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Random Walks Interacting with Evolving Random Environments and Related Kinetic Equations

AUTHOR:
Andrea Di Stefano

SUPERVISOR:
Prof. Yuri Kondratiev

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Introduction

In natural sciences we frequently confront with *open systems*, i.e. systems which are influenced by their surroundings through exchanges of energy and matters [38, 68, 122]. In physics, classical models for such systems are given by the so-called *system-plus-reservoir*, see e.g. [113]. These models describe the Hamiltonian dynamics of a test particle (system) moving in a fluid (reservoir). The fluid is formed by a large number of particles, hence one can usually neglect the influence that the system has on it. Clearly, the system can be interpreted both as the particle of a dilute gas or as a small part of the reservoir. In the latter case, a full understanding of the problem provides an understanding of the fluid itself. Classical examples of such models are the Rayleigh and Lorenz gas. They were originally proposed by Lord Rayleigh in 1891, see [105], and H. A. Lorenz in 1905, see [93], respectively, to describe the motion of a pendulum in the presence of friction and the diffusion of conduction electrons in metals, respectively. The subsequent century has seen a rapid grow of results concerning these and related models, which found applications in many different fields such as probability [3, 116], dynamical systems [115], statistical mechanics [30, 40], transport phenomena [28, 75] and so on. Particularly important are the applications in quantum mechanics, where the so-called open quantum systems revealed to be essential in the study of the approach to equilibrium [87, 92], quantum decoherence [11–13] and more recently in nanosystems [110]. For an extensive overview of all these subjects we refer to [29, 67, 90, 106, 113].

In this thesis we consider a specific class of system-plus-reservoir models, so-called *random walks in random environments* (RWREs). A RWRE describes the random walk (RW) of a particle jumping in some (phase) space X according to a transition kernel which depends on a random field on X , which we call random environment (RE). The study of these models has started in the early 1970s motivated by some problems in biology [24, 119] and disordered systems [41, 74]. Progressively, applications have spread out through different fields and nowadays RWRE represents a wide and very active area of research. The main reason of this rapid development lies in the fact that such models can differ drastically from ordinary RWs and new interesting phenomena appear. We refer to [22, 72] for an account of the history of the subject and a review of classical results. In the rigorous study of RWRE models one can distinguish two distinct situations depending if the RE is static or dynamic. In the first case, the environment is randomly chosen at time zero and remains fixed throughout all the time evolution. A mathematical analysis of such models was started in the '70s by Solomon [73, 112] and, at present, their behavior is fairly well understood even if some questions still remain open, especially in high dimensions. We refer to [118, 125] for a recent review of the topic. Contrarily, in RW in dynamic RE the environment changes over time according to a (stationary) stochastic process. The first models of this type appeared more recently in the '90s, see [15, 17], and in the last years they have been intensively studied under different assumptions, see e.g. [6, 18, 19, 35, 104, 124].

All the models of RW in static and dynamic RE mentioned above are defined on a lattice, namely one takes $X = \mathbb{Z}^d$ for some $d \geq 1$. Another possibility that has been analyzed in several papers concerns RWs evolving on some random graph, see e.g. [9, 101, 117]. In this work we concentrate on models of RWs in dynamic REs on \mathbb{R}^d , $d \geq 1$. Such models are not so widely studied in the literature. They were introduced for the first time in [16], where the

authors considered the RW of a particle moving in \mathbb{R}^d and interacting with a RE represented by a stationary Glauber-type dynamics in \mathbb{R}^d (see e.g. [82]). In particular, assuming a low activity-high temperature regime for the Glauber dynamics and small coupling between the tagged particle and the environment, they obtained the large time asymptotic for the particle position distribution. Here, we want to study the complementary situation where the RE is described by a non-equilibrium Markov process and the interaction between the particle and the environment is not necessarily weak. It is worth noting that in these two cases not much is known even on lattice, see e.g. [4, 26, 36, 37] for some results in this direction.

More concretely, we consider the RW of a tagged particle moving in \mathbb{R}^d according to a jump process and interacting with a RE formed by infinitely many identical particles. The evolution of these particles is described by some non-equilibrium Markov dynamics in \mathbb{R}^d , which can be a birth-and-death dynamics or the dynamics of jumping particles, see some examples in [60]. We assume that each particle of the model is completely characterized by its position in the space. Moreover, having in mind that any element has a physical size, we impose that in any bounded region there are only a finite number of particles of RE and, at the same time, we forbid them to occupy the same position in the space. As a consequence, the phase space of the model is given by $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$, where $\Gamma(\mathbb{R}^d)$ is the space of locally finite configurations over \mathbb{R}^d , see e.g. [1, 2, 77]. Any (microscopic) state of the system is represented by the pair (γ, y) , where $y \in \mathbb{R}^d$ identifies the position of the tagged particle, and $\gamma \in \Gamma(\mathbb{R}^d)$ is the configuration formed by all particles of the environment. On a microscopic level the dynamic of the RWRE is described by a heuristic Markov generator L acting on a proper space of functions on $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$. This generator can be written as

$$L := L_{RE} + L_{RW}(\gamma), \quad (1)$$

where the operator L_{RE} defines the Markov dynamics of the particles of RE, see [60], and $L_{RW}(\gamma)$ describes the RW of the tagged particle. The latter depends on the configuration of particles of the RE, $\gamma \in \Gamma(\mathbb{R}^d)$, due to some interaction of the tagged particle with the RE. This interaction is represented by a non-negative function $\lambda_{\text{int}}(\gamma, y, z)$ which modulates the density rate of a jump from a point $y \in \mathbb{R}^d$ to $z \in \mathbb{R}^d$. These models of RWREs can be considered as a stochastic version of the Rayleigh and Lorenz gases mentioned previously. They are inspired, in particular, by ecological systems: one can interpret them as a prey (RW) running away from a group of predators (RE) or as a predator (RW) moving in a group of prey (RE) depending on the form of λ_{int} , see e.g. [25] and reference therein.

The essential problem in the study of the models of RWRE introduced above is that, contrary to the lattice case, the construction of a spatial Markov process in \mathbb{R}^d , describing the (non-equilibrium) stochastic evolution of RE,

$$\Gamma(\mathbb{R}^d) \ni \gamma \mapsto X_t^\gamma \in \Gamma(\mathbb{R}^d), \quad t > 0,$$

is a difficult question which is not completely solved, in general, at present, see e.g. [65, 100]. Let us note that for systems on a lattice and systems in continuum with a finite number of particles this construction can be done for a wide class of systems, see e.g. [64, 91]. On the other hand, as it is well known in statistical mechanics [71], such a microscopic description of the dynamics is often too detailed to be really useful in concrete applications. Indeed, in the real-world systems the number of particles is so huge that, typically, we are not able to follow the trajectories of each them, but one can take into account just the statistical characteristics of the evolution. This leads to a statistical description of complex systems, see e.g. [55, 60, 62]. In such an approach, we study the evolution of states in the course of the (microscopic) stochastic evolution of RWRE. From the mathematical point of view a state corresponds to a measure $\mu(d\gamma, dy)$ on $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$. Heuristically, the evolution of a state is given by the Kolmogorov (or Fokker-Planck) equation

$$\frac{\partial}{\partial t} \mu_t = (L_{RE}^* + L_{RW}^*) \mu_t, \quad t \geq 0, \quad (2)$$

where $L^* := L_{RE}^* + L_{RW}^*$ is the adjoint operator of L w.r.t. to the pairing between functions and measures on $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$. Following the approach of statistical mechanics, in order to study (2) we reformulate the evolution of states in terms of the corresponding *correlation functions* $k_t^{(n)}(x_1, \dots, x_n, y)$ on $(\mathbb{R}^d)^n \times \mathbb{R}^d$, $n \geq 0$. As a result, the Markov dynamics of RWRE is described in terms of the hierarchy for correlation functions

$$\frac{\partial}{\partial t} k_t = (\widehat{L}_{RE}^* + \widehat{L}_{RW}^*) k_t, \quad t \geq 0, \quad (3)$$

where $\widehat{L}^* := \widehat{L}_{RE}^* + \widehat{L}_{RW}^*$ is the corresponding image of the operator L^* acting on sequences of functions $k_t = \{k_t^{(n)}(x_1, \dots, x_n, y)\}_{n=0}^\infty$. Equation (3) can be considered as a chain of infinite equations for $k_t^{(n)}$, which is an analog of the BBGKY hierarchy for Hamiltonian systems, see e.g. [34]. The correlation functions $\{k_t^{(n)}\}_{n=0}^\infty$ contain all the statistical quantities related to the evolution of RWRE, cf. [71, 109], and they play a key role in our analysis. In recent years this statistical approach has been successfully applied to study the non-equilibrium stochastic dynamics of many interacting particle systems in the continuum, see [10, 45, 49, 58, 81, 84] and references therein.

As in all hierarchical equations, we can attempt to study the existence and uniqueness of solutions to (3). However, in general, one cannot expect to obtain an explicit form of the solution or, at least, some information about its behavior. On the other hand, from kinetic theory we know that the dynamics of many-body systems (system-plus-environment models included) can be approximated through *kinetic equations*. The latter are differential equations of first-order in time which describe the evolution of the density (first-order correlation function) of particles distributions, see e.g. [5, 67, 113] for an account of the field. The kinetic equations can be easily guessed by truncating the BBGKY hierarchy at the first-order, see e.g. [71]. In many cases, this approximation gives a fairly good understanding of the underlying microscopic dynamics. One of the central problems in non-equilibrium statistical mechanics is to understand the approximate validity of these kinetic equations. From a mathematical point of view this is translated in finding a scaling limit leading to the kinetic equations starting from a microscopic model. Such limits have been the object of several studies from physicists, mathematicians and ecologists. We refer to [76, 95, 103, 114] for a review of rigorous results both for Hamiltonian dynamics and interacting particle systems.

In this thesis we focus on the mesoscopic limits, in particular, we will study a Vlasov-type limit [113]. In physics, the corresponding Vlasov equation describes the Hamiltonian motion of a system of infinitely many particles in the mean-field limit, namely taking into account the influence of weak and long-range interactions, see e.g. [113]. This equation was first suggested by A. A. Vlasov in the physics of plasmas, see [121]. The convergence of the Vlasov-type scaling limit was shown by W. Braun and K. Hepp [23], for the Hamiltonian systems, and later by R. L. Dobrushin [33], for general dynamical systems. Note that these two approaches cannot be applied to the model of RWREs considered here. The main reason is that, as already discussed above, we are not able to define the evolution of the particles in terms of a proper stochastic equation. Another problem is related to the possible variation of particle number in the RE during the evolution. As observed in [113], for Hamiltonian systems the correct Vlasov-type limit can be easily guessed from the BBGKY hierarchy. In [8], by a low density scaling limit of the evolution of correlation functions, the authors derive (point-wisely) a kinetic equation for systems of random number of particles with collision, fragmentation and coagulation both in \mathbb{R}^d and \mathbb{Z}^d . More recently, in [51] a general scheme to derive the Vlasov equation for Markov evolutions of infinite particle systems in continuum has been developed. This scheme, and in particular the convergence of the scaling limit, has been rigorously analyzed in different models: birth-and-death dynamics [45, 52, 54, 56–58, 61] and jumping particle systems [10, 43].

We use the scheme proposed in [51] to derive a mesoscopic limit for the stochastic dynamics

of RWREs. The basic idea is to rescale, by means of a parameter $\varepsilon > 0$, the generator \widehat{L}^* , making the interaction among all particles weak and the density of RE appropriately high. The result will be a renormalized operator, $\widehat{L}_{\varepsilon, \text{ren}}^{\Delta} := \widehat{L}_{RE, \varepsilon, \text{ren}}^{\Delta} + \widehat{L}_{RW, \varepsilon, \text{ren}}^{\Delta}$, which describes the rescaled evolution of the correlation functions. By means of the considered scaling we arrive to a limiting Vlasov hierarchy of the form

$$\frac{\partial}{\partial t} k_t = (L_{RE, V}^{\Delta} + L_{RW, V}^{\Delta}) k_t, \quad t \geq 0, \quad (4)$$

where the operator $L_V^{\Delta} := L_{RE, V}^{\Delta} + L_{RW, V}^{\Delta}$ is the point-wise limit of $\widehat{L}_{\varepsilon, \text{ren}}^{\Delta}$ as ε goes to zero. It is worth noting that we do not need to define a reduced dynamics of the tagged particle, see e.g. [113]. As consequence of this fact, in the mesoscopic limit we obtain a system of two kinetic equations, so-called *Vlasov equations*, for the densities of RE and of the tagged particle, $\rho_t, r_t : \mathbb{R}^d \rightarrow \mathbb{R}$, $t \geq 0$, respectively. Due to the special structure of the operator L , the kinetic equation for the density of RE, ρ_t , is independent, while the equation for the density of the jumping particle, r_t , depends on the solution of the previous one. In this approach, the kinetic equations follow from the chaos preservation property of the Vlasov hierarchy (4). The latter implies that uncorrelated states, $k_t^{(n)}(x_1, \dots, x_n, y) = \prod_{i=1}^n \rho_t(x_i) r_t(y)$, $t \geq 0$, are preserved in the course of the limiting evolution. The Vlasov equations turn out to describe a non-autonomous RW whose transition kernel is modulated by a function $\bar{\lambda}_t(y, z)$, $y, z \in \mathbb{R}^d$, which depends on the density of particle in RE ρ_t , $t \geq 0$. The function $\bar{\lambda}_t$ is an effective interaction in the mean-field theory which describes the "mean effect" that the original interaction λ_{int} has on the RW of the tagged particle.

Outline of the thesis

In this thesis we present a rigorous analysis of several models of RWREs according to the statistical approach described above. More precisely, we realize the following program:

- (1) we construct the evolution of correlation functions as the solution to hierarchy (3);
- (2) we use the Vlasov-type scaling to derive the mesoscopic evolution of the correlation functions. In particular, we prove that the rescaled evolution of correlation functions converges to the solution of the limiting hierarchy (4);
- (3) we show that Vlasov hierarchy (4) satisfies the chaos preservation property and derive the corresponding kinetic equations;
- (4) we prove the existence, uniqueness and some uniform bounds for the solutions to the Vlasov equations.

Clearly, the space where we study the evolution of correlation functions depends on the applications one has in mind. For interacting particle systems, it is rather natural to consider correlation functions which satisfy the so-called Ruelle bounds, see e.g. [107, 108]. In our case, this bound implies that for some $C > 0$

$$\left| k_t^{(n)}(x_1, \dots, x_n, y) \right| \leq M(y) C^n, \quad x_1, \dots, x_n, y \in \mathbb{R}^d, n \in \mathbb{N}. \quad (5)$$

Here, we distinguish two different situations depending whether the function $M(\cdot)$ is bounded on \mathbb{R}^d or integrable over the whole \mathbb{R}^d . We denote by \mathcal{K}_C^{∞} and \mathcal{K}_C^1 , respectively, the corresponding (Banach) spaces of sequences of functions $\{k_t^{(n)}\}_{n=0}^{\infty}$ (cf. (2.31) and (2.32)). Note that the analysis of existence problems in such spaces is quite non-trivial and requires deep techniques in infinite dimensional analysis. In recent years different methods to address these problems have been developed, see e.g. [45, 48, 54, 61], which we may also apply in the present

context. The Vlasov equations, instead, will be studied for measurable non-negative functions $\rho_t(x) \leq C$, a.a. $x \in \mathbb{R}^d$, and $r_t \in L^q(\mathbb{R}^d)$, with $q = \infty, 1$ depending on the class of functions M , cf. (5), we consider. Such solutions can be obtained by using the standard theory of first-order differential equations in Banach spaces, see e.g. [69, 85]. These results should be considered as a preliminary step for a future study of the properties of these kinetic equations. In our analysis, we consider different classes of REs, in particular birth-and-death processes, see e.g. [45, 50, 54, 57, 82], and jumping particle systems, see e.g. [10, 43], and we provide general conditions that the interaction λ_{int} should fulfill. Such conditions are satisfied in many specific situations, including also possibly unbounded interactions. Roughly speaking, we can consider functions λ_{int} which grow at most linearly with the number of particles of RE, namely $\lambda_{\text{int}}(\gamma, \cdot, \cdot) \leq \alpha_0 + \alpha_1 |\gamma|$ whenever $|\gamma| < \infty$, where $|\cdot|$ denotes the number of points (cardinality) of a configuration. For the sake of clarity, some of these models will be discussed in details and others will be just mentioned briefly through the work.

The thesis is organized as follows.

In Chapter 1 we introduce a general framework to describe models formed by one particle which interacts with an infinite particle system. For such models, we consider two different spaces: the phase space $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ and the additional product space $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$, where $\Gamma_0(\mathbb{R}^d)$ is the space of all finite configurations over \mathbb{R}^d . Section 1.1 is devoted to a description of these spaces. More precisely, we review the harmonic analysis on configurations spaces developed in [77] in the considered product spaces. This analysis is based on the so-called K -transform. The latter defines a mapping between functions on $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$ (quasi-observables) and functions on $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ (observables), see Section 1.1.2. In Section 1.2 we study measures (states) on the phase space. In particular, using the K -transform we define the concept of correlation functions associated to a given state, see Definition 1.30. The sequence of correlation functions $\{k^{(n)}\}_{n=0}^{\infty}$ can be represented as a function $k(\eta, y)$ on $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$. This representation allows us to establish a duality between correlation functions and quasi-observables, see Remark 1.49 and (2.24).

In Chapter 2 we describe the evolution of RWREs on the three different levels: microscopic, statistical and mesoscopic. We consider the case where the particles of RE evolve according to a birth-and-death dynamics or to a jumping stochastic dynamics. In Section 2.1 we give a detailed description of the Markov pre-generator L . We state minimal conditions on the parameters of the models and provide some concrete examples for the interaction λ_{int} . In Section 2.2, we determine the explicit form of the hierarchy for correlation functions (3). This is done in two steps. First, in Section 2.2.1 we calculate the image of the operator L under the K -transform, i.e. $\widehat{L} := K^{-1}LK$, which describes the evolution for quasi-observables. Then, the operator \widehat{L}^* is obtained by using the duality between quasi-observables and correlation functions, see Section 2.2.2. Afterwards, in Section 2.3 we give a general description of the Vlasov-type scaling proposed in [51] for the considered models of RWREs. An explicit realization of such a mesoscopic limit is done in Section 2.3.1. We state general conditions on the parameter of the model in order to derive the Vlasov equations of a RW moving in RE under a general interaction λ_{int} , cf. Lemma 2.25. This derivation is informal, in the sense that the limit is performed on the forms of the hierarchies rather than on their solutions. To conclude, in Section 2.3.2 we present a list of kinetic equations for different examples of interaction.

In the remaining three chapters we proceed with a rigorous study of some concrete models of RWREs.

In Chapter 3 we consider the case where the particles of RE evolve according to a birth-and-death dynamics. The evolution of correlation functions is constructed in Section 3.1. In particular, we provide sufficient conditions on the birth and death intensities as well as on λ_{int} for the existence of a strongly continuous semigroup with generator \widehat{L}^* on a proper subspace of $\mathcal{K}_{\mathcal{C}}^{\infty}$, see Theorem 3.8. In order to show this result we adapt the approach developed

in [54]. The idea is to use the duality (2.24) and consider the pre-dual evolution for quasi-observables in some pre-dual L^1 -space, see Theorem 3.3. In Section 3.2 we study the Vlasov-type scaling limit. In Section 3.2.1 we show a weak*-convergence of the rescaled evolution of correlation functions to the solution of the Vlasov hierarchy (4) in terms of the corresponding semigroup in \mathcal{K}_C^∞ . Namely, we prove the strong convergence of the pre-dual semigroups in the space of quasi-observables, see Theorem 3.35. The latter follows from a general result about the strong convergence of resolvent operators in Banach spaces, see Lemma 3.19 or also [54, Lemma 4.3]. Next, in Section 3.2.2 we study the solutions to the resulting system of Vlasov equations, see (3.124a)-(3.124b). For a birth-and-death dynamics it is natural to assume that the corresponding Vlasov equation (3.124a) has a non-negative and uniformly bounded solution, see e.g. [48, 57, 82]. Under these conditions in Theorem 3.41 we show that the kinetic equation of the RW (3.124b) also has a unique, non-negative and uniformly bounded solution. The general results obtained throughout this chapter are discussed for different interactions λ_{int} in the particular case of a RW moving in BDLP model of RE, see Section 3.1.3 and 3.2.3. The BDLP model is an individual based model in spatial ecology introduced by B. Bolker and S. Pacala [20, 21], U. Dieckmann and R. Law [31] to describe the competition among plants. The evolution of correlation functions and the Vlasov-type scaling for such a model have been studied in [50] and [56], respectively.

In Chapter 4 we consider the model of RW moving in a birth-and-death environment with a constant birth rate and a death rate inverse proportional to the number of particles in RE. This particular birth-and-death dynamics has been proposed in [45] and one can easily see that the smigroup techniques of the previous chapter cannot be applied. In Section 4.1, we study the evolution (3) in the scale of Banach spaces $\{\mathcal{K}_C^1 : 0 < C \leq C_0\}$. Under some general conditions on the interaction λ_{int} , in Theorem 4.3 we show that (3) has a unique solution on a proper space of the scale, but on a finite time interval. The proof of this result is given in Section 4.1.1 and it is carried out by using an Ovsjannikov-type result given in Theorem 4.2 and proved thoroughly in Appendix A. The assumptions on λ_{int} are verified for a number of particular examples in Section 4.1.2. Section 4.2 is devoted to the study of the Vlasov-type scaling. In Section 4.2.1, by using an Ovsjannikov-type result for families of initial value problems on a scale of Banach spaces, see Theorem 4.9 or [45, Theorem 3.3], we show that the rescaled evolution for correlation function converges to the solution to the Vlasov hierarchy (4) on a finite time interval. The precise result is stated in Theorem 4.17. The corresponding system of kinetic equations, see (4.76a)-(4.76b), is studied in Section 4.2.2. In Theorem 4.22 we show that, if the initial density of RE is small enough, the Vlasov equations have unique, non-negative and uniformly bounded solutions for any time interval. Moreover, if the initial density of RW is integrable on \mathbb{R}^d then it remains integrable in the course of the evolution. Note that the restriction on the initial density of RE prevents any type of aggregation in the environment, see [45]. These results will be also discussed for concrete forms of λ_{int} in Section 4.2.3.

Finally, in Chapter 5 we consider the model of a RW interacting with a jumping particle system which evolves according to a Kawasaki dynamics, see e.g. [43]. For this model, we do not study the statistical evolution in terms of correlation functions, but we reformulate the problem in terms of the corresponding Bogoliubov generating functionals, see e.g. [8, 14, 63, 79]. The analysis of the model will be performed for the concrete interactions $\lambda_{\text{int}} = \lambda^{(1)}$ and $\lambda_{\text{int}} = \lambda^{(2)}$ introduced in Section 2.1. In Section 5.1 we define and characterize the generating functionals for a RWRE. To any finite measure $\mu(\gamma, y)$ on $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ we may associate a Bogoliubov functional $Z_\mu(\theta, \psi)$ on $L^1(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d)$ according to Definition 5.1. In particular, we consider functionals which are entire and of bounded type. Under these assumptions, we can recover the notion of correlation functions introduced in Chapter 1, see Proposition 5.7. The latter specifies the connection between the Bogoliubov functional Z_μ and the correlation

functions k_μ associated to a state μ (cf. (5.23)),

$$B(\theta, y) := \frac{\delta Z_\mu(\theta, \psi)}{\delta \psi(y)} = \int_{\Gamma_0} d\lambda(\eta) \prod_{x \in \eta} \theta(x) k_\mu(\eta, y), \quad \theta \in L^1(\mathbb{R}^d), y \in \mathbb{R}^d. \quad (6)$$

In Section 5.2 we construct the statistical evolution of the considered models. Using the identity above, we rewrite the hierarchy for correlation functions (3) as an evolution equation for the generating functionals B_t of the form

$$\frac{\partial}{\partial t} B_t = (\tilde{L}'_{RE} + \tilde{L}'_{RW}) B_t, \quad t \geq 0. \quad (7)$$

The expression of operators \tilde{L}'_{RE} and \tilde{L}'_{RW} is given by (5.40) and (5.41), respectively. Let \mathcal{E}_α , $\alpha > 0$, be the (Banach) space of all functionals B_t associated with correlation functions $k_t \in \mathcal{K}_{1/\alpha}^1$ via (6) (cf. Definition 5.14). In Section 5.2.1 we study the solutions to (5.37) in \mathcal{E}_α . In particular, in Theorem 5.19 and 5.21 we show an existence and uniqueness result in the case where the interaction λ_{int} is given by $\lambda^{(1)}$ and $\lambda^{(2)}$, respectively. The analysis is carried out by using the Ovsjannikov-type Theorem 4.2 in the scale of Banach spaces $\{\mathcal{E}_\alpha : 0 < \alpha \leq \alpha_0\}$ leading to local in time solutions. In Section 5.2.2, we reformulate the Vlasov-type scaling in terms of the Bogoliubov functionals. As a result, we obtain a limiting evolution equation for the generating functionals B_t , which has also a chaos preservation property, cf. Lemma 5.27. The convergence of the rescaled evolution of generating functionals to their limiting evolution can be proved (on a finite time interval) by using Theorem 4.9. In Theorem 5.32 and 5.37 we show this convergence for the interactions $\lambda^{(1)}$ and $\lambda^{(2)}$, respectively, and derive the corresponding systems of Vlasov equations. Existence, uniqueness and uniform bounds of solutions to these two systems of kinetic equations can be proved by using a general result stated in Lemma 4.21, see Theorem 5.33 and 5.38, respectively.

In Appendix A which we give a detailed proof of Theorem 4.2 we use in Chapter 4 and 5. The latter is a slight generalization of the classical Ovsjannikov Theorem, see e.g. [44, 98, 120].

Chapter 1

One-particle system in a random environment

In this chapter we describe a mathematical framework for the study of the stochastic dynamics of a one-particle system in a random environment. In our case, the environment is represented by an interacting particle system consisting of an infinite number of identical particles. All particles of the composed model, i.e. system plus environment, are allowed to move in the space \mathbb{R}^d and each of them is completely identified by its position in the space.

In Section 1.1, we introduce the phase space of the model, describing its general structure and the main properties. Any (microscopic) state is represented by the position of the tagged particle and the configuration formed by all particles of the environment. Then, in order to arrive to a more convenient statistical description of the model, in Section 1.2 we introduce statistical states as measures on the phase space. Moreover, we will define the concept of correlation functions, which will be the main object of investigation in the (non-equilibrium) dynamics of our models.

1.1 Phase space of the model

Let us proceed to the mathematical realization of a complex systems formed by a particle interacting with an environment. In this context, environment refers to a system of infinitely many particles, identical to each other.

We assume that all particles of the complex system can move in the space \mathbb{R}^d and each of them can be characterized by its position in the space. Moreover, we impose that in any bounded region there are only a finite number of particles of the environment and, at the same time, we forbid them to occupy the same position in the space. Thus, as suggested in [60,62], in an abstract mathematical context we may conveniently describe the tagged-particle as a point in \mathbb{R}^d and the environment as a discrete set (configuration) of the underlying physical space. This leads to the following definition.

Definition 1.1. We consider as the phase space of the model the product space

$$\tilde{\Gamma}(\mathbb{R}^d) := \Gamma(\mathbb{R}^d) \times \mathbb{R}^d, \quad (1.1)$$

where $\Gamma(\mathbb{R}^d)$ denote the space of locally finite configurations over \mathbb{R}^d , namely

$$\Gamma(\mathbb{R}^d) := \Gamma = \left\{ \gamma \subset \mathbb{R}^d : |\gamma_\Lambda| := |\gamma \cap \Lambda| < \infty \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right\}. \quad (1.2)$$

Here $|\cdot|$ means the cardinality of a set and $\mathcal{B}_b(\mathbb{R}^d)$ denotes the collection of all bounded sets from the family $\mathcal{B}(\mathbb{R}^d)$ consisting of all Borel sets in \mathbb{R}^d .

The phase space contains all possible microscopic states of the model. Any microscopic state is given by a pair $(\gamma, y) \in \tilde{\Gamma}(\mathbb{R}^d)$, where $y \in \mathbb{R}^d$ corresponds to the position of the jumping particle, whereas $\gamma \in \Gamma(\mathbb{R}^d)$ is the configuration of points identified by the position of all the particles of the environment.

Configuration space (1.2) has been intensively studied in the last decades, see [1, 2, 77, 78, 80, 86]. The space (1.1) can be considered as a special case of two components configuration spaces [62], the aim of this section is to extend the theory of configuration spaces Γ to the phase space $\tilde{\Gamma}$. In the following, with a small abuse of notation, we keep the same notation commonly used in configuration space analysis.

Any configuration $\gamma \in \Gamma(\mathbb{R}^d)$ can be identified with a non-negative Random measure

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d),$$

where δ_x is the Dirac measure with unit mass in x and $\mathcal{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on $\mathcal{B}(\mathbb{R}^d)$. By definition $\sum_{x \in \emptyset} \delta_x$ indicates the zero measure. This embedding allows us to endow the configuration space $\Gamma(\mathbb{R}^d)$ with the topology induced by the vague topology on $\mathcal{M}(\mathbb{R}^d)$, i.e. the weakest topology with respect to which all mappings

$$\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle := \int_{\mathbb{R}^d} f(x) d\gamma(x) = \sum_{x \in \gamma} f(x),$$

are continuous for any $f \in C_{\text{bs}}(\mathbb{R}^d)$, where $C_{\text{bs}}(\mathbb{R}^d)$ is the set of all continuous functions on \mathbb{R}^d with bounded support. We denote by $\mathcal{B}(\Gamma)$ the corresponding Borel σ -algebra on $\Gamma(\mathbb{R}^d)$.

Remark 1.2. Independently of the topological structure of $\Gamma(\mathbb{R}^d)$, the Borel σ -algebra $\mathcal{B}(\Gamma)$ can be characterized as the smallest σ -algebra on $\Gamma(\mathbb{R}^d)$ for which all mappings

$$\Gamma \ni \gamma \mapsto N_{\Lambda}(\gamma) := |\gamma_{\Lambda}| \in \mathbb{N}_0$$

are measurable for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, namely

$$\mathcal{B}(\Gamma) = \sigma(\{N_{\Lambda} : \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\}).$$

See [77] for further details.

Remark 1.3. We can give a different description of the measurable space $(\Gamma(\mathbb{R}^d), \mathcal{B}(\Gamma))$, which will be useful later on in order to introduce a measure on this space. For any set $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, let us define the space of configurations contained in Λ given by

$$\Gamma(\Lambda) := \left\{ \gamma \in \Gamma(\mathbb{R}^d) : \left| \gamma \cap (\mathbb{R}^d \setminus \Lambda) \right| = 0 \right\}.$$

On this space we can introduce a Borel σ -algebra $\mathcal{B}(\Gamma(\Lambda))$ as in Remark 1.2, namely

$$\mathcal{B}(\Gamma(\Lambda)) := \sigma(\{N_{\Lambda'} \upharpoonright_{\Gamma(\Lambda)} : \Lambda' \in \mathcal{B}_b(\Lambda)\}).$$

Let us consider the measurable projection p_{Λ} defined by

$$\Gamma(\mathbb{R}^d) \ni \gamma \mapsto p_{\Lambda}(\gamma) := \gamma \cap \Lambda \in \Gamma(\Lambda). \quad (1.3)$$

It can be shown, see e.g. [111], that the space $(\Gamma, \mathcal{B}(\Gamma))$ coincides (up to an isomorphism) with the projective limit of the family of measurable spaces

$$\{\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda)) \mid \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\},$$

with respect to the projections p_{Λ} . In particular, $\mathcal{B}(\Gamma)$ coincides with the smallest σ -algebra for which all the projections (1.3) are measurable, namely

$$\mathcal{B}(\Gamma) = \sigma(\{p_{\Lambda} : \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\}).$$

Furthermore, we can introduce a filtration of $\Gamma(\mathbb{R}^d)$ given by

$$\mathcal{B}_{\Lambda}(\Gamma(\mathbb{R}^d)) := \sigma(\{N_{\Lambda'} : \Lambda' \in \mathcal{B}_b(\mathbb{R}^d), \Lambda' \subset \Lambda\}).$$

Note that the σ -algebras $\mathcal{B}(\Gamma(\Lambda))$ and $\mathcal{B}_{\Lambda}(\Gamma)$ are isomorphic, see e.g. [77].

As a result, we can equip the phase space $\tilde{\Gamma}(\mathbb{R}^d)$ with the topology induced by the product topological spaces $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ and with the corresponding Borel σ -algebra, namely $\mathcal{B}(\tilde{\Gamma}) := \mathcal{B}(\Gamma) \otimes \mathcal{B}(\mathbb{R}^d)$. It is worth noting that this topology may be metrizable in such a way that $\tilde{\Gamma}$ becomes a Polish space, see e.g. [80] and reference therein.

Functions on the phase space $\tilde{\Gamma}(\mathbb{R}^d)$ are called observables. This notion is borrowed from physics and it represents physical quantities which can be measured in course of empirical investigations. We denote by $L^0(\tilde{\Gamma}, \mathcal{B}(\tilde{\Gamma}))$ the space consisting of all $\mathcal{B}(\tilde{\Gamma})$ -measurable functions $F : \Gamma \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ such that $|F(\gamma, y)| < \infty$ whenever $|\gamma| < \infty$. In particular, a function $F \in L^0(\tilde{\Gamma}, \mathcal{B}(\tilde{\Gamma}))$ is called a cylinder function if it is also measurable with respect to $\mathcal{B}_\Lambda(\Gamma) \otimes \mathcal{B}(\mathbb{R}^d)$ for some $\Lambda \in \mathcal{B}_b(\Gamma)$, namely if

$$F(\gamma, y) = F|_{\Gamma(\Lambda) \times \mathbb{R}^d}(\gamma_\Lambda, y), \quad \text{for any } \gamma \in \Gamma(\mathbb{R}^d) \text{ and } y \in \mathbb{R}^d.$$

The class of cylinder functions is denoted by $\mathcal{F}_{\text{cyl}}(\tilde{\Gamma})$.

Let us introduce a subset of the phase space which contains only configurations of the environment with a finite number of particles.

Definition 1.4. We define the product space

$$\tilde{\Gamma}_0(\mathbb{R}^d) := \Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d, \quad (1.4)$$

where $\Gamma_0(\mathbb{R}^d)$ denote the space of finite configurations over \mathbb{R}^d , namely

$$\Gamma_0(\mathbb{R}^d) := \Gamma_0 := \{\eta \in \Gamma(\mathbb{R}^d) : |\eta| < \infty\}. \quad (1.5)$$

Remark 1.5. The space of finite configurations $\Gamma_0(\mathbb{R}^d)$ can be represented as a disjoint union of the spaces of n -point configurations $\Gamma^{(n)}(\mathbb{R}^d)$, namely

$$\Gamma_0(\mathbb{R}^d) = \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}(\mathbb{R}^d), \quad (1.6)$$

where for any $n \in \mathbb{N}$

$$\Gamma^{(n)}(\mathbb{R}^d) := \{\eta \in \Gamma(\mathbb{R}^d) : |\eta| = n\} \quad (1.7)$$

and $\Gamma^{(0)}(\mathbb{R}^d) := \{\emptyset\}$.

In order to introduce a topological structure on the space $\Gamma_0(\mathbb{R}^d)$, it is convenient to use the representation (1.6). The n -point configuration space $\Gamma^{(n)}(\mathbb{R}^d)$ can be easily constructed starting from the real space $(\mathbb{R}^d)^n$. More precisely, for any $n \in \mathbb{N}$ we can introduce a surjective mapping between the space

$$\widetilde{(\mathbb{R}^d)^n} := \{(x_1, \dots, x_n) : x_i \neq x_j \text{ if } i \neq j\} \quad (1.8)$$

and the n -point configuration space $\Gamma^{(n)}(\mathbb{R}^d)$, defined as the symmetrization map

$$\begin{aligned} \text{sym}_{\mathbb{R}^d}^n : \widetilde{(\mathbb{R}^d)^n} &\rightarrow \Gamma^{(n)}(\mathbb{R}^d) \\ (x_1, \dots, x_n) &\mapsto \{x_1, \dots, x_n\}. \end{aligned} \quad (1.9)$$

Clearly, the mapping (1.9) produces a one-to-one correspondence between $\Gamma^{(n)}(\mathbb{R}^d)$ and the symmetrized space $\widetilde{(\mathbb{R}^d)^n}/S_n$ ¹. Thus, we can use this bijection to induces a metrizable topology on $\Gamma^{(n)}(\mathbb{R}^d)$ and, thereafter, we endow $\Gamma_0(\mathbb{R}^d)$ with the topology of disjoint union of topological spaces. We denote by $\mathcal{B}(\Gamma^{(n)})$ and $\mathcal{B}(\Gamma_0)$ the corresponding Borel σ -algebras on $\Gamma^{(n)}(\mathbb{R}^d)$ and $\Gamma_0(\mathbb{R}^d)$, respectively. As a consequence, we can equip the space $\tilde{\Gamma}_0(\mathbb{R}^d)$ with the topology induced by the product topological spaces $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$ and with the corresponding Borel σ -algebra, namely $\mathcal{B}(\tilde{\Gamma}_0) := \mathcal{B}(\Gamma_0) \otimes \mathcal{B}(\mathbb{R}^d)$.

¹The symmetrization of the space $\widetilde{(\mathbb{R}^d)^n}$ is given by $\widetilde{(\mathbb{R}^d)^n}/S_n$ where S_n is the permutation group over the coordinate index $\{1, \dots, n\}$.

Remark 1.6. It is worth noting that, from a topological point of view, the space $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$ cannot be considered a subset of the phase space $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$. Indeed, the topology induced on Γ_0 by the vague topology on Γ does not coincide with the topology we have defined on the space of finite configurations. From our point of view the space $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$ should be considered as complementary mathematical object of the physical space $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$.

Functions on the space $\widetilde{\Gamma}_0$ are called quasi-observables. They are not observables themselves, but they can be used to construct observables, see Section 1.1.2. Let $L^0(\widetilde{\Gamma}_0, \mathcal{B}(\widetilde{\Gamma}_0))$ be the space of all $\mathcal{B}(\widetilde{\Gamma}_0)$ -measurable functions $G : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$. We denote by $L_{\text{ls}}^0(\widetilde{\Gamma}_0)$ the set of all measurable functions on $\widetilde{\Gamma}_0$ with local support on Γ_0 . Namely $G \in L_{\text{ls}}^0(\widetilde{\Gamma}_0)$ iff $G \in L^0(\widetilde{\Gamma}_0)$ and there exists a set $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that

$$G(\eta, y) \upharpoonright_{(\Gamma_0 \setminus \Gamma(\Lambda)) \times \mathbb{R}^d} = 0, \quad \text{for any } \eta \in \Gamma_0(\mathbb{R}^d) \text{ and } y \in \mathbb{R}^d.$$

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that

$$M \subset \bigsqcup_{n=0}^N \Gamma^{(n)}(\Lambda).$$

Let $\mathcal{B}_b(\Gamma_0)$ be the collection of all bounded sets in Γ_0 . We will also consider the set $B_{\text{bs}}(\widetilde{\Gamma}_0)$ formed by all bounded measurable functions which have a bounded support on Γ_0 . The latter means that there exists a bounded set $M \in \mathcal{B}_b(\Gamma_0)$ such that

$$G(\eta, y) \upharpoonright_{(\Gamma_0 \setminus M) \times \mathbb{R}^d} = 0, \quad \text{for any } \eta \in \Gamma_0(\mathbb{R}^d) \text{ and } y \in \mathbb{R}^d.$$

Remark 1.7. Let us consider quasi-observables $G : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$. Having in mind the representation given by (1.6)-(1.7), for any $n \in \mathbb{N}_0$ the restriction $G(\cdot; y) \upharpoonright_{\Gamma^{(n)} \times \mathbb{R}^d}$ can be written as

$$G(\eta; y) \upharpoonright_{\Gamma^{(n)} \times \mathbb{R}^d} = \tilde{g}^{(n)}(x_1, \dots, x_n; y), \quad \text{for any } \eta = \{x_1, \dots, x_n\} \in \Gamma^{(n)}, y \in \mathbb{R}^d,$$

where $\tilde{g}^{(n)} : (\widetilde{\mathbb{R}^d})^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function symmetric under permutations of points in $(\widetilde{\mathbb{R}^d})^n$. Moreover, the function $\tilde{g}^{(n)}$ we can extend it to all space $(\mathbb{R}^d)^n \times \mathbb{R}^d$, namely one can define the measurable function $g^{(n)} : (\mathbb{R}^d)^n \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$g^{(n)}(x_1, \dots, x_n, y) = \begin{cases} \tilde{g}^{(n)}(x_1, \dots, x_n; y), & \text{if } (x_1, \dots, x_n) \in (\widetilde{\mathbb{R}^d})^n, \\ 0, & \text{otherwise.} \end{cases} \quad (1.10)$$

As consequence, any quasi-observables on $\Gamma_0 \times \mathbb{R}^d$ can be represented as a sequence of functions of increasing number of variables (1.10), namely

$$\{g^{(n)}(x_1, \dots, x_n, y)\}_{n=0}^{\infty}.$$

For any $n \in \mathbb{N}_0$, we denote by $L_{\text{sym}^n}^0((\mathbb{R}^d)^n \times \mathbb{R}^d)$ the space of all functions of the form (1.10).

In the following subsections, we continue the analysis of the mathematical structure of the spaces introduced above. In particular, we consider and adapt to our case some important notions well-known in the theory of configuration spaces.

1.1.1 The phase space as a measure space

In this section we want to define the phase spaces $\tilde{\Gamma}(\mathbb{R}^d)$ and $\tilde{\Gamma}_0(\mathbb{R}^d)$ as measure spaces. For this purpose we need to introduce a measure on the measurable spaces $(\tilde{\Gamma}, \mathcal{B}(\tilde{\Gamma}))$ and $(\tilde{\Gamma}_0, \mathcal{B}(\tilde{\Gamma}_0))$.

We start for convenience with the space $(\tilde{\Gamma}_0, \mathcal{B}(\tilde{\Gamma}_0))$. On the space of finite configurations $(\Gamma_0, \mathcal{B}(\Gamma_0))$, we can introduce a measure by using the construction described in formulas (1.5)-(1.9), see [77] for details. More precisely, let σ be a non-degenerate and non-atomic Radon measure on the space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, for example, the Lebesgue measure on \mathbb{R}^d . For each $n \in \mathbb{N}$ we denote by $\sigma^{\otimes n}$ the product measure on $((\mathbb{R}^d)^n, \mathcal{B}((\mathbb{R}^d)^n))$. Since the measure σ is non-atomic it follows that $\sigma^{\otimes n}(\overline{(\mathbb{R}^d)^n} \setminus (\mathbb{R}^d)^n) = 0$. Then, one can consider the restriction of the measure $\sigma^{\otimes n}$ to the space $((\mathbb{R}^d)^n, \mathcal{B}((\mathbb{R}^d)^n))$ and use the mapping (1.9) to define a measure, denoted by $\sigma^{(n)}$, on the space on n -point configurations $(\Gamma^{(n)}(\mathbb{R}^d), \mathcal{B}(\Gamma^{(n)}(\mathbb{R}^d)))$ as

$$\sigma^{(n)} := \sigma^{\otimes n} \circ (\text{sym}_{\mathbb{R}^d}^n)^{-1}, \quad n \in \mathbb{N}, \quad (1.11)$$

where the r.h.s. denotes the image, or push-forward, measure under the mapping $\text{sym}_{\mathbb{R}^d}^n$. For $n = 0$ we put $\sigma^{(0)} := 1$. Finally, we can use representation (1.6) to define a measure on the space $(\Gamma_0, \mathcal{B}(\Gamma_0))$ as sum of the measures (1.11).

Definition 1.8. On the measurable space $(\Gamma_0 \times \mathbb{R}^d, \mathcal{B}(\Gamma_0) \otimes \mathcal{B}(\mathbb{R}^d))$ we can define the product measure

$$\tilde{\lambda}_\sigma = \lambda_\sigma \otimes dy, \quad (1.12)$$

where λ_σ is the Lebesgue-Poisson measure with intensity σ on the space $(\Gamma_0, \mathcal{B}(\Gamma_0))$, defined by

$$\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}. \quad (1.13)$$

Moreover, the triplet $(\tilde{\Gamma}_0, \mathcal{B}(\tilde{\Gamma}_0), \tilde{\lambda}_\sigma)$ define a measure space.

Remark 1.9.

- (i) The measure space $(\Gamma_0, \mathcal{B}(\Gamma_0), \lambda_\sigma)$ is called the Lebesgue-Poisson space.
- (ii) In the case that $d\sigma(x) = dx$, $x \in \mathbb{R}^d$, we denote the Lebesgue-Poisson measure λ_σ by λ .

Next, let us consider the phase space $(\tilde{\Gamma}, \mathcal{B}(\tilde{\Gamma}))$. On the configuration space $(\Gamma, \mathcal{B}(\Gamma))$ one can introduce the Poisson measure as a projective limit of the normalized finite-volume distributions of the Lebesgue-Poisson measure λ_σ , see Remark 1.3 and [77]. More precisely, given a $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ let us consider the restriction of the Lebesgue-Poisson measure λ_σ on the space $(\Gamma(\Lambda), \mathcal{B}_b(\Gamma(\Lambda)))$, namely

$$\lambda_{\sigma|_{\Gamma(\Lambda)}} = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_\Lambda^{(n)}, \quad \sigma_\Lambda^{(n)} := \sigma^{\otimes n} \circ (\text{sym}_\Lambda^n)^{-1}, \quad \sigma_\Lambda^{(0)} \equiv 1.$$

Note that, since σ is a Radon measure, the measure $\lambda_{\sigma|_{\Gamma(\Lambda)}}$ is finite with total mass

$$\lambda_{\sigma|_{\Gamma(\Lambda)}}(\Gamma(\Lambda)) = \lambda_\sigma(\Gamma(\Lambda)) = e^{\sigma(\Lambda)}.$$

Thus, one can define a probability measure on $(\Gamma(\Lambda), \mathcal{B}_b(\Gamma(\Lambda)))$ given by

$$\pi_\sigma^\Lambda = e^{-\sigma(\Lambda)} \lambda_{\sigma|_{\Gamma(\Lambda)}}. \quad (1.14)$$

The family of probability measures $\{\pi_\sigma^\Lambda : \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\}$ is consistent, namely

$$\forall \Lambda_1, \Lambda_2 \in \mathcal{B}_b(\mathbb{R}^d) : \Lambda_1 \subset \Lambda_2, \quad \pi_\sigma^{\Lambda_2} \circ (p_{\Lambda_2, \Lambda_1})^{-1} = \pi_\sigma^{\Lambda_1},$$

where, according to the notation (1.11), $\pi_\sigma^{\Lambda_2} \circ (p_{\Lambda_2, \Lambda_1})^{-1}$ is the image measure under the the projections p_{Λ_2, Λ_1} defined as (compare to (1.3))

$$\Gamma(\Lambda_2) \ni \gamma \mapsto p_{\Lambda_2, \Lambda_1}(\gamma) := \gamma \cap \Lambda_1 \in \Gamma(\Lambda_1). \quad (1.15)$$

Hence, by the Kolmogorov theorem for the projective limit spaces, see [99, Theorem 5.1], the family $\{\pi_\sigma^\Lambda : \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\}$ determines a unique probability measure π_σ on the configuration space $(\Gamma, \mathcal{B}(\Gamma))$ such that

$$\forall \Lambda \in \mathcal{B}_b(\Lambda), \quad \pi_\sigma^\Lambda = \pi_\sigma \circ (p_\Lambda)^{-1},$$

where $\pi_\sigma \circ (p_\Lambda)^{-1}$ is the image measure under the projection (1.3).

Definition 1.10. On the measurable space $(\Gamma \times \mathbb{R}^d, \mathcal{B}(\Gamma) \otimes \mathcal{B}(\mathbb{R}^d))$ we can define the product measure

$$\tilde{\pi}_\sigma = \pi_\sigma \otimes dy, \quad (1.16)$$

where the probability measure π_σ is called the Poisson measure with intensity σ . Moreover, the triplet $(\tilde{\Gamma}, \mathcal{B}(\tilde{\Gamma}), \tilde{\pi}_\sigma)$ define a measure space.

Remark 1.11. The probability space $(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma)$ is called the Poisson space.

Let us now present some important results concerning the Poisson-Lebesgue measure (1.8) which will be used later on. We refer to [97] for their proofs.

Lemma 1.12. *Let $n \in \mathbb{N}$, $n \geq 2$ be given. Then*

$$\begin{aligned} & \int_{\Gamma_0} d\lambda_\sigma(\eta_1) \dots \int_{\Gamma_0} d\lambda_\sigma(\eta_n) G(\eta_1 \cup \dots \cup \eta_n, y) H(\eta_1, \dots, \eta_n, y) \\ &= \int_{\Gamma_0} d\lambda_\sigma(\eta) G(\eta, y) \sum_{(\eta_1, \dots, \eta_n) \in \mathcal{P}_n(\eta)} H(\eta_1, \dots, \eta_n, y), \end{aligned} \quad (1.17)$$

for all positive measurable functions $G : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $H : (\Gamma_0)^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ with respect to which both sides make sense.

Let us note that for $n = 2$, Lemma 1.12 coincides with the well-known Minlos formula, see e.g. [83], which will play an important role in further calculations.

Corollary 1.13. *For all measurable positive functions $G : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$, $H : (\Gamma_0)^2 \times \mathbb{R}^d \rightarrow \mathbb{R}$ one has*

$$\int_{\Gamma_0} d\lambda_\sigma(\eta) G(\eta, y) \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, y) = \int_{\Gamma_0} d\lambda_\sigma(\eta) \int_{\Gamma_0} d\lambda_\sigma(\xi) G(\eta \cup \xi, y) H(\xi, \eta, y), \quad (1.18)$$

provided that both sides of the equality make sense.

Remark 1.14. Let us consider the particular case where $G(\eta, y) \equiv 1$ and

$$H(\xi, \eta, y) = \begin{cases} h(x, \eta, y), & \text{if } \xi = \{x\}, \\ 0, & \text{otherwise,} \end{cases}$$

for any $\eta, \xi \in \Gamma_0$ and $y \in \mathbb{R}^d$. Then, from Minlos formula (1.18) we obtain

$$\int_{\Gamma_0} d\lambda_\sigma(\eta) \sum_{x \in \eta} h(x, \eta \setminus x, y) = \int_{\Gamma_0} d\lambda_\sigma(\eta) \int_{\mathbb{R}^d} d\sigma(x) h(x, \eta, y), \quad (1.19)$$

for any measurable function $h : \mathbb{R}^d \times \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that both sides make sense. Identity (1.19) is an analog of the Mecke formula for Poisson measures.

1.1.2 The K -transform

In this section we want to investigate the connection between the phase space $(\Gamma \times \mathbb{R}^d, \mathcal{B}(\Gamma) \otimes \mathcal{B}(\mathbb{R}^d))$ and the space $(\Gamma_0 \times \mathbb{R}^d, \mathcal{B}(\Gamma_0) \otimes \mathcal{B}(\mathbb{R}^d))$. In particular, we introduce a mapping which transforms functions defined on $\Gamma_0 \times \mathbb{R}^d$ into functions on $\Gamma \times \mathbb{R}^d$, the so-called K -transform.

Definition 1.15. Let $G \in L_{\text{ls}}^0(\Gamma_0)$, we define the function $KG : \Gamma \rightarrow \mathbb{R}$ given by

$$(KG)(\gamma) := \sum_{\eta \in \gamma} G(\eta), \quad \gamma \in \Gamma(\mathbb{R}^d), \quad (1.20)$$

where the sum is taken over all finite sub-configurations η of the (infinite) configuration γ . The mapping K is called K -transform.

Remark 1.16. Note that the fact that $G \in L_{\text{ls}}^0(\Gamma_0)$ implies that the function KG is well-defined. Indeed, in this case, the sum in (1.20) has only a finite number of summands different from zero.

The K -transform introduced above can be extended to a mapping between functions on $\Gamma_0 \times \mathbb{R}^d$ and functions on $\Gamma \times \mathbb{R}^d$, namely for any $G \in L_{\text{ls}}^0(\Gamma_0 \times \mathbb{R}^d)$

$$(KG(\cdot, y))(\gamma) := \sum_{\eta \in \gamma} G(\eta, y), \quad (1.21)$$

for any fixed $y \in \mathbb{R}^d$ and all $\gamma \in \Gamma$.

The notion of K -transform first appears in the pioneering works of Lenard in statistical mechanics [88, 89] and has been formalized in [77]. From mathematical point of view, the main attraction of this transformation is the fact that it has a pure combinatorial nature independent of the measure under consideration. We will not go into details concerning the deriving combinatorial harmonic analysis. We just present the main properties of the K -transform which will be used throughout the thesis.

Proposition 1.17. *Let us consider the K -transform defined in (1.21). Then*

- (i) *The mapping K is linear and positivity preserving.*
- (ii) *The K -transform maps $L_{\text{ls}}^0(\widetilde{\Gamma}_0)$ into $\mathcal{F}_{\text{cyl}}(\widetilde{\Gamma})$. In particular, if $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ then there exists $C > 0$, $\Lambda \in \mathcal{B}_{\text{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that*

$$|(KG(\cdot, y))(\gamma)| \leq C(1 + |\gamma_\Lambda|)^N, \quad \gamma \in \Gamma, y \in \mathbb{R}^d.$$

- (iii) *The mapping $K : L_{\text{ls}}^0(\widetilde{\Gamma}_0) \rightarrow \mathcal{F}_{\text{cyl}}(\widetilde{\Gamma})$ is invertible with*

$$(K^{-1}F(\cdot, y))(\eta) := \sum_{\xi \subset \eta} (-1)^{\eta \setminus \xi} F(\xi, y), \quad \eta \in \Gamma_0, y \in \mathbb{R}^d. \quad (1.22)$$

Moreover, the inverse K -transform is well-defined for any $F \in L^0(\widetilde{\Gamma})$.

Proof. Let us consider a function $G \in L_{\text{ls}}^0(\widetilde{\Gamma}_0)$. For any fixed $y \in \mathbb{R}^d$ we can look at it as a function $G(\cdot; y) \in L_{\text{ls}}^0(\Gamma_0)$. Then, we can proceed as in the proof of Proposition 3.1 in [77]. \square

Before concluding, let us consider the K -transform in some concrete case which appear in applications.

Example 1.18. Let $f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a function such that there exists a bounded set $M \in \mathcal{B}(\mathbb{R}^d)$

$$f(x, y) = 0, \quad \text{for any } x \in \mathbb{R}^d \setminus M, y \in \mathbb{R}^d,$$

and consider the quasi-observable $G : \Gamma_0 \times \mathbb{R}^d$ defined by

$$G(\eta, y) = \begin{cases} f(x, y), & \text{if } \eta = \{x\}^2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the image of G under the K -transform is given by

$$(KG(\cdot, y))(\gamma) = \sum_{\eta \in \gamma} G(\eta, y) = \sum_{x \in \gamma} f(x, y), \quad \gamma \in \Gamma(\mathbb{R}^d), y \in \mathbb{R}^d.$$

Example 1.19. Given two $\mathcal{B}(\mathbb{R}^d)$ -functions f and g , let us consider the quasi-observable

$$G(\eta, y) = g(y) \prod_{x \in \eta} f(x),$$

where $B(\mathbb{R}^d)$ is the set of all bounded measurable functions on \mathbb{R}^d and $B_{\text{bs}}(\mathbb{R}^d)$ is the set of all bounded measurable functions on \mathbb{R}^d with bounded support. For brevity we introduce the so-called Lebesgue-Poisson exponent defined by

$$e_\lambda(f, \eta) = \prod_{x \in \eta} f(x), \quad (1.23)$$

with the convention that $e_\lambda(f, \emptyset) = 1$ and

$$e_\lambda(0, \eta) = 0^{|\eta|} = \begin{cases} 0, & \text{if } |\eta| \geq 1, \\ 1, & \text{if } |\eta| = 0. \end{cases}$$

Then, for any $f \in B_{\text{bs}}(\mathbb{R}^d)$ and $g \in B(\mathbb{R}^d)$, the image of G under the K -transform is given by

$$(KG(\cdot, y))(\gamma) = g(y) \sum_{\eta \in \gamma} e_\lambda(f, \eta) = g(y) \prod_{x \in \eta} (1 + f(x)),$$

for any $\gamma \in \Gamma(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. Let us note that for any $f \in L^1(\mathbb{R}^d, \sigma)$ the Lebesgue-Poisson exponent (1.23) is integrable with respect to the Lebesgue-Poisson measure λ_σ . In particular, one has

$$\int_{\Gamma_0} d\lambda_\sigma(\eta) e_\lambda(f, \eta) = e^{\int_{\mathbb{R}^d} d\sigma(x) f(x)}. \quad (1.24)$$

1.1.3 Algebraic product on phase spaces: \star -convolution

In this section we introduce some operation among functions on the space $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$. As it is known, see for example [77, 97], on the space of functions on $\Gamma_0(\mathbb{R}^d)$ one can introduce different algebraic products. Among them, we will focus on the so-called \star -convolution. This particular convolution has been introduced in [77] and it turns out to be a powerful tool for concrete calculations in applications.

Definition 1.20. Let us consider two functions $G_1, G_2 \in L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$. The \star -convolution of G_1 and G_2 is defined by

$$\begin{aligned} (G_1 \star G_2)(\eta) &:= \sum_{(\eta_1, \eta_2, \eta_3) \in \mathcal{P}_3(\eta)} G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3) \\ &= \sum_{\xi \subset \eta} G_1(\xi) \sum_{\zeta \subset \xi} G_2((\eta \setminus \xi) \cup \zeta), \quad \eta \in \Gamma_0, \end{aligned} \quad (1.25)$$

where $\mathcal{P}_3(\eta)$ denotes the set of all partitions of η in three parts which may be empty.

²It would be more precisely to write here $f((\text{sym}_{\mathbb{R}^d}^1)^{-1}(\eta), y)$, but we will use the notation above to simplify.

Remark 1.21. Note that the space $L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$ endowed with the \star -convolution has the structure of a commutative algebra with unit element given by $e_\lambda(0, \cdot)$,

$$(e_\lambda(0, \cdot) \star G)(\eta) = G(\eta), \quad \eta \in \Gamma_0.$$

Example 1.22. Let us consider the Lebesgue-Poisson exponents introduced in Example 1.19. For these functions the \star -convolution has a particular simple form, namely

$$(e_\lambda(f, \cdot) \star e_\lambda(g, \cdot))(\eta) = e_\lambda(f + g + fg, \eta), \quad \eta \in \Gamma_0(\mathbb{R}^d), \quad (1.26)$$

for all functions $f, g \in B_{\text{bs}}(\mathbb{R}^d)$. More generally, given a function $G \in L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$, we have

$$(G(\cdot) \star e_\lambda(f, \cdot))(\eta) = \sum_{\xi \subset \eta} G(\xi) e_\lambda(f + 1, \xi) e_\lambda(f, \eta \setminus \xi), \quad \eta \in \Gamma_0(\mathbb{R}^d). \quad (1.27)$$

In Definition 1.20, we have defined the \star -convolution as an algebraic product of functions in $L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$. However, we can extend it to functions on $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$. In this case, for any $G_1, G_2 \in L^0(\widetilde{\Gamma}_0, \mathcal{B}(\widetilde{\Gamma}_0))$ one has

$$\begin{aligned} (G_1(\cdot, y) \star G_2(\cdot, y))(\eta) &:= \sum_{(\eta_1, \eta_2, \eta_3) \in \mathcal{P}_3(\eta)} G_1(\eta_1 \cup \eta_2, y) G_2(\eta_2 \cup \eta_3, y) \\ &= \sum_{\xi \subset \eta} G_1(\xi, y) \sum_{\zeta \subset \xi} G_2((\eta \setminus \xi) \cup \zeta, y), \end{aligned} \quad (1.28)$$

for any fixed $y \in \mathbb{R}^d$ and all $\eta \in \Gamma_0$.

The main reason behind the introduction of \star -convolution resides on the following property: for any $F_1, F_2 \in L^0(\widetilde{\Gamma})$ one has, see e.g. [77],

$$(K^{-1}F_1) \star (K^{-1}F_2) = K^{-1}(F_1F_2). \quad (1.29)$$

Remark 1.23. The identity above enables us to consider the K -transform as a combinatorial Fourier transform on the configuration space, see e.g. [86].

1.2 Correlation measures and correlation functions

In the study of (non-equilibrium) stochastic dynamics in continuum, the main object of investigation is represented by the so-called correlation functions, see for example [55, 113] and references therein. In this section we will introduce the concept of correlation functions associated to a measure on the phase space $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$. In the following, by using a physical terminology, we will often call these measures *states* to stress their role in the description of a system.

We proceed similarly to what has been done in [77] on the configuration space $\Gamma(\mathbb{R}^d)$. First, by using the K -transform, we introduce the concept of correlation measure. Afterwards, we consider a special class of correlation measures to derive an explicit relation between correlation functions and states (in this special case).

1.2.1 Correlation measure of a state

In Section 1.1.1 we have already introduced a Poisson-type measure on the space $\Gamma \times \mathbb{R}^d$, see Definition 1.10. However, as is well known in statistical mechanics, Poisson measures describe the equilibrium of a system of particles without interactions. Then, in order to study the (non-equilibrium) dynamics of our model we should introduce a more general class of measures.

Let us denote by $\mathcal{M}(\Gamma \times \mathbb{R}^d)$ the space of all measures on the space $(\Gamma \times \mathbb{R}^d, \mathcal{B}(\Gamma) \otimes \mathcal{B}(\mathbb{R}^d))$. In the following we always assume that the states are measures with local finite moments.

Definition 1.24. A measure $\mu(d\gamma, dy) \in \mathcal{M}(\Gamma \times \mathbb{R}^d)$ has finite local moments (w.r.t. Γ -variable) of all orders if the following property holds

$$\int_A \int_{\Gamma} |\gamma_{\Lambda}|^n \mu(d\gamma, dy) < \infty, \quad (1.30)$$

for all $A, \Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $n \in \mathbb{N}$ ³. We denote by $\mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ the set of all such measures.

In the present work we study the evolution of two different types of states:

- (1) *bounded states*, corresponding to measures μ which are finite on $\Gamma \times \mathbb{R}^d$ and such that \mathbb{R}^d -marginal distributions have an integrable density w.r.t. the Lebesgue measure on \mathbb{R}^d , namely

$$\mu(\Gamma \times \mathbb{R}^d) = \int_{\mathbb{R}^d} \int_{\Gamma} d\mu(\gamma, y) < \infty \quad (1.31)$$

and

$$\mu(\Gamma, dy) = \int_{\Gamma} \mu(d\gamma, dy) = r(y) dy, \quad \text{with } r \in L^1(\mathbb{R}^d). \quad (1.32)$$

We denote by $\mathcal{M}^1(\Gamma \times \mathbb{R}^d)$ the set of all these measures. Similarly we can define the subspace $\mathcal{M}_{\text{fm}}^1(\Gamma \times \mathbb{R}^d)$ if, additionally, condition (1.30) is satisfied. Note that bounded states are non-normalized finite measures on $\Gamma \times \mathbb{R}^d$.

- (2) *locally-bounded states*, corresponding to all measures μ on $\Gamma \times \mathbb{R}^d$ which are finite w.r.t. the first variable $\gamma \in \Gamma$ and are Radon measures with bounded density w.r.t. the second variable $y \in \mathbb{R}^d$. Namely, for any set $A \in \mathcal{B}_b(\mathbb{R}^d)$ one has

$$\mu(\Gamma \times A) = \int_A \int_{\Gamma} d\mu(\gamma, y) < \infty$$

and

$$\int_{\Gamma} \mu(d\gamma, dy) = r(y) dy, \quad \text{with } r \in L^\infty(\mathbb{R}^d).$$

We denote by $\mathcal{M}^\infty(\Gamma \times \mathbb{R}^d)$ the set of all these measures. Analogously to the case above, we define the subspace $\mathcal{M}_{\text{fm}}^\infty(\Gamma \times \mathbb{R}^d)$.

Through the K -transform, introduced in Section 1.1.2, one can associate to each measure from $\mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ a measure on the space $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$.

Definition 1.25. Let $\mu \in \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ be given. Then, we can define a measure ρ_μ on $(\Gamma_0 \times \mathbb{R}^d, \mathcal{B}(\Gamma_0 \times \mathbb{R}^d))$ by

$$\begin{aligned} \rho_\mu(B \times A) &:= \int_{\mathbb{R}^d} \int_{\Gamma} (K \mathbb{1}_A(y) \mathbb{1}_B(\cdot))(\gamma) d\mu(\gamma, y) \\ &= \int_A \int_{\Gamma} (K \mathbb{1}_B)(\gamma) d\mu(\gamma, y), \end{aligned} \quad (1.33)$$

for all bounded sets $B \in \mathcal{B}_b(\Gamma_0)$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$. The measure ρ_μ is called the correlation measure corresponding to μ .

Remark 1.26. The fact that $\mu \in \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ insures that $\rho_\mu(B \times A) < \infty$ for all $B \in \mathcal{B}_b(\Gamma_0)$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$. Indeed, if $B \in \mathcal{B}_b(\Gamma_0)$ then there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}_0$ such that $B \subset \bigsqcup_{n=0}^N \Gamma^{(n)}(\Lambda)$. Thus, by using the definition of the K -transform, one can estimate

$$\rho_\mu(B \times A) \leq \sum_{n=0}^N \int_A \int_{\Gamma} (K \mathbb{1}_{\Gamma^{(n)}(\Lambda)})(\gamma) d\mu(\gamma, y)$$

³Cf. definition in [77, Proposition 4.1].

$$= \sum_{n=0}^N \int_A \int_{\Gamma} \binom{|\gamma_{\Lambda}|}{n} d\mu(\gamma, y) < \infty. \quad (1.34)$$

A measure ρ_{μ} which satisfies condition (1.34) is called locally finite. The set of all these measures is denoted by $\mathcal{M}_{\text{lf}}(\Gamma_0 \times \mathbb{R}^d)$.

Using standard measure theory, identity (1.33) may be proved for a wider class of functions on $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$.

Proposition 1.27. *Let us consider a measure $\mu \in \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$. Then, for any $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ one has*

$$\int_{\mathbb{R}^d} \int_{\Gamma_0} G(\eta, y) d\rho_{\mu}(\eta, y) = \int_{\mathbb{R}^d} \int_{\Gamma} (KG(\cdot, y))(\gamma) d\mu(\gamma, y). \quad (1.35)$$

Proof. We can show identity (1.35) for functions of the form $G_{\text{char}}(\eta, y) = \mathbb{1}_B(\eta) \mathbb{1}_A(y)$, $B \in \mathcal{B}_{\text{b}}(\Gamma_0)$, $A \in \mathcal{B}_{\text{b}}(\mathbb{R}^d)$, by using straightforward the definition of correlation measures (1.33). Hence, we can prove it for all functions $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ by approximating them with a linear combination of characteristic functions. \square

Let us note that identity (1.35) defines a relation between measures on $\Gamma \times \mathbb{R}^d$ and measures on $\Gamma_0 \times \mathbb{R}^d$. In particular, this provides a natural mapping between the spaces $\mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ and $\mathcal{M}_{\text{lf}}(\Gamma_0 \times \mathbb{R}^d)$ given by

$$\begin{aligned} K^* : \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d) &\mapsto \mathcal{M}_{\text{lf}}(\Gamma_0 \times \mathbb{R}^d) \\ \mu &\mapsto \rho_{\mu} := K^* \mu. \end{aligned} \quad (1.36)$$

As an example, we can apply the K^* -transform to the Poisson-type measure $\tilde{\pi}_{\sigma}$ in (1.16). Since $K^* \pi_{\sigma} = \lambda_{\sigma}$, see e.g. [77], we have

$$K^* \tilde{\pi}_{\sigma} = (K^* \pi_{\sigma}) \otimes dy = \lambda_{\sigma} \otimes dy := \tilde{\lambda}_{\sigma}, \quad (1.37)$$

where $\tilde{\lambda}_{\sigma}$ is the Lebesgue-Poisson-type measure defined in (1.12).

1.2.2 Correlation functions of a state

In this section we finally define the correlation functions associated to a state.

According to the definition in (1.3), for any $\Lambda \in \mathcal{B}_{\text{b}}(\mathbb{R}^d)$ we introduce the projection \tilde{p}_{Λ} defined by

$$\Gamma(\mathbb{R}^d) \times \mathbb{R}^d \ni (\gamma, y) \mapsto \tilde{p}_{\Lambda}(\gamma, y) := (\gamma \cap \Lambda, y) \in \Gamma(\Lambda) \times \mathbb{R}^d. \quad (1.38)$$

We use the notation $\mu_{\Lambda} = \mu \circ \tilde{p}_{\Lambda}^{-1}$ to indicate the projection of the measure μ onto the measurable space $(\Gamma(\Lambda) \times \mathbb{R}^d, \mathcal{B}(\Gamma(\Lambda)) \otimes \mathcal{B}(\mathbb{R}^d))$. Then, we say that a measure $\mu \in \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ is locally absolutely continuous with respect to $\tilde{\pi}_{\sigma} = \pi_{\sigma} \otimes dy$ if and only if $\mu_{\Lambda} = \mu \circ \tilde{p}_{\Lambda}^{-1}$ is absolutely continuous with respect to $\tilde{\pi}_{\sigma}^{\Lambda} = (\pi_{\sigma} \circ p_{\Lambda}^{-1}) \otimes dy$ for all $\Lambda \in \mathcal{B}_{\text{b}}(\mathbb{R}^d)$. Let us note that the absolute-continuity property of a measure μ is reflected on the corresponding correlation measure ρ_{μ} . This fact allows us to introduce the correlation functional k_{μ} associated to the measure μ .

Lemma 1.28. *Let $\mu \in \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ be a measure locally absolutely continuous with respect to $\tilde{\pi}_{\sigma} = \pi_{\sigma} \otimes dy$. Then, its correlation measure $\rho_{\mu} = K^* \mu$ is absolutely continuous with respect to $\tilde{\lambda}_{\sigma} = \lambda_{\sigma} \otimes dy$. Furthermore, for all $\Lambda \in \mathcal{B}_{\text{b}}(\mathbb{R}^d)$ we have*

$$k_{\mu}(\eta, y) := \frac{d\rho_{\mu}}{d\tilde{\lambda}_{\sigma}}(\eta, y) = \int_{\Gamma(\Lambda)} \frac{d\mu^{\Lambda}}{d\tilde{\pi}_{\sigma}^{\Lambda}}(\gamma \cup \eta, y) \pi_{\sigma}^{\Lambda}(d\gamma), \quad (1.39)$$

for $\tilde{\lambda}_{\sigma}$ -a.a. $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$.

Remark 1.29. Formula (1.39) is well known in statistical mechanics and in point process theory, see e.g. [107] and [27], respectively.

Proof. Let us consider the sets $B \in \mathcal{B}_b(\Gamma_0)$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$ such that $\tilde{\lambda}_\sigma(B \times A) = 0$. Since $\tilde{\lambda}_\sigma$ is the correlation measure of $\tilde{\pi}_\sigma$, we have

$$0 = \tilde{\lambda}_\sigma(B \times A) = \int_{\mathbb{R}^d} \int_{\Gamma(\Lambda)} \mathbf{1}_A(y) (K\mathbf{1}_B(\cdot))(\gamma) d\tilde{\pi}_\sigma^\Lambda(\gamma, y),$$

for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $B \subset \Gamma(\Lambda)$. Then, since $\mathbf{1}_A(y)(K\mathbf{1}_B(\cdot))(\gamma) \geq 0$, we must have

$$(K\mathbf{1}_{B \times A}(\cdot, y))(\gamma) \equiv 0, \quad \text{a.a.} - \tilde{\pi}_\sigma^\Lambda, \quad (1.40)$$

for $\tilde{\pi}_\sigma^\Lambda$ -a.a. $\gamma \in \Gamma_0(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. On the other hand, by hypothesis of the lemma, the correlation measure ρ_μ can be written as

$$\rho_\mu(B \times A) := \int_{\mathbb{R}^d} \int_{\Gamma} \mathbf{1}_A(y) (K\mathbf{1}_B(\cdot))(\gamma) d\mu(\gamma, y) \quad (1.41)$$

$$= \int_{\mathbb{R}^d} \int_{\Gamma(\Lambda)} \mathbf{1}_A(y) (K\mathbf{1}_B(\cdot))(\gamma) \frac{d\mu^\Lambda}{d\tilde{\pi}_\sigma^\Lambda}(\gamma, y) d\tilde{\pi}_\sigma^\Lambda(\gamma, y) = 0, \quad (1.42)$$

because of (1.40). The latter identity implies that $\rho_\mu = K^*\mu$ is absolutely continuous with respect to $\tilde{\lambda}_\sigma = \lambda_\sigma \otimes dy$.

Let us note that, by definition of the K -transform, for some $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $B \subset \Gamma(\Lambda)$ the correlation measure ρ_μ can be rewritten

$$\rho_\mu(B \times A) = \int_A \int_{\Gamma(\Lambda)} (K\mathbf{1}_B)(\gamma) \frac{d\mu^\Lambda}{d\tilde{\pi}_\sigma^\Lambda}(\gamma, y) d\pi_\sigma^\Lambda(\gamma) dy \quad (1.43)$$

$$= \int_A \int_{\Gamma(\Lambda)} \sum_{\eta \subset \gamma} \mathbf{1}_B(\eta) \frac{d\mu^\Lambda}{d\tilde{\pi}_\sigma^\Lambda}(\gamma, y) d\pi_\sigma^\Lambda(\gamma) dy. \quad (1.44)$$

Then, by applying the Minlos formula 1.18, one has

$$\rho_\mu(B \times A) = \int_A \int_{\Gamma(\Lambda)} \int_{\Gamma(\Lambda)} \mathbf{1}_B(\eta) \frac{d\mu^\Lambda}{d\tilde{\pi}_\sigma^\Lambda}(\gamma \cup \eta, y) d\pi_\sigma^\Lambda(\gamma) d\lambda_\sigma^\Lambda(\eta) dy \quad (1.45)$$

$$= \int_A \int_{\Gamma(\Lambda)} \mathbf{1}_B(\eta) \left[\int_{\Gamma(\Lambda)} \frac{d\mu^\Lambda}{d\tilde{\pi}_\sigma^\Lambda}(\gamma \cup \eta, y) d\pi_\sigma^\Lambda(\gamma) \right] d\tilde{\lambda}_\sigma^\Lambda(\eta, y). \quad (1.46)$$

Finally, by using the Radon-Nikodym Theorem we obtain (1.39). \square

From the lemma above we have the following natural definition.

Definition 1.30. Let $\mu \in \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ be a measure locally absolutely continuous with respect to $\tilde{\pi}_\sigma = \pi_\sigma \otimes dy$. We define the correlation functional associated to the measure μ as the measurable function $k_\mu : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ ⁴ given by the Radon-Nikodym derivative

$$k_\mu(\eta, y) := \frac{d\rho_\mu}{d\tilde{\lambda}_\sigma}(\eta, y). \quad (1.47)$$

According to Remark 1.7, the correlation functional k_μ can be represented as a sequence of non-negative measurable functions, so called correlation functions, defined as

$$\{k_\mu^{(n)}(x_1, \dots, x_n, y)\}_{n=0}^\infty, \quad (1.48)$$

where, $k_\mu^{(n)} \in L_{\text{sym}}^0((\mathbb{R}^d)^n \times \mathbb{R}^d)$ for any $n \in \mathbb{N}$.

⁴Hereinafter, \mathbb{R}_+ denotes the set of all non-negative real numbers, i.e. $\mathbb{R}_+ := [0, \infty)$.

Remark 1.31. As a result of Lemma 1.28 we have characterized a measure $\mu \in \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ with a sequence of real functions $\{k_\mu^{(n)}(x_1, \dots, x_n, y)\}_{n=0}^\infty$. Clearly, one may consider also the opposite problem, namely if it is possible to reconstruct a measure starting from a set of correlation functions. This interesting question is an analog of the well-known moment problem in classical analysis, but it will not be discussed in this work.

Let us note that, according to Definition 1.30, identity (1.35) can be rewritten as

$$\int_{\mathbb{R}^d} \int_{\Gamma} (KG(\cdot, y))(\gamma) d\mu(\gamma, y) = \int_{\mathbb{R}^d} \int_{\Gamma_0} G(\eta, y) k_\mu(\eta, y) d\lambda_\sigma(\eta) dy, \quad (1.49)$$

for any $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$, or equivalently

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\Gamma} \sum_{\{x_1, \dots, x_n\} \subset \gamma} G^{(n)}(x_1, \dots, x_n, y) d\mu(\gamma, y) \\ &= \frac{1}{n!} \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \dots, x_n, y) k_\mu^{(n)}(x_1, \dots, x_n, y) dx_1 \dots dx_n dy, \end{aligned} \quad (1.50)$$

for any bounded function $G^{(n)}$ on $(\mathbb{R}^d)^n \times \mathbb{R}^d$ with bounded support in $(\mathbb{R}^d)^n$. Equations (1.49) and (1.50) can be considered as alternative definitions of the correlation functional and correlation functions, respectively.

Remark 1.32. Let us mention that the r.h.s. of equation (1.49) defines a duality between quasi-observables and correlation functionals, see Section 2.2 for details.

Chapter 2

Random walks in Markov environments

In this chapter we present a general description of the random walk (RW) of a jumping particle moving in \mathbb{R}^d and interacting with an evolving random environment (RE). The latter is represented by an infinite particle system described by some non-equilibrium Markov dynamics in \mathbb{R}^d .

According to the general framework introduced in Chapter 1, we analyze these models of RWRE on three different levels. We start by defining a microscopic dynamics of the particles in terms of an heuristic Markov generator which describes an evolution of observables defined on the phase space. Afterwards, we derive a statistical description of the stochastic evolution in terms of a hierarchy for correlation functions associated with states of the model. Finally, we consider a mesoscopic limit of the evolution which leads to a system of two kinetic equations for the densities of the jumping particle and of particles of the environment.

General considerations will be illustrated in details in many concrete models which will be analyzed rigorously in the next chapters.

2.1 Microscopic description of the model

We study the RW of a particle moving in \mathbb{R}^d accordingly to a jump process and interacting with a RE formed by infinitely many particles. These particles are identical each other and are described by some non-equilibrium Markov dynamics in \mathbb{R}^d .

According to Definition 1.1, we consider as phase space of the model $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$, where \mathbb{R}^d is the phase space of the tagged particle, and $\Gamma(\mathbb{R}^d)$ is the phase space of the environment. Let us recall that an element of the phase space is given by a pair (γ, y) , where $y \in \mathbb{R}^d$ corresponds to the position of the jumping particle, whereas $\gamma \in \Gamma(\mathbb{R}^d)$ is the configuration of points representing the position of all the particles of the environment. We call functions $F(\gamma, y)$ on the space $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ observables.

Heuristically, the mechanism of evolution of the model can be described by a Markov pregenerator L acting on some proper space of observables. As we have two distinct systems, this pregenerator has the following general form

$$L = L_{RE} + L_{RW}, \tag{2.1}$$

where the operator L_{RE} defines the Markov dynamics of the particles of the environment and L_{RW} describes the RW of the tagged particle. Given an observable $F : \Gamma \times \mathbb{R}^d \rightarrow \mathbb{R}$: the operator L_{RE} acts only on the first argument $\gamma \in \Gamma(\mathbb{R}^d)$, namely $(L_{RE}F)(\gamma, y) = (L_{RE}F(\cdot, y))(\gamma)$ for any $y \in \mathbb{R}^d$; whereas, the operator L_{RW} acts on $y \in \mathbb{R}^d$, but depends, due to some interaction, on the configuration of RE, $\gamma \in \Gamma(\mathbb{R}^d)$. In formulas, $(L_{RW}F)(\gamma, y) := (L_{RW}(\gamma)F)(\gamma, y)$.

Remark 2.1. Note that as common characteristic of RWREs: the generator L_{RE} is Markovian by definition, the generator L_{RW} is not Markovian due to the presence of the interaction with RE, but, on the other hand, the whole generator L will still be Markovian.

The independent evolution of RE is described by a non-equilibrium Markov dynamic of infinitely many particles in continuum with phase space $\Gamma(\mathbb{R}^d)$. We consider two important classes of such interacting particle systems: birth-and-death models and hopping particle systems. Birth-and-death dynamics on the configuration space are non-conservative Markov dynamics where particles do not move, but randomly appear and disappear in \mathbb{R}^d with the rates

$$\begin{aligned} d(x, \gamma), & \quad \text{for death of a particle at } x \in \gamma \text{ of a configuration } \gamma; \\ b(x, \gamma), & \quad \text{for birth of a new particle at } x \in \mathbb{R}^d \text{ in a configuration } \gamma. \end{aligned}$$

The corresponding Markov generator L_{RE} on $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ has the following heuristic representation ¹

$$\begin{aligned} (L_{RE}F)(\gamma, y) := & \sum_{x \in \gamma} d(x, \gamma \setminus x) [F(\gamma \setminus x, y) - F(\gamma, y)] + \\ & \int_{\mathbb{R}^d} dx b(x, \gamma) [F(\gamma \cup x, y) - F(\gamma, y)], \end{aligned} \quad (2.2)$$

where we assume that

$$0 \leq d(x, \eta), b(x, \eta) < \infty, \quad \eta \in \Gamma_0 \setminus \{\emptyset\}, x \in \mathbb{R}^d \setminus \eta, \quad (2.3)$$

$$\int_M (d(x, \eta) + b(x, \eta)) d\lambda(\eta) < \infty, \quad M \in \mathcal{B}_b(\Gamma_0), \text{ a.a. } x \in \mathbb{R}^d, \quad (2.4)$$

$$\int_{\Lambda} (d(x, \eta) + b(x, \eta)) dx < \infty, \quad \eta \in \Gamma_0, \Lambda \in \mathcal{B}_b(\mathbb{R}^d). \quad (2.5)$$

Different birth-and-death models, corresponding to different choices of rates $d(x, \gamma)$ and $b(x, \gamma)$, may be found in [60] and references therein.

In contrast, hopping particles models are conservative Markov dynamics where different particles randomly jump over the space \mathbb{R}^d according to a rate

$$c(\gamma, x, x'), \quad \text{hop of the particle at } x \in \gamma \text{ to a site } x' \in \mathbb{R}^d.$$

The Markov generator L_{RE} describing this dynamics on $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ has the form

$$(L_{RE}F)(\gamma, y) := \sum_{x \in \gamma} \int_{\mathbb{R}^d} dx' c(\gamma, x, x') [F(\gamma \setminus x \cup x', y) - F(\gamma, y)], \quad (2.6)$$

where we assume that

$$0 \leq c(\eta, x, x') < \infty, \quad \eta \in \Gamma_0, x, x' \in \mathbb{R}^d \setminus \eta, \quad (2.7)$$

$$\int_M c(\eta, x, x') d\lambda(\eta) < \infty, \quad M \in \mathcal{B}_b(\Gamma_0), \text{ a.a. } x, x' \in \mathbb{R}^d, \quad (2.8)$$

$$\int_{\mathbb{R}^d} c(\eta, x, x') dx' < \infty, \quad \eta \in \Gamma_0, \text{ a. a. } x \in \mathbb{R}^d. \quad (2.9)$$

Many examples of hopping-particles systems, corresponding to different choice of the jump rate $c(\gamma, x, x')$ may be found in [60] and references therein.

Remark 2.2. It is possible to check that, under assumptions (2.3)-(2.5) and (2.7)-(2.9), the operator L_{RE} , both in (2.2) and (2.6), is such that

$$L_{RE}F \in L^0(\tilde{\Gamma}) \quad \text{for all } F \in K(B_{\text{bs}}(\tilde{\Gamma}_0)).$$

For further details see e.g. [49, 60].

¹In the following, for simplicity of notation, we will always write x instead of $\{x\}$.

The free RW of the tagged particle is a jump Markov process in \mathbb{R}^d . The intensity of the jump from a point $y \in \mathbb{R}^d$ to $z \in \mathbb{R}^d$ is given by a kernel $a(y - z)$, which we assume to be non-negative, even and integrable:

$$0 \leq a \in L^1(\mathbb{R}^d, dx), \quad a(-x) = a(x) \text{ for all } x \in \mathbb{R}^d.$$

We introduce an interaction of the tagged particle with the other particles of the environment through a non-negative function $\lambda_{\text{int}}(\gamma, y, z) \geq 0$ which modulates the intensity of jumps of the tagged particle in depending on RE, namely

$$a(y - z) \lambda_{\text{int}}(\gamma, y, z), \quad y, z \in \mathbb{R}^d \text{ and } \gamma \in \Gamma(\mathbb{R}^d).$$

Then, the corresponding generator L_{RW} in $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ becomes

$$(L_{RW}F)(\gamma, y) := \int_{\mathbb{R}^d} dz \lambda_{\text{int}}(\gamma, y, z) a(y - z) [F(\gamma, z) - F(\gamma, y)]. \quad (2.10)$$

Throughout this work, we assume some minimal condition on the interaction λ_{int} , in particular we demand that

$$\forall \eta \in \Gamma_0, \forall y \in \mathbb{R}^d, \quad \text{ess sup}_{z \in \mathbb{R}^d} \lambda_{\text{int}}(\eta, y, z) < \infty. \quad (2.11)$$

Remark 2.3. Note that condition (2.11) guarantees that

$$L_{RW}F \in L^0(\tilde{\Gamma}) \quad \text{for all } F \in K(B_{\text{bs}}(\tilde{\Gamma}_0)),$$

see e.g. (2.45) in the proof of Proposition 2.12.

In our analysis we discuss in details some concrete types of interaction λ_{int} , which correspond to different physical situations. We always assume that $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable non-negative even function, namely for any $x \in \mathbb{R}^d$

$$\phi(x) = \phi(-x), \quad \phi(x) \geq 0. \quad (2.12)$$

Case I. For any $\gamma \in \Gamma(\mathbb{R}^d)$ and $y, z \in \mathbb{R}^d$,

$$\lambda_{\text{int}}(\gamma, y, z) = \lambda^{(1)}(\gamma, y) = e^{-\sum_{x \in \gamma} \phi(x-y)}. \quad (2.13)$$

Such interaction implies that the tagged particle moves slower in regions with high concentration of points of the environment.

Case II. For any $\gamma \in \Gamma(\mathbb{R}^d)$ and $y, z \in \mathbb{R}^d$,

$$\lambda_{\text{int}}(\gamma, y, z) = \lambda^{(2)}(\gamma, y) = \lambda_0 + \sum_{x \in \gamma} \phi(x - y), \quad (2.14)$$

where λ_0 is some non-negative constant and the pair potential $\phi(\cdot)$ is additionally bounded. Such interaction implies that the tagged particle moves faster in regions with high concentration of points of the environment.

Case III. For any $\gamma \in \Gamma(\mathbb{R}^d)$ and $y, z \in \mathbb{R}^d$,

$$\lambda_{\text{int}}(\gamma, y, z) = \lambda^{(3)}(\gamma, z) = e^{-\sum_{x \in \gamma} \phi(x-z)}. \quad (2.15)$$

Such interaction implies that the tagged particle tends to move towards regions with low concentration of points of the environment.

Case IV. For any $\gamma \in \Gamma(\mathbb{R}^d)$ and $y, z \in \mathbb{R}^d$,

$$\lambda_{\text{int}}(\gamma, y, z) = \lambda^{(4)}(\gamma, z) = \lambda_0 + \sum_{x \in \gamma} \phi(x - z), \quad (2.16)$$

where λ_0 is some non-negative constant and the function $\phi(\cdot)$ is also bounded. Such interaction implies that the tagged particle tends to move towards regions with high concentration of points of the environment.

Remark 2.4. Let us note that all the interactions defined above satisfy condition (2.11), see Section 2.2.1 for details. In principle, many others are possible. For example one may consider a combination between the interaction I or III and the interaction II or IV, such as

$$\lambda_{\text{int}}(\gamma, y, z) = e^{-\sum_{x \in \gamma} \beta(x-y)} \sum_{x' \in \gamma} \phi(x' - z), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d,$$

where $\beta, \phi : \mathbb{R}^d \rightarrow \mathbb{R}$ are two non-negative even functions and ϕ is additionally bounded.

2.2 Statistical description of the model: evolution of correlation functions

In this section we study the stochastic dynamics of RWREs defined by the heuristic generator (2.1). Let us note that for such models there are essential difficulties for the construction of the corresponding spatial Markov process,

$$\Gamma(\mathbb{R}^d) \times \mathbb{R}^d \ni (\gamma, y) \longmapsto Z_t^{(\gamma, y)} = (X_t^\gamma, Y_t^y) \in \Gamma(\mathbb{R}^d) \times \mathbb{R}^d, \quad t > 0.$$

Indeed, Markov processes on the configuration space describing the evolution of RE can be constructed only in some special case, see e.g. [100] for a recent review and [65, 70] for more detailed results about birth-and-death processes. As an alternative approach, we consider the evolution of states associated with the underlying Markov dynamics. Such a statistical approach has been proposed in [55, 60, 62] and is often the only available technical tool to construct the non-equilibrium stochastic evolution of interacting particle systems in the continuum, see e.g. [10, 45, 49, 58, 84].

Let us recall that, for the considered models, a state corresponds to a measure on the phase space, i.e. $\mu(d\gamma, dy) \in \mathcal{M}(\Gamma \times \mathbb{R}^d)$. The evolution of states can be described by the forward Kolmogorov equation, or the Fokker-Planck equation. Let us consider the backward Kolmogorov equation associated to the pregenerator (2.1) for observables $F \in K(B_{\text{bs}}(\tilde{\Gamma}_0))$,

$$\frac{\partial}{\partial t} F_t = (L_{RE} + L_{RW}) F_t, \quad t \geq 0. \quad (2.17)$$

Using the pairing between functions and measures on $\Gamma \times \mathbb{R}^d$,

$$\langle F, \mu \rangle := \int_{\Gamma} \int_{\mathbb{R}^d} d\mu(\gamma, y) F(\gamma, y), \quad (2.18)$$

we can consider the evolution equation

$$\frac{\partial}{\partial t} \langle F, \mu_t \rangle = \langle (L_{RE} + L_{RW}) F, \mu_t \rangle, \quad t \geq 0, \quad (2.19)$$

in some proper space of measures on $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$. In fact, the solution to the equation (2.19) describes the time evolution of distributions instead of the initial points in the Markov process. We can rewrite it in the following heuristic form

$$\frac{\partial}{\partial t} \mu_t = (L_{RE}^* + L_{RW}^*) \mu_t, \quad t \geq 0, \quad (2.20)$$

where $L^* := L_{RE}^* + L_{RW}^*$ is (informally) adjoint operator of L w.r.t. the pairing (2.18).

An important technical observation to study the evolution equation (2.19) concerns the possibility to reformulate the problem in terms of the corresponding correlation functions. According to Section 1.2.2, let us suppose that $\mu_t \in \mathcal{M}_{\text{fm}}(\Gamma \times \mathbb{R}^d)$ is a solution to (2.19) which remains locally absolutely continuous w.r.t. the Poisson-type measure $\tilde{\pi}$, for all $t > 0$ provided that μ_0 has such a property. Then, we know that the measure μ_t can be characterized by a sequence of non-negative functions, the so-called correlation functions (cf. Definition 1.30),

$$\{k_{\mu_t}^{(n)}(x_1, \dots, x_n, y)\}_{n=0}^{\infty}, \quad x_i, y \in \mathbb{R}^d, i \geq 1. \quad (2.21)$$

In the following, for convenience, we set $k_t^{(n)} := k_{\mu_t}^{(n)}$, for any $n \geq 0$. For the present purpose, it is convenient to rewrite the sequence of correlation functions as a function, called correlation functional, on the space $\Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d$,

$$k_t(\eta, y) : \Gamma_0(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}_+, \quad t \geq 0. \quad (2.22)$$

The evolution equation for the correlation functional $k_t(\eta, y)$ can be derived by using harmonic analysis on the configuration space and, in particular, the K -transform introduced in Section 1.1.2. Using identity (1.49), we may rewrite equation (2.19) in the following way

$$\frac{d}{dt} \langle \langle K^{-1}F, k_t \rangle \rangle = \langle \langle K^{-1}LF, k_t \rangle \rangle, \quad L = L_{RE} + L_{RW}, \quad (2.23)$$

where we consider the pairing between functions on $\Gamma_0 \times \mathbb{R}^d$ given by

$$\langle \langle G, k \rangle \rangle = \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) G(\eta, y) k_{\mu}(\eta, y). \quad (2.24)$$

Remark 2.5. According to Remark 2.2 and 2.3, under conditions (2.3)-(2.5) (or (2.7)-(2.9)) and (2.11), we have

$$LF \in L^0(\tilde{\Gamma}) \quad \text{for all } F \in K(B_{\text{bs}}(\tilde{\Gamma}_0)).$$

Then, one can calculate $K^{-1}LF$ in (2.23) by using (1.22).

Next, if we substitute $F = KG$, $G \in B_{\text{bs}}(\tilde{\Gamma}_0)$, in (2.23), we find

$$\frac{d}{dt} \langle \langle G, k_t \rangle \rangle = \langle \langle (\hat{L}_{RE} + \hat{L}_{RW})G, k_t \rangle \rangle, \quad (2.25)$$

where the operator $\hat{L} := \hat{L}_{RE} + \hat{L}_{RW}$ is the image of L under the K -transform, namely

$$\hat{L} = K^{-1}LK = K^{-1}L_{RE}K + K^{-1}L_{RW}K =: \hat{L}_{RE} + \hat{L}_{RW}. \quad (2.26)$$

Let us recall that this operator is well defined (point-wise) for all $G \in B_{\text{bs}}(\tilde{\Gamma}_0)$, see Remark 2.5.

As a result, we are interested in a weak solution to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} k_t = (\hat{L}_{RE}^* + \hat{L}_{RW}^*) k_t, & t \geq 0, \\ k_t|_{t=0} = k_0 \end{cases}, \quad (2.27)$$

where the operator $\hat{L}^* := \hat{L}_{RE}^* + \hat{L}_{RW}^*$ is the adjoint operator of \hat{L} w.r.t. the duality (2.24), namely

$$\int_{\mathbb{R}^d} \int_{\Gamma_0} dy d\lambda(\eta) (\hat{L}G)(\eta, y) k(\eta, y) = \int_{\mathbb{R}^d} \int_{\Gamma_0} dy d\lambda(\eta) G(\eta, y) (\hat{L}^*k)(\eta, y). \quad (2.28)$$

Note that due to the linearity of the pairing (2.28), we have

$$\int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) G(\eta, y) (\hat{L}^*k)(\eta, y) =$$

$$\begin{aligned}
&= \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) (\widehat{L}_{REG})(\eta, y) k(\eta, y) + \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) (\widehat{L}_{RWG})(\eta, y) k(\eta, y) \\
&=: \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) G(\eta, y) (\widehat{L}_{RE}^* k)(\eta, y) + \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) G(\eta, y) (\widehat{L}_{RW}^* k)(\eta, y).
\end{aligned} \tag{2.29}$$

The procedure of deriving the operator \widehat{L} for a given L is fully combinatorial, while in obtaining the expression for the operator \widehat{L}^* we use the discrete integration by parts formula represented by the Minlos's formula (1.18).

Remark 2.6. It may be useful to consider not directly the evolution equation (2.27) for correlation functionals, but its pre-dual problem, w.r.t. (2.28), namely

$$\frac{\partial}{\partial t} G_t = (\widehat{L}_{RE} + \widehat{L}_{RW}) G_t, \quad t \geq 0, \tag{2.30}$$

for functions, so-called quasi-observables, $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$. Here the operator $\widehat{L} := \widehat{L}_{RE} + \widehat{L}_{RW}$ is given by (2.26).

Remark 2.7. Let us recall that any correlation functional $k(\eta, y)$ may be identified with an infinite vector of (correlation) functions of the growing number of variables, cf. (2.21). Thus, as matter of fact, instead of equation (2.20) for measure we deal with an infinite system of equations for functions of finite number of variables. This chain of evolution equations constitutes the so-called hierarchy which is an analog of the BBGKY hierarchy for Hamiltonian systems, see e.g. [34].

One of the main aims of the present thesis is to study the classical solution to (2.27) in a proper space. Clearly, the choice of such a space is motivated by the applications one has in mind. In this sense we consider two different cases:

- (1) *sub-Poissonian bounded functions*: for $C > 0$, we consider the Banach space \mathcal{K}_C^∞ consisting of all measurable functions $k : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\text{ess sup}_{(\eta, y) \in \Gamma_0 \times \mathbb{R}^d} C^{-|\eta|} |k(\eta, y)| < \infty. \tag{2.31}$$

We equip \mathcal{K}_C^∞ equipped with the norm $\|\cdot\|_{\mathcal{K}_C^\infty}$ given by the r.h.s. of (2.31).

- (2) *sub-Poissonian integrable functions*: for $C > 0$, we consider the Banach space \mathcal{K}_C^1 consisting of all measurable functions $k : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} dy \text{ess sup}_{\eta \in \Gamma_0} C^{-|\eta|} |k(\eta, y)| < \infty. \tag{2.32}$$

We equip \mathcal{K}_C^1 equipped with the norm $\|\cdot\|_{\mathcal{K}_C^1}$ given by the r.h.s. of (2.32).

We use the notation \mathcal{K}_C^q with $q = \infty, 1$ to indicate the two different spaces.

Remark 2.8. Note that in both cases listed above, we consider measurable functions which satisfies the so-called Ruelle bounds [107] in the space Γ_0 . Indeed, for any $k \in \mathcal{K}_C^q$, $q = \infty, 1$, one has

$$|k(\eta, y)| \leq M(y) C^{|\eta|}, \quad \tilde{\lambda} - \text{a.a. } \eta \in \Gamma_0, y \in \mathbb{R}^d, \tag{2.33}$$

with $M(y) := \text{ess sup}_{\eta \in \Gamma_0} C^{-|\eta|} |k(\eta, y)|$. For interacting particle systems this is a rather natural assumption, for example, for the correlation functions associated to a Gibbs measure inequality (2.33) holds true, see e.g. [108]. On the other hand, concerning the dependence of $k(\eta, y)$ on $y \in \mathbb{R}^d$, we have more choices and we can consider measurable functions which are essential bounded or integrable in $y \in \mathbb{R}^d$, for λ -a.a. $\eta \in \Gamma_0$. This second case appears to be more interesting from a physical point of view since such correlation functions are, in general, associated to some probability measure.

The study of the evolution equation for correlation functions (2.27) in the Banach spaces \mathcal{K}_C^q is a highly non trivial problem in infinite dimensional analysis. Nevertheless, one can consider three alternative methods to perform this analysis, see e.g. [45, 49, 61] and references therein.

The first technical possibility is to construct a strongly continuous semigroup in the space \mathcal{K}_C^q , $q = \infty, 1$, with generator \widehat{L}^* . However, by the Lotz's theorem [94] we know that in a L^∞ -space any strongly continuous semigroup is associated to a bounded generator. Note that in our case the operator \widehat{L}^* is unbounded in \mathcal{K}_C^q . To avoid this difficulty one can use a trick which goes back to Phillips [102]. The main idea of this method is to consider a semigroup in L^∞ as the dual semigroup to a strongly continuous semigroup in the pre-dual L^1 -space. As result we will have a strongly continuous semigroup not on the whole space L^∞ but only on the closure of the domain of its generator. We will apply successfully this approach on the space \mathcal{K}_C^∞ only, see Chapter 3. The use of this method on the space \mathcal{K}_C^1 presents some technical difficulties that have not been solved completely at present.

For the study of the initial value problem (2.27) on the space \mathcal{K}_C^∞ , the Philips's technique leads to the following scheme. By means of duality (2.28), we consider the evolution equation for quasi-observables (2.30) in the pre-dual space

$$\mathcal{L}_C := L^1(\Gamma_0 \times \mathbb{R}^d, C^{|\eta|} d\lambda(\eta) dy).$$

If we can construct a strongly continuous semigroup $\widehat{T}(t)$ in \mathcal{L}_C with generator \widehat{L} , then, from Philip's result it will follow that the restriction of the dual semigroup $\widehat{T}^*(t)$ onto $\overline{\text{Dom}(\widehat{L}^*)}$ is a strongly continuous semigroup with generator which is a part of \widehat{L}^* . The dual semigroup $\widehat{T}^*(t)$ will provide a solution to (2.27) on the space $\overline{\text{Dom}(\widehat{L}^*)}$. Afterwards, we would like to find a more useful universal subspace of \mathcal{K}_C^∞ where to formulate this result.

Another possibility for the study of the Cauchy problem (2.27) is to consider this evolution equation in a proper scale of Banach spaces $\{\mathcal{K}_C^q\}_{C_* \leq C \leq C^*}$, $q = \infty, 1$. This approach can be realized by using the so-called Ovsyannikov's method see e.g. [98, 120]. Compared with the semigroup approach, this method provides less restrictions on the parameters of the system, but leads to a local in time solution only. The Ovsyannikov's method can be applied on both Banach spaces \mathcal{K}_C^∞ and \mathcal{K}_C^1 . In this thesis, see Chapter 4, we consider only the latter space. In this case we can show that there exists $T > 0$ such that for any $t \in [0, T)$ the initial value problem (2.27) has an unique solution k_t which lies in the space $\mathcal{K}_{C_t}^1$ for some $C_t \in [C_*, C^*]$.

Finally, a third possibility consists in expressing the dynamic of correlation functions in terms of the corresponding generating functionals, the so-called Bogoliubov functionals [14, 79], defined by

$$Z_\mu(\theta, \psi) = \int_{\mathbb{R}^d} dy \psi(y) \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_\mu(\eta, y), \quad \theta \in L^1(\mathbb{R}^d), \psi \in L^\infty(\mathbb{R}^d). \quad (2.34)$$

Informally, this new formalism allows us to rewrite the infinite system of equations (2.27) as a functional equation on the Banach space of all (entire) generating functionals, see e.g. [79]. This new evolution equation can be again solved using Ovsyannikov's method in the framework of a scale of Banach spaces. We refer to Chapter 5 for a detailed description of this approach in the study of the initial value problem (2.27) on the space \mathcal{K}_C^1 .

Remark 2.9. Let us also mention a further possibility to study the evolution equation (2.27), which will not be considered in this work. This approach consists in a combination of semigroup techniques and the Ovsyannikov's method described above in the spirit of [47, 48]. In this case, we will have again a solution local in time, but under different physical assumptions on the parameters of the model.

In the next subsections we will derive explicitly the evolution equation for quasi-observables and correlation functions, respectively, for concrete models of REs and RWs.

2.2.1 Generator for quasi-observables

Informally, the evolution for quasi-observables, $G : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$, is described by the initial value problem, cf. equation (2.30),

$$\begin{cases} \frac{\partial}{\partial t} G_t = (\widehat{L}_{RE} + \widehat{L}_{RW}) G_t, & t \geq 0, \\ G_t|_{t=0} = G_0 \end{cases}, \quad (2.35)$$

where the \widehat{L}_{RE} and \widehat{L}_{RW} are the image of the operators L_{RE} and L_{RW} , respectively, under the K -transform, see (2.26),

$$\widehat{L}_{RE} = K^{-1} L_{RE} K \quad \text{and} \quad \widehat{L}_{RW} = K^{-1} L_{RW} K.$$

The expression of the generator for the quasi-observables \widehat{L}_{RE} , associated to the environment, in the case of a birth-and-death process, as well as a hopping particle system, may be found in [60]. For completeness we formulate general results.

Proposition 2.10. *Let us consider an operator L_{RE} as in (2.2) and such that conditions (2.3)-(2.5) hold. Then, the action of \widehat{L}_{RE} on functions $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by*

$$\begin{aligned} (\widehat{L}_{RE} G)(\eta, y) = & - \sum_{\xi \subset \eta} G(\xi, y) \sum_{x \in \xi} (K^{-1} d(x, \cdot \cup (\xi \setminus x))) (\eta \setminus \xi) + \\ & \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} dx G(\xi \cup x, y) (K^{-1} b(x, \cdot \cup \xi)) (\eta \setminus \xi). \end{aligned} \quad (2.36)$$

Moreover, one has $\widehat{L}_{RE} G \in L^0(\widetilde{\Gamma}_0)$ for any $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$.

Proposition 2.11. *Let us consider an operator L_{RE} as in (2.6) and such that conditions (2.7)-(2.9) hold. Then, the action of \widehat{L}_{RE} on functions $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by*

$$(\widehat{L}_{RE} G)(\eta, y) = \int_{\mathbb{R}^d} dx' \sum_{\xi \subset \eta} \sum_{x \in \xi} [G(\xi \setminus x \cup x', y) - G(\xi, y)] (K^{-1} c_{x,x'}(\cdot \cup (\xi \setminus x))) (\eta \setminus \xi). \quad (2.37)$$

Moreover, one has $\widehat{L}_{RE} G \in L^0(\widetilde{\Gamma}_0)$ for any $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$.

Let us now consider the generator of quasi-observables \widehat{L}_{RW} associated to the RW of the tagged particle. First we compute an explicit formula for a generic interaction λ_{int} and after we specify it for each of the concrete examples I-IV in Section 2.1.

Proposition 2.12. *Let us consider the RW of a jumping particle described by an operator L_{RW} as in (2.10) and such that condition (2.11) holds. Then, the action of \widehat{L}_{RW} on functions $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by*

$$(\widehat{L}_{RW} G)(\eta, y) = \int_{\mathbb{R}^d} dz a(y - z) \sum_{\xi \subset \eta} (G(\xi, z) - G(\xi, y)) (K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, y, z)) (\eta \setminus \xi). \quad (2.38)$$

Moreover, one has $\widehat{L}_{RW} G \in L^0(\widetilde{\Gamma}_0)$ for any $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$.

Proof. We start from the heuristic Markov generator of the RW defined in equation (2.10), namely for any $F \in \mathcal{F}_{\text{cyl}}(\widetilde{\Gamma})$

$$(L_{RW} F)(\gamma, y) = \int_{\mathbb{R}^d} dz \lambda_{\text{int}}(\gamma, y, z) a(y - z) [F(\gamma, z) - F(\gamma, y)], \quad \gamma \in \Gamma, y \in \mathbb{R}^d.$$

For $F = KG$, with $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$, by using the linearity of the K -transform, we can write it as

$$(L_{RW} F)(\gamma, y) = \int_{\mathbb{R}^d} dz \lambda_{\text{int}}(\gamma, y, z) a(y - z) [(KG)(\gamma, z) - (KG)(\gamma, y)]$$

$$= \int_{\mathbb{R}^d} dz \lambda_{\text{int}}(\gamma, y, z) a(y-z) (K[G(\cdot, z) - G(\cdot, y)])(\gamma),$$

and by applying the inverse K -transform, we obtain

$$\begin{aligned} (\widehat{L}_{RW}G)(\eta, y) &= (K^{-1}L_{RW}F)(\eta, y) \\ &= \int_{\mathbb{R}^d} dz a(y-z) \left(K^{-1} \left[\lambda(\cdot, y, z) K(G(\cdot, z) - G(\cdot, y)) \right] \right)(\eta). \end{aligned} \quad (2.39)$$

If for any $\eta \in \Gamma_0$ and $y, z \in \mathbb{R}^d$ we define

$$A(\eta, y, z) := (K^{-1}\lambda(\cdot, y, z))(\eta), \quad (2.40)$$

then, from the identity (1.29), it follows that

$$(\widehat{L}_{RW}G)(\eta, y) = \int_{\mathbb{R}^d} dz a(y-z) (A(\cdot, y, z) \star [G(\cdot, z) - G(\cdot, y)])(\eta). \quad (2.41)$$

Next, using the definition of the \star -convolution, see formula (1.28), one has

$$(\widehat{L}_{RW}G)(\eta, y) = \int_{\mathbb{R}^d} dz a(y-z) \left[\sum_{\xi \subset \eta} (G(\xi, z) - G(\xi, y)) \sum_{\zeta \subset \xi} A((\eta \setminus \xi) \cup \zeta, y, z) \right]. \quad (2.42)$$

Note that the second sum in the square brackets can be rewritten in terms of K -transform, i.e.

$$(\widehat{L}_{RW}G)(\eta, y) = \int_{\mathbb{R}^d} dz a(y-z) \left[\sum_{\xi \subset \eta} (G(\xi, z) - G(\xi, y)) (KA((\eta \setminus \xi) \cup \cdot, y, z))(\xi) \right].$$

Hence, we can use of the simple identity

$$(K^{-1}F(\cdot \cup \eta_2, y, z))(\xi_1) = (KG(\xi_1 \cup \cdot, y, z))(\eta_2), \quad \xi_1 \cap \eta_2 = \emptyset \quad (2.43)$$

and obtain (2.38). Note that in this formula the term $(K^{-1}\lambda_{\text{int}}(\cdot \cup \xi, y, z))(\eta)$ has sense due to (2.11).

Finally, by Proposition 1.17 for any $F = KG$, with $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$, there exist $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, $N \in \mathbb{N}$ and $C > 0$ such that

$$|F(\gamma, z) - F(\gamma, y)| \leq C(1 + |\gamma_\Lambda|)^N, \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d. \quad (2.44)$$

Therefore, by using (2.11) one has

$$\begin{aligned} |(L_{RW}F)(\eta, y)| &\leq C(1 + |\eta_\Lambda|)^N \int_{\mathbb{R}^d} dz \lambda_{\text{int}}(\eta, y, z) a(y-z) \\ &\leq \langle a \rangle C(1 + |\eta_\Lambda|)^N \sup_{z \in \mathbb{R}^d} \lambda_{\text{int}}(\eta, y, z) < \infty, \quad \eta \in \Gamma_0, y \in \mathbb{R}^d \end{aligned} \quad (2.45)$$

and, consequently,

$$\left| (\widehat{L}_{RW}G)(\eta, y) \right| = \left| (K^{-1}L_{RW}F)(\eta, y) \right| < \infty, \quad \eta \in \Gamma_0, y \in \mathbb{R}^d. \quad (2.46)$$

In particular, the latter justifies all the interchange order of summations and integrations in the previous computations. \square

Case I. Let us consider a RW whose interaction with the RE is given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(1)}(\gamma, y) = e^{-\sum_{x' \in \gamma} \phi(x' - y)}, \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d, \quad (2.47)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is some non-negative measurable even function. Clearly condition (2.11) is satisfied.

Having in mind Example 1.19, we can rewrite this interaction as

$$\lambda^{(1)}(\eta, y) = e_\lambda(e^{-\phi(\cdot - y)}, \eta) = (K e_\lambda(e^{-\phi(\cdot - y)} - 1, \cdot))(\eta), \quad (2.48)$$

for any $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Hence, from equality (2.40) we can identify

$$A(\eta, y) = e_\lambda(e^{-\phi(\cdot - y)} - 1, \eta) = \prod_{x' \in \eta} (e^{-\phi(x' - y)} - 1) \quad (2.49)$$

and, by means of the identity (2.43), we find

$$\begin{aligned} (K^{-1} \lambda^{(1)}(\cdot \cup \xi, y))(\eta \setminus \xi) &= (K A((\eta \setminus \xi) \cup \cdot, y))(\xi) \\ &= (K e_\lambda(e^{-\phi(\cdot - y)} - 1, (\eta \setminus \xi) \cup \cdot))(\xi), \end{aligned}$$

for any $\xi \subset \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Next, we can use straightforward definition of K -transform to rewrite it as

$$\begin{aligned} (K^{-1} \lambda^{(1)}(\cdot \cup \xi, y))(\eta \setminus \xi) &= \sum_{\zeta \subset \xi} \prod_{x' \in (\eta \setminus \xi) \cup \zeta} (e^{-\phi(x' - y)} - 1) \\ &= \prod_{x' \in (\eta \setminus \xi)} (e^{-\phi(x' - y)} - 1) \sum_{\zeta \subset \xi} \prod_{x' \in \zeta} (e^{-\phi(x' - y)} - 1) \\ &= \lambda^{(1)}(\xi, y) e_\lambda(e^{-\phi(\cdot - y)} - 1, (\eta \setminus \xi)), \quad \xi \subset \eta \in \Gamma_0, y \in \mathbb{R}^d. \end{aligned} \quad (2.50)$$

Thus, according to the result of Proposition 2.12, for interaction (2.47) the generator \widehat{L}_{RW} for quasi-observables $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by

$$(\widehat{L}_{RW} G)(\eta, y) = \sum_{\xi \subset \eta} \lambda^{(1)}(\xi, y) e_\lambda(e^{-\phi(\cdot - y)} - 1, (\eta \setminus \xi)) \int_{\mathbb{R}^d} dz a(y - z) [G(\xi, z) - G(\xi, y)]. \quad (2.51)$$

Case II. Let us consider the interaction given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(2)}(\gamma, y) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - y), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d, \quad (2.52)$$

where λ_0 is some non-negative constant and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is some non-negative measurable even function such that

$$C_\infty^\phi := \text{ess sup}_{x \in \mathbb{R}^d} \phi(x) < \infty. \quad (2.53)$$

Under conditions above, it is easy to check that condition (2.11) holds. Indeed

$$|\lambda^{(2)}(\eta, y)| \leq \lambda_0 + |\eta| \|\phi\|_\infty < \infty, \quad \text{for any } \eta \in \Gamma_0, y \in \mathbb{R}^d. \quad (2.54)$$

In this case, by using the results in Example 1.18, we can rewrite this interaction in the following way

$$\lambda^{(2)}(\eta, y) = \lambda_0 + \sum_{x' \in \eta} \phi(x' - y)$$

$$\begin{aligned}
&= (K\lambda_0 e_\lambda(0, \cdot))(\eta) + (K\phi(\cdot - y))(\eta) \\
&= (K(\lambda_0 e_\lambda(0, \cdot) + \phi(\cdot - y)))(\eta), \quad \eta \in \Gamma_0, y \in \mathbb{R}^d,
\end{aligned}$$

where, cf. Example 1.19,

$$e_\lambda(0, \eta) = 0^{|\eta|} = \begin{cases} 0, & \text{if } |\eta| \geq 1, \\ 1, & \text{if } |\eta| = 0. \end{cases} \quad (2.55)$$

Similarly one has

$$\begin{aligned}
\lambda^{(2)}((\eta \setminus \xi) \cup \xi, y) &= \lambda_0 + \sum_{x' \in (\eta \setminus \xi) \cup \xi} \phi(x' - y) \\
&= \lambda^{(2)}(\xi, y) + (K\phi(\cdot - y))(\eta \setminus \xi), \quad \xi \subset \eta \in \Gamma_0, y \in \mathbb{R}^d.
\end{aligned}$$

Let us apply the inverse K -transform to it. By linearity we find

$$(K^{-1}\lambda^{(2)}(\cdot \cup \xi, y))(\eta \setminus \xi) = (K^{-1}\lambda^{(2)}(\xi, y))(\eta \setminus \xi) + \left(K^{-1} \sum_{x' \in \cdot} \phi(x' - y) \right)(\eta \setminus \xi),$$

for any $\xi \subset \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Concerning the first term, we simply have

$$(K^{-1}\lambda^{(2)}(\xi, y))(\eta \setminus \xi) = \lambda^{(2)}(\xi, y) (K^{-1}1)(\eta \setminus \xi) = \lambda^{(2)}(\xi, y) 0^{|\eta \setminus \xi|},$$

where we used the notation (2.55). On the other hand, the second term can be computed by directly applying the definition of the K -transform, as a result we find

$$\left(K^{-1} \sum_{x' \in \cdot} \phi(x' - y) \right)(\eta \setminus \xi) = \phi(x' - y) \mathbb{1}_{\Gamma(1)}((\eta \setminus \xi) = \{x'\}).$$

By combing the last two equalities we find

$$(K^{-1}\lambda^{(2)}(\cdot \cup \xi, y))(\eta \setminus \xi) = \lambda^{(2)}(\xi, y) 0^{|\eta \setminus \xi|} + \phi(x' - y) \mathbb{1}_{\Gamma(1)}((\eta \setminus \xi) = \{x'\}), \quad (2.56)$$

for any $\xi \subset \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Hence, according to the result of Proposition 2.12, for interaction (2.52) the generator \widehat{L}_{RW} for quasi-observables $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by

$$\begin{aligned}
(\widehat{L}_{RW}G)(\eta, y) &= \lambda^{(2)}(\eta, y) \int_{\mathbb{R}^d} dz a(y - z) [G(\eta, z) - G(\eta, y)] + \\
&\quad \sum_{x' \in \eta} \phi(x' - y) \int_{\mathbb{R}^d} dz a(y - z) [G((\eta \setminus x'), z) - G((\eta \setminus x'), y)]. \quad (2.57)
\end{aligned}$$

Case III. Let us consider

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(3)}(\gamma, z) = e^{-\sum_{x' \in \gamma} \phi(x' - z)}, \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d, \quad (2.58)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is some non-negative measurable even function. Clearly condition (2.11) is again satisfied.

Analogously to Case I, cf. formula (2.50), we have

$$(K^{-1}\lambda^{(3)}(\cdot \cup \xi, z))(\eta \setminus \xi) = \lambda^{(3)}(\xi, z) e_\lambda(e^{-\phi(\cdot - z)} - 1, (\eta \setminus \xi)), \quad (2.59)$$

for any $\xi \subset \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Thus, according to the result of Proposition 2.12, for the considered interaction the generator \widehat{L}_{RW} for quasi-observables $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ has the form

$$(\widehat{L}_{RW}G)(\eta, y) = \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} dz a(y - z) \lambda^{(3)}(\xi, z) e_\lambda(e^{-\phi(\cdot - z)} - 1, (\eta \setminus \xi)) [G(\xi, z) - G(\xi, y)]. \quad (2.60)$$

Case IV. Consider the interaction given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(2)}(\gamma, z) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - z), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d, \quad (2.61)$$

where λ_0 is some non-negative constant and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is some non-negative bounded even function in $L^\infty(\mathbb{R}^d)$ with

$$C_\infty^\phi := \|\phi\|_\infty = \int_{\mathbb{R}^d} dx \phi(x). \quad (2.62)$$

One can check that condition (2.11) holds (compare to (2.54)). Moreover, according to (2.56), we have

$$(K^{-1}\lambda^{(4)}(\cdot \cup \xi, z))(\eta \setminus \xi) = \lambda^{(4)}(\xi, z) 0^{|\eta \setminus \xi|} + \phi(x' - z) \mathbf{1}_{\Gamma(1)}((\eta \setminus \xi) = \{x'\}), \quad (2.63)$$

for any $\xi \subset \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Thus, by using the result of Proposition 2.12 for the considered interaction, the generator \widehat{L}_{RW} for quasi-observables $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ has the form

$$\begin{aligned} (\widehat{L}_{RW}G)(\eta, y) &= \int_{\mathbb{R}^d} dz \lambda^{(4)}(\eta, z) a(y - z) [G(\eta, z) - G(\eta, y)] + \\ &\quad \int_{\mathbb{R}^d} dz a(y - z) \sum_{x' \in \eta} \phi(x' - z) [G((\eta \setminus x'), z) - G((\eta \setminus x'), y)]. \end{aligned} \quad (2.64)$$

2.2.2 Generator for correlation functions

As already discussed, the evolution of correlation functionals, $k : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}_+$, is described by the following hierarchy

$$\begin{cases} \frac{\partial}{\partial t} k_t = (\widehat{L}_{RE}^* + \widehat{L}_{RW}^*) k_t \\ k_t|_{t=0} = k_0 \end{cases}, \quad t \geq 0, \quad (2.65)$$

where the operators \widehat{L}_{RE}^* and \widehat{L}_{RW}^* are the adjoint operators of \widehat{L}_{RE} and \widehat{L}_{RW} , respectively, w.r.t. the duality (2.28), see (2.29).

The precise form of the operator \widehat{L}_{RE}^* , associated to the environment, can be found in [60] for different types of birth-and-death processes as well as hopping particles systems. Below we formulate these general results.

Proposition 2.13. *Let us consider an operator L_{RE} as in (2.2) and such that conditions (2.3)-(2.5) hold. Then the action of \widehat{L}_{RE}^* on functions $k \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by*

$$\begin{aligned} (\widehat{L}_{RE}^*k)(\eta, y) &= - \sum_{x \in \eta} \int_{\Gamma_0} d\lambda(\zeta) k(\zeta \cup \eta, y) (K^{-1}d(x, \cdot \cup \eta \setminus x))(\zeta) + \\ &\quad \sum_{x \in \eta} \int_{\Gamma_0} d\lambda(\zeta) k(\zeta \cup (\eta \setminus x), y) (K^{-1}b(x, \cdot \cup \eta \setminus x))(\zeta), \end{aligned} \quad (2.66)$$

where \widehat{L}_{RE}^*k is defined by (2.29).

Proposition 2.14. *Let us consider an operator L_{RE} as in (2.6) and such that conditions (2.7)-(2.9) hold. Then the action of \widehat{L}_{RE}^* on functions $k \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by*

$$\begin{aligned} (\widehat{L}_{RE}^*k)(\eta, y) &= \sum_{x' \in \eta} \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup (\eta \setminus x') \cup x, y) (K^{-1}c_{x,x'}(\cdot \cup (\eta \setminus x')))(\xi) - \\ &\quad \int_{\Gamma_0} d\lambda(\xi), k(\xi \cup \eta, y) \sum_{x \in \eta} \int_{\mathbb{R}^d} dx' (K^{-1}c_{x',x}(\cdot \cup (\eta \setminus x')))(\xi), \end{aligned} \quad (2.67)$$

where \widehat{L}_{RE}^*k is defined by (2.29).

In the following proposition we compute explicitly the exact form of the operator \widehat{L}_{RW}^* for a generic interaction λ_{int} .

Proposition 2.15. *Let us consider the RW of a jumping particle described by an operator L_{RW} as in equation (2.10) and such that condition (2.11) holds. Then the action of \widehat{L}_{RW}^* on functions $k \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by*

$$\begin{aligned} (\widehat{L}_{RW}^* k)(\eta, y) = \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) & \left[k(\zeta \cup \eta, z) \left(K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, z, y) \right) (\zeta) - \right. \\ & \left. k(\zeta \cup \eta, y) \left(K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, y, z) \right) (\zeta) \right], \end{aligned} \quad (2.68)$$

where $\widehat{L}_{RW}^* k$ is defined by (2.29).

Proof. The operator \widehat{L}_{RW}^* is defined through the identity

$$\int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) (\widehat{L}_{RW} G)(\eta, y) k(\eta, y) = \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) G(\eta, y) (\widehat{L}_{RW}^* k)(\eta, y), \quad (2.69)$$

for any $G, k \in B_{\text{bs}}(\widetilde{\Gamma}_0)$. By using the result of Proposition 2.12, the double integral in the l.h.s can be rewritten explicitly as

$$\begin{aligned} \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) (\widehat{L}_{RW} G)(\eta, y) k(\eta, y) = \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) & \left[\right. \\ & \left. \int_{\mathbb{R}^d} dz a(y-z) k(\eta, y) \sum_{\xi \subset \eta} [G(\xi, z) - G(\xi, y)] \left(K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, y, z) \right) (\eta \setminus \xi) \right]. \end{aligned}$$

Using the Minlos formula (1.18) we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) (\widehat{L}_{RW} G)(\eta, y) k(\eta, y) & = \\ = \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\xi) \int_{\Gamma_0} d\lambda(\eta) \int_{\mathbb{R}^d} dz a(y-z) & k(\eta \cup \xi, y) G(\xi, z) \left(K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, y, z) \right) (\eta) - \\ \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\xi) \int_{\Gamma_0} d\lambda(\eta) \int_{\mathbb{R}^d} dz a(y-z) & k(\eta \cup \xi, y) G(\xi, y) \left(K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, y, z) \right) (\eta). \end{aligned}$$

Note that in the first integral we exchange variables $y \leftrightarrow z$ and write

$$\begin{aligned} \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) (\widehat{L}_{RW} G)(\eta, y) k(\eta, y) & = \\ = \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) G(\eta, y) \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) & \times \\ & \left[k(\zeta \cup \eta, z) \left(K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, z, y) \right) (\zeta) - k(\zeta \cup \eta, y) \left(K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, y, z) \right) (\zeta) \right]. \end{aligned}$$

Thus, by comparing the expression above with (2.69), we obtain the statement (2.38). To conclude, let us note that the correctness of using (1.18) follows from the assumptions $G, k \in B_{\text{bs}}(\Gamma_0 \times \mathbb{R}^d)$ and (2.11), which make all integrals above to be finite. \square

Let us apply the result of the proposition above to each of the interactions I-IV in Section 2.1.

Case I. Let us consider the interaction

$$\lambda_{\text{int}}(\gamma, y, z) = \lambda^{(1)}(\gamma, y) = e^{-\sum_{x' \in \gamma} \phi(x' - y)}, \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d,$$

defined as in (2.47). In this case we have (cf. (2.50)),

$$\left(K^{-1} \lambda^{(1)}(\cdot \cup \eta, y) \right) (\zeta) = \lambda^{(1)}(\eta, y) e_\lambda \left(e^{-\phi(\cdot - y)} - 1, \zeta \right),$$

for any $\eta, \zeta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Therefore, according to Proposition 2.15, the operator \widehat{L}_{RW}^* for $k \in B_{\text{bs}}(\Gamma_0)$ is given by

$$\begin{aligned} \left(\widehat{L}_{RW}^* k \right) (\eta, y) &= \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y - z) \left[k(\zeta \cup \eta, z) \lambda^{(1)}(\eta, z) e_\lambda \left(e^{-\phi(\cdot - z)} - 1, \zeta \right) - \right. \\ &\quad \left. k(\zeta \cup \eta, y) \lambda^{(1)}(\eta, y) e_\lambda \left(e^{-\phi(\cdot - y)} - 1, \zeta \right) \right]. \end{aligned} \quad (2.70)$$

Case II. Consider the interaction

$$\lambda_{\text{int}}(\gamma, y, z) = \lambda^{(2)}(\eta, y) = \lambda_0 + \sum_{x' \in \eta} \phi(x' - y), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d,$$

defined as in (2.52)-(2.53). In this case, see (2.56),

$$\left(K^{-1} \lambda^{(2)}(\cdot \cup \eta, y) \right) (\zeta) = \lambda^{(2)}(\eta, y) 0^{|\zeta|} + \mathbb{1}_{\Gamma(1)}(\zeta = \{x'\}) \phi(x' - y),$$

for any $\eta, \zeta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Then, according to Proposition 2.15, the operator \widehat{L}_{RW}^* for $k \in B_{\text{bs}}(\Gamma_0)$ is given by

$$\begin{aligned} \left(\widehat{L}_{RW}^* k \right) (\eta, y) &= \int_{\mathbb{R}^d} dz a(y - z) \lambda^{(2)}(\eta, z) k(\eta, z) + \\ &\quad \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dz a(y - z) k(x' \cup \eta, z) \phi(x' - z) - \\ &\quad k(\eta, y) \lambda^{(2)}(\eta, y) \int_{\mathbb{R}^d} dz a(y - z) - \\ &\quad \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} da(y - z) k(x' \cup \eta, y) \phi(x' - y). \end{aligned} \quad (2.71)$$

Case III. For $\lambda_{\text{int}}(\gamma, y, z) = \lambda^{(3)}(\eta, z)$, $\gamma \in \Gamma$, $y, z \in \mathbb{R}^d$, as in (2.58) we have, see (2.59),

$$\left(K^{-1} \lambda^{(3)}(\cdot \cup \eta, z) \right) (\zeta) = \lambda^{(3)}(\eta, z) e_\lambda \left(e^{-\phi(\cdot - z)} - 1, \zeta \right), \quad \eta \in \Gamma, z \in \mathbb{R}^d.$$

Then, according to Proposition 2.15, in this case the operator \widehat{L}_{RW}^* for $k \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by

$$\begin{aligned} \left(\widehat{L}_{RW}^* k \right) (\eta, y) &= \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y - z) \left[k(\zeta \cup \eta, z) \lambda^{(3)}(\eta, z) e_\lambda \left(e^{-\phi(\cdot - y)} - 1, \zeta \right) - \right. \\ &\quad \left. k(\zeta \cup \eta, y) \lambda^{(3)}(\eta, z) e_\lambda \left(e^{-\phi(\cdot - z)} - 1, \zeta \right) \right]. \end{aligned} \quad (2.72)$$

Case IV. Consider the interaction $\lambda_{\text{int}}(\gamma, y, z) = \lambda^{(4)}(\eta, z)$ defined as in (2.61)-(2.62). In correspondence, we have (cf. (2.63)),

$$\left(K^{-1} \lambda^{(4)}(\cdot \cup \eta, z) \right) (\zeta) = \lambda^{(4)}(\eta, z) 0^{|\zeta|} + \mathbb{1}_{\Gamma(1)}(\zeta = \{x'\}) \phi(x' - z), \quad \gamma \in \Gamma, z \in \mathbb{R}^d.$$

Then, by Proposition 2.15 the operator \widehat{L}_{RW}^* for $k \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ is given by

$$\begin{aligned} (\widehat{L}_{RW}^* k)(\eta, y) &= \lambda^{(4)}(\eta, y) \int_{\mathbb{R}^d} dz a(y - z) k(\eta, z) + \\ &\int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dz a(y - z) k(x' \cup \eta, z) \phi(x' - y) - \\ &k(\eta, y) \int_{\mathbb{R}^d} dz a(y - z) \lambda^{(4)}(\eta, z) - \\ &\int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dz a(y - z) k(x' \cup \eta, y) \phi(x' - z). \end{aligned} \quad (2.73)$$

2.3 Kinetic description of the model: mesoscopic limit

In the previous section we gave an effective description of the (non-equilibrium) stochastic dynamics for models of RWREs in continuum in terms of hierarchical equations for correlation functions. As we have already discussed, the study of existence and uniqueness of solutions to these hierarchical chain-equations is a delicate problem that can be resolved by using different techniques, which will be considered in next chapters.

In order to get information about the evolution of the system we can consider a proper scaling limit of the stochastic dynamics and derive the corresponding kinetic equations. In [51] the authors proposed a general scheme to derive the Vlasov-type equations for Markov dynamics of interacting particle systems on configuration spaces. This approach is based on a scaling of the hierarchy of correlation functions which takes into account the influence of weak long-range interactions. It has been realized for several models, see e.g. [10, 45, 54, 58]. In this section we apply the general scheme proposed in [51] to study the mesoscopic limit for a RWRE described by the heuristic generator

$$(LF)(\gamma, y) = (L_{RE}F)(\gamma, y) + (L_{RW}F)(\gamma, y), \quad (2.74)$$

where the operator $L_{RW} := L_{RW}(\lambda_{\text{int}})$ is defined as in (2.10) and L_{RE} is of the form (2.2) or (2.6), depending on which model of environment we want to consider. In order to keep the discussion as general as possible, in the following we denote $L_{RE} := L_{RE}(b, d, c)$. For the moment we aim to present an heuristic description of this mesoscopic scaling in the general case, postponing a more rigorous analysis to Section 2.3.1.

For RWREs the evolution of correlation functional, $k_t : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R}_+$, is described by the hierarchical equations:²

$$\begin{cases} \frac{dk_t}{dt} = (L_{RE}^\Delta + L_{RW}^\Delta) k_t \\ k_t|_{t=0} = k_0 \end{cases}, \quad t \geq 0, \quad (2.75)$$

where the generator $L^\Delta = L_{RE}^\Delta + L_{RW}^\Delta$ is defined as in Section 2.2.2. The construction of the Vlasov-type scaling of the hierarchy (2.75) can be summarized in three main steps.

As the first step, we chose an initial function $k_{0,\varepsilon}$ in (2.75) which has a singularity in ε , namely such that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{|\eta|} k_{0,\varepsilon}(\eta, y) = r_0(\eta, y), \quad \eta \in \Gamma_0, y \in \mathbb{R}^d, \varepsilon > 0, \quad (2.76)$$

²Note that, for brevity, in this section we will use the notation $L^\Delta := \widehat{L}^*$.

for some (correlation) function r_0 . Intuitively, we are considering initial states of the system where the particle of RE becomes more and more dense (singular) as the parameter ε goes to zero. It is convenient to introduce the renormalized correlation functional defined as

$$k_{0,\varepsilon,\text{ren}}(\eta, y) := \varepsilon^{|\eta|} k_{0,\varepsilon}(\eta, y) = (R_\varepsilon k_{0,\varepsilon})(\eta, y), \quad (2.77)$$

where for any $\varepsilon > 0$ we have defined the mapping of functions on $\Gamma_0 \times \mathbb{R}^d$

$$(R_\varepsilon k_{0,\varepsilon})(\eta, y) := \varepsilon^{|\eta|} k_{0,\varepsilon}(\eta, y). \quad (2.78)$$

Note that this mapping is self-dual w.r.t. duality (2.28). Moreover, $R_\varepsilon^{-1} = R_{\varepsilon^{-1}}$.

In correspondence with the previous scaling of initial condition, as the second step, we rescale also the generator L^Δ in (2.75):

$$L^\Delta \longmapsto L_\varepsilon^\Delta, \quad \varepsilon > 0. \quad (2.79)$$

The family of operators L_ε^Δ , $\varepsilon > 0$, is being built by a proper scaling of the generator L , namely

$$\begin{aligned} L = L_{RE}(b, d, c) + L_{RW}(\lambda) &\longmapsto L_\varepsilon := L_{RE,\varepsilon}(b, d, c) + L_{RW,\varepsilon}(\lambda_{\text{int}}) \\ &= L_{RE}(\varepsilon^{-1}b_\varepsilon, d_\varepsilon, c_\varepsilon) + L_{RW}(\lambda_{\text{int},\varepsilon}). \end{aligned} \quad (2.80)$$

Clearly the choice of the scaling (2.80) depends on the model under consideration: intuitively, the idea is to reduce the strength of the interaction among all particles (weak-coupling limit). The precise form of the rescaled rates b_ε , d_ε and c_ε may be found in [51] for different models, while, the rescaled interaction $\lambda_\varepsilon := \lambda_{\text{int},\varepsilon}$ will be discussed later on in each of the cases I-IV of Section 2.1.

As a result of the two scalings defined above, we arrive to a "renormalized version" of the hierarchy (2.75), namely

$$\begin{cases} \frac{dk_{t,\varepsilon,\text{ren}}}{dt} = (L_{RE,\varepsilon,\text{ren}}^\Delta + L_{RW,\varepsilon,\text{ren}}^\Delta) k_{t,\varepsilon,\text{ren}} & , \quad t \geq 0, \\ k_{t,\varepsilon,\text{ren}}|_{t=0} = k_{0,\varepsilon,\text{ren}} \end{cases} \quad (2.81)$$

where we have defined the renormalized generator

$$L_{\varepsilon,\text{ren}}^\Delta = R_{\varepsilon^{-1}} L_\varepsilon^\Delta R_\varepsilon = R_{\varepsilon^{-1}} L_{RE,\varepsilon}^\Delta R_\varepsilon + R_{\varepsilon^{-1}} L_{RW,\varepsilon}^\Delta R_\varepsilon =: L_{RE,\varepsilon,\text{ren}}^\Delta + L_{RW,\varepsilon,\text{ren}}^\Delta, \quad (2.82)$$

and (cf. (2.77))

$$k_{t,\varepsilon,\text{ren}}(\eta, y) := \varepsilon^{|\eta|} k_{t,\varepsilon}(\eta, y), \quad \eta \in \Gamma_0, y \in \mathbb{R}^d. \quad (2.83)$$

The explicit expression for the operators $L_{RE,\varepsilon,\text{ren}}^\Delta$, $L_{RW,\varepsilon,\text{ren}}^\Delta$ will be given later in Proposition 2.17, 2.18 and 2.19.

We are interested in the limit for ε going to zero of the renormalized hierarchy (2.81). Then, in the third and last step, we consider this limit by imposing two conditions. First, we require that the scaling preserves the order of the singularity of the initial function $k_{0,\varepsilon}$ during the evolution. Namely, given (2.83) for any $t > 0$ we impose that

$$\lim_{\varepsilon \rightarrow 0} k_{t,\varepsilon,\text{ren}}(\eta, y) = r_t(\eta, y), \quad \eta \in \Gamma_0, y \in \mathbb{R}^d. \quad (2.84)$$

The second condition we want to demand, it is the so-called chaos-preservation property of the limiting evolution. More precisely, if at time $t = 0$ we start from an (uncorrelated) state,

$$r_0(\eta, y) = \left(\prod_{x \in \eta} \rho_0(x) \right) r_0(y), \quad (2.85)$$

then, at time $t > 0$ we want still to have a state of the same type,

$$r_t(\eta, y) = \left(\prod_{x \in \eta} \rho_t(x) \right) r_t(y), \quad (2.86)$$

for any $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. In applications the functions $\rho, r : \mathbb{R}^d \rightarrow \mathbb{R}$ may be interpreted as the density distribution of RE and density distribution of RW, respectively.

Informally, under condition (2.84), we want to show that the solution of the renormalized hierarchy (2.81) converges (in a proper sense) to some function $r_t(\eta, y)$ which is solution to the Vlasov hierarchy

$$\begin{cases} \frac{dr_t}{dt} = (L_{RE,V}^\Delta + L_{RW,V}^\Delta) r_t, & t \geq 0, \\ r_t|_{t=0} = r_0 \end{cases}, \quad (2.87)$$

where

$$L_V^\Delta = \lim_{\varepsilon \rightarrow 0} L_{\varepsilon, \text{ren}}^\Delta = \lim_{\varepsilon \rightarrow 0} L_{RE, \varepsilon, \text{ren}}^\Delta + \lim_{\varepsilon \rightarrow 0} L_{RW, \varepsilon, \text{ren}}^\Delta =: L_{RE,V}^\Delta + L_{RW,V}^\Delta. \quad (2.88)$$

Note that the limiting hierarchy (2.87) can be obtained (point-wise) under general conditions, see Proposition 2.20, 2.21 and 2.22 for details.

Finally, if the limiting evolution satisfies the chaos preservation property (2.85)-(2.86), it follows that Vlasov hierarchy (2.87) with initial condition (2.85) has a solution of the form (2.86), where the pair of functions (ρ_t, r_t) are solutions of a system of coupled differential equations

$$\frac{\partial}{\partial t} \rho_t(x) = (\mathcal{V}_{RE} \rho_t)(x), \quad (2.89a)$$

$$\frac{\partial}{\partial t} r_t(y) = (\mathcal{V}_{RW}(\rho_t) r_t)(y), \quad (2.89b)$$

which we will call the Vlasov equations of RWRE. Here, \mathcal{V}_{RE} and $\mathcal{V}_{RW}(\rho_t)$ are two operators acting on a proper space of functions on \mathbb{R}^d .

The kinetic equations (2.89a) and (2.89b) will be derived point-wise in Lemma 2.25.(i), for the moment let us just stress some remarkable common features. The first equation (2.89a), describing the evolution of the density of RE ρ_t , is non-linear, but independent of the RW. Whereas, the second equation (2.89b), describing the evolution of the density distribution of the RW r_t , is linear and depends on the solution of the first equation, ρ_t , due to the presence of the interaction λ_{int} . In our model of RWRE, Vlasov equations (2.89a)-(2.89b) have a nice probabilistic interpretation. More precisely, cf. Lemma 2.25.ii, equation (2.89b) describes a non-autonomous RW with the heuristic generator

$$\begin{aligned} (\bar{L}_{RW}(t)f)(y) &:= (L_{RW}(\bar{\lambda}_t)f)(y) \\ &= \int_{\mathbb{R}^d} dz \bar{\lambda}_t(y, z) a(y-z) [f(z) - f(y)], \end{aligned} \quad (2.90)$$

for a proper class of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$, which we call reduced observables. The function $\bar{\lambda}_t(y, z)$ depends on the solution of (2.89a) and represents some effective interaction which approximates the interaction with the particles of RE, see formula (2.126) in next section.

Remark 2.16. It may be convenient to consider the Vlasov-type scaling in terms of quasi-observables G , which are the pre-duals of correlation functionals k w.r.t. (2.28). By means of the scaling (2.80), we can define the rescaled generator for quasi-observables as

$$\hat{L}_\varepsilon = K^{-1} L_\varepsilon K = K^{-1} L_{RE, \varepsilon} K + K^{-1} L_{RW, \varepsilon} K =: \hat{L}_{RE, \varepsilon} + \hat{L}_{RW, \varepsilon}. \quad (2.91)$$

Then, since the map R_ε in (2.78) is self-dual w.r.t. (2.28), we can define the renormalized hierarchy for quasi-observables

$$\begin{cases} \frac{dG_t}{dt} = (\widehat{L}_{RE,\varepsilon,\text{ren}} + \widehat{L}_{RW,\varepsilon,\text{ren}}) G_t \\ G_t|_{t=0} = G_0 \end{cases}, \quad t \geq 0, \quad (2.92)$$

with

$$\widehat{L}_{\varepsilon,\text{ren}} = R_{\varepsilon^{-1}} \widehat{L}_\varepsilon R_\varepsilon = R_{\varepsilon^{-1}} \widehat{L}_{RE,\varepsilon} R_\varepsilon + R_{\varepsilon^{-1}} \widehat{L}_{RW,\varepsilon} R_\varepsilon =: \widehat{L}_{RE,\varepsilon,\text{ren}} + \widehat{L}_{RW,\varepsilon,\text{ren}}, \quad (2.93)$$

and the corresponding limiting hierarchy

$$\begin{cases} \frac{dG_t}{dt} = (\widehat{L}_{RE,V} + \widehat{L}_{RW,V}) G_t \\ G_t|_{t=0} = G_0 \end{cases}, \quad t \geq 0, \quad (2.94)$$

where

$$\widehat{L}_V = \lim_{\varepsilon \rightarrow 0} \widehat{L}_{\varepsilon,\text{ren}} = \lim_{\varepsilon \rightarrow 0} \widehat{L}_{RE,\varepsilon,\text{ren}} + \lim_{\varepsilon \rightarrow 0} \widehat{L}_{RW,\varepsilon,\text{ren}} =: \widehat{L}_{RE,V} + \widehat{L}_{RW,V}. \quad (2.95)$$

2.3.1 Derivation of Vlasov equations

In this section we apply the general scheme of the Vlasov-type scaling to a wide class of RWREs described by the heuristic generator (2.74). We state general conditions on the parameter of the model which give a point-wise convergence of the rescaled generator to the limiting generator of the related hierarchies. Finally, we derive informally the Vlasov equations for the considered RWREs.

For any $\varepsilon > 0$ using the scaling (2.76)-(2.80) we can define the renormalized hierarchy for correlation functions given by (2.81) with the generators

$$L_{RE,\varepsilon,\text{ren}}^\Delta := R_{\varepsilon^{-1}} L_{RE,\varepsilon}^\Delta R_\varepsilon \quad \text{and} \quad L_{RW,\varepsilon,\text{ren}}^\Delta := R_{\varepsilon^{-1}} L_{RW,\varepsilon}^\Delta R_\varepsilon. \quad (2.96)$$

The explicit expression for the operator $L_{RE,\varepsilon,\text{ren}}^\Delta$ can be found in [51] for different types of birth-and-death processes and hopping particle systems. Below we describe general results.

Proposition 2.17. *Let us consider an operator L_{RE} as in (2.2). Assume that we have some scaling of birth and death rates b_ε and d_ε , respectively, such that conditions (2.3)-(2.5) are satisfied for any $\varepsilon > 0$. Then, for any $k \in B_{\text{bs}}(\widetilde{\Gamma}_0)$, $L_{RE,\varepsilon,\text{ren}}^\Delta k \in L^0(\widetilde{\Gamma}_0)$ and*

$$\begin{aligned} (L_{RE,\varepsilon,\text{ren}}^\Delta k)(\eta, y) = & - \sum_{x \in \eta} \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup \eta, y) \varepsilon^{-|\xi|} (K^{-1} d_\varepsilon(x, \cdot \cup \eta \setminus x))(\xi) + \\ & \sum_{x \in \eta} \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup (\eta \setminus x), y) \varepsilon^{-|\xi|} (K^{-1} b_\varepsilon(x, \cdot \cup \eta \setminus x))(\xi). \end{aligned} \quad (2.97)$$

Moreover, for any $G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ it follows that $\widehat{L}_{RE,\varepsilon,\text{ren}} G \in L^0(\widetilde{\Gamma}_0)$, where

$$\begin{aligned} (\widehat{L}_{RE,\varepsilon,\text{ren}} G)(\eta, y) = & - \sum_{\xi \subset \eta} \varepsilon^{-|\eta \setminus \xi|} G(\xi, y) \sum_{x \in \xi} (K^{-1} d_\varepsilon(x, \cdot \cup (\eta \setminus x))) (\eta \setminus \xi) + \\ & \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} dx \varepsilon^{-|\eta \setminus \xi|} G(\xi \cup x, y) (K^{-1} b_\varepsilon(x, \cdot \cup \xi)) (\eta \setminus \xi). \end{aligned} \quad (2.98)$$

Proposition 2.18. *Let us consider an operator L_{RE} as in (2.6). Assume that we have some scaling of the jump rate c_ε such that conditions (2.7)-(2.9) are satisfied for any $\varepsilon > 0$. Then, for any $k \in B_{\text{bs}}(\tilde{\Gamma}_0)$, $L_{RE,\varepsilon,\text{ren}}^\Delta k \in L^0(\tilde{\Gamma}_0)$ and*

$$\begin{aligned} (L_{RE,\varepsilon,\text{ren}}^\Delta k)(\eta, y) &= \\ &= \sum_{x' \in \eta} \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup (\eta \setminus x') \cup x, y) \varepsilon^{-|\xi|} (K^{-1} c_{x,x'}(\cdot \cup (\eta \setminus x')))(\xi) - \\ &\quad \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup \eta, y) \sum_{x \in \eta} \int_{\mathbb{R}^d} dx' \varepsilon^{-|\xi|} (K^{-1} c_{x',x}(\cdot \cup (\eta \setminus x')))(\xi). \end{aligned} \quad (2.99)$$

Moreover, for any $G \in B_{\text{bs}}(\tilde{\Gamma}_0)$ it follows that $\hat{L}_{RE,\varepsilon,\text{ren}} G \in L^0(\tilde{\Gamma}_0)$, where

$$\begin{aligned} (\hat{L}_{RE,\varepsilon,\text{ren}} G)(\eta, y) &= \int_{\mathbb{R}^d} dx' \sum_{\xi \subset \eta} \sum_{x \in \xi} \varepsilon^{-|\eta/\xi|} [G(\xi \setminus x \cup x', y) - G(\xi, y)] \times \\ &\quad (K^{-1} c_{x,x'}(\cdot \cup (\xi \setminus x)))(\eta \setminus \xi). \end{aligned} \quad (2.100)$$

In the following proposition we present the expression of the renormalized the operators $L_{RW,\varepsilon,\text{ren}}^\Delta$ and $\hat{L}_{RW,\varepsilon,\text{ren}}$ for a general rescaled interaction λ_ε .

Proposition 2.19. *Let us consider the RW of a jumping particle described by an operator L_{RW} as in (2.10). Assume that we have some scaling of the interaction $\lambda_\varepsilon \geq 0$ such that condition (2.11) holds for any $\varepsilon > 0$. Then, for any $k \in B_{\text{bs}}(\tilde{\Gamma}_0)$, $L_{RW,\varepsilon,\text{ren}}^\Delta k \in L^0(\tilde{\Gamma}_0)$ and*

$$\begin{aligned} (L_{RW,\varepsilon,\text{ren}}^\Delta k)(\eta, y) &= \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) \varepsilon^{-|\xi|} [k(\xi \cup \eta, z) (K^{-1} \lambda_\varepsilon(\cdot \cup \eta, z, y))(\xi) - \\ &\quad k(\xi \cup \eta, y) (K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z))(\xi)]. \end{aligned} \quad (2.101)$$

Moreover, for any $G \in B_{\text{bs}}(\tilde{\Gamma}_0)$ one has $L_{RW,\varepsilon,\text{ren}}^\Delta G \in L^0(\tilde{\Gamma}_0)$, where

$$(\hat{L}_{RW,\varepsilon,\text{ren}} G)(\eta, y) = \int_{\mathbb{R}^d} dz a(y-z) \sum_{\xi \subset \eta} \varepsilon^{-|\eta/\xi|} (G(\xi, z) - G(\xi, y)) (K^{-1} \lambda_\varepsilon(\cdot \cup \xi, y, z))(\eta \setminus \xi). \quad (2.102)$$

Proof. Let us first compute the renormalized operator for correlation function $L_{RW,\varepsilon,\text{ren}}^\Delta$. According to Proposition 2.15, by means of the scaling (2.80), we have

$$\begin{aligned} (L_{RW,\varepsilon,\text{ren}}^\Delta k)(\eta, y) &= \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) [k(\zeta \cup \eta, z) (K_0^{-1} \lambda_\varepsilon(\cdot \cup \eta, z, y))(\zeta) - \\ &\quad k(\zeta \cup \eta, y) (K_0^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z))(\zeta)]. \end{aligned}$$

For $R_{\varepsilon^{-1}} k$ we can write it as

$$\begin{aligned} (L_{RW,\varepsilon,\text{ren}}^\Delta R_{\varepsilon^{-1}} k)(\eta, y) &= \\ &= \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) \varepsilon^{-|\zeta \cup \eta|} [k(\zeta \cup \eta, z) (K_0^{-1} \lambda_\varepsilon(\cdot \cup \eta, z, y))(\zeta) - \\ &\quad k(\zeta \cup \eta, y) (K_0^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z))(\zeta)]. \end{aligned}$$

Finally by applying the map R_ε to it, since $\eta \cap \zeta = \emptyset$ for λ -a.a. $\zeta \in \Gamma_0$, we obtain

$$\begin{aligned} (L_{RW,\varepsilon,\text{ren}}^\Delta k)(\eta, y) &= (R_\varepsilon L_{RW,\varepsilon}^\Delta R_{\varepsilon^{-1}} k)(\eta, y) \\ &= \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) \varepsilon^{-|\zeta|} \left[k(\zeta \cup \eta, z) (K_0^{-1} \lambda_\varepsilon(\cdot \cup \eta, z, y))(\zeta) - \right. \\ &\quad \left. k(\zeta \cup \eta, y) (K_0^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z))(\zeta) \right], \end{aligned}$$

as we wanted to show. Similarly, using Proposition 2.12 we can compute the renormalized generator for quasi-observables in (2.102). \square

Next, we want to show that for ε going to zero the generators in the renormalized hierarchy (2.81) converge point-wise to the generators in the Vlasov hierarchy (2.87), namely

$$L_{RE,V}^\Delta := \lim_{\varepsilon \rightarrow 0} L_{RE,\varepsilon,\text{ren}}^\Delta, \quad \text{and} \quad L_{RW,V}^\Delta := \lim_{\varepsilon \rightarrow 0} L_{RW,\varepsilon,\text{ren}}^\Delta. \quad (2.103)$$

Let us stress that, at present, we are considering a general scaling of the rates b_ε , d_ε , c_ε and interaction λ_ε . However, in order to calculate the two limits above, we need to impose some assumptions on the limiting behavior (in ε) of these rescaled parameters, which should be verified in each concrete model. In the next two propositions we give the precise expression for the limiting operator $L_{RE,V}^\Delta$ for a birth-and-death process and hopping particle system, respectively. These general results, as well as applications in specific models, can be found in [51].

Proposition 2.20. *Let us consider the operators $L_{RE,\varepsilon,\text{ren}}^\Delta$ and $\widehat{L}_{RE,\varepsilon,\text{ren}}$, $\varepsilon > 0$, given by (2.97) and (2.98), respectively. Suppose that for all $\eta, \xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$ the following limits exist and coincide*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} d_\varepsilon(x, \cdot \cup \eta))(\xi) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} d_\varepsilon(x, \cdot))(\xi) =: D_x^V(\xi), \quad (2.104)$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} b_\varepsilon(x, \cdot \cup \eta))(\xi) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} b_\varepsilon(x, \cdot))(\xi) =: B_x^V(\xi) \quad (2.105)$$

and let $D_x^V(\xi)$ and $B_x^V(\xi)$ satisfy conditions to (2.3)-(2.5). Then, for any $k, G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ the following formulas hold:

$$(L_{RE,V}^\Delta k)(\eta, y) = - \sum_{x \in \eta} \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup \eta, y) D_x^V(\xi) + \sum_{x \in \eta} \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup (\eta \setminus x), y) B_x^V(\xi) \quad (2.106)$$

and

$$(\widehat{L}_{RE,V} G)(\eta, y) = - \sum_{\xi \subset \eta} G(\xi, y) \sum_{x \in \xi} D_x^V(\eta \setminus \xi) + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} dx G(\xi \cup x, y) B_x^V(\eta \setminus \xi). \quad (2.107)$$

Moreover, $L_{RE,V}^\Delta k, \widehat{L}_{RE,V} G \in L^0(\widetilde{\Gamma}_0)$.

Proposition 2.21. *Let us consider the operators $L_{RE,\varepsilon,\text{ren}}^\Delta$ and $\widehat{L}_{RE,\varepsilon,\text{ren}}$, $\varepsilon > 0$, given by (2.99) and (2.100), respectively. Suppose that for all $\eta, \xi \in \Gamma_0$ and a.a. $x, x' \in \mathbb{R}^d$ the following limits exist and coincide*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} c_\varepsilon(\cdot \cup \eta, x, x'))(\xi) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} c_\varepsilon(\cdot, x, x'))(\xi) =: C_{x,x'}^V(\xi) \quad (2.108)$$

and let $C_{x,x'}^V(\xi)$ satisfies conditions (2.7)-(2.9). Then, for any $k, G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ we have

$$(L_{RE,V}^\Delta k)(\eta, y) = \sum_{x' \in \eta} \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup (\eta \setminus x') \cup x, y) C_{x,x'}^V(\xi)$$

$$- \int_{\Gamma_0} d\lambda(\xi) k(\xi \cup \eta, y) \sum_{x \in \eta} \int_{\mathbb{R}^d} dx' C_{x',x}^V(\xi) \quad (2.109)$$

and

$$(\widehat{L}_{RE,V} G)(\eta, y) = \int_{\mathbb{R}^d} dx' \sum_{\xi \subset \eta} \sum_{x \in \xi} [G(\xi \setminus x \cup x', y) - G(\xi, y)] C_{x',x}^V(\eta \setminus \xi). \quad (2.110)$$

Moreover, $L_{RE,V}^\Delta k, \widehat{L}_{RE,V} G \in L^0(\widetilde{\Gamma}_0)$.

Now, under some general conditions on the rescaled interaction λ_ε we derive the explicit form for the limiting operators $L_{RW,V}^\Delta$ and $\widehat{L}_{RW,V}$.

Proposition 2.22. *Let us consider the operators $L_{RW,\varepsilon,\text{ren}}^\Delta$ and $\widehat{L}_{RW,\varepsilon,\text{ren}}$, $\varepsilon > 0$, given by (2.101) and (2.102), respectively. Suppose that for all $\eta, \xi \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$ the following limits exist and coincide*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z))(\xi) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} \lambda_\varepsilon(\cdot, y, z))(\xi) =: A_V(\xi, y, z) \quad (2.111)$$

with

$$\sup_{z \in \mathbb{R}^d} A(\eta, y, z) < \infty, \quad \forall \eta \in \Gamma_0, \forall y \in \mathbb{R}^d. \quad (2.112)$$

Then, for any $k, G \in B_{\text{bs}}(\widetilde{\Gamma}_0)$ the following formulas hold

$$(L_{RW,V}^\Delta k)(\eta, y) = \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) [k(\xi \cup \eta, z) A_V(\xi, z, y) - k(\xi \cup \eta, y) A_V(\xi, y, z)] \quad (2.113)$$

and

$$(\widehat{L}_{RW,V} G)(\eta, y) = \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} dz a(y-z) (G(\xi, z) - G(\xi, y)) A_V(\eta \setminus \xi, y, z). \quad (2.114)$$

Moreover, $L_{RW,V}^\Delta k, \widehat{L}_{RW,V} G \in B_0(\widetilde{\Gamma}_0)$.

Proof. The limiting generator $L_{RW,V}^\Delta$ is defined as

$$(L_{RW,V}^\Delta k)(\eta, y) = \lim_{\varepsilon \rightarrow 0} (L_{RW,\varepsilon,\text{ren}}^\Delta k)(\eta, y).$$

By (2.101) we can write this limit explicitly as

$$(L_{RW,V}^\Delta k)(\eta, y) = \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) \left[k(\xi \cup \eta, z) \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K_0^{-1} \lambda_\varepsilon(\cdot \cup \eta, z, y))(\xi) - k(\xi \cup \eta, y) \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K_0^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z))(\xi) \right].$$

Hence, assumption (2.111) yields (2.113). Similarly, starting from (2.102), one can derive (2.114). \square

Finally, let us show that the Vlasov hierarchy (2.87), specified by Proposition 2.20, 2.21 and 2.22, has the chaos preservation property and, then, derive the corresponding kinetic equations. Firstly, let us present the Vlasov equations for a generic birth-and-death dynamics and hopping particle system, see [51, 54] and reference therein for details.

Proposition 2.23. *Let us consider the generator $L_{\text{bad},V}^\Delta$ defined as in (2.106). Assume that the Vlasov equation*

$$\frac{\partial \rho_t(x)}{\partial t} = -\rho_t(x) \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) D_x^V(\xi) + \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) B_x^V(\xi), \quad (2.115)$$

has a point-wise solution $\rho_t : \mathbb{R}^d \rightarrow \mathbb{R}$, $t \geq 0$. Then, for any time $t \geq 0$ the evolution equation

$$\begin{cases} \frac{\partial}{\partial t} k_t(\eta) = (L_{\text{bad},V}^\Delta k_t)(\eta) \\ k_t(\eta)|_{t=0} = e_\lambda(\rho_0, \eta) \end{cases}, \quad k_t : \Gamma_0 \rightarrow \mathbb{R}, \quad (2.116)$$

has a solution of the form $k_t = e_\lambda(\rho_t)$. Namely,

$$\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) = (L_{\text{bad},V}^\Delta e_\lambda(\rho_t, \cdot))(\eta), \quad \eta \in \Gamma_0, t \geq 0. \quad (2.117)$$

Proposition 2.24. *Let us consider a generator L_{hp}^Δ of the form (2.109). Assume that the Vlasov equation*

$$\begin{aligned} \frac{\partial \rho_t(x)}{\partial t} &= \int_{\mathbb{R}^d} dx' \rho_t(x') \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) C_{x',x}^V(\xi) - \\ &\rho_t(x) \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \int_{\mathbb{R}^d} dx' e_\lambda(\rho_t, \xi) C_{x,x'}^V(\xi), \end{aligned} \quad (2.118)$$

has a point-wise solution $\rho_t : \mathbb{R}^d \rightarrow \mathbb{R}$, $t \geq 0$. Then, for any time $t \geq 0$ the evolution equation

$$\begin{cases} \frac{\partial}{\partial t} k_t(\eta) = (L_{\text{hp},V}^\Delta k_t)(\eta) \\ k_t(\eta)|_{t=0} = e_\lambda(\rho_0, \eta) k_0(\eta) \end{cases}, \quad k_t : \Gamma_0 \rightarrow \mathbb{R}, \quad (2.119)$$

has a solution of the form $k_t = e_\lambda(\rho_t)$. Namely,

$$\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) = (L_{\text{hp},V}^\Delta e_\lambda(\rho_t, \cdot))(\eta), \quad \eta \in \Gamma_0, t \geq 0. \quad (2.120)$$

Next, we derive the Vlasov equations for RWRE described by the limiting generator $L_V^\Delta := L_{RE,V}^\Delta + L_{RW,V}^\Delta$, where $L_{RW,V}^\Delta$ is given by (2.113) and $L_{RE,V}^\Delta$ is of the form (2.106) or (2.109). According to results of Proposition 2.23 and 2.24, in order to keep the discuss as general as possible, we assume that for any $\eta \in \Gamma_0$ and $t \geq 0$

$$\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) = (L_{RE,V}^\Delta e_\lambda(\rho_t, \cdot))(\eta), \quad (2.121)$$

where $\rho_t : \mathbb{R}^d \rightarrow \mathbb{R}$ is a point-wise solution to the kinetic equation

$$\frac{\partial}{\partial t} \rho_t(x) = (\mathcal{V}_{RE} \rho_t)(x), \quad t \geq 0. \quad (2.122)$$

Lemma 2.25. *Let us consider the generator $L_{RW,V}^\Delta$ defined as in (2.113) and let $L_{RE,V}^\Delta$ be an operator such that conditions (2.121) and (2.122) hold.*

(i) *Assume that the Vlasov equation*

$$\frac{\partial r_t(y)}{\partial t} = \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) A_V(\xi, z, y) - r_t(y) A_V(\xi, y, z)], \quad (2.123)$$

has a point-wise solution $r_t : \mathbb{R}^d \rightarrow \mathbb{R}$, $t \geq 0$. Then, for any time $t \geq 0$ the limiting Vlasov hierarchy (2.87), with initial condition of the form $k_0(\eta, y) = e_\lambda(\rho_0, \eta)r_0(y)$, has a solution of the form $k_t = e_\lambda(\rho_t, \eta)r_t(y)$ and

$$\frac{\partial}{\partial t} [e_\lambda(\rho_t, \eta)r_t(y)] = ([L_{RE,V}^\Delta + L_{RW,V}^\Delta]e_\lambda(\rho_t, \cdot)r_t)(\eta, y), \quad (2.124)$$

for any $\eta \in \Gamma_0$, $y \in \mathbb{R}^d$ and $t \geq 0$.

(ii) The system of Vlasov equations (2.122)-(2.123) describes a non-autonomous RW with heuristic generator $\bar{L}_{RW}(t)$ acting on proper functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$(\bar{L}_{RW}(t)f)(y) = \int_{\mathbb{R}^d} dz \bar{\lambda}_t(y, z) a(y-z) [f(z) - f(y)], \quad (2.125)$$

where

$$\bar{\lambda}_t(y, z) = \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) A_V(\xi, y, z), \quad t \geq 0, \quad (2.126)$$

and $\rho_t(x)$ is solution to (2.122).

Remark 2.26. The Vlasov equation (2.123) can be rewritten in terms of the effective interaction (2.126) as follow

$$\frac{\partial r_t(y)}{\partial t} = \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) \bar{\lambda}_t(z, y) - r_t(y) \bar{\lambda}_t(y, z)].$$

Proof of (i). In order to derive the Vlasov equations we need to check that, if ρ_t and r_t satisfies (2.122) and (2.123), then

$$\frac{\partial}{\partial t} [e_\lambda(\rho_t, \eta)r_t(y)] = (L_{RE,V}^\Delta e_\lambda(\rho_t, \cdot)r_t(\cdot))(\eta, y) + (L_{RW,V}^\Delta e_\lambda(\rho_t, \cdot)r_t(\cdot))(\eta, y), \quad (2.127)$$

for any $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. By using the product rule of derivatives, since

$$(L_{RE,V}^\Delta e_\lambda(\rho_t)r_t)(\eta, y) = r_t(y) (L_{RE,V}^\Delta e_\lambda(\rho_t, \cdot))(\eta),$$

we can rewrite (2.127) explicitly as

$$r_t(y) \left[\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) \right] + e_\lambda(\rho_t, \eta) \frac{\partial r_t(y)}{\partial t} = r_t(y) (L_{RE,V}^\Delta e_\lambda(\rho_t, \cdot))(\eta) + (L_{RW,V}^\Delta e_\lambda(\rho_t, \cdot)r_t(\cdot))(\eta, y). \quad (2.128)$$

Note that, under hypothesis (2.121), this condition is equivalent to

$$e_\lambda(\rho_t, \eta) \frac{\partial r_t(y)}{\partial t} = (L_{RW,V}^\Delta e_\lambda(\rho_t, \cdot)r_t(\cdot))(\eta, y). \quad (2.129)$$

Then, by using the result of Proposition 2.22, one has

$$\begin{aligned} e_\lambda(\rho_t, \eta) \frac{\partial r_t(y)}{\partial t} &= \\ &= \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) e_\lambda(\rho_t, \xi \cup \eta) [r_t(z) A_V(\xi, z, y) - r_t(y) A_V(\xi, y, z)] \\ &= e_\lambda(\rho_t, \eta) \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) A_V(\xi, z, y) - r_t(y) A_V(\xi, y, z)], \end{aligned} \quad (2.130)$$

where in the last step we used the fact that $\xi \cap \eta = \emptyset$ for λ -a.a. $\xi \in \Gamma_0$ and the identity $e_\lambda(f, \xi \cup \eta) = e_\lambda(f, \xi)e_\lambda(f, \eta)$. Finally, we can easily deduce that (2.130) is satisfied if r_t satisfies (2.123). \square

Proof of (ii). The kinetic equation (2.123) can be seen as the Fokker-Planck equation for the density r_t . We rewrite it formally as

$$\frac{\partial r_t(y)}{\partial t} = (\bar{L}_{RW}^*(t)r_t)(y), \quad t \geq 0, \quad (2.131)$$

where the operator $\bar{L}_{RW}^*(t)$ is given by

$$(\bar{L}_{RW}^*(t)r_t)(y) := \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) \bar{\lambda}_t(z,y) - r_t(y) \bar{\lambda}_t(y,z)], \quad (2.132)$$

with $\bar{\lambda}_t$ defined as in (2.126). Next, by using the duality between functions and densities on \mathbb{R}^d , defined by the pairing

$$\langle f, p \rangle = \int_{\mathbb{R}^d} f(x) p(x) dx, \quad (2.133)$$

we can derive the corresponding backward Kolmogorov equation for (reduced) observables $f_t : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\frac{\partial f_t(y)}{\partial t} = (\bar{L}_{RW}(t)f_t)(y), \quad t \geq 0, \quad (2.134)$$

where $\bar{L}_{RW}(t)$ is the pre-dual operator to $\bar{L}_{RW}^*(t)$ w.r.t. the pairing (2.133), namely

$$\langle \bar{L}_{RW}(t)f, r \rangle = \langle f, \bar{L}_{RW}^*(t)r \rangle. \quad (2.135)$$

By using (2.132), the r.h.s. of the above identity can be written explicitly as

$$\begin{aligned} \langle f, \bar{L}_{RW}^*(t)r \rangle &= \int_{\mathbb{R}^d} dy f(y) \int_{\mathbb{R}^d} dz r_t(z) a(y-z) \bar{\lambda}_t(z,y) - \\ &\quad \int_{\mathbb{R}^d} dy f(y) r_t(y) \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_t(y,z). \end{aligned}$$

In the first integral in the r.h.s. we can just exchange y with z to obtain

$$\langle f, \bar{L}_{RW}^*(t)r \rangle = \int_{\mathbb{R}^d} dy r_t(y) \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_t(y,z) [f(z) - f(y)]. \quad (2.136)$$

Finally, comparing this expression with the l.h.s. of (2.135), we obtain the desired result. \square

Remark 2.27.

- (i) In this work we aim to study the Vlasov-type scaling in the Banach spaces \mathcal{K}_C^∞ and \mathcal{K}_C^1 , defined in (2.31) and (2.32), respectively. In particular, we discuss the existence of the renormalized and the limiting evolutions, see equations (2.81) and (2.87), as well as the convergence for the solutions of the considered hierarchies. This analysis can be performed by using one the three methods explained at the end of Section 2.2, see also [45, 54, 61] and references therein.
- (ii) The question about existence and uniqueness of solutions of Vlasov equations shall be considered as a separated problem. The space of these solutions depends on the class of correlation functions that has been considered. In particular, if $k \in \mathcal{K}_C^q$, with $q = \infty$ or 1, then, $\rho \in L^\infty$ and $r \in L^q$. In this thesis we will study existence and uniqueness problem of these equations and prove some uniform estimate of their solutions.

2.3.2 Examples of Vlasov equations

In this section, we present the explicit form of the Vlasov equations of RWRE for each of the interaction I-IV introduced in Section 2.1. In what follows we always assume that the density of particles of RE, $\rho_t : \mathbb{R}^d \rightarrow \mathbb{R}$, is the solution of some Vlasov-type equation given by

$$\frac{\partial}{\partial t} \rho_t(x) = (\mathcal{V}_{RE} \rho)(x), \quad t \geq 0. \quad (2.137)$$

Clearly, the exact form of this equation depends on the model of RE one is considering, cf. (2.115) and (2.118).

Case I. Let us consider the interaction $\lambda_{\text{int}} = \lambda^{(1)}$ defined by (2.47). For any $\varepsilon > 0$ we consider the following scaling of $\lambda^{(1)}$,

$$\lambda_\varepsilon^{(1)}(\gamma, y) := e^{-\varepsilon \sum_{x' \in \gamma} \phi(x' - y)} = e_\lambda(e^{-\varepsilon \phi(\cdot - y)}, \gamma), \quad (2.138)$$

for any $\gamma \in \Gamma$ and $y \in \mathbb{R}^d$. In correspondence to such a scaling, analogously to (2.50), we have

$$(K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y))(\eta) = \lambda_\varepsilon^{(1)}(\xi, y) e_\lambda(e^{-\varepsilon \phi(\cdot - y)} - 1, \eta) \quad (2.139)$$

and letting ε go to zero, one obtains

$$\begin{aligned} A_V^{(1)}(\eta, y) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y))(\eta) \\ &= \lim_{\varepsilon \rightarrow 0} e^{-\varepsilon \sum_{x \in \xi} \phi(x - y)} e_\lambda\left(\frac{e^{-\varepsilon \phi(\cdot - y)} - 1}{\varepsilon}, \eta\right) \\ &= e_\lambda(-\phi(\cdot - y), \eta), \end{aligned} \quad (2.140)$$

for all $\eta, \xi \in \Gamma_0$ and a.a. $y \in \mathbb{R}^d$. Note that in the last step we used the identity

$$\lim_{\varepsilon \rightarrow 0} \frac{e^{\varepsilon f} - 1}{\varepsilon} = f, \quad f > 0. \quad (2.141)$$

In this case, we can define the effective interaction (compare to (2.126)),

$$\begin{aligned} \bar{\lambda}_t^{(1)}(y) &:= \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) A_V^{(1)}(\xi, y) \\ &= e^{-\int_{\mathbb{R}^d} dx' \rho_t(x') \phi(x' - y)}, \quad y \in \mathbb{R}^d. \end{aligned} \quad (2.142)$$

Then, according to Remark 2.26, the corresponding Vlasov equation can be written as

$$\frac{\partial r_t}{\partial t} = -\langle a \rangle e^{-(\rho_t * \phi)} r_t + \left((e^{-(\rho_t * \phi)} r_t) * a \right), \quad (2.143)$$

where we have defined

$$\langle a \rangle := \int_{\mathbb{R}^d} dz a(z). \quad (2.144)$$

Here and below, the symbol $*$ indicates the usual notion of convolution on \mathbb{R}^d , namely

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x') g(x' - x) dx, \quad f, g \in \mathbb{R}^d.$$

Case II. Let us consider the interaction $\lambda^{(2)}$ defined as in (2.52)-(2.53). For any $\varepsilon > 0$ we rescale this interaction as follows

$$\lambda_\varepsilon^{(2)}(\gamma, y) := \lambda_0 + \varepsilon \sum_{x' \in \gamma} \phi(x' - y), \quad (2.145)$$

for any $\gamma \in \Gamma$ and $y \in \mathbb{R}^d$. In correspondence to such a scaling, analogously to (2.56), we have

$$(K_0^{-1} \lambda_\varepsilon^{(2)}(\cdot \cup \xi, y))(\eta) = \lambda_\varepsilon^{(2)}(\xi, y) 0^{|\eta|} + \varepsilon \phi(x' - y) \mathbb{1}_{\Gamma(1)}(\eta = \{x'\}) \quad (2.146)$$

and considering the limit for ε going to zero, one finds

$$\begin{aligned} A_V^{(2)}(\eta, y) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K^{-1} \lambda_\varepsilon^{(2)}(\cdot \cup \xi, y))(\eta) \\ &= \lim_{\varepsilon \rightarrow 0} \left[\lambda_0 0^{|\eta|} + \varepsilon \sum_{x \in \xi} \phi(x - y) 0^{|\eta|} + \varepsilon^{-|\eta|} \varepsilon \phi(x' - y) \mathbb{1}_{\Gamma(1)}(\eta = \{x'\}) \right] \\ &= \lambda_0 0^{|\eta|} + \phi(x' - y) \mathbb{1}_{\Gamma(1)}(\eta = \{x'\}), \end{aligned} \quad (2.147)$$

for all $\eta, \xi \in \Gamma_0$ and a.a. $y \in \mathbb{R}^d$. Then, one can define the effective interaction (compare to (2.126)),

$$\begin{aligned} \bar{\lambda}_t^{(2)}(y) &:= \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) A_V^{(2)}(\xi, y) \\ &= \lambda_0 + \int_{\mathbb{R}^d} dx' \rho_t(x') \phi(x' - y), \quad y \in \mathbb{R}^d. \end{aligned} \quad (2.148)$$

and from Remark 2.26 the corresponding Vlasov equation can be written as

$$\frac{\partial r_t}{\partial t} = (a * [r_t(\lambda_0 + (\rho_t * \phi))]) - \langle a \rangle r_t[\lambda_0 + (\rho_t * \phi)]. \quad (2.149)$$

Case III. For any $\varepsilon > 0$, we consider the following scaling of the interaction (2.58),

$$\lambda_\varepsilon^{(3)}(\gamma, z) := e^{-\varepsilon \sum_{x' \in \gamma} \phi(x' - z)} = e_\lambda(e^{-\varepsilon \phi(\cdot - z)}, \gamma), \quad (2.150)$$

for any $\gamma \in \Gamma$ and $z \in \mathbb{R}^d$. In correspondence, analogously to equations (2.139) and (2.140), we find

$$(K^{-1} \lambda_\varepsilon^{(3)}(\cdot \cup \xi, z))(\eta) = \lambda_\varepsilon^{(3)}(\xi, z) e_\lambda(e^{-\varepsilon \phi(\cdot - z)} - 1, \eta) \quad (2.151)$$

and, letting ε go to zero,

$$\begin{aligned} A_V^{(3)}(\eta, z) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K^{-1} \lambda_\varepsilon^{(3)}(\cdot \cup \xi, z))(\eta) \\ &= e_\lambda(-\phi(\cdot - z), \eta), \end{aligned} \quad (2.152)$$

for all $\eta, \xi \in \Gamma_0$ and a.a. $z \in \mathbb{R}^d$. We can introduce the effective interaction (cf. (2.126)),

$$\begin{aligned} \bar{\lambda}_t^{(3)}(z) &:= \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) A_V^{(3)}(\xi, z) \\ &= e^{-\int_{\mathbb{R}^d} dx' \rho_t(x') \phi(x' - z)}, \quad z \in \mathbb{R}^d. \end{aligned} \quad (2.153)$$

Then, according to Remark 2.26 the corresponding Vlasov equation has the form

$$\frac{\partial r_t}{\partial t} = e^{-(\rho_t * \phi)}(r_t * a) - r_t(e^{-(\rho_t * \phi)} * a). \quad (2.154)$$

Case IV. For any $\varepsilon > 0$ we rescale interaction (2.61) as follows

$$\lambda_\varepsilon^{(4)}(\gamma, z) := \lambda_0 + \varepsilon \sum_{x' \in \gamma} \phi(x' - z), \quad (2.155)$$

for any $\gamma \in \Gamma$ and $z \in \mathbb{R}^d$. Then, analogously to (2.146) and (2.147), we have

$$(K_0^{-1} \lambda_\varepsilon^{(4)}(\cdot \cup \xi, z))(\eta) = \lambda_\varepsilon^{(4)}(\xi, z) 0^{|\eta|} + \varepsilon \phi(x' - z) \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) \quad (2.156)$$

and, in the limit $\varepsilon \rightarrow 0$,

$$\begin{aligned} A_V^{(4)}(\eta, z) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K^{-1} \lambda_\varepsilon^{(4)}(\cdot \cup \xi, z))(\eta) \\ &= \lambda_0 0^{|\eta|} + \phi(x' - z) \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}), \end{aligned} \quad (2.157)$$

for all $\eta, \xi \in \Gamma_0$ and a.a. $z \in \mathbb{R}^d$. Next we define the effective interaction (cf. (2.126)) given by

$$\begin{aligned} \bar{\lambda}_t^{(4)}(z) &:= \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) A_V^{(4)}(\xi, z) \\ &= \lambda_0 + \int_{\mathbb{R}^d} dx' \rho_t(x') \phi(x' - z), \quad \text{for a.a. } z \in \mathbb{R}^d. \end{aligned} \quad (2.158)$$

Hence, from Remark 2.26 the corresponding Vlasov equation can be written as

$$\frac{\partial r_t}{\partial t} = (r_t * a) [\lambda_0 + (\rho_t * \phi)] - r_t [\lambda_0 \langle a \rangle + ((\rho_t * \phi) * a)]. \quad (2.159)$$

Remark 2.28. In all examples discussed in this section, the effective interaction $\bar{\lambda}_t$ can be obtained from the original interaction λ_{int} through the substitution

$$\sum_{x \in \gamma} \phi(x - \cdot) \longrightarrow \int_{\mathbb{R}^d} dx \rho_t(x) \phi(x - \cdot), \quad \gamma \in \Gamma.$$

This simple observation emphasize the nature of the mesoscopic limit that we realized. Indeed, on the microscopic level we have a many-body interaction λ_{int} which describes the interaction between the tagged particle and other particles of RE. Then, by means of the Vlasov-type scaling limit, we arrive to the two-body interaction $\bar{\lambda}_t$ which takes into account the mean effect that particles of RE has on the RW of the tagged particle.

Chapter 3

Random walks in birth-and-death environments

We study the model introduced in Chapter 2 of the RW of a jumping particle which interacts with an evolving RE described by a non-equilibrium birth-and-death dynamics [54].

Following the general approach described in [49, 54], we present conditions on the birth-and-death intensities as well as on the interaction between the target particle and the other particles of the environment, which are sufficient for the existence of an (statistical) evolution as a strongly continuous semigroup (C_0 -semigroup) on a proper Banach space of correlation functions satisfying a Ruelle-type bound. Moreover, by using the Vlasov-type scaling introduced in [51] and described in Section 2.3 we study the corresponding dynamics in the mesoscopic limit.

Our analysis can be applied to different classes of RE, see e.g. [42, 50, 57, 82], as well as to different types of interactions of RW with RE. As an example of these applications, we will discuss in details the concrete model of RW evolving in a Bolker-Dieckmann-Law-Pacala (BDLP) model of RE under each of the interactions I-IV introduced in Section 2.1.

3.1 Non-equilibrium evolutions

In this section we want to study the statistical evolution corresponding to the Markov dynamics of RWRE described by the heuristic generator

$$(LF)(\gamma, y) = (L_{RE}F)(\gamma, y) + (L_{RW}F)(\gamma, y), \quad (3.1)$$

where, for any $F \in K(B_{\text{bs}}(\Gamma_0 \times \mathbb{R}^d))$, the generators L_{RE} and L_{RW} are defined by (2.2) and (2.10), respectively, such that conditions (2.3)-(2.5) as well as (2.11) hold.

From Section 2.2 we know that the evolution of correlation functions is described by a hierarchy of the form

$$\begin{cases} \frac{\partial}{\partial t} k_t(\eta, y) = (\widehat{L}_{RE}^* k_t)(\eta, y) + (\widehat{L}_{RW}^* k_t)(\eta, y) \\ k_t(\eta, y)|_{t=0} = k_0(\eta, y) \end{cases}, \quad t \geq 0, \quad (3.2)$$

where the operators \widehat{L}_{RE}^* and \widehat{L}_{RW}^* are given by (2.66) and (2.68), respectively. We also define $\widehat{L}^* = \widehat{L}_{RE}^* + \widehat{L}_{RW}^*$.

For any fixed $C > 1$, we study the solution to the initial value problem (3.2) in the Banach space (cf. (2.31)),

$$\mathcal{K}_C^\infty = \left\{ k : \Gamma_0 \times \mathbb{R}^d \rightarrow \mathbb{R} \mid C^{-|\eta|} k(\eta, y) \in L^\infty(\Gamma_0 \times \mathbb{R}^d, d\lambda dy) \right\}, \quad (3.3)$$

equipped with the norm

$$\|k\|_{\mathcal{K}_C^\infty} := \operatorname{ess\,sup}_{(\eta, y) \in \Gamma_0 \times \mathbb{R}^d} C^{-|\eta|} |k(\eta, y)|. \quad (3.4)$$

In order to solve (3.2) we will use the Phillips' trick [54, 102] as explained in Section 2.2. By means of the duality (2.24), we consider the hierarchy for quasi-observables $G \in B_{\text{bs}}(\Gamma_0 \times \mathbb{R}^d)$ given by (cf. Remark 2.6)

$$\begin{cases} \frac{\partial}{\partial t} G_t(\eta, y) = (\widehat{L}_{RE} G_t)(\eta, y) + (\widehat{L}_{RW} G_t)(\eta, y) \\ G_t(\eta, y)|_{t=0} = G_0(\eta, y) \end{cases}, \quad t \geq 0, \quad (3.5)$$

where the operators \widehat{L}_{RE} and \widehat{L}_{RW} are defined in (2.36) and (2.38), respectively. This initial value problem will be solved in the (pre-dual) Banach space

$$\mathcal{L}_C := L^1(\Gamma_0 \times \mathbb{R}^d, C^{|\eta|} d\lambda(\eta) dy), \quad C > 1, \quad (3.6)$$

equipped with the norm

$$\|G\|_C := \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |G(\eta, y)|. \quad (3.7)$$

In Section 3.1.1 we construct a holomorphic semigroup $\widehat{U}(t)$ on the space \mathcal{L}_C which gives a solution to (3.5) in this space. The dual semigroup $\widehat{U}^*(t)$ provides a weak*-solution to (3.2) in the space \mathcal{K}_C^∞ . Then, in Section 3.1.2 we find a proper subspace of \mathcal{K}_C^∞ in which a strong solution of (3.2) exists.

3.1.1 Evolution of quasi-observables

In this section, we want to construct a semigroup on the Banach space \mathcal{L}_C , defined in (3.6), with generator $\widehat{L} = \widehat{L}_{RE} + \widehat{L}_{RW}$ given by formulas (2.36) and (2.38).

In order to accomplish this task we will use a modification of the approach proposed in [54] to construct the state evolution associated to a birth-and-death dynamics on the configuration space. More precisely, following [54] we assume that there exist $a_1 \geq 1$, $a_2 > 0$ such that for all $\xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$

$$\sum_{x \in \xi} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |K^{-1} d(x, \cdot \cup \xi \setminus x)|(\eta) \leq a_1 \sum_{x \in \xi} d(x, \xi \setminus x), \quad (3.8)$$

$$\sum_{x \in \xi} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |K^{-1} b(x, \cdot \cup \xi \setminus x)|(\eta) \leq a_2 \sum_{x \in \xi} d(x, \xi \setminus x). \quad (3.9)$$

In addition, we make the following assumption on the interaction λ_{int} .

Assumption 3.1. *Suppose that there exist $\alpha_0, \alpha_1 \geq 0$ such that for all $\xi \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$*

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, y, z)|(\eta) \leq \alpha_1 \sum_{x \in \xi} d(x, \xi \setminus x) + \alpha_0. \quad (3.10)$$

Remark 3.2. As we will see in the discussion of concrete models in Section 3.1.3, conditions (3.8)-(3.10) imply that the mortality of RE should be large enough "to dominate" both the birth of new particles and the interaction λ_{int} .

Let us consider the generator \widehat{L} with domain given by

$$\mathcal{D} = \{G \in \mathcal{L}_C : D(\eta)G(\eta, y) \in \mathcal{L}_C\}, \quad D(\xi) := \sum_{x \in \xi} d(x, \xi \setminus x). \quad (3.11)$$

Note that the set \mathcal{D} is a dense set in \mathcal{L}_C . Thus, under the assumptions above, we can formulate the following theorem about the existence of the corresponding semigroup in \mathcal{L}_C . The proof of this result is given in Section 3.1.1.1.

Theorem 3.3. *Suppose that the hypotheses (3.8)-(3.10) hold with*

$$a_1 + \frac{a_2}{C} + 2\langle a \rangle \alpha_1 < \frac{3}{2}, \quad (3.12)$$

then the operator $(\widehat{L}, \mathcal{D})$ is the generator of a holomorphic semigroup $\widehat{T}(t)$ on \mathcal{L}_C .

3.1.1.1 Proof of Theorem 3.3

We proceed with a rigorous analysis of the operator \widehat{L} in the Banach space \mathcal{L}_C . According to the results obtained in Proposition 2.10 and 2.12, for any $G \in B_{\text{bs}}(\Gamma_0 \times \mathbb{R}^d)$, we can rewrite the operator \widehat{L} as

$$(\widehat{L}G)(\eta, y) = (L_0G)(\eta, y) + (L_1G)(\eta, y) + (\widehat{L}_{RW}G)(\eta, y), \quad (3.13)$$

where \widehat{L}_{RW} is given by formula (2.38),

$$(L_0G)(\eta, y) = -D(\eta)G(\eta, y) \quad (3.14)$$

and

$$\begin{aligned} (L_1G)(\eta, y) = & - \sum_{\xi \subseteq \eta} G(\xi, y) \sum_{x \in \xi} (K^{-1}d(x, \cdot \cup (\xi \setminus x))) (\eta \setminus \xi) \\ & + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} dx G(\xi \cup x, y) (K^{-1}b(x, \cdot \cup \xi)) (\eta \setminus \xi). \end{aligned} \quad (3.15)$$

Let us first consider the multiplication operator (L_0, \mathcal{D}) , with \mathcal{D} given by (3.11). Multiplication operators are quite simple to define and analyze, see e.g. [39]. In the present case we can show the following lemma.

Lemma 3.4. *The operator (L_0, \mathcal{D}) is a generator of a contraction semigroup on \mathcal{L}_C . Moreover, L_0 is a sectorial operator of any angle $\omega \in (0, \pi/2)$, i.e. for each $\varepsilon \in (0, \omega)$ and $z \in \mathbb{C} \setminus \{0\}$, with $|\arg z| \leq \pi/2 + \omega - \varepsilon$, there exists $M_\varepsilon > 0$ such that*

$$\|(z\mathbb{1} - L_0)^{-1}\|_C \leq \frac{M_\varepsilon}{|z|}. \quad (3.16)$$

Furthermore, condition (3.16) holds with $M_\varepsilon = 1/\cos \omega$ for all $\varepsilon \in (0, \omega)$.

Proof of Lemma 3.4. The statement can be shown by using the same arguments used in the proof of Lemma 3.3 in [54]. \square

As it is well known, see e.g. [39, Theorem II.4.6], any sectorial operator of angle ω generates a bounded semigroup which is holomorphic in the sector

$$\text{Sect}(\omega) = \{z \in \mathbb{C} \mid |\arg z| < \omega\},$$

for all $\omega \in (0, \pi/2)$. Then, in order to construct a semigroup associated to the generator $(\widehat{L}, \mathcal{D})$, we can use the theorem about perturbations of holomorphic semigroups, see e.g. [39, Theorem III.2.10]. For this purpose, it is enough to show that both the operators L_1 and \widehat{L}_{RW} are, at least, relatively bounded by the operator L_0 (L_0 -bounded) in the space \mathcal{L}_C .

The operator L_1 has been already studied in [54] and we know therefrom that, under assumptions (3.8) and (3.9), it is L_0 -bounded in \mathcal{L}_C . In particular, we can formulate the following result.

Proposition 3.5. *Let us suppose that conditions (3.8) and (3.9) hold. Then, the operator (L_1, \mathcal{D}) is a well-defined operator in \mathcal{L}_C , such that*

$$\|L_1 G\|_C \leq \left(a_1 - 1 + \frac{a_2}{C}\right) \|L_0 G\|_C, \quad G \in \mathcal{D}. \quad (3.17)$$

Proof of Proposition 3.5. The proof is similar to that one of Lemma 3.4 in [54]. \square

Now let us consider the operator \widehat{L}_{RW} . In the space \mathcal{L}_C we can prove the following estimate.

Proposition 3.6. *Let us suppose that hypothesis (3.10) is fulfilled. Then, the operator $(\widehat{L}_{RW}, \mathcal{D})$ is a well-defined operator in \mathcal{L}_C satisfying*

$$\|\widehat{L}_{RW} G\|_C \leq 2\langle a \rangle \alpha_1 \|L_0 G\|_C + 2\langle a \rangle \alpha_0 \|G\|_C, \quad G \in \mathcal{D}. \quad (3.18)$$

Proof of Proposition 3.6. We calculate the \mathcal{L}_C -norm of the operator \widehat{L}_{RW} in (2.38) explicitly. By definition, for any $G \in \mathcal{D}$ we have

$$\begin{aligned} \|\widehat{L}_{RW} G\|_C &\leq \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \int_{\mathbb{R}^d} dz a(y-z) \times \\ &\quad \sum_{\xi \subset \eta} \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, y, z) \right| (\eta \setminus \xi) |G(\xi, z) - G(\xi, y)|. \end{aligned}$$

Applying the Minlos identity (1.18) we can rewrite it as

$$\begin{aligned} \|\widehat{L}_{RW} G\|_C &\leq \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \int_{\mathbb{R}^d} dz a(y-z) \times \\ &\quad \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, y, z) \right| (\eta) |G(\xi, z) - G(\xi, y)|. \end{aligned}$$

Then, one can use condition (3.10) and estimate

$$\begin{aligned} \|\widehat{L}_{RW} G\|_C &\leq \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) (\alpha_1 D(\xi) + \alpha_0) \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) |G(\xi, z) - G(\xi, y)| \\ &\leq 2\langle a \rangle \alpha_1 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) D(\xi) |G(\xi, y)| + \\ &\quad 2\langle a \rangle \alpha_0 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |G(\xi, y)|. \end{aligned} \quad (3.19)$$

Finally, by using the definition of the generator L_0 in (3.14) we obtain the desired result. \square

Remark 3.7. Note that the first inequality in (3.19) implies

$$\begin{aligned} \|\widehat{L}_{RW} G\|_C &\leq 2\langle a \rangle \alpha_1 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) D(\xi) |G(\xi, y)| + \\ &\quad \alpha_0 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) |G(\xi, z) - G(\xi, y)| \\ &\leq 2\langle a \rangle \alpha_1 \|L_0 G\|_C + \alpha_0 \|L_{RW}^{(0)} G\|_C, \quad G \in \mathcal{D}, \end{aligned} \quad (3.20)$$

where $L_{RW}^{(0)}$ is the generator of the free RW of the tagged particle, i.e.

$$\left(L_{RW}^{(0)} G\right)(\eta, y) := \int_{\mathbb{R}^d} dz a(y-z) [G(\xi, z) - G(\xi, y)]. \quad (3.21)$$

for any $G \in \mathcal{D}$. In other words, through assumption (3.10) we can incorporate all dependence of \widehat{L}_{RW} on λ_{int} in the operator L_0 .

We are now in the position to show the result stated in Theorem 3.3.

Proof of Theorem 3.3. Let us set

$$\theta := a_1 + \frac{a_2}{C} - 1 + 2\langle a \rangle \alpha_1. \quad (3.22)$$

From Lemma 3.4, we know that (L_0, \mathcal{D}) is a sectorial operator of angle $\omega \in (0, \pi/2)$ such that

$$\|(z\mathbf{1} - L_0)^{-1}\|_C \leq \frac{M_\varepsilon}{|z|}, \quad M_\varepsilon = \frac{1}{\cos \omega}, \quad (3.23)$$

for any $z \neq 0$ with $|z| \leq \pi/2 + \omega$. Moreover, by combining results of Proposition 3.5 and 3.6 we obtain the following estimate in \mathcal{L}_C

$$\|(L_1 + L_{RW})G\|_C \leq \left[\left(a_1 - 1 + \frac{a_2}{C} \right) + 2\langle a \rangle \alpha_1 \right] \|L_0 G\|_C + 2\langle a \rangle \alpha_0 \|G\|_C. \quad (3.24)$$

Hence, by the first part of the proof of Theorem III.2.10 in [39], we know that the operator $(\widehat{L} = L_0 + L_1 + \widehat{L}_{RW}, \mathcal{D})$ is the generator of an holomorphic semigroup on \mathcal{L}_C , denoted by $\widehat{T}(t)$, for all values of θ such that

$$\theta \leq \alpha \quad \text{with} \quad \alpha := \frac{1}{1 + M_\varepsilon} = \frac{1}{1 + \frac{1}{\cos \omega}} > 0. \quad (3.25)$$

Note that for $\omega \in (0, \pi/2)$ we have $\alpha \leq 1/2$. Since we are not interested in the domain of analyticity of the semigroup $\widehat{T}(t)$, we can choose the value of ω in (3.25) to get the weakest bound on θ , namely $\theta \leq 1/2$. The latter together with (3.22) yields condition (3.12) in Theorem 3.3. \square

3.1.2 Evolution of correlation functions

In this section we want to use the semigroup $\widehat{T}(t)$, acting on the space \mathcal{L}_C , to construct a solution to the evolution equation (3.2) on the Banach space \mathcal{K}_C^∞ defined in (3.3). This construction is adapted from [49, 54]. Let us first outline the general idea of this method which allows us to formulate the main result of this section.

According to (2.24), the dual space $(\mathcal{L}_C)' = L^\infty(\Gamma_0 \times \mathbb{R}^d, d\lambda_C dy)$, with $\lambda_C := C^{|\cdot|} \lambda$, is isometrically isomorphic to the Banach space \mathcal{K}_C^∞ , where the isomorphism is given by the isometry R_C

$$(\mathcal{L}_C)' \ni k \mapsto R_C k := C^{-|\cdot|} k \in \mathcal{K}_C^\infty. \quad (3.26)$$

Let $(\widehat{L}', \text{Dom}(\widehat{L}'))$ on $(\mathcal{L}_C)'$ be the dual operator of $(\widehat{L}, \mathcal{D})$ on \mathcal{L}_C . Its image on \mathcal{K}_C^∞ , under the isometry R_C , will be the operator $\widehat{L}^* = R_C \widehat{L}' R_C^{-1}$ with the domain $\text{Dom}(\widehat{L}^*) = R_C \text{Dom}(\widehat{L}')$. Similarly, one can consider the adjoint semigroup $\widehat{T}'(t)$ on $(\mathcal{L}_C)'$ and its image $\widehat{T}^*(t)$ on \mathcal{K}_C^∞ .

Let us note that the space \mathcal{L}_C is not reflexive, hence, $\widehat{T}^*(t)$ is not a C_0 -semigroup on \mathcal{K}_C^∞ , but it is weak*-continuous, weak*-differentiable in 0 and with weak*-generator \widehat{L}^* , see e.g. [39, Section II.2.5]. As a matter of fact, one has an evolution for correlation functions in \mathcal{K}_C^∞ only in a weak*-sense.

To overcome this handicap, we can restrict the semigroup $\widehat{T}^*(t)$ to its subspace of strong continuity, see e.g. [39, Section II.2.6]. Namely, we consider the restriction $\widehat{T}^\circ(t)$ of the semigroup $\widehat{T}^*(t)$ onto its sun dual subspace $\mathcal{K}_C^\circ = \overline{\text{Dom}(\widehat{L}^*)} \subseteq \mathcal{K}_C^\infty$ ¹. According to [39, Section II.2.6], $\widehat{T}^\circ(t)$ is a strongly continuous semigroup on $\overline{\text{Dom}(\widehat{L}^*)}$ whose generator, \widehat{L}° , is a part of the dual operator $(\widehat{L}^*, \text{Dom}(\widehat{L}^*))$, i.e.

$$\widehat{L}^\circ k = \widehat{L}^* k \quad \text{for any} \quad k \in \text{Dom}(\widehat{L}^\circ) = \left\{ k \in \text{Dom}(\widehat{L}^*) : \widehat{L}^* k \in \overline{\text{Dom}(\widehat{L}^*)} \right\}. \quad (3.27)$$

¹Here and below all closures are w.r.t. to the norm (3.4) of the space Banach \mathcal{K}_C^∞ .

However, we would like to have a strongly continuous semigroup on an universal subspace of \mathcal{K}_C^∞ which does not depend on the operator \widehat{L}^* . As in [54], the next step is to restrict the semigroup $\widehat{T}^\circ(t)$ on the subspace $\mathcal{K}_{\alpha C}^\infty \subseteq \mathcal{K}_C^\infty$, for some $\alpha \in (0, 1)$. It is not difficult to see that, under some simple conditions, $\mathcal{K}_{\alpha C}^\infty \subseteq \text{Dom}(\widehat{L}^*)$, cf. Proposition 3.10. Moreover, the set $\overline{\mathcal{K}_{\alpha C}^\infty}$ is a \widehat{T}° -invariant subspace of \mathcal{K}_C^∞ , see Lemma 3.12. Therefore, according to [39, Section II.2.3], the restriction $\widehat{T}^{\circ\alpha}$ of the semigroup \widehat{T}° to $\overline{\mathcal{K}_{\alpha C}^\infty}$ will be a strongly continuous semigroup with generator $\widehat{L}^{\circ\alpha}$, which is the part of \widehat{L}° on $\overline{\mathcal{K}_{\alpha C}^\infty}$. As a consequence, we can formulate the following theorem about the evolution (3.2) in the space \mathcal{K}_C^∞ . We refer to Section 3.1.2.1 for a detailed proof.

Theorem 3.8. *Let us assume that hypotheses (3.8)-(3.10) hold. Suppose that there exist $A > 0$, $N \in \mathbb{N}_0$ and $\nu \geq 1$ such that for $\xi \in \Gamma_0$ and $x \notin \xi$*

$$d(x, \xi) \leq A(1 + |\xi|)^N \nu^{|\xi|}, \quad (3.28)$$

with

$$\nu < \frac{C}{a_2} \left(\frac{3}{2} - a_1 - 2\langle a \rangle \alpha_1 \right). \quad (3.29)$$

Then, for any

$$\alpha \in \left(\frac{a_2}{C \left(\frac{3}{2} - a_1 - 2\langle a \rangle \alpha_1 \right)}, \frac{1}{\nu} \right), \quad (3.30)$$

the evolution equation (3.2), with initial condition $k_0 \in \mathcal{K}_{\alpha C}^\infty$, has a unique solution in the space $\mathcal{K}_{\alpha C}^\infty$ given by $k_t = \widehat{T}^{\circ\alpha}(t)k_0$.

Remark 3.9. Let us note that condition (3.29) implies (3.12), as $\nu \geq 1$.

3.1.2.1 Proof of Theorem 3.8

In this section we want to carry out the proof of Theorem 3.8 outlined above. In order to accomplish this task we will show some auxiliary results which are used later to prove the theorem.

First of all, we establish for which values of $\alpha \in (0, 1)$ the subspace $\mathcal{K}_{\alpha C}^\infty \subseteq \mathcal{K}_C^\infty$ belongs to the domain of the generator \widehat{L}^* , specified by (2.66) and (2.68).

Proposition 3.10. *Let us assume that hypotheses (3.8)-(3.10), as well as condition (3.28) hold. Then for any $\alpha \in (0, 1/\nu)$*

$$\mathcal{K}_{\alpha C}^\infty \subset \text{Dom}(\widehat{L}^*). \quad (3.31)$$

Proof of Proposition 3.10. In order to prove the statement, it is enough to verify that for any $k \in \mathcal{K}_{\alpha C}^\infty$ there exist $k^* \in \mathcal{K}_C^\infty$ such that for any $G \in \text{Dom}(\widehat{L})$

$$\langle \widehat{L}G, k \rangle = \langle G, k^* \rangle. \quad (3.32)$$

We know that (3.32) holds for any $k \in \mathcal{K}_{\alpha C}^\infty$ with $k^* = \widehat{L}^*k$, provided $k^* \in \mathcal{K}_C^\infty$. The latter condition is equivalent to

$$C^{-|\cdot|}(\widehat{L}^*k) \in L^\infty(\Gamma_0 \times \mathbb{R}^d, d\lambda dy). \quad (3.33)$$

In our case, we have

$$C^{-|\eta|} \left| (\widehat{L}^*k)(\eta, y) \right| \leq C^{-|\eta|} \left| (\widehat{L}_{RE}^*k)(\eta, y) \right| + C^{-|\eta|} \left| (\widehat{L}_{RW}^*k)(\eta, y) \right|. \quad (3.34)$$

From [54, Proposition 3.5], we know that, under the conditions (3.8) and (3.9), for any $k \in \mathcal{K}_{\alpha C}^\infty$, $C^{-|\cdot|} \widehat{L}_{RE}^* k \in L^\infty(\Gamma_0 \times \mathbb{R}^d, d\lambda dy)$. In particular, one can show that for any $k \in \mathcal{K}_{\alpha C}^\infty$,

$$\operatorname{ess\,sup}_{(\eta, y) \in \Gamma_0 \times \mathbb{R}^d} C^{-|\eta|} \left| (\widehat{L}_{RE}^* k)(\eta, y) \right| \leq \|k\|_{\mathcal{K}_{\alpha C}^\infty} \left(a_1 + \frac{a_2}{\alpha C} \right) \frac{A}{\alpha \nu^2} \left(\frac{N+1}{-e \ln(\alpha \nu)} \right)^{N+1} < \infty. \quad (3.35)$$

Let us now consider the second term in the r.h.s of equation (3.34). For any $k \in \mathcal{K}_{\alpha C}^\infty$ we have

$$\begin{aligned} C^{-|\eta|} \left| (\widehat{L}_{RW}^* k)(\eta, y) \right| &\leq \\ &\leq C^{-|\eta|} \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) |k(\zeta \cup \eta, z)| \left| (K^{-1} \lambda(\cdot \cup \eta, z, y))(\zeta) \right| + \\ &\quad C^{-|\eta|} \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) |k(\zeta \cup \eta, y)| \left| (K^{-1} \lambda(\cdot \cup \eta, y, z))(\zeta) \right| \\ &\leq \alpha^{|\eta|} \|k\|_{\mathcal{K}_{\alpha C}^\infty} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} (\alpha C)^{|\zeta|} d\lambda(\zeta) \left| (K_0^{-1} \lambda(\cdot \cup \eta, z, y))(\zeta) \right| + \\ &\quad \alpha^{|\eta|} \|k\|_{\mathcal{K}_{\alpha C}^\infty} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} (\alpha C)^{|\zeta|} d\lambda(\zeta) \left| (K_0^{-1} \lambda(\cdot \cup \eta, y, z))(\zeta) \right|. \end{aligned}$$

Then, by using hypothesis (3.10), one can estimate

$$\begin{aligned} C^{-|\eta|} \left| (\widehat{L}_{RW}^* k)(\eta, y) \right| &\leq 2\alpha^{|\eta|} \|k\|_{\mathcal{K}_{\alpha C}^\infty} \int_{\mathbb{R}^d} dz a(y-z) [\alpha_1 D(\eta) + \alpha_0] \\ &\leq 2\langle a \rangle \|k\|_{\mathcal{K}_{\alpha C}^\infty} \left[\alpha_1 \alpha^{|\eta|} D(\eta) + \alpha_0 \alpha^{|\eta|} \right]. \end{aligned}$$

Note that from condition (3.28) it follows that

$$D(\eta) = \sum_{x \in \eta} d(x, \eta \setminus x) \leq A \sum_{x \in \eta} |\eta|^N \nu^{|\eta|-1} \leq A(1 + |\eta|)^{N+1} \nu^{|\eta|-1}, \quad (3.36)$$

therefore, we get

$$C^{-|\eta|} \left| (\widehat{L}_{RW}^* k)(\eta, y) \right| \leq 2\langle a \rangle \|k\|_{\mathcal{K}_{\alpha C}^\infty} \left[A\alpha_1 \alpha^{|\eta|} (1 + |\eta|)^{N+1} \nu^{|\eta|-1} + \alpha_0 \alpha^{|\eta|} \right].$$

At this point we can use the elementary inequality

$$a^t (1+t)^b \leq \frac{1}{a} \left(\frac{b}{-e \ln a} \right)^b, \quad t \geq 0, a \in (0, 1), b \geq 1, \quad (3.37)$$

to obtain for any $\alpha \in (0, 1/\nu)$

$$\operatorname{ess\,sup}_{(\eta, y) \in \Gamma_0 \times \mathbb{R}^d} C^{-|\eta|-1} \left| (\widehat{L}_{RW}^* k)(\eta, y) \right| \leq 2\langle a \rangle \|k\|_{\mathcal{K}_{\alpha C}^\infty} \left[A\alpha_1 \frac{1}{\alpha \nu^2} \left(\frac{N+1}{-e \ln(\alpha \nu)} \right)^{N+1} + \alpha_0 \right] < \infty,$$

which concludes the proof of the proposition. \square

Next, let us consider the pre-dual space $\mathcal{L}_{\alpha C}$. In this space, we define the set

$$\mathcal{D}_\alpha = \{G \in \mathcal{L}_{\alpha C} \mid D(\cdot) G \in \mathcal{L}_{\alpha C}\}, \quad \alpha \in (0, 1). \quad (3.38)$$

Note that, if condition (3.28) holds, then for any $\alpha \in (0, 1/\nu)$ we have the following inclusions $\mathcal{D} \subset \mathcal{L}_C \subset \mathcal{D}_\alpha \subset \mathcal{L}_{\alpha C}$, compare to [54, Lemma 3.6]. Proceeding as in Theorem 3.3, one can construct a C_0 -semigroup on $\mathcal{L}_{\alpha C}$ with generator $(\widehat{L}, \mathcal{D}_\alpha)$.

Proposition 3.11. *Let us assume that condition (3.28) as well as hypotheses (3.8)-(3.10) hold with*

$$a_1 + \frac{a_2}{\alpha C} + 2\langle a \rangle \alpha_1 < \frac{3}{2}, \quad (3.39)$$

for some $\alpha \in (0, 1)$. Then, the operator $(\widehat{L}, \mathcal{D}_\alpha)$ is the generator of a holomorphic semigroup $\widehat{T}_\alpha(t)$ on $\mathcal{L}_{\alpha C}$.

Proof. For any $\alpha \in (0, 1)$, it is easy to check that the domain \mathcal{D}_α is dense in $\mathcal{L}_{\alpha C}$ and that the densely defined operator L_0 is closed in $\mathcal{L}_{\alpha C}$. Then for the operator $(L_0, \mathcal{D}_\alpha)$ we can show the statement of Lemma 3.4, but with C replaced by αC . Moreover, by using the results of Proposition 3.5 and 3.6, for αC instead of C , we obtain the following estimate in $\mathcal{L}_{\alpha C}$ (compare with (3.24))

$$\|(L_1 + L_{RW})G\|_{\alpha C} \leq \left(a_1 - 1 + \frac{a_2}{\alpha C} + 2\langle a \rangle \alpha_1 \right) \|L_0 G\|_{\alpha C} + 2\langle a \rangle \alpha_0 \|G\|_{\alpha C}, \quad G \in \mathcal{D}_\alpha.$$

Hence, we can proceed as in the Proof of Theorem 3.3 and get the statement. \square

Now, we can come back to the space $\mathcal{K}_{\alpha C}^\infty$ and show the following result.

Lemma 3.12. *Let us assume that hypotheses (3.8)-(3.10) are satisfied. Suppose, additionally, that condition (3.28) holds with (3.29). Then, for any*

$$\alpha \in \left(\frac{a_2}{C \left(\frac{3}{2} - a_1 - 2\langle a \rangle \alpha_1 \right)}, \frac{1}{\nu} \right), \quad (3.40)$$

both the sets $\mathcal{K}_{\alpha C}^\infty$ and $\overline{\mathcal{K}_{\alpha C}^\infty}$ are \widehat{T}° -invariant subspaces of \mathcal{K}_C^∞ .

Proof. We show that $\mathcal{K}_{\alpha C}^\infty \subset \text{Dom}(\widehat{L}^*)$ is \widehat{T}° -invariant subspace of \mathcal{K}_C^∞ . Then, the statement for $\overline{\mathcal{K}_{\alpha C}^\infty}$ follows from the continuity of the family $\widehat{T}^*(t)$. First of all, let us note that condition (3.40) implies that the operator $(\widehat{L}, \mathcal{D}_\alpha)$ generates an holomorphic semigroup $\widehat{T}_\alpha(t)$ on $\mathcal{L}_{\alpha C}$, see Proposition 3.11. Next, following the proof of [54, Proof of Theorem 3.8], we prove that for any $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$

$$\widehat{T}_\alpha G = \widehat{T}G. \quad (3.41)$$

According to [39, Corollary V.5.5], it is sufficient to show that the resolvents of \widehat{T} and \widehat{T}_α , $R(z, \widehat{L})$ and $R(z, \widehat{L}_\alpha)$, respectively, coincide on \mathcal{L}_C for elements $z \in \rho(\widehat{L}) \cap \rho(\widehat{L}_\alpha)$ real and large enough. Here $\rho(\widehat{L})$ denotes the resolvent set of $(\widehat{L}, \mathcal{D})$ and $\rho(\widehat{L}_\alpha)$ is the resolvent set of $(\widehat{L}_\alpha, \mathcal{D}_\alpha)$. From [39, Lemma III.2.6], we know that there exists a constant $r \geq 0$ such that $(r, \infty) \in \rho(\widehat{L}) \cap \rho(\widehat{L}_\alpha)$. Let us consider some fixed $z \in (r, \infty)$, then for any $G \in \mathcal{L}_C$ we have $R(z, \widehat{L})G \in \mathcal{D} \subset \mathcal{D}_\alpha$. Moreover, since $\widehat{L}_\alpha = \widehat{L}$ on \mathcal{D} , we find

$$R(z, \widehat{L})G - R(z, \widehat{L}_\alpha)G = R(z, \widehat{L}_\alpha) \left[(z\mathbf{1} - \widehat{L}_\alpha) - (z\mathbf{1} - \widehat{L}) \right] R(z, \widehat{L})G = 0$$

and, consequently, $\widehat{T}_\alpha G = \widehat{T}G$ on \mathcal{L}_C . The latter implies that for any $G \in \mathcal{L}_C$ and $k \in \mathcal{K}_C^\infty$,

$$\langle \widehat{T}_\alpha(t)G, k \rangle = \langle \widehat{T}(t)G, k \rangle = \langle G, \widehat{T}^*(t)k \rangle. \quad (3.42)$$

On the other hand, due to the duality (2.24), for any $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$ and $k \in \mathcal{K}_{\alpha C}^\infty \subset \mathcal{K}_C^\infty$, we have

$$\langle \widehat{T}_\alpha(t)G, k \rangle = \langle G, \widehat{T}_\alpha^*(t)k \rangle, \quad (3.43)$$

where $\widehat{T}_\alpha(t)G \in \mathcal{L}_{\alpha C}$ and $\widehat{T}_\alpha^*(t)k \in \mathcal{K}_{\alpha C}^\infty$. Hence, by combining equations (3.42) and (3.43), for any $k \in \mathcal{K}_{\alpha C}^\infty$ we find

$$\widehat{T}^*(t)k = \widehat{T}_\alpha^*(t)k \in \mathcal{K}_{\alpha C}^\infty. \quad (3.44)$$

Finally, since $T^\circ(t)$ is the restriction of $\widehat{T}^*(t)$ on $\text{Dom}(\widehat{L}^*)$ and $\mathcal{K}_{\alpha C}^\infty \subset \text{Dom}(\widehat{L}^*)$, it follows that $\mathcal{K}_{\alpha C}^\infty$ is $T^\circ(t)$ -invariant. \square

Having disposed of these preliminary steps, we can now return to the proof of Theorem 3.8.

Proof of Theorem 3.8. From Theorem 3.3 we know that the operator $(\widehat{L}, \mathcal{D})$ is the generator of the holomorphic semigroup $\widehat{T}(t)$. By [39, Section II.2.6], the restriction \widehat{T}° of the dual semigroup \widehat{T}^* on the subspace of strong continuity $\overline{\text{Dom}(\widehat{L}^*)}$ is again a holomorphic semigroup with generator \widehat{L}° , which is the part of \widehat{L}^* on $\text{D}(\widehat{L}^\circ) = \overline{\text{Dom}(\widehat{L}^*)}$. On the other hand, in Lemma 3.12 we showed that $\overline{\mathcal{K}_{\alpha C}^\infty} \subset \text{Dom}(\widehat{L}^*)$, is $T^\circ(t)$ -invariant. Then, the restriction $\widehat{T}^{\circ\alpha}$ of the dual semigroup \widehat{T}° onto $\overline{\mathcal{K}_{\alpha C}^\infty}$ is also a strongly continuous semigroup. Moreover, its generator $\widehat{L}^{\circ\alpha}$ is the restriction of \widehat{L}° on $\overline{\mathcal{K}_{\alpha C}^\infty}$, see e.g. [39, Corollary II.2.3]. As a result, the evolution equation (3.2) with initial condition $k_0 \in \overline{\mathcal{K}_{\alpha C}^\infty}$ has a unique solution $k_t = \widehat{T}^{\circ\alpha}(t)k_0 \in \overline{\mathcal{K}_{\alpha C}^\infty}$. In order to complete the proof of the theorem we can just note that, by Lemma 3.12, the subspace $\mathcal{K}_{\alpha C}^\infty \subseteq \overline{\mathcal{K}_{\alpha C}^\infty}$, is also $T^\circ(t)$ -invariant. \square

3.1.3 Examples: random walks in a spatial ecological model of environment

Let us consider the case where RE is described by a BDLP model, see for instance [50] and reference therein. Heuristically, the dynamics of this birth-and-death process is described by a Markov pregenerator of the form (2.2) with

$$d(x, \gamma \setminus x) = m + \chi^- \sum_{x' \in (\gamma \setminus x)} a^-(x - x'), \quad x \in \gamma, \gamma \in \Gamma, \quad (3.45)$$

$$b(x, \gamma) = \chi^+ \sum_{x' \in (\gamma \setminus x)} a^+(x - x'), \quad x \in \mathbb{R}^d \setminus \gamma, \gamma \in \Gamma, \quad (3.46)$$

where $m > 0$, $\chi^\pm \geq 0$ are some positive constants and $0 \leq a^\pm \in L^1(\mathbb{R}^d, dx) \cap L^\infty(\mathbb{R}^d, dx)$ are even non-negative functions such that

$$\int_{\mathbb{R}^d} dx a^\pm(x) = 1. \quad (3.47)$$

In this case, for any $\eta, \xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$ we have (cf. derivation of (2.56))

$$(K^{-1}d(x, \cdot \cup \xi \setminus x))(\eta) = d(x, \xi \setminus x) 0^{|\eta|} + \chi^- \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) a^-(x - x') \quad (3.48)$$

and

$$(K^{-1}b(x, \cdot \cup \xi \setminus x))(\eta) = b(x, \xi \setminus x) 0^{|\eta|} + \chi^+ \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) a^+(x - x'). \quad (3.49)$$

As a consequence,

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| K^{-1}d(x, \cdot \cup \xi \setminus x) \right|(\eta) = d(x, \xi \setminus x) + C\chi^-$$

and

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| K^{-1}b(x, \cdot \cup \xi \setminus x) \right|(\eta) = b(x, \xi \setminus x) + C\chi^+.$$

Now if we assume that there exists a constant $\delta > 0$ such that

$$(4 + \delta) C\chi^- \leq m, \quad (3.50)$$

$$4\chi^+ a^+(x) \leq C\chi^- a^-(x), \quad \text{a.a. } x \in \mathbb{R}^d, \quad (3.51)$$

one finds

$$\int_{\Gamma_0} \left| K^{-1}d(x, \cdot \cup \xi \setminus x) \right|(\eta) C^{|\eta|} d\lambda(\eta) \leq \left(1 + \frac{1}{4 + \delta}\right) d(x, \xi \setminus x)$$

and ²

$$\int_{\Gamma_0} \left| K^{-1} b(x, \cdot \cup \xi \setminus x) \right|(\eta) C^{|\eta|} d\lambda(\eta) < \frac{C}{4} d(x, \xi \setminus x).$$

Hence, conditions (3.8) and (3.9) are satisfied with

$$a_1 = \left(1 + \frac{1}{4 + \delta}\right), \quad a_2 = \frac{C}{4}. \quad (3.52)$$

Moreover,

$$d(x, \xi) < m \left(1 + \frac{\|a^-\|_\infty}{4C}\right) (1 + |\xi|).$$

Thus, (3.28) holds with $\nu = 1$.

3.1.3.1 RW in a BDLP model of environment: Case I and III

Let us consider the RW of a jumping particle whose interaction with RE is given by

$$\begin{aligned} \lambda_{\text{int}}(\gamma, y, z) &:= \lambda_{\text{int}}(\gamma, w) = e^{-\sum_{x' \in \gamma} \phi(x' - w)} \\ &= e_\lambda \left(e^{-\phi(\cdot - w)}, \gamma \right), \quad \gamma \in \Gamma, w \in \mathbb{R}^d, \end{aligned} \quad (3.53)$$

with $w = y$ or z . Note that for $w = y$ we have the interaction $\lambda_{\text{int}}(\gamma, w) = \lambda^{(1)}(\gamma, y)$ given by (2.13), whereas for $w = z$ we obtain $\lambda_{\text{int}}(\gamma, w) = \lambda^{(3)}(\gamma, z)$ defined by (2.15).

In the following we assume that $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that

$$C'_\phi := \int_{\mathbb{R}^d} (1 - e^{-\phi(x)}) dx < \infty. \quad (3.54)$$

In this case, according to (2.50), for any $\xi \in \Gamma_0$ and $w \in \mathbb{R}^d$ one has

$$\left(K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, w) \right) (\eta) = \lambda_{\text{int}}(\xi, w) e_\lambda \left(e^{-\phi(\cdot - w)} - 1, \eta \right), \quad (3.55)$$

and we can prove the following estimate.

Proposition 3.13. *Suppose that condition (3.54) holds. Then*

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, w) \right| (\eta) \leq e^{CC'_\phi}, \quad (3.56)$$

for any $\xi \in \Gamma_0$, a.a. $w \in \mathbb{R}^d$ and $C > 0$.

Proof. By definition we can write

$$\int_{\Gamma_0} \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, w) \right| (\eta) C^{|\eta|} d\lambda(\eta) \leq e^{-\sum_{x' \in \xi} \phi(x' - w)} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) e_\lambda \left(\left| e^{-\phi(\cdot - w)} - 1 \right|, \eta \right) \quad (3.57)$$

and since the potential ϕ is non-negative one has

$$\int_{\Gamma_0} \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, w) \right| (\eta) C^{|\eta|} d\lambda(\eta) \leq \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) e_\lambda \left(1 - e^{-\phi(\cdot - w)}, \eta \right). \quad (3.58)$$

Then, by using identity (1.24) and condition (3.54), the r.h.s. can be estimated as follows

$$\begin{aligned} \int_{\Gamma_0} \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, w) \right| (\eta) C^{|\eta|} d\lambda(\eta) &\leq \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) e_\lambda \left(C \left(1 - e^{-\phi(\cdot - w)} \right), \eta \right) \\ &\leq e^{C \int_{\mathbb{R}^d} dx (1 - e^{-\phi(x-w)})} \\ &\leq e^{CC'_\phi}, \end{aligned} \quad (3.59)$$

which concludes the proof of the proposition. \square

² Note that by integrating both sides of (3.51) over \mathbb{R}^d we get $4\chi^+ \leq C\chi^- < m/4$.

From the proposition above it follows that both interactions $\lambda^{(1)}$ and $\lambda^{(3)}$ fulfill Assumption 3.1 with

$$\alpha_1 \equiv 0, \quad \alpha_0 = e^{CC'_\phi}. \quad (3.60)$$

Moreover,

$$a_1 + \frac{a_2}{C} + 2\langle a \rangle \alpha_1 = 1 + \frac{1}{4+\delta} + \frac{1}{4} < \frac{3}{2}. \quad (3.61)$$

Let us consider the operator \widehat{L}_{RE} defined by (2.36) with (3.45)-(3.46) and let \widehat{L}_{RW} be an operator given by (2.51) or (2.60). Then, according to Theorem 3.3 and 3.8, we have the following existence and uniqueness result for the initial value problem (3.2) in the Banach space \mathcal{K}_C^∞ .

Corollary 3.14. *Suppose that condition (3.54) holds. Assume that the functions a^\pm and the constants χ^\pm , m and C satisfy conditions (3.50)-(3.51). Then*

- (i) *The operator $(\widehat{L}, \mathcal{D})$ is the generator of a holomorphic semigroup $\widehat{U}(t)$ in \mathcal{L}_C .*
- (ii) *For any $\alpha \in (1/2, 1)$, the evolution equation (3.2) with initial condition $k_0 \in \mathcal{K}_{\alpha C}^\infty$ has a unique solution in the space $\mathcal{K}_{\alpha C}^\infty$ given by $k_t = \widehat{T}^{\odot \alpha}(t)k_0$.*

3.1.3.2 RW in a BDLP model of environment: Case II and IV

Let us consider now the interaction given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda_{\text{int}}(\gamma, w) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - w), \quad \gamma \in \Gamma, w \in \mathbb{R}^d, \quad (3.62)$$

where $w = y$ or z depending if we want to analyze the interaction $\lambda^{(2)}(\gamma, y)$ or $\lambda^{(4)}(\gamma, z)$, see (2.14) and (2.16), respectively. In what follows, we assume that $\lambda_0 \geq 0$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that

$$C_1^\phi := \|\phi\|_1 = \int_{\mathbb{R}^d} dx \phi(x) < \infty \quad (3.63)$$

and

$$C_\infty^\phi := \|\phi\|_\infty = \text{ess sup}_{x \in \mathbb{R}^d} \phi(x) < \infty. \quad (3.64)$$

According to equation (2.56) one has

$$(K^{-1}\lambda_{\text{int}}(\cdot \cup \xi, w))(\eta) = \lambda_{\text{int}}(\xi, w) 0^{|\eta|} + \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) \phi(x' - w), \quad (3.65)$$

for any $\xi \in \Gamma_0$ and $w \in \mathbb{R}^d$. Then, we can prove the following estimate.

Proposition 3.15. *Let us assume that conditions (3.63) and (3.64) hold. Then*

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| K^{-1}\lambda_{\text{int}}(\cdot \cup \xi, w) \right|(\eta) \leq \lambda_0 + CC_1^\phi + C_\infty^\phi |\xi|, \quad (3.66)$$

for any $\xi \in \Gamma_0$, a.a. $w \in \mathbb{R}^d$ and $C > 0$.

Proof. By (3.65) one can write

$$\begin{aligned} \int_{\Gamma_0} \left| K^{-1}\lambda_{\text{int}}(\cdot \cup \xi, w) \right|(\eta) C^{|\eta|} d\lambda(\eta) &\leq \lambda_{\text{int}}(\xi, w) \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) 0^{|\eta|} + \\ &\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) \phi(x' - w) \end{aligned}$$

$$\leq \lambda_0 + \sum_{x \in \xi} \phi(x - w) + C \int_{\mathbb{R}^d} dx \phi(x - w).$$

Then, by using conditions (3.63) and (3.64), we find

$$\int_{\Gamma_0} \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \xi, w) \right| (\eta) C^{|\eta|} d\lambda(\eta) \leq \lambda_0 + C \|\phi\|_1 + \|\phi\|_\infty |\xi|,$$

which concludes the proof of the proposition. \square

Note that from (3.45) it follows that for any $\xi \in \Gamma_0$

$$\sum_{x \in \xi} d(x, \xi \setminus x) = m |\xi| + \chi^- E^{a^-}(\xi), \quad E^{a^-}(\xi) = \sum_{x \in \xi} \sum_{x' \in \xi \setminus x} a^-(x - x') \geq 0. \quad (3.67)$$

Therefore, Proposition 3.15 implies that Assumption 3.1 is satisfied with

$$\alpha_1 = \frac{C_\infty^\phi}{m}, \quad \alpha_0 = \lambda_0 + C C_1^\phi. \quad (3.68)$$

Hence, for the interactions $\lambda^{(2)}$ and $\lambda^{(4)}$ we can apply Theorem 3.3 and 3.8 and formulate the following result for the evolution of correlation functions in the Banach space \mathcal{K}_C^∞ .

Corollary 3.16. *Let us consider the operator \widehat{L}_{RE} defined by (2.36) with (3.45)-(3.46). Let \widehat{L}_{RW} be an operator of the form (2.57) or (2.64) such that conditions (3.63) and (3.64) hold. Suppose that there exists $\delta > 0$ such that the functions a^\pm and the constants χ^\pm , m and C satisfy the conditions*

$$C \chi^- a^-(x) \geq 4 \chi^+ a^+(x), \quad a.a. \ x \in \mathbb{R}^d, \quad (3.69)$$

$$m \geq (4 + \delta) (C \chi^- + 2 \langle a \rangle C_\infty^\phi). \quad (3.70)$$

Then

(i) *The operator $(\widehat{L}, \mathcal{D})$ is the generator of a holomorphic semigroup $\widehat{U}(t)$ in \mathcal{L}_C .*

(ii) *For any $\alpha \in (1/2, 1)$, the evolution equation (3.2) with initial condition $k_0 \in \mathcal{K}_{\alpha C}^\infty$ has a unique solution in the space $\mathcal{K}_{\alpha C}^\infty$ given by $k_t = \widehat{T}^{\odot \alpha}(t) k_0$.*

Remark 3.17. Note that conditions (3.69)-(3.70) are stronger than (3.50)-(3.51) and one can easily check that hypotheses (3.8), (3.9) and (3.12) are still satisfied.

Remark 3.18. Let us stress that in all models of RW in BDLP model of environment considered above we found a unique solution to the evolution equation for correlation functions (3.2) on the space $\mathcal{K}_{\alpha C}^\infty$, $\alpha \in (1/2, 1)$, only if the mortality m is large enough, see conditions (3.50) and (3.70). Actually, it is possible to remove this restriction on the parameter m by studying the initial value problem (3.2) with the method mentioned in Remark 2.9. This approach was introduced in [47, 48] and consists of a combination of the semigroup techniques described in this chapter and the Ovsyannikov's method, see e.g. [98, 120], which will be considered in next chapter. In this approach we get a solution k_t to (3.2) on some space $\mathcal{K}_{C_t}^\infty$ for any mortality $m > 0$, but for a finite time interval only.

3.2 Mesoscopic evolutions: Vlasov-type scaling

Let us consider the Vlasov-type scaling of the statistical evolution for RW in a birth-and-death environment studied in the previous section.

According to the general scheme discussed in Section 2.3, for any $\varepsilon > 0$ we introduce the renormalized hierarchy

$$\begin{cases} \frac{dk_{t,\varepsilon,\text{ren}}}{dt} = (L_{RE,\varepsilon,\text{ren}}^\Delta + L_{RW,\varepsilon,\text{ren}}^\Delta) k_{t,\varepsilon,\text{ren}} & , \quad t \geq 0, \\ k_t|_{t=0} = k_0 \end{cases} \quad (3.71)$$

where the generator $L_{RE,\varepsilon,\text{ren}}^\Delta$ and $L_{RW,\varepsilon,\text{ren}}^\Delta$ are given by by formulas (2.97) and (2.101), respectively. We also define $L_{\varepsilon,\text{ren}}^\Delta := L_{RE,\varepsilon,\text{ren}}^\Delta + L_{RW,\varepsilon,\text{ren}}^\Delta$.

Next let us assume that the limits (2.104), (2.105) and (2.111) exist. Then, letting ε go to zero, we have the Vlasov hierarchy

$$\begin{cases} \frac{dr_t}{dt} = (L_{RE,V}^\Delta + L_{RW,V}^\Delta) r_t & , \quad t \geq 0, \\ r_t|_{t=0} = r_0 \end{cases} \quad (3.72)$$

where the generators $L_{RE,V}^\Delta$ and $L_{RW,V}^\Delta$ are defined by (2.106) and (2.113), respectively. Also, we set $L_V^\Delta := L_{RE,V}^\Delta + L_{RW,V}^\Delta$. Moreover, the hierarchy (3.72) has the so-called chaos preservation property, see (2.85)-(2.86). As a result, we obtain a system of kinetic equations for the densities ρ_t and r_t given by (2.115) and (2.123).

In next sections we study this mesoscopic limit in the Banach space \mathcal{K}_C^∞ , defined by (3.3). In Section 3.2.1, we show the convergence of the Vlasov-type scaling for the considered evolutions. Then, in Section 3.2.2 we derive the corresponding Vlasov equations and study their solutions.

3.2.1 Convergence of the Vlasov-type scaling

In this section we study the convergence of the solutions of the family of renormalized hierarchies (3.71) to the solution of the limiting Vlasov hierarchy (3.72) in the Banach space $\mathcal{K}_{\alpha C}^\infty$. As in the construction of the time evolution of correlation functions in Section 3.1, we consider the duality (2.24) and study the convergence of Vlasov-type scaling in the pre-dual space \mathcal{L}_C defines in (3.6), see Remark 2.16. Our approach is adapted from [54]. Let us outline the strategy that we will follow.

First, we show that for any $\varepsilon > 0$ the renormalized operator $\widehat{L}_{\varepsilon,\text{ren}} := \widehat{L}_{RE,\varepsilon,\text{ren}} + \widehat{L}_{RW,\varepsilon,\text{ren}}$, defined by (2.97) and (2.102), is the generator of a strongly continuous contraction semigroup $\widehat{U}_\varepsilon(t)$ on \mathcal{L}_C , see Lemma 3.22.1. Then, we show that the limiting operator \widehat{L}_V is also the generator of a strongly continuous contraction semigroup $\widehat{U}_V(t)$ on \mathcal{L}_C , see Lemma 3.30.1. Finally, we prove that the semigroup $\widehat{U}_\varepsilon(t)$ converges to $\widehat{U}_V(t)$ strongly in \mathcal{L}_C as ε goes to zero, see Theorem 3.35. The last statement will be a consequence of an abstract result about the strong convergence of resolvent operators, see for instance [54, Lemma 4.3]. For the reader's convenience, we will formulate this general result below.

Lemma 3.19. *Let X be a Banach space, and let $(A_\varepsilon, D_\varepsilon)$, $(B_\varepsilon, D_\varepsilon)$, with $\varepsilon \geq 0$, be closed, densely defined operators on X . Suppose that there exist a constant $\bar{u} > 0$ and $u \in \mathbb{C}$ with ³ $\Re u > \bar{u}$ such that $u \in \rho(A_\varepsilon)$ for all $\varepsilon \geq 0$ and that the following conditions hold*

$$\kappa := \sup_{\varepsilon > 0} \left\| (A_\varepsilon - u\mathbb{1})^{-1} \right\| < \infty, \quad (3.73)$$

$$\sigma := \sup_{\varepsilon > 0} \left\| B_\varepsilon (A_\varepsilon - u\mathbb{1})^{-1} \right\| < 1, \quad (3.74)$$

³We denote by $\Re u$ the real part of $u \in \mathbb{C}$.

$$(A_\varepsilon - u\mathbf{1})^{-1} \xrightarrow{s} (A_0 - u\mathbf{1})^{-1}, \quad \varepsilon \rightarrow 0, \quad (3.75)$$

$$B_\varepsilon (A_\varepsilon - u\mathbf{1})^{-1} \xrightarrow{s} B_0 (A_0 - u\mathbf{1})^{-1}, \quad \varepsilon \rightarrow 0. \quad (3.76)$$

Then, u belongs to the resolvent set of $L_\varepsilon := A_\varepsilon + B_\varepsilon$, for any $\varepsilon \geq 0$, and

$$(L_\varepsilon - u\mathbf{1})^{-1} \xrightarrow{s} (L_0 - u\mathbf{1})^{-1}, \quad \varepsilon \rightarrow 0. \quad (3.77)$$

Transferring the general theory about adjoint semigroups, see e.g. [39, Section II.2.5], onto the semigroups $U_\varepsilon^\Delta(t)$ and $U_V^\Delta(t)$ we deduce that they will be weak*-continuous, weak*-differentiable at 0 and with weak*-generators $L_{\varepsilon, \text{ren}}^\Delta$ and L_V^Δ , respectively, on \mathcal{K}_C^∞ . Moreover, we obtain the weak*-convergence of the semigroups $U_\varepsilon^\Delta(t)$ to $U_V^\Delta(t)$ in \mathcal{K}_C^∞ .

Remark 3.20. The question about the strong convergence to the solution of the limiting hierarchy is still open. Indeed, we can show that the restrictions $\widehat{U}_{\varepsilon, \text{ren}}^{\odot\alpha}(t)$ and $\widehat{U}_V^{\odot\alpha}(t)$ of the semigroups $U_{\varepsilon, \text{ren}}^\Delta(t)$ and $U_V^\Delta(t)$, respectively, are C_0 -semigroups on the subspace $\mathcal{K}_{\alpha C}^\infty \subseteq \mathcal{K}_C^\infty$, for some $\alpha \in (0, 1)$, see Lemma 3.22.2 and 3.30.2. The main problem consists of the fact that we have an explicit expression for the sun dual generator $L_V^{\odot\alpha} = L_V^\Delta$ only on the core $\{k \in \mathcal{K}_{\alpha C}^\infty \mid L_V^\Delta \in \overline{\mathcal{K}_{\alpha C}^\infty}\}$. In [53] the strong convergence of the Vlasov-type scaling has been shown for the Glauber-type dynamics in continuum by using an approximative approach, see [53, 59] for details. Unfortunately, such a technique can be successfully applied only in that particular model.

Renormalized evolutions. According to Proposition 2.17 and 2.19, for any $G \in B_{\text{bs}}(\tilde{\Gamma}_0)$ the operator $\widehat{L}_{\varepsilon, \text{ren}}$, $\varepsilon > 0$, can be written in the following form

$$(\widehat{L}_{\varepsilon, \text{ren}} G)(\eta, y) = (L_0^{(\varepsilon)} G)(\eta, y) + (L_1^{(\varepsilon)} G)(\eta, y) + (\widehat{L}_{RW}^{(\varepsilon)} G)(\eta, y), \quad (3.78)$$

where $\widehat{L}_{RW}^{(\varepsilon)} := \widehat{L}_{RW, \varepsilon, \text{ren}}$ is given by (2.102),

$$(L_0^{(\varepsilon)} G)(\eta, y) = -D_\varepsilon(\eta) G(\eta, y), \quad D_\varepsilon(\eta) = \sum_{x \in \eta} d_\varepsilon(x, \eta \setminus x) \quad (3.79)$$

and

$$\begin{aligned} (L_1^{(\varepsilon)} G)(\eta, y) = & - \sum_{\xi \subseteq \eta} G(\xi, y) \varepsilon^{-|\eta \setminus \xi|} \sum_{x \in \xi} (K^{-1} d_\varepsilon(x, \cdot \cup (\xi \setminus x))) (\eta \setminus \xi) + \\ & \sum_{\xi \subseteq \eta} \int_{\mathbb{R}^d} dx G(\xi \cup x, y) \varepsilon^{-|\eta \setminus \xi|} (K^{-1} b_\varepsilon(x, \cdot \cup \xi)) (\eta \setminus \xi). \end{aligned} \quad (3.80)$$

Let us analyze the operator (3.78) in the space Banach space \mathcal{L}_C . For any $\varepsilon > 0$ we define as domain the set

$$\mathcal{D}^{(\varepsilon)} = \{G \in \mathcal{L}_C \mid D_\varepsilon(\cdot) G \in \mathcal{L}_C\}. \quad (3.81)$$

Following the approach in [54, Proposition 4.1], for any $\varepsilon \in (0, 1]$ we assume that there exist $\tilde{a}_1 \geq 1$, $\tilde{a}_2 > 0$ such that for all $\xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$

$$\sum_{x \in \xi} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \varepsilon^{-|\eta|} |K^{-1} d_\varepsilon(x, \cdot \cup \xi \setminus x)|(\eta) \leq \tilde{a}_1 \sum_{x \in \xi} d_\varepsilon(x, \xi \setminus x), \quad (3.82)$$

$$\sum_{x \in \xi} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \varepsilon^{-|\eta|} |K^{-1} b_\varepsilon(x, \cdot \cup \xi \setminus x)|(\eta) \leq \tilde{a}_2 \sum_{x \in \xi} d_\varepsilon(x, \xi \setminus x). \quad (3.83)$$

Moreover, analogously to the non-rescaled case (cf. (3.10)), we make the following assumption on the rescaled interaction λ_ε .

Assumption 3.21. *Suppose that for any $\varepsilon \in (0, 1]$ there exist $\tilde{\alpha}_0, \tilde{\alpha}_1 \geq 0$ such that for all $\xi \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$*

$$\int_{\Gamma_0} C^{|\eta|} d\lambda_\varepsilon(\eta) \varepsilon^{-|\eta|} \left| K^{-1} \lambda_\varepsilon(\cdot \cup \xi, y, z) \right|(\eta) \leq \tilde{\alpha}_1 \sum_{x \in \xi} d_\varepsilon(x, \xi \setminus x) + \tilde{\alpha}_0. \quad (3.84)$$

Under the conditions listed above, we can formulate the following results about the existence of the semigroups \widehat{U}_V and U_V^Δ on the spaces \mathcal{L}_C and \mathcal{K}_C^∞ , respectively.

Lemma 3.22.

1. *Suppose that (3.82)-(3.84) hold with*

$$\tilde{a}_1 + \frac{\tilde{a}_2}{C} + 2\langle a \rangle \tilde{\alpha}_1 < \frac{3}{2}. \quad (3.85)$$

Then, for any $\varepsilon \in (0, 1]$ the operator $(\widehat{L}_{\varepsilon, \text{ren}}, \mathcal{D}^{(\varepsilon)})$ is the generator of a holomorphic semigroup $\widehat{U}_\varepsilon(t)$ on \mathcal{L}_C .

2. *Assume, additionally, that for any $\varepsilon \in (0, 1]$ there exist $A > 0$, $N \in \mathbb{N}_0$ and $\nu \geq 1$ such that for any $\xi \in \Gamma_0$ and $x \in \xi$*

$$d_\varepsilon(x, \xi) \leq A(1 + |\xi|)^N \nu^{|\xi|}, \quad (3.86)$$

with

$$1 \leq \nu < \frac{C}{\tilde{a}_2} \left(\frac{3}{2} - \tilde{a}_1 - 2\langle a \rangle \tilde{\alpha}_1 \right). \quad (3.87)$$

Then, for any $\varepsilon \in (0, 1]$ and for any

$$\alpha \in \left(\frac{\tilde{a}_2}{C(\frac{3}{2} - \tilde{a}_1 - 2\langle a \rangle \tilde{\alpha}_1)}; \frac{1}{\nu} \right), \quad (3.88)$$

there exists a strongly continuous semigroup $\widehat{U}_\varepsilon^{\circ\alpha}(t)$ on the space $\overline{\mathcal{K}_{\alpha C}}$ with generator $\widehat{L}_\varepsilon^{\circ\alpha} = L_{\varepsilon, \text{ren}}^\Delta$ on the domain $\text{Dom}(\widehat{L}_\varepsilon^{\circ\alpha}) = \{k \in \overline{\mathcal{K}_{\alpha C}} : \widehat{L}_{\varepsilon, \text{ren}}^ k \in \overline{\mathcal{K}_{\alpha C}}\}$.*

Proof of Lemma 3.22.1. We can proceed as in Section 3.1.1.1. Indeed, it is easy to check that, for any $\varepsilon \in (0, 1]$, the operator $(L_0^{(\varepsilon)}, \mathcal{D}^{(\varepsilon)})$ has the same properties of the corresponding non-renormalized operator (L_0, D) . In particular, we can show the following result (cf. Lemma 3.4).

Lemma 3.23. *For any $\varepsilon \in (0, 1]$, the operator $(L_0^{(\varepsilon)}, \mathcal{D}^{(\varepsilon)})$ is the generator of a contraction semigroup on \mathcal{L}_C . Moreover, $L_0^{(\varepsilon)}$ is a sectorial operator for any $\omega \in (0, \pi/2)$ and for all $z \in \mathbb{C} \setminus \{0\}$ with $|\arg z| \leq \pi/2 + \omega - \varepsilon$,*

$$\left\| \left(z\mathbf{1} - L_0^{(\varepsilon)} \right)^{-1} \right\|_C \leq \frac{M_\varepsilon}{|z|}, \quad (3.89)$$

where $M_\varepsilon = 1/\cos \omega$ for all $\varepsilon \in (0, \omega)$.

Moreover, from [54] we know that for any $\varepsilon \in (0, 1]$ the operator $L_1^{(\varepsilon)}$ is relatively bounded by $\widehat{L}_0^{(\varepsilon)}$ in \mathcal{L}_C .

Proposition 3.24. *Let us suppose that conditions (3.82) and (3.83) hold. Then, the operator $(L_1^{(\varepsilon)}, \mathcal{D}^{(\varepsilon)})$, $\varepsilon \in (0, 1]$, is a well-defined operator in \mathcal{L}_C satisfying*

$$\left\| L_1^{(\varepsilon)} G \right\|_C \leq \left(\tilde{a}_1 - 1 + \frac{\tilde{a}_2}{C} \right) \|L_0 G\|_C, \quad G \in \mathcal{D}^{(\varepsilon)}. \quad (3.90)$$

Next we can show that the operator $\widehat{L}_{RW}^{(\varepsilon)}$ is also $L_0^{(\varepsilon)}$ -bounded in \mathcal{L}_C .

Proposition 3.25. *Let us suppose that assumption (3.84) is satisfied. Then, the operator $(\widehat{L}_{RW}^{(\varepsilon)}, \mathcal{D}^{(\varepsilon)})$, $\varepsilon \in (0, 1]$, is a well-defined operator on \mathcal{L}_C and the following bound holds*

$$\left\| \widehat{L}_{RW}^{(\varepsilon)} G \right\|_C \leq 2\langle a \rangle \tilde{\alpha}_1 \|L_0 G\|_C + 2\langle a \rangle \tilde{\alpha}_0 \|G\|_C, \quad G \in \mathcal{D}^{(\varepsilon)}. \quad (3.91)$$

Proof of Proposition 3.25. We need to estimate the following norm in \mathcal{L}_C

$$\begin{aligned} \left\| \widehat{L}_{RW}^{(\varepsilon)} G \right\|_C &\leq \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \int_{\mathbb{R}^d} dz a(y-z) \times \\ &\quad \sum_{\xi \subset \eta} \varepsilon^{-|\eta \setminus \xi|} \left| K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z) \right| (\eta \setminus \xi) |G(\xi, z) - G(\xi, y)|. \end{aligned}$$

One can proceed as in the proof of Proposition 3.6, by using condition (3.84) instead of (3.10). As a result, we obtain

$$\begin{aligned} \left\| \widehat{L}_{RW}^{(\varepsilon)} G \right\|_C &\leq 2\langle a \rangle \tilde{\alpha}_1 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) D_\varepsilon(\xi) |G(\xi, y)| + \\ &\quad 2\langle a \rangle \tilde{\alpha}_0 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |G(\xi, y)|, \end{aligned}$$

which concludes the proof of the proposition. \square

Having disposed this preliminary results, we can prove the first part of the lemma by applying the same arguments used in the proof of Theorem 3.3. \square

Proof of Lemma 3.22.2. In order to construct a strongly continuous semigroup on the space \mathcal{K}_C^∞ we follow the same strategy outlined in Section 3.1.2. From the first part of the lemma, we can deduce that the operator $L_{\varepsilon, \text{ren}}^\Delta$ is the weak*-generator of the semigroups $U_\varepsilon^\Delta(t)$ which is weak*-continuous and weak*-differentiable at 0 in the Banach space \mathcal{K}_C^∞ , for any $\varepsilon \in (0, 1]$. Hence, by [39, Section II.2.6], it follows that the restrictions $U_\varepsilon^\circ(t)$ of the semigroups $U_\varepsilon^\Delta(t)$ onto its sun-dual subspaces $\overline{\text{Dom}(L_{\varepsilon, \text{ren}}^\Delta)}$ is a strongly continuous semigroup whose generator $L_{\varepsilon, \text{ren}}^\circ$ is a part of the operator $L_{\varepsilon, \text{ren}}^\Delta$ onto $\text{Dom}(L_{\varepsilon, \text{ren}}^\circ) = \overline{\text{Dom}(L_{\varepsilon, \text{ren}}^\Delta)}$, $\varepsilon \in (0, 1]$. Next, we restrict the C_0 -semigroup $U_\varepsilon^\circ(t)$ onto the subspace $\mathcal{K}_{\alpha C}^\infty \subseteq \mathcal{K}_C^\infty$ for some $\alpha \in (0, 1)$. For this purpose, it sufficient to show that the set $\mathcal{K}_{\alpha C}^\infty$ satisfies the inclusion

$$\mathcal{K}_{\alpha C}^\infty \subset \left(\bigcap_{\varepsilon > 0} \overline{\text{Dom}(L_{\varepsilon, \text{ren}}^\Delta)} \right)$$

and is a U_ε^Δ -invariant subspace of \mathcal{K}_C^∞ , for any $\varepsilon \in (0, 1)$. These two conditions follows directly from the following propositions which will be proved at the end of the proof.

Proposition 3.26. *Let us assume that hypothesis (3.82)-(3.84) are satisfied. Suppose that for any $\varepsilon \in [0, 1)$ there exist $A > 0$, $N \in \mathbb{N}_0$ and $\nu \geq 1$ such that for $\xi \in \Gamma_0$ and $x \notin \xi$*

$$d_\varepsilon(x, \xi) \leq A(1 + |\xi|)^N \nu^{|\xi|}. \quad (3.92)$$

Then for any $\alpha \in (0, 1/\nu)$

$$\mathcal{K}_{\alpha C}^\infty \subset \bigcap_{\varepsilon > 0} \text{Dom}(L_{\varepsilon, \text{ren}}^\Delta). \quad (3.93)$$

Proposition 3.27. *Suppose that the hypotheses of Lemma 3.22 are satisfied. Then, the set $\overline{\mathcal{K}_{\alpha C}^\infty}$ is a $\widehat{U}_\varepsilon^\circ$ -invariant subspace of \mathcal{K}_C^∞ .*

As a result of the two propositions above, for any $\varepsilon \in (0, 1]$ the restriction $U_\varepsilon^{\circ\alpha}(t)$ of the semigroup $U_\varepsilon^\circ(t)$ to the closed subspace $\overline{\mathcal{K}_{\alpha C}^\infty}$ is a strongly continuous semigroup with generator $L_{\varepsilon, \text{ren}}^{\circ\alpha}$ is the restriction of $L_{\varepsilon, \text{ren}}^\circ$ to $\overline{\mathcal{K}_{\alpha C}^\infty}$, see e.g. [39, Section II.2.3]. This concludes the proof of Lemma 3.22.2. \square

Proof of Proposition 3.26. As discussed in Proposition 3.10, in order to prove the statement, we need to show that for any $\varepsilon \in (0, 1]$ and $k \in \mathcal{K}_{\alpha C}^\infty$

$$C^{-|\cdot|} L_{\varepsilon, \text{ren}}^\Delta k \in L^\infty(\Gamma_0 \times \mathbb{R}^d, d\lambda dy).$$

By definition, we have

$$C^{-|\eta|} \left| \left(L_{\varepsilon, \text{ren}}^\Delta k \right) (\eta, y) \right| \leq C^{-|\eta|} \left| \left(L_{RE, \varepsilon, \text{ren}}^\Delta k \right) (\eta, y) \right| + C^{-|\eta|} \left| \left(L_{RW, \varepsilon, \text{ren}}^\Delta k \right) (\eta, y) \right|.$$

From the proof of Proposition 4.1.2 in [54], we know that $C^{-|\cdot|} L_{RE, \varepsilon, \text{ren}}^\Delta k \in L^\infty(\Gamma_0 \times \mathbb{R}^d, d\lambda dy)$, i.e. for any $k \in \mathcal{K}_{\alpha C}^\infty$, $\alpha \in (0, 1/\nu)$,

$$\text{ess sup}_{(\eta, y) \in \Gamma_0 \times \mathbb{R}^d} C^{-|\eta|} \left| \left(\widehat{L}_{RE, \varepsilon, \text{ren}}^\Delta k \right) (\eta, y) \right| < \infty.$$

Let us now consider the operator $L_{RW, \varepsilon, \text{ren}}^\Delta$, $\varepsilon \in (0, 1]$. As in the proof of Proposition 3.10, for any $k \in \mathcal{K}_{\alpha C}^\infty$, $\alpha \in (0, 1/\nu)$, one can estimate

$$\begin{aligned} C^{-|\eta|} \left| \left(L_{RW, \varepsilon, \text{ren}}^\Delta k \right) (\eta, y) \right| &\leq \\ &\leq \alpha^{|\eta|} \|k\|_{\mathcal{K}_{\alpha C}^\infty} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} (\alpha C)^{|\zeta|} d\lambda(\zeta) \varepsilon^{-|\zeta|} \left| \left(K^{-1} \lambda_\varepsilon(\cdot \cup \eta, z, y) \right) (\zeta) \right| + \\ &\quad \alpha^{|\eta|} \|k\|_{\mathcal{K}_{\alpha C}^\infty} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} (\alpha C)^{|\zeta|} d\lambda(\zeta) \varepsilon^{-|\zeta|} \left| \left(K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z) \right) (\zeta) \right| \\ &\leq 2\langle a \rangle \|k\|_{\mathcal{K}_{\alpha C}^\infty} \left[\tilde{\alpha}_1 \alpha^{|\eta|} D_\varepsilon(\eta) + \tilde{\alpha}_0 \alpha^{|\eta|} \right], \end{aligned}$$

where in the last step we use (3.84). Note that from hypothesis (3.92) it follows that

$$D_\varepsilon(\eta) = \sum_{x \in \eta} d_\varepsilon(x, \eta \setminus x) \leq A(1 + |\eta|)^{N+1} \nu^{|\eta|-1}, \quad (3.94)$$

then, by using elementary inequality (3.37) we find

$$\begin{aligned} C^{-|\eta|} \left| \left(L_{RW, \varepsilon, \text{ren}}^\Delta k \right) (\eta, y) \right| &\leq 2\langle a \rangle \|k\|_{\mathcal{K}_{\alpha C}^\infty} \left[A \tilde{\alpha}_1 \alpha^{|\eta|} (1 + |\eta|)^{N+1} \nu^{|\eta|-1} + \tilde{\alpha}_0 \alpha^{|\eta|} \right] \\ &\leq 2\langle a \rangle \|k\|_{\mathcal{K}_{\alpha C}^\infty} \left[A \tilde{\alpha}_1 \frac{1}{\alpha \nu^2} \left(\frac{N+1}{-\ln(\alpha \nu)} \right)^{N+1} + \tilde{\alpha}_0 \right] < \infty, \end{aligned}$$

which concludes the proof of the proposition. \square

Proof of Proposition 3.27. Similarly to the proof of Lemma 3.22.1, we can show that the operator $(\widehat{L}_{\varepsilon, \text{ren}}, \mathcal{D}_\alpha^{(\varepsilon)})$, $\varepsilon \in (0, 1]$, with $\mathcal{D}_\alpha^{(\varepsilon)} = \{G \in \mathcal{L}_{\alpha C} : D_\varepsilon G \in \mathcal{L}_{\alpha C}\}$, is the generator of an holomorphic semigroup $\widehat{U}_{\varepsilon, \text{ren}}^\alpha$ on $\mathcal{L}_{\alpha C}$ for any $\alpha \in (0, 1]$ such that

$$\tilde{\alpha}_1 + \frac{\tilde{\alpha}_2}{\alpha C} + 2\langle a \rangle \tilde{\alpha}_1 < \frac{3}{2}. \quad (3.95)$$

Note that under hypothesis (3.88) of the lemma the condition above is always satisfied. Then, we can proceed as in the proof of Lemma 3.12. \square

Limiting evolutions. Using Proposition 2.20 and 2.22, for any $G \in B_{\text{bs}}(\Gamma_0 \times \mathbb{R}^d)$ we can rewrite the operator \widehat{L}_V as

$$(\widehat{L}_V G)(\eta, y) = (L_0^V G)(\eta, y) + (L_1^V G)(\eta, y) + (\widehat{L}_{RW}^V G)(\eta, y), \quad (3.96)$$

where $\widehat{L}_{RW}^V := \widehat{L}_{RW,V}$ is defined in (2.114),

$$(L_0^V G)(\eta, y) = -D_V(\eta) G(\eta, y), \quad D_V(\eta) = \sum_{x \in \eta} D_x^V(\emptyset) \quad (3.97)$$

and

$$(L_1^V G)(\eta, y) = - \sum_{\xi \subset \eta} G(\xi, y) \sum_{x \in \xi} D_x^V(\eta \setminus \xi) + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} dx G(\xi \cup x, y) B_x^V(\eta \setminus \xi). \quad (3.98)$$

Similarly to [54, Proposition 4.2], we define the set

$$\mathcal{D}^V = \{G \in \mathcal{L}_C \mid D_V(\cdot) G \in \mathcal{L}_C\} \quad (3.99)$$

and we assume that for a.a. $x \in \mathbb{R}^d$

$$\int_{\Gamma_0} |D_x^V(\eta)| C^{|\eta|} d\lambda(\eta) \leq \tilde{a}_1 D_x^V(\emptyset), \quad (3.100)$$

$$\int_{\Gamma_0} |B_x^V(\eta)| C^{|\eta|} d\lambda(\eta) \leq \tilde{a}_2 D_x^V(\emptyset), \quad (3.101)$$

where the constants are the same as in (3.82)-(3.83). In addition, we make the following assumption on the limiting interaction (2.111).

Assumption 3.28. *Suppose that for a.a. $y, z \in \mathbb{R}^d$ the following bound holds*

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |A_V(\eta, y, z)| \leq \tilde{\alpha}_0, \quad (3.102)$$

with the same constant of (3.84).

Note that, under Assumption 3.28, the operator \widehat{L}_{RW}^V is bounded in \mathcal{L}_C .

Proposition 3.29. *Suppose that condition (3.102) holds. Then the operator \widehat{L}_{RW}^V satisfies the following bound in \mathcal{L}_C*

$$\|\widehat{L}_{RW}^V G\|_C \leq 2\langle a \rangle \tilde{\alpha}_0 \|G\|_C, \quad G \in \mathcal{L}_C. \quad (3.103)$$

Proof. Let us compute the norm of \widehat{L}_{RW}^V in \mathcal{L}_C , by definition we have

$$\|\widehat{L}_{RW}^V G\|_C \leq \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \int_{\mathbb{R}^d} dza (y-z) \sum_{\xi \subset \eta} |A_V(\eta \setminus \xi, y, z)| |G(\xi, z) - G(\xi, y)|.$$

We can proceed as in the proof of Proposition 3.6, using (3.102) instead of (3.10). As a result we have

$$\|\widehat{L}_{RW}^V G\|_C \leq 2\langle a \rangle \tilde{\alpha}_0 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |G(\xi, y)|,$$

which concludes the proof of the proposition. \square

As a consequence, we can show the following results about the existence of renormalized and limiting semigroups on the spaces \mathcal{L}_C and \mathcal{K}_C^∞ .

Lemma 3.30.

1. Suppose that hypotheses (3.100)-(3.102) hold with

$$\tilde{a}_1 + \frac{\tilde{a}_2}{C} < \frac{3}{2}, \quad (3.104)$$

then the operator $(\widehat{L}_V, \mathcal{D}^V)$ is the generator of a holomorphic semigroup $\widehat{U}_V(t)$ on \mathcal{L}_C .

2. Assume, additionally, that there exist $A > 0$ such that for a.a. $x \in \mathbb{R}^d$

$$D_x^V(\emptyset) \leq A. \quad (3.105)$$

Then, for any

$$\alpha \in \left(\frac{\tilde{a}_2}{C(3/2 - \tilde{a}_1)}; 1 \right), \quad (3.106)$$

there exists a strongly continuous semigroup $\widehat{U}_V^{\circ\alpha}(t)$ on the space $\overline{\mathcal{K}_{\alpha C}^\infty}$ with generator $\widehat{L}_V^{\circ\alpha} = L_V^\Delta$ on $\text{Dom}(\widehat{L}_V^{\circ\alpha}) = \{k \in \overline{\mathcal{K}_{\alpha C}^\infty} : L_V^\Delta k \in \overline{\mathcal{K}_{\alpha C}^\infty}\}$.

Proof of Lemma 3.30.1. From [54, Proposition 4.2], we know that, under hypotheses (3.100) and (3.101), the operator $(L_0^V + L_1^V, \mathcal{D}^V)$ generates a holomorphic semigroup in \mathcal{L}_C . On the other hand, from Proposition 3.29 we know the operator \widehat{L}_{RW}^V is bounded in \mathcal{L}_C , under hypothesis (3.102). Hence, we can apply [39, Theorem III.1.3] and obtain the desired result. \square

Proof of Lemma 3.30.2. In order to show the second statement we follow the proof of Lemma 3.22.2. In this case, we make use of the following auxiliary results. Their proof is given at the end of this paragraph.

Proposition 3.31. *Let us assume that conditions (3.100)-(3.102) and (3.105) are satisfied. Suppose, additionally, that there exists $A > 0$ such that for a.a. $x \in \mathbb{R}^d$*

$$D_x^V(\emptyset) \leq A. \quad (3.107)$$

Then, for any $\alpha \in (0, 1/\nu)$,

$$\mathcal{K}_{\alpha C}^\infty \subset \text{Dom}(L_V^\Delta). \quad (3.108)$$

Proposition 3.32. *Suppose that the hypotheses of the Lemma 3.30 are satisfied. Then, the set $\overline{\mathcal{K}_{\alpha C}^\infty}$ is a $\widehat{U}_V^{\circ\alpha}$ -invariant subspace of \mathcal{K}_C^∞ .*

Let us now consider the restriction $U_V^{\circ\alpha}(t)$ of the semigroups $U_V^{\circ}(t)$ onto the closed subspace $\overline{\mathcal{K}_{\alpha C}^\infty}$. As a consequence of the two propositions above, $U_V^{\circ\alpha}(t)$ is a strongly continuous semigroup with generators $L_V^{\circ\alpha}$ which is the restriction of the corresponding operator L_V° . This proves Lemma 3.30.2. \square

Proof of Proposition 3.31. In order to prove the statement, it is enough to verify that for any $k \in \mathcal{K}_{\alpha C}^\infty$,

$$C^{-|\cdot|} L_V^\Delta k \in L^\infty(\Gamma_0 \times \mathbb{R}^d, d\lambda dy).$$

In general we can write

$$C^{-|\eta|} \left| (L_V^\Delta k)(\eta, y) \right| \leq C^{-|\eta|} \left| (L_{RE, V}^\Delta k)(\eta, y) \right| + C^{-|\eta|} \left| (L_{RW, V}^\Delta k)(\eta, y) \right|.$$

From [54, Proposition 4.2.2], we know that $C^{-|\cdot|}L_{RE,V}^\Delta k \in L^\infty(\Gamma_0 \times \mathbb{R}^d, d\lambda dy)$, namely

$$\operatorname{ess\,sup}_{(\eta,y) \in \Gamma_0 \times \mathbb{R}^d} C^{-|\eta|} \left| (L_{RE,V}^\Delta k)(\eta, y) \right| < \infty.$$

Moreover, by using (3.102) one can estimate

$$\begin{aligned} C^{-|\eta|} \left| (\widehat{L}_{RW,V}^\Delta k)(\eta, y) \right| &\leq C^{-|\eta|} \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) |k(\zeta \cup \eta, z)| |A_V(\eta, z, y)| + \\ &\quad C^{-|\eta|} \int_{\Gamma_0} d\lambda(\zeta) \int_{\mathbb{R}^d} dz a(y-z) |k(\zeta \cup \eta, y)| |A_V(\eta, y, z)| \\ &\leq \alpha^{|\eta|} \|k\|_{\mathcal{K}_{\alpha C}^\infty} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} (\alpha C)^{|\zeta|} d\lambda(\zeta) |A_V(\eta, z, y)| + \\ &\quad \alpha^{|\eta|} \|k\|_{\mathcal{K}_{\alpha C}^\infty} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} (\alpha C)^{|\zeta|} d\lambda(\zeta) |A_V(\eta, y, z)| \\ &\leq 2\langle a \rangle \tilde{\alpha}_0 \|k\|_{\mathcal{K}_{\alpha C}^\infty} \alpha^{|\eta|} \\ &\leq 2\langle a \rangle \tilde{\alpha}'_0 \|k\|_{\mathcal{K}_{\alpha C}^\infty} < \infty, \end{aligned}$$

and the proof follows. \square

Proof of Proposition 3.32. Similarly to the proof of Lemma 3.30.1 one can show that the operator $(\widehat{L}_V, \mathcal{D}_\alpha^V)$, with $\mathcal{D}_\alpha^V = \{G \in \mathcal{L}_{\alpha C} : D_V G \in \mathcal{L}_{\alpha C}\}$, is the generator of a holomorphic semigroup \widehat{U}_V^α on $\mathcal{L}_{\alpha C}$ for any $\alpha \in (0, 1]$ such that

$$\tilde{a}_1 + \frac{\tilde{a}_2}{\alpha C} < \frac{3}{2}.$$

Then, we can proceed as in the proof of Lemma 3.12 and obtain the desired result. \square

Convergence to the limiting evolution. Now we show the main result of about the convergence of the semigroups $\widehat{U}_\varepsilon(t)$, $\varepsilon \in (0, 1]$, to $\widehat{U}_V(t)$ in the space \mathcal{L}_C . In order to prove this convergence we need to impose stronger conditions than (2.104), (2.105) and (2.111). As in [54, Theorem 4.4.], we demand that the convergences (2.104)-(2.105) hold also in $L^1(\Gamma_0(\mathbb{R}^d), C^{|\eta|} d\lambda(\eta))$ and we make the following assumption on limit (2.111).

Assumption 3.33. *Suppose that limit (2.111), explicitly*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon(\cdot \cup \xi, y, z) \right) (\eta) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon(\cdot, y, z) \right) (\eta) = A_V(\eta, y, z), \quad (3.109)$$

holds for any $\eta \in \Gamma_0$ as well as in $L^1(\Gamma_0(\mathbb{R}^d), C^{|\eta|} \lambda(\eta))$, for all $\xi \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$.

Remark 3.34. Let us note that the convergence in $L^1(\Gamma_0(\mathbb{R}^d), C^{|\eta|} \lambda(\eta))$ of (2.104) and (2.105) together with conditions (3.82) and (3.83) yields (3.100) and (3.101), respectively.

Theorem 3.35. *Let conditions (3.82)-(3.85) and (3.102) hold. Suppose that the convergences (2.104), (2.105) and (2.111) take places for all $\eta \in \Gamma_0$ as well as in $L^1(\Gamma_0(\mathbb{R}^d), C^{|\eta|} \lambda(\eta))$. Assume also that there exists $\sigma > 0$ such that (cf. (2.104)) either*

$$d_\varepsilon(x, \xi) \leq \sigma D_x^V(\emptyset) \quad \text{or} \quad d_\varepsilon(x, \xi) \geq \sigma D_x^V(\emptyset) \quad (3.110)$$

is satisfied for all $\xi \in \Gamma_0$ and for a.a. $x \in \mathbb{R}^d$. Then, the semigroup $\widehat{U}_\varepsilon(t)$ converges strongly to $\widehat{U}_V(t)$ in \mathcal{L}_C as $\varepsilon \rightarrow 0$ uniformly on any finite interval of time.

Proof of Theorem 3.35. From Lemma 3.22.1 and 3.30.1, we already know that $\widehat{U}_\varepsilon(t)$ and $\widehat{U}_V(t)$ are holomorphic semigroups in \mathcal{L}_C . Then, to prove their convergence, it is enough to show the strong convergence of the resolvents corresponding to their generators, namely

$$\left(\widehat{L}_{\varepsilon,\text{ren}} - u\mathbb{1}\right)^{-1} G \xrightarrow{s} \left(\widehat{L}_V - u\mathbb{1}\right)^{-1} G, \quad (3.111)$$

for any $G \in \mathcal{L}_C$, see e.g. [39, Theorem III.4.8]. In order to verify the latter assertion, we apply Lemma 3.19 taking $L_\varepsilon = \widehat{L}_{\varepsilon,\text{ren}}$ and $\mathcal{D}_\varepsilon = \mathcal{D}^{(\varepsilon)}$ with $A_\varepsilon = L_0^{(\varepsilon)}$, $B_\varepsilon = L_1^{(\varepsilon)} + \widehat{L}_{RW}^{(\varepsilon)}$. Of course, $L_{\varepsilon=0} = \widehat{L}_V$ and $\mathcal{D}_0 \equiv \mathcal{D}^V$ with $A_0 = L_0^V$, $B_0 = L_1^V + \widehat{L}_{RW}^V$. Then, the statement of Theorem 3.35 will be proved once we check the conditions of Lemma 3.19 are satisfied.

Condition (3.73) is straightforward. Indeed for any $u \in \mathbb{C}$ with $\Re u > 0$ fixed, we have

$$\left\| \left(L_0^{(\varepsilon)} - u\mathbb{1}\right)^{-1} G \right\|_C \leq \left\| \frac{G}{D_\varepsilon + u} \right\|_C \leq \frac{1}{\Re u} \|G\|_C, \quad (3.112)$$

for all $\varepsilon \in (0, 1]$. Condition (3.75) follows directly from (2.104) (cf. proof of Theorem 4.4 in [54]). In particular, we can show the following result.

Lemma 3.36. *Let us assume that condition (3.82) as well as convergence (2.104) hold. Then for any $u \in \mathbb{C}$ with $\Re u > 0$ fixed, we have*

$$\left(L_0^{(\varepsilon)} - u\mathbb{1}\right)^{-1} \xrightarrow{s} \left(L_0^V - u\mathbb{1}\right)^{-1}, \quad (3.113)$$

in \mathcal{L}_C as $\varepsilon \rightarrow 0$.

It remains to verify conditions (3.74) and (3.76). They are consequence of Proposition 3.37 and Lemma 3.38, respectively. The proof of the following two auxiliary results is given at the end of this section.

Proposition 3.37. *Set, for brevity, $B_\varepsilon := L_1^{(\varepsilon)} + \widehat{L}_{RW}^{(\varepsilon)}$ and suppose that conditions (3.82)-(3.84) hold. Then, given $u \in \mathbb{C}$ with $\Re u > \bar{u} > 0$, for any $\varepsilon \in (0, 1]$ the following estimate holds*

$$\left\| B_\varepsilon (u\mathbb{1} - L_0)^{-1} \right\|_C \leq \tilde{a}_1 - 1 + \frac{\tilde{a}_2}{C} + 2\langle a \rangle \tilde{\alpha}_1 + \frac{2\langle a \rangle \tilde{\alpha}_0}{\bar{u}}. \quad (3.114)$$

Lemma 3.38. *Set for brevity $B_\varepsilon := L_1^{(\varepsilon)} + L_{RW}^{(\varepsilon)}$ and $B_0 := L_1^V + L_{RW}^V$. Then, under the hypotheses of the main theorem, for any $u \in \mathbb{C}$ with $\Re u > 0$, we have*

$$B_\varepsilon \left(L_0^{(\varepsilon)} - u\mathbb{1}\right)^{-1} \xrightarrow{s} B_0 \left(L_0^V - u\mathbb{1}\right)^{-1}, \quad (3.115)$$

in \mathcal{L}_C as $\varepsilon \rightarrow 0$.

Note that, under hypothesis (3.85), Proposition 3.37 gives us the following bound

$$\left\| \left(L_1^{(\varepsilon)} + L_{RW}^{(\varepsilon)}\right) \left(u\mathbb{1} - L_0^{(\varepsilon)}\right)^{-1} \right\|_C \leq \frac{1}{2} + \frac{2\langle a \rangle \tilde{\alpha}_0}{\bar{u}}, \quad (3.116)$$

for any $u \in \mathbb{C}$ with $\Re u > \bar{u} > 0$. Hence, we can choose \bar{u} big enough to make the r.h.s. smaller than 1, as condition (3.74) requires.

Finally, by combining (3.112) and (3.116) together with the results of Lemma 3.36 and 3.38 we obtain the desired result. \square

Proof of Proposition 3.37. Let us calculate the \mathcal{L}_C -norm of the operator $B_\varepsilon(u\mathbb{1} - L_0^{(\varepsilon)})^{-1}$. By definition we can write

$$\left\| B_\varepsilon \left(u\mathbb{1} - L_0^{(\varepsilon)}\right)^{-1} \right\|_C \leq \left\| L_1^{(\varepsilon)} \left(u\mathbb{1} - L_0^{(\varepsilon)}\right)^{-1} \right\|_C + \left\| \widehat{L}_{RW}^{(\varepsilon)} \left(u\mathbb{1} - L_0^{(\varepsilon)}\right)^{-1} \right\|_C. \quad (3.117)$$

From [54, Proposition 4.1], it follows that (cf. (4.14) in [54])

$$\left\| L_1^{(\varepsilon)} \left(u\mathbf{1} - L_0^{(\varepsilon)} \right)^{-1} \right\|_C \leq \tilde{a}_1 - 1 + \frac{\tilde{a}_2}{C}.$$

Now let us consider the second term in the r.h.s. of (3.117). For any $G \in \mathcal{L}_C$, by using the result of Proposition 3.25, for $(L_0^{(\varepsilon)} - u\mathbf{1})^{-1}G$ instead of G , we obtain

$$\begin{aligned} \left\| \widehat{L}_{RW}^{(\varepsilon)} \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G \right\|_C &\leq 2\langle a \rangle \tilde{\alpha}_1 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\xi) \frac{D_\varepsilon(\eta)}{D_\varepsilon(\eta) + \mathfrak{R}u} |G(\xi, y)| + \\ &\quad 2\langle a \rangle \tilde{\alpha}_0 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\xi) \frac{|G(\xi, y)|}{D_\varepsilon(\eta) + \mathfrak{R}u}. \end{aligned}$$

Note that for $\mathfrak{R}u > \bar{u} > 0$ one has

$$\frac{D_\varepsilon(\eta)}{D_\varepsilon(\eta) + \mathfrak{R}u} \leq 1, \quad \frac{1}{D_\varepsilon(\eta) + \mathfrak{R}u} \leq \frac{1}{\mathfrak{R}u} < \frac{1}{\bar{u}}, \quad (3.118)$$

yielding

$$\begin{aligned} \left\| \widehat{L}_{RW}^{(\varepsilon)} \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G \right\|_C &\leq 2\langle a \rangle \tilde{\alpha}_1 \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\xi) |G(\xi, y)| + \\ &\quad \frac{2\langle a \rangle \tilde{\alpha}_0}{\bar{u}} \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\xi) |G(\xi, y)|, \end{aligned}$$

which concludes the proof of the proposition. \square

Proof of Lemma 3.38. For any $G \in \mathcal{L}_C$, let us consider the following norm

$$\begin{aligned} \left\| B_\varepsilon \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G - B_0 \left(L_0^V - u\mathbf{1} \right)^{-1} G \right\|_C &\leq \\ &\leq \left\| L_1^{(\varepsilon)} \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G - L_1^V \left(L_0^V - u\mathbf{1} \right)^{-1} G \right\|_C + \\ &\leq \left\| \widehat{L}_{RW}^{(\varepsilon)} \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G - \widehat{L}_{RW}^V \left(L_0^V - u\mathbf{1} \right)^{-1} G \right\|_C. \end{aligned} \quad (3.119)$$

We want to show that the r.h.s. of (3.119) vanishes as ε goes to zero. From the proof of Theorem 4.4 in [54], we easily deduce that

$$\left\| L_1^{(\varepsilon)} \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G - L_1^V \left(L_0^V - u\mathbf{1} \right)^{-1} G \right\|_C \xrightarrow{s} 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (3.120)$$

if either $d_\varepsilon(x, \xi) \leq \sigma D_x^V(\emptyset)$ or $d_\varepsilon(x, \xi) \geq \sigma D_x^V(\emptyset)$ holds for all $\xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$.

Next let us consider the second term in the r.h.s of (3.119). We can estimate it as follows

$$\begin{aligned} \left\| \widehat{L}_{RW}^{(\varepsilon)} \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G - \widehat{L}_{RW}^V \left(L_0^V - u\mathbf{1} \right)^{-1} G \right\|_C &\leq \\ &\leq \left\| \left(L_{RW}^{(\varepsilon)} - \widehat{L}_{RW}^V \right) \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G \right\|_C + \left\| L_{RW}^V \left[\left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} - \left(L_0^V - u\mathbf{1} \right)^{-1} \right] G \right\|_C, \end{aligned}$$

more explicitly, we can write it

$$\begin{aligned} \left\| \widehat{L}_{RW}^{(\varepsilon)} \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G - \widehat{L}_{RW}^V \left(L_0^V - u\mathbf{1} \right)^{-1} G \right\|_C &\leq \\ &\leq \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \int_{\mathbb{R}^d} dz a(y-z) \times \end{aligned}$$

$$\begin{aligned} & \sum_{\xi \subset \eta} \left| \varepsilon^{-|\eta \setminus \xi|} \left(K_0^{-1} \lambda_\varepsilon (\cdot \cup \xi, y, z) \right) (\eta \setminus \xi) - A_V (\eta \setminus \xi, y, z) \right| \frac{|G(\xi, z) - G(\xi, y)|}{D_\varepsilon(\xi) + |u|} + \\ & \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \int_{\mathbb{R}^d} dz a(y-z) \times \\ & \sum_{\xi \subset \eta} |A_V(\eta \setminus \xi, y, z)| \frac{|D_\varepsilon(\xi) - D_V(\xi)|}{(D_\varepsilon(\xi) + |u|)(D_V(\xi) + |u|)} |G(\xi, z) - G(\xi, y)|. \end{aligned}$$

By using the Minlos formula (1.18), we find

$$\begin{aligned} & \left\| \widehat{L}_{RW}^{(\varepsilon)} \left(L_0^{(\varepsilon)} - u\mathbf{1} \right)^{-1} G - \widehat{L}_{RW}^V \left(L_0^V - u\mathbf{1} \right)^{-1} G \right\|_C \leq \\ & \leq \int_{\mathbb{R}^d} dy \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |G(\xi, y)| \times \\ & \left[\int_{\mathbb{R}^d} dz \frac{a(y-z)}{D_\varepsilon(\xi) + |u|} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon (\cdot \cup \xi, y, z) \right) (\eta) - A_V(\eta, y, z) \right| + \right. \\ & \int_{\mathbb{R}^d} dz \frac{a(y-z)}{D_\varepsilon(\xi) + |u|} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon (\cdot \cup \xi, z, y) \right) (\eta) - A_V(\eta, z, y) \right| + \\ & \frac{|D_\varepsilon(\xi) - D_V(\xi)|}{(D_\varepsilon(\xi) + |u|)(D_V(\xi) + |u|)} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |A_V(\eta, y, z)| + \\ & \left. \frac{|D_\varepsilon(\xi) - D_V(\xi)|}{(D_\varepsilon(\xi) + |u|)(D_V(\xi) + |u|)} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |A_V(\eta, z, y)| \right]. \end{aligned} \quad (3.121)$$

Note that by (2.104), for all $\eta \in \Gamma_0$ we have

$$D_\varepsilon(\eta) \rightarrow D_V(\eta), \quad \varepsilon \rightarrow 0. \quad (3.122)$$

The latter together with the convergence in $L^1(\Gamma_0(\mathbb{R}^d), C^{|\eta|} \lambda(\eta))$ for (2.111) implies that all four integrands, as function of (ξ, y) , appearing in (3.121) converge to zero $\tilde{\lambda}$ -a.s., as $\varepsilon \rightarrow 0$. Then, in order to use the dominated convergence theorem and show the convergence of the corresponding integrals, it is enough to show that these functions are uniformly bounded.

Let us consider the first integrand

$$\begin{aligned} I_1(\xi, y) & := \int_{\mathbb{R}^d} dz \frac{a(y-z)}{D_\varepsilon(\xi) + |u|} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon (\cdot \cup \xi, y, z) \right) (\eta) - A_V(\eta, y, z) \right| \\ & \leq \int_{\mathbb{R}^d} dz \frac{a(y-z)}{D_\varepsilon(\xi) + |u|} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left[\left| \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon (\cdot \cup \xi, y, z) \right) (\eta) \right| + |A_V(\eta, y, z)| \right]. \end{aligned}$$

By using conditions (3.84) and (3.102), one can estimate

$$\begin{aligned} I_1(\xi, y) & \leq \int_{\mathbb{R}^d} dz \frac{a(y-z)}{D_\varepsilon(\xi) + |u|} [\tilde{\alpha}_1 D_\varepsilon(\xi) + 2\tilde{\alpha}_0] \\ & \leq \langle a \rangle \tilde{\alpha}_1 \frac{D_\varepsilon(\xi)}{D_\varepsilon(\xi) + |u|} + \frac{2\langle a \rangle \tilde{\alpha}_0}{D_\varepsilon(\xi) + |u|} \\ & \leq \langle a \rangle \tilde{\alpha}_1 + \frac{2\langle a \rangle \tilde{\alpha}_0}{\mathfrak{R}u} < \infty. \end{aligned}$$

Similarly, for the second integrand in equation (3.119) we obtain

$$\begin{aligned} I_2(\xi, y) & := \int_{\mathbb{R}^d} dz \frac{a(y-z)}{D_\varepsilon(\xi) + |u|} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| \varepsilon^{-|\eta|} \left(K_0^{-1} \lambda_\varepsilon (\cdot \cup \xi, z, y) \right) (\eta) - A_V(\eta, z, y) \right| \\ & \leq \langle a \rangle \tilde{\alpha}_1 + \frac{2\langle a \rangle \tilde{\alpha}_0}{\mathfrak{R}u} < \infty. \end{aligned}$$

Next let us consider the third integrand in equation (3.119), namely

$$I_3(\xi, y) := \frac{|D_\varepsilon(\xi) - D_V(\xi)|}{(D_\varepsilon(\xi) + |u|)(D_V(\xi) + |u|)} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |A_V(\eta, y, z)|.$$

By using condition (3.102), we have

$$\begin{aligned} I_3(\xi, y) &\leq \langle a \rangle \tilde{\alpha}_0 \frac{|D_\varepsilon(\xi) - D_V(\xi)|}{(D_\varepsilon(\xi) + |u|)(D_V(\xi) + |u|)} \\ &\leq \langle a \rangle \tilde{\alpha}_0 \left[\frac{D_\varepsilon(\xi)}{(D_\varepsilon(\xi) + |u|)(D_V(\xi) + |u|)} + \frac{D_V(\xi)}{(D_\varepsilon(\xi) + |u|)(D_V(\xi) + |u|)} \right] \\ &\leq 2 \frac{\langle a \rangle \tilde{\alpha}_0}{\mathfrak{R}u} < \infty. \end{aligned}$$

Analogously for the last integrand in equation (3.119) we find

$$\begin{aligned} I_4(\xi, y) &:= \frac{|D_\varepsilon(\xi) - D_V(\xi)|}{(D_\varepsilon(\xi) + |u|)(D_V(\xi) + |u|)} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) |A_V(\eta, z, y)| \\ &\leq 2 \frac{\langle a \rangle \tilde{\alpha}_0}{\mathfrak{R}u} < \infty, \end{aligned}$$

which concludes the proof of the lemma. \square

3.2.2 Vlasov equations

In this section we want to study the Vlasov equations that appear in the Vlasov-type scaling limit. We begin with a rigorous derivation of the Vlasov equations starting from the Vlasov hierarchy (3.72) in the space \mathcal{K}_C^∞ defined by (3.3).

Given $C > 0$, we denote by \overline{B}_C^∞ a closed ball of radius C in the Banach space $L^\infty(\mathbb{R}^d)$.

Lemma 3.39. *For any $\alpha \in (\alpha_0, 1)$ as in (3.106), let ρ_0 and r_0 be two functions belonging to $\overline{B}_{\alpha C}^\infty$ and $L^\infty(\mathbb{R}^d)$, respectively. Suppose the hypotheses (3.100)-(3.102) hold. Then, the evolution equation (3.72) with initial conditions $k_0(\eta, y) = e_\lambda(\rho_0, \eta)r_0(y) \in \overline{\mathcal{K}_{\alpha C}^\infty}$ has a unique solution of the form*

$$k_t(\eta, y) = e_\lambda(\rho_t, \eta) r_t(y) \quad \text{in } \overline{\mathcal{K}_{\alpha C}^\infty}, \quad t \geq 0, \quad (3.123)$$

provided that $\rho_t \in \overline{B}_{\alpha C}^\infty$ and $r_t \in L^\infty(\mathbb{R}^d)$ satisfy the system of equations, so-called Vlasov equations,

$$\frac{\partial \rho_t(x)}{\partial t} = -\rho_t(x) \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) D_x^V(\xi) + \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) B_x^V(\xi), \quad (3.124a)$$

$$\frac{\partial r_t(y)}{\partial t} = \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) \bar{\lambda}_t(z, y) - r_t(y) \bar{\lambda}_t(y, z)], \quad (3.124b)$$

with initial conditions $\rho_t|_{t=0} = \rho_0$ and $r_t|_{t=0} = r_0$, respectively, where $\bar{\lambda}_t$ is defined by (2.126).

Proof. From Proposition 2.23 and Lemma 2.25.(i), it follows that if ρ_t and r_t solve (3.124a) and (3.124b), then $k_t = e_\lambda(\rho_t, \cdot)r_t$ solves Vlasov hierarchy (3.72). On the other hand, $\rho_0 \in \overline{B}_{\alpha C}^\infty$, $r_0 \in L^\infty(\mathbb{R}^d)$ implies $k_0 \in \overline{\mathcal{K}_{\alpha C}^\infty}$, hence, the uniqueness of solution (3.123) follows directly from Lemma 3.30.2. \square

Next let us consider the existence and uniqueness problem for the solutions to the system of Vlasov equations (3.124a)-(3.124b). The pair of functions (ρ_t, r_t) is defined to be a solution of the Vlasov equations on \mathbb{R}_+ if and only if

$$\rho, r \in C^1((0, +\infty) \rightarrow L^\infty(\mathbb{R}^d)) \cap C([0, +\infty) \rightarrow L^\infty(\mathbb{R}^d)) := \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$$

and they solve (3.124a) and (3.124b) on \mathbb{R}_+ . We are mostly interested in non-negative solutions, $\rho_t(x), r_t(x) \geq 0$, a.a. $x \in \mathbb{R}^d$, to have that $k_t = e_\lambda(\rho_t, \cdot)r_t$ is a correlation functional. In the following we also assume that if $\rho_t(x) \geq 0$ for a.a. $x \in \mathbb{R}^d$ and $t \geq 0$, then

$$\bar{\lambda}_t(y, z) = \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) A_V(\xi, y, z) \geq 0, \quad y, z \in \mathbb{R}^d. \quad (3.125)$$

Remark 3.40. Note that condition (3.125) is satisfied by each of the interaction I-IV introduced in Section 2.1, see e.g. Section 2.3.2.

The kinetic equation of RE (3.124a) coincides with the Vlasov equation of a birth-and-death model, see e.g. [54]. At present there are no general results regarding its solutions. However, Vlasov equations for a birth-and-death dynamics have been intensively studied concerning the existence, uniqueness and behavior of its solution ρ_t , in many concrete models, see e.g. [52, 56, 57] and reference therein. In the following theorem we assume that equation (3.124a) has a unique solution $\rho \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ which is non-negative and uniformly bounded, then we show that the solution to (3.124b) exists and it is unique, non-negative and uniformly bounded in L^∞ .

Theorem 3.41. *Let us consider the system of equations (3.124a)-(3.124b). Given $C > 0$, suppose that equation (3.124a) with initial condition $0 \leq \rho_0 \in \bar{B}_C^\infty$ has a unique solution $\rho \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ such that $0 \leq \rho_t \in \bar{B}_C^\infty$, for any $t > 0$. Then, if we assume that (3.102) and (3.125) hold, equation (3.124b) with initial condition $0 \leq r_0 \in L^\infty$ has a unique solution $0 \leq r \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$. Moreover, if additionally $r_0 \in \bar{B}_{C'}^\infty$, $C' > 0$, then $r_t \in \bar{B}_{C'}$ on \mathbb{R}_+ .*

Proof. The proof will be divided in 2 steps.

Step.1. Existence and uniqueness of the solution. We have to establish existence and uniqueness of solutions $r \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ for the non-autonomous Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} r_t(y) = \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) \bar{\lambda}_t(z, y) - r_t(y) \bar{\lambda}_t(y, z)] \\ r_t(y)|_{t=0} = r_0(y) \end{cases}, \quad (3.126)$$

where $\rho \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ with $0 \leq \rho_t(x) \leq C$ for a.a. $x \in \mathbb{R}^d$.

For any $t \geq 0$ we consider the linear operator

$$\begin{aligned} (\bar{L}_{RW}^*(t)r_t)(y) &= \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) \bar{\lambda}_t(z, y) - r_t(y) \bar{\lambda}_t(y, z)] \\ &= \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) A_V(\xi, z, y) - r_t(y) A_V(\xi, y, z)]. \end{aligned} \quad (3.127)$$

Let us first note that the operator (3.127) is bounded in $L^\infty(\mathbb{R}^d)$ for any $t \geq 0$. Indeed, for $r \in L^\infty(\mathbb{R}^d)$ we can write

$$\begin{aligned} |(\bar{L}_{RW}^*(t)r)(y)| &\leq \|r\|_\infty \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \int_{\mathbb{R}^d} dz a(y-z) |A_V(\xi, z, y)| + \\ &\quad \|r\|_\infty \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \int_{\mathbb{R}^d} dz a(y-z) |A_V(\xi, y, z)|. \end{aligned}$$

As $0 \leq \rho_t \in \overline{B}_C^\infty$, by using condition (3.102) one may estimate

$$\begin{aligned} \left| (\overline{L}_{RW}^*(t)r)(y) \right| &\leq \|r\|_\infty \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |A_V(\xi, z, y)| + \\ &\quad \|r\|_\infty \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |A_V(\xi, y, z)| \\ &\leq 2 \|r\|_\infty \int_{\mathbb{R}^d} dz a(y-z) [\tilde{\alpha}_0] \\ &\leq 2\tilde{\alpha}_0 \langle a \rangle \|r\|_\infty, \end{aligned}$$

which proves the claim. Next, since ρ_t is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, one can easily check that $\overline{L}_{RW}^*(t) \in C(\mathbb{R}_+; \mathfrak{L}(L^\infty))$, where $\mathfrak{L}(L^\infty)$ is the Banach algebra of all linear operators on $L^\infty(\mathbb{R}^d)$. Then, we can conclude that the Cauchy problem (3.126) has a unique solution in $L^\infty(\mathbb{R}^d)$, see e.g. [85, Section II.2].

Step.2. Non-negativity and uniform bound of the solution. In order to show that the solution $r \in C^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ to (3.126) is non-negative and uniformly bounded, we will construct a sequence of functions which converges to r_t in $L^\infty(\mathbb{R}^d)$ for any finite time interval.

Let us fix a moment of time $T > 0$ and define the space $X_{T,\infty} = C([0, T]; L^\infty(\mathbb{R}^d))$ of all continuous functions on $[0, T]$ with values in $L^\infty(\mathbb{R}^d)$. As usual we introduce a norm into this space according to the formula

$$\|r\|_T = \max_{t \in [0, T]} \|r_t\|_\infty. \quad (3.128)$$

In this norm the space $X_{T,\infty}$ is a Banach space. We denote by $X_{T,\infty}^+$ the cone of all non-negative functions from $X_{T,\infty}$. Note that the set $X_{T,\infty}^+$ with a metric induced by the norm (3.128) constitutes a complete metric space.

For any $0 \leq r_0 \in L^\infty$, let Φ be a mapping which assigns to any $v \in X_{T,\infty}$ the solution u_t of the (local) Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u_t(y) = \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_t(z, y) v_t(z) - u_t(y) \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_t(y, z) \\ u_t(y)|_{t=0} = r_0(y) \end{cases}. \quad (3.129)$$

Therefore, we can write, see e.g. [69],

$$\begin{aligned} (\Phi v)_t(y) &= e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y, z)} r_0(y) + \\ &\quad \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y, z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z, y) v_s(z). \end{aligned} \quad (3.130)$$

Let us show some basic properties of the mapping Φ . First, we can note that $\Phi v \in X_{T,\infty}^+$ for any $v \in X_{T,\infty}^+$. Indeed, $v \in X_{T,\infty}^+$ and (3.125) imply that $\Phi v \geq 0$. Moreover, since $0 \leq \rho_t \in \overline{B}_C^\infty$, from condition (3.102) it follows that

$$0 \leq \bar{\lambda}_t(y, z) \leq \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) |A_V(\xi, y, z)| \leq \tilde{\alpha}_0, \quad (3.131)$$

for $t \geq 0$ and a.a. $y, z \in \mathbb{R}^d$. Then, one can estimate

$$\begin{aligned} (\Phi v)_t(y) &\leq r_0(y) + \|v\|_T \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y, z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z, y) \\ &\leq r_0(y) + \tilde{\alpha}_0 \langle a \rangle \|v\|_T \int_0^t ds e^{-\tilde{\alpha}_0 \langle a \rangle (t-s)}. \end{aligned} \quad (3.132)$$

Note that

$$\int_0^t ds (\tilde{\alpha}_0 \langle a \rangle) e^{-\tilde{\alpha}_0 \langle a \rangle (t-s)} = \int_0^t ds \frac{\partial}{\partial s} (e^{-\tilde{\alpha}_0 \langle a \rangle (t-s)}) = 1 - e^{-\tilde{\alpha}_0 \langle a \rangle t}, \quad (3.133)$$

thus,

$$\begin{aligned} (\Phi v)_t(y) &\leq r_0(y) + \|v\|_T (1 - e^{-\tilde{\alpha}_0 \langle a \rangle t}) \\ &\leq r_0(y) + \|v\|_T < \infty. \end{aligned} \quad (3.134)$$

Next, we show that the mapping Φ is a contraction on the cone $X_{T,\infty}^+$. For any $v, w \in X_{T,\infty}^+$, let us consider the following difference

$$\begin{aligned} |(\Phi v)_t(y) - (\Phi w)_t(y)| &\leq \\ &\leq \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y,z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z,y) |v_s(z) - w_s(z)| \\ &\leq \|v - w\|_T \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y,z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z,y). \end{aligned}$$

By repeating the same estimation done in (3.131)-(3.133), we find

$$|(\Phi v)_t(y) - (\Phi w)_t(y)| \leq \|v - w\|_T (1 - e^{-\tilde{\alpha}_0 \langle a \rangle t}).$$

Then, for any $t \in [0, T]$ we can write

$$\begin{aligned} |(\Phi v)_t(y) - (\Phi w)_t(y)| &\leq \|v - w\|_T (1 - e^{-\tilde{\alpha}_0 \langle a \rangle T}) \\ &< \|v - w\|_T. \end{aligned}$$

Given $n \geq 1$ and $v^{(0)} \in X_{T,\infty}^+$, we use the contraction mapping Φ to define the iterative scheme $v^{(n)} = \Phi^n v^{(0)}$. Namely,

$$\begin{aligned} v_t^{(n)}(y) &= e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y,z)} r_0(y) + \\ &\int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y,z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z,y) v_s^{(n-1)}(z), \end{aligned} \quad (3.135)$$

with

$$v_t^{(0)}(y) = e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y,z)} r_0(y). \quad (3.136)$$

According to the classical Banach fixed point theorem, since Φ is a contraction and the cone X_T^+ is a complete metric space, the sequence $\{v^{(n)}\} \subset X_T^+$ has a unique fixed point $v \in X_T^+$. The limiting point $v_t \in L^\infty$ corresponds to the non-negative solution of (3.126) on the interval $[0, T]$.

Now given $C' > 0$ and $0 \leq r \in \bar{B}_{C'}^\infty$, let us show that the solution v_t , $t \in [0, T]$, is uniformly bounded. We proceed by induction: by hypothesis we know that

$$v_t^{(0)}(y) \leq r_0(y) \leq C', \quad \text{for a.a. } y \in \mathbb{R}^d. \quad (3.137)$$

We assume that the same bound holds at step $n-1$, $n \geq 1$, i.e.

$$v_t^{(n-1)}(y) \leq C', \quad \text{for a.a. } y \in \mathbb{R}^d, \quad (3.138)$$

and, then, we prove it at the n -th step. In this case we have

$$\begin{aligned} v_t^{(n)}(y) &\leq e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y,z)} r_0(y) + \\ &\int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y,z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z,y) v_s^{(n-1)}(z). \end{aligned}$$

By definition of the induction scheme, see equation (3.137) and (3.138), for a.a. $y \in \mathbb{R}^d$ we have

$$v_t^{(n)}(y) \leq C' e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y,z)} +$$

$$C' \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y,z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z,y).$$

Then, by using (3.131) and (3.133), one can estimate

$$\begin{aligned} v_t^{(n)}(y) &\leq C' e^{-\langle a \rangle \tilde{\alpha}_0 t} + C' \int_0^t ds \frac{\partial}{\partial s} (e^{-\langle a \rangle \tilde{\alpha}_0 (t-s)}) \\ &\leq C' e^{-\langle a \rangle \tilde{\alpha}_0 t} + C' (1 - e^{-\langle a \rangle \tilde{\alpha}_0 t}) \\ &\leq C', \quad \text{for a.a. } y \in \mathbb{R}^d. \end{aligned} \quad (3.139)$$

Letting n go to infinity it follows that $v_t(y) \leq C'$, for any $t \in [0, T]$ and a.a. $y \in \mathbb{R}^d$. Clearly, changing the initial condition in (3.126) to $r_t|_{t=T} = v_T$, we may extend all our considerations on the time interval $[T, 2T]$ and so on. As a result, the initial value problem (3.126) has a unique global bounded non-negative solution $r_t(y) \in L^\infty(\mathbb{R}^d)$ on \mathbb{R}_+ . \square

Remark 3.42. In concrete applications we are mainly interested in finding a solution to Vlasov equation (3.124b) in the space $L^1(\mathbb{R}^d)$, namely $r \in \mathcal{C}^1(\mathbb{R}_+; L^1(\mathbb{R}^d))$. Indeed, in this case the corresponding correlation functions $k_t = e_\lambda(\rho_t)r_t$ are associated to some finite measure. This problem will be considered in Section 4.2.2.

In the next section we will apply this general result to study the existence and uniqueness of solutions to the Vlasov equations in the concrete models of RW in a BDLP model of environment, under each of the interactions I-IV introduced in Section 2.1.

3.2.3 Examples: random walks in a spatial ecological model of environment

Let us consider the case where the environment is described by the BDLP model introduced in Section 3.1.3. The Vlasov-type scaling for this model has been studied in details in [54, 56].

For any $\varepsilon \in (0, 1]$ we consider the following scaling of rates (3.45) and (3.46)

$$d_\varepsilon(x, \gamma \setminus x) = m + \varepsilon \chi^- \sum_{x' \in (\gamma \setminus x)} a^-(x - x'), \quad x \in \gamma, \gamma \in \Gamma, \quad (3.140)$$

$$b_\varepsilon(x, \gamma) = \varepsilon \chi^+ \sum_{x' \in (\gamma \setminus x)} a^+(x - x'), \quad x \in \mathbb{R}^d \setminus \gamma, \gamma \in \Gamma. \quad (3.141)$$

In correspondence we have (cf. (3.48)-(3.49))

$$(K^{-1}d_\varepsilon(x, \cdot \cup (\xi \setminus x))) (\eta) = d_\varepsilon(x, \xi) 0^{|\eta|} + \varepsilon \chi^- \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) a^-(x - x')$$

and

$$(K^{-1}b_\varepsilon(x, \cdot \cup \xi)) (\eta) = b_\varepsilon(x, \xi) 0^{|\eta|} + \chi^+ \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) a^+(x - x').$$

Therefore, under assumptions (3.50)-(3.51), hypotheses (3.82), (3.83) and (3.86) are satisfied with

$$\tilde{a}_1 = 1 + \frac{1}{4 + \delta}, \quad \tilde{a}_2 = \frac{C}{4}, \quad \nu = 1, \quad (3.142)$$

for some $\delta > 0$. Moreover, it is easy to check that the following limits hold in $L^1(\Gamma_0, C^{|\eta|} d\lambda(\eta))$

- (i) $D_x^V(\eta) = \lim_{\varepsilon \rightarrow 0} (K^{-1}d_\varepsilon(x, \cdot \cup \xi)) (\eta) = m 0^{|\eta|} + \chi^- \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) a^-(x - x')$;
- (ii) $B_x^V(\eta) = \lim_{\varepsilon \rightarrow 0} (K^{-1}b_\varepsilon(x, \cdot \cup \xi)) (\eta) = \chi^+ \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) a^+(x - x')$,

for a.a. $x \in \mathbb{R}^d$. See [54] for further details. According to these results the Vlasov equation (3.124a) has now the form

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m \rho_t - \chi^- \rho_t (\rho_t * a^-) + \chi^+ (\rho_t * a^+) \\ \rho_t|_{t=0} = \rho_0 \end{cases}, \quad t \geq 0. \quad (3.143)$$

This kinetic equation was studied in [47, 48, 56]. For technical reasons, it is convenient to consider solutions $\rho \in C^1(\mathbb{R}_+; C_b(\mathbb{R}^d))$. Here, $C_b(\mathbb{R}^d)$ is the Banach space of all bounded continuous functions on \mathbb{R}^d equipped with norm

$$\|f\|_\infty := \sup_{x \in \mathbb{R}^d} |f(x)|, \quad f \in C_b(\mathbb{R}^d).$$

Note that the Banach space $C_b(\mathbb{R}^d)$ can be isomorphically embedded into $L^\infty(\mathbb{R}^d)$. For simplicity of notation, we define $\tilde{B}_C^\infty := C_b(\mathbb{R}^d) \cap \bar{B}_C^\infty$.

In the Banach space $C_b(\mathbb{R}^d)$ we can formulate the following existence and uniqueness result for the evolution equation (3.143). We refer to [48] for a detailed proof.

Theorem 3.43. *Given $C > 0$, let us consider a function $0 \leq \rho_0 \in \tilde{B}_C^\infty$. Suppose that there exists $\theta > 0$ such that*

$$a^+(x) \leq \theta a^-(x), \quad a.a. \ x \in \mathbb{R}^d. \quad (3.144)$$

Then, the Cauchy problem (3.143) has a unique solution $\rho \in C^1(\mathbb{R}_+; \tilde{B}_C^\infty)$ such that $0 \leq \rho_t \in \tilde{B}_C^\infty$ for any $t > 0$.

3.2.3.1 RW in BDLP model of environment: Case I

Let us consider the RW of a jumping particle whose interaction with RE is given by

$$\begin{aligned} \lambda_{\text{int}}(\gamma, y, z) &:= \lambda^{(1)}(\gamma, y) = e^{-\sum_{x' \in \gamma} \phi(x'-y)} \\ &= e_\lambda(e^{-\phi(\cdot-y)}, \gamma), \quad \gamma \in \Gamma, y \in \mathbb{R}^d, \end{aligned} \quad (3.145)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that

$$C_1^\phi := \int_{\mathbb{R}^d} \phi(x) dx < \infty. \quad (3.146)$$

Following the analysis in Section 2.3.2, for any $\varepsilon \in (0, 1]$ we rescale interaction (3.145) as follows

$$\lambda_\varepsilon^{(1)}(\gamma, y) := e^{-\varepsilon \sum_{x' \in \gamma} \phi(x'-y)} = e_\lambda(e^{-\varepsilon \phi(\cdot-y)}, \gamma) \quad \gamma \in \Gamma, y \in \mathbb{R}^d. \quad (3.147)$$

According to (2.139), we have

$$(K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y))(\eta) = \lambda_\varepsilon^{(1)}(\xi, y) e_\lambda(e^{-\varepsilon \phi(\cdot-y)} - 1, \eta), \quad (3.148)$$

for any $\eta, \xi \in \Gamma_0$ and $y \in \mathbb{R}^d$. Thus, one can show the following estimate.

Proposition 3.44. *Suppose that condition (3.146) holds. Then*

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \varepsilon^{-|\eta|} \left| K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y) \right|(\eta) \leq e^{CC_1^\phi}, \quad (3.149)$$

for any $\xi \in \Gamma_0$, a.a. $y \in \mathbb{R}^d$ and $C, \varepsilon > 0$.

Proof. By repeating exactly the same estimates done in the proof of Proposition 3.13 we arrive to

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \varepsilon^{-|\eta|} \left| K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y) \right|(\eta) \leq e^{C\varepsilon^{-1} \int_{\mathbb{R}^d} dx (1 - e^{-\varepsilon \phi(x-y)})}. \quad (3.150)$$

Next, we can use the inequalities $\phi \geq 0$ and

$$\frac{1 - e^{-\varepsilon \phi(x)}}{\varepsilon} \leq \phi(x), \quad x \in \mathbb{R}^d, \quad (3.151)$$

to get the statement. \square

Note that from the proposition above, it follows that Assumption 3.21 is trivially satisfied with

$$\tilde{\alpha}_1 \equiv 0, \quad \tilde{\alpha}_0 = e^{CC_1^\phi}. \quad (3.152)$$

Now let us consider the limit for $\varepsilon \rightarrow 0$ of (3.148). We can show the following result.

Proposition 3.45. *Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a non-negative even function. Then*

$$\begin{aligned} A_V^{(1)}(\eta, y) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y) \right) (\eta) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon^{(1)}(\cdot, y) \right) (\eta) \\ &= e_\lambda(-\phi(\cdot - y), \eta), \end{aligned} \quad (3.153)$$

for all $\eta, \xi \in \Gamma_0$ and a.a. $y \in \mathbb{R}^d$. Moreover, if condition (3.146) holds then we have the convergence in $L^1(\Gamma_0, C^{|\eta|} d\lambda(\eta))$ of (3.153) and

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| A_V^{(1)}(\eta, y) \right| \leq e^{CC_1^\phi}, \quad (3.154)$$

for a.a. $y \in \mathbb{R}^d$ and $C > 0$.

Proof. From Section 2.3.2, see in particular (2.140), we know that the limit (3.153) holds point-wise. In order to prove that this convergence takes place also in the sense of $L^1(\Gamma_0, C^{|\eta|} d\lambda(\eta))$, we need to show that for all $\xi \in \Gamma_0$ and a.a. $y \in \mathbb{R}^d$

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y) \right) (\eta) - A_V^{(1)}(\eta, y) \right| \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

From the dominated convergence theorem, it is enough to show that the integrand above is dominated by a function belonging to $L^1(\Gamma_0, C^{|\eta|} d\lambda(\eta))$. By definition, we can write

$$\begin{aligned} \left| \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y) \right) (\eta) - A_V^{(1)}(\eta, y) \right| &\leq \\ &\leq \left| e^{-\varepsilon \sum_{x \in \xi} \phi(x-y)} e_\lambda \left(\frac{e^{-\varepsilon \phi(\cdot - y)} - 1}{\varepsilon}, \eta \right) - e_\lambda(-\phi(\cdot - y), \eta) \right| \\ &\leq \left| e^{-\varepsilon \sum_{x \in \xi} \phi(x-y)} e_\lambda \left(\frac{e^{-\varepsilon \phi(\cdot - y)} - 1}{\varepsilon}, \eta \right) \right| + |e_\lambda(-\phi(\cdot - y), \eta)|. \end{aligned}$$

Since the potential ϕ is non-negative, one has

$$\left| \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y) \right) (\eta) - A_V^{(1)}(\eta, y) \right| \leq e_\lambda \left(\frac{1 - e^{-\varepsilon \phi(\cdot - y)}}{\varepsilon}, \eta \right) + e_\lambda(\phi(\cdot - y), \eta)$$

and by using the inequality (3.151) we obtain

$$\left| \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon^{(1)}(\cdot \cup \xi, y) \right) (\eta) - A_V^{(1)}(\eta, y) \right| \leq 2e_\lambda(\phi(\cdot - y), \eta),$$

for any $\xi, \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. It is easy to check that the function on the r.h.s. belongs to $L^1(\Gamma_0, C^{|\eta|} d\lambda(\eta))$. Indeed, by using identity (1.24), we find

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) e_\lambda(\phi(\cdot - y), \eta) = e^{C \int_{\mathbb{R}^d} dx \phi(x-y)} = e^{CC_1^\phi}, \quad (3.155)$$

which concludes the proof of the proposition. \square

From the proposition above, it follows that both Assumption 3.28 and 3.33 are satisfied with

$$\tilde{\alpha}_0 = e^{CC_1^\phi}. \quad (3.156)$$

As a consequence of the analysis above, we can apply Lemma 3.22 and 3.30, Theorem 3.35 and Lemma 3.39, to formulate the following result about the Vlasov-type scaling for the considered model.

Corollary 3.46. *Suppose that condition (3.146) holds. Assume that the functions a^\pm and the constants χ^\pm , m and C satisfy conditions (3.50)-(3.51). Then*

- 1a. *for any $\varepsilon \in (0, 1]$, the operator $(\widehat{L}_{\varepsilon, \text{ren}}, \mathcal{D}^{(\varepsilon)})$ is the generator of a holomorphic semigroup $\widehat{U}_\varepsilon(t)$ on \mathcal{L}_C ;*
- 1b. *for any $\alpha \in (1/2, 1)$, the operator $(\widehat{L}_{\varepsilon, \text{ren}}^{\circ\alpha}, \text{Dom}(\widehat{L}_{\varepsilon, \text{ren}}^{\circ\alpha}))$ is the generator of a strongly continuous semigroup $\widehat{U}_\varepsilon^{\circ\alpha}(t)$ on the space $\overline{\mathcal{K}}_{\alpha C}^\infty$;*
- 2a. *the operator $(\widehat{L}_V, \mathcal{D}^V)$ is the generator of a holomorphic semigroup $\widehat{U}_V(t)$ on \mathcal{L}_C ;*
- 2b. *for any $\alpha \in (1/2, 1)$, the operator $(\widehat{L}_V^{\circ\alpha}, \text{Dom}(\widehat{L}_V^{\circ\alpha}))$ is the generator of a strongly continuous semigroup $\widehat{U}_V^{\circ\alpha}(t)$ on the space $\overline{\mathcal{K}}_{\alpha C}^\infty$;*
3. *$\widehat{U}_\varepsilon(t) \rightarrow \widehat{U}_V(t)$, as $\varepsilon \rightarrow 0$, strongly in \mathcal{L}_C uniformly on finite time intervals;*
4. *for any $\alpha \in (1/2, 1)$, given $\rho_0 \in \overline{B}_{\alpha C}^\infty$ and $r_0 \in L^\infty(\mathbb{R}^d)$, the Vlasov hierarchy (3.72), with $k_0(\eta, y) = \rho_0(\eta)r(y) \in \overline{\mathcal{K}}_{\alpha C}^\infty$, has a unique solution $k_t(\eta, y) = e_\lambda(\rho_t, \eta)r_t(y)$ in $\overline{\mathcal{K}}_{\alpha C}^\infty$ provided that $\rho_t \in \overline{B}_{\alpha C}^\infty$ and $r_t \in L^\infty(\mathbb{R}^d)$, $t > 0$, satisfy the Vlasov equations*

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m\rho_t - \chi^- \rho_t (\rho_t * a^-) + \chi^+ (\rho_t * a^+) \\ \frac{\partial r_t}{\partial t} = -\langle a \rangle e^{-(\rho_t * \phi)} r_t + ((e^{-(\rho_t * \phi)} r_t) * a) \end{cases}, \quad (3.157)$$

with initial conditions $\rho_t|_{t=0} = \rho_0$ and $r_t|_{t=0} = r_0$.

Moreover, by combining the results of Theorem 3.41 and 3.43, we have the following existence and uniqueness result of the solution of system of Vlasov equations.

Theorem 3.47. *Assume that conditions (3.146) and (3.144) hold. Given $C, C' > 0$, let $0 \leq \rho_0 \in \overline{B}_C^\infty(\mathbb{R}^d)$ and $0 \leq r_0 \in \overline{B}_{C'}^\infty(\mathbb{R}^d)$. Then, the system of equations (3.157) has a unique solution (ρ_t, r_t) , with $\rho \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ and $r \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ such that $0 \leq \rho_t \in \overline{B}_C^\infty$ and $0 \leq r_t \in \overline{B}_{C'}^\infty$, for any time $t > 0$.*

3.2.3.2 RW in BDLP model of environment: Case II

Let us consider the interaction given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(2)}(\gamma, y) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - y), \quad \gamma \in \Gamma, y \in \mathbb{R}^d, \quad (3.158)$$

where $\lambda_0 \geq 0$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that conditions (3.63) and (3.64) hold. In this case, according to Section 2.3.2, for any $\varepsilon \in (0, 1]$ we consider the following scaling

$$\lambda_\varepsilon^{(2)}(\gamma, y) := \lambda_0 + \varepsilon \sum_{x' \in \gamma} \phi(x' - y), \quad \gamma \in \Gamma, y \in \mathbb{R}^d. \quad (3.159)$$

As a consequence one has

$$(K^{-1}\lambda_\varepsilon^{(2)}(\cdot \cup \xi, y))(\eta) = \lambda_\varepsilon(\xi, y) 0^{|\eta|} + \varepsilon \phi(x' - y) \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}), \quad (3.160)$$

for any $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$.

Proposition 3.48. *Suppose that conditions (3.63) and (3.64) hold. Then, given $\varepsilon \in (0, 1]$ we have*

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \varepsilon^{-|\eta|} |K^{-1}\lambda_\varepsilon^{(2)}(\cdot \cup \xi, y)|(\eta) \leq \lambda_0 + CC_1^\phi + C_\infty^\phi, \quad (3.161)$$

for any $\xi \in \Gamma_0$, a.a. $y \in \mathbb{R}^d$ and $C > 0$.

Proof. Following the proof of Proposition 3.15 one can show

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| K^{-1} \lambda_\varepsilon^{(2)}(\cdot \cup \xi, y) \right|(\eta) \leq \lambda_0 + \varepsilon \|\phi\|_\infty |\xi| + \varepsilon C \|\phi\|_1. \quad (3.162)$$

Then, since $\varepsilon \leq 1$ we get the desired result. \square

According to (3.140), from the above proposition it follows that Assumption 3.21 is satisfied with

$$\tilde{\alpha}_1 = \frac{C_\infty^\phi}{m}, \quad \tilde{\alpha}_0 = \lambda_0 + C C_1^\phi. \quad (3.163)$$

Next let ε go to zero in (3.160). We can show the following result.

Proposition 3.49. *Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a non-negative even function. Then*

$$\begin{aligned} A_V^{(2)}(\eta, y) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon^{(2)}(\cdot \cup \xi, y) \right)(\eta) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} \left(K^{-1} \lambda_\varepsilon^{(2)}(\cdot, y) \right)(\eta) \\ &= \lambda_0 0^{|\eta|} + \phi(x' - y) \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}), \end{aligned} \quad (3.164)$$

for all $\eta, \xi \in \Gamma_0$ and a.a. $y \in \mathbb{R}^d$. If additionally we assume that condition (3.63) holds, in (3.164) we have a $L^1(\Gamma_0, C^{|\eta|} d\lambda(\eta))$ -convergence and, moreover, the following estimate holds

$$\int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| A_V^{(2)}(\eta, y) \right| \leq \lambda_0 + C C_1^\phi, \quad (3.165)$$

for a.a. $y \in \mathbb{R}^d$ and $C > 0$.

Proof. We already know that limit (3.164) holds point-wise, see (2.147) in Section 2.3.2. Since $\phi \in L^1(\mathbb{R}^d)$, the $L^1(\Gamma_0, C^{|\eta|} d\lambda(\eta))$ -convergence follows directly from the dominated convergence theorem. Finally, we can note that

$$\begin{aligned} \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left| A_V^{(2)}(\eta, y) \right| &= \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) \left[\lambda_0 0^{|\eta|} + \phi(x' - y) \mathbf{1}_{\Gamma(1)}(\eta = \{x'\}) \right] \\ &= \lambda_0 + C \int_{\mathbb{R}^d} dx \phi(x - y) \\ &= \lambda_0 + C C_1^\phi, \end{aligned} \quad (3.166)$$

which concludes the proof of the proposition. \square

As a consequence of the above proposition, both Assumption 3.28 and 3.33 are fulfilled with

$$\tilde{\alpha}_0 = \lambda_0 + C C_1^\phi. \quad (3.167)$$

We are now able to apply the general results about the Vlasov-type scaling stated in Theorem 3.35 and Lemma 3.39.

Corollary 3.50. *Let us assume that the functions a^\pm , ϕ and the constants χ^\pm , m and C satisfy conditions (3.63) and (3.64) as well as (3.69) and (3.70). Then, for any $\varepsilon \in (0, 1]$ and $\alpha \in (1/2, 1)$:*

- 1a. the operator $(\widehat{L}_{\varepsilon, \text{ren}}, \mathcal{D}^{(\varepsilon)})$ is the generator of a holomorphic semigroup $\widehat{U}_\varepsilon(t)$ on \mathcal{L}_C ;
- 1b. the operator $(\widehat{L}_{\varepsilon, \text{ren}}^{\circ\alpha}, \text{Dom}(\widehat{L}_{\varepsilon, \text{ren}}^{\circ\alpha}))$ is the generator of a strongly continuous semigroup $\widehat{U}_\varepsilon^{\circ\alpha}(t)$ on the space $\overline{\mathcal{K}_{\alpha C}^\infty}$;
- 2a. the operator $(\widehat{L}_V, \mathcal{D}^V)$ is the generator of a holomorphic semigroup $\widehat{U}_V(t)$ on \mathcal{L}_C ;

- 2b. the operator $(\widehat{L}_V^{\circ\alpha}, \text{Dom}(\widehat{L}_V^{\circ\alpha}))$ is the generator of a strongly continuous semigroup $\widehat{U}_V^{\circ\alpha}(t)$ on the space $\overline{\mathcal{K}}_{\alpha C}^{\infty}$;
3. for $\varepsilon > 0$, $\widehat{U}_\varepsilon(t) \rightarrow \widehat{U}_V(t)$ strongly in \mathcal{L}_C uniformly on any finite interval of time;
4. for $\rho_0 \in \overline{B}_{\alpha C}^{\infty}$ and $r_0 \in L^\infty(\mathbb{R}^d)$, the evolution equation (3.72) with initial conditions $k_0(\eta, y) = \rho_0(\eta)r(y) \in \overline{\mathcal{K}}_{\alpha C}^{\infty}$, has a unique solution $k_t(\eta, y) = e_\lambda(\rho_t, \eta)r_t(y)$ in $\overline{\mathcal{K}}_{\alpha C}^{\infty}$ provided that $\rho_0 \in \overline{B}_{\alpha C}^{\infty}$ and $r_0 \in L^\infty(\mathbb{R}^d)$ satisfy the system of equations

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m\rho_t - \chi^- \rho_t (\rho_t * a^-) + \chi^+ (\rho_t * a^+) \\ \frac{\partial r_t}{\partial t} = (a * [r_t (\lambda_0 + (\rho_t * \phi))]) - \langle a \rangle r_t [\lambda_0 + (\rho_t * \phi)] \end{cases}, \quad (3.168)$$

with initial conditions $\rho_t|_{t=0} = \rho_0$ and $r_t|_{t=0} = r_0$.

Finally, we can apply Theorem 3.41 and 3.43 to establish existence and uniqueness of solutions to Vlasov equations.

Theorem 3.51. *Assume that conditions (3.63) and (3.144) hold. Given $C, C' > 0$, let $0 \leq \rho_0 \in \overline{B}_C^\infty(\mathbb{R}^d)$ and $0 \leq r_0 \in \overline{B}_{C'}^\infty(\mathbb{R}^d)$. Then, the system of equations (3.168), has a unique solution $\rho_t \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ and $r_t \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ with $0 \leq \rho_t \in \overline{B}_C^\infty$ and $0 \leq r_t \in \overline{B}_{C'}^\infty$, on \mathbb{R}_+ .*

3.2.3.3 RW in BDLP model of environment: Case III

Let us consider the interaction

$$\begin{aligned} \lambda_{\text{int}}(\gamma, y, z) &:= \lambda^{(3)}(\gamma, z) = e^{-\sum_{x' \in \gamma} \phi(x' - z)} \\ &= e_\lambda(e^{-\phi(\cdot - z)}, \gamma), \quad \gamma \in \Gamma, z \in \mathbb{R}^d, \end{aligned} \quad (3.169)$$

where ϕ is a non-negative even function on \mathbb{R}^d such that condition (3.146) holds.

The scaling of such an interaction has been discussed in the see Case III of Section 2.3.2. Using results of Proposition 3.44 and 3.45, one can see that Assumption 3.21, 3.28 and 3.33 hold with constants identical to those found in Section 3.2.3.1. Therefore, we can show the same results as in Lemma 3.46. In this case, the Vlasov equation has the form

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m\rho_t - \chi^- \rho_t (\rho_t * a^-) + \chi^+ (\rho_t * a^+) \\ \frac{\partial r_t}{\partial t} = e^{-(\rho_t * \phi)} (r_t * a) - r_t (e^{-(\rho_t * \phi)} * a) \end{cases}. \quad (3.170)$$

Then, by applying Theorem 3.41 and 3.43 as has been done in Theorem 3.47, we have the following existence and uniqueness result.

Theorem 3.52. *Assume that conditions (3.146) and (3.144) hold. Then, given $C, C' > 0$ the system of Vlasov equations (3.170), with initial conditions $0 \leq \rho_0 \in \overline{B}_C^\infty$ and $0 \leq r_0 \in \overline{B}_{C'}^\infty$, has a unique solution $\rho_t \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$, $r_t \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ such that $0 \leq \rho_t \in \overline{B}_C^\infty$, $0 \leq r_t \in \overline{B}_{C'}^\infty$ for any $t > 0$.*

3.2.3.4 RW in BDLP model of environment: Case IV

Let us consider the interaction

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(2)}(\gamma, z) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - z), \quad (3.171)$$

where $\lambda_0 \geq 0$ and ϕ are defined as in Section 3.2.3.2.

In this case, according to general results obtained in Section 2.3.2 (see Case IV), we can repeat straightforward the analysis done in Section 3.2.3.2 and so prove the same results as in Lemma 3.50. The Vlasov equations has now the form

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m\rho_t - \chi^- \rho_t (\rho_t * a^-) + \chi^+ (\rho_t * a^+) \\ \frac{\partial r_t}{\partial t} = (r_t * a) [\lambda_0 + (\rho_t * \phi)] - r_t [\lambda_0 \langle a \rangle + ((\rho_t * \phi) * a)] \end{cases} \quad (3.172)$$

Since Assumption 3.28 is still satisfied with (3.167), we can apply Theorem 3.41 and 3.43 and show the following existence and uniqueness result for the solution of the system of equations above.

Theorem 3.53. *Assume that conditions (3.63) and (3.144) hold. Then, the system of Vlasov equations (3.172) with initial conditions $0 \leq \rho_0 \in \overline{B}_C^\infty$ and $0 \leq r_0 \in \overline{B}_{C'}^\infty$, $C, C' > 0$, has a unique solution $\rho_t \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ and $r_t \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ such that $0 \leq \rho_t \in \tilde{B}_C^\infty$, $0 \leq r_t \in \overline{B}_{C'}^\infty$ for any $t > 0$.*

Chapter 4

Random walks in a birth-and-death environment with aggregation

Let us consider the RW of a jumping particle moving in a birth-and-death environment. In this chapter we study the particular case of a non-equilibrium birth-and-death dynamics which has constant birth rate and density dependent decreasing death rate. This birth-and-death model has been introduced in [45], where the authors constructed the corresponding evolution of states and derived the limiting mesoscopic dynamics that appear in the Vlasov-type scaling. We present some general conditions on the interaction λ_{int} to extend this analysis to the considered model of RWRE. These conditions can be satisfied by different types of interactions, it will be seen in detail in some concrete examples.

It is worth noting that for these models of RWRE cannot be included in the general class of those studied in the previous chapter. Indeed, in this case RE does not satisfy the conditions stated in Theorem 3.8. Our approach is based on an Ovsjannikov-type result, which yields to an evolution in a scale of Banach spaces and restricted to a finite time interval.

Let us also mention that the same technique can be applied to study RW in different REs which are not included in this chapter, for instance Glauber-type dynamics [46] and systems of hopping particles [10].

4.1 Evolution for correlation functions

In this section we study the statistical evolution of a tagged particle jumping in \mathbb{R}^d and interacting with other particles which evolve according to the birth-and-death dynamics described in [45].

Heuristically, the dynamics of the model is specified by the heuristic Markov generator

$$(LF)(\gamma, y) = (L_{RE}F)(\gamma, y) + (L_{RW}F)(\gamma, y), \quad F \in K(B_{\text{bs}}(\Gamma_0 \times \mathbb{R}^d)), \quad (4.1)$$

where the generator L_{RW} is defined by (2.10), whereas L_{RE} is given by (2.2) with

$$b(x, \gamma) \equiv b > 0, \quad (4.2)$$

for any $x \in \mathbb{R}^d \setminus \gamma$, $\gamma \in \Gamma$, and

$$d(x, \gamma \setminus x) = m e^{-\sum_{x' \in \gamma \setminus x} V(x-x')}, \quad m > 0, \quad (4.3)$$

for any $x \in \gamma$, $\gamma \in \Gamma$. We always assume that $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is an even non-negative function,

$$V(x) = V(-x) > 0, \quad \forall x \in \mathbb{R}^d, \quad (4.4)$$

which satisfies the following integrability condition

$$C'_V := \int_{\mathbb{R}^d} (1 - e^{-V(x)}) dx < \infty. \quad (4.5)$$

We also adhere to the convention that $d(x, \gamma) = 0$ if $\sum_{x' \in \gamma} V(x-x') = \infty$.

Remark 4.1. Let us note that the death rate in (4.3) is a decreasing function in γ . Namely given $\gamma' \subset \gamma$ then

$$d(x, \gamma \setminus x) \leq d(x, \gamma' \setminus x).$$

In other words the probability to die for a particle of a configuration γ is lower if the particle lies in a dense area. This represent some kind of attraction interaction between the particles in the terminology of interacting particle systems theory, see e.g. [91]. Heuristically, for a large enough initial number of particle an aggregation effect is expected in the environment, see [45] for further details.

According to the results of Section 2.2, the evolution of correlation functions for the current model of RWRE is described by the hierarchy, $t \geq 0$,

$$\begin{cases} \frac{\partial}{\partial t} k_t(\eta, y) = \left(\widehat{L}_{RE}^* k_t \right)(\eta, y) + \left(\widehat{L}_{RW}^* k_t \right)(\eta, y) \\ k_t(\eta, y)|_{t=0} = k_0 \end{cases}, \quad (4.6)$$

where the operator \widehat{L}_{RW}^* is defined by (2.68), whereas \widehat{L}_{RE}^* is of the form (cf. Proposition 2.13 and [45])

$$\begin{aligned} \left(\widehat{L}_{RE}^* k \right)(\eta, y) &= -m \sum_{x \in \eta} e^{-\sum_{x' \in \eta \setminus x} V(x-x')} \int_{\Gamma_0} d\lambda(\xi) e_\lambda \left(e^{-V(x-\cdot)} - 1, \xi \right) k(\eta \cup \xi, y) + \\ & b \sum_{x \in \eta} k(\eta \setminus x, y). \end{aligned} \quad (4.7)$$

Given $C > 0$, we want to solve the initial value problem (4.6) in the Banach space \mathcal{K}_C^1 defined by (2.32). In this space we introduce the norm

$$\|k\|_{\mathcal{K}_C^1} := \int_{\mathbb{R}^d} dy \|k(\cdot, y)\|_{\mathcal{K}_C},$$

where we defined

$$\|k(\cdot, y)\|_{\mathcal{K}_C} := \operatorname{ess\,sup}_{\eta \in \Gamma_0} C^{-|\eta|} |k(\eta, y)|.$$

Following [45] we study this evolution problem in the framework of scales of Banach spaces. Indeed, we can easily see that for any $0 < C \leq C'$

$$\mathcal{K}_C^1 \subset \mathcal{K}_{C'}^1, \quad \|\cdot\|_{\mathcal{K}_C^1} \geq \|\cdot\|_{\mathcal{K}_{C'}^1}. \quad (4.8)$$

Hence, for any $C_0 \geq 0$ the family $\{\mathcal{K}_C^1 : 0 < C \leq C_0\}$ is a scale of Banach spaces. Within this framework, we can use the following existence and uniqueness result. We refer to Appendix A for a detailed proof.

Theorem 4.2. *Let $\{\mathbb{B}_s : 0 < s \leq s_0\}$ be a one parameter family of Banach spaces such that $\mathbb{B}_{s''} \subset \mathbb{B}_{s'}$, $\|\cdot\|_{s'} \leq \|\cdot\|_{s''}$, for any pair (s', s'') such that $s \leq s' < s'' \leq s_0$, where $\|\cdot\|_s$ denotes the norm of \mathbb{B}_s . Consider the initial value problem*

$$\begin{cases} \frac{du(t)}{dt} = Au(t) \\ u(0) = u_0 \in \mathbb{B}_{s_0} \end{cases}, \quad (4.9)$$

where, for each $s \in (0, s_0)$ fixed and for each pair (s', s'') such that $s \leq s' < s'' \leq s_0$, the mapping $A : \mathbb{B}_{s''} \rightarrow \mathbb{B}_{s'}$ is linear satisfying

$$\|Au\|_{s'} \leq M_0 \|u\|_{s''} + \frac{M_1}{s'' - s'} \|u\|_{s''}, \quad (4.10)$$

for some $M_0, M_1 \geq 0$ and for all $u \in \mathbb{B}_{s''}$. Here, M_0 and M_1 are two constant independent of s', s'' and u , however they might depend continuously on s and s_0 .

Then, for each $s \in (0, s_0)$ there is a constant $\delta = (eM_1)^{-1} > 0$ such that there exists a unique function $u : [0, \delta(s_0 - s)) \mapsto \mathbb{B}_s$ which is continuously differentiable on $(0, \delta(s_0 - s))$ in \mathbb{B}_s , $Au \in \mathbb{B}_s$ and solves the initial value problem (4.9) in the time interval $0 \leq t < \delta(s_0 - s)$.

The application of this general theorem to hierarchy (4.6) is stated below. The proof of this result is given in Section 4.1.1

Theorem 4.3. *Suppose that condition (4.5) holds. Assume that there exist $\alpha_0, \alpha_1 \geq 0$ such that for all $\eta \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$*

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, y, z) \right|(\xi) \leq \alpha_1 |\eta| + \alpha_0. \quad (4.11)$$

Then, given $C_0 > 0$ arbitrary and fixed, for each $C > C_0$ there exists a moment of time

$$T(C_0, C) := \frac{C_0(C - C_0)}{C^2 \left(me^{CC'_v} + \frac{b}{C_0} + 2\langle a \rangle \alpha_1 \right)} > 0, \quad (4.12)$$

such that the initial value problem (4.6), with $k_0 \in \mathcal{K}_{C_0}^1$, has a unique solution k_t in the space \mathcal{K}_C^1 on the time interval $[0, T(C_0, C))$.

The theorem above ensures the existence and uniqueness of solutions to the evolution equation (4.6) on a finite time interval only. Moreover, starting with an initial condition from a certain Banach space $\mathcal{K}_{C_0}^1$ in general this solution evolves on a larger Banach space \mathcal{K}_C^1 with $C > C_0 > 0$.

4.1.1 Proof of Theorem 4.3

First we study the operator $\widehat{L}^* := \widehat{L}_{RE}^* + \widehat{L}_{RW}^*$ given by (2.68) and (4.7) on the Banach space \mathcal{K}_C^1 . In general, the operator \widehat{L}^* is unbounded on \mathcal{K}_C^1 , however it might be bounded as an operator between two different spaces of functions. From [45] one can easily deduce that \widehat{L}_{RE}^* is a bounded operator from $\mathcal{K}_{C'}^1$ to \mathcal{K}_C^1 for any $0 < C' < C$. In particular we can show the following result.

Proposition 4.4. *Suppose that condition (4.5) holds. Let $C > C_0 > 0$ be arbitrary and fixed. Then for any C', C'' such that $C_0 \leq C' < C'' \leq C$, the operator \widehat{L}_{RE}^* is a linear operator acting from $\mathcal{K}_{C'}^1$ to $\mathcal{K}_{C''}^1$ such that*

$$\left\| \widehat{L}_{RE}^* k \right\|_{\mathcal{K}_{C''}^1} \leq \frac{1}{C'' - C'} \frac{C}{e} \left(me^{CC'_v} + \frac{b}{C_0} \right) \|k\|_{\mathcal{K}_{C'}^1}, \quad (4.13)$$

for any $k \in \mathcal{K}_{C'}^1$.

Proof of Proposition 4.4. The statement follows immediately from Proposition 3.2 in [45]. \square

Next we can use condition (4.11) to prove a similar estimate for the operator \widehat{L}_{RW}^* .

Proposition 4.5. *Suppose that there exist $\alpha_0, \alpha_1 \geq 0$ such that for all $\eta \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$ condition (4.11) holds. Let $C > C_0 > 0$ be arbitrary and fixed. Then for any C', C'' such that $C_0 \leq C' < C'' \leq C$, the operator \widehat{L}_{RW}^* is a linear operator acting from $\mathcal{K}_{C'}^1$ to $\mathcal{K}_{C''}^1$ and for any $k \in \mathcal{K}_{C'}^1$*

$$\left\| \widehat{L}_{RW}^* k \right\|_{\mathcal{K}_{C''}^1} \leq 2\langle a \rangle \left(\frac{1}{C'' - C'} \frac{C}{e} \alpha_1 + \alpha_0 \right) \|k\|_{\mathcal{K}_{C'}^1}. \quad (4.14)$$

Proof of Proposition 4.5. According to (2.68), for any $k \in \mathcal{K}_{C'}^1$, we have

$$(C'')^{-|\eta|} \left| \left(\widehat{L}_{RW}^* k \right) (\eta, y) \right| \leq$$

$$\begin{aligned}
&\leq (C'')^{-|\eta|} \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) |k(\xi \cup \eta, z)| \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, z, y) \right|(\xi) + \\
&\quad (C'')^{-|\eta|} \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) |k(\xi \cup \eta, y)| \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, y, z) \right|(\xi) \\
&\leq \left(\frac{C'}{C''} \right)^{|\eta|} \int_{\mathbb{R}^d} dz a(y-z) \|k(\cdot, z)\|_{\mathcal{K}_{C'}} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, z, y) \right|(\xi) + \\
&\quad \left(\frac{C'}{C''} \right)^{|\eta|} \|k(\cdot, y)\|_{\mathcal{K}_{C'}} \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| K^{-1} \lambda_{\text{int}}(\cdot \cup \eta, y, z) \right|(\xi).
\end{aligned}$$

Then, by using assumption (4.11) one may estimate

$$\begin{aligned}
(C'')^{-|\eta|} \left| (\widehat{L}_{RW}^* k)(\eta, y) \right| &\leq \\
&\leq \left(\frac{C'}{C''} \right)^{|\eta|} (\alpha_1 |\eta| + \alpha_0) \int_{\mathbb{R}^d} dz a(y-z) \left[\|k(\cdot, z)\|_{\mathcal{K}_{C'}} + \|k(\cdot, y)\|_{\mathcal{K}_{C'}} \right].
\end{aligned}$$

Since $C'/C'' < 1$, we can use the elementary inequality

$$a^t t \leq \left(\frac{1}{-e \ln a} \right), \quad t \geq 0, \quad a \in (0, 1), \quad (4.15)$$

to obtain

$$\begin{aligned}
(C'')^{-|\eta|} \left| (\widehat{L}_{RW}^* k)(\eta, y) \right| &\leq \\
&\leq \left[\frac{\alpha_1}{e(\ln C'' - \ln C')} + \alpha_0 \right] \int_{\mathbb{R}^d} dz a(y-z) \left[\|k(\cdot, z)\|_{\mathcal{K}_{C'}} + \|k(\cdot, y)\|_{\mathcal{K}_{C'}} \right].
\end{aligned}$$

The latter implies that

$$\left\| \widehat{L}_{RW}^* k \right\|_{\mathcal{K}_{C''}^1} \leq 2\langle a \rangle \|k\|_{\mathcal{K}_{C'}^1} \left[\frac{\alpha_1}{e(\ln C'' - \ln C')} + \alpha_0 \right].$$

On the other hand, there exists a constant $c \in [C', C''] \subset [C_0, C]$ such that

$$\ln C'' - \ln C' = \frac{1}{c} (C'' - C') \geq \frac{1}{C} (C'' - C') > 0, \quad (4.16)$$

which concludes the proof of the proposition. \square

Let us note that by combining the results of Proposition 4.4 and 4.5, for any C', C'' such that $C_0 \leq C' < C'' \leq C$ and $k \in \mathcal{K}_{C'}^1$, we obtain the following bound for the operator \widehat{L}^*

$$\left\| \widehat{L}^* k \right\|_{\mathcal{K}_{C''}^1} \leq 2\langle a \rangle \alpha_0 \|k\|_{\mathcal{K}_{C'}^1} + \frac{1}{C'' - C'} \frac{C}{e} \left[\left(m e^{C C'} + \frac{b}{C_0} \right) + 2\langle a \rangle \alpha_1 \right] \|k\|_{\mathcal{K}_{C'}^1}. \quad (4.17)$$

Having established of this preliminary result, we can now proceed to prove Theorem 4.3.

Proof of Theorem 4.3. Let us apply Theorem 4.2 to the scale of Banach spaces $\{\mathbb{B}_s : 0 \leq s \leq s_0\}$ with

$$\mathbb{B}_s := \mathcal{K}_{1/s}^1 \quad (4.18)$$

and $s_0 := 1/C_0$. For $s = 1/C$ and

$$s'' := \frac{1}{C''} < \frac{1}{C'} =: s',$$

one can rewrite (4.17) in the following way

$$\begin{aligned} \|\widehat{L}^* k\|_{\mathbb{B}_{s''}} &\leq 2\langle a \rangle \alpha_0 \|k\|_{\mathbb{B}_{s'}} + \frac{s' s''}{s' - s''} \frac{1}{se} \left[me^{\frac{c'_V}{s}} + bs_0 + 2\langle a \rangle \alpha_1 \right] \|k\|_{\mathbb{B}_{s'}} \\ &\leq 2\langle a \rangle \alpha_0 \|k\|_{\mathbb{B}_{s'}} + \frac{1}{s' - s''} \frac{s_0^2}{se} \left[me^{\frac{c'_V}{s}} + bs_0 + 2\langle a \rangle \alpha_1 \right] \|k\|_{\mathbb{B}_{s'}} \\ &\leq M_0 \|k\|_{\mathbb{B}_{s'}} + \frac{M_1(s, s_0)}{s' - s''} \|k\|_{\mathbb{B}_{s'}}, \end{aligned} \quad (4.19)$$

where

$$M_0 = 2\langle a \rangle \alpha_0 \quad (4.20)$$

and

$$M_1 := M_1(s, s_0) = \frac{s_0^2}{se} \left[me^{\frac{c'_V}{s}} + bs_0 + 2\langle a \rangle \alpha_1 \right]. \quad (4.21)$$

Then, directly applying Theorem 4.2, we obtain an evolution

$$\mathcal{K}_{C_0}^1 := \mathbb{B}_{s_0} \in k_0 \mapsto k_t \in \mathbb{B}_s := \mathcal{K}_C^1, \quad (4.22)$$

for any time $t \geq 0$ less than

$$\begin{aligned} T(C_0, C) &:= \frac{s_0 - s}{eM_1(s, s_0)} = \left[\frac{s_0^2}{s} \left(me^{\frac{c'_V}{s}} + bs_0 + 2\langle a \rangle \alpha_1 \right) \right]^{-1} (s_0 - s) \\ &= \frac{C_0(C - C_0)}{C^2 \left(me^{CC'_V} + \frac{b}{C_0} + 2\langle a \rangle \alpha_1 \right)}, \end{aligned} \quad (4.23)$$

and the proof is complete. \square

4.1.2 Examples

In this section we apply the general result stated in Theorem 4.3 to each of the interactions I-IV introduced in Section 2.1. For this purpose it is enough to show that these interactions satisfy condition (4.11).

4.1.2.1 RW in an aggregation model of environment: Case I and III

Let us consider the interactions defined in Section 3.1.3.1 given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(1)}(\gamma, y) = e^{-\sum_{x' \in \gamma} \phi(x' - y)}, \quad \gamma \in \Gamma, y \in \mathbb{R}^d, \quad (4.24)$$

and

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(3)}(\gamma, z) = e^{-\sum_{x' \in \gamma} \phi(x' - z)}, \quad \gamma \in \Gamma, z \in \mathbb{R}^d, \quad (4.25)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that

$$C'_\phi := \int_{\mathbb{R}^d} dx (1 - e^{-\phi(x)}) < \infty. \quad (4.26)$$

From Proposition 3.13 we know that both interactions (4.24) and (4.25) satisfy condition (4.11) with

$$\alpha_0 = e^{CC'_\phi} \quad \text{and} \quad \alpha_1 \equiv 0. \quad (4.27)$$

Therefore, we can apply Theorem 4.3 to construct the evolution of correlation functions.

Corollary 4.6. *Let us consider the operator \widehat{L}_{RE}^* defined by (4.7). Let \widehat{L}_{RW}^* be an operator of the form (2.70) or (2.72) and such that condition (4.26) holds. Then, given a $C_0 > 0$ arbitrary and fixed, for each $C > C_0$ the evolution equation (4.6) with initial condition $k_0 \in \mathcal{K}_{C_0}^1$ has a unique solution $k_t \in \mathcal{K}_C^1$ on the time interval $[0, T(C, C_0))$ with*

$$T(C, C_0) = \frac{C_0(C - C_0)}{C^2 \left(me^{CC'_V} + \frac{b}{C_0} \right)}. \quad (4.28)$$

4.1.2.2 RW in an aggregation model of environment: Case II and IV

Let us consider a RW whose interaction with RE is given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(2)}(\gamma, y) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - y), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d. \quad (4.29)$$

As in Section 3.1.3.2 we assume that $\lambda_0 \geq 0$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that the following two conditions hold

$$C_1^\phi := \|\phi\|_1 = \int_{\mathbb{R}^d} dx \phi(x) < \infty, \quad (4.30)$$

$$C_\infty^\phi := \|\phi\|_\infty = \text{ess sup}_{x \in \mathbb{R}^d} \phi(x) < \infty. \quad (4.31)$$

Using the result of Proposition 3.15, we see that for interaction (4.29) condition (4.11) holds with

$$\alpha_0 = \lambda_0 + CC_1^\phi \quad \text{and} \quad \alpha_1 = C_\infty^\phi. \quad (4.32)$$

Then, according to Theorem 4.3 we have the following existence and uniqueness result.

Corollary 4.7. *Let us assume that conditions (4.30) and (4.31) hold. Given $C_0 > 0$, for each $C > C_0$ hierarchy (4.6) specified by (4.7) and (2.71) with initial condition $k_0 \in \mathcal{K}_{C_0}^1$ has a unique solution $k_t \in \mathcal{K}_C^1$ on the time interval $0 \leq t < T(C, C_0)$, where*

$$T(C, C_0) = \frac{C_0(C - C_0)}{C^2 \left(me^{CC'_V} + \frac{b}{C_0} + 2\langle a \rangle C_\infty^\phi \right)}. \quad (4.33)$$

Clearly the same result can be also proved for the interaction

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(4)}(\gamma, z) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - z), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d. \quad (4.34)$$

Remark 4.8. Let us note that, compared with the semigroup approach discussed in Chapter 3, the Ovsjannikov's method requires weaker conditions on the interaction λ_{int} . Indeed in this case we do not have any constraint on the parameters α_0 and α_1 .

4.2 Mesoscopic evolution: Vlasov-type scaling

In this section we want to study the mesoscopic evolution of RWs in an aggregation model of RE.

The mesoscopic limit of the considered stochastic dynamics can be obtained by applying the general scheme of the Vlasov-type scaling, described in Section 2.3, to the hierarchy for correlation functions (4.6). A detailed discussion of the scaling for the generator (2.2) with (4.2)-(4.3) can be found in [45]; throughout the next sections we will intensively use the results of this paper.

Following the general construction, for any $\varepsilon > 0$ we consider a renormalized hierarchy given by

$$\begin{cases} \frac{dk_{t,\varepsilon}}{dt} = \left(L_{RE,\varepsilon,\text{ren}}^\Delta + L_{RW,\varepsilon,\text{ren}}^\Delta \right) k_{t,\varepsilon} \\ k_{t,\varepsilon}|_{t=0} = k_{0,\varepsilon} \end{cases}, \quad (4.35)$$

where $L_{RW,\varepsilon,\text{ren}}^\Delta$ is defined as in (2.101) and (cf. [45, Proposition 4.1])

$$\left(\widehat{L}_{RE,\varepsilon,\text{ren}}^\Delta k \right) (\eta, y) = -m \sum_{x \in \eta} e^{-\varepsilon \sum_{x' \in \eta \setminus x} V(x-x')} \int_{\Gamma_0} d\lambda(\xi) e_\lambda \left(\frac{e^{-\varepsilon V(x-\cdot)} - 1}{\varepsilon}, \xi \right) k(\eta \cup \xi, y) +$$

$$b \sum_{x \in \eta} k(\eta \setminus x, y). \quad (4.36)$$

Then, if we assume that the condition (2.111) holds in the limit $\varepsilon \rightarrow 0$ we obtain the Vlasov hierarchy

$$\begin{cases} \frac{dk_{t,V}}{dt} = (L_{RE,V}^\Delta + L_{RW,V}^\Delta) k_{t,V} \\ k_{t,V}|_{t=0} = k_{0,V} \end{cases}, \quad (4.37)$$

where $L_{RW,V}^\Delta$ is given by (2.113) and (cf. [45])

$$\begin{aligned} (L_{RE,V}^\Delta k)(\eta, y) &= \lim_{\varepsilon \rightarrow 0} (\widehat{L}_{RE,\varepsilon,\text{ren}}^\Delta k)(\eta, y) \\ &= -m \sum_{x \in \eta} \int_{\Gamma_0} d\lambda(\xi) e_\lambda(-V(x - \cdot), \xi) k(\eta \cup \xi, y) + b \sum_{x \in \eta} k(\eta \setminus x, y). \end{aligned} \quad (4.38)$$

In the remaining part of this chapter we study the Vlasov-type scaling in the Banach space \mathcal{K}_C^1 . In Section 4.2.1 we prove the convergence of the renormalized evolution to the solution of the limiting hierarchy (4.37). Finally, in Section 4.2.2 we present the explicit form of the Vlasov equations and study the properties of their solutions.

4.2.1 Convergence of the Vlasov-type scaling

We proceed by studying the convergence of the Vlasov-type scaling for the considered statistical evolution in the Banach space \mathcal{K}_C^1 . We follow the same strategy realized in [45].

First, we use Theorem 4.2 in the scale of Banach space $\{\mathcal{K}_C^1 : 0 < C \leq C_0\}$, $C_0 > 0$, to find a solution to the renormalized and to the limiting evolution equations, (4.35) and (4.37), respectively. It is important that these two solutions are defined on the same time interval and with values in the same Banach space. Indeed, in this case it is natural to study under which conditions we have a convergence. The latter will be done by using a general result presented in [61], which we formulate below for the reader's convenience.

Theorem 4.9. *Let the family of Banach spaces $\{\mathbb{B}_s : 0 < s \leq s_0\}$ be such as in Theorem 4.2. For any $\varepsilon > 0$ consider the family of initial value problems*

$$\frac{du_\varepsilon}{dt} = A_\varepsilon u_\varepsilon(t), \quad u_\varepsilon(0) = u_\varepsilon, \quad (4.39)$$

where, for each $s \in (0, s_0)$ fixed and for each pair (s', s'') such that $s \leq s' < s'' \leq s_0$, $A_\varepsilon : \mathbb{B}_{s''} \mapsto \mathbb{B}_{s'}$ is a linear mapping and there is a constant $M > 0$ such that for all $u \in \mathbb{B}_{s''}$

$$\|A_\varepsilon u\|_{s'} \leq \frac{M}{s'' - s'} \|u\|_{s''}. \quad (4.40)$$

Here M is independent of ε , s' , s'' and u , however, it might depend continuously on s and s_0 . In addition, assume that for each $\varepsilon > 0$ there is a N_ε and a $p \in \mathbb{N}$ such that for each pair (s', s'') with $s \leq s' < s'' \leq s_0$ and for all $u \in \mathbb{B}_{s''}$

$$\|A_\varepsilon u - A_0 u\|_{s'} \leq \sum_{k=1}^p \frac{N_\varepsilon}{(s'' - s')^k} \|u\|_{s''}, \quad (4.41)$$

with

$$\lim_{\varepsilon \rightarrow 0} N_\varepsilon = 0. \quad (4.42)$$

Then, if

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(0) - u_0(0)\|_{s_0} = 0, \quad (4.43)$$

for each $s \in (0, s_0)$ there is a constant $\delta > 0$, depending on M , such that the family of initial value problems (4.39) has a unique solution $u_\varepsilon : [0, \delta(s_0 - s)) \mapsto \mathbb{B}_s$ for each $\varepsilon > 0$. Moreover, for all $t \in [0, \delta(s_0 - s))$ we have

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(t) - u_0(t)\|_s = 0. \quad (4.44)$$

Following the scheme outlined above, let us consider the hierarchies (4.35) and (4.37). As in Theorem 4.3, in order to apply the Ovsjannikov's method, see Theorem 4.2, to these two initial value problems, we should impose some conditions on the rescaled interaction λ_ε and on the function A_V defined in (2.111).

Assumption 4.10. *Suppose that for any $\varepsilon > 0$ there exist $\tilde{\alpha}_0, \tilde{\alpha}'_0 > 0$ and $\tilde{\alpha}_1 \geq 0$ such that for all $\eta \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$*

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \varepsilon^{-|\xi|} \left| K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z) \right|(\xi) \leq \tilde{\alpha}_0 + \tilde{\alpha}_1 |\eta|, \quad (4.45)$$

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |A_V(\xi, y, z)| \leq \tilde{\alpha}'_0. \quad (4.46)$$

Without loss of generality we can assume that $\tilde{\alpha}'_0 = \tilde{\alpha}_0$.

Under the assumption above we can prove the following result.

Lemma 4.11. *Let V be a non-negative even function such that*

$$C_1^V := \int_{\mathbb{R}^d} dx V(x) < \infty. \quad (4.47)$$

Suppose that conditions (4.45) and (4.46) are satisfied. Then, given $C_0 > 0$ arbitrary and fixed, for any $C > C_0$ there exists a moment of time

$$T_1(C_0, C) := \frac{C_0(C - C_0)}{C^2 \left(me^{CC_1^V} + \frac{b}{C_0} + 2\langle a \rangle \tilde{\alpha}_1 \right)}, \quad (4.48)$$

such that for any $\varepsilon > 0$ the evolution equations (4.35) and (4.37) with initial conditions $\{k_{0,\varepsilon}, k_{0,V}\} \subset \mathcal{K}_{C_0}^1$, have unique solutions $k_{t,\varepsilon}$ and $k_{t,V}$, respectively, in the space \mathcal{K}_C^1 on the time interval $0 \leq t < T(C_0, C_1)$.

Proof of Lemma 4.11. Let us first analyze the generators that appear in (4.35) and (4.37). From [45, Proposition 4.2] we have the following result for the operators $L_{RE,\varepsilon,\text{ren}}^\Delta$ and $L_{RE,V}^\Delta$.

Proposition 4.12. *Suppose that conditions (4.47) holds. Let $C > C_0 > 0$ arbitrary and fixed. Then for any C', C'' such that $C_0 \leq C' < C'' \leq C$, and for any $k \in \mathcal{K}_{C'}^1$, one has*

$$\left\| L_{RE,\#}^\Delta k \right\|_{\mathcal{K}_{C''}^1} \leq \frac{1}{C'' - C'} \frac{C}{e} \left(me^{CC_1^V} + \frac{b}{C_0} \right) \|k\|_{\mathcal{K}_{C'}^1}, \quad (4.49)$$

where $\# = \text{"}\varepsilon, \text{ren"}$ or $\text{"}V\text{"}$.

Next, under hypotheses (4.45) and (4.46), we can show the following two estimates for the operators $L_{RW,\varepsilon,\text{ren}}^\Delta$ and $L_{RW,V}^\Delta$, respectively.

Proposition 4.13. *Let us assume that condition (4.45) holds. Then, given $C > 0$ arbitrary and fixed, for any C', C'' such that $0 < C' < C'' \leq C$, and for any $k \in \mathcal{K}_{C'}^1$, the following estimate holds*

$$\left\| L_{RW,\varepsilon,\text{ren}}^\Delta k \right\|_{\mathcal{K}_{C''}^1} \leq 2\langle a \rangle \left(\frac{1}{C'' - C'} \frac{C}{e} \tilde{\alpha}_1 + \tilde{\alpha}_0 \right) \|k\|_{\mathcal{K}_{C'}^1}, \quad \varepsilon > 0. \quad (4.50)$$

Proof of Proposition 4.13. We can proceed as in the proof of Proposition 4.5 using condition (4.45) instead of (4.11). This is equivalent to replace (α_0, α_1) with $(\tilde{\alpha}_0, \tilde{\alpha}_1)$ in (4.14). \square

Proposition 4.14. *Suppose that (4.46) holds. Let $C > 0$ be arbitrary and fixed. Then, for any C', C'' such that $0 < C' < C'' \leq C$, and for any $k \in \mathcal{K}_{C'}^1$, we have*

$$\left\| L_{RW,V}^\Delta k \right\|_{\mathcal{K}_{C''}^1} \leq 2\langle a \rangle \tilde{\alpha}_0 \|k\|_{\mathcal{K}_{C'}^1}. \quad (4.51)$$

Proof of Proposition 4.14. We can follow the proof of Proposition 4.5 setting $\alpha_0 = \tilde{\alpha}_0$ and $\tilde{\alpha}_1 \equiv 0$. \square

Let us come back to the proof of Lemma 4.11. According to Proposition 4.13 and 4.14, for any C', C'' such that $0 < C_0 < C' < C'' \leq C$ and $k \in \mathcal{K}_{C'}^1$, one has

$$\left\| L_{RW,\#}^\Delta k \right\|_{\mathcal{K}_{C''}^1} \leq 2\langle a \rangle \tilde{\alpha}_0 \|k\|_{\mathcal{K}_{C'}^1} + \frac{1}{C'' - C'} \frac{C}{e} 2\langle a \rangle \tilde{\alpha}_1 \|k\|_{\mathcal{K}_{C'}^1}, \quad (4.52)$$

where $\# = \text{"}\varepsilon, \text{ren"}$ or $\text{"}V\text{"}$. Then, by combining this estimate with that in Proposition 4.12 we obtain

$$\left\| L_{\#}^\Delta k \right\|_{\mathcal{K}_{C''}^1} \leq 2\langle a \rangle \tilde{\alpha}_0 \|k\|_{\mathcal{K}_{C'}^1} + \frac{1}{C'' - C'} \frac{C}{e} \left[\left(me^{CC_1^V} + \frac{b}{C_0} \right) + 2\langle a \rangle \tilde{\alpha}_1 \right] \|k\|_{\mathcal{K}_{C'}^1}, \quad (4.53)$$

for any $k \in \mathcal{K}_{C'}^1$, and with $\# = \text{"}\varepsilon, \text{ren"}$ or $\text{"}V\text{"}$. Finally, an application of Theorem 4.2 similar to the one in the proof of Theorem 4.3 leads us to desired result. \square

Next let us study the convergence of the solutions of (4.35) to the solution of (4.37) on the time interval $[0, T_1(C_0, C_1))$ as ε goes to zero. Following [45] we assume that

$$\bar{V} := \operatorname{ess\,sup}_{x \in \mathbb{R}^d} V(x) < \infty. \quad (4.54)$$

In addition we need to impose a stronger condition than (2.111).

Assumption 4.15. *Assume that the limits*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} \left(K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z) \right) (\xi) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} \left(K^{-1} \lambda_\varepsilon(\cdot, y, z) \right) (\xi) = A_V(\xi, y, z), \quad (4.55)$$

hold point-wise in $\xi \in \Gamma_0$ as well as in $L^1(\mathbb{R}^d, C^{|\xi|} \lambda(\xi))$, for all $\eta \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$.

In particular, given $\varepsilon > 0$, we demand that there exist $\tilde{\sigma}_0, \tilde{\sigma}_1 \geq 0$ such that for all $\eta \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} \left(K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z) \right) (\xi) - A_V(\xi, y, z) \right| \leq \varepsilon (\sigma_0 + \sigma_1 |\eta|). \quad (4.56)$$

Remark 4.16. We can always write

$$\left(K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z) \right) (\xi) := A_\varepsilon(\xi, y, z) + \tilde{A}_\varepsilon(\xi, \eta, y, z), \quad (4.57)$$

where

$$A_\varepsilon(\xi, y, z) := \left(K^{-1} \lambda_\varepsilon(\cdot, y, z) \right) (\xi) \quad (4.58)$$

and

$$\tilde{A}_\varepsilon(\xi, \eta, y, z) := \left(K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z) \right) (\xi) - A_\varepsilon(\xi, y, z). \quad (4.59)$$

Then assumption (4.56) is satisfied if

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} A_\varepsilon(\xi, y, z) - A_V(\xi, y, z) \right| \leq \varepsilon \sigma_0, \quad (4.60)$$

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} \tilde{A}_\varepsilon(\xi, \eta, y, z) \right| \leq \varepsilon \sigma_1 |\eta|, \quad (4.61)$$

for all $\eta \in \Gamma_0$ and a.a. $y, z \in \mathbb{R}^d$. Note that in concrete models conditions (4.60) and (4.61) are, in general, easier to verify than (4.56).

In the following theorem we show the main result of this section about the convergence of the Vlasov-type scaling.

Theorem 4.17. *Let $T_1(C_0, C)$ be given by (4.48) and V be a even non-negative function on \mathbb{R}^d such that (4.47) and (4.54) hold. Suppose that conditions (4.45)-(4.46) as well as (4.56) are satisfied. Then, given $C > C_0 > 0$ fixed, the evolution equations (4.35) and (4.37) with initial conditions $\{k_{0,\varepsilon}, k_{0,V}\}_{\varepsilon>0} \subset \mathcal{K}_{C_0}^1$ have unique solutions, $k_{t,\varepsilon}$ and $k_{t,V}$, respectively, in \mathcal{K}_C^1 on the time interval $[0, T_1(C_0, C)]$. If, additionally, we assume that*

$$\lim_{\varepsilon \rightarrow 0} \|k_{0,\varepsilon} - k_{0,V}\|_{\mathcal{K}_{C_0}^1} = 0, \quad (4.62)$$

then

$$\lim_{\varepsilon \rightarrow 0} \|k_{t,\varepsilon} - k_{t,V}\|_{\mathcal{K}_C^1} = 0. \quad (4.63)$$

on the time interval $0 \leq t < T_1(C_0, C)$.

Proof of Theorem 4.17. The existence of solutions to (4.35) and (4.37) in \mathcal{K}_C^1 has been already proved in Lemma 4.11. In order to show the convergence in (4.63), we apply Theorem 4.9 to (4.35). Below we check that the conditions of this general theorem, see (4.40)-(4.42), are fulfilled.

We already know that condition (4.40) is satisfied. Indeed, from the proof of Lemma 4.11 we have (cf. (4.53))

$$\|L_{\#}^\Delta k\|_{\mathcal{K}_{C''}^1} \leq \frac{1}{C'' - C'} \frac{C}{e} \left[m e^{CC_1^V} + \frac{b}{C_0} + 2\langle a \rangle \tilde{\alpha}_1 + c(C, C_0) \right] \|k\|_{\mathcal{K}_{C'}^1}, \quad (4.64)$$

for any $k \in \mathcal{K}_{C'}^1$, where $\# = \varepsilon, \text{ren}$ or V and

$$c(C, C_0) := 2\langle a \rangle \tilde{\alpha}_0 \frac{e}{C} (C - C_0). \quad (4.65)$$

On the other hand, conditions (4.41) and (4.42) are a direct consequence of the following proposition. Its proof is given at the end of this section.

Proposition 4.18. *Assume that hypotheses of Theorem 4.17 are satisfied. Given $C > C_0 > 0$ fixed, let us consider any C', C'' such that $C_0 \leq C' < C'' \leq C$. Then for any $k \in \mathcal{K}_{C'}^1$, the following estimate holds*

$$\begin{aligned} \|L_{\varepsilon, \text{ren}}^\Delta k - L_V^\Delta k\|_{\mathcal{K}_{C''}^1} &\leq \varepsilon \left(\frac{\tilde{M}_1}{C'' - C'} + \frac{\tilde{M}_2}{(C'' - C')^2} \right) \|k\|_{\mathcal{K}_{C'}^1} + \\ &2\varepsilon \langle a \rangle \left[\tilde{\sigma}_0 + \frac{C \tilde{\sigma}_1}{e(C'' - C')} \right] \|k\|_{\mathcal{K}_{C'}^1}, \end{aligned} \quad (4.66)$$

with

$$\tilde{M}_1 = \frac{mC_1^V C^2}{2e} \bar{V} e^{CC_1^V}, \quad \tilde{M}_2 = \frac{4mC^2}{e^2} \bar{V} e^{CC_1^V}. \quad (4.67)$$

According to the result stated above, for any $C_0 \leq C' < C'' \leq C$ and all $\varepsilon > 0$, for any $k \in \mathcal{K}_{C'}^1$, we have

$$\|L_{\varepsilon, \text{ren}}^\Delta k - L_V^\Delta k\|_{\mathcal{K}_{C''}^1} \leq \varepsilon \frac{\|k\|_{\mathcal{K}_{C'}^1}}{(C'' - C')} \left[\tilde{M}_1 + \frac{2C}{e} \langle a \rangle \tilde{\sigma}_1 + \tilde{c}(C, C_0) \right] + \varepsilon \frac{\|k\|_{\mathcal{K}_{C'}^1}}{(C'' - C')^2} \tilde{M}_2, \quad (4.68)$$

where

$$\tilde{c}(C, C_0) := 2 \langle a \rangle \tilde{\sigma}_0 (C - C_0). \quad (4.69)$$

Hence, by (4.64) and (4.68) we can apply Theorem 4.9 for $p = 2$ and

$$N_\varepsilon = \varepsilon \max \left\{ \tilde{M}_1 + \frac{2C}{e} \langle a \rangle \tilde{\sigma}_1 + \tilde{c}(C, C_0), \tilde{M}_2 \right\}. \quad (4.70)$$

This concludes the proof of Theorem 4.17. \square

Proof of Proposition 4.18. For any $k \in \mathcal{K}_{C'}^1$, let us estimate

$$\|L_{\varepsilon, \text{ren}}^\Delta k - L_V^\Delta k\|_{\mathcal{K}_{C''}^1} \leq \|L_{RE, \varepsilon, \text{ren}}^\Delta k - L_{RE, V}^\Delta k\|_{\mathcal{K}_{C''}^1} + \|L_{RW, \varepsilon, \text{ren}}^\Delta k - L_{RW, V}^\Delta k\|_{\mathcal{K}_{C''}^1}. \quad (4.71)$$

By using [45, Proposition 4.6], for all $\varepsilon > 0$ and for any C', C'' such that $C_0 \leq C' < C'' \leq C$ one can show

$$\|L_{RE, \varepsilon, \text{ren}}^\Delta k - L_{RE, V}^\Delta k\|_{\mathcal{K}_{C''}^1} \leq \varepsilon \left(\frac{\tilde{M}_1}{C'' - C'} + \frac{\tilde{M}_2}{(C'' - C')^2} \right) \|k\|_{\mathcal{K}_{C'}^1}, \quad (4.72)$$

where the constants M_1 and M_2 are given by (4.67).

Next let us estimate the second term in the r.h.s. of (4.71). By (2.101) and (2.113) we have

$$\begin{aligned} (C'')^{-|\eta|} & \left| (L_{RW, \varepsilon, \text{ren}}^\Delta k)(\eta, y) - (L_{RW, V}^\Delta k)(\eta, y) \right| \leq \\ & \leq (C'')^{-|\eta|} \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) |k(\xi \cup \eta, z)| \times \\ & \quad \left| \varepsilon^{-|\xi|} (K^{-1} \lambda_\varepsilon(\cdot \cup \eta, z, y))(\xi) - A_V(\xi, z, y) \right| + \\ & (C'')^{-|\eta|} \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) |k(\xi \cup \eta, y)| \times \\ & \quad \left| \varepsilon^{-|\xi|} (K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z))(\xi) - A_V(\xi, y, z) \right|. \end{aligned}$$

Then, since $k \in \mathcal{K}_{C'}^1$, we obtain

$$\begin{aligned} (C'')^{-|\eta|} & \left| (L_{RW, \varepsilon, \text{ren}}^\Delta k)(\eta, y) - (L_{RW, V}^\Delta k)(\eta, y) \right| \leq \\ & \leq \left(\frac{C'}{C''} \right)^{|\eta|} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) \|k(\cdot, z)\|_{\mathcal{K}_{C'}} \times \\ & \quad \left| \varepsilon^{-|\xi|} (K^{-1} \lambda_\varepsilon(\cdot \cup \eta, z, y))(\xi) - A_V(\xi, z, y) \right| + \\ & \left(\frac{C'}{C''} \right)^{|\eta|} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \int_{\mathbb{R}^d} dz a(y-z) \|k(\cdot, y)\|_{\mathcal{K}_{C'}} \times \\ & \quad \left| \varepsilon^{-|\xi|} (K^{-1} \lambda_\varepsilon(\cdot \cup \eta, y, z))(\xi) - A_V(\xi, y, z) \right|. \end{aligned}$$

By using condition (4.56) in Assumption 4.15, one may estimate

$$\begin{aligned}
(C'')^{-|\eta|} & \left| (L_{RW,\varepsilon,\text{ren}}^\Delta k)(\eta, y) - (L_{RW,V}^\Delta k)(\eta, y) \right| \leq \\
& \leq \left(\frac{C'}{C''} \right)^{|\eta|} \int_{\mathbb{R}^d} dz a(y-z) \|k(\cdot, z)\|_{\mathcal{K}_{C'}} \varepsilon (\sigma_0 + \sigma_1 |\eta|) + \\
& \quad \left(\frac{C'}{C''} \right)^{|\eta|} \int_{\mathbb{R}^d} dz a(y-z) \|k(\cdot, y)\|_{\mathcal{K}_{C'}} \varepsilon (\sigma_0 + \sigma_1 |\eta|) \\
& \leq \varepsilon \left(\frac{C'}{C''} \right)^{|\eta|} (\sigma_0 + \sigma_1 |\eta|) \int_{\mathbb{R}^d} dz a(y-z) [\|k(\cdot, z)\|_{\mathcal{K}_{C'}} + \|k(\cdot, y)\|_{\mathcal{K}_{C'}}].
\end{aligned}$$

Now following the proof of Proposition 4.5 we can use inequality (4.15) to find

$$\begin{aligned}
(C'')^{-|\eta|} & \left| (L_{RW,\varepsilon,\text{ren}}^\Delta k)(\eta, y) - (L_{RW,V}^\Delta k)(\eta, y) \right| \leq \\
& \leq \varepsilon \left[\sigma_0 + \frac{\sigma_1}{e(\ln C'' - \ln C')} \right] \int_{\mathbb{R}^d} dz a(y-z) [\|k(\cdot, z)\|_{\mathcal{K}_{C'}} + \|k(\cdot, y)\|_{\mathcal{K}_{C'}}].
\end{aligned}$$

As a result, we have

$$\left\| L_{RW,\varepsilon,\text{ren}}^\Delta k - L_{RW,V}^\Delta k \right\|_{\mathcal{K}_{C''}^1} \leq 2\varepsilon \langle a \rangle \left[\sigma_0 + \frac{\sigma_1}{e(\ln C'' - \ln C')} \right] \|k\|_{\mathcal{K}_{C'}^1}.$$

But there exists a constant $c \in [C', C''] \subset [C_0, C]$ such that

$$0 < \ln C'' - \ln C' = \frac{1}{c} (C'' - C') \geq \frac{1}{C} (C'' - C'), \quad (4.73)$$

therefore

$$\left\| L_{RW,\varepsilon,\text{ren}}^\Delta k - L_{RW,V}^\Delta k \right\|_{\mathcal{K}_{C''}^1} \leq 2\varepsilon \langle a \rangle \|k\|_{\mathcal{K}_{C'}^1} \left[\sigma_0 + \frac{C\sigma_1}{e(C'' - C')} \right], \quad (4.74)$$

which concludes the proof of the proposition. \square

4.2.2 Vlasov equations

This section is devoted to the study of Vlasov equations for the considered model of RWRE.

First, let us consider the Vlasov hierarchy (4.37). We want to show that this limiting evolution has the chaos preservation property, see (2.85)-(2.86), and, then, we derive the corresponding kinetic equations.

Lemma 4.19. *Given $C > C_0 > 0$, let us consider functions $\rho_0 \in \overline{B}_{C_0}^\infty$ and $r_0 \in L^1(\mathbb{R}^d)$. Suppose that conditions (4.46) and (4.47) hold. Then, the initial value problem (4.37) with initial condition $k_0 = e_\lambda(\rho_0, \cdot)r_0 \in \mathcal{K}_{C_0}^1$ has a unique solution*

$$k_{t,V} = e_\lambda(\rho_t, \cdot)r_t \in \mathcal{K}_C^1, \quad (4.75)$$

provided that $\rho_t \in \overline{B}_C^\infty$ and $r_t \in L^1(\mathbb{R}^d)$ are solutions to the Vlasov equations

$$\frac{\partial \rho_t(x)}{\partial t} = -m\rho_t(x) e^{-(\rho_t * V)(x)} + b, \quad (4.76a)$$

$$\frac{\partial r_t(y)}{\partial t} = \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) \bar{\lambda}_t(z, y) - r_t(y) \bar{\lambda}_t(y, z)], \quad (4.76b)$$

with initial conditions $\rho_t|_{t=0} = \rho_0$ and $r_t|_{t=0} = r_0$, on the time interval $[0, T_1)$, where $T_1 := T_1(C, C_0)$ is given by (4.48) and $\bar{\lambda}_t$ is defined as in (2.126).

Proof. By combining the results of Proposition 2.23 and Lemma 2.25.(i), we can easily show that $k_{t,V} = e_\lambda(\rho_t, \cdot)r_t$ is a solution to the initial value problem (4.37) with initial conditions $k_0 = e_\lambda(\rho_0, \cdot)r_0$ provided that ρ_t and r_t are solutions to (4.76a)-(4.76b). The uniqueness of solution (4.75) follows directly from the result of Lemma 4.11. Indeed, we can note that for $\rho_0 \in \overline{B}_{C_0}^\infty$ and $r_0 \in L^1(\mathbb{R}^d)$ the function $k_0 = e_\lambda(\rho_0, \cdot)r_0$ belongs to $\mathcal{K}_{C_0}^1$. \square

In the remainder of this section we want to study the question about existence and uniqueness of solutions to the Vlasov equations (4.76a)-(4.76b). We will be interested in non-negative solutions (ρ_t, r_t) on \mathbb{R}_+ with

$$\rho \in \mathcal{C}^1(\mathbb{R}_+; C_b(\mathbb{R}^d)), \quad r \in \mathcal{C}^1(\mathbb{R}_+; L^1(\mathbb{R}^d)),$$

where let us recall that, given $X = C_b(\mathbb{R}^d), L^1(\mathbb{R}^d)$,

$$\mathcal{C}^1(\mathbb{R}_+; X) := \mathcal{C}^1((0, +\infty) \rightarrow X) \cap \mathcal{C}([0, +\infty) \rightarrow X).$$

The kinetic equation (4.76a) has been studied in [45]. Below we briefly recall some results about properties of its solutions. Consider the stationary homogeneous equation for (4.76a), namely

$$b = m\rho e^{-C_1^V \rho}. \quad (4.77)$$

It is easy to check that, under condition

$$b < \frac{m}{C_1^V e}, \quad (4.78)$$

equation (4.77) has two solutions, denoted by κ_1 and κ_2 , such that $0 < \kappa_1 < (1/C_1^V) < \kappa_2$. Then we can formulate the following result for solutions to (4.76a). For more details we refer the reader to [45].

Theorem 4.20. *Assume that (4.47) holds. Suppose that condition (4.78) is satisfied and let κ_1 and κ_2 be constant solutions to (4.77). Then, given $0 \leq \rho_0 \in C_b(\mathbb{R}^d)$ with $\|\rho_0\|_\infty \leq \kappa_2$, equation (4.76a) has a unique non-negative solution $0 \leq \rho \in \mathcal{C}^1(\mathbb{R}_+; C_b(\mathbb{R}^d))$ with $\|\rho_t\|_\infty \leq \kappa_2$, for all $t \geq 0$. Moreover, for an arbitrary $c \in [\kappa_1, \kappa_2]$, the condition $\kappa_1 \leq \rho_0(x) \leq c$, $x \in \mathbb{R}^d$, yields $\kappa_1 \leq \rho_0(x) \leq c$, $x \in \mathbb{R}^d$.*

Next let us consider the kinetic equation (4.76b). As in Section 3.2.2, for the moment, we assume that $\rho_t \in L^\infty(\mathbb{R}^d)$, $t \geq 0$, is a non-negative bounded function. Then, we can show the following existence and uniqueness result.

Lemma 4.21. *Let $r_0 \in L^1(\mathbb{R}^d)$ be such that $r_0(y) \geq 0$ for a.a. $y \in \mathbb{R}^d$. Given $C > 0$, suppose that $\rho \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ with $0 \leq \rho_t(x) \leq C$ for a.a. $x \in \mathbb{R}^d$, $t \geq 0$. Then, if we assume that (3.125) and (4.46) hold, the evolution equation (4.76b) has a unique non-negative solution $0 \leq r \in \mathcal{C}^1(\mathbb{R}_+; L^1(\mathbb{R}^d))$. Moreover, given $C' > 0$*

(i) *if $\|r_0\|_1 \leq C'$ then $\|r_t\|_1 \leq C'$ for any $t > 0$;*

(ii) *if $r_0(x) \leq C'$, a.a. $x \in \mathbb{R}^d$, then $r_t(x) \leq C'$, a.a. $x \in \mathbb{R}^d$, for any $t > 0$.*

Proof. The proof of the statement will be divided in 2 steps. First we show that there exists a unique solution $r \in \mathcal{C}^1(\mathbb{R}_+; L^1(\mathbb{R}^d))$ to (4.76b). Afterwards, we prove that, for non-negative initial conditions, this solution is non-negative and satisfies the bounds (i) and (ii). To perform this task we follow the same strategy as in the proof of Theorem 3.41.

Step.1. Existence and uniqueness of solutions. As we have already seen in the proof of Theorem 3.41, the evolution equation (4.76b) can be written as

$$\begin{cases} \frac{\partial}{\partial t} r_t(y) = (\overline{L}_{RW}^*(t)) r_t(y) \\ r_t(y)|_{t=0} = r_0(y) \end{cases}, \quad (4.79)$$

where $\bar{L}_{RW}^*(t)$ is a linear operator defined by (3.127). We can easily show that this operator is bounded in $L^1(\mathbb{R}^d)$ on \mathbb{R}_+ . Indeed for any $r \in L^1(\mathbb{R}^d)$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} dy \left| (\bar{L}_{RW}^*(t)r)(y) \right| &\leq \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\xi) e_{\lambda}(\rho_t, \xi) \int_{\mathbb{R}^d} dz a(y-z) |A_V(\xi, z, y)| |r(z)| + \\ &\quad \int_{\Gamma_0} d\lambda(\xi) e_{\lambda}(\rho_t, \xi) \int_{\mathbb{R}^d} dz a(y-z) |A_V(\xi, y, z)| |r(y)|. \end{aligned}$$

By hypothesis we know that $0 \leq \rho_t \in \bar{B}_C^\infty$, then we can use (4.46) to estimate

$$\begin{aligned} \int_{\mathbb{R}^d} dy \left| (\bar{L}_{RW}^*(t)r)(y) \right| &\leq \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) |r(z)| \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |A_V(\xi, z, y)| + \\ &\quad \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) |r(y)| \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) |A_V(\xi, y, z)| \\ &\leq \tilde{\alpha}_0 \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) |r(z)| + \tilde{\alpha}_0 \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) |r(y)| \\ &\leq 2\tilde{\alpha}_0 \langle a \rangle \|r\|_1. \end{aligned}$$

Therefore, from [85, Section II.2] we know that the initial value problem (4.76b) has a unique solution $r \in \mathcal{C}^1(\mathbb{R}_+; L^1(\mathbb{R}^d))$.

Step.2. Non-negativity and uniform bounds of the solution. Let us first show that for initial conditions $0 \leq r_0 \in L^1(\mathbb{R}^d)$ with $\|r_0\|_1 \leq C'$ we have a non-negative solution $0 \leq r_t \in L^1(\mathbb{R}^d)$ such that $\|r_t\|_1 \leq C'$, for any time $t > 0$.

Given $T > 0$ let us consider the Banach space $X_{T,1} = C([0, T]; L^1(\mathbb{R}^d))$ equipped with the norm

$$\|r\|_{T,1} = \max_{t \in [0, T]} \|r_t\|_1 = \max_{t \in [0, T]} \int_{\mathbb{R}^d} dy r_t(y). \quad (4.80)$$

We denote by $X_{T,1}^+$ the cone of all non-negative functions from $X_{T,1}$. This cone forms a complete metric space with a metric induced by the norm in $X_{T,1}$.

Following Step.2 in the proof of Theorem 3.41, for any $0 \leq r_0 \in L^1(\mathbb{R}^d)$ let us consider the mapping Ψ which assigns to any $v \in X_{T,1}$ the solution u_t of the (local) Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u_t(y) = \int_{\mathbb{R}^d} dz a(y-z)_t(z, y) v_t(z) - u_t(y) \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_t(y, z) \\ u_t(y)|_{t=0} = r_0(y) \end{cases}, \quad (4.81)$$

namely, see e.g. [69],

$$\begin{aligned} (\Psi v)_t(y) &= e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y, z)} r_0(y) + \\ &\quad \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y, z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z, y) v_s(z). \end{aligned} \quad (4.82)$$

for all $t > 0$ and a.a. $y \in \mathbb{R}^d$. Let us recall that, under condition (3.125) and (4.46), for $0 \leq \rho_t \in \bar{B}_C^\infty$ we have (cf. (3.131))

$$0 \leq \bar{\lambda}_t(y, z) \leq \tilde{\alpha}_0, \quad \text{for all } t > 0 \text{ and a.a. } y, z \in \mathbb{R}^d. \quad (4.83)$$

It is easy to check that Ψ maps $X_{T,1}^+$ into itself $X_{T,1}^+$. Indeed, for any $v \in X_{T,1}^+$ one has $\Psi v \geq 0$. Furthermore, using (4.83) one can estimate

$$\begin{aligned} \|(\Psi v)_t\|_1 &\leq \int_{\mathbb{R}^d} dy e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y, z)} r_0(y) + \\ &\quad \int_{\mathbb{R}^d} dy \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y, z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z, y) v_s(z) \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}} dy r_0(y) + \tilde{\alpha}_0 \int_0^t ds e^{-\langle a \rangle \tilde{\alpha}_0(t-s)} \int_{\mathbb{R}} dy \int_{\mathbb{R}^d} dz a(y-z) v_s(z) \\
&\leq \|r_0\|_1 + \tilde{\alpha}_0 \langle a \rangle \int_0^t ds e^{-\tilde{\alpha}_0 \langle a \rangle (t-s)} \|v_s\|_1 \\
&\leq \|r_0\|_1 + \|v\|_{T,1} \tilde{\alpha}_0 \langle a \rangle \int_0^t ds e^{-\tilde{\alpha}_0 \langle a \rangle (t-s)}.
\end{aligned} \tag{4.84}$$

Then from identity (3.133) it follows that

$$\begin{aligned}
\|(\Psi v)_t\|_1 &\leq \|r_0\|_1 + \|v\|_{T,1} (1 - e^{-\tilde{\alpha}_0 \langle a \rangle t}) \\
&\leq \|r_0\|_1 + \|v\|_{T,1} < \infty.
\end{aligned} \tag{4.85}$$

Hence, we can conclude that $\Psi v \in X_{T,1}^+$. Actually, we can also prove that Ψ defines a contraction mapping on $X_{T,1}^+$. Indeed, for any $v, w \in X_{T,1}^+$ we have

$$\begin{aligned}
\|(\Psi v)_t - (\Psi w)_t\|_1 &\leq \int_{\mathbb{R}^d} dy \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y,z)} \times \\
&\quad \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z,y) |v_s(z) - w_s(z)|.
\end{aligned}$$

Then by repeating the same arguments used to estimate $\|(\Psi v)_t\|_1$, for any $t \in [0, T]$ we obtain

$$\begin{aligned}
\|(\Psi v)_t - (\Psi w)_t\|_1 &\leq \|v - w\|_{T,1} (1 - e^{-\tilde{\alpha}_0 \langle a \rangle T}) \\
&< \|v - w\|_{T,1}.
\end{aligned} \tag{4.86}$$

Next, for any $n \geq 1$ let us consider the iterative scheme $v^{(n)} = \Psi^n v^{(0)}$, $v^{(0)} \in X_{T,1}^+$, defined by (3.135) and (3.136). From the Banach fixed point theorem, it follows that the sequence $\{v^{(n)}\} \subset X_{T,1}^+$ has a unique fixed point $v \in X_{T,1}^+$, which provides a unique non-negative solution to (4.76b).

Assume additionally that $\|r_0\|_1 \leq C'$, $C' > 0$, then proceeding by induction we can also show that $\|v_t\|_1 \leq C'$, for any $t \in [0, T]$. Indeed, from (3.136) we have

$$\begin{aligned}
\|v_t^{(0)}\|_1 &= \int_{\mathbb{R}^d} dy e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y,z)} r_0(y) \\
&\leq \int_{\mathbb{R}^d} dy r_0(y) \leq C', \quad t \in [0, T].
\end{aligned} \tag{4.87}$$

Suppose that the same bound holds also for $v_t^{(n-1)}$, $n \geq 1$, namely

$$\|v_t^{(n-1)}\|_1 \leq C', \quad t \in [0, T]. \tag{4.88}$$

Let us show it for $v_t^{(n)}$. According to (3.135), for any $t \in [0, T]$ we can write

$$\begin{aligned}
\|v_t^{(n)}\|_1 &\leq \int_{\mathbb{R}^d} dy e^{-\int_0^t ds \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(y,z)} r_0(y) + \\
&\quad \int_{\mathbb{R}^d} dy \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y,z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z,y) v_s^{(n-1)}(z).
\end{aligned}$$

By using (4.83) together with (4.87) and (4.88), one can estimate

$$\|v_t^{(n)}\|_1 \leq e^{-\langle a \rangle \tilde{\alpha}_0 t} \int_{\mathbb{R}^d} dy r_0(y) + \tilde{\alpha}_0 \int_0^t ds e^{-\langle a \rangle \tilde{\alpha}_0(t-s)} \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) v_s^{(n-1)}(z)$$

$$\leq C' e^{-\langle a \rangle \tilde{\alpha}_0 t} + C' \langle a \rangle \tilde{\alpha}_0 \int_0^t ds e^{-\langle a \rangle \tilde{\alpha}_0 (t-s)}.$$

Then, identity (3.133) yields

$$\|v_t^{(n)}\|_1 \leq C' e^{-\langle a \rangle \tilde{\alpha}_0 t} + C' (1 - e^{-\langle a \rangle \tilde{\alpha}_0 t}) \leq C', \quad t \in [0, T]. \quad (4.89)$$

Clearly, taking the limit $n \rightarrow \infty$ we obtain that $\|v_t\|_1 \leq C'$, for any $t \in [0, T]$. Note that we can repeat the above proof for the initial condition $0 \leq r_T \leq C'$ and extend all our results to the time interval $[T, 2T]$ and hence to \mathbb{R}_+ .

It remains to prove claim (ii). Let us take initial conditions $r_0 \in L^1(\mathbb{R}^d)$ such that $0 \leq r_0(y) \leq C'$, $C' > 0$, for a.a. $y \in \mathbb{R}^d$. For brevity, we introduce the set $B_{T, C'}^+$ consisting of all functions $v \in X_{T, 1}^+$ such that $v(y) \leq C'$ for a.a. $y \in \mathbb{R}^d$. It is easy to show that this set constitutes a complete metric space with the metric induced by the norm (4.80).

Consider the map Ψ defined by (4.81)-(4.82). Then, for any $v \in B_{T, C'}^+$ we have $\Psi v \in B_{T, C'}^+$. Indeed, by (4.83) and (3.133) one can estimate

$$\begin{aligned} 0 \leq (\Psi v)_t(y) &= r_0(y) e^{-\int_0^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y, z)} + \\ &\quad \int_0^t ds e^{-\int_s^t ds' \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_{s'}(y, z)} \int_{\mathbb{R}^d} dz a(y-z) \bar{\lambda}_s(z, y) v_s(z) \\ &\leq C' e^{-\langle a \rangle \tilde{\alpha}_0 t} + C' (1 - e^{-\langle a \rangle \tilde{\alpha}_0 t}) \\ &\leq C'. \end{aligned} \quad (4.90)$$

Furthermore, from (4.86) it follows that $\Psi : B_{T, C'}^+ \rightarrow B_{T, C'}^+$ is a contraction mapping on the complete metric space $B_{T, C'}^+$. Now we can apply the Banach fixed point theorem as done above. As a result, the mapping Ψ has a unique fixed point $v \in B_{T, C'}^+$ which solves (3.124b) on $[0, T]$. By definition, this solution is non-negative and essentially bounded by $C' > 0$. Obviously, taking as initial condition $v_T \in B_{T, C'}^+$, the same considerations can be extended on the time interval $[T, 2T]$ and, consequently, to all \mathbb{R}_+ . This concludes the proof of the lemma. \square

Finally, we can combine Theorem 4.20 with Lemma 4.21 to formulate our main result about the existence and uniqueness of non-negative solutions to the system of Vlasov equations (4.76a)-(4.76b).

Theorem 4.22. *Let*

$$b \leq \frac{m}{C_1^V e} \quad (4.91)$$

and let κ_1 and κ_2 be constant solutions to equation (4.77). Suppose that $0 \leq \rho_0 \in C_b(\mathbb{R}^d)$ with $\|\rho_0\|_\infty \leq \kappa_2$ and $0 \leq r_0 \in L^1(\mathbb{R}^d)$. Then, if we assume that conditions (3.125), (4.46) and (4.47) hold, the system of equations (4.76a)-(4.76b) has a unique solution $0 \leq \rho_t \in C^1(\mathbb{R}_+; C_b(\mathbb{R}^d))$ and $0 \leq r_t \in C^1(\mathbb{R}_+; L^1(\mathbb{R}^d))$ such that $\|\rho_t\|_\infty \leq \kappa_2$, for any $t > 0$. Moreover, given $C' > 0$

(i) if $\|r_0\|_1 \leq C'$ then $\|r_t\|_1 \leq C'$ for any $t > 0$;

(ii) if $r_0(x) \leq C'$, a.a. $x \in \mathbb{R}^d$, then $r_t(x) \leq C'$, a.a. $x \in \mathbb{R}^d$, for any $t > 0$.

Remark 4.23. Let us note that the theorem above provides a solution to the Vlasov equations only if the initial density ρ_0 is small enough, i.e. if $\|\rho_0\|_\infty \leq \kappa_2$. If not, an aggregation effect is expected in RE. More precisely, it is possible to show that if ρ_0 is large enough in some volume then the solution to (4.76a) grows (point-wise) to infinity in this volume, see [45, Section 5.2] for further details. In this case, the study of existence and uniqueness of solutions to (4.76a) requires a more subtle analysis.

4.2.3 Examples

In this section we study the Vlasov-type scaling and the corresponding Vlasov equations in the considered model of RWRE for each of the interactions I-IV described in Section 2.1.

4.2.3.1 RW in an aggregation model of environment: Case I and III

Let us study the interaction given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(1)}(\gamma, y) = e_\lambda(e^{-\phi(\cdot-y)}, \gamma), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d, \quad (4.92)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that

$$C_1^\phi := \|\phi\|_1 = \int_{\mathbb{R}^d} dx \phi(x) < \infty, \quad (4.93)$$

$$C_\infty^\phi := \|\phi\|_\infty = \text{ess sup}_{x \in \mathbb{R}^d} \phi(x) < \infty. \quad (4.94)$$

For any $\varepsilon > 0$ we consider the following scaling of interaction (4.92),

$$\lambda_\varepsilon^{(1)}(\gamma, y) := e_\lambda(e^{-\varepsilon\phi(\cdot-y)}, \gamma), \quad \gamma \in \Gamma_0, y \in \mathbb{R}^d. \quad (4.95)$$

According to the results obtained in Section 3.2.3.1, we have

$$(K^{-1}\lambda_\varepsilon^{(1)}(\cdot \cup \eta, y))(\xi) = e^{-\varepsilon \sum_{x' \in \eta} \phi(x'-y)} e_\lambda(e^{-\varepsilon\phi(\cdot-y)} - 1, \xi) \quad (4.96)$$

and letting ε go to zero

$$\begin{aligned} A_V^{(1)}(\xi, y) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1}\lambda_\varepsilon(\cdot \cup \eta, y))(\xi) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1}\lambda_\varepsilon(\cdot, y))(\xi) \\ &= e_\lambda(-\phi(\cdot-y), \xi), \end{aligned} \quad (4.97)$$

for any $\xi, \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. From Proposition 3.44 and 3.45, we know that the limit (4.97) holds in $L^1(\Gamma_0, C^{|\xi|}d\lambda(\xi))$, for a.a. $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$, and Assumption 4.10 is satisfied with

$$\tilde{\alpha}_0 = \tilde{\alpha}'_0 = e^{CC_1^\phi}, \quad \tilde{\alpha}_1 \equiv 0. \quad (4.98)$$

Next let us check Assumption 4.15. In order to do that we will show that conditions (4.60) and (4.61) hold, see Remark 4.16. In this case, the rescaled interaction (4.95) can be rewritten in the following form (cf. (4.57))

$$(K^{-1}\lambda_\varepsilon^{(1)}(\cdot \cup \eta, y))(\xi) = A_\varepsilon^{(1)}(\xi, y) + \tilde{A}_\varepsilon^{(1)}(\xi, \eta, y), \quad (4.99)$$

where

$$A_\varepsilon^{(1)}(\xi, y) = e_\lambda(e^{-\varepsilon\phi(\cdot-y)} - 1, \xi) \quad (4.100)$$

and

$$\tilde{A}_\varepsilon^{(1)}(\xi, \eta, y) = [e_\lambda(e^{-\varepsilon\phi(\cdot-y)}, \eta) - 1] e_\lambda(e^{-\varepsilon\phi(\cdot-y)} - 1, \xi), \quad (4.101)$$

for any $\xi, \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. For functions (4.100) and (4.101) we can show the following estimates.

Proposition 4.24. *Suppose that conditions (4.93) and (4.94) hold. Then one has*

$$\int_{\Gamma_0} C^{|\xi|}d\lambda(\xi) \left| \varepsilon^{-|\xi|} A_\varepsilon^{(1)}(\xi, y) - A_V^{(1)}(\xi, y) \right| \leq \varepsilon \frac{C}{2} e^{CC_1^\phi} C_1^\phi C_\infty^\phi \quad (4.102)$$

and

$$\int_{\Gamma_0} C^{|\xi|}d\lambda(\xi) \left| \varepsilon^{-|\xi|} \tilde{A}_\varepsilon^{(1)}(\xi, \eta, w) \right| \leq \varepsilon e^{CC_1^\phi} C_\infty^\phi |\eta|, \quad (4.103)$$

for a.a. $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$.

Proof. First let us show (4.102). By using inequalities $\phi(x) \geq 0$ and

$$\frac{1 - e^{-\varepsilon\phi(x)}}{\varepsilon} \leq \phi(x), \quad x \in \mathbb{R}^d, \quad (4.104)$$

we can write

$$\begin{aligned} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} A_\varepsilon^{(1)}(\xi, y) - A_V^{(1)}(\xi, y) \right| &= \\ &= \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| e_\lambda \left(\frac{e^{-\varepsilon\phi(\cdot-y)} - 1}{\varepsilon}, \xi \right) - e_\lambda(-\phi(\cdot-y), \xi) \right| \\ &= \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left[e_\lambda(\phi(\cdot-y), \xi) - e_\lambda \left(\frac{1 - e^{-\varepsilon\phi(\cdot-y)}}{\varepsilon}, \xi \right) \right], \end{aligned} \quad (4.105)$$

for a.a. $y \in \mathbb{R}^d$. Note that

$$\begin{aligned} 0 &\leq e_\lambda(\phi(\cdot-y), \xi) - e_\lambda \left(\frac{1 - e^{-\varepsilon\phi(\cdot-y)}}{\varepsilon}, \xi \right) \\ &\leq \sum_{x' \in \xi} \left(\phi(x'-y) - \frac{1 - e^{-\varepsilon\phi(x'-y)}}{\varepsilon} \right) e_\lambda(\phi(\cdot-y), \xi \setminus x'), \end{aligned} \quad (4.106)$$

for any $\xi \in \Gamma_0$, $y \in \mathbb{R}^d$, and

$$\phi(x'-y) - \frac{1 - e^{-\varepsilon\phi(x'-y)}}{\varepsilon} = \frac{1}{\varepsilon^2 \phi^2(x'-y)} (e^{-\varepsilon\phi(x'-y)} + \varepsilon\phi(x'-y) - 1) \varepsilon \phi^2(x'-y), \quad (4.107)$$

for any $y, x' \in \mathbb{R}^d$. Then, since

$$0 < \frac{e^{-t} + t - 1}{t^2} < \frac{1}{2}, \quad t > 0, \quad (4.108)$$

we find

$$\begin{aligned} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} A_\varepsilon^{(1)}(\xi, y) - A_V^{(1)}(\xi, y) \right| &\leq \\ &\leq \frac{\varepsilon}{2} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \sum_{x' \in \xi} \phi^2(x'-y) e_\lambda(\phi(\cdot-y), \xi \setminus x'). \end{aligned}$$

Finally, using the Minlos formula (1.18) together with conditions (4.93)-(4.94), one can estimate

$$\begin{aligned} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} A_\varepsilon^{(1)}(\xi, y) - A_V^{(1)}(\xi, y) \right| &\leq \\ &\leq \frac{\varepsilon}{2} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \int_{\mathbb{R}^d} C dx' \phi^2(x'-y) e_\lambda(\phi(\cdot-y), \xi) \\ &\leq \frac{C}{2} e^{CC_1^\phi} \int_{\mathbb{R}^d} dx' \phi^2(x'-y) \\ &\leq \frac{C}{2} e^{CC_1^\phi} C_1^\phi C_\infty^\phi. \end{aligned}$$

Next, let us consider the l.h.s of (4.103). Since $\phi \geq 0$, we can write it as

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} \tilde{A}_\varepsilon^{(1)}(\xi, \eta, y) \right| \leq \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left[1 - e_\lambda(e^{-\varepsilon\phi(\cdot-y)}, \eta) \right] e_\lambda \left(\frac{1 - e^{-\varepsilon\phi(\cdot-y)}}{\varepsilon}, \xi \right).$$

By using the elementary inequality

$$\prod_{i=1}^n b_i - \prod_{i=1}^n a_i \leq \sum_{i=1}^n (b_i - a_i) \prod_{j=1, j \neq i}^n b_j, \quad b_i \geq a_i > 0, 1 \leq i \leq n, n \in \mathbb{N},$$

it is easy to check that

$$\left[1 - e_\lambda(e^{-\varepsilon\phi(\cdot-y)}, \eta)\right] e_\lambda\left(\frac{1 - e^{-\varepsilon\phi(\cdot-y)}}{\varepsilon}, \xi\right) \leq \sum_{x' \in \eta} \varepsilon\phi(x' - y) e_\lambda(\phi(\cdot - y), \xi), \quad (4.109)$$

for any $\xi, \eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. By using this estimate we get

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} \tilde{A}_\varepsilon^{(1)}(\xi, \eta, y) \right| \leq \varepsilon \sum_{x' \in \eta} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \phi(x' - y) e_\lambda(\phi(\cdot - y), \xi).$$

Then, taking into account hypotheses (4.93) and (4.94), one can estimate

$$\begin{aligned} \int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} \tilde{A}_\varepsilon^{(1)}(\xi, \eta, y) \right| &\leq \varepsilon e^{CC_1^\phi} \sum_{x' \in \eta} \phi(x' - y) \\ &\leq \varepsilon e^{CC_1^\phi} C_\infty^\phi |\eta|, \quad \text{for a.a. } \eta \in \Gamma_0, \end{aligned}$$

which concludes the proof of the proposition. \square

From the above proposition it follows that Assumption 4.15 is satisfied with

$$\tilde{\sigma}_0 = \frac{C}{2} e^{CC_1^\phi} C_1^\phi C_\infty^\phi, \quad \tilde{\sigma}_1 = e^{CC_1^\phi} C_\infty^\phi. \quad (4.110)$$

Now we are in the position to apply Theorem 4.17 and Lemma 4.19 to show the convergence of the Vlasov-type scaling and derive the corresponding kinetic equations. The formulation of these results for this particular model is given in the corollary below.

Corollary 4.25. *Let us assume that conditions (4.47), (4.54), (4.93) and (4.94) are satisfied. Given $C_0 > 0$ arbitrary and fixed let us consider $k_{0,\varepsilon}, k_{V,\varepsilon} \in \mathcal{K}_{C_0}^1$, $\varepsilon > 0$, with*

$$\lim_{\varepsilon \rightarrow 0} \|k_{0,\varepsilon} - k_{0,V}\|_{\mathcal{K}_{C_0}^1} = 0. \quad (4.111)$$

Then, for any $C > C_0$ there exists a time $T(C, C_0) \geq 0$ given by (4.28) such that the evolution equations (4.35) and (4.37) have unique solutions, $k_{t,\varepsilon}$ and $k_{t,V}$, respectively, in \mathcal{K}_C^1 such that

$$\lim_{\varepsilon \rightarrow 0} \|k_{t,\varepsilon} - k_{t,V}\|_{\mathcal{K}_C^1} = 0. \quad (4.112)$$

on the time interval $[0, T_1(C_0, C))$.

Moreover, given $\rho_0 \in \overline{B}_{C_0}^\infty$ and $r_0 \in L^1(\mathbb{R}^d)$, if we take

$$k_{0,V} = e_\lambda(\rho_0, \cdot) r_0 \in \mathcal{K}_{C_0}^1, \quad (4.113)$$

then we have

$$k_{t,V} = e_\lambda(\rho_t, \cdot) r_t \in \mathcal{K}_C^1, \quad (4.114)$$

provided that $\rho_t \in \overline{B}_C^\infty$ and $r_t \in L^1(\mathbb{R}^d)$, are solutions to the Vlasov equations

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m\rho_t e^{-\beta(\rho_t * V)} + b \\ \frac{\partial r_t}{\partial t} = -\langle a \rangle e^{-(\rho_t * \phi)} r_t + \left((e^{-(\rho_t * \phi)} r_t) * a \right) \end{cases}, \quad (4.115)$$

with initial conditions $\rho_t|_{t=0} = \rho_0$ and $r_t|_{t=0} = r_0$, on the time interval $[0, T_1(C, C_0))$.

Let us study the solution to the Vlasov equations (4.115). We know that condition (4.46) holds, then from Theorem 4.22 we have the following existence and uniqueness result.

Corollary 4.26. *Assume that conditions (4.91) (4.47), (4.93) and (4.94) hold. Let κ_1 and κ_2 be constant solutions to (4.77) and consider $0 \leq \rho_0 \in C_b(\mathbb{R}^d)$ and $0 \leq r_0 \in L^1(\mathbb{R}^d)$ with $\|\rho_0\|_\infty \leq \kappa_2$. Then, the system of equations (4.115) has a unique non-negative solution given by $0 \leq \rho_t \in C^1(\mathbb{R}_+; C_b(\mathbb{R}^d))$ and $0 \leq r_t \in C^1(\mathbb{R}_+; L^1(\mathbb{R}^d))$ with $\|\rho_t\|_\infty \leq \kappa_2$ for any $t > 0$.*

Moreover, if there exists a constant $C' > 0$ such that $\|r_0\|_1 \leq C'$, or $r_0(y) \leq C'$, a.a. $y \in \mathbb{R}^d$, then we have $\|r_t\|_1 \leq C'$, or $r_t(y) \leq C'$, a.a. $y \in \mathbb{R}^d$, for any $t > 0$.

Note that all the results obtained in Corollary 4.25 and 4.26 can be also extended to the interaction

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(3)}(\gamma, z) = e_\lambda(e^{-\phi(\cdot-z)}, \gamma), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d. \quad (4.116)$$

Let us just mention that in this case Lemma 4.19 leads to the following Vlasov equations (see Case III in Section 2.3.2)

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m\rho_t e^{-\beta(\rho_t * V)} + b \\ \frac{\partial r_t}{\partial t} = e^{-(\rho_t * \phi)}(r_t * a) - r_t((e^{-(\rho_t * \phi)}) * a) \end{cases}, \quad (4.117)$$

with initial conditions $\rho_t|_{t=0} = \rho_0$ and $r_t|_{t=0} = r_0$.

4.2.3.2 RW in an aggregation model of environment: Case II and IV

Let us study the interaction defined by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(2)}(\gamma, y) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - y), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d, \quad (4.118)$$

where $\lambda_0 \geq 0$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that conditions (4.93) and (4.94) hold.

As discussed in Section 3.2.3.2, for any $\varepsilon > 0$ we introduce the rescaled interaction

$$\lambda_\varepsilon^{(2)}(\gamma, y) = \lambda_0 + \varepsilon \sum_{x' \in \gamma} \phi(x' - y), \quad \gamma \in \Gamma, y \in \mathbb{R}^d. \quad (4.119)$$

In correspondence we have, see (3.160),

$$(K^{-1}\lambda_\varepsilon^{(2)}(\cdot \cup \eta, y))(\xi) = A_\varepsilon(\xi, y) + \tilde{A}_\varepsilon(\xi, \eta, y), \quad (4.120)$$

where

$$A_\varepsilon^{(2)}(\xi, y) = \lambda_0 0^{|\xi|} + \varepsilon \phi(x' - y) \mathbf{1}_{\Gamma^{(1)}}(\xi = \{x'\}) \quad (4.121)$$

and

$$\tilde{A}_\varepsilon^{(2)}(\xi, \eta, y) = \varepsilon \sum_{x' \in \eta} \phi(x' - y) 0^{|\xi|}, \quad (4.122)$$

for any $\varepsilon > 0$, $\eta, \xi \in \Gamma_0$ and $y \in \mathbb{R}^d$. Then, as $\varepsilon \rightarrow 0$ we have the following limit in $L^1(\Gamma_0, C^{|\xi|} d\lambda(\xi))$

$$\begin{aligned} A_V^{(2)}(\xi, y) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1}\lambda_\varepsilon^{(2)}(\cdot \cup \eta, y))(\xi) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1}\lambda_\varepsilon^{(2)}(\cdot, y))(\xi) \\ &= \lambda_0 0^{|\xi|} + \phi(x' - y) \mathbf{1}_{\Gamma^{(1)}}(\xi = \{x'\}), \end{aligned} \quad (4.123)$$

for a.a. $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$, see Proposition 3.49.

From Proposition 3.48 and 3.49, we also know that conditions (4.45) and (4.46) in Assumption 4.10 are satisfied with

$$\tilde{\alpha}_0 = \tilde{\alpha}'_0 = \lambda_0 + CC_1^\phi, \quad \tilde{\alpha}_1 = C_\infty^\phi. \quad (4.124)$$

It remains to check Assumption 4.15. The latter will be a consequence of the following proposition.

Proposition 4.27. *Let us assume that condition (4.94) is satisfied. Then, we have*

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} A_\varepsilon^{(2)}(\xi, y) - A_V^{(2)}(\xi, y) \right| \equiv 0 \quad (4.125)$$

and

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} \tilde{A}_\varepsilon^{(2)}(\xi, \eta, y) \right| \leq \varepsilon C_\infty^\phi |\eta|, \quad (4.126)$$

for a.a. $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$.

Proof. Identity (4.125) follows immediately by (4.121) and (4.123). Let us now show (4.126). By (4.122) we have

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} \tilde{A}_\varepsilon^{(2)}(\xi, \eta, y) \right| = \varepsilon \sum_{x' \in \eta} \phi(x' - y), \quad (4.127)$$

for a.a. $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Then, from (4.94) it follows that

$$\int_{\Gamma_0} C^{|\xi|} d\lambda(\xi) \left| \varepsilon^{-|\xi|} \tilde{A}_\varepsilon^{(2)}(\xi, \eta, y) \right| \leq \varepsilon C_\infty^\phi |\eta|, \quad (4.128)$$

which concludes the proof of the proposition. \square

According to the above proposition and Remark 4.16, we can easily see that condition (4.56) holds with

$$\tilde{\sigma}_0 \equiv 0, \quad \tilde{\sigma}_1 = C_\infty^\phi. \quad (4.129)$$

As a consequence, by Theorem 4.17 and Lemma 4.19 we can show the following result about the Vlasov-type scaling.

Corollary 4.28. *Assume that conditions (4.47), (4.54), (4.93) and (4.94) are satisfied. Let $C > C_0 > 0$ be fixed and $T_1(C_0, C)$ be given by (4.33). Suppose that $\{k_{0,\varepsilon}, k_{V,\varepsilon}\}_{\varepsilon>0} \subset \mathcal{K}_{C_0}^1$, and, moreover,*

$$\lim_{\varepsilon \rightarrow 0} \|k_{0,\varepsilon} - k_{0,V}\|_{\mathcal{K}_{C_0}^1} = 0. \quad (4.130)$$

Then, the evolution equations (4.35) and (4.37) have unique solutions in \mathcal{K}_C^1 and

$$\lim_{\varepsilon \rightarrow 0} \|k_{t,\varepsilon} - k_{t,V}\|_{\mathcal{K}_C^1} = 0, \quad (4.131)$$

on the time interval $0 \leq t < T_1(C_0, C)$. Moreover, let $\rho_0 \in \overline{B_{C_0}^\infty}$ and $r_0 \in L^1(\mathbb{R}^d)$. If $k_{0,V} = e_\lambda(\rho_0, \cdot)r_0 \in \mathcal{K}_{C_0}^1$ then $k_{t,V} = e_\lambda(\rho_t, \cdot)r_t \in \mathcal{K}_C^1$ provided that $\rho_t \in \overline{B_C^\infty}$ and $r_t \in L^1(\mathbb{R}^d)$, are solutions to system of equations

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m\rho_t e^{-\beta(\rho_t * V)} + b, & \rho_t|_{t=0} = \rho_0, \\ \frac{\partial r_t}{\partial t} = (a * [r_t(\lambda_0 + (\rho_t * \phi))]) - \langle a \rangle r_t[\lambda_0 + (\rho_t * \phi)], & r_t|_{t=0} = r_0, \end{cases} \quad (4.132)$$

on the time interval $0 \leq t < T_1(C, C_0)$.

Clearly the same analysis can be repeated for the interaction

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(4)}(\gamma, z) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - z), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d. \quad (4.133)$$

Indeed, Assumption 4.10 and 4.15 still hold with constants (4.124) and (4.129) and we can show the same result as in Corollary 4.28. The corresponding Vlasov equations are now given by (cf. case IV in Section 2.3.2)

$$\begin{cases} \frac{\partial \rho_t}{\partial t} = -m\rho_t e^{-\beta(\rho_t * V)} + b, & \rho_t|_{t=0} = \rho_0, \\ \frac{\partial r_t}{\partial t} = (r_t * a)[\lambda_0 + (\rho_t * \phi)] - r_t[\lambda_0 \langle a \rangle + ((\rho_t * \phi) * a)], & r_t|_{t=0} = r_0. \end{cases} \quad (4.134)$$

The solutions to the systems of equations (4.132) and (4.134) can be studied by using Theorem 4.22. Note that for both interactions (4.118) and (4.133), under condition (4.94), we can show the same result as in Corollary 4.26.

Chapter 5

Random walks in a Kawasaki model of environment via generating functionals

We present an alternative approach to the study of the statistical dynamics of RWREs described in Chapter 2 in terms of the corresponding Bogoliubov generating functionals. A description of this approach for interacting particle systems in continuum can be found in [43, 58, 61, 63].

First we introduce the Bogoliubov functionals associated to the states of the considered models. In the study of interacting particle systems it is natural to consider entire functionals allowing us to recover the definition of correlation functions given in Section 1.2.2. Then, we reformulate the evolution of correlation functions of RWREs in terms of the corresponding generating functionals. In this case, the statistical dynamics of the system will be described by an evolution equation for holomorphic functionals over an infinite-dimensional space. Moreover, we apply this new formalism to study the Vlasov-type scaling, introduced in Section 2.3 for correlation functions, through Bogoliubov functionals.

For concreteness, we employ this general method to study some models of RWREs, in the case where the environment is formed by infinitely many particles which jump according to a Kawasaki dynamics, see e.g. [43]. However, the Bogoliubov functional approach can be also applied to other types of RE such as Glauber-type of environment, see e.g. [61]. Note that the analysis of all these cases is carried out by using the Ovsjannikov's method in the scale of Banach space presented in Chapter 4.

5.1 The Bogoliubov generating functionals

The Bogoliubov generating functionals were originally introduced by N.N. Bogoliubov [14] to study correlation functions in statistical mechanics. Over the years, these functionals and their generalizations found many others application in classical and quantum statistical mechanics, theory of point process and so on. We refer to [96] for an extensive review of the subject.

In this section we define and characterize the Bogoliubov functional for a one-particle system interacting with an environment consisting of infinitely many particles. For the moment we do not specify any dynamics for such model. In what follows, we use the general framework introduced in Chapter 1. In particular, we restrict our attention to the so-called bounded states, namely to measures $\mu \in \mathcal{M}^1(\Gamma \times \mathbb{R}^d)$, see (1.31)-(1.32). Bogoliubov functionals for interacting particle systems in continuum have been studied in [79] and [62] for one-component and two-component systems, respectively. These definitions can be modified to fit our case.

Definition 5.1. The Bogoliubov generating functional Z_μ corresponding to a finite measure $\mu \in \mathcal{M}^1(\Gamma \times \mathbb{R}^d)$ is a functional defined at each pair of $\mathcal{B}(\mathbb{R}^d)$ -measurable functions (θ, ψ) by

$$Z_\mu(\theta, \psi) = \int_{\mathbb{R}^d} \psi(y) \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) d\mu(\gamma, y), \quad (5.1)$$

provided the right-hand side exists.

From Definition 5.1 it is clear that the set of (θ, ψ) for which Z_μ is well-defined depends on the measure μ . Conversely, this set reflects properties of the underlying measure. Let $\mathcal{M}_{\text{fexp}}^1(\Gamma \times \mathbb{R}^d)$ denote the set of all measures $\mu \in \mathcal{M}^1(\Gamma \times \mathbb{R}^d)$ which have finite local exponential moments (w.r.t. Γ -variables), namely such that

$$\int_{\mathbb{R}^d} \int_{\Gamma} e^{\alpha|\gamma\Lambda|} d\mu(\gamma, y) < \infty,$$

for all $\alpha > 0$ and all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. Then, for $\mu \in \mathcal{M}_{\text{fexp}}^1(\Gamma \times \mathbb{R}^d)$ one can easily see that the Bogoliubov functional Z_μ is well defined, for instance, for all pairs of functions (θ, ψ) such that ψ is bounded and θ is bounded with bounded support. The converse is also true and follows from the following identity

$$\int_{\mathbb{R}^d} \int_{\Gamma} e^{\alpha|\gamma\Lambda|} d\mu(\gamma, y) = Z_\mu((e^\alpha - 1) \mathbb{1}_\Lambda, 1), \quad \alpha > 0, \Lambda \in \mathcal{B}_b(\mathbb{R}^d).$$

Throughout this chapter we consider Bogoliubov functionals defined on the whole space $L^1(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d)$. Furthermore, we assume that the following representation holds

$$Z_\mu(\theta, \psi) = \int_{\mathbb{R}^d} dy \psi(y) B_\mu(\theta, y), \quad \theta \in L^1(\mathbb{R}^d), \psi \in L^\infty(\mathbb{R}^d), \quad (5.2)$$

where for a.a. $y \in \mathbb{R}^d$, $B_\mu(\cdot; y)$ is an entire functional on $L^1(\mathbb{R}^d)$.

Remark 5.2. Note that representation (5.2) allows us to identify the generating functional $B_\mu(\theta, y)$ as the functional derivative of $Z_\mu(\theta, \psi)$ w.r.t $\psi(y)$, i.e.

$$B_\mu(\theta, y) := \frac{\delta Z_\mu(\theta, \psi)}{\delta \psi(y)}, \quad (5.3)$$

for any $\theta \in L^1(\mathbb{R}^d)$, $\psi \in L^\infty(\mathbb{R}^d)$ and a.a. $y \in \mathbb{R}^d$.

For a comprehensive review of the general theory of holomorphic functionals we refer to [7, 32]. Here we just recall that for a.a. $y \in \mathbb{R}^d$, the functional $B_\mu(\cdot; y) : L^1(\mathbb{R}^d) \rightarrow \mathbb{C}$ is said to be entire on $L^1(\mathbb{R}^d)$ if it is locally bounded and for all $\theta, \theta_0 \in L^1(\mathbb{R}^d)$ the mapping $\mathbb{C} \ni z \mapsto B_\mu(\theta_0 + z\theta; y) \in \mathbb{C}$ is entire. Thus, at each $\theta_0 \in L^1(\mathbb{R}^d)$ the entire functional $B_\mu(\cdot; y)$ can be expressed in terms of its Taylor expansion,

$$B_\mu(\theta_0 + z\theta; y) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n B_\mu(\theta_0; y; \dots, \theta), \quad z \in \mathbb{C}, \theta \in L^1(\mathbb{R}^d), \quad (5.4)$$

where $d^n B_\mu(\theta_0; y; \theta, \dots, \theta)$ denotes the differential of order n w.r.t. θ evaluated in θ_0 . The main properties of these differentials are specified in the theorem below. For a proof we refer the reader to [97, Theorem 9.2], where an analogous result has been showed.

Lemma 5.3. For a.a. $y \in \mathbb{R}^d$, let $B_\mu(\cdot; y)$ be an entire functional on $L^1(\mathbb{R}^d)$. Then, for each $n \in \mathbb{N}$ the differential $d^n B_\mu(\theta_0; y; \theta, \dots, \theta)$, $\theta_0 \in L^1(\mathbb{R}^d)$, is defined by a symmetric kernel in $L^\infty((\mathbb{R}^d)^n)$ denoted by

$$\frac{\delta^n B_\mu(\theta_0; y)}{\delta \theta_0(x_1) \dots \delta \theta_0(x_n)} \quad (5.5)$$

and called the functional derivative of n th order of $B_\mu(\theta_0; y)$ w.r.t. θ_0 . More precisely,

$$d^n B_\mu(\theta_0; y; \theta_1, \dots, \theta_n) = \left. \frac{\partial^n}{\partial z_1 \dots \partial z_n} B_\mu \left(\theta_0 + \sum_{i=1}^n z_i \theta_i; y \right) \right|_{z_1 = \dots = z_n = 0}$$

$$= \int_{\mathbb{R}^d} dx_1 \dots \int_{\mathbb{R}^d} dx_n \left(\prod_{i=1}^n \theta_i(x_i) \right) \frac{\delta^n B_\mu(\theta_0; y)}{\delta\theta_0(x_1) \dots \delta\theta_0(x_n)}, \quad (5.6)$$

for all $\theta_1, \dots, \theta_n \in L^1(\mathbb{R}^d)$. Furthermore, for any $r > 0$ one has

$$\left\| \frac{\delta B_\mu(\theta_0; y)}{\delta\theta_0(\cdot)} \right\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{r} \sup_{\|\theta'\|_{L^1} \leq r} |B_\mu(\theta_0 + \theta'; y)| \quad (5.7)$$

and, for $n \geq 2$,

$$\left\| \frac{\delta^n B_\mu(\theta_0; y)}{\delta\theta_0(\cdot) \dots \delta\theta_0(\cdot)} \right\|_{L^\infty((\mathbb{R}^d)^n)} \leq n! \left(\frac{1}{r}\right)^n \sup_{\|\theta'\|_{L^1} \leq r} |B_\mu(\theta_0 + \theta'; y)|. \quad (5.8)$$

Remark 5.4. Note that the fact $B_\mu(\cdot; y)$, a.a. $y \in \mathbb{R}^d$, is entire on $L^1(\mathbb{R}^d)$ does not ensure that the supremum in (5.7) and (5.8) is always finite. For simplicity, we assume that this entire functional is also bounded in $L^1(\mathbb{R}^d)$, that is,

$$\sup_{\|\theta'\|_{L^1} \leq r} |B_\mu(\theta_0 + \theta', y)| < \infty, \quad (5.9)$$

for all $r > 0$, $\theta_0 \in L^1(\mathbb{R}^d)$ and a.a. $y \in \mathbb{R}^d$.

According to Lemma 5.3 and the Taylor expansion (5.4), the Bogoliubov functional Z_μ can be written in the form

$$\begin{aligned} Z_\mu(\theta_0 + \theta, \psi) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} dy \psi(y) \int_{\mathbb{R}^d} dx_1 \dots \int_{\mathbb{R}^d} dx_n \left(\prod_{i=1}^n \theta_i(x_i) \right) \frac{\delta^n B_\mu(\theta_0, y)}{\delta\theta_0(x_1) \dots \delta\theta_0(x_n)} \\ &= \int_{\mathbb{R}^d} dy \psi(y) \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) (D^{|\eta|} B_\mu)(\theta_0, y; \eta), \end{aligned} \quad (5.10)$$

where we introduced the notation

$$(D^{|\eta|} B_\mu)(\theta_0, y; \eta) := \frac{\delta^n B_\mu(\theta_0, y)}{\delta\theta_0(x_1) \dots \delta\theta_0(x_n)}, \quad (5.11)$$

for $\eta = \{x_1, \dots, x_n\} \in \Gamma^{(n)}$, $n \in \mathbb{N}$ and a.a. $y \in \mathbb{R}^d$. In the simple case where $\eta = \{x\} \in \Gamma^{(1)}$, we simply write $(D^{|\eta|} B_\mu)(\theta_0, y; \eta) := (DB_\mu)(\theta_0, y; x)$.

Having in mind the representation in (5.10), Lemma 5.3 yields the following result.

Corollary 5.5. *Let Z_μ be a Bogoliubov functional on the space $L^1(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d)$ corresponding to some measure $\mu \in \mathcal{M}^1(\Gamma \times \mathbb{R}^d)$. If we assume that for a.a. $y \in \mathbb{R}^d$ the functional $B_\mu(\cdot; y)$ is entire on $L^1(\mathbb{R}^d)$, then the measure μ is locally absolutely continuous with respect to the Poisson-type measure $\tilde{\pi} = \pi \otimes dy$. Namely, for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ the measure $\mu^\Lambda = \mu \circ (\tilde{p}_\Lambda)^{-1}$ is absolutely continuous with respect to $\tilde{\pi}^\Lambda = (\pi \circ p_\Lambda^{-1}) \otimes dy$. In particular, for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ one has*

$$\frac{d\mu^\Lambda}{d\pi^\Lambda}(\gamma, y) = e^{\sigma(\Lambda)} (D^{|\gamma|} B_\mu)(-\mathbf{1}_\Lambda, y; \gamma), \quad \tilde{\pi}^\Lambda - \text{a.a. } (\gamma, y) \in \Gamma(\Lambda) \times \mathbb{R}^d,$$

where $\sigma(\Lambda) = \int_{\mathbb{R}^d} dx \mathbf{1}_\Lambda(x)$.

Remark 5.6. Note that if the entire functional $B_\mu(\cdot; y)$, a.a. $y \in \mathbb{R}^d$, is of bounded type in $L^1(\mathbb{R}^d)$, i.e. (5.9) holds, then for each $r > 0$ there exists a constant $C \geq 0$ such that

$$\left| \frac{d\mu^\Lambda}{d\pi^\Lambda}(\gamma, y) \right| \leq e^{\sigma(\Lambda)} C |\gamma|! \left(\frac{e}{r}\right)^{|\gamma|},$$

for $\tilde{\pi}^\Lambda$ -a.a. $(\gamma, y) \in \Gamma(\Lambda) \times \mathbb{R}^d$.

Proof of Corollary 5.5. Let us consider functions $\theta \in L^1(\mathbb{R}^d)$ with support contained in some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $\psi \in L^\infty(\mathbb{R}^d)$. Then, the Bogoliubov functional Z_μ can be written as

$$\begin{aligned} Z_\mu(\theta, \psi) &= Z_\mu(-\mathbf{1}_\Lambda + (\theta + \mathbf{1}_\Lambda), \psi) \\ &= \int_{\mathbb{R}^d} dy \psi(y) \int_{\Gamma_\Lambda} d\pi^\Lambda(\eta) \prod_{x \in \gamma} (1 + \theta(x)) e^{\sigma(\Lambda)} (D^{|\eta|} B_\mu)(-\mathbf{1}_\Lambda, y; \eta), \end{aligned} \quad (5.12)$$

where in the second inequality we apply (5.10) with $\theta_0 = -\mathbf{1}_\Lambda$ and identity (1.14).

On the other hand, one can easily see that for functions $\theta \in L^1(\mathbb{R}^d)$ with support contained in some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, the definition of a Bogoliubov functional Z_μ reduces to the Bogoliubov functional Z_{μ^Λ} , namely

$$Z_\mu(\theta, \psi) = \int_{\mathbb{R}^d} \psi(y) \int_{\Gamma(\Lambda)} \prod_{x \in \gamma} (1 + \theta(x)) d\mu^\Lambda(\gamma, y), \quad \psi \in L^\infty(\mathbb{R}^d). \quad (5.13)$$

Therefore, by comparing (5.12) with (5.13) we find

$$\begin{aligned} \int_{\mathbb{R}^d} \psi(y) \int_{\Gamma(\Lambda)} \prod_{x \in \gamma} (1 + \theta(x)) d\mu^\Lambda(\gamma, y) &= \\ \int_{\mathbb{R}^d} dy \psi(y) \int_{\Gamma_\Lambda} d\pi^\Lambda(\eta) \prod_{x \in \gamma} (1 + \theta(x)) e^{\sigma(\Lambda)} (D^{|\eta|} B_\mu)(-\mathbf{1}_\Lambda, y; \eta), \end{aligned} \quad (5.14)$$

for all functions $\theta \in L^1(\mathbb{R}^d)$ with support contained in some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $\psi \in L^\infty(\mathbb{R}^d)$. Then the statement follows by a monotone class argument. \square

As discussed at the beginning of the section, a Bogoliubov functional Z_μ that is well defined on the whole space $L^1(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d)$ is necessarily associated to measures $\mu \in \mathcal{M}_{\text{fexp}}^1$. On the other hand, for such measures one can define the corresponding correlation measure ρ_μ , see Remark 1.26. Consequently, taking into account Proposition 1.27 and Example 1.19, we can rewrite the Bogoliubov functional Z_μ in terms of the measure ρ_μ . More precisely, we have

$$\begin{aligned} Z_\mu(\theta, \psi) &= \int_{\mathbb{R}^d} \int_{\Gamma} (K\psi(y) e_\lambda(\theta, \cdot))(\gamma) d\mu(\gamma, y) \\ &= \int_{\mathbb{R}^d} \psi(y) \int_{\Gamma_0} e_\lambda(\theta, \eta) d\rho_\mu(\eta, y), \quad \theta \in L^1(\mathbb{R}^d), \psi \in L^\infty(\mathbb{R}^d). \end{aligned} \quad (5.15)$$

In this new formalism Lemma 5.3 leads to the following result (cf. Lemma 1.28).

Proposition 5.7. *Let Z_μ be a Bogoliubov functional on the space $L^1(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d)$. Suppose that the $B_\mu(\cdot; y)$ is an entire functional of bounded type on $L^1(\mathbb{R}^d)$, for a.a. $y \in \mathbb{R}^d$. Then, the measure $\rho_\mu = K^*\mu$ is absolutely continuous with respect to $\tilde{\lambda}_\sigma = \lambda_\sigma \otimes dy$ and the Radon-Nikodym derivative, or correlation functional, is given by*

$$k_\mu(\eta, y) := \frac{d\rho_\mu}{d\tilde{\lambda}_\sigma}(\eta, y) = (D^{|\eta|} B_\mu)(0, y; \eta), \quad (5.16)$$

for $\tilde{\lambda}$ -a.a. $(\eta, y) \in \Gamma_0 \times \mathbb{R}^d$. Furthermore, for each $r > 0$ there exists a constant $C \geq 0$ such that

$$|k_\mu(\eta, y)| \leq C |\eta|! \left(\frac{e}{r}\right)^{|\eta|}, \quad (5.17)$$

for $\tilde{\lambda}$ -a.a. $(\eta, y) \in \Gamma_0 \times \mathbb{R}^d$.

Remark 5.8. Consider a Bogoliubov functional Z_μ defined as in the proposition above. From (5.17) it follows that the correlation functionals associated to it satisfy a generalized Ruelle bound, that is, for any $\varepsilon \in [0, 1]$ and $r > 0$ there exists some constant $C \geq 0$ such that

$$|k_\mu(\eta, y)| \leq C (|\eta|!)^{1-\varepsilon} \left(\frac{e}{r}\right)^{|\eta|}, \quad \tilde{\lambda} - \text{a.a. } (\eta, y) \in \Gamma_0 \times \mathbb{R}^d. \quad (5.18)$$

Clearly if condition (5.18) holds for $\varepsilon = 1$ and for some $r > 0$, then it coincides with the classical Ruelle bound, see (2.33). In our case, (5.18) holds for $\varepsilon = 0$.

Proof of Proposition 5.7. Using representation (5.10) for $\theta_0 = 0$, we can rewrite the Bogoliubov functional Z_μ in the following form

$$Z_\mu(\theta, \psi) = \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) \psi(y) e_\lambda(\theta, \eta) (D^{|\eta|} B_\mu)(0, y; \eta). \quad (5.19)$$

Comparing the formula above with (5.15) we can identify $k_\mu(\eta, y)$ with $(D^{|\eta|} B_\mu)(0, y; \eta)$. Then, in order to conclude the proof we can just note that, since B_μ is of bounded type, from Lemma 5.3 we have

$$\left| (D^{|\eta|} B_\mu)(0, y; \eta) \right| \leq C |\eta|! \left(\frac{e}{r}\right)^{|\eta|}, \quad (5.20)$$

$\tilde{\lambda}$ -a.a. $(\eta, y) \in \Gamma_0 \times \mathbb{R}^d$ for some $C \geq 0$ depending on $r > 0$. \square

Having in mind (5.15), Proposition 5.7 yields to a description of the Bogoliubov functional in terms of the correlation functionals k_μ :

$$Z_\mu(\theta, \psi) = \int_{\mathbb{R}^d} dy \psi(y) \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_\mu(\eta, y), \quad (5.21)$$

for all $\theta \in L^1(\mathbb{R}^d)$ and $\psi \in L^\infty(\mathbb{R}^d)$.

Let us now specify the connection between the Bogoliubov functional Z_μ and the corresponding correlation functionals k_μ . By (5.3) and (5.11), the Radon-Nykodym derivative (5.16) can be written explicitly as

$$\begin{aligned} k_\mu(\eta, y) &:= k_\mu^{(n)}(\eta, y) = \frac{\delta^{(n)} B_\mu(\theta, y)}{\delta\theta(x_1) \dots \delta\theta(x_n)} \Big|_{\theta=0} \\ &= \frac{\delta^{(n,1)} Z_\mu(\theta, \psi)}{\delta\theta(x_1) \dots \delta\theta(x_n) \delta\psi(y)} \Big|_{(\theta, \psi)=(0,0)}, \end{aligned} \quad (5.22)$$

for $\eta = \{x_1, \dots, x_n\} \in \Gamma^{(n)}$, $n \in \mathbb{N}$, and a.a. $y \in \mathbb{R}^d$. Hence, we can see that the Bogoliubov functional Z_μ is the generating functional for the correlation functions $k_\mu^{(n)}$. Indeed, this was the reason why N. N. Bogoliubov introduced these functionals, see e.g. [14]. However, it is worth noting that, for the present model of a particle moving in RE, all the information about the correlation functionals $k_\mu^{(n)}$ is included in the generating functionals B_μ . Indeed, by using the representation (5.21) the latter can be written as

$$B_\mu(\theta, y) := \frac{\delta Z_\mu(\theta, \psi)}{\delta\psi(y)} = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_\mu(\eta, y), \quad \theta \in L^1(\mathbb{R}^d), \quad (5.23)$$

for a.a. $y \in \mathbb{R}^d$. For this reason, the generating functionals B_μ turns out to be the main object of investigation in the study of evolution of correlation functions in the models of RWREs.

Before concluding this section, let us show some useful relation between functional derivatives of the generating functional B_μ .

Proposition 5.9. *Assume that for a.a. $y \in \mathbb{R}^d$, $B_\mu(\cdot; y)$ is an entire functional of bounded type on $L^1(\mathbb{R}^d)$. Then the following relation between variational derivatives holds*

$$(D^{|\eta|} B_\mu)(\theta; y; \eta) = \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\theta, \xi) k_\mu(\eta \cup \xi, y) \quad (5.24)$$

and, more generally,

$$(D^{|\eta|} B_\mu)(\theta_1 + \theta_2; y; \eta) = \int_{\Gamma_0} d\lambda(\xi) (D^{|\eta \cup \xi|} B_\mu)(\theta_1; y; \eta \cup \xi) e_\lambda(\theta_2, \xi), \quad (5.25)$$

for any $\theta_1, \theta_2 \in L^1(\mathbb{R}^d)$ and $\tilde{\lambda}$ -a.a. $(\eta, y) \in \Gamma_0 \times \mathbb{R}^d$.

Proof. First of all let us note that (5.24) is a special case of (5.25) when $\theta_1 = 0$ and $\theta_2 = \theta$. Therefore, we only need to show (5.25).

For $\theta_0 := \theta_1 + \theta_2$ formula (5.10) reads

$$Z_\mu(\theta_1 + \theta_2 + \theta, \psi) = \int_{\mathbb{R}^d} dy \psi(y) \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) (D^{|\eta|} B_\mu)(\theta_1 + \theta_2, y; \eta), \quad (5.26)$$

for any $\theta_1, \theta_2, \theta \in L^1(\mathbb{R}^d)$ and $\psi \in L^\infty(\mathbb{R}^d)$. On the other hand, if we replace θ_0 by θ_1 and θ by $\theta_2 + \theta$, the same formula gives

$$Z_\mu(\theta_1 + \theta_2 + \theta, \psi) = \int_{\mathbb{R}^d} dy \psi(y) \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta_2 + \theta, \eta) (D^{|\eta|} B_\mu)(\theta_1, y; \eta). \quad (5.27)$$

Note that for all $\eta \in \Gamma_0$ one has

$$\begin{aligned} e_\lambda(\theta_2 + \theta, \eta) &= \prod_{x \in \eta} (\theta_2(x) + \theta(x)) \\ &= \sum_{\xi \subset \eta} \left(\prod_{x \in \xi} \theta(x) \right) \left(\prod_{x' \in \eta \setminus \xi} \theta_2(x') \right) = \sum_{\xi \subset \eta} e_\lambda(\theta, \xi) e_\lambda(\theta_2, \eta \setminus \xi). \end{aligned}$$

Then, by applying the Minlos formula (1.18) to (5.27) we find

$$\begin{aligned} Z_\mu(\theta_1 + \theta_2 + \theta, \psi) &= \int_{\mathbb{R}^d} dy \psi(y) \int_{\Gamma_0} d\lambda(\eta) \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\theta, \xi) \times \\ &\quad e_\lambda(\theta_2, \eta) (D^{|\eta \cup \xi|} B_\mu)(\theta_1, y; \eta \cup \xi). \end{aligned} \quad (5.28)$$

Finally, by comparing this expression with (5.26) we obtain the desired result. \square

Remark 5.10. Given $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, for $\theta_1 = -\mathbf{1}_\Lambda$ and $\theta_2 = \mathbf{1}_\Lambda$ identity (5.25) yields

$$k_\mu(\eta, y) = (D^{|\eta|} B_\mu)(0, y; \eta) = \int_{\Gamma_\Lambda} d\lambda(\xi) (D^{|\eta \cup \xi|} B_\mu)(-\mathbf{1}_\Lambda; y; \eta \cup \xi).$$

Then, from Corollary 5.5 we obtain

$$k_\mu(\eta, y) = \int_{\Gamma(\Lambda)} \frac{d\mu^\Lambda}{d\tilde{\pi}_\sigma^\Lambda}(\gamma \cup \eta, y) \pi_\sigma^\Lambda(d\gamma), \quad \tilde{\lambda}\text{-a.a. } (\eta, y) \in \Gamma_0 \times \mathbb{R}^d.$$

In other words, the definition of correlation functionals given by (5.16) is equivalent to that stated in Lemma 1.28.

5.2 Random Walk in a Kawasaki model of environment

Let us consider the RW of a tagged particle jumping in \mathbb{R}^d and interacting with an infinite system of other hopping particles. For such a model, on the phase space $\Gamma(\mathbb{R}^d) \times \mathbb{R}^d$ we introduce a stochastic dynamics defined by the Markov pregenerator, $F \in K(B_{\text{bs}}(\Gamma_0 \times \mathbb{R}^d))$,

$$(LF)(\gamma, y) = (L_{RE}F)(\gamma, y) + (L_{RW}F)(\gamma, y), \quad (5.29)$$

where L_{RW} is defined by (2.10), whereas L_{RE} is the pregenerator of the Kawasaki dynamics given by (2.6) with, see e.g. [43],

$$c(x, x', \gamma) := \tilde{c}(x - x') e^{-\sum_{x'' \in \gamma} V(x' - x'')}, \quad (5.30)$$

for any $x, x' \in \mathbb{R}^d \setminus \gamma$ and $\gamma \in \Gamma$. In what follows, we assume that $\tilde{c}, V : \mathbb{R}^d \rightarrow \mathbb{R}$ are two non-negative even functions which are integrable in \mathbb{R}^d . For convenience, we denote

$$\langle \tilde{c} \rangle := \int_{\mathbb{R}^d} dx \tilde{c}(x) < \infty \quad (5.31)$$

and

$$C_V^1 := \int_{\mathbb{R}^d} dx V(x) < \infty. \quad (5.32)$$

In Section 2.2 we have seen that the evolution of states $\mu_t \in \mathcal{M}_{\text{fm}}^1(\Gamma \times \mathbb{R}^d)$ for models of RWRE can be reformulated in terms of the corresponding correlation functional $k_t := k_{\mu_t}$, $t \geq 0$. This leads to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} k_t(\eta, y) = (\widehat{L}_{RE}^* k_t)(\eta, y) + (\widehat{L}_{RW}^* k_t)(\eta, y) \\ k_t(\eta, y)|_{t=0} = k_0 \end{cases}, \quad \eta \in \Gamma_0, y \in \mathbb{R}^d, \quad (5.33)$$

where the generator $\widehat{L}^* = \widehat{L}_{RE}^* + \widehat{L}_{RW}^*$ is defined by (2.26) and (2.28).

Identity (5.21) allows us to express the dynamics of correlation functionals k_t in terms of the Bogoliubov functional $Z_t := Z_{\mu_t}$ associated to the measure μ_t , provided that this functional exists. Informally, by using (2.29), for all $\theta \in L^1(\mathbb{R}^d)$, $\psi \in L^\infty(\mathbb{R}^d)$ and $t \geq 0$ we can write

$$\begin{aligned} \frac{\partial}{\partial t} Z_t(\theta, \psi) &= \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) \psi(y) \left(\frac{\partial}{\partial t} k_t(\eta, y) \right) \\ &= \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) \psi(y) [(\widehat{L}_{RE}^* k_t)(\eta, y) + (\widehat{L}_{RW}^* k_t)(\eta, y)] \\ &= \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) [(\widehat{L}_{RE} e_\lambda(\theta, \eta) \psi)(\eta, y) + (\widehat{L}_{RW} e_\lambda(\theta, \eta) \psi)(\eta, y)] k_t(\eta, y). \end{aligned}$$

Then, if we introduce the operator $\tilde{L} := \tilde{L}_{RE} + \tilde{L}_{RW}$ such that

$$(\tilde{L}_{RE} Z_t)(\theta, \psi) := \int_{\Gamma_0} d\lambda(\eta) \int_{\mathbb{R}^d} dy (\widehat{L}_{RE} e_\lambda(\theta) \psi)(\eta, y) k_t(\eta, y) \quad (5.34)$$

and

$$(\tilde{L}_{RW} Z_t)(\theta, \psi) := \int_{\Gamma_0} d\lambda(\eta) \int_{\mathbb{R}^d} dy (\widehat{L}_{RW} e_\lambda(\theta) \psi)(\eta, y) k_t(\eta, y), \quad (5.35)$$

the evolution of the Bogoliubov functional Z_t is obtained as the solution of the equation

$$\left(\frac{\partial}{\partial t} Z_t \right)(\theta, \psi) = (\tilde{L} Z_t)(\theta, \psi) = (\tilde{L}_{RE} Z_t)(\theta, \psi) + (\tilde{L}_{RW} Z_t)(\theta, \psi), \quad (5.36)$$

for all $\theta \in L^1(\mathbb{R}^d)$, $\psi \in L^\infty(\mathbb{R}^d)$ and $t \geq 0$.

As already pointed out in Section 5.1, we do not study the Bogoliubov functional Z_t directly, but its first-order derivative in ψ , namely the generating functional $B_t(\cdot, y) := B_{\mu_t}(\cdot, y)$, a.a. $y \in \mathbb{R}^d$, on $L^1(\mathbb{R}^d)$, see identity (5.23). According to representation (5.2), the evolution equation for B_t , $t \geq 0$, can be written as

$$\begin{cases} \frac{\partial}{\partial t} B_t(\theta, y) = (\tilde{L}'_{RE} B_t)(\theta, y) + (\tilde{L}'_{RW} B_t)(\theta, y) \\ B_t(\theta, y)|_{t=0} = B_0(\theta, y) \end{cases}, \quad (5.37)$$

where the operators \tilde{L}'_{RE} and \tilde{L}'_{RW} are defined by the identities

$$(\tilde{L}'_{RE} Z_t)(\theta, \psi) = \int_{\mathbb{R}^d} dy \psi(y) (\tilde{L}'_{RE} B_t)(\theta, y) \quad (5.38)$$

and

$$(\tilde{L}'_{RW} Z_t)(\theta, \psi) = \int_{\mathbb{R}^d} dy \psi(y) (\tilde{L}'_{RW} B_t)(\theta, y), \quad (5.39)$$

for any $\theta \in L^1(\mathbb{R}^d)$ and $\psi \in L^\infty(\mathbb{R}^d)$.

In the case of a RW in Kawasaki model of RE associated to the Markov generator (5.29): the operator \tilde{L}'_{RE} is given by, see e.g. [43, 60],

$$\begin{aligned} (\tilde{L}'_{RE} B)(\theta, y) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dx' \tilde{c}(x-x') e^{-V(x-x')} (\theta(x') - \theta(x)) \times \\ \frac{\delta B(\theta e^{-V(\cdot-x')} + (e^{-V(\cdot-x')} - 1), y)}{\delta \theta(x)}, \end{aligned} \quad (5.40)$$

while the explicit form of the \tilde{L}'_{RW} will be computed in the following proposition.

Proposition 5.11. *Let us consider the operator \hat{L}_{RW} defined by (2.38). Suppose that $B(\cdot; y)$ is an entire functional of bounded type on $L^1(\mathbb{R}^d)$, for a.a. $y \in \mathbb{R}^d$. Then, the following formula holds*

$$\begin{aligned} (\tilde{L}'_{RW} B)(\theta, y) = \\ = \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta+1, \eta) \left[(K^{-1} \lambda_{\text{int}}(\cdot, z, y))(\eta) (D^{|\eta|} B)(\theta, z; \eta) - \right. \\ \left. (K^{-1} \lambda_{\text{int}}(\cdot, y, z))(\eta) (D^{|\eta|} B)(\theta, y; \eta) \right], \end{aligned} \quad (5.41)$$

provided that the right-hand side makes sense.

Remark 5.12. It may be useful to reformulate the result above in some special case that appears in concrete applications. In particular, if

$$(K^{-1} \lambda_{\text{int}}(\cdot, y, z))(\eta) = e_\lambda(\ell_{y,z}(\cdot), \eta), \quad \eta \in \Gamma_0, y, z \in \mathbb{R}^d,$$

for some measurable function $\ell_{y,z} : \mathbb{R}^d \rightarrow \mathbb{R}$, $y, z \in \mathbb{R}^d$. Then, according to formula (5.10), the operator \tilde{L}'_{RW} reduces to

$$(\tilde{L}'_{RW} B)(\theta, y) = \int_{\mathbb{R}^d} dz a(y-z) [B(\theta + (1+\theta)\ell_{z,y}(\cdot), z) - B(\theta + (1+\theta)\ell_{y,z}(\cdot), y)], \quad (5.42)$$

for all $\theta \in L^1(\mathbb{R}^d, dx)$ and $y \in \mathbb{R}^d$. It is worth noting that in contrast to (5.41), in this case the operator \tilde{L}'_{RW} doesn't depend on any functional derivative of B .

Proof of Proposition 5.11. Let us derive the expression for the operator \tilde{L}_{RW} defined by (5.35), that is

$$(\tilde{L}_{RW}Z)(\theta, \psi) = \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) (\hat{L}_{RW}e_\lambda(\theta)\psi)(\eta, y) k(\eta, y),$$

for all $\theta \in L^1(\mathbb{R}^d)$ and $\psi \in L^\infty(\mathbb{R}^d)$. From (2.40) and (2.41) we have

$$(\hat{L}_{RW}e_\lambda(\theta)\psi)(\eta, y) = \int_{\mathbb{R}^d} dz a(y-z) [A(\cdot, y, z) \star (e_\lambda(\theta, \cdot)\psi(z) - e_\lambda(\theta, \cdot)\psi(y))] (\eta).$$

Due to the linearity of the \star -convolution, by using (1.27) we can write

$$\begin{aligned} (\hat{L}_{RW}e_\lambda(\theta)\psi)(\eta, y) &= \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) [A(\cdot, y, z) \star e_\lambda(\theta, \cdot)] (\eta) \\ &= \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) \sum_{\xi \subset \eta} A(\xi, y, z) e_\lambda(\theta + 1, \xi) e_\lambda(\theta, \eta \setminus \xi). \end{aligned}$$

Consequently,

$$\begin{aligned} (\tilde{L}_{RW}Z)(\theta, \psi) &= \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\eta) \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) \times \\ &\quad \sum_{\xi \subset \eta} A(\xi, y, z) e_\lambda(\theta + 1, \xi) e_\lambda(\theta, \eta \setminus \xi) k(\eta, y). \end{aligned}$$

Next we can apply the Minlos formula 1.18 to get

$$\begin{aligned} (\tilde{L}_{RW}Z)(\theta, \psi) &= \int_{\mathbb{R}^d} dy \int_{\Gamma_0} d\lambda(\xi) \int_{\mathbb{R}^d} dz e_\lambda(\theta + 1, \xi) a(y-z) (\psi(z) - \psi(y)) \times \\ &\quad A(\xi, y, z) \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k(\xi \cup \eta, y). \end{aligned}$$

Having in mind (5.2), by using identity (5.24) we have

$$\begin{aligned} (\tilde{L}_{RW}Z)(\theta, \psi) &= \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) \times \\ &\quad \int_{\Gamma_0} d\lambda(\xi) A(\xi, y, z) e_\lambda(\theta + 1, \xi) (D^{|\eta|}B)(\theta, y; \eta), \end{aligned}$$

and (5.41) follows. \square

Remark 5.13. Let us note that the Cauchy problems (5.33) and (5.37) are closely connected to each other. More precisely, if k_t , $t \geq 0$, is a solution to (5.33) then the functional

$$B_t(\theta, y) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_t(\eta, y), \quad \theta \in L^1(\mathbb{R}^d), y \in \mathbb{R}^d,$$

solves (5.37). Conversely, given a solution B_t , $t \geq 0$, of (5.37) one can construct the measurable function

$$k_t(\eta, y) = (D^{|\eta|}B_t)(0, y; \eta), \quad \eta \in \Gamma_0, y \in \mathbb{R}^d, \quad (5.43)$$

which solves (5.33).

The next sections are devoted to the study of solutions to the Cauchy problem (5.37) on a proper Banach space. According to the considerations above, the nature of this space depends directly on the class of correlation functions we are interested in. In this chapter we consider correlation functionals from the Banach space \mathcal{K}_C^1 defined by (2.32). Let us recall that every $k \in \mathcal{K}_C^1$ is a measurable function such that

$$|k(\eta, y)| \leq M(y)C^{|\eta|}, \quad \tilde{\lambda} - \text{a.a. } \eta \in \Gamma_0, y \in \mathbb{R}^d,$$

where $C > 0$ and $M(\cdot)$ is a positive and integrable function on \mathbb{R}^d . As a consequence, by using (5.23) together with (1.24), one has

$$\begin{aligned} B(\theta, y) &\leq M(y) \int_{\Gamma_0} C^{|\eta|} d\lambda(\eta) e_\lambda(\theta, \eta) \\ &\leq M(y) e^{C\|\theta\|_1}, \quad \theta \in L^1(\mathbb{R}^d), \text{ a.a. } y \in \mathbb{R}^d. \end{aligned}$$

This estimate motivates the definition of the following family of Banach spaces.

Definition 5.14. Given $\alpha > 0$, let \mathcal{E}_α be the Banach space of all generating functionals $B(\theta, y)$ on $L^1(\mathbb{R}^d) \times \mathbb{R}^d$ such that for a.a. $y \in \mathbb{R}^d$, $B(\cdot; y)$ is entire on $L^1(\mathbb{R}^d)$ and

$$\|B\|_\alpha := \int_{\mathbb{R}^d} dy \|B(\cdot, y)\|_\infty^\alpha < \infty, \quad (5.44)$$

where

$$\|B(\cdot, y)\|_\infty^\alpha := \sup_{\theta \in L^1} \left(|B(\theta, y)| e^{-\frac{1}{\alpha}\|\theta\|_1} \right). \quad (5.45)$$

Remark 5.15. The fact that $(\mathcal{E}_\alpha, \|B\|_\alpha)$, $\alpha > 0$, defines a Banach space follows from the result in [79, Proposition 23].

In Section 5.2.1 we construct the time evolution of the generating functional B_t , $t \geq 0$, as the solution of the initial value problem (5.37) in the Banach space \mathcal{E}_α . Afterwards, in Section 5.2.2 we apply the general scheme of Vlasov-type scaling described in Section 2.3 to (5.37): we study the convergence to the corresponding mesoscopic evolution and derive the kinetic equations for the considered model.

5.2.1 Non-equilibrium evolution of generating functionals

In this section we study the existence and uniqueness of solutions to initial value problem (5.37) in the Banach space \mathcal{E}_α introduced in Definition 5.14.

The time evolution of Bogoliubov functionals for the Kawasaki dynamics in (5.29) has been constructed in [43]. This construction was carried out by using an Ovsjannikov-type result in a scale of Banach spaces. In the present case, from Definition 5.14 it follows that, for each $\alpha_0 > 0$, the family $\{\mathcal{E}_\alpha : 0 < \alpha \leq \alpha_0\}$ defines a scale of Banach spaces. In fact, for any $\alpha' \geq \alpha > 0$ we have

$$\mathcal{E}_{\alpha'} \subseteq \mathcal{E}_\alpha, \quad \|\cdot\|_\alpha \leq \|\cdot\|_{\alpha'}.$$

In this settings, the solutions to the Cauchy problem (5.37) can be analyzed by applying the existence and uniqueness result stated in Theorem 4.2.

In the following two subsections we consider explicitly the cases for the interactions $\lambda^{(1)}$ and $\lambda^{(2)}$ defined in (2.13) and (2.14), respectively. Clearly, similar considerations can be also applied to $\lambda^{(3)}$ and $\lambda^{(4)}$, see (2.15) and (2.16), respectively. Before proceeding with this analysis, let us recall a general result from [43, Proposition 3.2] regarding the operator \tilde{L}'_{RE} , which we will use later on.

Lemma 5.16. *Suppose that condition (5.32) holds. Let $0 < \alpha < \alpha_0$ be given. If $B \in \mathcal{E}_{\alpha'}$ for some $\alpha' \in (\alpha, \alpha_0]$, then $\tilde{L}'_{RE}B \in \mathcal{E}_{\alpha'}$, and*

$$\left\| \tilde{L}'_{RE}B \right\|_{\alpha'} \leq 2 \frac{\alpha_0}{\alpha'' - \alpha'} \langle \tilde{c} \rangle e^{\frac{C_V^1}{\alpha}} \|B\|_{\alpha}, \quad (5.46)$$

for all $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$.

Remark 5.17. Note that a similar estimate can be shown in the case where \tilde{L}'_{RE} is the generator associated to a Glauber-type dynamics, see e.g. [61, Proposition 3.2]. As a consequence, the scheme illustrated above can be also applied to study RWs moving in a Glauber-type environment.

5.2.1.1 RW in a Kawasaki model of environment: Case I

Let us consider a RW whose interaction with RE is given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(1)}(\gamma, y) = e^{-\sum_{x' \in \gamma} \phi(x' - y)}, \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d, \quad (5.47)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that

$$C_{\phi}^1 := \int_{\mathbb{R}^d} dx \phi(x) < \infty. \quad (5.48)$$

For interaction (5.47) we have, compare to (2.49),

$$(K^{-1}\lambda^{(1)}(\cdot, y))(\eta) = e_{\lambda}(\ell_y(\cdot), \eta), \quad \ell_y(\cdot) = e^{-\phi(\cdot - y)} - 1,$$

for any $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$. Then, taking into account (5.42), the initial value problem (5.37) reads

$$\begin{cases} \frac{\partial}{\partial t} B_t(\theta, y) = (\tilde{L}'_{RE}B_t)(\theta, y) + (\tilde{L}_{RW}^{(1)}B_t)(\theta, y) \\ B_t(\theta, y)|_{t=0} = B_0(\theta, y) \end{cases}, \quad (5.49)$$

where the operator \tilde{L}'_{RE} is defined by (5.40), while $\tilde{L}_{RW}^{(1)}$ is given by

$$\begin{aligned} (\tilde{L}_{RW}^{(1)}B)(\theta, y) := & \int_{\mathbb{R}^d} dz a(y - z) [B(\theta e^{-\phi(\cdot - z)} + (e^{-\phi(\cdot - z)} - 1), z) - \\ & B(\theta e^{-\phi(\cdot - y)} + (e^{-\phi(\cdot - y)} - 1), y)]. \end{aligned} \quad (5.50)$$

Note that under assumption (5.48) the operator $\tilde{L}_{RW}^{(1)}$ turns out to be bounded in \mathcal{E}_{α} . In particular we can show the following result.

Lemma 5.18. *Suppose that condition (5.48) holds. Then, given an $\alpha > 0$, for all $B \in \mathcal{E}_{\alpha}$ and $\alpha' \leq \alpha$ we have $\tilde{L}_{RW}^{(1)}B \in \mathcal{E}_{\alpha'}$ and*

$$\left\| \tilde{L}_{RW}^{(1)}B \right\|_{\alpha'} \leq 2\langle a \rangle e^{\frac{C_{\phi}^1}{\alpha}} \|B\|_{\alpha}. \quad (5.51)$$

Proof. Let us fix a $y \in \mathbb{R}^d$ and denote

$$\varphi(\cdot; y) = e^{-\phi(\cdot - y)}, \quad \chi(\cdot; y) = e^{-\phi(\cdot - y)} - 1.$$

As the potential ϕ is non-negative and integrable, $\varphi(\cdot; y) \in L^{\infty}(\mathbb{R}^d)$ and $\chi(\cdot; y) \in L^1(\mathbb{R}^d)$. In particular, one has

$$\|\varphi(\cdot; y)\|_{\infty} \leq 1 \quad (5.52)$$

and

$$\|\chi(\cdot; y)\|_1 = \int_{\mathbb{R}^d} dx' |e^{-\phi(x'-y)} - 1| \leq \int_{\mathbb{R}^d} dx' |\phi(x)| =: C_\phi^1. \quad (5.53)$$

In this notation the operator $\tilde{L}_{RW}^{(1)}$ can be written as

$$\left(\tilde{L}_{RW}^{(1)} B\right)(\theta, y) = \int_{\mathbb{R}^d} dz a(y-z) B(\theta\varphi(\cdot, z) + \chi(\cdot, z), z) - \langle a \rangle B(\theta\varphi(\cdot, y) + \chi(\cdot, y), y).$$

Since $B \in \mathcal{E}_\alpha$, for all $\theta \in L^1$ and $y \in \mathbb{R}^d$ one has

$$\begin{aligned} |B(\theta\varphi(\cdot, y) + \chi\phi(\cdot, y), y)| &\leq \|B(\cdot, y)\|_\infty^\alpha e^{\frac{\|\theta\varphi(\cdot, y)\|_1 + \|\chi(\cdot, y)\|_1}{\alpha}} \\ &\leq \|B(\cdot, y)\|_\infty^\alpha e^{\frac{\|\theta\|_1 + C_\phi^1}{\alpha}}, \end{aligned}$$

where the last inequality follows from (5.52) and (5.53). As a result, we find

$$\left| \left(\tilde{L}_{RW}^{(1)} B\right)(\theta, y) \right| \leq e^{\frac{\|\theta\|_1 + C_\phi^1}{\alpha}} \left[\int_{\mathbb{R}^d} dz a(y-z) \|B(\cdot, z)\|_\infty^\alpha - \langle a \rangle \|B(\cdot, y)\|_\infty^\alpha \right]. \quad (5.54)$$

Hence, the norm of $\tilde{L}_{RW}^{(1)}$ in the space $\mathcal{E}_{\alpha'}$ can be estimated as follows

$$\begin{aligned} \left\| \tilde{L}_{RW}^{(1)} B \right\|_{\alpha'} &= \int_{\mathbb{R}^d} dy \sup_{\theta \in L^1} e^{-\frac{\|\theta\|_1}{\alpha'}} \left| \left(\tilde{L}'_{RW} B\right)(\theta, y) \right| \\ &\leq e^{\frac{C_\phi^1}{\alpha}} \left(\sup_{\theta \in L^1} e^{-(\frac{1}{\alpha'} - \frac{1}{\alpha})\|\theta\|_1} \right) \left[\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) \|B(\cdot, z)\|_\infty^\alpha - \langle a \rangle \|B(\cdot, y)\|_\infty^\alpha \right] \\ &\leq 2\langle a \rangle e^{\frac{C_\phi^1}{\alpha}} \|B\|_\alpha \left(\sup_{\theta \in L^1} e^{-(\frac{1}{\alpha'} - \frac{1}{\alpha})\|\theta\|_1} \right). \end{aligned}$$

Note that the latter supremum is finite for all $\alpha' \leq \alpha$. In such cases we simply have

$$\left\| \tilde{L}_{RW}^{(1)} B \right\|_{\alpha'} \leq 2\langle a \rangle e^{\frac{C_\phi^1}{\alpha}} \|B\|_\alpha, \quad (5.55)$$

which concludes the proof of the lemma. \square

By Proposition 5.16 and 5.18, given $0 < \alpha < \alpha_0$ for any $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$ and $B \in \mathcal{E}_{\alpha''}$ we have the following estimate

$$\left\| \left(\tilde{L}'_{RE} + \tilde{L}_{RW}^{(1)}\right) B \right\|_{\alpha'} \leq 2 \frac{\alpha_0}{\alpha'' - \alpha'} \langle \tilde{c} \rangle e^{\frac{C_V^1}{\alpha}} \|B\|_{\alpha''} + 2\langle a \rangle e^{\frac{C_\phi^1}{\alpha}} \|B\|_{\alpha''}. \quad (5.56)$$

Then a direct application of Theorem 4.2 leads to the following existence and uniqueness result for (5.49).

Theorem 5.19. *Suppose that conditions (5.32) and (5.48) hold. Then, given an $\alpha_0 > 0$, for each $\alpha \in (0, \alpha_0)$ there exists a moment of time*

$$T_1 := T_1(\alpha, \alpha_0) = \frac{e^{-\frac{C_V^1}{\alpha}} - 1}{2\alpha_0 \langle \tilde{c} \rangle}, \quad (5.57)$$

such that the Cauchy problem (5.49) with initial condition $B_0 \in \mathcal{E}_{\alpha_0}$ has a unique solution $B_t \in \mathcal{E}_\alpha$, on the time interval $[0, T_1)$.

Proof. The result follows from Theorem 4.2 by (5.56). \square

5.2.1.2 RW in a Kawasaki model of environment: Case II

Let us consider the interaction given by

$$\lambda_{\text{int}}(\gamma, y, z) := \lambda^{(2)}(\gamma, y) = \lambda_0 + \sum_{x' \in \gamma} \phi(x' - y), \quad \gamma \in \Gamma, y, z \in \mathbb{R}^d, \quad (5.58)$$

where $\lambda_0 \geq 0$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-negative even function such that

$$C_\phi^1 := \int_{\mathbb{R}^d} dx \phi(x) < \infty \quad (5.59)$$

and

$$C_\phi^\infty := \text{ess sup}_{x \in \mathbb{R}^d} \phi(x) < \infty. \quad (5.60)$$

According to (5.41) and (2.56), in this case the operator \tilde{L}' has the form

$$(\tilde{L}'B)(\theta, y) = (\tilde{L}'_{RE}B)(\theta, y) + (\tilde{L}'_{RW}B)(\theta, y), \quad (5.61)$$

where \tilde{L}'_{RE} is given by (5.40) and

$$\begin{aligned} (\tilde{L}'_{RW}B)(\theta, y) &= \lambda_0 \int_{\mathbb{R}^d} dz a(y-z) (B(\theta, z) - B(\theta, y)) + \\ &\int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx [\phi(x-z) (DB)(\theta, z; x) - \phi(x-y) (DB)(\theta, y; x)] + \\ &\int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx \theta(x) [\phi(x-z) (DB)(\theta, z; x) - \phi(x-y) (DB)(\theta, y; x)]. \end{aligned} \quad (5.62)$$

For the latter we can show the following estimate in the Banach space \mathcal{E}_α .

Lemma 5.20. *Assume that conditions (5.59) and (5.60) hold. Then, given $\alpha > 0$, for all $B \in \mathcal{E}_\alpha$ and $\alpha' < \alpha$ we have $\tilde{L}'_{RW}B \in \mathcal{E}_{\alpha'}$. Moreover,*

$$\|\tilde{L}'_{RW}B\|_{\alpha'} \leq 2 \left(\lambda_0 \langle a \rangle + \frac{e \langle a \rangle}{\alpha} C_\phi^1 + \frac{\alpha'}{\alpha - \alpha'} \langle a \rangle C_\phi^\infty \right) \|B\|_\alpha. \quad (5.63)$$

Proof. For brevity, we write

$$(\tilde{L}'_{RW}B)(\theta, y) := (\tilde{L}'_{RW,1}B)(\theta, y) + (\tilde{L}'_{RW,2}B)(\theta, y) + (\tilde{L}'_{RW,3}B)(\theta, y),$$

where

$$(\tilde{L}'_{RW,1}B)(\theta, y) = \lambda_0 \int_{\mathbb{R}^d} dz a(y-z) (B(\theta, z) - B(\theta, y)), \quad (5.64)$$

$$(\tilde{L}'_{RW,2}B)(\theta, y) = \int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx [\phi(x-z) (DB)(\theta, z; x) - \phi(x-y) (DB)(\theta, y; x)] \quad (5.65)$$

and

$$\begin{aligned} (\tilde{L}'_{RW,3}B)(\theta, y) &= \int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx \theta(x) [\phi(x-z) (DB)(\theta, z; x) - \\ &\phi(x-y) (DB)(\theta, y; x)]. \end{aligned} \quad (5.66)$$

Let us estimate these three terms separately. We start with the operator $\tilde{L}_{RW,1}^{(2)}$. From Definition 5.14, given a $B \in \mathcal{E}_\alpha$ for all $\theta \in L^1$ and a.a. $y \in \mathbb{R}^d$ we have

$$|B(\theta, y)| \leq e^{\frac{\|\theta\|_1}{\alpha}} \|B(\cdot, y)\|_\infty^\alpha.$$

Thus, one may estimate

$$\begin{aligned} \left| \left(\tilde{L}_{RW,1}^{(2)} B \right) (\theta, y) \right| &\leq \lambda_0 \int_{\mathbb{R}^d} dz a(y-z) |B(\theta, z)| + \langle a \rangle |B(\theta, y)| \\ &\leq \lambda_0 e^{\frac{\|\theta\|_1}{\alpha}} \left[\int_{\mathbb{R}^d} dz a(y-z) \|B(\cdot, z)\|_\infty^\alpha + \langle a \rangle \|B(\cdot, y)\|_\infty^\alpha \right]. \end{aligned}$$

As a consequence, its norm in $\mathcal{E}_{\alpha'}$ can be bounded as follows

$$\begin{aligned} \left\| \tilde{L}_{RW,1}^{(2)} B \right\|_{\alpha'} &= \int_{\mathbb{R}^d} dy \sup_{\theta \in L^1} e^{-\frac{\|\theta\|_1}{\alpha'}} \left| \left(\tilde{L}_{RW,2}^{(1)} B \right) (\theta, y) \right| \\ &\leq \lambda_0 \left(\sup_{\theta \in L^1} e^{-\left(\frac{1}{\alpha'} - \frac{1}{\alpha}\right)\|\theta\|_1} \right) \left[\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) \|B(\cdot, z)\|_\infty^\alpha + \langle a \rangle \int_{\mathbb{R}^d} dy \|B(\cdot, y)\|_\infty^\alpha \right] \\ &\leq 2\lambda_0 \langle a \rangle \|B\|_\alpha \left(\sup_{\theta \in L^1} e^{-\left(\frac{1}{\alpha'} - \frac{1}{\alpha}\right)\|\theta\|_1} \right). \end{aligned}$$

In particular, if we chose $\alpha' \leq \alpha$ we simply find

$$\left\| \tilde{L}_{RW,1}^{(2)} \right\|_{\alpha'} \leq 2\lambda_0 \langle a \rangle \|B\|_\alpha. \quad (5.67)$$

Next let us consider the operator $\tilde{L}_{RW,3}^{(2)}$ defined by (5.66). By (5.60), for all $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$ we have

$$\begin{aligned} \left| \left(\tilde{L}_{RW,3}^{(2)} B \right) (\theta, y) \right| &\leq \int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx |\theta(x)| \left[\phi(x-z) |(DB)(\theta, z; x)| + \right. \\ &\quad \left. \phi(x-y) |(DB)(\theta, y; x)| \right] \\ &\leq \|\theta\|_1 C_\phi^\infty \int_{\mathbb{R}^d} dz a(y-z) \left[\|(DB)(\theta, z; \cdot)\|_\infty + \|(DB)(\theta, y; \cdot)\|_\infty \right]. \end{aligned}$$

From Lemma 5.3, we know that for all $r > 0$,

$$\|(DB)(\theta, z; \cdot)\|_\infty \leq \frac{1}{r} \sup_{\|\theta_0\|_1 \leq r} |B(\theta + \theta_0, z)|. \quad (5.68)$$

On the other hand, since $B \in \mathcal{E}_\alpha$ for all $\theta_0 \in L^1$ such that $\|\theta_0\|_1 \leq r$ and a.a. $y \in \mathbb{R}^d$ we have

$$|B(\theta + \theta_0, z)| \leq e^{\frac{r}{\alpha} + \frac{\|\theta\|_1}{\alpha}} \|B(\cdot, z)\|_\infty^\alpha. \quad (5.69)$$

These two estimates yield

$$\left| \left(\tilde{L}_{RW,3}^{(2)} B \right) (\theta, y) \right| \leq C_\phi^\infty \frac{e^{\frac{r}{\alpha}}}{r} \|\theta\|_1 e^{\frac{\|\theta\|_1}{\alpha}} \left[\int_{\mathbb{R}^d} dz a(y-z) \|B(\cdot, z)\|_\infty^\alpha + \langle a \rangle \|B(\cdot, y)\|_\infty^\alpha \right]. \quad (5.70)$$

Then, by (5.70) the norm of $\tilde{L}_{RW,3}^{(2)}$ in $\mathcal{E}_{\alpha'}$ can be estimated as follows

$$\left\| \tilde{L}_{RW,3}^{(2)} B \right\|_{\alpha'} = \int_{\mathbb{R}^d} dy \sup_{\theta \in L^1} e^{-\frac{\|\theta\|_1}{\alpha'}} \left| \left(\tilde{L}_{RW,3}^{(2)} B \right) (\theta, y) \right|$$

$$\begin{aligned}
&\leq C_\phi^\infty \frac{e^{\frac{r}{\alpha}}}{r} \left(\sup_{\theta \in L^1} \|\theta\|_1 e^{-(\frac{1}{\alpha'} - \frac{1}{\alpha})\|\theta\|_1} \right) \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) \left[\|B(\cdot, z)\|_\infty^\alpha + \|B(\cdot, y)\|_\infty^\alpha \right] \\
&\leq 2\langle a \rangle C_\phi^\infty \frac{e^{\frac{r}{\alpha}}}{r} \|B\|_\alpha \sup_{\theta \in L^1} \left(\|\theta\|_1 e^{-(\frac{1}{\alpha'} - \frac{1}{\alpha})\|\theta\|_1} \right),
\end{aligned}$$

where the last supremum is finite provided that $\alpha' < \alpha$. In such a situation, by using the elementary inequality

$$x e^{-mx} \leq \frac{1}{em}, \quad x \geq 0, m > 0, \quad (5.71)$$

we find

$$\left\| \tilde{L}_{RW,3}^{(2)} B \right\|_{\alpha'} \leq 2\langle a \rangle C_\phi^\infty \frac{e^{\frac{r}{\alpha}}}{r} \frac{\alpha' \alpha}{e(\alpha - \alpha')} \|B\|_\alpha. \quad (5.72)$$

Note that the best bound can be obtained by minimizing the term in the r.h.s. with respect to the parameter r , that is, for $r = \alpha$. With this choice it follows

$$\left\| \tilde{L}_{RW,3}^{(2)} B \right\|_{\alpha'} \leq \frac{\alpha'}{(\alpha - \alpha')} 2\langle a \rangle C_\phi^\infty \|B\|_\alpha. \quad (5.73)$$

Finally let us consider the operator $\tilde{L}_{RW,2}^{(2)}$ given by (5.65). Since the potential ϕ is integrable, see (5.59), one has

$$\begin{aligned}
\left| \left(\tilde{L}_{RW,2}^{(2)} B \right) (\theta, y) \right| &\leq \int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx \left[\phi(x-z) |(DB)(\theta, z; x)| + \right. \\
&\quad \left. \phi(x-y) |(DB)(\theta, y; x)| \right] \\
&\leq C_\phi^1 \int_{\mathbb{R}^d} dz a(y-z) \left[\|(DB)(\theta, z; \cdot)\|_\infty + \|(DB)(\theta, y; \cdot)\|_\infty \right],
\end{aligned}$$

for all $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. Next, by using estimates (5.68) and (5.69), we find

$$\left| \left(\tilde{L}_{RW,2}^{(2)} B \right) (\theta, y) \right| \leq C_\phi^1 \frac{e^{\frac{r}{\alpha}}}{r} e^{\frac{\|\theta\|_1}{\alpha}} \left[\int_{\mathbb{R}^d} dz a(y-z) \|B(\cdot, z)\|_\infty^\alpha + \langle a \rangle \|B(\cdot, y)\|_\infty^\alpha \right].$$

Thus,

$$\begin{aligned}
\left\| \tilde{L}_{RW,2}^{(2)} B \right\|_{\alpha'} &= \int_{\mathbb{R}^d} dy \sup_{\theta \in L^1} e^{-\frac{\|\theta\|_1}{\alpha'}} \left| \left(\tilde{L}_{RW,2}^{(2)} B \right) (\theta, y) \right| \\
&\leq C_\phi^\infty \frac{e^{\frac{r}{\alpha}}}{r} \left(\sup_{\theta \in L^1} e^{-(\frac{1}{\alpha'} - \frac{1}{\alpha})\|\theta\|_1} \right) \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) \left[\|B(\cdot, z)\|_\infty^\alpha + \|B(\cdot, y)\|_\infty^\alpha \right] \\
&\leq 2\langle a \rangle C_\phi^\infty \frac{e^{\frac{r}{\alpha}}}{r} \|B\|_\alpha \left(\sup_{\theta \in L^1} e^{-(\frac{1}{\alpha'} - \frac{1}{\alpha})\|\theta\|_1} \right).
\end{aligned}$$

Note that the latter supremum is finite for any $\alpha' \leq \alpha$. In such a case, if we take again $r = \alpha$, we obtain

$$\left\| \tilde{L}_{RW,2}^{(2)} B \right\|_{\alpha'} \leq \frac{2e}{\alpha} \langle a \rangle C_\phi^1 \|B\|_\alpha. \quad (5.74)$$

Finally, the combination of (5.67), (5.73) and (5.74) provides (5.63). \square

Now we can apply Theorem 4.2 and formulate the following results about the evolution of the generating functionals.

Theorem 5.21. *Suppose that conditions (5.32), (5.59) and (5.60) hold. Given an $\alpha_0 > 0$, let $B_0 \in \mathcal{E}_{\alpha_0}$. Then, for each $\alpha \in (0, \alpha_0)$ there exists a moment of time*

$$T_2 := T_2(\alpha, \alpha_0) = \frac{2\alpha_0}{\left(\langle \tilde{c} \rangle e^{\frac{C_V^1}{\alpha}} + \langle a \rangle C_\phi^\infty\right)}, \quad (5.75)$$

such that the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} B_t = (\tilde{L}'_{RE} B_t) + (\tilde{L}'_{RW} B_t) \\ B_t|_{t=0} = B_0 \end{cases}, \quad (5.76)$$

has a unique solution $B_t \in \mathcal{E}_\alpha$ on the time interval $[0, T_2]$.

Proof. By combining the results stated in Proposition 5.16 and 5.20, for any $B \in \mathcal{E}_{\alpha''}$ and $0 < \alpha \leq \alpha' < \alpha'' \leq \alpha_0$ we have

$$\left\| (\tilde{L}'_{RE} + \tilde{L}'_{RW,2}) B \right\|_{\alpha'} \leq \frac{M_1}{\alpha'' - \alpha'} \|B\|_{\alpha''} + M_0 \|B\|_{\alpha''}, \quad (5.77)$$

where

$$M_1 := M_1(\alpha, \alpha_0) = 2\alpha_0 \langle \tilde{c} \rangle e^{\frac{C_V^1}{\alpha}} + 2\alpha_0 \langle a \rangle C_\phi^\infty$$

and

$$M_0 := M_0(\alpha, \alpha_0) = 2\langle a \rangle \left(\lambda_0 + \frac{eC_\phi^1}{\alpha} \right).$$

Then the statement is a direct consequence of Lemma 4.2. \square

5.2.2 Vlasov-type scaling via generating functionals

In this section we want to investigate the Vlasov-type scaling limit for the stochastic dynamics associated to the heuristic Markov generator (5.29)-(5.32). Through this section we assume that

$$C_V^\infty := \operatorname{ess\,sup}_{x \in \mathbb{R}^d} V(x) < \infty. \quad (5.78)$$

In Section 2.3 we described a general scheme to construct the Vlasov-type scaling of the evolution of correlation functions for RWREs. Here, we reformulate the problem in terms of generating functionals. For the Kawasaki dynamics (5.29) this mesoscopic limit has been studied in [10] and [43] through correlation functions and generating functionals, respectively.

Let us recall that the Vlasov-type scaling can be realized in three steps starting from the hierarchy for correlation functions (5.33). First we rescale the initial condition of (5.33) in order to increase the density of particles of RE. Namely, for any $\varepsilon > 0$ we introduce a rescaled correlation function $k_0^{(\varepsilon)}$ such that

$$k_{0,\varepsilon,\text{ren}}(\eta, y) := \varepsilon^{|\eta|} k_0^{(\varepsilon)}(\eta, y) \longrightarrow r_0(\eta, y),$$

as ε goes to zero, for any $\eta \in \Gamma_0(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. In correspondence, for each ε we define the rescaled generating functional

$$B_0^{(\varepsilon)}(\theta, y) := \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_0^{(\varepsilon)}(\eta, y)$$

and the renormalized generating functional

$$B_{0,\varepsilon,\text{ren}}(\theta, y) := \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_{0,\varepsilon,\text{ren}}(\eta, y)$$

$$= \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\varepsilon\theta, \eta) k_0^{(\varepsilon)}(\eta, y) = B_0^{(\varepsilon)}(\varepsilon\theta, y), \quad (5.79)$$

for any $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. Afterwards, we consider a proper scaling of the generators in (5.33), which makes all interactions among particles weak. Formally, we write

$$L^\Delta = L_{RE}^\Delta(V) + L_{RW}^\Delta(\lambda_{\text{int}}) \longmapsto L_\varepsilon^\Delta = L_{RE}^\Delta(\varepsilon V) + L_{RW}^\Delta(\lambda_\varepsilon). \quad (5.80)$$

The form of the rescaled interaction λ_ε has been already discussed in Section 2.3.2 in different cases. As in Section 5.2, we can use (5.21) to construct a scaling of the operator $\tilde{L}' = \tilde{L}'_{RE} + \tilde{L}'_{RW}$. Consequently, scaling (5.79) and (5.80) yield the renormalized initial value problem

$$\begin{cases} \frac{\partial}{\partial t} B_{t,\varepsilon,\text{ren}}(\theta, y) = (\tilde{L}'_{RE,\varepsilon,\text{ren}} B_{t,\varepsilon,\text{ren}})(\theta, y) + (\tilde{L}'_{RW,\varepsilon,\text{ren}} B_{t,\varepsilon,\text{ren}})(\theta, y) \\ B_{t,\varepsilon,\text{ren}}|_{t=0} = B_{0,\varepsilon,\text{ren}} \end{cases}, \quad (5.81)$$

where for any $\theta \in L^1(\mathbb{R}^d)$, $y \in \mathbb{R}^d$ and $t > 0$,

$$B_{t,\varepsilon,\text{ren}}(\theta, y) := B_t^{(\varepsilon)}(\varepsilon\theta, y) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\varepsilon\theta, \eta) k_t^{(\varepsilon)}(\eta, y). \quad (5.82)$$

According to [43, Proposition 4.1], the operator $\tilde{L}'_{RE,\varepsilon,\text{ren}}$ is given by

$$\begin{aligned} (\tilde{L}'_{RE,\varepsilon,\text{ren}} B)(\theta, y) &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dx' \tilde{c}(x - x') e^{-\varepsilon V(x-x')} (\theta(x') - \theta(x)) \times \\ &\quad \frac{\delta B\left(\theta e^{-\varepsilon V(\cdot-x')} + \frac{e^{-\varepsilon V(\cdot-x')}-1}{\varepsilon}, y\right)}{\delta\theta(x)}. \end{aligned} \quad (5.83)$$

In the following proposition we compute the precise form of $\tilde{L}'_{RW,\varepsilon,\text{ren}}$ for a generic rescaled interaction λ_ε .

Proposition 5.22. *Given $\varepsilon > 0$ the following formula holds*

$$\begin{aligned} (\tilde{L}'_{RW,\varepsilon,\text{ren}} B)(\theta, y) &= \int_{\mathbb{R}^d} dz a(y - z) \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\varepsilon\theta + 1, \xi) \times \\ &\quad \left[\varepsilon^{-|\xi|} A_\varepsilon(\xi, z, y) (D^{|\xi|} B)(\theta, z; \xi) - \varepsilon^{-|\xi|} A_\varepsilon(\xi, y, z) (D^{|\xi|} B)(\theta, y; \xi) \right], \end{aligned} \quad (5.84)$$

for all $\theta \in L^1(\mathbb{R}^d, dx)$ and $y \in \mathbb{R}^d$.

Remark 5.23. According to Remark 5.12, in the case

$$(K^{-1}\lambda_\varepsilon(\cdot, y, z))(\xi) = e_\lambda(\ell_{y,z}^{(\varepsilon)}(\cdot), \xi), \quad \xi \in \Gamma_0(\mathbb{R}^d), y, z \in \mathbb{R}^d, \quad (5.85)$$

we have

$$\begin{aligned} (\tilde{L}'_{RW,\varepsilon,\text{ren}} B)(\theta, y) &= \int_{\mathbb{R}^d} dz a(y - z) \left[B\left(\theta(1 + \ell_{z,y}^{(\varepsilon)}) + \varepsilon^{-1}\ell_{z,y}^{(\varepsilon)}, z\right) - \right. \\ &\quad \left. B\left(\theta(1 + \ell_{y,z}^{(\varepsilon)}) + \varepsilon^{-1}\ell_{y,z}^{(\varepsilon)}, y\right) \right]. \end{aligned} \quad (5.86)$$

Proof of Proposition 5.22. The renormalized operator $\tilde{L}'_{RW,\varepsilon,\text{ren}}$ is defined by

$$(\tilde{L}_{RW,\varepsilon,\text{ren}}Z_t)(\theta, \psi) = \int_{\mathbb{R}^d} dy \psi(y) (\tilde{L}'_{RW,\varepsilon,\text{ren}}B_t)(\theta, y), \quad (5.87)$$

where

$$(\tilde{L}_{RW,\varepsilon,\text{ren}}Z)(\theta, \psi) := \int_{\Gamma_0} d\lambda(\eta) \int_{\mathbb{R}^d} dy (\hat{L}_{RW,\varepsilon,\text{ren}}e_\lambda(\theta)\psi)(\eta, y) k(\eta, y),$$

for any $\theta \in L^1(\mathbb{R}^d)$ and $\psi \in L^\infty(\mathbb{R}^d)$. On the other hand, according to (2.41) and (2.91) we have

$$\begin{aligned} (\hat{L}_{RW,\varepsilon,\text{ren}}e_\lambda(\theta)\psi)(\eta, y) &= \varepsilon^{-|\eta|} (\hat{L}_{RW,\varepsilon}e_\lambda(\varepsilon\theta, \cdot)\psi(\cdot))(\eta, y) \\ &= \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) \varepsilon^{-|\eta|} [(K^{-1}\lambda_\varepsilon(\cdot, y, z))(\xi) \star e_\lambda(\varepsilon\theta, \cdot)](\eta). \end{aligned}$$

In particular, by using (1.27) we can write

$$\begin{aligned} (\hat{L}_{RW,\varepsilon,\text{ren}}e_\lambda(\theta)\psi)(\eta, y) &= \\ &= \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) \varepsilon^{-|\eta|} \sum_{\xi \subset \eta} (K^{-1}\lambda_\varepsilon(\cdot, y, z))(\xi) e_\lambda(\varepsilon\theta + 1, \xi) e_\lambda(\varepsilon\theta, \eta \setminus \xi), \end{aligned}$$

which yields

$$\begin{aligned} (\tilde{L}_{RW,\varepsilon,\text{ren}}Z_t)(\theta, \psi) &= \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) \times \\ &\quad \int_{\Gamma_0} d\lambda(\eta) k_t(\eta, y) \sum_{\xi \subset \eta} \varepsilon^{-|\xi|} (K^{-1}\lambda_\varepsilon(\cdot, y, z))(\xi) e_\lambda(\varepsilon\theta + 1, \xi) e_\lambda(\varepsilon\theta, \eta \setminus \xi). \end{aligned}$$

Next, using Minlos identity (1.18) together with (5.24), we find

$$\begin{aligned} (\tilde{L}_{RW,\varepsilon,\text{ren}}Z_t)(\theta, \psi) &= \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) \times \\ &\quad \int_{\Gamma_0} d\lambda(\xi) \varepsilon^{-|\xi|} (K^{-1}\lambda_\varepsilon(\cdot, y, z))(\xi) e_\lambda(\varepsilon\theta + 1, \xi) \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_t(\eta, y) \\ &= \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz a(y-z) (\psi(z) - \psi(y)) \times \\ &\quad \int_{\Gamma_0} d\lambda(\xi) \varepsilon^{-|\xi|} (K^{-1}\lambda_\varepsilon(\cdot, y, z))(\xi) e_\lambda(\varepsilon\theta + 1, \xi) (D^{|\xi|}B)(\theta, y; \xi). \quad (5.88) \end{aligned}$$

Having in mind (5.87), from (5.88) we can easily obtain (5.84). \square

As the next step we consider the limit when ε goes to zero of the two generators which appear in (5.81). From [43, Proposition 4.2] we have the following result about the point-wise convergence of $\tilde{L}'_{RE,\varepsilon,\text{ren}}$.

Proposition 5.24. *Suppose that condition (5.32) is satisfied. Then,*

(i) *given $B \in \mathcal{E}_\alpha$, for some $\alpha > 0$, the following limit holds*

$$\begin{aligned} (\tilde{L}'_{RE,V}B)(\theta, y) &= \lim_{\varepsilon \rightarrow 0} (\tilde{L}'_{RE,\varepsilon,\text{ren}}B)(\theta, y) \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dx' \tilde{c}(x-x') (\theta(x') - \theta(x)) \frac{\delta B(\theta - V(\cdot - x'), y)}{\delta \theta(x)}, \quad (5.89) \end{aligned}$$

for any $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$.

(ii) given $\alpha_0 > \alpha > 0$, if $B \in \mathcal{E}_{\alpha''}$, for some $\alpha'' \in (\alpha, \alpha_0]$, we have

$$\{\tilde{L}'_{RE,\varepsilon,\text{ren}}B, \tilde{L}'_{RE,V}B\} \subset \mathcal{E}_{\alpha'}, \quad \text{for all } \alpha \leq \alpha' < \alpha'',$$

moreover,

$$\|\tilde{L}'_{RE,\#}\|_{\alpha'} \leq \frac{\alpha_0}{\alpha'' - \alpha'} 2\langle \tilde{c} \rangle e^{\frac{c_V^1}{\alpha}} \|B\|_{\alpha''}, \quad (5.90)$$

where $\# = \text{"}\varepsilon, \text{ren"}$ or $\text{"}V\text{"}$.

In the proposition below we consider the point-wise limit of $\tilde{L}'_{RW,\varepsilon,\text{ren}}$.

Proposition 5.25. *Assume that*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\xi|} (K^{-1} \lambda_\varepsilon(y, z, \cdot))(\xi) = A_V(\xi, y, z), \quad (5.91)$$

for a.a. $\xi \in \Gamma_0(\mathbb{R}^d)$ and $y, z \in \mathbb{R}^d$. Then, for any $B \in \mathcal{E}_\alpha$, $\alpha > 0$, the following limit holds:

$$\begin{aligned} (\tilde{L}'_{RW,V}B)(\theta, y) &= \lim_{\varepsilon \rightarrow 0} (\tilde{L}'_{RW,\varepsilon,\text{ren}}B)(\theta, y) \\ &= \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} d\lambda(\xi) [A_V(\xi, z, y) (D^{|\xi|}B)(\theta, z; \xi) - \\ &\quad A_V(\xi, y, z) (D^{|\xi|}B)(\theta, y; \xi)], \end{aligned} \quad (5.92)$$

for all $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$.

Proof. The limit in (5.92) follows directly from (5.84). Indeed, we can note that for $\varepsilon \rightarrow 0$

$$e_\lambda(\varepsilon\theta + 1, \xi) \longrightarrow e_\lambda(1, \xi) \equiv 1$$

and, due to hypothesis (5.91),

$$\varepsilon^{-|\xi|} (K^{-1} \lambda_\varepsilon(\cdot, y, z))(\xi) \longrightarrow A_V(\xi, y, z).$$

□

Remark 5.26. As in Remark 5.23, let us consider the case where (5.85) holds. In this case, if we assume that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \ell_{y,z}^{(\varepsilon)}(x) = \ell_{y,z}^{(V)}(x), \quad (5.93)$$

for all $x \in \mathbb{R}^d$ and a.a. $y, z \in \mathbb{R}^d$, the operator $\tilde{L}'_{RW,\varepsilon,\text{ren}}$ in (5.86), converges as ε goes to zero, to

$$(\tilde{L}'_{RW,V}B)(\theta, y) = \int_{\mathbb{R}^d} dz a(y-z) [B(\theta + \ell_{z,y}^{(V)}, z) - B(\theta + \ell_{y,z}^{(V)}, y)]. \quad (5.94)$$

Indeed, in the case (5.85), the operator $\tilde{L}'_{RW,\varepsilon,\text{ren}}$ is given by (5.86). Condition (5.93) implies that

$$\ell_{y,z}^{(\varepsilon)}(x) \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

and the statement follows.

According to Proposition 5.24 and 5.25, we can define the limiting Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} B_{t,V}(\theta, y) = (\tilde{L}'_{RE,V}B_{t,V})(\theta, y) + (\tilde{L}'_{RW,V}B_{t,V})(\theta, y) \\ B_{t,V}|_{t=0} = B_{0,V} \end{cases}. \quad (5.95)$$

Following the analysis in Section 2.3.1, in the next lemma we show that the limiting evolution (5.95) preserves chaos, see (2.85)-(2.86), and, therefore, we derive (point-wisely) the

kinetic equations associated to the Markov dynamics (5.29). Note that for states of the form $r_0(\eta, y) = e_\lambda(\rho_0, \eta)r_0(y)$, $\eta \in \Gamma_0$ and $y \in \mathbb{R}^d$, the corresponding generating functional reads

$$B_{0,V}(\theta, y) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) e_\lambda(\rho_0(\cdot), \eta) r_0(y) = r_0(y) e^{\int_{\mathbb{R}^d} dx \theta(x) \rho_0(x)}, \quad (5.96)$$

for any $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. Then, the chaos preservation property (2.85)-(2.86) implies that for initial conditions $B_{0,V}$ of the form (5.96) the evolution $B_{0,V} \mapsto B_{t,V}$, $t > 0$, is such that

$$B_{t,V}(\theta, y) = r_t(y) e^{\int_{\mathbb{R}^d} dx \theta(x) \rho_t(x)} =: B_{t,V}^{(\text{coh})}(\theta, y), \quad (5.97)$$

for any $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$.

Lemma 5.27. *Given $0 < \alpha < \alpha_0$, let $\rho_t \in \overline{B}_{1/\alpha_0}^\infty$ and $r_t \in L^1(\mathbb{R}^d)$, $t \geq 0$, be solutions to the Vlasov equations*

$$\frac{\partial \rho_t(x)}{\partial t} = (\rho_t * \tilde{c})(x) e^{-(\rho_t * V)(x)} - \rho_t(x) (\tilde{c} * e^{-(\rho_t * V)})(x), \quad (5.98)$$

$$\frac{\partial r_t(y)}{\partial t} = \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \int_{\mathbb{R}^d} dz a(y-z) [r_t(z) A_V(\xi, z, y) - r_t(y) A_V(\xi, y, z)], \quad (5.99)$$

with initial conditions $\rho_t|_{t=0} = \rho_0$ and $r_t|_{t=0} = r_0$. Then, $B_{t,V}^{(\text{coh})} \in \mathcal{E}_{\alpha_0} \subset \mathcal{E}_\alpha$, $t \geq 0$, solves the Cauchy problem (5.95) with initial condition $B_{0,V}^{(\text{coh})} \in \mathcal{E}_{\alpha_0}$.

Proof. First let us note that $\rho_t \in \overline{B}_{1/\alpha_0}^\infty$ and $r_t \in L^1(\mathbb{R}^d)$ implies that $B_{t,V}^{(\text{coh})} \in \mathcal{E}_{\alpha_0} \subset \mathcal{E}_\alpha$, $t \geq 0$. Next let us show that $B_{t,V}^{(\text{coh})}(\theta, y)$ solves equation (5.95), namely

$$\frac{\partial}{\partial t} B_{t,V}^{(\text{coh})}(\theta, y) = (\tilde{L}'_{RE,V} B_{t,V}^{(\text{coh})})(\theta, y) + (\tilde{L}'_{RW,V} B_{t,V}^{(\text{coh})})(\theta, y), \quad (5.100)$$

for any $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. By (5.97), the derivative in the l.h.s. can be written explicitly as

$$\begin{aligned} \frac{\partial}{\partial t} B_{t,V}^{(\text{coh})}(\theta, y) &= \frac{\partial}{\partial t} [r_t(y) e^{\int_{\mathbb{R}^d} dx \theta(x) \rho_t(x)}] \\ &= \frac{\partial r_t(y)}{\partial t} B_{t,V,RE}^{(\text{coh})}(\theta) + B_{t,V}^{(\text{coh})}(\theta, y) \int_{\mathbb{R}^d} dx \theta(x) \frac{\partial \rho_t(x)}{\partial t}, \end{aligned} \quad (5.101)$$

where

$$B_{t,V,RE}^{(\text{coh})}(\theta) := e^{\int_{\mathbb{R}^d} dx \theta(x) \rho_t(x)}. \quad (5.102)$$

On the other hand, according to (5.89) and (5.92), we have

$$\begin{aligned} (\tilde{L}'_V B_{t,V}^{(\text{coh})})(\theta, y) &:= (\tilde{L}'_{RE,V} B_{t,V}^{(\text{coh})})(\theta, y) + (\tilde{L}'_{RW,V} B_{t,V}^{(\text{coh})})(\theta, y) \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dx' \tilde{c}(x' - x) (\theta(x') - \theta(x)) (DB_{t,V}^{(\text{coh})})(\theta - V(x' - \cdot), y; x) + \\ &\quad \int_{\mathbb{R}^d} dz a(y-z) \int_{\Gamma_0} d\lambda(\xi) [A_V(\xi, z, y) (D^{|\xi|} B_{t,V}^{(\text{coh})})(\theta, z; \xi) - \\ &\quad A_V(\xi, y, z) (D^{|\xi|} B_{t,V}^{(\text{coh})})(\theta, y; \xi)]. \end{aligned}$$

Next, since

$$(D^{|\xi|} B_{t,V,RE}^{(\text{coh})})(\theta; \xi) = e_\lambda(\rho_t, \xi) B_{t,V,RE}^{(\text{coh})}(\theta),$$

one has

$$(D^{|\xi|} B_{t,V}^{(\text{coh})})(\theta, y; \xi) = r_t(y) (D^{|\xi|} B_{t,V,RE}^{(\text{coh})})(\theta; \xi)$$

$$\begin{aligned}
&= r_t(y) e_\lambda(\rho_t, \xi) B_{t,V,RE}^{(\text{coh})}(\theta) \\
&= e_\lambda(\rho_t, \xi) B_{t,V}^{(\text{coh})}(\theta, y)
\end{aligned}$$

and we can write

$$\begin{aligned}
\left(\tilde{L}'_V B_{t,V}^{(\text{coh})}\right)(\theta, y) &= B_{t,V}^{(\text{coh})}(\theta, y) \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dx' \tilde{c}(x' - x) (\theta(x') - \theta(x)) \rho_t(x) e^{-(\rho_t * V)(x')} + \\
&\quad B_{t,V,RE}^{(\text{coh})}(\theta) \int_{\mathbb{R}^d} dz a(y - z) \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \left[r_t(z) A_V(\xi, z, y) - \right. \\
&\quad \left. r_t(y) A_V(\xi, y, z) \right] \\
&= B_{t,V}^{(\text{coh})}(\theta, y) \int_{\mathbb{R}^d} dx \theta(x) \left[(\tilde{c} * \rho_t)(x) e^{-(\rho_t * V)(x)} - \rho_t(x) (\tilde{c} * e^{-(\rho_t * V)})(x) \right] \\
&\quad B_{t,V,RE}^{(\text{coh})}(\theta) \int_{\mathbb{R}^d} dz a(y - z) \int_{\Gamma_0} d\lambda(\xi) e_\lambda(\rho_t, \xi) \left[r_t(z) A_V(\xi, z, y) - \right. \\
&\quad \left. r_t(y) A_V(\xi, y, z) \right].
\end{aligned} \tag{5.103}$$

Finally, by comparing the r.h.s. of (5.101) and (5.103), we can conclude that $B_{t,V}^{(\text{coh})}(\theta, y)$ solves (5.95) if ρ_t and r_t are solutions to the system of equations (5.98)-(5.99). \square

In the remaining part of this chapter we want to give a rigorous meaning to the Vlasov-type scaling described above in some specific model. In particular, in Section 5.2.2.1 and 5.2.2.2 we consider the cases where the interaction λ_{int} is given by (5.47) and (5.58), respectively.

In the analysis that follows, we will use the same techniques as in Section 4.2.1 and 4.2.2. More precisely, in the scale of Banach spaces $\{\mathcal{E}_\alpha : 0 < \alpha \leq \alpha_0\}$ by using Ovsjannikov-type result in Theorem 4.2 we show that the initial value problems (5.81) and (5.95) have unique solutions, $B_{t,\varepsilon}$ and $B_{t,V}$, respectively, on the same time interval and with values in the same Banach space. Hence, by means of Theorem 4.9 we prove that solutions $B_{t,\varepsilon}$ converge to $B_{t,V}$. This will lead to a rigorous derivation of the Vlasov equations for the considered models. The existence and uniqueness of their solutions can be established by using the same techniques as in the proof of Lemma 4.21. Let us recall that properties of solutions to (5.98) have been studied in [10]. In particular, one can show the following existence and uniqueness result.

Lemma 5.28. *Given $C > 0$, let $\rho_0 \in L^\infty(\mathbb{R}^d)$ with $0 \leq \rho_0(x) \leq C$ for a.a. $x \in \mathbb{R}^d$. Suppose that condition (5.32) holds. Then, equation (5.98) with initial condition $\rho_t|_{t=0} = \rho_0$ has a unique non-negative solution $0 \leq \rho \in C^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$. Moreover, the solution ρ_t is uniformly bounded on any finite time interval.*

5.2.2.1 RW in a Kawasaki model of environment: Case I

Let us consider interaction $\lambda^{(1)}$ given by (5.47) and such that condition (5.48) holds and

$$C_\phi^\infty := \text{ess sup}_{x \in \mathbb{R}^d} \phi(x) < \infty. \tag{5.104}$$

The scaling of this interaction has been discussed in Case I of Section 2.3.2. From (2.139) and (2.140), we can write

$$\left(K^{-1} \lambda_\varepsilon^{(1)}(\cdot, y)\right)(\xi) = e_\lambda(\ell_y^{(\varepsilon)}(\cdot), \xi), \quad \ell_y^{(\varepsilon)}(\cdot) = e^{-\varepsilon \phi(\cdot - y)} - 1, \quad \varepsilon > 0,$$

and

$$A_V^{(1)}(\xi, y) = e_\lambda(\ell_y^{(V)}(\cdot), \xi), \quad \ell_y^{(V)}(\cdot) = -\phi(\cdot - y), \tag{5.105}$$

for any $\xi \in \Gamma_0$ and a.a. $y \in \mathbb{R}^d$. Then, according to Remark 5.23, given $\alpha > 0$ for any $\varepsilon > 0$ and $B \in \mathcal{E}_\alpha$, we define the renormalized operator $\tilde{L}'_{\varepsilon, \text{ren}}$ by

$$(\tilde{L}'_{\varepsilon, \text{ren}} B)(\theta, y) = (\tilde{L}'_{RE, \varepsilon, \text{ren}} B)(\theta, y) + (\tilde{L}'_{RW, \varepsilon, \text{ren}}(1) B)(\theta, y), \quad (5.106)$$

where $\tilde{L}'_{RE, \varepsilon, \text{ren}}$ is given by (5.83) and

$$\begin{aligned} (\tilde{L}'_{RW, \varepsilon, \text{ren}}(1) B)(\theta, y) = \int_{\mathbb{R}^d} dz a(y-z) & \left[B \left(\theta e^{-\varepsilon \phi(\cdot-z)} + \frac{e^{-\varepsilon \phi(\cdot-z)} - 1}{\varepsilon}, z \right) - \right. \\ & \left. B \left(\theta e^{-\varepsilon \phi(\cdot-y)} + \frac{e^{-\varepsilon \phi(\cdot-y)} - 1}{\varepsilon}, y \right) \right]. \end{aligned} \quad (5.107)$$

Moreover, as $\varepsilon \rightarrow 0$, by combining (5.94) and (5.105), we have the point-wise limit

$$(\tilde{L}'_V B)(\theta, y) = (\tilde{L}'_{RE, V} B)(\theta, y) + (\tilde{L}'_{RW, V}(1) B)(\theta, y), \quad (5.108)$$

where $\tilde{L}'_{RE, V}$ is defined by (5.89) and

$$(\tilde{L}'_{RW, V}(1) B)(\theta, y) = \int_{\mathbb{R}^d} dz a(y-z) [B(\theta - \phi(\cdot-z), z) - B(\theta - \phi(\cdot-y), y)]. \quad (5.109)$$

As in Proposition 5.18, one can show that both $\tilde{L}'_{RW, \varepsilon, \text{ren}}(1)$ and $\tilde{L}'_{RW, V}(1)$ are bounded operators on the Banach space \mathcal{E}_α .

Proposition 5.29. *Suppose that condition (5.48) is satisfied. Given $\alpha > 0$, for all $B \in \mathcal{E}_\alpha$ and $\alpha' \leq \alpha$ we have*

$$\left\{ \tilde{L}'_{RW, \varepsilon, \text{ren}}(1) B, \tilde{L}'_{RW, V}(1) B \right\}_{\varepsilon > 0} \subset \mathcal{E}_{\alpha'}.$$

Moreover, the following inequality holds

$$\left\| \tilde{L}'_{RW, \#}(1) \right\|_{\alpha'} \leq 2 \langle a \rangle e^{\frac{C_\phi^1}{\alpha}} \|B\|_\alpha, \quad (5.110)$$

where $\# = \text{"}\varepsilon, \text{ren"}$ or $\text{"}V\text{"}$.

Proof. Let us first consider the renormalized operator $\tilde{L}'_{RW, \varepsilon, \text{ren}}(1)$. We can proceed as in the proof of Lemma 5.18 replacing φ and χ by

$$\varphi_\varepsilon(\cdot, y) = e^{-\varepsilon \phi(\cdot-y)}, \quad \chi_\varepsilon(\cdot, y) = \frac{e^{-\varepsilon \phi(\cdot-y)} - 1}{\varepsilon},$$

for any $y \in \mathbb{R}^d$ and $\varepsilon > 0$, respectively. Note that for any $\varepsilon > 0$ and $y \in \mathbb{R}^d$ we still have $\varphi_\varepsilon(\cdot; y) \in L^\infty(\mathbb{R}^d)$ and $\chi_\varepsilon(\cdot; y) \in L^1(\mathbb{R}^d)$, with

$$\|\varphi_\varepsilon(\cdot; y)\|_\infty \leq 1$$

and

$$\|\chi_\varepsilon(\cdot; y)\|_1 = \varepsilon^{-1} \int_{\mathbb{R}^d} dx' |e^{-\varepsilon \phi(x'-y)} - 1| \leq \int_{\mathbb{R}^d} dx' |\phi(x')| =: C_\phi^1.$$

For the limiting operator $\tilde{L}'_{RW, V}(1)$ we can still use the same arguments as in Lemma 5.18, but choosing this time

$$\varphi(\cdot; y) \equiv 1 \in L^\infty(\mathbb{R}^d)$$

and

$$\chi(\cdot; y) = -\phi(\cdot-y) \in L^1(\mathbb{R}^d).$$

□

Let us now proceed with the analysis of the Vlasov-type scaling outlined in the previous section. First, we show the existence of the rescaled and limiting evolution associated to the generators (5.106) and (5.108), respectively. This will be a consequence of the general result stated in Theorem 4.2.

Lemma 5.30. *Assume that conditions (5.32) and (5.48) are satisfied. Given $\alpha_0 > 0$, let $\{B_{0,\varepsilon}, B_{0,V}\}_{\varepsilon>0} \subset \mathcal{E}_{\alpha_0}$. Then, for each $\alpha \in (0, \alpha_0)$ the initial value problems (5.81) and (5.95) have unique solutions $B_{t,\varepsilon} \in \mathcal{E}_\alpha$ and $B_{t,V} \in \mathcal{E}_\alpha$, respectively, on the time interval $[0, T_1)$, where T_1 is given by (5.57).*

Proof. From Proposition 5.24 and 5.29, given $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$ for any $B \in \mathcal{E}_{\alpha''}$ we have

$$\left\| \tilde{L}'_{\#} B \right\|_{\alpha'} \leq 2 \frac{\alpha_0}{\alpha'' - \alpha'} \langle \tilde{c} \rangle e^{\frac{C_V^1}{\alpha}} \|B\|_{\alpha''} + 2 \langle a \rangle e^{\frac{C_\phi^1}{\alpha}} \|B\|_{\alpha''}, \quad (5.111)$$

where $\# = \text{"}\varepsilon, \text{ren"}$ or $\text{"}V\text{"}$. Then, we can apply Theorem 4.2 to each of the Cauchy problems (5.81), (5.95) and get the statement. \square

Note that as desired, the solutions to (5.81) and (5.95) are defined on the same time interval and on the same Banach space, namely $B_{t,\varepsilon}, B_{t,V} : [0, T_1) \rightarrow \mathcal{E}_\alpha$, $\varepsilon > 0$. Next, we analyze conditions under which $B_{t,\varepsilon}$ converges to $B_{t,V}$. In order to apply Theorem 4.9, one needs the following estimate.

Proposition 5.31. *Assume that conditions (5.32), (5.78), (5.48) and (5.104) are satisfied. Given $0 < \alpha < \alpha_0$, let α', α'' such that $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$. Then, for all $B \in \mathcal{E}_{\alpha''}$ and $\varepsilon > 0$ the following estimate holds*

$$\begin{aligned} \left\| \tilde{L}'_{\varepsilon, \text{ren}} B - \tilde{L}'_V B \right\|_{\alpha'} &\leq 2\varepsilon \langle \tilde{c} \rangle C_V^\infty \frac{e\alpha_0}{\alpha} \|B\|_{\alpha''} e^{\frac{C_V^1}{\alpha}} \left(\left(2eC_V^1 + \frac{\alpha_0}{e} \right) \frac{1}{\alpha'' - \alpha'} + \frac{8\alpha_0^2}{(\alpha'' - \alpha')^2} \right) + \\ &2\varepsilon C_\phi^\infty \langle a \rangle e^{\frac{C_\phi^1}{\alpha}} \|B\|_{\alpha''} \left(\frac{\alpha_0}{\alpha'' - \alpha'} + \frac{e}{\alpha} C_\phi^1 \right). \end{aligned} \quad (5.112)$$

Proof. For any $B \in \mathcal{E}_{\alpha''}$ we have

$$\left\| \tilde{L}'_{\varepsilon, \text{ren}} B - \tilde{L}'_V B \right\|_{\alpha'} \leq \left\| \tilde{L}'_{RE, \varepsilon, \text{ren}} B - \tilde{L}'_{RE, V} B \right\|_{\alpha'} + \left\| \tilde{L}'_{RW, \varepsilon, \text{ren}} B - \tilde{L}'_{RW, V} B \right\|_{\alpha'}. \quad (5.113)$$

From [43, Proposition 4.3], we know that for any $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$ and $\varepsilon > 0$

$$\left\| \tilde{L}'_{RE, \varepsilon, \text{ren}} B - \tilde{L}'_{RE, V} B \right\|_{\alpha'} \leq 2\varepsilon \langle \tilde{c} \rangle C_V^\infty \frac{e\alpha_0}{\alpha} \|B\|_{\alpha''} e^{\frac{C_V^1}{\alpha}} \left(\left(2eC_V^1 + \frac{\alpha_0}{e} \right) \frac{1}{\alpha'' - \alpha'} + \frac{8\alpha_0^2}{(\alpha'' - \alpha')^2} \right). \quad (5.114)$$

Next, let us consider the second term in the r.h.s of (5.113). By (5.107) and (5.109) we can write

$$\begin{aligned} &\left| \left(\tilde{L}'_{RW, \varepsilon, \text{ren}} B \right) (\theta, y) - \left(\tilde{L}'_{RW, V} B \right) (\theta, y) \right| \leq \\ &\leq \int_{\mathbb{R}^d} dz a(y-z) \left| B \left(\theta e^{-\varepsilon\phi(\cdot-z)} + \frac{e^{-\varepsilon\phi(\cdot-z)} - 1}{\varepsilon}, z \right) - B(\theta - \phi(\cdot-z), z) \right| + \\ &\int_{\mathbb{R}^d} dz a(y-z) \left| B \left(\theta e^{-\varepsilon\phi(\cdot-y)} + \frac{e^{-\varepsilon\phi(\cdot-y)} - 1}{\varepsilon}, y \right) - B(\theta - \phi(\cdot-y), y) \right|. \end{aligned} \quad (5.115)$$

In order to estimate the two integrand functions that appear in (5.115), given $\theta_1, \theta_2 \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$ let us define the function

$$C_{\theta_1, \theta_2, y}(t) := B(t\theta_1 + (1-t)\theta_2, y), \quad t \in [0, 1].$$

According to the result of Lemma 5.3, one has

$$\begin{aligned} \frac{\partial}{\partial t} C_{\theta_1, \theta_2, y}(t) &= \frac{\partial}{\partial s} C_{\theta_1, \theta_2, y}(t+s) \Big|_{s=0} \\ &= \frac{\partial}{\partial s} B(\theta_2 + t(\theta_1 - \theta_2) + s(\theta_1 - \theta_2), y) \Big|_{s=0} \\ &= dB(\theta_2 + t(\theta_1 - \theta_2), y; \theta_1 - \theta_2) \\ &= \int_{\mathbb{R}^d} dx (\theta_1(x) - \theta_2(x)) (DB)(\theta_2 + t(\theta_1 - \theta_2), y; x), \end{aligned}$$

which yields

$$\begin{aligned} |B(\theta_1, y) - B(\theta_2, y)| &= |C_{\theta_1, \theta_2, y}(1) - C_{\theta_1, \theta_2, y}(0)| \leq \max_{t \in [0, 1]} \left| \frac{\partial}{\partial t} C_{\theta_1, \theta_2, y}(t) \right| \\ &\leq \|\theta_1 - \theta_2\|_1 \max_{t \in [0, 1]} \|(DB)(\theta_2 + t(\theta_1 - \theta_2), y; \cdot)\|_{L^\infty(\mathbb{R}^d)}. \end{aligned}$$

Moreover, since $B \in \mathcal{E}_{\alpha''}$, by using estimate (5.7) we obtain, for any $r > 0$,

$$\begin{aligned} |B(\theta_1, y) - B(\theta_2, y)| &\leq \|\theta_1 - \theta_2\|_1 \frac{1}{r} \max_{t \in [0, 1]} \sup_{\|\theta'\|_1 \leq r} |B(\theta_2 + t(\theta_1 - \theta_2) + \theta', y)| \\ &\leq \|\theta_1 - \theta_2\|_1 \frac{e}{\alpha''} \|B(\cdot, y)\|_\infty^{\alpha''} \max_{t \in [0, 1]} e^{\frac{\|\theta_2 + t(\theta_1 - \theta_2)\|_1}{\alpha''}}. \end{aligned}$$

The latter implies that for any $\theta \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$

$$\begin{aligned} &\left| B\left(\theta e^{-\varepsilon\phi(\cdot-y)} + \frac{e^{-\varepsilon\phi(\cdot-y)} - 1}{\varepsilon}, y\right) - B(\theta - \phi(\cdot-y), y) \right| \leq \\ &\leq \frac{e}{\alpha''} \|B(\cdot, y)\|_\infty^{\alpha''} \left[\left\| \theta (1 - e^{\varepsilon\phi(\cdot-y)}) \right\|_1 + \left\| \phi(\cdot-y) - \frac{1 - e^{-\varepsilon\phi(\cdot-y)}}{\varepsilon} \right\|_1 \right] \times \\ &\quad \max_{t \in [0, 1]} \exp \left\{ \frac{1}{\alpha''} \left(\left\| \theta e^{-\varepsilon\phi(\cdot-y)} + \frac{1 - e^{-\varepsilon\phi(\cdot-y)}}{\varepsilon} \right\|_1 t + (1-t) \|\theta - \phi(\cdot-y)\|_1 \right) \right\} \\ &\leq \varepsilon \frac{e}{\alpha''} \|\phi\|_\infty \|B(\cdot, y)\|_\infty^{\alpha''} (\|\theta\|_1 + \|\phi\|_1) e^{\frac{1}{\alpha''}(\|\theta\|_1 + \|\phi\|_1)}, \end{aligned} \quad (5.116)$$

where the last inequality follows from (4.104) and estimates (4.107)-(4.108). As a result, we find

$$\begin{aligned} \left| \left(\tilde{L}_{RW, \varepsilon, \text{ren}}^{(1)} B \right) (\theta, y) - \left(\tilde{L}_{RW, V}^{(1)} B \right) (\theta, y) \right| &\leq \varepsilon \frac{e}{\alpha''} C_\phi^\infty (\|\theta\|_1 + C_\phi^1) e^{\frac{1}{\alpha''}(\|\theta\|_1 + C_\phi^1)} \times \\ &\quad \left[\int_{\mathbb{R}^d} dz a(y-z) \|B(\cdot, z)\|_\infty^{\alpha''} + \langle a \rangle \|B(\cdot, y)\|_\infty^{\alpha''} \right]. \end{aligned}$$

Thus, the corresponding norm in $\mathcal{E}_{\alpha'}$ can be bounded as follows

$$\begin{aligned} \left\| \tilde{L}_{RW, \varepsilon, \text{ren}}^{(1)} B - \tilde{L}_{RW, V}^{(1)} B \right\|_{\alpha'} &\leq \int_{\mathbb{R}^d} dy \sup_{\theta \in L^1} e^{-\frac{\|\theta\|_1}{\alpha'}} \left| \left(\tilde{L}'_{RW, \varepsilon, \text{ren}} B \right) (\theta, y) - \left(\tilde{L}'_{RW, V} B \right) (\theta, y) \right| \\ &\leq 2\varepsilon \frac{e}{\alpha''} C_\phi^\infty \langle a \rangle \|B\|_{\alpha''} e^{\frac{C_\phi^1}{\alpha''}} \left\{ \sup_{\theta \in L^1} \|\theta\|_1 e^{-\|\theta\|_1 \left(\frac{1}{\alpha'} - \frac{1}{\alpha''} \right)} + C_\phi^1 \sup_{\theta \in L^1} e^{-\|\theta\|_1 \left(\frac{1}{\alpha'} - \frac{1}{\alpha''} \right)} \right\} \end{aligned}$$

$$\leq 2\varepsilon C_\phi^\infty \langle a \rangle e^{\frac{c_\phi^1}{\alpha}} \|B\|_{\alpha''} \left(\frac{\alpha_0}{\alpha'' - \alpha'} + \frac{e}{\alpha''} C_\phi^1 \right), \quad (5.117)$$

where in the last step we use inequality (5.71). Finally, by combining (5.114) with (5.117) we obtain the desired result. \square

We are now in the position to show the following result about the convergence of the Vlasov-type scaling.

Theorem 5.32. *Assume that conditions (5.32), (5.78), (5.48) and (5.104) are satisfied. Let $0 < \alpha < \alpha_0$ be fixed and T_1 be given by (5.57). Suppose also that $\{B_{0,\varepsilon}, B_{0,V}\}_{\varepsilon>0} \subset \mathcal{E}_{\alpha_0}$ and*

$$\lim_{\varepsilon \rightarrow 0} \|B_{0,\varepsilon} - B_{0,V}\|_{\alpha_0} = 0. \quad (5.118)$$

Then initial value problems (5.81) and (5.95) have unique solutions $B_{t,\varepsilon}, B_{t,V} : [0, T_1] \rightarrow \mathcal{E}_\alpha$ such that for each $t \in [0, T_1]$,

$$\lim_{\varepsilon \rightarrow 0} \|B_{t,\varepsilon} - B_{t,V}\|_\alpha = 0. \quad (5.119)$$

Moreover, if

$$B_{0,V}(\theta, y) = B_{0,V}^{(\text{coh})}(\theta, y) = r_0(y) e^{\int_{\mathbb{R}^d} dx \theta(x) \rho_0(x)}, \quad \theta \in L^1(\mathbb{R}^d), y \in \mathbb{R}^d, \quad (5.120)$$

for some function $\rho_0 \in \overline{B}_{1/\alpha_0}^\infty$ and $r_0 \in L^1(\mathbb{R}^d)$, then for each $t \in [0, T_1]$,

$$B_{t,V}(\theta, y) = B_{t,V}^{(\text{coh})}(\theta, y) = r_t(y) e^{\int_{\mathbb{R}^d} dx \theta(x) \rho_t(x)}, \quad \theta \in L^1(\mathbb{R}^d), y \in \mathbb{R}^d, \quad (5.121)$$

provided that $\rho_t \in \overline{B}_{1/\alpha_0}^\infty$ and $r_t \in L^1(\mathbb{R}^d)$ are solutions to the Vlasov equations

$$\frac{\partial \rho_t}{\partial t} = (\rho_t * \tilde{c})(x) e^{-(\rho_t * V)(x)} - \rho_t(x) (\tilde{c} * e^{-(\rho_t * V)})(x), \quad (5.122a)$$

$$\frac{\partial r_t}{\partial t} = -\langle a \rangle e^{-(\rho_t * \phi)} r_t + \left((e^{-(\rho_t * \phi)} r_t) * a \right), \quad (5.122b)$$

with initial conditions $\rho_t|_{t=0} = \rho_0 \in \overline{B}_{1/\alpha_0}^\infty$ and $r_t|_{t=0} = r_0 \in L^1(\mathbb{R}^d)$.

Proof. In Lemma 5.30 we have already showed the existence of solutions to the initial value problems (5.81) and (5.95). In order to prove convergence (5.119) we apply Theorem 4.9 to (5.81). For this purpose it is enough to verify the validity of conditions (4.40)-(4.42) for the considered model. As shown in the proof of Lemma 5.30, see in particular (5.111), condition (4.40) is satisfied with

$$M = 2\alpha_0 \langle \tilde{c} \rangle e^{\frac{c_V^1}{\alpha}} + c_{\alpha, \alpha_0}, \quad \text{with} \quad c_{\alpha, \alpha_0} = 2\langle a \rangle e^{\frac{c_\phi^1}{\alpha}} (\alpha_0 - \alpha).$$

Moreover, by Proposition 5.31 for any $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$ and all $\varepsilon > 0$ we have

$$\begin{aligned} \|\tilde{L}'_{\varepsilon, \text{ren}} B - \tilde{L}'_V B\|_{\alpha'} &\leq 2\varepsilon \alpha_0 \frac{\|B\|_{\alpha''}}{\alpha'' - \alpha'} \left[\langle \tilde{c} \rangle C_V^\infty \frac{e}{\alpha} e^{\frac{c_V^1}{\alpha}} \left(2e C_V^1 + \frac{\alpha_0}{e} \right) + \langle a \rangle C_\phi^\infty e^{\frac{c_\phi^1}{\alpha}} (1 + c'_{\alpha, \alpha_0}) \right] + \\ &2\varepsilon \alpha_0 \frac{\|B\|_{\alpha''}}{(\alpha'' - \alpha')^2} \left[8\langle \tilde{c} \rangle C_V^\infty \frac{e}{\alpha} e^{\frac{c_V^1}{\alpha}} (\alpha_0)^2 \right], \end{aligned} \quad (5.123)$$

where

$$c'_{\alpha, \alpha_0} = \frac{e}{\alpha \alpha_0} C_\phi^1 (\alpha_0 - \alpha).$$

Hence, conditions (4.41)-(4.42) are satisfied with $p = 2$ and

$$N_\varepsilon = 2\varepsilon \alpha_0 \max \left\{ \langle \tilde{c} \rangle C_V^\infty \frac{e}{\alpha} e^{\frac{c_V^1}{\alpha}} \left(2e C_V^1 + \frac{\alpha_0}{e} \right) + \langle a \rangle C_\phi^\infty e^{\frac{c_\phi^1}{\alpha}} (1 + c'_{\alpha, \alpha_0}), 8\langle \tilde{c} \rangle C_V^\infty \frac{e}{\alpha} e^{\frac{c_V^1}{\alpha}} (\alpha_0)^2 \right\}.$$

Let us now prove the second part of the statement. From Lemma 5.27 and (5.105) one can easily see that $B_{t,V}^{(\text{coh})} \in \mathcal{E}_\alpha$ is solution to (5.95) with initial condition $B_{0,V}^{(\text{coh})} \in \mathcal{E}_{\alpha_0}$ if ρ_t and r_t solve (5.122a)-(5.122b), on the time interval $[0, T_1]$. The uniqueness of this solution is ensured by Lemma 5.30. \square

The existence and uniqueness of solutions to the kinetic equations (5.122a)-(5.122b) is considered in the following theorem.

Theorem 5.33. *Given $C > 0$, let us consider $\rho_0 \in L^\infty(\mathbb{R}^d)$ and $r_0 \in L^1(\mathbb{R}^d)$ such that $0 \leq \rho_0(x) \leq C$, for a.a. $x \in \mathbb{R}^d$, and $r_0(y) \geq 0$, for a.a. $y \in \mathbb{R}^d$, respectively. Suppose that conditions (5.32) and (5.48) and hold. Then, the system of Vlasov equations (5.122a)-(5.122b) have a unique non-negative solution $0 \leq \rho \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ and $0 \leq r \in \mathcal{C}^1(\mathbb{R}_+; L^1(\mathbb{R}^d))$. Moreover, given $C' > 0$ for any time $t > 0$*

(i) *if $\|r_0\|_1 \leq C'$ then $\|r_t\|_1 \leq C'$;*

(ii) *if $r_0(y) \leq C'$, a.a. $y \in \mathbb{R}^d$, then $r_t(y) \leq C'$, a.a. $y \in \mathbb{R}^d$.*

Proof. First let us note that, according to Lemma 5.28, equation (5.122a) has a unique non-negative solution $0 \leq \rho \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ which is uniformly bounded on any finite time interval. Then, the statement can be shown proceeding as in Step.2 of the proof of Lemma 4.21. \square

5.2.2.2 RW in a Kawasaki model of environment: Case II

Consider interaction $\lambda^{(2)}$ defined by (5.58)-(5.60). According to the results of Section 2.3.2, see (2.146) and (2.147), in this case we have

$$(K^{-1}\lambda_\varepsilon^{(2)}(\cdot, y))(\xi) = \lambda_0 0^{|\xi|} + \varepsilon \phi(x' - y) \mathbf{1}_{\Gamma(1)}(\xi = \{x'\}), \quad \varepsilon > 0, \quad (5.124)$$

and

$$A_V^{(2)}(\xi, y) = \lambda_0 0^{|\xi|} + \phi(x' - y) \mathbf{1}_{\Gamma(1)}(\xi = \{x'\}), \quad (5.125)$$

for any $\xi \in \Gamma_0$ and a.a. $y \in \mathbb{R}^d$. Then, by Proposition 5.22 and 5.25, given $\alpha > 0$ for any $B \in \mathcal{E}_\alpha$ the operators $\tilde{L}'_{RW, \varepsilon, \text{ren}}$ and $\tilde{L}'_{RW, V}$ have the form

$$\begin{aligned} (\tilde{L}'_{RW, \varepsilon, \text{ren}} B)(\theta, y) &= (\tilde{L}_{RW, \varepsilon, \text{ren}}^{(2)} B)(\theta, y) := \lambda_0 \int_{\mathbb{R}^d} dz a(y-z) (B(\theta, z) - B(\theta, y)) + \\ &\int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx' [\phi(x'-z) (DB)(\theta, z; x') - \phi(x'-y) (DB)(\theta, y; x')] + \\ &\varepsilon \int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx' \theta(x) [\phi(x'-z) (DB)(\theta, z; x') - \phi(x'-y) (DB)(\theta, y; x')] \end{aligned} \quad (5.126)$$

and

$$\begin{aligned} (\tilde{L}'_{RW, V} B)(\theta, y) &= (\tilde{L}_{RW, V}^{(2)} B)(\theta, y) = \\ &= \lambda_0 \int_{\mathbb{R}^d} dz a(y-z) (B(\theta, z) - B(\theta, y)) + \\ &\int_{\mathbb{R}^d} dz a(y-z) \int_{\mathbb{R}^d} dx' [\phi(x'-z) (DB)(\theta, z; x') - \phi(x'-y) (DB)(\theta, y; x')]. \end{aligned} \quad (5.127)$$

For the two operators above we can show the following estimate in the scale of Banach spaces $\{\mathcal{E}_\alpha, 0 < \alpha \leq \alpha_0\}$.

Proposition 5.34. *Assume that conditions (5.59) and (5.60) hold. Given $\alpha_0 > \alpha > 0$, let $B \in \mathcal{E}_{\alpha''}$ for some $\alpha'' \in (\alpha, \alpha_0]$. Then,*

$$\left\| \tilde{L}_{RW,\varepsilon,\text{ren}}^{(2)} \right\|_{\alpha'} \leq 2 \left(\lambda_0 \langle a \rangle + \frac{e \langle a \rangle}{\alpha} C_\phi^1 + \frac{\alpha_0}{\alpha'' - \alpha'} \langle a \rangle C_\phi^\infty \right) \|B\|_{\alpha''}, \quad (5.128)$$

for all $\alpha \leq \alpha' < \alpha''$ and $\varepsilon \in (0, 1]$, and

$$\left\| \tilde{L}_{RW,V}^{(2)} \right\|_{\alpha'} \leq 2 \left(\lambda_0 \langle a \rangle + \frac{e \langle a \rangle}{\alpha} C_\phi^1 \right) \|B\|_{\alpha''}, \quad (5.129)$$

for all $\alpha \leq \alpha' \leq \alpha''$.

Proof. The operators $\tilde{L}_{RW,\varepsilon,\text{ren}}^{(2)}$ and $\tilde{L}_{RW,V}^{(2)}$ can be written as

$$\begin{aligned} \left(\tilde{L}_{RW,\varepsilon,\text{ren}}^{(2)} B \right) (\theta, y) &= \left(\tilde{L}_{RW,1}^{(2)} B \right) (\theta, y) + \left(\tilde{L}_{RW,2}^{(2)} B \right) (\theta, y) + \varepsilon \left(\tilde{L}_{RW,3}^{(2)} B \right) (\theta, y), \\ \left(\tilde{L}_{RW,V}^{(2)} B \right) (\theta, y) &= \left(\tilde{L}_{RW,1}^{(2)} B \right) (\theta, y) + \left(\tilde{L}_{RW,2}^{(2)} B \right) (\theta, y), \end{aligned}$$

where $\tilde{L}_{RW,1}^{(2)}$, $\tilde{L}_{RW,2}^{(2)}$ and $\tilde{L}_{RW,3}^{(2)}$ are defined by (5.64)-(5.66). Then, we can repeat the same estimate as in Lemma 5.20. and obtain the desired results. \square

An application of Theorem 4.2 yields the following existence and uniqueness result for the solutions of the initial value problems (5.81) and (5.95).

Lemma 5.35. *Assume that conditions (5.32), (5.59) and (5.60) are satisfied. Given $\alpha_0 > 0$, for each $\alpha \in (0, \alpha_0)$ the Cauchy problems (5.81) and (5.95) with $B_{0,\varepsilon}, B_{0,V} \in \mathcal{E}_{\alpha_0}$, $\varepsilon \in (0, 1]$, have unique solutions $B_{t,\varepsilon}, B_{t,V} : [0, T_1) \rightarrow \mathcal{E}_\alpha$ on the time interval $[0, T_2)$, where T_2 is given by (5.75).*

Proof. By combining the results stated in Proposition 5.24 and 5.34, for any $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$ we have

$$\left\| \tilde{L}'_{\#} B \right\|_{\alpha'} \leq 2 \frac{\alpha_0}{\alpha'' - \alpha'} \left(\langle \tilde{c} \rangle e^{\frac{C_V^1}{\alpha}} + \langle a \rangle C_\phi^\infty \right) \|B\|_{\alpha''} + 2 \left(\lambda_0 \langle a \rangle + \frac{e \langle a \rangle}{\alpha} C_\phi^1 \right) \|B\|_{\alpha''}, \quad (5.130)$$

where $\# = \text{"}\varepsilon, \text{ren"}$ or $\text{"}V\text{"}$. Then the statement follows from Theorem 4.2. \square

We can now ask whether solutions $B_{t,\varepsilon}$ converge to $B_{t,V}$ on the interval $[0, T_2)$ as ε goes to zero. For this purpose we first derive an auxiliary result which provides an estimate analogous to that one in Proposition 5.31.

Proposition 5.36. *Assume that conditions (5.32), (5.78), (5.59) and (5.60) are satisfied. Given $0 < \alpha < \alpha_0$, let α', α'' such that $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$. Then, for any $B \in \mathcal{E}_{\alpha''}$ and $\varepsilon \in (0, 1]$ the following estimate holds*

$$\begin{aligned} \left\| \tilde{L}'_{\varepsilon,\text{ren}} B - \tilde{L}'_V B \right\|_{\alpha'} &\leq 2\varepsilon \langle \tilde{c} \rangle C_V^\infty \frac{e\alpha_0}{\alpha} \|B\|_{\alpha''} e^{\frac{C_V^1}{\alpha}} \left(\left(2eC_V^1 + \frac{\alpha_0}{e} \right) \frac{1}{\alpha'' - \alpha'} + \frac{8\alpha_0^2}{(\alpha'' - \alpha')^2} \right) + \\ &2\varepsilon \langle a \rangle C_\phi^\infty \frac{\alpha_0}{\alpha'' - \alpha'} \|B\|_{\alpha''}. \end{aligned} \quad (5.131)$$

Proof. For any $B \in \mathcal{E}_{\alpha''}$ one has

$$\left\| \tilde{L}'_{\varepsilon,\text{ren}} B - \tilde{L}'_V B \right\|_{\alpha'} \leq \left\| \tilde{L}'_{RE,\varepsilon,\text{ren}} B - \tilde{L}'_{RE,V} B \right\|_{\alpha'} + \left\| \tilde{L}'_{RW,\varepsilon,\text{ren}} B - \tilde{L}'_{RW,V} B \right\|_{\alpha'}. \quad (5.132)$$

From (5.126) and (5.127), for any $\varepsilon \in (0, 1]$ we have

$$\left| \left(\tilde{L}_{RW,\varepsilon,\text{ren}}^{(2)} B \right) (\theta, y) - \left(\tilde{L}_{RW,V}^{(2)} B \right) (\theta, y) \right| \leq \varepsilon \left| \left(\tilde{L}_{RW,3}^{(2)} B \right) (\theta, y) \right|,$$

where the operator $\tilde{L}_{RW,3}^{(2)}$ is given by (5.66). Then, by using (5.73), one may estimate

$$\left\| \tilde{L}_{RW,\varepsilon,\text{ren}}^{(2)} B - \tilde{L}_{RW,V}^{(2)} B \right\|_{\alpha'} \leq 2\varepsilon \langle a \rangle C_\phi^\infty \frac{\alpha_0}{\alpha'' - \alpha'} \|B\|_{\alpha''}, \quad (5.133)$$

for any $\alpha \leq \alpha' < \alpha'' \leq \alpha_0$ and $\varepsilon \in (0, 1]$. The latter inequality together with (5.114) yields the desired result. \square

Having established this preliminary result, the convergence of solutions of (5.81) to the solution of (5.95) can be shown by using Theorem 4.9.

Theorem 5.37. *Suppose that conditions (5.32), (5.78), (5.59) and (5.60) are satisfied. Let $0 < \alpha < \alpha_0$ be fixed and consider $B_{0,\varepsilon}, B_{0,V} \in \mathcal{E}_{\alpha_0}$, $\varepsilon \in (0, 1]$, such that*

$$\lim_{\varepsilon \rightarrow 0} \|B_{0,\varepsilon} - B_{0,V}\|_{\alpha_0} = 0. \quad (5.134)$$

Then, given T_2 defined by (5.75), the Cauchy problems (5.81) and (5.95) have unique solutions $B_{t,\varepsilon}, B_{t,V} : [0, T_2] \rightarrow \mathcal{E}_\alpha$ such that for each $t \in [0, T_2]$,

$$\lim_{\varepsilon \rightarrow 0} \|B_{t,\varepsilon} - B_{t,V}\|_\alpha = 0. \quad (5.135)$$

Moreover, given $\rho_0 \in \overline{B}_{1/\alpha_0}^\infty$ and $r_0 \in L^1(\mathbb{R}^d)$, for $B_{0,V}(\theta, y) = r_0(y) e^{\int_{\mathbb{R}^d} dx \theta(x) \rho_0(x)}$, $y \in \mathbb{R}^d$ and $\theta \in L^1(\mathbb{R}^d)$, we have

$$B_{t,V}(\theta, y) = B_{t,V}^{(\text{coh})}(\theta, y) = r_t(y) e^{\int_{\mathbb{R}^d} dx \theta(x) \rho_t(x)}, \quad (5.136)$$

provided that $\rho_t \in \overline{B}_{1/\alpha_0}^\infty$ and $r_t \in L^1(\mathbb{R}^d)$ are solutions to the system of equations

$$\frac{\partial \rho_t}{\partial t} = (\rho_t * \tilde{c})(x) e^{-(\rho_t * V)(x)} - \rho_t(x) (\tilde{c} * e^{-(\rho_t * V)})(x), \quad (5.137a)$$

$$\frac{\partial r_t}{\partial t} = (a * [r_t (\lambda_0 + (\rho_t * \phi))]) - \langle a \rangle r_t [\lambda_0 + (\rho_t * \phi)], \quad (5.137b)$$

with initial conditions $\rho_t|_{t=0} = \rho_0 \in \overline{B}_{1/\alpha_0}^\infty$ and $r_t|_{t=0} = r_0 \in L^1(\mathbb{R}^d)$, on the time interval $[0, T_2]$.

Proof. The proof is similar to that of Theorem 5.32. Having in mind the results of Lemma 5.35 and Proposition 5.36, in order to show the first part we can apply Theorem 4.9 taking $p = 2$ and

$$N_\varepsilon = 2\varepsilon \alpha_0 \max \left\{ \langle \tilde{c} \rangle C_V^\infty \frac{e}{\alpha} e^{\frac{C_V^1}{\alpha}} \left(2e C_V^1 + \frac{\alpha_0}{e} \right) + \langle a \rangle C_\phi^\infty, 8 \langle \tilde{c} \rangle C_V^\infty \frac{e}{\alpha} e^{\frac{C_V^1}{\alpha}} (\alpha_0)^2 \right\}.$$

Concerning the second part, by uniqueness of solutions to (5.95), it is enough to show that $B_{t,V}$ given in (5.136) solves (5.95). The latter follows from Lemma 5.27 by (5.125). \square

Finally, we can give the following existence and uniqueness result for the solutions to the systems of Vlasov equations (5.137a)-(5.137b).

Theorem 5.38. *Assume that conditions (5.32), (5.59) and (5.60) are satisfied. Given $C > 0$, let us consider $\rho_0 \in L^\infty(\mathbb{R}^d)$ and $r_0 \in L^1(\mathbb{R}^d)$ such that $0 \leq \rho_0(x) \leq C$, a.a. $x \in \mathbb{R}^d$, and $r_0(y) \geq 0$, a.a. $y \in \mathbb{R}^d$, respectively. Then, the system of Vlasov equations (5.137a)-(5.137b) has a unique non-negative solution $0 \leq r \in \mathcal{C}^1(\mathbb{R}_+; L^1(\mathbb{R}^d))$ and $0 \leq \rho \in \mathcal{C}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$. Moreover, given $C' > 0$, for any time $t > 0$*

- (i) if $\|r_0\|_1 \leq C'$ then $\|r_t\|_1 \leq C'$;
- (ii) if $r_0(y) \leq C'$, a.a. $y \in \mathbb{R}^d$, then $r_t(y) \leq C'$, a.a. $y \in \mathbb{R}^d$.

The proof of the above theorem is analogous to that one of Theorem 5.33. The details are left to the reader.

Appendix A

Ovsjannikov-type theorem

In this appendix we present a detailed proof of the Ovsjannikov-type result stated in Theorem 4.2 which we used to construct statistical dynamics of RWREs in Chapter 4 and 5.

The Ovsjannikov theorem provides existence and uniqueness of solutions to certain singular Cauchy problems in a scale of Banach spaces. This result is based on Picard-type approximations and a method by A. G. Kostyuchenko and G. E. Shilov presented in [66, Appendix 2, A2.1]. This method, originally considered for equations with time independent coefficients, has been extended to an abstract and general framework by T. Yamanaka in [123] and L. V. Ovsjannikov in [98] in the linear case, and many applications were exposed by F. Trèves in [120]. For a recent review of the Ovsjannikov's method we refer to [44].

In Theorem 4.2 we consider a bounded perturbation of singular operators in a scale of Banach spaces. The proof of this theorem is similar to the original proof of the Ovsjannikov theorem see [61] and [120, Chapter 3] for existence and uniqueness, respectively.

A.1 Proof of Theorem 4.2

We divide the proof of the theorem in two steps. First we find a solution to the initial value problem (4.9). Then we show that this solution is unique.

Step.1 *Existence of the solution.*

For some $t > 0$ which will be properly chosen later on, let us consider the sequence of functions $(u_n)_{n \in \mathbb{N}_0}$ with $u_0(t) = u_0 \in \mathbb{B}_{s_0}$ and

$$u_n(t) := u_0 + \int_0^t ds (Au_{n-1})(s), \quad n \in \mathbb{N}. \quad (\text{A.1})$$

By an induction argument, it is easy to check that $u_n(t) \in \mathbb{B}_s$ for any $s < s_0$, see e.g. [120]. Moreover, in an equivalent way, the sequence above may be rewritten as

$$u_n(t) := u_0 + \sum_{m=1}^n \frac{t^m}{m!} A^m u_0. \quad (\text{A.2})$$

Fix $0 < s < s_0$. We want to show the convergence of sequence (A.2) in \mathbb{B}_s . Let us consider a partition of the interval $[s, s_0)$ into m equal parts, $[s_l, s_{l+1})$, with

$$s_l := s_0 - \frac{l(s_0 - s)}{m}, \quad l = 0, \dots, m.$$

By assumption for each $l = 0, \dots, m$ we have

$$\|A\|_{s_l s_{l+1}} := \|A\|_{\mathbb{B}_l \rightarrow \mathbb{B}_{l+1}} \leq \frac{mM_1}{s_0 - s_1} + M_0 \quad (\text{A.3})$$

and thus

$$\|A^m\|_{s_0s} \leq \|A\|_{s_0s_1} \cdots \|A\|_{s_{m-1}s} \leq \left(\frac{mM_1}{s_0 - s_1} + M_0 \right)^m. \quad (\text{A.4})$$

From this follows that the norm in \mathbb{B}_s of the series in (A.2) can be written as

$$\sum_{m=1}^n \frac{t^m}{m!} \|A^m u_0\|_s \leq \|u_0\|_{s_0} \sum_{m=1}^n \frac{t^m}{m!} \left(\frac{mM_1}{s_0 - s_1} + M_0 \right)^m. \quad (\text{A.5})$$

In order to establish its convergence we use the Cauchy's root criterion, yielding

$$\sqrt[m]{\frac{t^m}{m!} \left(\frac{mM_1}{s_0 - s_1} + M_0 \right)^m} = \frac{tM_1}{s_0 - s} \frac{m}{\sqrt[m]{m!}} + tM_0 \frac{1}{\sqrt[m]{m!}}.$$

Letting m go to infinity we find

$$\lim_{m \rightarrow \infty} \sqrt[m]{\frac{t^m}{m!} \left(\frac{mM_1}{s_0 - s_1} + M_0 \right)^m} = \frac{tM_1 e}{s_0 - s} < 1,$$

whenever

$$t < \frac{s_0 - s}{M_1 e}. \quad (\text{A.6})$$

This means that under condition (A.6), the series (A.2) converges in \mathbb{B}_s to the function

$$u(t) := u_0 + \sum_{m=1}^{\infty} \frac{t^m}{m!} A^m u_0. \quad (\text{A.7})$$

Moreover, setting $\delta := 1/(eM_1)$ this convergence is uniform on any interval $[0, T] \subset [0, \delta(s_0 - s))$.

Similar arguments show that an analogous situation occurs for the time derivative of (A.2),

$$\frac{du_n(t)}{dt} = \sum_{m=1}^n \frac{t^{m-1}}{(m-1)!} A^m u_0. \quad (\text{A.8})$$

Indeed, by using (A.4) we obtain

$$\sum_{m=1}^n \frac{t^{m-1}}{(m-1)!} \|A^m u_0\|_s \leq \|u_0\|_{s_0} \sum_{m=1}^n \frac{t^{m-1}}{(m-1)!} \left(\frac{mM_1}{s_0 - s_1} + M_0 \right)^m. \quad (\text{A.9})$$

Then, by using again the Cauchy criterion we have

$$\sqrt[m]{\frac{t^{m-1}}{(m-1)!} \left(\frac{mM_1}{s_0 - s_1} + M_0 \right)^m} = \frac{1}{\sqrt[m]{t}} \left[\frac{tM_1}{s_0 - s} \frac{m}{\sqrt[m]{(m-1)!}} + tM_0 \frac{1}{\sqrt[m]{(m-1)!}} \right]$$

and thus

$$\lim_{m \rightarrow \infty} \sqrt[m]{\frac{t^{m-1}}{(m-1)!} \left(\frac{mM_1}{s_0 - s_1} + M_0 \right)^m} = \frac{tM_1 e}{s_0 - s} < 1,$$

whenever condition (A.6) holds. This means that on the time interval $(0, \delta(s_0 - s))$ the function $u(t)$ is continuously differentiable in \mathbb{B}_s . Finally, it is also possible to check, see e.g. [61], that $Au(t) \in \mathbb{B}_s$, showing that $u(t)$ is a solution to the initial value problem (4.9).

Step.2 *Uniqueness of the solution.*

It is enough to show that if a function $u(t) : [0, \delta(s_0 - s)) \rightarrow \mathbb{B}_{s_0}$ satisfies

$$\begin{cases} \frac{du(t)}{dt} = Au(t) \\ u(0) = 0 \end{cases}, \quad (\text{A.10})$$

then, it is identically zero. The set of points $N = \{t \in [0, \delta[s - s_0)) : u(t) = 0\}$ where the function u vanishes is closed; we are going to show that it is also open. This will imply that $N \equiv [0, \delta(s - s_0))$ and complete the proof.

Let $\tau \in N$. Then one has

$$u(t) = \int_{\tau}^t dt' (Au)(t'). \quad (\text{A.11})$$

Given an $0 < s < s_0$ such that $0 < \varepsilon < s_0 - s$, from assumption (4.10) it follows that

$$\begin{aligned} \|u(t)\|_s &\leq \int_{\tau}^t dt' \|(Au)(t')\|_s \\ &\leq (\varepsilon^{-1}M_1 + M_0) \int_{\tau}^t dt' \|u(t')\|_{s+\varepsilon}. \end{aligned} \quad (\text{A.12})$$

We want to show that, given $0 < s < s_0$, for any $k \geq 1$ whenever t is in some compact neighborhood K of τ contained in the set $[0, \delta(s - s_0))$, the following inequality holds

$$\|u(t)\|_s \leq M \left[\left(\frac{eM_1}{s_0 - s} + \frac{eM_0}{k} \right) |t - \tau| \right]^k, \quad (\text{A.13})$$

where

$$M := \max_{t \in K} \|u(t)\|_{s_0}.$$

Note that for $k = 1$ it follows directly from (A.12). Indeed, since $\|\cdot\|_{s'} \leq \|\cdot\|_{s_0}$ for any $s' < s_0$, we have

$$\begin{aligned} \|u(t)\|_s &\leq (\varepsilon^{-1}M_1 + M_0) \int_{\tau}^t \|u(t')\|_{s_0} dt' \\ &\leq M (\varepsilon^{-1}M_1 + M_0) |t - \tau|. \end{aligned}$$

In particular, if we choose $\varepsilon = (s_0 - s)/e$ we obtain

$$\begin{aligned} \|u(t)\|_s &\leq M \left(\frac{eM_1}{s_0 - s} + M_0 \right) |t - \tau| \\ &\leq M \left(\frac{eM_1}{s_0 - s} + eM_0 \right) |t - \tau|. \end{aligned} \quad (\text{A.14})$$

For $k \geq 2$ we prove (A.13) by induction. We know that for $k = 1$ it is satisfied. We assume that it holds for $k - 1$, with $k \geq 2$, namely

$$\|u(t)\|_s \leq M \left[\left(\frac{eM_1}{s_0 - s} + \frac{eM_0}{k-1} \right) |t - \tau| \right]^{k-1} \quad (\text{A.15})$$

and we check it for k . Condition (A.15) can be rewritten in the following form

$$\|u(t)\|_s \leq M \left[\frac{eM_1 |t - \tau|}{s_0 - s} \right]^{k-1} \left[1 + \frac{M_0(s_0 - s)}{M_1(k-1)} \right]^{k-1}. \quad (\text{A.16})$$

By inserting (A.16) in (A.12) we find

$$\|u(t)\|_s \leq M\varepsilon^{-1} (M_1 + \varepsilon M_0) \left[\frac{eM_1}{s_0 - s - \varepsilon} \right]^{k-1} \left[1 + \frac{M_0(s_0 - s - \varepsilon)}{M_1(k-1)} \right]^{k-1} \int_{\tau}^t |t' - \tau|^{k-1} dt'$$

$$\leq M\varepsilon^{-1}M_1 \left(1 + \frac{\varepsilon M_0}{M_1}\right) \left[\frac{eM_1}{s_0 - s - \varepsilon}\right]^{k-1} \left[1 + \frac{M_0(s_0 - s - \varepsilon)}{M_1(k-1)}\right]^{k-1} \frac{|t - \tau|^k}{k}.$$

As a matter of fact we can take

$$\varepsilon = \frac{s_0 - s}{k}, \quad s_0 - s - \varepsilon = (s_0 - s) \left(1 - \frac{1}{k}\right). \quad (\text{A.17})$$

With this choice we have

$$\|u(t)\|_s \leq M e^{k-1} \left[\frac{M_1}{s_0 - s}\right]^k \left(\frac{1}{1 - 1/k}\right)^{k-1} \left[1 + \frac{M_0(s_0 - s)}{M_1 k}\right]^k |t - \tau|^k \quad (\text{A.18})$$

and since

$$\left(\frac{1}{1 - 1/k}\right)^{k-1} = \left(1 + \frac{1}{k-1}\right)^{k-1} \leq e,$$

we obtain

$$\begin{aligned} \|u(t)\|_s &\leq M \left[\frac{eM_1}{s_0 - s}\right]^k \left[1 + \frac{M_0(s_0 - s)}{M_1 k}\right]^k |t - \tau|^k \\ &\leq M \left[\left(\frac{eM_1}{s_0 - s} + \frac{eM_0}{k}\right) |t - \tau|\right]^k. \end{aligned} \quad (\text{A.19})$$

Then, taking

$$|t - \tau| < \frac{1}{eM_1} (s_0 - s)$$

and $k \rightarrow \infty$, we see that $u(t)$, as an element of \mathbb{B}_s , vanishes in a neighborhood of τ . Clearly, as \mathbb{B}_{s_0} is naturally injected in \mathbb{B}_s , this must be true also when we consider $u(t)$ as an element of \mathbb{B}_{s_0} . This means that the set N is open.

This concludes the proof of the theorem. □

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