$. It is well known that there the simplicies piece together into a cube. We will refer to x as the *bottom* and to y as the *top* of the cube $[x, y]$. It is clear that face of cubes are again cubes and that the intersection of cubes is either empty or again a cube. So X_V carries the structure of a cubical complex. Furthermore it is clear, that the action of V on X_V preserves the cube structure.

Recall that we have the function $h\colon \mathcal{P}_1 \to \mathbb{N}$, that assigns each split-permute-merge diagram its number of feet and that it is invariant under dangling and the V action.

Lemma 4.2.9. *Let $x = [\sigma_x] \in \mathcal{P}_{1,n}$ be a vertex of X_V. Then $\mathrm{Stab}_V(x)$ is isomorphic to S_n.*

Proof. Let $(\mathcal{T}_-, \rho, \mathcal{F}_+)$ be a reduced representative for σ_x, where $\mathcal{T}_-$ is a tree with ℓ leaves, $\mathcal{F}_+$ a forest with n roots and ℓ leaves and $\rho \in S_\ell$. Its inverse is then given by $(\mathcal{F}_+, \rho^{-1}, \mathcal{T}_-)$, call that σ_x^{-1}. Now let $g \in \mathrm{Stab}_V(x)$. We then have $[g\sigma_x] = [\sigma_x]$. In particular this implies that $\sigma_x^{-1} g \sigma_x = (1_n, \xi, 1_n)$ for some $\xi \in S_n$. So, define the homomorphism $\psi\colon \mathrm{Stab}_V(x) \to S_n$ by $g \mapsto \sigma_x^{-1} g \sigma_x$. This is an isomorphism with inverse $\rho \mapsto \sigma_x(1_n, \rho, 1_n)\sigma_x^{-1}$, that clearly depends on the choice of σ_x. □

Corollary 4.2.10. *Let $J \subseteq \{1, \ldots, n\}$. Let $x = [\sigma]$ be a vertex of X_V with $h(x) = n$. Let $F_J^{(n)}$ be an elementary forest. If $y = [\sigma(F_J^{(n)}, \mathrm{id}, 1_{n+|J|})]$, then the stabilizer in V of the cube $[x, y]$ is isomorphic to the subgroup of S_n that stabilizes the set J. In particular all cell stabilizers are finite and hence of type F_∞.*

Proof. Note first, that $g \in V$ stabilizes the cube $[x, y]$ if and only if it stabilizes x and y. So for $g \in \mathrm{Stab}_V(x)$ let ξ be the permutation in S_n as constructed in the proof of Lemma 4.2.9. Then g stabilizes y if and only if

$$[\sigma(1_n, \xi, 1_n)(F_J^{(n)}, \mathrm{id}, 1_{n+|J|})] = [\sigma(F_J^{(n)}, \mathrm{id}, 1_{n+|J|})]$$

By Observation 4.2.4 this tells us, that

$$[(1_n, \xi, 1_n)(F_J^{(n)}, \mathrm{id}, 1_{n+|J|})] = [(F_J^{(n)}, \mathrm{id}, 1_{n+|J|})]$$

But this is equivalent to ξ stabilizing the set J. This proves the corollary. □

Again we will use the filtration of X_V into the h-sublevel sets $X_V^{\leq n}$ and Brown's Criterion (Proposition 1.1.8).

Lemma 4.2.11. *For each $n \geq 1$, the sublevel set $X_V^{\leq n}$ is finite modulo the action of V.*

Proof. As in the situation of F, we observe, that V acts transitively on $\mathtt{S}_{1,k}$, the set of $(1,k)$-split-permute-merge diagrams. Thus there is only one orbit of vertices x with $h(x) = k$ in $X_V^{\leq n}$ for each $1 \leq k \leq n$. Since X_V consists of the elementary simplices, there are only finitely many cubes $C_1, \ldots, C_r$ in $X_V^{\leq n}$ having x as bottom. Hence, if C is a cube in $X_V^{\leq n}$ such that its bottom is in the same orbit as x, then C is in the same orbit as C_i for some $1 \leq i \leq r$. It follows that there are only finitely many orbits of cubes in the sublevel set $X_V^{\leq n}$. □

The last assertion of Brown's Criterion we need to verify is the increasing connectivity properties of the pairs $(X_V^{\leq n+1}, X_V^{\leq n})$. We will do this using the Morse theoretic tools of Section 1.2 with the function h as height and a matching complex (Section 2.1) as model for the descending links.

Descending Links

As in the situation for F, the vertices of the descending link $\mathrm{lk}{\downarrow}(x)$ for $x \in \mathrm{vt}(X_V)$ with $h(x) = n$ are given by the dangling split-permute-merge diagrams y obtained from x by an elementary merging. But we have to be careful and deal with the permutations and dangling.

Let x be a vertex of X_V with $h(x) = n$. Then the descending link $\mathrm{lk}{\downarrow}(x)$ consists of cubes having x as top. The possible elementary mergings now are given by split-permute-merge diagrams of the form $(1_n, \rho, F_J^{(m)})$, where $F_J^{(m)}$ is a forest with n leaves and $m \leq n$ roots and $\rho \in S_n$. If we label the feet of x by $1, \ldots, n$, then the elementary merging does not need to attach carets to a pair of leaves of the form $(i, i+1)$, but rather (i,j), with $i \neq j$, thanks to the permutation ρ. So instead of giving rise to a matching of the linear graph, an elementary merging now corresponds to a matching of an oriented version of the complete graph K_n. Oriented since we need to keep track of whether we merge the ordered pair (i,j) or (j,i).

For any graph Γ there is a version of Γ that we call *oriented*. It has the same vertex set as Γ and for each edge e with ends v, w, the oriented version of Γ has two edges, one pointing from v to w, and one pointing from w to v. If we consider the matching complex of this oriented version of Γ, we will speak of the *oriented matching complex* of Γ and denote it by $\mathcal{M}^o(\Gamma)$.

Recall from Subsection 2.1 that a matching of the oriented complete graph K_n is a collection $\{e_1, \ldots, e_k\}$ of k pairwise disjoint edges. Given an elementary merging as above, the forest $F_J^{(m)}$ consists of $n - m = |J|$ carets. The leaves of each of these are numbered by consecutive numbers, say $(i, i+1)$ and correspond to the leaves labeled $(\rho^{-1}(i), \rho^{-1}(i+1))$ of x. So each of these carets corresponds to an edge in the oriented K_n, namely the edge pointing from the vertex $\rho^{-1}(i)$ to the vertex $\rho^{-1}(i+1)$. See Figure 4.5 for a better idea of the correspondence between elementary mergings and simplices of $\mathcal{M}^o(K_n)$.

Observation 4.2.12. *Let $x \in X_V^{=n}$. The descending link $\mathrm{lk}{\downarrow}(x)$ with respect to h is isomorphic to the oriented matching complex $\mathcal{M}^o(K_n)$ of the complete graph K_n.*

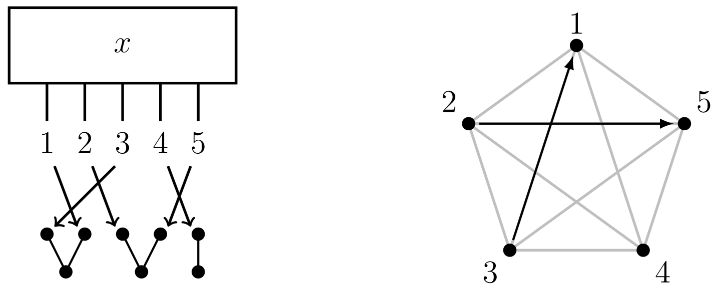

FIGURE 4.5: The correspondence between $\mathrm{vt}(\mathrm{lk}\!\downarrow\!(x))$ and $\mathcal{M}^o(K_n)$.

Proof. The construction of the isomorphism is described in the previous paragraph. □

We are left with verifying the connectivity properties of the space $\mathcal{M}^o(K_n)$. There is an obvious projection $\pi\colon \mathcal{M}^o(K_n) \twoheadrightarrow \mathcal{M}(K_n)$ given by forgetting the orientation on edges. The fiber of this map over a vertex, i.e. an single edge, is clearly discrete. As a k-simplex σ of $\mathcal{M}(K_n)$ consists of $k+1$ disjoint edges, the fiber $\pi^{-1}(\sigma)$ is $(k-1)$-connected, as it is the join of $k+1$ discrete sets. The link $\mathrm{lk}(\sigma)$ is given by the poset of cofaces of σ, i.e. all the collections of pairwise disjoint edges of K_n, that are also disjoint to σ. So $\mathrm{lk}(\sigma)$ is isomorphic to the matching complex of $K_{n'}$, where $n' = n-2(k+1)$. Hence by Proposition 2.1.3 $\mathrm{lk}(\sigma)$ is at least $(\nu(n')-1)$-connected, where $\nu(\ell) = \lfloor(\ell-2)/3\rfloor$.

Corollary 4.2.13. *The oriented matching complex $\mathcal{M}^o(K_n)$ of the complete graph is at least $(\nu(n)-1)$-connected.*

Proof. By the above considerations we only need to verify that $\nu(n') - 1 \geq \nu(n) - k - 2$. Since then we have all the assumptions of Proposition 1.3.5 in place to deduce the Corollary. As $n' = n - 2k - 2$ we have:

$$\left\lfloor \frac{n'-2}{2} \right\rfloor - 1 = \left\lfloor \frac{n-2k-2-2}{2} \right\rfloor - 1 \geq \left\lfloor \frac{n-3k-3-2}{2} \right\rfloor - 1 = \left\lfloor \frac{n-2}{2} \right\rfloor - k - 2$$

□

We are now in the position to prove the Theorem of this subsection:

Theorem 4.2.14. *Thompson's group V is of type* F_∞.

Proof. We use Brown's Criterion as stated in Proposition 1.1.8. By Corollary 4.2.8 X_V is contractible and by Corollary 4.2.10 cell stabilizers are of type F_∞. The filtration $(X_F^{\leq n})$ is a filtration in cocompact subspaces by Lemma 4.2.11.

It follows from the first part of the Morse Lemma 1.2.2 and Observation 4.2.12 together with Corollary 4.2.13 that the connectivity of the pair $(X_V^{\leq n+1}, X_V^{\leq n})$ tends to ∞ as n tends to ∞.

We conclude that Thompson's group V is of type F_∞. □

5 Finiteness Properties of the Braided Thompson's Groups

We will now start to prove our Main Theorem. In this section we deal with the braided Thompson's groups.

As V_{br} and F_{br} are closely related to V and F, we use the same approach as in Section 4. So the first part of this section will be very similar to Subsections 4.1 and 4.2. In particular we will construct a "Stein space" in analogy to the classical case. The main difficulty here will be the analysis of descending links in the Stein space. Contrary to before, matching complexes of graphs will not suffice. That is why we introduced the concept of matching complexes of arcs on a surface in Subsection 2.2. This will allow us to prove our main theorem for the braided Thompson's groups:

Main Theorem (V_{br} and F_{br})**.** *The braided Thompson's groups V_{br} and F_{br} are of type* F_∞.

We start this section by thoroughly introducing the groups V_{br} and F_{br} and remark that this section is based on the article [BFM+14] by Bux, Fluch, Witzel, Zaremsky and the author.

5.1 The Groups V_{br}, F_{br} and Basic Definitions

In analogy to Sections 3 and 4, we define a *braided paired tree diagram* to be a triple $(\mathcal{T}_-, b, \mathcal{T}_+)$ of rooted binary trees $\mathcal{T}_-$ and $\mathcal{T}_+$ with the same number of leaves n and a braid $b \in B_n$. As before, we draw a braided paired tree diagram with $\mathcal{T}_+$ upside down and below $\mathcal{T}_-$ and the braid b connecting the leaves.

We can again define an equivalence relation on braided paired tree diagrams using the notions of reduction and expansion. Denote by ρ_b the permutation in S_n corresponding to the braid $b \in B_n$ and let $(\mathcal{T}_-, b, \mathcal{T}_+)$ be a braided paired tree diagram. We label the leaves of $\mathcal{T}_-$ by $1, \dots, n$ from left to right. Then the $\rho_b^{-1}(i)^{\text{th}}$ leaf of $\mathcal{T}_+$ is labeled i. An *expansion* of $(\mathcal{T}_-, b, \mathcal{T}_+)$ then amounts to the following operation: Pick $1 \leq i \leq n$ and add a caret to the i^{th} leaf of $\mathcal{T}_-$ and to the $\rho_b^{-1}(i)^{\text{th}}$ leaf of $\mathcal{T}_+$ and call the resulting trees $\mathcal{T}'_\pm$. Let $b' \in B_{n+1}$ be the braid that arises from b by "doubling" the i^{th} strand of b. This is saying that we add a strand to b that runs parallel and to the right of the i^{th} strand of b all throughout b. We then call $(\mathcal{T}'_-, b', \mathcal{T}'_+)$ an expansion of $(\mathcal{T}_-, b, \mathcal{T}_+)$. A *reduction* of a braided paired tree diagram is the reverse operation of an expansion. See Figure 5.1 for an example of a reduction.

Now two braided paired tree diagrams are equivalent if and only if one is obtained from the other by a sequence of reductions. It is easy to see, that there is a unique reduced representative of each equivalence class.

Given two braided paired tree diagrams $(\mathcal{T}_-, b, \mathcal{T}_+)$ and $(\mathcal{S}_-, c, \mathcal{S}_+)$, we define a multiplication the following way. By applying repeated expansions, we can find equivalent diagrams $(\mathcal{T}'_-, b', \mathcal{T}'_+)$ and $(\mathcal{S}'_-, c', \mathcal{S}'_+)$ such that $\mathcal{T}'_+ = \mathcal{S}'_-$. The product is then given by the diagram $(\mathcal{T}'_-, b'c', \mathcal{S}'_+)$. This is a well defined operation on equivalence classes of braided paired tree diagrams and a group operation, cf. [Bri07].

Definition 5.1.1. The braided Thompson's group V_{br} is the group of equivalence classes of braided paired tree diagrams with the above multiplication.

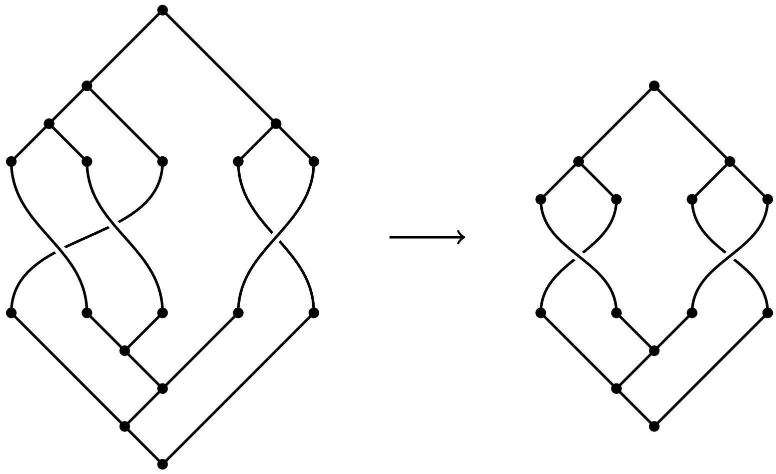

Figure 5.1: A reduction of a braided paired tree diagram.

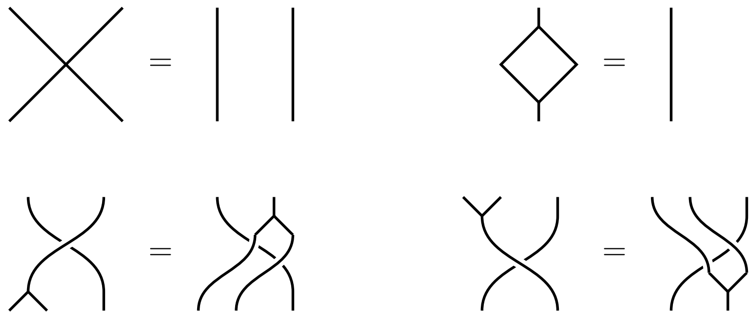

Figure 5.2: Moves to reduce braided paired tree diagrams after stacking.

One can again visualize the multiplication $gh \in V_{br}$ by stacking the picture for g on top of h and the reducing the diagram by certain moves. As in Section 3 a merge followed immediately by a split is the same as doing nothing, also a split immediately followed by a merge. Additionally we can move splits or merges through braids as indicated in Figure 5.2.

If we restrict ourselves to pure braids instead of braids, we end up with a subgroup of V_{br}, namely the group of *pure braided paired tree diagrams*. This group is the braided Thompson's group F_{br}.

Split-Braid-Merge Diagrams

In order to define a Stein space for V_{br} to act on, we will again use a more general class of diagrams than braided paired tree diagrams. As before, we generalize trees in the triples to forests and speak of *split-braid-merge diagrams*. All of this is analogous to the split-permute-merge diagrams for V, cf. Subsection 4.2.

Definition 5.1.2. A *braided paired forest diagram* on n *heads* with m *feet* is a triple $(\mathcal{F}_-, b, \mathcal{F}_+)$, where $\mathcal{F}_-$ is a forest with n roots and $\mathcal{F}_+$ is a forest on m roots. Additionally $\mathcal{F}_-$ and $\mathcal{F}_+$ have the same number of leaves, say ℓ. Then b is a braid in B_ℓ. We equivalently call such an braided paired tree diagram an (n, m)-split-braid-merge diagram and denote the set of (n, m)-split-braid-merge diagrams again

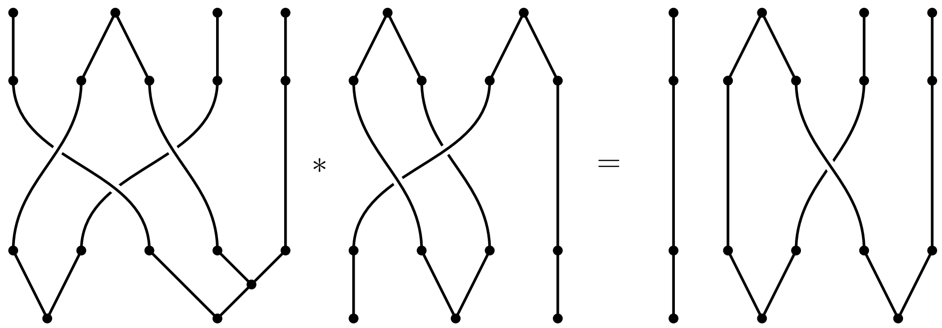

FIGURE 5.3: Multiplication of split-braid-merge diagrams.

by $\mathtt{S}_{n,m}$. An *n-split-braid-merge diagram* is a diagram in $\mathtt{S}_{n,m}$ for some m, and an *split-braid-merge diagram* is in $\mathtt{S}_{n,m}$ for some n and m. The symbols $\mathtt{S}$ and $\mathtt{S}_n$ are defined accordingly.

It should be clear how to generalize the notions of expansion and reduction from braided paired tree diagrams to split-braid-merge diagrams. Again these give an equivalence relation with unique reduced representatives. We will call an equivalence class of split-braid-merge diagrams under reduction again a split-braid-merge diagram. So the elements of V_{br} are in bijection to the $(1,1)$-split-braid-merge diagrams.

The multiplication defined for V_{br} also readily generalizes to general split-braid-merge diagrams, if we make sure that the number of roots match up. This is that we can only multiply two elements, say σ and τ, of $\mathtt{S}$ if $\sigma \in \mathtt{S}_{n_1,m}$ and $\tau \in \mathtt{S}_{m,n_2}$. We then obtain $\sigma\tau \in \mathtt{S}_{n_1,n_2}$. See Figure 5.3 for examples of split-braid-merge diagrams and a multiplication.

We remark that for all $n \in \mathbb{N}$ there is an identity split-braid-merge diagram for the multiplication, it is represented by the braided paired forest diagram $(1_n, \mathrm{id}, 1_n)$, where 1_n denotes the trivial forest on n roots. Given a split-braid-merge diagram $\sigma = (\mathcal{F}_-, b, \mathcal{F}_+)$, its inverse is given by $(\mathcal{F}_+, b^{-1}, \mathcal{F}_-)$. With this notions it is easily verified that $\mathtt{S}$ is a groupoid.

Recall from Subsection 4.1 the notion of an elementary forest, i.e. a forest $\mathcal{F}_J^{(n)}$ for $n \in \mathbb{N}$ and $J \subset \{1, \ldots, n\}$, having n roots and a single caret attached to the i^{th} root for each $i \in J$. We also adopt the notions of *splitting* (resp. *merging*) by a forest $\mathcal{F}$ to the present setting of split-braid-merge diagrams. So for example, an elementary splitting of $\sigma = (\mathcal{F}_-, b, \mathcal{F}_+) \in \mathtt{S}_{n,m}$ by $\mathcal{F}_J^{(m)}$ will be the split-braid-merge diagram $\tau = (\mathcal{F}_-, b, \mathcal{F}_+)(\mathcal{F}_J^{(m)}, \mathrm{id}, 1_\ell)$, where $\ell = m + |J|$.

A particular class of split-braid-merge diagrams will become important later, so we define it here. A *braid-merge diagram* will be a split-braid-merge diagram with no splits. So an n-braid-merge diagram will be a split-braid-merge diagram of the form $\sigma = (1_n, b, \mathcal{F})$, where $b \in B_n$ and $\mathcal{F}$ is a forest with n leaves. If $\mathcal{F}$ is elementary we will call σ an *elementary braid-merge diagram.*

When dealing with F_{br} instead of V_{br}, we will again restrict the braid groups to the subgroups of pure braids. Whenever we want to restrict ourselves to the pure setting, we will add the modifier "pure", i.e. we speak of *pure split-braid-merge diagrams* or *elementary pure n-braid-merge diagrams.*

Dangling Split-Braid-Merge Diagrams

In analogy to the analysis of V (Subsection 4.2), we introduce a right action of the braid group B_m on $\mathtt{S}_{n,m}$ and refer to it as *dangling.* In order to do this, we identify the braid group B_m with a subgroup of $\mathtt{S}_{m,m}$ by the morphism $b \mapsto (1_m, b, 1_m)$ and let this subgroup act by right multiplication. Again we will denote the orbit under the action of B_m by $[\sigma]$, for $\sigma \in \mathtt{S}_{n,m}$ and speak of a *dangling split-braid-merge diagram.*

The set of all dangling split-braid-merge diagrams will be denoted by $\mathcal{P}$ and we copy all the notation from the setting of dangling split-permute-merge diagrams to the current setting. This includes the partial ordering $\leq$ and the relation $\preceq$ on $\mathcal{P}$, given by (elementary) splitting. In particular Observation 4.2.4 holds true in the present setup. We record it again for reference.

Observation 5.1.3. *Let $\sigma \in \mathtt{S}_{n,m}$ and $\tau_1, \tau_2 \in \mathtt{S}_{m,\ell}$. If $[\sigma\tau_1] = [\sigma\tau_2]$, then $[\tau_1] = [\tau_2]$.* □

Clearly all of this also works in the pure case for F_{br}.

5.2 The Stein Space

In this subsection we study the geometric realization $|\mathcal{P}_1|$ of $\mathcal{P}_1$, the V_{br}-set of all dangling $(1, n)$-split-braid-merge diagrams and introduce a Stein space $X_{V_{br}}$ that will prove to be the correct space to determine the finiteness properties of V_{br}. At the end of the subsection we will remark how the construction has to be altered in order to obtain a Stein space $X_{F_{br}}$ for F_{br}.

Lemma 5.2.1. *Let $x, y \in \mathcal{P}_1$. Then x and y have a least upper bound. If they have a lower bound then they have a greatest lower bound.*

Proof. The same proof as for Lemma 4.2.5 works. □

Corollary 5.2.2. *The poset $(\mathcal{P}_1, \leq)$ is directed and hence $|\mathcal{P}_1|$ is contractible by Lemma 1.3.3.* □

Definition 5.2.3. The *Stein space* $X_{V_{br}}$ for V_{br} is the subcomplex of $|\mathcal{P}_1|$ consisting of elementary simplices.

Recall from the setting of F in Subsection 4.1, that for $x \leq y \in \mathcal{P}_1$, the *elementary core* y_0 of y, is the maximal element of the interval $[x, y]$ in $\mathcal{P}_1$, such that $x \preceq y_0$. By the discussion in Subsection 4.2 this notion is again invariant under dangling. Hence the arguments given in the proofs of Lemma 4.1.8 and Corollary 4.1.9 carry over to the current setting of V_{br} and we get:

Corollary 5.2.4. *The space $X_{V_{br}}$ is contractible.* □

Moreover the ideas introduced in the analysis of the Stein space X_V for V carry over to this setup. We only need to be careful when adopting arguments involving the symmetric group S_n, since we now deal with the braid groups B_n instead.

Without any change in the arguments, we still have the structure of a cubical complex on $X_{V_{br}}$, where the cubes are given by the closed intervals $[x, y]$, where

$x \preceq y$. We will again refer to x as the *bottom* and y as the *top* of the cube. We also have an obvious simplicial action of V_{br} on the $X_{V_{br}}$ that respects the cube structure, given by $gx = [g\sigma_x]$ for $g \in V_{br}$ and a dangling $(1,n)$-split-braid-merge diagram $x = [\sigma_x]$.

Let $h\colon \mathrm{vt}(X_{V_{br}}) \to \mathbb{N}$ be the map, that assigns each dangling split-braid-merge diagram its number of feet. Define $X_{V_{br}}^{\leq n}$ to be the full subcomplex of $X_{V_{br}}$ spanned by vertices with $h(x) \leq n$. Analogously define the subcomplexes $X_{V_{br}}^{<n}$ and $X_{V_{br}}^{=n}$. These subcomplexes are invariant under the action of V_{br}.

The proof of the following Lemma is the same as the proof of Lemma 4.2.11, but now it is crucial to the argument that we consider equivalence classes under dangling, which we did not necessarily need in the proof there, since the groups S_n are finite.

Lemma 5.2.5. *For each $n \geq 1$, the sublevel set $X_{V_{br}}^{\leq n}$ is finite modulo V_{br}.* □

This gives the first assertion of Brown's Criterion (Proposition 1.1.8) in order to prove our Main Theorem for the braided Thompson's groups. The second assertion, namely that all cell-stabilizers are of type F_∞ also holds true.

Lemma 5.2.6. *Let $x = [\sigma_x]$ be a vertex of $X_{V_{br}}$, such that $h(x) = n$. Then the stabilizer $\mathrm{Stab}_{V_{br}}(x)$ is isomorphic to B_n.*

Proof. We identify the braid group B_n with its image in $\mathtt{S}_{n,n}$ under the inclusion $b \mapsto (1_n, b, 1_n)$. Let $g \in \mathrm{Stab}_{V_{br}}(x)$. By definition we have $[g\sigma_x] = [\sigma_x]$ and hence $\sigma_x^{-1} g \sigma_x \in B_n$. This gives rise to an morphism

$$\begin{aligned} \Psi\colon \mathrm{Stab}_{V_{br}}(x) &\longrightarrow B_n \\ g &\longmapsto \sigma_x^{-1} g \sigma_x \end{aligned}$$

This is an isomorphism with inverse $b \mapsto \sigma_x b \sigma_x^{-1}$ that depends on σ_x. But this dependence is, thanks to dangling, only up to inner automorphism of B_n. □

Definition 5.2.7. Let $J \subseteq \{1, \ldots, n\}$. Let $b \in B_n$ and ρ_b be the permutation in S_n induced by b. If ρ_b stabilizes J set-wise, call b a *J-stabilizing braid.* Denote by B_n^J the subgroup of B_n of J-stabilizing braids.

Before we continue we need a well known result on braid groups due to work of Arnol′d ([Arn69]), Brieskorn ([Bri73]) and Deligne ([Del72]). The Theorem can also be deduced from work by Brady ([Bra01]).

Theorem 5.2.8. *For all $n \geq 2$, the braid group B_n is of type* F_∞. □

Observation 5.2.9. *The subgroup B_n^J is of finite index in B_n and hence of type F_∞ by Lemma 1.1.5.* □

Corollary 5.2.10. *Let x be a vertex in $X_{V_{br}}$ with $h(x) = n$ and $x = [\sigma_x]$. Let further $\mathcal{F}_J^{(n)}$ be an elementary forest. If y is obtained from x by a splitting by $\mathcal{F}_J^{(n)}$, then the stabilizer of the cube $[x, y]$ is isomorphic to B_n^J. In particular all cell stabilizers are of type F_∞.*

Proof. The second statement follows immediately from the previous observation once we have established the first statement.

Observe the $g \in V_{br}$ stabilizes the cube $[x, y]$ if and only if it stabilizes x and y. For $g \in \mathrm{Stab}_{V_{br}}(x)$ let $b_g \in B_n$ be the braid given by $b_g = \sigma_x^{-1} g \sigma_x$ as in the proof of Lemma 5.2.6. Then g stabilizes y if and only if

$$[\sigma_x b_g(F_J^{(n)}, \mathrm{id}, 1_m)] = [\sigma_x(F_J^{(n)}, \mathrm{id}, 1_m)].$$

By Observation 5.1.3 this is equivalent to

$$[b_g(F_J^{(n)}, \mathrm{id}, 1_m)] = [(F_J^{(n)}, \mathrm{id}, 1_m)].$$

This in turn is equivalent to $b_g \in B_n^J$. So the cube stabilizer equals $\Psi^{-1}(B_n^J)$, where Ψ is the map from the proof of Lemma 5.2.6. Since Ψ is an isomorphism this establishes the first assertion and finishes the proof. $\square$

We are at this point left with analyzing the connectivity properties of pairs $(X_{V_{br}}^{\leq n+1}, X_{V_{br}}^{\leq n})$. We will do this by once more using the Morse theoretic tools from Subsection 1.2 and showing descending links to be highly connected. Contrary to the proofs in Section 4 the correct model for the descending will not be a matching complex of a graph, but rather a *matching complex on a surface*, see Subsection 2.2.

Note that everything we have done so far goes through without essential changes if we consider F_{br} instead of V_{br}.

5.3 Connectivity of Descending Links

In order to analyze the connectivity properties of the filtration steps of the Stein spaces for V_{br}, respectively F_{br}, note that every cube of the respective spaces has a unique vertex that maximizes the function h, namely the top vertex. This leads us to analyzing descending links with respect to h to deduce the connectivity properties using the Morse Lemma 1.2.2. We will do this in the case of V_{br} and mention the necessary changes in argument for the pure case, i.e. the F_{br} case, along the way.

Recall that we identify the vertex set of $X_{V_{br}}$ with the poset $\mathcal{P}_1$ of dangling $(1, n)$-split-braid-merge diagrams. The cubes in $X_{V_{br}}$ are (geometric realizations of) intervals $[y, x]$ where $y \preceq x$, i.e. x is obtained from y by an elementary splitting. For a fixed $x \in \mathcal{P}_1$ the descending star $\mathrm{st}{\downarrow}(x)$ in $X_{V_{br}}$ is given by cubes $[y, x]$ with top x. Given such a cube $C = [y, x]$, let $\mathrm{bot}(C) := y$ denote the map giving the bottom of the cube. This is a bijection from the set of cubes in $\mathrm{st}{\downarrow}(x)$ to the set

$$D(x) := \{y \in \mathcal{P}_1 \mid y \preceq x\}.$$

The cube $C' = [y', x]$ is a face of C if and only if $y' \in [y, x]$ if and only if $y' \geq y$. So bot is an order-reversing poset map. We obtain a description of $\mathrm{lk}{\downarrow}(x)$ with respect to h by considering cubes $[y, x]$ with $y \neq x$ and restricting to $D(x) \setminus \{x\}$. Namely, a simplex in $\mathrm{lk}{\downarrow}(x)$ is a dangling split-braid-merge diagram y with $y \prec x$, the rank of the simplex is given by the number of elementary splits needed to get from y to x (equivalently the number of elementary merges needed to get from x to y). The face relation of $\mathrm{lk}{\downarrow}(x)$ is the reverse of the relation $<$ on $D(x) \setminus \{x\}$. Since $X_{V_{br}}$ is a cubical complex, $\mathrm{lk}{\downarrow}(x)$ is a simplicial complex.

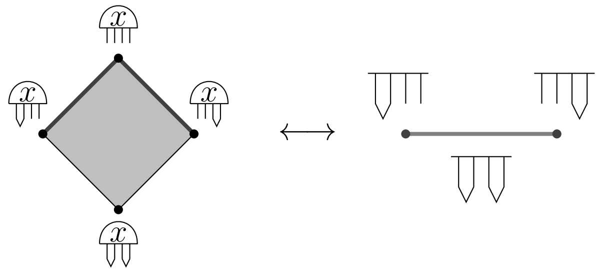

FIGURE 5.4: The correspondence between $\mathrm{lk}{\downarrow}(x)$ and $\mathcal{EB}_n$.

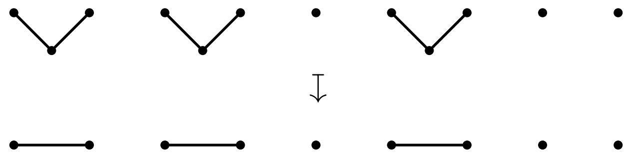

FIGURE 5.5: An example of the bijective correspondence between elementary forests with 9 leaves and simplices of $\mathcal{M}(L_8)$.

Now suppose $h(x) = n$, so x is a dangling $(1, n)$-split-braid-merge diagram. By Observation 5.1.3, dangling and the above considerations, $\mathrm{lk}{\downarrow}(x)$ is isomorphic to the simplicial complex $\mathcal{EB}_n$ of dangling elementary n-braid-merge diagrams $[(1_n, b, \mathcal{F}_J^{(n-|J|)})]$, for $J \neq \emptyset$. The face relation is the reverse of the relation $\leq$ in $\mathcal{P}_n$. See Figure 5.4 for an example.

An analogous argument shows, that in the case of F_{br}, the descending link of an dangling pure $(1, n)$-split-braid-merge diagram x is isomorphic to the simplicial complex $\mathcal{EPB}_n$ of dangling elementary pure braid-merge diagrams.

Using our results on matching complexes on surfaces from Subsection 2.2, we will verify the connectivity properties of the complexes $\mathcal{EB}_n$ (resp. $\mathcal{EPB}_n$). To do so, we will construct projections $\mathcal{EB}_n \to \mathcal{MA}(K_n)$ and use the methods of Quillen, as introduced in Subsection 1.3.

Recall that L_n denotes the linear graph, i.e. the graph with $n+1$ nodes, labeled 1 to $n+1$, and n edges, one connecting the node i to the node $i+1$ for $1 \leq i \leq n$. Let $\mathcal{M}(L_n)$ denote the matching complex of L_n. Note that we changed the numbering of vertices from Subsection 2.1, this is for easier notation later.

Observation 5.3.1. *Elementary forests with n leaves correspond bijectively to simplices of $\mathcal{M}(L_{n-1})$. Under this identification carets correspond to edges. See Figure 5.5.* □

This is already contained in Observation 4.1.10, but we record it again for reference. So for an elementary n-braid-merge diagram $[(1_n, b, \mathcal{F}_J^{(n-|J|)})]$, we can write $[(b, \Gamma)]$, where Γ is a simplex in $\mathcal{M}(L_{n-1})$.

Let $\mathcal{S}$ denote the unit disk. Fix an embedding of L_{n-1} into $\mathcal{S}$. Denote by P the image of the vertex set, so P is a set of n points in $\mathcal{S}$ labeled 1 through n. With this setup we can consider the complex $\mathcal{MA}(K_n)$, the matching complex on the surface $(\mathcal{S}, P)$, and have an induced embedding of simplicial complexes $\mathcal{M}(L_{n-1}) \hookrightarrow \mathcal{MA}(K_n)$. Denote by D_n the n-punctured disc. It is clear that $\mathcal{S} \setminus P = D_n$. It is

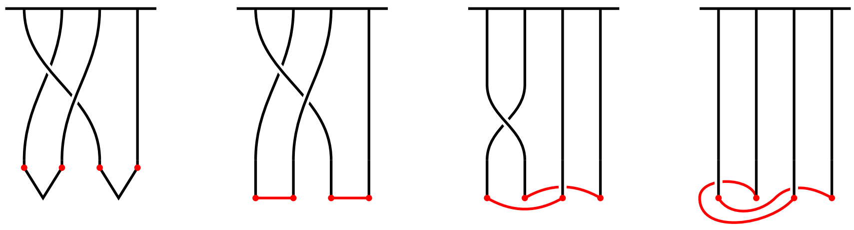

FIGURE 5.6: From braid-merge-diagrams to arc systems. From left to right the pictures show the process of "combing straight" the braid.

a well-known result, that the braid group B_n is the mapping class group of D_n relative ∂D_n (cf. for example [Bir74] or [KT08]). Hence we have an action of B_n on $\mathcal{MA}(K_n)$. It will be convenient to consider this action as a right action, much as dangling, so for $b \in B_n$ and $\sigma \in \mathcal{MA}(K_n)$ we write $(\sigma)b$ for the image of σ under the action of b.

We are now ready to define our desired projection as follows. Viewing $\mathcal{M}(L_{n-1})$ as a subcomplex of $\mathcal{MA}(K_n)$, we can associate to any elementary n-braid-merge-diagram (b,Γ) the arc system $(\Gamma)b^{-1}$ in $\mathcal{MA}(K_n)$. This map is well defined on equivalence classes under dangling, since the arc systems are homotopy classes and B_n is the mapping class group. So we obtain a simplicial map

$$\begin{aligned}\pi\colon \mathcal{EB}_n &\to \mathcal{MA}(K_n)\\ [(b,\Gamma)] &\mapsto (\Gamma)b^{-1}\end{aligned}$$

Note that π is surjective, but not injective.

One can visualize this map by considering the merges as arcs, then "combing straight" the braid and seeing where the arcs are taken, as in Figure 5.6. Note that the resulting simplex $(\Gamma)b^{-1}$ of $\mathcal{MA}(K_n)$ has the same dimension as the simplex $[(b,\Gamma)]$ of $\mathcal{EB}_n$, namely one less than the number of edges in Γ.

The next lemma and proposition are concerned with the fibers of π.

Lemma 5.3.2. *Let E and Γ be simplices in $\mathcal{M}(L_{n-1})$, such that E has one edge and Γ has $e(\Gamma)$ edges. Let $[(b,E)]$ and $[(c,\Gamma)]$ be dangling elementary n-braid-merge diagrams. Suppose that their images under the map π are contained in a simplex of $\mathcal{MA}(K_n)$. Then there exists a simplex in $\mathcal{EB}_n$ that contains $[(b,E)]$ and $[(c,\Gamma)]$.*

Proof. We may assume that $[(b,E)]$ is not contained in $[(c,\Gamma)]$.

There is an action of B_n on $\mathcal{EB}_n$ ("from above"), given by $b'[(c',\Gamma')] = [(b'c',\Gamma')]$. One can check that for each $k \geq 0$, this action is transitive on the k-simplices of $\mathcal{EB}_n$. We can therefore assume without loss of generality that $c = \mathrm{id}$, and Γ is the subgraph of L_{n-1} whose edges are precisely those connecting j to $j+1$, for $j \in \{1,3,\ldots,2e(\Gamma)-1\}$.

Now there is an arc α representing $\pi([(b,E)])$ that is disjoint from Γ. This disjointness ensures that, after dangling, we can assume the following condition on b: for each edge of Γ, say with endpoints j and $j+1$, b can be represented as a braid in such a way that the j^{th} and $(j+1)^{\text{st}}$ strands of b run straight down, parallel to each other, and no strands cross between them. Otherwise the images of $[(b,E)]$ and $[(\mathrm{id},\Gamma)]$ would not form a valid arc system in $\mathcal{MA}(K_n)$. In particular $[(b,\Gamma)] = [(\mathrm{id},\Gamma)]$, so $[(b,\Gamma\cup E)]$ is a simplex in $\mathcal{EB}_n$ with $[(b,E)]$ and $[(\mathrm{id},\Gamma)]$ as faces. □

Proposition 5.3.3. *Let σ be a k-simplex in $\mathcal{MA}(K_n)$ with vertices $v_0,\dots,v_k$. Then*

$$\pi^{-1}(\sigma) = \mathop{\ast}_{j=0}^{k} \pi^{-1}(v_j).$$

In particular $\pi^{-1}(\sigma)$ is k-spherical.

Proof. The equation expresses an equality of abstract simplicial complexes with the same vertex set.

"$\subseteq$": This inclusion is just saying that vertices in $\pi^{-1}(\sigma)$ that are connected by an edge map to distinct vertices under π, which is clear.

"$\supseteq$": The 0-skeleton of $\ast_{j=0}^{k}\pi^{-1}(v_j)$ is automatically contained in $\pi^{-1}(\sigma)$. Now assume that the same is true of the r-skeleton, for some $r \geq 0$. Let τ be an $(r+1)$-simplex in $\ast_{j=0}^{k}\pi^{-1}(v_j)$, and decompose τ as the join of a vertex $[(b,E)]$ and an r-simplex $[(c,\Gamma)]$. By induction, these are both in $\pi^{-1}(\sigma)$, and by Lemma 5.3.2 they share a simplex in $\mathcal{EB}_n$. The minimal dimensional such simplex maps to σ under π, so we are done. □

Recall the numbers $\nu(n) = \lfloor \frac{n-2}{3} \rfloor$ and $\eta(n) = \lfloor \frac{n-1}{4} \rfloor$.

Corollary 5.3.4. *The complex $\mathcal{EB}_n$ is $(\nu(n)-1)$-connected. Hence for any x in $X_{V_{br}}$ with $h(x) = n$, $\mathrm{lk}{\downarrow}(x)$ is $(\nu(n)-1)$-connected.*

Proof. By Theorem 2.2.8 $\mathcal{MA}(K_n)$ is $(\nu(n)-1)$-connected and by Proposition 5.3.3 $\pi^{-1}(\sigma)$ is $(k-1)$-connected, for every k-simplex σ in $\mathcal{MA}(K_n)$. We have argued in Subsection 2.2 that $\mathrm{lk}(\sigma)$ is isomorphic to $\mathcal{MA}(K_{n-2k-2})$, which is $(\nu(n-2k-2)-1)$-connected, again by Theorem 2.2.8. As

$$\nu(n-2k-2)-1 = \left\lfloor \frac{n-2-2(k+1)}{3} \right\rfloor - 1 \geq \left\lfloor \frac{n-2}{3} \right\rfloor - k - 1 - 1 = \nu(n) - k - 2$$

we conclude from Proposition 1.3.5 that $\mathcal{EB}_n$ is $(\nu(n)-1)$-connected. □

Having verified the connectivity properties of $\mathcal{EB}_n$, we turn to the pure case and the complexes $\mathcal{EPB}_n$. As usual, everything runs very similar to the non-pure case. Except we have to be careful with our indices, as L_n has n edges and $n+1$ vertices, contrary to K_n having n vertices.

As in the non-pure case, $\mathrm{lk}{\downarrow}(x)$ is isomorphic to $\mathcal{EPB}_{n+1}$ for $x \in X_{F_{br}}$ having $n+1$ feet. Since in this setting we only consider pure braids, this complex projects onto the complex $\mathcal{MA}(L_n)$, instead of $\mathcal{MA}(K_{n+1})$, using the same construction as before. The rest of the proof, namely the analysis of fibers, goes through without major changes. Hence we get:

Corollary 5.3.5. *The complex $\mathcal{EPB}_{n+1}$ is $(\eta(n)-1)$-connected. Hence for any x in $X_{F_{br}}$ with $h(x)=n+1$, $\mathrm{lk}{\downarrow}(x)$ is $(\eta(n)-1)$-connected.* □

To summarize we conlcude from the Morse Lemma 1.2.2 and the above corollaries:

Corollary 5.3.6. *For each $n \geq 1$, the pair $(X_{V_{br}}^{\leq n}, X_{V_{br}}^{<n})$ is $\nu(n)$-connected and the pair $(X_{F_{br}}^{\leq n}, X_{F_{br}}^{<n})$ is $(\eta(n)-1)$-connected.* □

5.4 Proof of the Main Theorem for V_{br} and F_{br}

We are now ready to proof the Main Theorem of this section:

Main Theorem (V_{br} and F_{br})**.** *The braided Thompson's groups V_{br} and F_{br} are of type* F_∞.

Proof. First consider the action of V_{br} on the space $X_{V_{br}}$, which is contractible by Corollary 5.2.4. By Corollary 5.2.10 we know, that all cell stabilizers for this action are of type F_∞. Finally, each $X_{V_{br}}^{\leq n}$ is finite modulo the action of V_{br} by Lemma 5.2.5 and the connectivity of the pairs $(X_{V_{br}}^{\leq n}, X_{V_{br}}^{<n})$ tends to ∞ as n tends to ∞, by Corollary 5.3.6. From Brown's Criterion (Proposition 1.1.8) we conclude that V_{br} is of type F_∞.

As for F_{br}, a similar argument applies to the action of F_{br} on $X_{F_{br}}$ and hence Brown's Criterion (Proposition 1.1.8) implies that F_{br} is of type F_∞. □

6 Finiteness Properties of the Groups sV

Finally we turn our attention to the higher-dimensional Brin-Thompson groups, that we denote sV for $s \in \mathbb{N}$. Recall from Section 3 that elements of V can be thought of as maps from a sliced up unit interval $[0,1]$ to another unit interval, that is cut into the same number of pieces. For the higher-dimensional groups we no longer think of the unit interval but rather unit s-cubes $[0,1]^s$. The cube $[0,1]^s$ can be halved by hyperplanes in s different directions, as can any resulting piece of such an operation. Analogously to V an element of sV can be described as a sequence of halvings of the domain and codomain and an identification of the resulting pieces by a permutation, maybe affinely deforming the pieces. We will give a formal definition and further intuition for the groups sV in the next subsection.

Using once more a Stein space for the group sV to act on, we will prove

Main Theorem (sV)**.** *The Brin-Thompson group sV is of type* F_∞ *for all s.*

The rest of this section is mainly [FMWZ13] and organized as follows. In Subsection 6.1 we give a formal definition of the groups sV and introduce the poset $\widetilde{\mathcal{P}}$ of dyadic maps. The Stein space sX is defined in Subsection 6.2. In Subsection 6.3 we will use discrete Morse theory to analyze a natural filtration of sX. Finally we will gather all results and deduce the Main Theorem for sV in Subsection 6.4.

6.1 The Groups sV and Basic Definitions

The elements of the Brin-Thompson group sV can be described as dyadic self-maps of s-dimensional cubes. In order to describe and formally define the groups sV, fix from now on a natural number s.

Dyadic Maps and the Groups sV

Recall that a real number is called *dyadic* if it is of the form $k/2^\ell$, where $k \in \mathbb{Z}$ and $\ell \in \mathbb{N}_0$. The *non-dyadic interval* I will be the subspace of $[0,1]$ of non-dyadic numbers. By a *dyadic interval* we will mean a set of the form $\left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right] \cap I$, i.e. it is the intersection of I with a standard dyadic interval (cf. Section 3). Note that a dyadic interval consists entirely of non-dyadic numbers and in particular it is open in I. The *length* of the dyadic interval above is defined to be $1/2^\ell$. Now a *simple dyadic map* is a bijection $f\colon A \to B$ of dyadic intervals, that is affine and of positive slope. Necessarily that slope will be a power of two.

Consider the subspace of non-dyadic points I^s of the standard s-cube, that is the s-fold product of I. We call a subset C of I^s that is a product of s dyadic intervals, a *brick*. The *edges* of C are the individual dyadic intervals, the *volume* of C is the product of their lengths. By definition this will always be a negative power of two. A *dyadic covering* of the cube I^s will be a disjoint covering of I^s by finitely many bricks. Such a dyadic covering is the model for the sequence of halvings of the unit s-cube described before.

For a natural number m we denote by $I^s(m)$ the disjoint union of m copies of I^s.

$$I^s(m) = B_1 \sqcup \cdots \sqcup B_m.$$

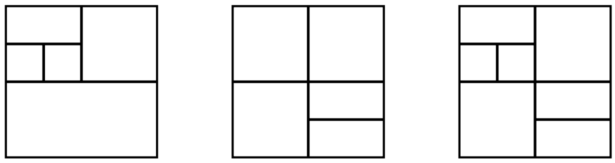

FIGURE 6.1: The dyadic coverings $\mathcal{U}$ and $\mathcal{V}$ and their coarsest common refinement $\mathcal{U} \vee \mathcal{V}$.

Each cube $B_i = I^s$ in this union is a *block*. Note that for now the blocks have a fixed order. A covering $\mathcal{U}$ of $I^s(m)$ is called *dyadic* if $\mathcal{U} = \mathcal{U}_1 \sqcup \cdots \sqcup \mathcal{U}_m$, where $\mathcal{U}_i$ is a dyadic covering of B_i. We denote by $\mathcal{T}_m$ the *trivial* dyadic covering of $I^s(m)$, where each brick is one of the blocks itself. So $\mathcal{T}_m = \{B_1, \ldots, B_m\}$.

Given two dyadic coverings $\mathcal{U}$ and $\mathcal{V}$ of $I^s(m)$ we call $\mathcal{V}$ a *refinement* of $\mathcal{U}$ if $\mathcal{V}$ arises from $\mathcal{U}$ by an additional sequence of halvings or, equivalently, if the bricks of $\mathcal{V}$ disjointly cover the bricks in $\mathcal{U}$. Clearly the set of dyadic coverings of $I^s(m)$ is partially ordered by the refinement relation. There is a unique minimum, namely $\mathcal{T}_m$, and for any two dyadic coverings their join, i.e. a coarsest common refinement, exists. Hence we obtain from Lemma 1.3.2:

Observation 6.1.1. *The set of dyadic coverings of $I^s(m)$ is a lattice with respect to the refinement relation.* □

We say that a pair of dyadic coverings $(\mathcal{U}, \mathcal{V})$ of $I^s(m)$, respectively $I^s(n)$, is *compatible* with a map $f\colon I^s(m) \to I^s(n)$, if for every brick $C \in \mathcal{U}$ the map $f|_C$ is a product of simple dyadic maps and $f(C)$ is a brick in $\mathcal{V}$. This means that f maps every brick in the domain affinely to a brick in the codomain. If such a compatible pair $(\mathcal{U}, \mathcal{V})$ exists, we call f a *dyadic map*. It is easy to see that the set of dyadic maps together with composition forms a group.

Note that every dyadic map f induces a bijection of dyadic coverings $\mathcal{U} \to \mathcal{V}$ for a compatible pair $(\mathcal{U}, \mathcal{V})$. On the other hand a bijection of dyadic coverings gives rise to a dyadic map. Now it is possible for two bijections of dyadic coverings, say $\mathcal{U}_1 \to \mathcal{V}_1$ and $\mathcal{U}_2 \to \mathcal{V}_2$, to determine the same dyadic map. This is the case if and only if $\mathcal{U}_1$ and $\mathcal{U}_2$ have a common refinement $\mathcal{U}$ and $\mathcal{V}_1$ and $\mathcal{V}_2$ have a common refinement $\mathcal{V}$, such that the bijections $\mathcal{U} \to \mathcal{V}$ induced by f_1 and f_2 are the same. See Figure 6.2 for an example.

Definition 6.1.2. The Brin-Thompson group sV is the group of dyadic self maps of I^s with multiplication given by composition.

The Poset $\mathcal{P}_1$

Next we want to define a natural poset $\mathcal{P}_1$ on which sV acts and that has been studied before. In particular it is the space used in [KMPN13] to prove that $2V$ and $3V$ are of type F_∞. We need some further notation.

Denote by $\widetilde{\mathcal{P}}_{m,n}$ the set of dyadic maps $f\colon I^s(m) \to I^s(n)$. $\widetilde{\mathcal{P}}$ will denote the union of the $\widetilde{\mathcal{P}}_{m,n}$ where m and n range over the positive integers. Further $\widetilde{\mathcal{P}}_m$ will

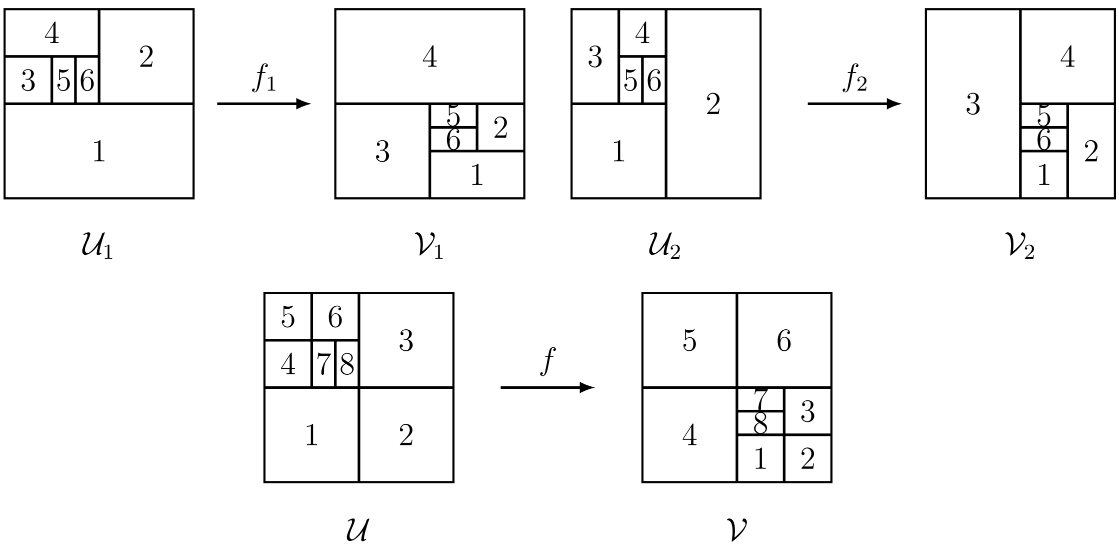

FIGURE 6.2: Two pairs of dyadic coverings and the induced maps. The numbers on the bricks represent the bijections $\mathcal{U}_i \to \mathcal{V}_i$. The common refinements $\mathcal{U}$ and $\mathcal{V}$ showing that the dyadic maps f_1 and f_2 coincide.

denote the subset of $\widetilde{\mathcal{P}}$ consisting of all dyadic maps where the domain consists of m blocks. In particular $sV = \widetilde{\mathcal{P}}_{1,1}$.

Clearly the group sV acts on $\widetilde{\mathcal{P}}_1$ by precomposition, i.e. $f^g = f \circ g$ for $g \in sV$ and $f \in \widetilde{\mathcal{P}}_1$. We will think of this as a left-action. There is also a right-action on $\widetilde{\mathcal{P}}_{m,n}$ by the symmetric group S_n, for each n, permuting the blocks in the codomain. This is analogous to the notion of "dangling" introduced for V and V_{br}. Denote the quotient $\widetilde{\mathcal{P}}_{m,n}/S_n$ by $\mathcal{P}_{m,n}$. We obtain an element of $\mathcal{P}_{m,n}$ from an element in $\widetilde{\mathcal{P}}_{m,n}$ by forgetting the order of the blocks in the codomain. Again we set

$$\mathcal{P} := \bigcup_{n,m \geq 1} \mathcal{P}_{m,n} \quad \text{and} \quad \mathcal{P}_m := \bigcup_{n \geq 1} \mathcal{P}_{m,n}$$

Hence the poset $\mathcal{P}_1$ is the set of all dyadic maps where the domain consists of a single block, and the codomain of arbitrarily many unordered blocks.

We observe that $\widetilde{\mathcal{P}}_{1,n}$ is an sV-invariant subset of $\widetilde{\mathcal{P}}_1$, and that the action of sV commutes with the action of S_n. Hence we get an action of sV on $\mathcal{P}_{1,n}$ for all n. In particular the action of sV on $\widetilde{\mathcal{P}}_1$ induces an sV-action on $\mathcal{P}_1$.

In order to define a poset structure on $\mathcal{P}_1$ we introduce the notion of "splitting". A dyadic map $z\colon I(m) \to I(n)$ is called a *splitting (along $\mathcal{U}$)* if it is compatible with a pair of dyadic coverings of the form $(\mathcal{U}, \mathcal{T}_n)$. The splitting z is *non-trivial* if $n > m$. In other words a non-trivial splitting is obtained by splitting up some cubes in the domain along a dyadic covering and not putting them back together in the codomain. See Figure 6.3 for an example. The inverse of a splitting (along $\mathcal{U}$) is called a *merging (along $\mathcal{U}$)*.

We define a partial oder $\leq$ on $\widetilde{\mathcal{P}}$ by saying that $x < y$ if there exists a non-trivial splitting z such that $y = z \circ x$. That is, $x < y$ if y is obtained from x by a non-trivial

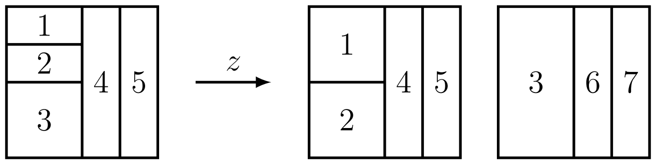

FIGURE 6.3: A splitting $z\colon I^2(1) \to I^2(2)$ along a horizontal line.

splitting. The induced order on $\mathcal{P}$ will also be denoted $\leq$, in particular $\mathcal{P}_1$ is ordered by $\leq$.

Definition 6.1.3. The function $t\colon \mathcal{P} \to \mathbb{N}$ counts for each $x \in \mathcal{P}$ the number of blocks in the codomain. That is $t(x) = n$ if $x \in \mathcal{P}_{m,n}$ for some m.

The poset $\mathcal{P}_1$ is filtered by the function t in sublevel sets of the form

$$\mathcal{P}_1^{\leq n} = \bigcup_{1 \leq k \leq n} \mathcal{P}_{1,k}.$$

Note that for elements of $\mathcal{P}_1^{\leq n}$ the number of blocks in the codomain is limited to n and that there are only finitely many splittings into n blocks for any $I^s(m)$ with $m \leq n$. Therefore the geometric realization $|\mathcal{P}_1^{\leq n}|$ is locally finite.

We observe

Observation 6.1.4. *The poset $\widetilde{\mathcal{P}}_1$ is directed. Hence by Lemma 1.3.3 $|\widetilde{\mathcal{P}}_1|$ and $|\mathcal{P}_1|$ are contractible.* □

Observation 6.1.5. *The action of sV on $\widetilde{\mathcal{P}}_1$ is free. Thus for each vertex x in $|\mathcal{P}_1|$ the stabilizer $\mathrm{Stab}_{sV}(x)$ is a symmetric group and hence finite. Consequently all cell stabilizers are finite and of type F_∞.* □

Observation 6.1.6. *The action of sV on $\mathcal{P}_1^{=1}$ is transitive and $|\mathcal{P}_1^{\leq n}|$ is locally finite. Hence $|\mathcal{P}_1^{\leq n}|$ is finite modulo sV.* □

These observations suggest that the filtration in t-sublevel sets of $|\mathcal{P}_1|$ can be used to show that sV is of type F_∞ by invoking Brown's Criterion (Proposition 1.1.8). It would suffice to show that the connectivity of the pair $(|\mathcal{P}_1^{\leq n+1}|, |\mathcal{P}_1^{\leq n}|)$ tends to ∞ as n tends to ∞. This is precisely what the authors of [KMPN13] did in the cases of $s = 2, 3$. For increasing s it turns out, that the space $|\mathcal{P}_1|$ is too big to efficiently analyze the connectivity properties of the filtration steps. The main point in the approach here is once more to restrict to a Stein space sX of $|\mathcal{P}_1|$.

6.2 The Stein Space

Contrary to all other cases considered so far, we need not only the notion of elementary splittings, but also of very elementary splittings. The reason will become clear in Subsection 6.3. In the present case an elementary splitting of an s-cube will amount to halving the cube at most once in any given direction. A very elementary splitting will be such that the s-cube is halved at most once.

Definition 6.2.1. We call a brick C of a dyadic covering $\mathcal{U}$ *elementary* if every edge of C has length at least $1/2$. An elementary brick is *very elementary* if it has volume at least $1/2$. The dyadic covering $\mathcal{U}$ will be called *(very) elementary* if each brick of $\mathcal{U}$ has this property. And a splitting along $\mathcal{U}$ is *(very) elementary* if $\mathcal{U}$ is.

Suppose $x, y \in \mathcal{P}$ such that y can be obtained from x by an elementary splitting, then we denote this by $x \preceq y$. If the splitting is non-trivial, we write $x \prec y$. For the very elementary relations we will use $\sqsubseteq$ and $\sqsubset$. Note that the relations $\preceq$ and $\sqsubseteq$ are not transitive. The length of a chain of very elementary splittings for example is bounded by the number of blocks, since we may split each block only once. But if $x_1 \leq x_2 \leq x_3$ and $x_1 \preceq x_3$ then we have $x_1 \preceq x_2$ and $x_2 \preceq x_3$. The same holds for $\sqsubseteq$.

Observation 6.2.2. *It is clear that the action of sV on $\mathcal{P}_1$, being precompositions of maps, respects the relations $\leq, \preceq, \sqsubseteq$.* □

Clearly $I^s(m)$ has a unique maximal elementary covering $\mathcal{E}$ by $m \cdot 2^s$ bricks all of which have volume 2^{-s}. It arises by splitting each block once in each of the s dimensions. An arbitrary covering of $I^s(m)$ is elementary if and only if $\mathcal{E}$ is a refinement of it.

Recall that the closed interval $[x, y]$ in $\mathcal{P}$ is defined to be

$$[x, y] := \{w \in \mathcal{P} \mid x \leq w \leq y\}.$$

Analogously the open and half-open intervals are defined. Call an interval $[x, y]$ *(very) elementary* if $x \preceq y$ (resp. $x \sqsubseteq y$). A simplex of $|\mathcal{P}_1|$ is *(very) elementary* if there is a (very) elementary interval that contains each of its vertices.

Definition 6.2.3. The *Stein space sX for sV* is the subcomplex of $|\mathcal{P}_1|$ consisting of the elementary simplices.

The following Lemma is the key to the contractibility of the Stein space.

Lemma 6.2.4. *Let $x, y \in \mathcal{P}_1$ with $x \leq y$. There exists a unique $y_0 \in [x, y]$ such that $x \preceq y_0$ and for any $x \preceq w \leq y$, we have $w \leq y_0$. Moreover if $x < y$, then $x < y_0$.*

Proof. Let $m := t(x)$ and $\widetilde{x}$ be a representative of x in $\widetilde{\mathcal{P}}_1$. Since y is obtained from $\widetilde{x}$ by a splitting, there is a dyadic covering $\mathcal{U}$ of $I^s(m)$ such that the splitting is along $\mathcal{U}$. Let $\mathcal{E}$ denote the maximal elementary covering of $I^s(m)$. By Observation 6.1.1 the meet $\mathcal{E} \wedge \mathcal{U}$ exists and is unique. The element y_0 is obtained from $\widetilde{x}$ by splitting along $\mathcal{E} \wedge \mathcal{U}$. Since $\mathcal{E}$ is a refinement of $\mathcal{E} \wedge \mathcal{U}$, we have $x \preceq y_0$. If $x \preceq w \leq y$ then w is obtained by a splitting along a coarsening $\mathcal{V}$ of $\mathcal{E}$ and $\mathcal{U}$. But since $\mathcal{E} \wedge \mathcal{U}$ is the meet, it is a refinement of $\mathcal{V}$ and hence $w \leq y_0$. Lastly $\mathcal{E} \wedge \mathcal{U}$ is non-trivial if $\mathcal{U}$ is. □

For $x \leq y$ we call the y_0 from the lemma the *elementary core of y with respect to x* and write $\mathrm{core}_x(y) := y_0$. We will omit the subscript if it is understood from the context. Note that for $y_1 \leq y_2$ we have $\mathrm{core}(y_1) \leq \mathrm{core}(y_2)$. That is taking elementary cores respects the poset relation. See Figure 6.4 for an example of an elementary core.

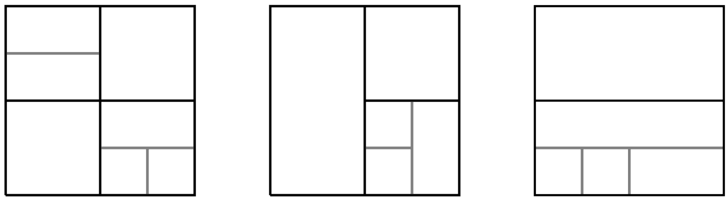

FIGURE 6.4: A non-elementary dyadic covering, for $s = 2$. The non-gray lines indicate the elementary core.

In order to show that sX is contractible we will take [Bro92] as an orientation. The proof of the following Lemma is essentially the same as the proof of the Lemma in Section 4 of [Bro92].

Lemma 6.2.5. *Suppose $x < y$ such that $x \not\preceq y$. Then $|(x, y)|$ is contractible.*

Proof. Let $w \in (x, y]$. Then $\mathrm{core}(w) \neq y$ since $x \not\preceq y$. On the other hand $\mathrm{core}(w) \neq x$ since $x < w$. In fact we have $\mathrm{core}(w) \in (x, y)$. By the previous paragraph we also know that $\mathrm{core}(w) \leq \mathrm{core}(y)$. Now the inequalities $w \geq \mathrm{core}(w) \leq \mathrm{core}(y)$ provide a contraction of $|(x, y)|$ by Lemma 1.3.4. □

As Brown did for the Stein space of V in [Bro92], cf. Section 4, we will now build up from sX to $|\mathcal{P}_1|$ to show that sX is contractible.

Corollary 6.2.6. *The Stein space sX is contractible for all s.*

Proof. By Observation 6.1.4, $|\mathcal{P}_1|$ is contractible. We will now build up from sX to $|\mathcal{P}_1|$ without changing the homotopy type.

For a closed interval $[x, y]$ with $x \not\preceq y$ define $r := t(y) - t(x)$. We attach the contractible (intervals are directed subposets) subspaces $|[x, y]|$ in increasing order of r-value. Then, when we attach $|[x, y]|$, we attach it along $|[x, y)| \cup |(x, y]|$. But this is the suspension of $|(x, y)|$ and so it is contractible by Lemma 6.2.5. Hence we attach only contractible subspaces along contractible subspaces and conclude that we never change the homotopy type. Since $|\mathcal{P}_1|$ is contractible, so is sX. □

As before we have a filtration of sX by t-sublevel sets $(sX^{\leq n})_n$, where t is counting blocks in the codomain. Note that our sublevel sets are all invariant under the action of sV, since it does not change the value of t.

We will continue to show that the filtration $(sX^{\leq n})_n$ of sX satisfies the hypotheses of Brown's Criterion (Proposition 1.1.8). Thanks to Observation 6.1.5, Observation 6.1.6 and Corollary 6.2.6, we need only to verify that the connectivity of the pairs $(sX^{\leq n+1}, sX^{\leq n})$ tends to ∞ as n tends to ∞.

We will verify this by using discrete Morse Theory (cf. Subsection 1.2). The idea is to treat t as a height function and inspect descending links.

6.3 Connectivity of Descending Links

Recall the basic Morse-theoretic setup from Subsection 1.2. Fix a vertex x in the Stein space sX, say with $t(x) = n$. We call n the *height* of x. The *descending link* $\mathrm{lk}{\downarrow}(x)$ of x is the intersection of $\mathrm{lk}(x)$ with $X^{<n}$. By definition of the relation $\leq$

each neighbor of x in sX has a different t-value, so t is indeed a height function. Therefore we may obtain $sX^{\leq n}$ from $sX^{<n}$ by gluing in each vertex at height n along its descending link.

Fix now a vertex x in sX of height n and let $L(x) := \mathrm{lk}{\downarrow}(x)$. Considering $L(x)$ as a subcomplex of $|\mathcal{P}_1|$, a simplex in $L(x)$ is given as a chain $y_k < \cdots < y_0 < x$ with $y_k \prec x$, as sX is the subcomplex of elementary simplices. We first consider the subcomplex $L_0(x)$ of $L(x)$ consisting of the very elementary simplices, i.e. $y_k \sqsubset x$. There is a natural projection of $L_0(x)$ to a matching complex.

As discussed in Subsection 2.1 the *matching complex* $\mathcal{M}(\Gamma)$ of a graph Γ is the simplicial complex with a k-simplex for every collection $\{e_0, \ldots, e_k\}$ of $k+1$ pairwise disjoint edges. The face relation in $\mathcal{M}(\Gamma)$ is given by inclusion. If the edges of Γ are oriented, we obtain an *oriented matching complex* $\mathcal{M}^o(\Gamma)$. Clearly there is a projection $\mathcal{M}^o(\Gamma) \twoheadrightarrow \mathcal{M}(\Gamma)$ of matching complexes for every oriented graph Γ by forgetting the orientation on the edges. We discussed this already in Subsection 4.2.

The specific graphs that we need to consider here are generalizations of complete graphs. For $s \in \mathbb{N}$, let sK_n be the graph with n nodes and s edges between any two distinct nodes. Color the edges of sK_n with colors 1 to s such that any two distinct nodes have exactly one edge of each color between them. If we fix a numbering of the nodes of sK_n we obtain a projection $s\pi\colon sK_n \to K_n$ by mapping an edge with endpoints i and j to the unique edge in K_n with endpoints i and j. As disjoint edges map to disjoint edges under $s\pi$, this induces a map $\mathcal{M}(s\pi)$ between the matching complexes.

Recall that $\mathcal{M}(K_n)$ is $(\nu(n)-1)$-connected, where $\nu(\ell) := \lfloor \frac{\ell-2}{3} \rfloor$, by Proposition 2.1.3.

Lemma 6.3.1. *$\mathcal{M}(sK_n)$ is $(\nu(n)-1)$-connected, as is $\mathcal{M}^o(sK_n)$.*

Proof. Consider the map $\mathcal{M}(s\pi)\colon \mathcal{M}(sK_n) \to \mathcal{M}(K_n)$. Let σ be a k-simplex of $\mathcal{M}(K_n)$. The fiber $\mathcal{M}(s\pi)^{-1}(\sigma)$ is the join of the fibers of the vertices of σ, that are discrete. So in particular $\mathcal{M}(s\pi)^{-1}(\sigma)$ is homotopy equivalent to a wedge of k-spheres. In particular it is $(k-1)$-connected. We observe that links in $\mathcal{M}(K_n)$ are again matching complexes of complete graphs. In case of σ the link is $\mathcal{M}(K_m)$, where $m = n - 2(k+1)$ and hence is $(\nu(m)-1)$-connected by Proposition 2.1.3. The hypotheses of Proposition 1.3.5 are satisfied once we verify that $\nu(m) - 1 \geq \nu(n) - k - 2$. A quick calculation shows this to be true, hence $\mathcal{M}(sK_n)$ is $(\nu(n)-1)$-connected.

For the second claim, we consider the map $\mathcal{M}^o(sK_n) \twoheadrightarrow \mathcal{M}(sK_n)$. The fibers of this map are similarly spherical of the right dimension, as are the links again of the form $\mathcal{M}(sK_m)$. So we conclude again by Proposition 1.3.5 that $\mathcal{M}^o(sK_n)$ is $(\nu(n)-1)$-connected. $\square$

Now every vertex $y \in L_0(x)$, say with $t(y) = m$, is obtained from x by a non-trivial very elementary merging. This merging is given by a non-trivial very elementary covering $\mathcal{U}$ of m blocks whose n bricks are numbered by the blocks of x. Two such coverings define the same element y if and only if they differ by a permutation of the blocks (recall that we factored out the action of the symmetric group on the blocks). We denote by VE_n the set of very elementary dyadic coverings consisting of n labeled bricks up to permutation of the blocks. By the previous discussion we

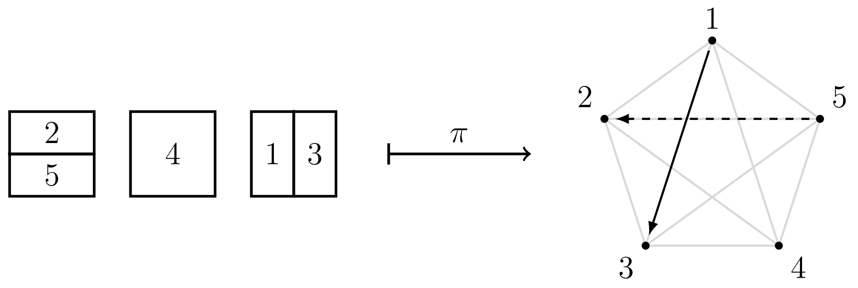

FIGURE 6.5: An example of $\pi\colon VE_n \to \mathcal{M}^o(sK_n)$ in the case $n = 5$ and $s = 2$. The solid arrow corresponds to a merge along a vertical face, and the dashed arrow corresponds to a merge along a horizontal face.

have a one-to-one correspondence between $L_0(x)$ and VE_n. We turn VE_n into a poset using the order induced by this identification.

Corollary 6.3.2. *VE_n, and therefore $L_0(x)$, is isomorphic to $\mathcal{M}^o(sK_n)$. Hence both are $(\nu(n) - 1)$-connected.*

Proof. The connectivity statement follows from Lemma 6.3.1 once we define an isomorphism of ordered sets $VE_n \to \mathcal{M}^o(sK_n)$.

Let $\mathcal{U} \in VE_n$ be a covering of $I^s(m)$ with the n bricks labeled 1 to n. Since $\mathcal{U}$ is very elementary each of the blocks consists of at most two bricks. Each such block defines an oriented edge of sK_n as follows. The bricks of such a block are given by halving exactly one of the dyadic intervals I in the product I^s, say the k-th. If the first brick, corresponding to the half $[0, \frac{1}{2}]$, is labeled i and the second is labeled j, then the block defines the edge of sK_n that points from i to j and has color k. See Figure 6.5 for an example.

This procedure yields the desired isomorphism of ordered sets. $\square$

Next we have to show that $L(x)$ is highly connected. We will do this by building up from $L_0(x)$ to $L(x)$ along highly connected links to apply the Morse Lemma 1.2.2. If $s = 1$ we have $L_0(x) = L(x)$, so we assume $s > 1$ in what follows.

Note first that there is an analogous combinatorial description of $L(x)$ as for $L_0(x)$. Each vertex in $L(x)$ is obtained from x by an non-trivial elementary merging. Replacing “very elementary” by “elementary” above, we obtain that the poset E_n of elementary coverings by n labeled bricks is isomorphic to $L(x)$.

We now describe the height function, that tells us in which order to glue in simplices to build up from $L_0(x)$ to $L(x)$. For any $\mathcal{U} \in E_n$, the volume of any brick is at least $1/2^s$, since $\mathcal{U}$ is elementary. Let c_i be the number of bricks in $\mathcal{U}$ of volume $1/2^i$ for each $0 \leq i \leq s$. We define c to be the lexicographically ordered function $c = (c_s, c_{s-1}, \ldots, c_3, c_2)$. Note that we do not include the bricks of volume 1 or 1/2. This will be crucial to the arguments. Denote by b the number of blocks of $\mathcal{U}$. The *height* h of $\mathcal{U}$ is defined to be $h = (c, b)$, ordered lexicographically.

Observation 6.3.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be in E_n such that $\mathcal{X} < \mathcal{Y}$. By the induced order on E_n this means that $\mathcal{Y}$ is obtained by a splitting from $\mathcal{X}$. In particular we have*

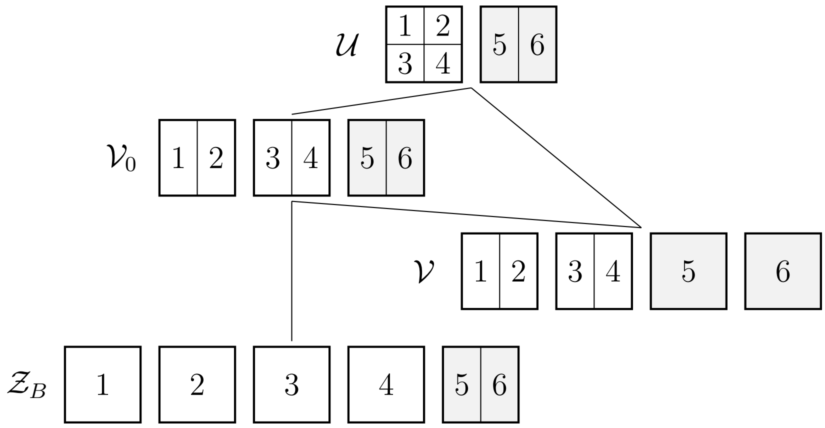

FIGURE 6.6: A step in building up from VE_6 to E_6 as described in the proof of Lemma 6.3.5. The block B of the covering $\mathcal{U}$ and its images under the various splittings are highlighted.

$c(\mathcal{X}) \geq c(\mathcal{Y})$ and $b(\mathcal{X}) < b(\mathcal{Y})$. Hence $h(\mathcal{X}) < h(\mathcal{Y})$ if and only if $c(\mathcal{X}) = c(\mathcal{Y})$ and $h(\mathcal{X}) > h(\mathcal{Y})$ if and only if $c(\mathcal{X}) > c(\mathcal{Y})$. □

Note that equality in c-value is only possible since we excluded c_0 and c_1 in the definition of c.

Fix a vertex $\mathcal{U}$ in $E_n \backslash VE_n$. We will denote the descending link of $\mathcal{U}$ with respect to h by $\mathrm{lk}{\downarrow_h}(\mathcal{U})$. By Observation 6.3.3 there are two types of vertices in $\mathrm{lk}{\downarrow_h}(\mathcal{U})$. First we could have $\mathcal{U} > \mathcal{V}$ which implies $c(\mathcal{U}) = c(\mathcal{V})$. We will call the full subcomplex of $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ spanned by these vertices the *downlink*. Secondly we can have $\mathcal{U} < \mathcal{V}$ which implies $c(\mathcal{U}) > c(\mathcal{V})$. The full subcomplex of $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ spanned by these vertices will be called the *uplink*. Compare this to the setup of the proof of Proposition 2.1.3.

Observation 6.3.4. *Vertices $\mathcal{V}$ in the downlink and $\mathcal{W}$ in the uplink automatically satisfy $\mathcal{V} < \mathcal{W}$. Hence $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is a join of uplink and downlink.* □

So we may consider uplink and downlink separately.

Lemma 6.3.5. *If $\mathcal{U}$ has a block with precisely two bricks, then the uplink of $\mathcal{U}$ is contractible. Hence $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is contractible.*

Proof. Let B be a block of $\mathcal{U}$ with two bricks. Note that splitting just B does not yield a vertex with lower height than $\mathcal{U}$ by definition of the function c. For an arbitrary vertex $\mathcal{V}$ of the uplink we have $\mathcal{V} > \mathcal{U}$ and $c(\mathcal{V}) < c(\mathcal{U})$ and it is obtained from $\mathcal{U}$ by a splitting. Define the covering $\mathcal{V}_0$ as follows (see Figure 6.6). $\mathcal{V}_0$ is obtained from $\mathcal{U}$ by doing the same splittings as for $\mathcal{V}$ except that B is not split (whether it was split for $\mathcal{V}$ or not). Then clearly $\mathcal{V}_0 > \mathcal{U}$ and $c(\mathcal{V}_0) < c(\mathcal{U})$ since the same hold for $\mathcal{V}$ and whether B is split or not does not change the c-value. Hence $\mathcal{V}_0$ is a vertex in the uplink of $\mathcal{U}$. Let $\mathcal{Z}_B$ be the maximal elementary splitting of $\mathcal{U}$ that does not split B, which is clearly a vertex in the uplink. We have $\mathcal{V}_0 \leq \mathcal{Z}_B$ for all vertices $\mathcal{V}$ in the uplink. We obtain the inequalities $\mathcal{V} \geq \mathcal{V}_0 \leq \mathcal{Z}_B$, which provide a contraction of the uplink of $\mathcal{U}$ by Lemma 1.3.4. □

For $\ell \in \mathbb{Z}$ define $\chi(\ell) := \lfloor \frac{\ell - 2}{2^s} \rfloor$. For a fixed s, note that $\chi(\ell)$ increases monotonically to ∞ as ℓ tends to ∞.

Lemma 6.3.6. *If $\mathcal{U}$ has no block with precisely two bricks, then $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is at least $(\chi(n)-2)$-connected.*

Proof. We call a block of $\mathcal{U}$ *big* if it has more than two bricks and *small* if it has only one brick. Let k_b be the number of big blocks and k_s the number of small blocks. By assumption $k_s + k_b = m$ equals the number of blocks of $\mathcal{U}$.

The uplink of $\mathcal{U}$ is at least $(k_b - 2)$-connected, since splitting any big block in any way produces a vertex with lower height. So each big block contributes a non-empty join factor to the uplink. The downlink of $\mathcal{U}$ consists of vertices that are obtained from $\mathcal{U}$ by merges and have lower height. By Observation 6.3.3 this amounts to merging small blocks, since a merge involving a big block would change the c-value. For the same reason each vertex in the downlink arises as a very elementary merging. So the downlink of $\mathcal{U}$ is isomorphic to VE_{k_s} and hence by Corollary 6.3.2 $(\nu(k_s)-1)$-connected. This implies that $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is $(k_b+\nu(k_s)-1)$-connected. As n is the number of bricks in $\mathcal{U}$, we have $n \leq 2^s k_b + k_s$.

Since we assumed $s > 1$, we have $2^s > 3$ and obtain

$$\begin{aligned} k_b + \nu(k_s) - 1 \geq k_b + \left\lfloor \frac{k_s - 2}{2^s} \right\rfloor - 1 \geq k_b + \frac{k_s - 2}{2^s} - 2 \\ = \frac{2^s k_b + k_s - 2}{2^s} - 2 \geq \frac{n-2}{2^s} - 2 \geq \chi(n) - 2\,. \end{aligned}$$

We conclude that $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is at least $(\chi(n)-2)$-connected. □

Corollary 6.3.7. *If $s = 1$ then E_n and hence $L(x)$ is $(\nu(n)-1)$-connected. If $s > 1$ then E_n and hence $L(x)$ is at least $(\chi(n)-1)$-connected.*

Proof. The case $s = 1$ is done, since then $E_n = VE_n$. Suppose $s > 1$. Then $\chi \leq \nu$, so VE_n is at least $(\chi(n)-1)$-connected (Corollary 6.3.2). By Lemmas 6.3.5 and 6.3.6, $\mathrm{lk}{\downarrow_h}(\mathcal{U})$ is $(\chi(n)-2)$-connected for all $\mathcal{U} \in E_n \setminus VE_n$. We conclude from the first part of the Morse Lemma 1.2.2 that E_n is at least $(\chi(n)-1)$-connected. □

Corollary 6.3.8. *For each $n \geq 1$, the pair $(sX^{\leq n}, sX^{<n})$ is $\chi(n)$-connected for $s > 1$ and the pair $(1X^{\leq n}, 1X^{<n})$ is $\nu(n)$-connected.*

Proof. Let x be a vertex in $sX^{=n}$. By Corollary 6.3.7 the descending link $\mathrm{lk}{\downarrow}(x)$ of x in sX is at least $(\chi(n)-1)$-connected for $s > 1$ or $(\nu(n)-1)$-connected for $s = 1$. Our claim now follows from the Morse Lemma 1.2.2. □

6.4 Proof of the Main Theorem for sV

We are now ready to prove the Main Theorem for the groups sV.

Main Theorem (sV)**.** *The Brin-Thompson group sV is of type F_∞ for all s.*

Proof. Consider the action of sV on the Stein space sX. By Corollary 6.2.6 sX is contractible., by Observation 6.1.5 the stabilizer of every cell is finite, and by Observation 6.1.6 each $sX^{\leq n}$ is finite modulo the action of sV. By Proposition 6.3.8 the connectivity of the pair $(sX^{\leq n}, sX^{<n})$ tends to ∞ as n tends to ∞. Hence sV is of type F_∞ by Brown's Criterion (Proposition 1.1.8). □

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