Overgroups of elementary block-diagonal subgroups in even unitary groups over quasi-finite rings

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0.1 Introduction

This dissertation describes the overgroups H of an elementary block-diagonal subgroup $\mathrm{EU}(\nu)$ of an even unitary group U over a quasi-finite form ring, under the assumption that the minimal size of a self-conjugate block of $\mathrm{EU}(\nu)$ is at least 6 (or 4 in the case that the form parameter of the ground ring is sufficiently large) and the minimal size of a non-self-conjugate block of $\mathrm{EU}(\nu)$ is at least 5. The main result is the following sandwich classification theorem: let H, $\mathrm{EU}(\nu)$ and U be as above. Then there exists a unique major exact form net of ideals (σ, Γ) over the ground form ring such that H fits into the sandwich

$$\mathrm{EU}(\sigma, \Gamma) \leq H \leq \mathrm{N_U}(\mathrm{U}(\sigma, \Gamma)),$$

where $\mathrm{N_U}(\mathrm{U}(\sigma, \Gamma))$ denotes the normalizer in U of the form net subgroup $\mathrm{U}(\sigma, \Gamma)$ of U of level (σ, Γ) and $\mathrm{EU}(\sigma, \Gamma)$ the elementary form net subgroup of $\mathrm{U}(\sigma, \Gamma)$.

To put this result into context we provide a brief overview of related results describing the subgroup structure of linear groups over fields. In [Dyn57a, Dyn57b] Dynkin determined the maximal closed connected subgroups of classical algebraic groups over $\mathbb{C}$. In particular, he showed that all reductive maximal closed connected subgroups are precisely the stabilizers of totally isotropic or non-degenerate subspaces. Similar results for classical groups over algebraically closed fields of positive characteristic were obtained by Gary Seitz [Sei87] and for exceptional groups by Donna Testerman [Tes88]. In the papers [Asc84, Asc85, Asc86] Michael Aschbacher described the maximal subgroups of finite simple classical groups. The subgroup structure theorem of Aschbacher says that every maximal subgroup of a finite simple classical group belongs to either one of 8 explicitly defined classes $\mathcal{C}_1$—$\mathcal{C}_8$ of large subgroups or to the class $\mathcal{S}$ of almost simple groups in irreducible representations. An exposition of results regarding the members of Aschbacher classes in finite classical groups can be found in the book of Kleidman and Liebeck [KL90]. The Aschbacher classes which are relevant for us are the classes $\mathcal{C}_1$ and $\mathcal{C}_2$. The subgroups of class $\mathcal{C}_1$ are stabilizers of proper totally isotropic or non-degenerate submodules of the module on which the group is acting. The subgroups of class $\mathcal{C}_2$ are the stabilizers of direct decompositions of that module into the summands of a fixed dimension. Given a member H of an Aschbacher class the book of Kleidman and Liebeck provides a recipe for constructing a maximal overgroup of H that is in turn also a member of some not necessarily the same Aschbacher class. Unfortunately, a similar result for classical groups over arbitrary commutative rings or more generally unitary groups over form rings has not yet been obtained. However, it is possible to describe the lattice structure of the set of all overgroups of a given member of an Aschbacher class or of an appropriate modified notion thereof in terms of the structure of the ground ring.

The literature contains several modifications of the notion of Aschbacher classes. The current dissertation focuses on a specific simultaneous modification of the Aschbacher classes $\mathcal{C}_1$ and $\mathcal{C}_2$ which we call *block-diagonal subgroups*. These subgroups are the stabilizers of certain direct decompositions of the quadratic module on which the unitary group is acting into totally isotropic or non-degenerate submodules. As our methods only employ the elementary unitary matrices contained in these block-diagonal subgroups, we will describe the overgroups of elementary block-diagonal subgroups instead of full block-diagonal subgroups. The problem of describing overgroups of elementary block-diagonal matrices was first considered for the case of the general linear group in papers [BVN70, BV82, Vav83, BV84, Vav87] of Z.I. Borewicz, N. A. Vavilov and W. Narkiewicz over commutative rings and rings satisfying a stable rank condition. These papers do not use localization. Over quasi-finite rings the problem is solved in [BS01] using localization. For the case of other classical groups over a commutative ring with 2 invertible, this classification was generalized in chapter V of the habilitation of Nikolai Vavilov, although complete proofs were only published much later in [Vav04] for the split orthogonal case and [Vav08] for the symplectic case. Important auxiliary results can be also found in [Vav93], [Vav88] and in the references therein. Roughly the main results in the above references can be described as follows. Let G denote a Chevalley group of type A_l, B_l, C_l or D_l over a commutative ring R or the general linear group $\mathrm{GL}(n, R)$ over a quasi-finite ring R. If $\mathrm{G} \neq \mathrm{G}(A_l, R)$ or $\mathrm{GL}(n, R)$, assume that $2 \in R^*$. Let H be a subgroup of G containing a group of elementary block-diagonal matrices whose minimal block size is sufficiently large. Then there exists a unique net of ideals σ such that H fits into the sandwich

$$\mathrm{E}(\sigma) \leq H \leq \mathrm{N}(\sigma),$$

where $\mathrm{E}(\sigma)$ is the elementary subgroup associated with σ and $\mathrm{N}(\sigma)$ is the normalizer in G of the net subgroup $\mathrm{G}(\sigma)$. In [BV84] such a description was called *standard*. We shall refer to it as *standard sandwich classification*.

Unfortunately, due to known counterexamples the standard sandwich classification in [Vav08] for the symplectic group $\mathrm{Sp}(2n, R)$ when 2 is invertible in the ring R does not generalize to the case of an arbitrary commutative ring. The obstacle is that the notion of a net of ideals is not fine enough when 2 is not invertible in the ring and has to be replaced by the notion of a form net of ideals. This is analogous to the situation encountered in the sandwich classification of subgroups of Bak unitary groups [Bak69], which are normalized by the elementary subgroup. Here the notion of ideal had to be refined by the notion of form ideal when 2 is not invertible in the ground ring. Significant work on developing the concept of a form net of ideals in the context of even unitary groups over fields and simple Artinian rings was done already by E. Dybkova [Dyb98, Dyb99, Dyb06, Dyb07, Dyb09]. Further review of known results on the problem of describing overgroups of subsystem subgroups in classical-like and some exceptional groups can be found in [VS13]. As mentioned there, no steps have yet been taken towards describing overgroups of subsystem subgroups in classical (other than GL) groups over an arbitrary commutative ring or Bak unitary groups over form rings other than simple Artinian form rings.

Historically, results regarding the subgroup structure of even unitary groups were usually proved first for the classical symplectic group, then for the classical orthogonal group, and only after that carried over to the general case of an arbitrary even dimensional Bak unitary group. Also, most of the results in this area involve versions of the localization-completion method, first introduced by Bak in [Bak91] for the general linear group and further developed by Bak, Golubchik, Hazrat, Mikhalev, Stepanov, Suslin, Vaserstein, Vavilov and many others. Following both of these trends, this dissertation consists first of analysing the case of the classical symplectic group over a commutative ring and then the general case of an even unitary group over a quasi-finite ring. This leaves open the problem of describing the overgroups of a block-diagonal subgroup in classical odd dimensional orthogonal groups, namely Chevalley groups of type B_n, over an arbitrary commutative ring and more generally its generalization the Bak odd dimensional unitary groups over form rings.

We sketch now in more detail the main results in this dissertation. Given an abstract group G and two subgroups A and B of G, call the subset $\mathrm{Transp}_G(A,B) = \{g \in G \mid {}^gA \leq B\}$ the *transporter in G from the subgroup A to the subgroup B*. In general, a transporter is not necessarily a subgroup. In fact, it contains the identity element if and only if $A \leq B$ and is in general not closed under taking products or multiplicative inverses. However, in some situations it turns out to be a subgroup, for example, when $A \leq B$ and B is normal. Furthermore, the normalizer $\mathrm{N}_G(B)$ of B in G, which is a group, is by definition $\mathrm{Transp}_G(B,B)$ and in general $\mathrm{N}_G(B) \leq \mathrm{Transp}_G(A,B)$ whenever $A \leq B$.

Given an exact form net of ideals (σ,Γ) over a form ring (R,Λ) denote by $\mathrm{U}(\sigma,\Gamma)$ the net subgroup of the even dimensional unitary group $\mathrm{U}(2n,R,\Lambda)$ defined by the form net (σ,Γ). Let $\mathrm{EU}(\sigma,\Gamma)$ denote the subgroup of $\mathrm{U}(\sigma,\Gamma)$ generated by the (σ,Γ)-elementary unitary matrices. For a unitary equivalence relation ν on the index set $\{1,\dots,n,-n,\dots,-1\}$ denote by $\mathrm{EU}(\nu)$ the group of elementary block-diagonal unitary matrices defined by ν. We will call (σ,Γ) a major form net of ideals with respect to ν if $\mathrm{EU}(\nu) \leq \mathrm{EU}(\sigma,\Gamma)$. Finally, let $h(\nu)$ denote the ordered pair (p,q), where p is the minimal size of a self-conjugate equivalence class of ν and q is the minimal size of a non-self-conjugate equivalence class of ν.

Theorem 1. *Let (R,Λ) be a quasi-finite form ring. Let $h(\nu) \geq (4,5)$ and suppose either $h(\nu) \geq (6,5)$ or $R\Lambda + \Lambda R = R$. Let H be a subgroup of $\mathrm{U}(2n,R,\Lambda)$ such that $\mathrm{EU}(\nu) \leq H$. Then there exists a unique major form net of ideals (σ,Γ) over (R,Λ) such that*

$$\mathrm{EU}(\sigma,\Gamma) \leq H \leq \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma,\Gamma),\mathrm{U}(\sigma,\Gamma)).$$

Furthermore, (σ,Γ) is the maximal form net of ideals such that $\mathrm{EU}(\sigma,\Gamma) \leq H$.

The next theorem shows that indeed the transporter $\mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma,\Gamma),\mathrm{U}(\sigma,\Gamma))$ in Theorem 1 is a group which is equal to the normalizer $\mathrm{N}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{U}(\sigma,\Gamma))$ of the net subgroup $\mathrm{U}(\sigma,\Gamma)$. Moreover it can be defined in terms of congruences as in the theorem below.

Theorem 2. *Let (R,Λ) be an arbitrary form ring with respect to an involution $\bar{\ }$ with symmetry λ. Let $h(\nu) \geq (4,3)$ and suppose either $h(\nu) \geq (6,4)$ or $R\Lambda + \Lambda R = R$. Let (σ,Γ) be an exact major form net over (R,Λ). Then $\mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma,\Gamma),\mathrm{U}(\sigma,\Gamma))$ coincides with the normalizer $\mathrm{N}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{U}(\sigma,\Gamma))$ and consists precisely of all matrices a in $\mathrm{U}(2n,R,\Lambda)$ that satisfy the following three conditions: for an invertible matrix a let $a'_{ij} = (a^{-1})_{ij}$ and $S_{k,-k}(a^{-1})$ be the length of the k'th row of a^{-1}.*

(T1) $a_{ij}\sigma_{jk}a'_{kl} \le \sigma_{il}$ *for all* $i,j,k,l \in I$

(T2) $a_{ij}\xi S_{k,-k}(a^{-1})\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \le \Gamma_i$ *for all* $i,j,k \in I$ *and* $\xi \in \sigma_{jk}$.

(T3) $a_{ij}\Gamma_j a'_{-j,-i} \le \Gamma_i$ *for all* $i,j \in I$.

The rest of this dissertation is organized as follows. Basic notation regarding matrices and equivalence relations is given in Section 0.2. After that, the dissertation is divided into three chapters. Each chapter begins with an outline of the results of that chapter. Chapter 1 treats the special case of Theorem 1 for the classical symplectic group and assumes that the elementary block-diagonal subgroup has no non-self-conjugate blocks. This is designed to make the proofs more transparent, while at the same time to highlight certain basic constructions, such as major and exact form nets of ideals. Chapter 2 presents the general symplectic case. Whereas the treatment in Chapter 1 does not use localization, Chapter 2 does and paves the way for its extensive use in Chapter 3. Chapter 3 proves the general case of Theorems 1 and 2.

0.2 Notations

Throughout this paper we will adhere to the following notations and conventions. By *ring* we will always mean associative unital ring. Given a ring R and a natural number n we will denote by $\mathrm{M}(n,R)$ the full matrix ring of rank n over R and by $\mathrm{GL}(n,R)$ the group of invertible elements of $\mathrm{M}(n,R)$. For any matrix a in $\mathrm{M}(n,R)$ we will denote by a_{ij} the entry of a at the position (i,j) and by a'_{ij} the corresponding entry of the matrix inverse to a. We will denote by a^t the matrix transpose of a, i.e. the matrix in $\mathrm{M}(n,R)$ such that $(a^t)_{ij} = a_{ji}$. By a_{i*} and a_{*j} we will denote the i'th row and j'th column of a respectively. Naturally, a'_{i*} and a'_{*j} should be read as the i'th row and j'th column of a^{-1} respectively. We will also use the notation $\mathrm{diag}(\xi_1,\ldots,\xi_n)$ for the diagonal matrix with entries $\xi_1,\ldots\xi_n$ reading from the top-left corner and $\mathrm{sdiag}(\xi_1,\ldots,\xi_n)$ for the skew-diagonal matrix with entries $\xi_1,\ldots\xi_n$ reading from the top-right corner. When the rank of the matrix ring is clear from the context, we will also denote by diag the diagonal embedding of R into $\mathrm{M}(n,R)$, i.e. $\mathrm{diag} : R \to \mathrm{M}(n,R)$ is a ring homomorphism sending each $\xi \in R$ to the diagonal matrix $\mathrm{diag}(\xi) = \mathrm{diag}(\xi,\ldots,\xi)$. We will denote by e_n the identity of the matrix ring $\mathrm{M}(n,R)$. When the rank is clear from the context, we will simply write e. The entries of e, as an exception from the above convention, will be denoted by δ_{ij} (Kronecker delta), while e_{ij} will stand for the corresponding standard matrix unit, i.e the matrix in $\mathrm{M}(n,R)$ whose (i,j)'th entry equals 1 and whose other entries are zero. Given a ring morphism $\varphi : R \to Q$ we will denote by $\mathrm{M}_n(\varphi) = \mathrm{M}(\varphi)$ the induced ring morphism of the matrix rings $\mathrm{M}(n,R)$ and $\mathrm{M}(n,Q)$. If we consider $\mathrm{M}(\varphi)$ as a morphism of the multiplicative monoids of $\mathrm{M}(n,R)$ and $\mathrm{M}(n,Q)$ then its kernel is precisely the set $\mathrm{M}(n,R,\ker(\varphi)) = \{a \in \mathrm{M}(n,R) \mid a_{ij} \equiv \delta_{ij} \mod \ker(\varphi) \text{ for all } i,j\}$. Note that $\mathrm{GL}(2n,R,\ker(\varphi)) = \mathrm{GL}(2n,R) \cap \mathrm{M}(2n,R,\ker(\varphi))$ is a normal subgroup in $\mathrm{GL}(2n,R)$.

In our applications it is convenient to index the rows and columns of $2n \times 2n$ matrices by the ordered set $I = I_{2n} = \{1, \dots, n, -n, \dots, -1\}$. We equip the poset I with the sign map $\varepsilon : I \longrightarrow \{\pm 1\}$, defined by

$$\varepsilon(i) = \begin{cases} +1 & i > 0 \\ -1 & i < 0 \end{cases}.$$

For the sake of shortening formulas we will also denote $\varepsilon(i)$ by ε_i.

Now consider an equivalence relation ν on the set I. If two indices i and j are equivalent under ν, we will write $i \sim^{\nu} j$ or just $i \sim j$ when the equivalence relation is clear from the context. The equivalence class of an index i will be denoted by $\nu(i)$. Call ν *unitary* if for any equivalent indices i and j the indices $-i$ and $-j$ are also equivalent. All the equivalence relations mentioned in this paper are unitary and thus we will sometimes omit the word "unitary". The index set I can be decomposed as a disjoint union of equivalence classes:

$$I = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_t.$$

We can introduce a left action of $\{\pm 1\}$ on the set $\mathcal{C}l(\nu) = \{C_1, \dots, C_t\}$ of all equivalence classes C_l of ν by putting: $1 \cdot \nu(i) = \nu(i)$ and $-1 \cdot \nu(i) = \nu(-i)$, for any $i \in I$. Following [Vav08] we will call the classes stable under this action *self-conjugate* (i.e. the classes C_l such that for every $i \in C_l$ one has also $-i \in C_l$). Accordingly the non-stable classes will be called *non-self-conjugate*. We will denote by $h(\nu)$ the ordered pair consisting of the minimum size (as a set) of all self-conjugate equivalence classes of ν and the minimum size (also as a set) of all non-self-conjugate equivalence classes of ν. Note, that an arbitrary equivalence relation does not necessary have equivalent classes of both types; therefore $h(\nu)$ is an element in $\mathbb{N} \cup \{\infty\} \times \mathbb{N} \cup \{\infty\}$. We will always view $\mathbb{N} \cup \{\infty\} \times \mathbb{N} \cup \{\infty\}$ as a partially ordered set with the *product order*, i.e. $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$.

We call a k-tuple $(i_1, \dots, i_k)$ of indices in I a C-*type base k-tuple* [*of indices*] if for each $1 \leq r \neq s \leq k$, we have $i_r \neq \pm i_s$ and $i_r \sim i_s \sim -i_s \sim -i_r$. Similarly, we call a k-tuple $(i_1, \dots, i_k)$ of indices in I an A-*type base k-tuple* [*of indices*] if for each $1 \leq r \neq s \leq k$, we have $i_r \neq \pm i_s$ and $i_r \sim i_s$. The condition $h(\nu) \geq (a, b)$ is equivalent to the condition that every index $i \in I$ can be included in either an A-type base b-tuple, or a C-type base $\lceil \frac{a}{2} \rceil$-tuple. This simple observation will be used repeatedly without specific mention in the rest of the paper.

Finally, by angular brackets $\langle \cdot \rangle$ we will denote the subgroups and ideals defined in terms of generators. The rest of the notations are standard for the field of this research.

1 Standard sandwich classification for symplectic subsystems in symplectic subgroups

In this chapter we classify the overgroups of elementary block-diagonal subgroups in symplectic groups over arbitrary commutative rings provided that all the blocks are self-conjugate. The methods we use to prove the standard sandwich classification theorem in this case closely resemble those used in [Vav08] with two main differences. First, we use long root unipotents, that are conjugates of long elementary symplectic transvections, rather than long root involutions, that are conjugates of long elementary diagonal matrices, as in [Vav08]. Second, we remove the assumption that 2 is invertible in the ground ring. As a result, we will have to deal with form nets of ideals instead of just nets of ideals.

Let R denote a commutative ring and $\mathrm{Sp}(2n, R)$ the classical simplectic group with coefficients in R. Let $\mathrm{Ep}(2n, R)$ denote the elementary subgroup of $\mathrm{Sp}(2n, R)$. Given a form net of ideals (σ, Γ) denote by $\mathrm{Ep}^L(\sigma, \Gamma)$ the subgroup of $\mathrm{Ep}(2n, R)$ generated by all long symplectic transvections $T_{i,-i}(\alpha)$, where $i \in I$ and $\alpha \in \Gamma_i$. The central result of this chapter is the following theorem.

Theorem 3. *Let ν be a unitary equivalence relation on the index set I such that $h(\nu) \geq (4, 5)$. Let $\mathrm{Ep}(\nu, R)$ denote the block-diagonal elementary subgroup of $\mathrm{Sp}(2n, R)$ defined by ν. Let H be a subgroup of $\mathrm{Sp}(2n, R)$ such that $\mathrm{Ep}(\nu, R) \leq H$. Denote by (σ, Γ) the form net of ideals associated with H. Then*

$$H \leq \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}^L(\sigma, \Gamma), \mathrm{Sp}(\sigma, \Gamma)).$$

Another important result is that the transporter $\mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma, \Gamma), \mathrm{Sp}(\sigma, \Gamma))$ coincides with the normalizer in $\mathrm{Sp}(2n, R)$ of $\mathrm{Sp}(\sigma, \Gamma)$ and can be described in terms of congruences.

Theorem 4. *Let ν be a unitary equivalence relation on the index set I such that all the equivalence classes of ν contain at least 3 elements. Let (σ, Γ) be a form net of ideals over R such that $[\nu]_R \leq (\sigma, \Gamma)$. Then the transporter $\mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma, \Gamma), \mathrm{Sp}(\sigma, \Gamma))$ coincides with the normalizer $\mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\sigma, \Gamma))$ and consists precisely of all matrices a in $\mathrm{Sp}(2n, R)$ such that the following three properties hold:*

(T1) $a_{ij}\sigma_{jk}a'_{kl} \leq \sigma_{il}$ *for all* $i, j, k, l \in I$

(T2) $a_{ij}^2\sigma_{jk}^{\boxed{2}}S_{k,-k}(a^{-1}) \in \Gamma_i$ *for all* $i, j, k \in I$

(T3) $a_{ij}^2\Gamma_j \leq \Gamma_i$ *for all* $i, j \in I$.

Note that both Theorem 3 and Theorem 4 hold without the assumption that all the equivalence classes of ν are self-conjugate. However, the conclusion of Theorem 3 is too weak for us. We want the stronger conclusion of Theorem 5 below. This is where we use the assumption that all the equivalence classes of ν are self-conjugate. With this assumption we will show that Theorem 5 follows already from Theorem 3. Theorem 5 is already a generalization of the main result in [Vav08].

Theorem 5. *Let* ν *be a unitary equivalence relation on the index set* I *such that all the equivalence classes of* ν *are self-conjugate and contain at least* 4 *elements. Let* H *be a subgroup of* $\mathrm{Sp}(2n, R)$ *such that* $\mathrm{Ep}(\nu, R) \leq H$. *Then there exists a unique exact form net of ideals* $(\sigma, \Gamma) \geq [\nu]_R$ *such that*

$$\mathrm{Ep}(\sigma, \Gamma) \leq H \leq \mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\sigma, \Gamma)).$$

An important difference in the methodology of the current chapter compared to that of successive ones is the absence of localizing, radical reduction, direct decomposition, scaling, and Morita equivalence. Theorem 5 is the frontier of results that can be proved without using localization methods.

The rest of this chapter is organized as follows. In Section 1.1 we give all the required definitions. In Section 1.2 we define the net associated with a subgroup and prove Theorem 4. In Section 1.3 we collect the results on extracting transvections from parabolic subgroups. Finally, in Section 1.4 we describe the method of extracting transvections from a long root element, which mimics the extracting transvections from a long root involution which is used in [Vav08], but without the assumption that 2 is invertible in the ground ring. Theorems 3 and 5 are proved there.

1.1 Preliminaries

Symplectic group Let R be a commutative ring. Fix a natural number n. Let $\mathrm{Sp}(2n, R)$ denote the classical symplectic group of rank $2n$. It is known that a matrix a in $\mathrm{GL}(2n, R)$ belongs to $\mathrm{Sp}(2n, R)$ if and only if the equality

$$a'_{ij} = \varepsilon_i \varepsilon_j a_{-j,-i}$$

holds for all possible indices. In the future, this property will be used without reference. We will also use the extension to a column or a row rather then single matrix element. The extension is given in the following proposition and can be checked by a straightforward calculation.

Proposition 1.1.1. *Let* $\cdot^t$ *denote the transpose operator in matrices. Let* a *be a matrix in* $\mathrm{Sp}(2n, R)$. *Fix arbitrary indices* i *and* j. *Then*

$$a_{i*} = \varepsilon_i (\mathfrak{p} a'_{*,-i})^t \qquad \textit{and} \qquad a_{*j} = -\varepsilon_j (a_{-j,*}\mathfrak{p})^t,$$

where $\mathfrak{p} = \mathrm{sdiag}(1, .., 1, -1, .., -1)$. *Obviously*

$$\mathfrak{p}^{-1} = \mathfrak{p}^t = -\mathfrak{p}.$$

Given an element $\xi \in R$ and an index $i \in I$ we will call the matrix

$$T_{i,-i}(\xi) = e + \xi e_{i,-i}$$

the [*elementary*] *long* [*symplectic*] *transvection.* Given an additional index $j \neq \pm i$ we will call the matrix

$$T_{ij}(\xi) = e + \xi e_{ij} - \varepsilon_i \varepsilon_j \xi e_{-j,-i}$$

the [*elementary*] *short* [*symplectic*] *transvection.*

It's a well known fact, that all the long and short elementary symplectic transvections are containd in $\mathrm{Sp}(2n, R)$ and satisfy the following relations known as the *Steinberg relations* for all ξ, ζ in R:

(R1) $T_{ij}(\xi) = T_{-j,-i}(-\varepsilon_i \varepsilon_j \xi)$ for all $i \neq \pm j$

(R2) $T_{ij}(\xi) T_{ij}(\zeta) = T_{ij}(\xi + \zeta)$ for all $i \neq j$

(R3) $[T_{ij}(\xi), T_{hk}(\zeta)] = e$ for all $h \neq j, -i$ and $k \neq i, -j$

(R4) $[T_{ij}(\xi), T_{jh}(\zeta)] = T_{ih}(\xi\zeta)$ for all $i, h \neq \pm j$ and $i \neq \pm h$

(R5) $[T_{ij}(\xi), T_{j,-i}(\zeta)] = T_{i,-i}(2\xi\zeta)$ for all i

(R6) $[T_{i,-i}(\xi), T_{-i,j}(\zeta)] = T_{ij}(\xi\zeta) T_{-j,j}(\varepsilon_i \varepsilon_j \xi \zeta^2)$ for all $i \neq \pm j$.

In future we will sometimes use these relations without a reference.

For any matrix g in $\mathrm{Sp}(2n, R)$, any short symplectic transvection $T_{sr}(\xi)$ and any long symplectic transvection $T_{s,-s}(\zeta)$, we call the matrices ${}^{g}T_{sr}(\xi) = gT_{sr}(\xi)g^{-1}$ and ${}^{g}T_{s,-s}(\zeta) = gT_{s,-s}(\zeta)g^{-1}$ [*elementary*] *short and long root elements* respectively.

Equivalence relations and block-diagonal subgroups Given a unitary equivalence relation ν on the index set I we will call the subgroup

$$\mathrm{Ep}(\nu) = \mathrm{Ep}(\nu, R) = \langle T_{i,-i}(\xi), T_{jk}(\xi) \mid i \sim -i, j \sim k, j \neq \pm k, \xi \in R \rangle$$

the *elementary block-diagonal subgroup of type* ν in $\mathrm{Sp}(2n, R)$. Note that $\mathrm{Ep}(\nu, R)$ does not necessarily consist of block-diagonal matrices. It can only be the case if all (or at least all but one of) the equivalence classes of ν are non-self-conjugate. However, these groups behave like block-diagonal matrices because for each such group $\mathrm{Ep}(\nu, R)$ there exists a permutation matrix B in $\mathrm{GL}(2n, R)$, but not necessarily in $\mathrm{Sp}(2n, R)$, such that $B \cdot \mathrm{Ep}(\nu, R) \cdot B^{-1}$ is block-diagonal in the usual sense.

From the point of view of Chevalley groups, the elementary block-diagonal subgroup is precisely *the elementary subsystem subgroup.* Namely, let $C_1, -C_1, \ldots C_s, -C_s$ be all the non-self-conjugate classes, and $C_{s+1}, \ldots, C_t$ be the self-conjugate ones. Set $n_i = |C_i|$ for $1 \leq i \leq s$ and $l_i = |C_i|/2$ for $s+1 \leq i \leq t$. Then $n_1 + \cdots + n_s + l_{s+1} + \cdots + l_t = n$ and

$$\mathrm{Ep}(\nu, R) \approx \mathrm{E}(n_1, R) \times \cdots \times \mathrm{E}(n_s, R) \times \mathrm{Ep}(2l_{s+1}, R) \times \cdots \times \mathrm{Ep}(2l_t, R),$$

where $\mathrm{E}(n_i, R)$ denotes the usual elementary subgroup of $\mathrm{GL}(n_i, R)$ appearing in the hyperbolic embedding and the product is meant as a product of linear groups. From the viewpoint of algebraic groups this is exactly the elementary Chevalley group of type Δ, where

$$\Delta = A_{n_1-1} + \cdots + A_{n_s-1} + C_{l_{s+1}} + \cdots + C_{l_t}.$$

The reader is referred to [Vav08] and the references therein for further details.

This analogy allows the following geometric interpretation of concepts of A-type and C-type base tuples By choosing an equivalence relation ν we fix a subsystem $\Delta \leq C_n$ consisting of all roots $\alpha_{ij} \in C_n$ where $i \sim^{\nu} j$. Irreducible components of Δ are in one to one correspondence with equivalence classes of ν, namely the components of type C_l correspond to self-conjugate equivalence classes and the components of type A_l correspond to pairs of non-self-conjugate classes. Then an A- or C- type base k-tuple $(i_1, \ldots, i_k)$ provides us with a root subsystem in Δ of type A_{k-1} or C_k respectively, namely $\langle \alpha_{i_1,i_2}, \ldots, \alpha_{i_{k-1},i_k} \rangle$ or $\langle \alpha_{i_1,i_2}, \ldots, \alpha_{i_{k-1},i_k}, \alpha_{i_k} \rangle$, respectively. Moreover, both generating sets above can be chosen as systems (or bases) of simple roots in the subsystems they generate.

Form nets of ideals and corresponding groups Consider a square array $\sigma = (\sigma_{ij})_{i,j \in I}$ of $(2n)^2$ ideals of the ring R. We will call it *a net of ideals over* R if for any indices i, j and k, we have the following inclusions:

$$\sigma_{ik}\sigma_{kj} \leq \sigma_{ij}.$$

A net of ideals σ is called *unitary*, if $\sigma_{ij} = \sigma_{-j,-i}$ for each i and j. We will call σ a D-*net*, if $\sigma_{ii} = R$ for every i in I. Equip the net of ideals σ with $2n$ additive subgroups $\Gamma = (\Gamma_i)_{i \in I}$ of R such that for any indices $i, j \in I$ the following inclusions hold:

1. $2\sigma_{i,-i} \leq \Gamma_i \leq \sigma_{i,-i}$
2. $\sigma_{ij}^{\boxed{2}}\Gamma_j \leq \Gamma_i$,

where $2\sigma_{i,-i} = \{2\alpha \mid \alpha \in \sigma_{i,-i}\}$ and $\sigma_{ij}^{\boxed{2}} = \{\xi^2 | \xi \in \sigma_{ij}\}$. In this situation Γ is called *a column of form parameters for* σ and the pair (σ, Γ) *a form net of ideals*. It is the analogue for nets of ideals of the concept of form ideal of Bak [Bak69] for form rings. A form net of ideals (σ, Γ) is said to be *exact* if for any index i the equality

$$\sigma_{i,-i} = \sum_{k \neq \pm i} \sigma_{ik}\sigma_{k,-i} + \langle \Gamma_i \rangle$$

holds. Any form net of ideals is assumed to be a unitary exact form D-net of ideals.

Introduce a partial ordering on the set of all form nets of ideals over R by setting $(\sigma', \Gamma') \leq (\sigma'', \Gamma'')$ whenever for all $i, j \in I$ the inclusions $\sigma'_{ij} \leq \sigma''_{ij}$ and $\Gamma'_i \leq \Gamma''_i$ hold. As a matter of convenience, given an element $\xi \in R$ and indices s and r we will write "$\xi \in (\sigma, \Gamma)_{sr}$" instead of "$\xi \in \sigma_{sr}$ if $r \neq -s$ and $\xi \in \Gamma_s$ otherwise".

We can associate two kinds of subgroups of $\mathrm{Sp}(2n, R)$ to each form net of ideals (σ, Γ) over R. We call the subgroup

$$\mathrm{Ep}(\sigma, \Gamma) = \langle T_{ij}(\xi), T_{i,-i}(\alpha) \mid i \neq \pm j, \xi \in \sigma_{ij}, \alpha \in \Gamma_i \rangle$$

the elementary form net subgroup of level (σ, Γ). We call the above generators of $\mathrm{Ep}(\sigma, \Gamma)$ *the short and long* (σ, Γ)*-elementary symplectic transvections*, respectively. Note that any equivalence relation ν on the set of indices I defines a form net

$$[\nu]_R = (\sigma_\nu, \Gamma_\nu),$$

where

$$(\sigma_\nu)_{ij} = \begin{cases} R, & \text{if } i \sim^\nu j \\ 0, & \text{if } i \nsim^\nu j \end{cases} \qquad (\Gamma_\nu)_i = \begin{cases} R, & \text{if } i \sim^\nu -i \\ 0 & i \nsim^\nu -i \end{cases}.$$

This is clearly a D-net. Clearly the elementary block-diagonal subgroup is a special case of an elementary form net subgroup. We will call a form net of ideals (σ, Γ) *major* [*with respect to* ν] if $[\nu]_R \leq (\sigma, \Gamma)$. The notion of elementary form net subgroup is a generalization of that of relative elementary subgroup of even unitary groups introduced in Bak [Bak81, p. 66]. The concept of the relative principal congruence subgroups therein is generalized as follows. We will call the subgroup

$$\mathrm{Sp}(\sigma, \Gamma) = \{g \in \mathrm{Sp}(2n, R) \mid \forall i, j \in I \; g_{ij} \in \sigma_{ij}, S_{i,-i}(g) \in \Gamma_i\},$$

the form net subgroup of level (σ, Γ), where

$$S_{i,-i}(g) = \sum_{j>0} g_{ij} g'_{j,-i}$$

is the so called *length of the row* g_{i*}. The element $S_{i,-i}(g)$ is clearly in $\sigma_{i,-i}$, by definition of a net of ideals, and insisting that $S_{i,-i}(g) \in \Gamma_i \leq \sigma_{i,-i}$ is a further restrinction on g. The word "length" was introduced by You in [You12].

In the situation when $\Gamma_i = R$ for all i the form net subgroup coincides with the regular *net subgroup* $\mathrm{Sp}(\sigma) = \{g \in \mathrm{Sp}(2n, R) \mid \forall i, j \in I \; g_{ij} \in \sigma_{ij}\}$. The next proposition allows us to compute lengths of rows of products of two matrices in terms of lengths of rows of factors.

Proposition 1.1.2. *Let a and b be two matrices in* $\mathrm{Sp}(2n, R)$. *Then*

$$\begin{aligned} S_{i,-i}(ab) = S_{i,-i}(a) &+ \sum_k a_{ik} S_{k,-k}(b) a'_{-k,-i} - \\ &- 2 \sum_{j,k,l>0} a_{i,l} b_{l,-j} b'_{-j,k} a'_{k,-i} - \\ &- 2 \sum_{j,k>0} \sum_{l>k} (a_{i,-k} b_{-k,-j} b'_{-j,l} a'_{l,-i} + a_{ik} b_{k,-j} b'_{-j,-l} a'_{-l,-i}). \end{aligned}$$

Proof. By the definition of the length of a row and the formula for the coefficients of the product of two matrices we get:

$$S_{i,-i}(ab) = \sum_{j>0} (ab)_{ij}(ab)'_{j,-i} = \sum_{j>0}\sum_{k,l} a_{ik}b_{kj}b'_{jl}a'_{l,-i}.$$

Next we collect the summands corresponding to the four different combinations of signs of the indices k and l, rewrite them in a way that all the sums are taken over positive indices and then group them in the following way:

$$\begin{aligned} S_{i,-i}(ab) = & \sum_{j,k,l>0} (a_{ik}b_{kj}b'_{jl}a'_{l,-i} + a_{i,-k}b_{-k,j}b'_{j,-l}a'_{-l,-i}) + \\ & + \sum_{j,k,l>0} (a_{i,-k}b_{-k,j}b'_{jl}a'_{l,-i} + a_{ik}b_{kj}b'_{j,-l}a'_{-l,-i}). \end{aligned} \tag{1.1}$$

Denote the first sum of (1.1) by X and the second by Y. In X swap the indices k and l in the second summand in brackets and then replace every matrix entry in the same summand with the corresponding one of the inverse matrix. Doing this, we get

$$X = \sum_{j,k,l>0} (a_{ik}b_{kj}b'_{jl}a'_{l,-i} - a_{i,k}b_{k,-j}b'_{-j,l}a'_{l,-i}).$$

Now as $\sum_j b_{kj}b'_{jl} = (b \cdot b^{-1})_{kl} = \delta_{kl}$ we have

$$X = \sum_{k>0} a_{ik}a'_{k,-i} - 2\sum_{j,k,l>0} a_{i,k}b_{k,-j}b'_{-j,l}a'_{l,-i} = S_{i,-i}(a) - 2\sum_{j,k,l>0} a_{i,k}b_{k,-j}b'_{-j,l}a'_{l,-i}. \tag{1.2}$$

Consider the summand Y. We group together the summands of Y such that $l > k$, $l = k$ and $l < k$ to obtain three summands. In the last summand swap the indices l and k. Finally we join the first and last summands together. Summing up,

$$\begin{aligned} Y = & \sum_{j,k>0}\sum_{l>k} \big((a_{i,-k}b_{-k,j}b'_{jl}a'_{l,-i} + a_{i,-l}b_{-l,j}b'_{jk}a'_{k,-i}) \\ & + (a_{ik}b_{kj}b'_{j,-l}a'_{-l,-i} + a_{il}b_{lj}b'_{j,-k}a'_{-k,-i})\big) \\ & + \sum_{j,k>0} \left(a_{i,-k}b_{-k,j}b'_{jk}a'_{k,-i} + a_{ik}b_{kj}b'_{j,-k}a'_{-k,-i}\right). \end{aligned} \tag{1.3}$$

Denote the first sum in (1.3) by Z and the second by W. Clearly

$$W = \sum_k a_{ik}\left(\sum_{j>0} b_{kj}b'_{j,-k}\right)a'_{-k,-i} = \sum_k a_{ik}S_{k,-k}(b)a'_{-k,-i}. \tag{1.4}$$

As for Z, passing where necessary to the elements of inverse matrices we get:

$$\begin{aligned} Z = & \sum_{j,k>0}\sum_{l>k} ((a_{i,-k}b_{-k,j}b'_{jl}a'_{l,-i} - a_{i,-k}b_{-k,-j}b'_{-j,l}a'_{l,-i}) \\ & + (a_{ik}b_{kj}b'_{j,-l}a'_{-l,-i} - a_{ik}b_{k,-j}b'_{-j,-l}a'_{-l,-i})) \end{aligned}$$

and using the formula $\sum_j b_{-k,j}b'_{jl} = 0$ whenever $k, l > 0$ we get

$$Z = \sum_{j,k>0} \sum_{l>k} (-2a_{i,-k}b_{-k,-j}b'_{-j,l}a'_{l,-i} - 2a_{ik}b_{k,-j}b'_{-j,-l}a'_{-l,-i}). \tag{1.5}$$

Combining the equalities (1.1), (1.2), (1.4) and (1.5) we get the required formula. □

Corollary 1.1.3. *Let (σ, Γ) be a form net of ideals over R. Suppose $a, b \in \mathrm{Sp}(\sigma)$, then*

$$S_{i,-i}(ab) \equiv S_{i,-i}(a) + \sum_k a_{ik}^2 S_{k,-k}(b) \mod \Gamma_i. \tag{1.6}$$

In particular, the form net subgroup $\mathrm{Sp}(\sigma, \Gamma)$ is indeed a group.

Proof. Clearly $e \in \mathrm{Sp}(\sigma, \Gamma)$. Next the congruences (1.6) together with the condition $\sigma_{ik}^{2}\Gamma_k \leq \Gamma_i$ show that $\mathrm{Sp}(\sigma, \Gamma)$ is closed under taking products. Let $a \in \mathrm{Sp}(\sigma, \Gamma)$. By (1.6) we get that

$$0 = S_{i,-i}(e) = S_{i,-i}(a^{-1}a) \equiv S_{i,-i}(a^{-1}) + \sum_k (a'_{ik})^2 S_{k,-k}(a) \equiv S_{i,-i}(a^{-1}) \mod \Gamma_i.$$

Therefore $a^{-1} \in \mathrm{Sp}(\sigma, \Gamma)$ and $\mathrm{Sp}(\sigma, \Gamma)$ is a group. □

The following two corollaries allow us to compute the lengths of rows in products of matrices in $\mathrm{Sp}(\sigma)$ and short (σ, Γ)-symplectic transvection as well as lengths of rows of some root elements.

Corollary 1.1.4. *Let $a \in \mathrm{Sp}(\sigma)$ and $T_{pq}(\xi)$ be a short symplectic transvection in $\mathrm{Ep}(\sigma, \Gamma)$. Then*

$$S_{i,-i}(T_{pq}(\xi)a) \equiv \begin{cases} S_{i,-i}(a) & \text{if } i \neq p, -q \\ S_{p,-p}(a) + \xi^2 S_{q,-q}(a) & \text{if } i = p \\ S_{-q,q}(a) + \xi^2 S_{-p,p}(a) & \text{if } i = -q \end{cases} \mod \Gamma_i$$

and for all $i \in I$

$$S_{i,-i}(aT_{pq}(\xi)) \equiv S_{i,-i}(a) \mod \Gamma_i.$$

Corollary 1.1.5. *Let $a \in \mathrm{Sp}(\sigma)$, $T_{sr}(\xi), T_{st}(\zeta)$ be short root elements in $\mathrm{Ep}(\sigma)$ and $s \neq \pm r, \pm t$ and $r \neq \pm t$. Then*

$$\begin{aligned} S_{i,-i}(aT_{sr}(\xi)T_{st}(\zeta)a^{-1}) \equiv\ & a_{is}^2\zeta^2 S_{t,-t}(a^{-1}) + a_{is}^2\xi^2 S_{r,-r}(a^{-1}) + \\ & + a_{i,-t}^2\zeta^2 S_{-s,s}(a^{-1}) + a_{i,-r}^2\xi^2 S_{-s,s}(a^{-1}) \mod \Gamma_i. \end{aligned}$$

In particular if $\zeta = 0$ then

$$S_{i,-i}(aT_{sr}(\xi)a^{-1}) \equiv a_{is}^2\xi^2 S_{r,-r}(a^{-1}) + a_{i,-r}^2\xi^2 S_{-s,s}(a^{-1}) \mod \Gamma_i.$$

Proof. Clearly $aT_{sr}(\xi)T_{st}(\zeta)a^{-1} \in \mathrm{Sp}(\sigma)$. Then combining Proposition 1.1.2 and Corollary 1.1.4 we get

$$\begin{aligned}
S_{i,-i}(aT_{sr}(\xi)T_{st}(\zeta)a^{-1}) &\equiv S_{i,-i}(a) + \sum_k \varepsilon_i\varepsilon_k a_{ik}^2 S_{k,-k}(T_{sr}(\xi)T_{st}(\zeta)a^{-1}) \\
&\equiv S_{i,-i}(a) + \sum_k \varepsilon_i\varepsilon_k a_{ik}^2 (S_{k,-k}(a^{-1}) + \\
&\quad + \delta_{ks}\zeta^2 S_{t,-t}(a^{-1}) + \delta_{ks}\xi^2 S_{r,-r}(a^{-1}) + \\
&\quad + \delta_{k,-t}\zeta^2 S_{-s,s}(a^{-1}) + \delta_{k,-r}\xi^2 S_{-s,s}(a^{-1})) \\
&\equiv S_{i,-i}(a\cdot a^{-1}) + \varepsilon_i\varepsilon_s a_{is}^2\zeta^2 S_{t,-t}(a^{-1}) + \varepsilon_i\varepsilon_s a_{is}^2\xi^2 S_{r,-r}(a^{-1}) + \\
&\quad + \varepsilon_i\varepsilon_{-t} a_{i,-t}^2\zeta^2 S_{-s,s}(a^{-1}) + \varepsilon_i\varepsilon_{-r} a_{i,-r}^2\xi^2 S_{-s,s}(a^{-1}).
\end{aligned}$$

Clearly $S_{i,-i}(e) = 0$ for all i and the signs are insignificant due to the inclusion $2\sigma_{i,-i} \leq \Gamma_i$. Therefore we get the required congruence. □

We finish this section with two technical results which will be used repeatedly and without reference in proofs throughout the paper. The first one shows that major form nets of ideals are partitioned into blocks in which all ideals are equal and all form parameters are equal. The second one allows simplifying reasoning dealing with case-by-case analysis of small equivalence classes. The proofs of both results can be checked straightforwardly and are left to the reader.

Proposition 1.1.6. *Let (σ, Γ) be a form net of ideal such that $\mathrm{Ep}(\nu) \leq \mathrm{Ep}(\sigma, \Gamma)$, with $h(\nu) \geq (4,3)$. Then for any indices i, j, k, l such that $k \sim i$, $l \sim j$, we have:*

1. $\sigma_{kj} = \sigma_{ij} = \sigma_{il}$
2. $\Gamma_i = \Gamma_k$.

Proposition 1.1.7. *Let ν be a unitary equivalence relation on the index set I such that $h(\nu) \geq (4,3)$. Let i, j be two indices in I such that $i \neq j$. Then one of the following holds:*

1. $\nu(i) = \{i, -i, j, -j\}$
2. *There exists an index k in I such that $k \neq \pm i, \pm j$ and $k \sim^{\nu} i$.*

1.2 Form net associated with a subgroup and the description of the transporter

Form net of ideals associated with a subgroup. Let ν be a unitary equivalence relation on the index set I such that $h(\nu) \geq (4,3)$. Let H be a subgroup of $\mathrm{Sp}(2n, R)$ such that $\mathrm{Ep}(\nu, R) \leq H$. An exact form net of ideal (σ, Γ) is called *the net associated with H* if $\mathrm{Ep}(\sigma, \Gamma) \leq H$ and if for any exact form net of ideals (σ', Γ') such that $\mathrm{Ep}(\sigma', \Gamma') \leq H$, it follows that $(\sigma', \Gamma') \leq (\sigma, \Gamma)$. Clearly, if (σ, Γ) exists then it is unique. The next lemma shows that (σ, Γ) exists.

Lemma 1.2.1. *Let ν be a unitary equivalence relation on I such that $h(\nu) \geq 3$ and H a subgroup of* $\mathrm{Sp}(2n, R)$ *that contains the subgroup* $\mathrm{Ep}(\nu, R)$. *For each $i \neq \pm j \in I$ set*

$$\sigma_{ij} = \{\xi \in R \mid T_{ij}(\xi) \in H\}, \quad \Gamma_i = \{\alpha \in R \mid T_{i,-i}(\alpha) \in H\}, \quad \sigma_{ii} = R, \\ \sigma_{i,-i} = \sum_{j \neq \pm i} \sigma_{ij}\sigma_{j,-i} + \langle \Gamma_i \rangle \,. \tag{1.7}$$

Then (σ, Γ) is the form net of ideals associated with H.

Proof. First, we will prove that (σ, Γ) is a form net of ideals. We will split the proof of this into the following steps:

1. σ_{ij} is an ideal of R for any indices $i, j \in I$ and Γ_i is an additive subgroup for all $i \in I$.

2. $2\sigma_{i,-i} \leq \Gamma_i$ for any $i \in I$.

3. $\sigma_{ij}\sigma_{jk} \leq \sigma_{ik}$ for any i, j and k.

4. $\Gamma_i \leq \sigma_{i,-i}$ for all $i \in I$.

5. $\sigma_{ij}^{2}\Gamma_j \leq \Gamma_i$ for any i and j.

The rest of the properties required for (σ, Γ) being a form net of ideals are obviously fulfilled.

1. Due to the Steinberg relation (R2) it's clear that all σ_{ij} and Γ_i are additive subgroups in R. Moreover, for any $i, j \in I$ such that $i \sim^{\nu} j$ as well as $j = \pm i$, σ_{ij} is an ideal by definition. To prove, that all σ_{ij} are ideals it only remains to handle the case when $i \neq \pm j$ and $i \not\sim^{\nu} j$. In this situation, according to Proposition 1.1.7, it's possible to pick an index $k \neq \pm i$ such that (j, k) is an A-type base pair. Take any ξ in σ_{ij} and any ζ in $R = \sigma_{jk} = \sigma_{kj}$. By the Steinberg relation (R4) we get

$$T_{ij}(\xi\zeta) = [[T_{ij}(\xi), T_{jk}(\zeta)], T_{kj}(1)] \in H$$

and thus $\xi\zeta \in R$. Therefore σ_{ij} is an ideal in R.

2. Next we will show that for every index i the ideal $2\sigma_{i,-i}$ is contained in Γ_i. As we already know that Γ_i is an additive subgroup, it's sufficient to check this property separately for each summand in the definition of $\sigma_{i,-i}$. Let $\alpha \in \sigma_{i,-i}$. First suppose α admits a decomposition $\alpha = \alpha_1\alpha_2$, where $\alpha_1 \in \sigma_{ij}$ and $\alpha_2 \in \sigma_{j,-i}$ for some $j \neq \pm i$. This means that $T_{ij}(\alpha_1)$ and $T_{j,-i}(\alpha_2)$ are contained in H. Then, by the Steinberg relation (R5):

$$T_{i,-i}(2\alpha_1\alpha_2) = [T_{ij}(\alpha_1), T_{j,-i}(\alpha_2)] \in H,$$

and thus $2\alpha \in \Gamma_i$. Now take α in Γ_i. We will show that $2\alpha\xi$ lies in Γ_i for any $\xi \in R$. Take any index j such that $j \neq \pm i$ and $\zeta \in \sigma_{ji}$. According to the relation (R6) we get

$$T_{i,-j}(\alpha\zeta)T_{j,-j}(\varepsilon_i\varepsilon_j\alpha\zeta^2) = [T_{i,-i}(\alpha), T_{-i,-j}(\zeta)] \in H.$$

Then if $i \not\sim -j$, it follows from Proposition 1.1.7 that there exists an index $k \neq \pm j$ such that (i, k) is an A-type base pair. Then

$$T_{i,-j}(\alpha\zeta) = [T_{ik}(1), [T_{ki}(1), T_{i,-j}(\alpha\zeta)T_{j,-j}(\varepsilon_i\varepsilon_j\alpha\zeta^2)]] \in H,$$

hence $\alpha\zeta \in \sigma_{i,-j}$. In the case when $i \sim -j$ this inclusion is trivial. Thus we have proved the following useful relation:

$$\sigma_{ji}\Gamma_i \leq \sigma_{j,-i} \tag{1.8}$$

for all $j \neq \pm i$.

Now take j such that (i, j) is an A-type base pair. Then (1.8) yields the inclusion $\alpha \in \sigma_{j,-i}$ and it's only left to notice that

$$T_{i,-i}(2\alpha\xi) = [T_{ij}(\xi), T_{j,-i}(\alpha)] \in H.$$

3. Now we prove the inclusions $\sigma_{ij}\sigma_{jk} \leq \sigma_{ik}$ for all indices i, j and k in I.

For indices i, j, k such that $i \neq \pm j, \pm k$ and $j \neq \pm k$ these relations obviously follow from the Steinberg relation (R4). Note also, that for $k \sim^{\nu} i$ the corresponding inclusions are redundant as the right-hand side equals R by definition. The inclusions are also obvious for the case $k = -i$ due to the definition of $\sigma_{i,-i}$. If $j = i$ or $j = k$, the inclusion follows from the fact that every σ_{ij} is an ideal. Thus, we can assert that $k \neq \pm i, j \neq i, j \neq k$ and $k \not\sim^{\nu} i$. Therefore it's enough to consider the case $j = -i, k \neq \pm i$. But in this case, as $j \not\sim k$ there exists an extra index $l \sim i$ such that $l \neq \pm i, \pm k$. Thus

$$\sigma_{i,-i}\sigma_{-i,k} = \sum_{t \neq \pm i} \sigma_{it}\sigma_{t,-i}\sigma_{-i,k} + \langle \Gamma_i \rangle\, \sigma_{-i,k}.$$

We already know that $\sigma_{it}\sigma_{t,-i}\sigma_{-i,k} \leq \sigma_{ik}$ for all $t \neq \pm i, \pm k$. By (1.8) we get that $\langle \Gamma_i \rangle\, \sigma_{-i,k} = \Gamma_i\sigma_{-i,k} \leq \sigma_{ik}$. Obviously $\sigma_{ik}\sigma_{k,-i}\sigma_{-i,k} \leq \sigma_{ik}$ and so it's only left to prove that $\sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,k} \leq \sigma_{ik}$. But

$$\sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,k} = \sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,k}\sigma_{kl} \leq \sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,l} \leq \sigma_{i,-k}\sigma_{-k,l} \leq \sigma_{il}$$

and it's clear that $\sigma_{il} \leq \sigma_{il}R = \sigma_{il}\sigma_{lk} \leq \sigma_{ik}$. So $\sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,k} \leq \sigma_{ik}$.

4. Fix an index $i \in I$. Pick another index j such that (i, j) is an A-type base pair. By (1.8) we get $\sigma_{ji}\Gamma_i \leq \sigma_{j,-i}$. We have already proved that $\sigma_{j,-i} \leq R\sigma_{j,-i} = \sigma_{ij}\sigma_{j,-i} \leq \sigma_{i,-i}$. As σ_{ji} contains the identity it follows that $\Gamma_i \leq \sigma_{i,-i}$ for all i in I.

5. Finally we prove that for any indices i, j we have the following inclusion: $\sigma_{ij}^{\boxed{2}}\Gamma_j \leq \Gamma_i$. Let $\alpha \in \Gamma_j$ and $\xi \in \sigma_{ij}$. If $j \neq \pm i$, then by the Steinberg relation (R6) we get

$$T_{j,-i}(\alpha\xi)T_{i,-i}(\varepsilon_i\varepsilon_j\alpha\xi^2) = [T_{j,-j}(\alpha), T_{-j,-i}(\xi)] \in H.$$

Recall that according to (1.8) we have $T_{j,-i}(\alpha\xi) \in H$ and thus also $T_{i,-i}(\varepsilon_i\varepsilon_j\alpha\xi^2) \in H$ which proves that $\sigma_{ij}^{\boxed{2}}\Gamma_j \leq \Gamma_i$ for all $j \neq \pm i$.

Now the case when $i = j$ is quite obvious. Pick any index k such that (i, k) is an A-type base pair. By (1.8) we get

$$\sigma_{ii}^{\boxed{2}}\Gamma_i = R^{\boxed{2}}\Gamma_i = \sigma_{ki}^{\boxed{2}}\Gamma_i \leq \Gamma_k \leq R^{\boxed{2}}\Gamma_k = \sigma_{ik}^{\boxed{2}}\Gamma_k \leq \Gamma_i.$$

Finally it remains to consider the case $j = -i$. Again pick an index j such that (i, j) is an A-base pair. Then

$$\sigma^{\boxed{2}}_{i,-i}\Gamma_{-i} = (\sigma_{ji}\sigma_{i,-i})^{\boxed{2}}\Gamma_{-i} \le \sigma^{\boxed{2}}_{j,-i}\Gamma_{-i} \le \Gamma_j \le \sigma^{\boxed{2}}_{ij}\Gamma_j \le \Gamma_i.$$

Therefore $\sigma^{\boxed{2}}_{ij}\Gamma_j \le \Gamma_i$ for all i and j. Hence, (σ, Γ) is an exact major form net of ideals. By construction, $\mathrm{Ep}(\sigma, \Gamma) \le H$ and for any exact form net of ideals (σ', Γ') such that $\mathrm{Ep}(\sigma', \Gamma') \le H$, it follows that $(\sigma', \Gamma') \le (\sigma, \Gamma)$. This completes the proof. □

Description of the transporter. The rest of this section is devoted to the proof of Theorem 4. The following proposition shows that the lengths of rows of matrices in $\mathrm{Sp}(2n, R)$ that already satisfy the property (T1) look relatively simple modulo minimal form parameters. This is an analogue and a generalization of Proposition 1.1.2.

Proposition 1.2.2. *Let (σ, Γ) be an exact form net. Suppose $a \in \mathrm{Sp}(2n, R)$ satisfies the condition*

$$a_{ij}\sigma_{jk}a'_{kl} \le \sigma_{il}$$

for all $i, j, k, l \in I$. Then for any matrix $g \in \mathrm{Sp}(\sigma, \Gamma)$ and any $i \in I$ the following congruence holds:

$$S_{i,-i}(aga^{-1}) \equiv \sum_{k\in I} a_{ik}^2 \left(S_{k,-k}(g) + S_{k,-k}(a^{-1}) + \sum_{t\in I} g_{kt}^2 S_{t,-t}(a^{-1}) \right) \mod \Gamma_i.$$

Proof. By Proposition 1.1.2 we get

$$\begin{aligned} S_{i,-i}(aga^{-1}) = S_{i,-i}(a) &+ \sum_{k\in I} \varepsilon_i\varepsilon_k a_{ik}^2 S_{k,-k}(ga^{-1}) \\ &- 2\sum_{j,k,l>0} a_{il}(ga^{-1})_{l,-j}(ag^{-1})_{-j,k}a'_{k,-i} \\ &- 2\sum_{j,k>0}\sum_{l>k}(a_{i,-k}(ga^{-1})_{-k,-j}(ag^{-1})_{-j,l}a'_{l,-i} + \\ &\qquad a_{ik}(ga^{-1})_{k,-j}(ag^{-1})_{-j,-l}a'_{-l,-i}). \end{aligned} \tag{1.9}$$

Consider the summand of the second big sum above. By the assumption that $a_{ij}\sigma_{jk}a'_{kl} \le \sigma_{il}$ we get

$$a_{il}(ga^{-1})_{l,-j}(ag^{-1})_{-j,k}a'_{k,-i} = \sum_{p,q\in I}(a_{il}g_{lp}a'_{p,-j})(a_{-j,q}g'_{qk}a'_{k,-i}) \le \sigma_{i,-j}\sigma_{-j,-i} \le \sigma_{i,-i} \tag{1.10}$$

and therefore the doubled second big sum in (1.9) is contained in $2\sigma_{i,-i} \le \Gamma_i$. Applying the same principle to the last summand in (1.9) we get the inclusion

$$a_{i,-k}(ga^{-1})_{-k,-j}(ag^{-1})_{-j,l}a'_{l,-i} + a_{ik}(ga^{-1})_{k,-j}(ag^{-1})_{-j,-l}a'_{-l,-i}) \in \sigma_{i,-i}. \tag{1.11}$$

Combining (1.9), (1.10) and (1.11) we get

$$S_{i,-i}(aga^{-1}) \equiv S_{i,-i}(a) + \sum_{k\in I} \varepsilon_i\varepsilon_k a_{ik}^2 S_{k,-k}(ga^{-1}) \mod \Gamma_i. \tag{1.12}$$

Expand (1.12) further using Proposition 1.1.2:

$$\begin{aligned}
S_{i,-i}(aga^{-1}) &\equiv S_{i,-i}(a) + \sum_{k\in I} \varepsilon_i\varepsilon_k a_{ik}^2 S_{k,-k}(ga^{-1}) \\
&\equiv S_{i,-i}(a) + \sum_{k\in I} \varepsilon_i\varepsilon_k a_{ik}^2 \Bigg(S_{k,-k}(g) + \sum_{t\in I} \varepsilon_k\varepsilon_t g_{kt}^2 S_{t,-t}(a^{-1}) \\
&\quad -2 \sum_{j,t,l>0} g_{kl} a'_{l,-j} a_{-j,t} g'_{t,-k} \\
&\quad -2 \sum_{j,t>0} \sum_{l>t} \big(g_{k,-t} a'_{-t,-j} a_{-j,l} g'_{l,-k} + g_{it} a'_{t,-j} a_{-j,-l} g'_{-l,-l} \big) \Bigg) \\
&\equiv S_{i,-i}(a) + \sum_{k\in I} \varepsilon_i\varepsilon_k a_{ik}^2 \Bigg(S_{k,-k}(g) + \sum_{t\in I} \varepsilon_k\varepsilon_t g_{kt}^2 S_{t,-t}(a^{-1}) \Bigg) \\
&\quad - 2 \sum_{j,t,l>0} a_{ik} g_{kl} a'_{l,-j} a_{-j,t} g'_{t,-k} a'_{-k,-i} \\
&\quad - 2 \sum_{j,t>0} \sum_{l>t} \big(a_{ik} g_{k,-t} a'_{-t,-j} a_{-j,l} g'_{l,-k} a'_{-k,-i} \\
&\qquad\qquad + a_{ik} g_{it} a'_{t,-j} a_{-j,-l} g'_{-l,-l} a'_{-k,-i} \big) \\
&\quad \mod \Gamma_i.
\end{aligned} \tag{1.13}$$

Using the same trick as before we may conclude that both doubled terms of (1.13) are contained in $2\sigma_{i,-i} \leq \Gamma_i$. Summing up, we get the congruence

$$S_{i,-i}(aga^{-1}) \equiv S_{i,-i}(a) + \sum_{k\in I} \varepsilon_i\varepsilon_k a_{ik}^2 \Bigg(S_{k,-k}(g) + \sum_{t\in I} \varepsilon_k\varepsilon_t g_{kt}^2 S_{t,-t}(a^{-1}) \Bigg) \mod \Gamma_i. \tag{1.14}$$

Finally it's easy to see that

$$\varepsilon_i\varepsilon_k a_{ik}^2 S_{k,-k}(g) = \sum_{l>0, p\in I} (a_{ik} g_{kl} a'_{lp})(a_{pl} g'_{l,-k} a'_{-k,-i}) \in \sigma_{i,-i}$$

and

$$\varepsilon_i\varepsilon_k\varepsilon_k\varepsilon_t a_{ik}^2 g_{kt}^2 S_{t,-t}(a^{-1}) = \sum_{l>0} (a_{ik} g_{kt} a'_{tl})(a'_{l,-t} g'_{-t,-k} a'_{-k,-i}) \in \sigma_{i,-i}.$$

Therefore the choice of signs in (1.14) is insignificant and we can rewrite (1.14) as follows

$$S_{i,-i}(aga^{-1}) \equiv S_{i,-i}(a) + \sum_{k\in I} a_{ik}^2 \Bigg(S_{k,-k}(g) + \sum_{t\in I} g_{kt}^2 S_{t,-t}(a^{-1}) \Bigg) \mod \Gamma_i. \tag{1.15}$$

It' clear that $S_{i,-i}(e) = 0$ for all i. Rewrite the formula (1.15) for $g = e$. As $S_{i,-i}(a) = \sum_{j>0} a_{ij}\delta_{jj}a'_{j,-i} \leq \sigma_{i,-i}$, we can also change the sign at the first term:

$$0 = S_{i,-i}(a \cdot a^{-1}) \equiv -S_{i,-i}(a) + \sum_{k\in I} a_{ik}^2 S_{k,-k}(a^{-1}) \mod \Gamma_i. \tag{1.16}$$

Finally, adding (1.16) to (1.15) we get the required inclusion

$$S_{i,-i}(aga^{-1}) \equiv \sum_{k\in I} a_{ik}^2 \left(S_{k,-k}(g) + S_{k,-k}(a^{-1}) + \sum_{t\in I} g_{kt}^2 S_{t,-t}(a^{-1}) \right) \mod \Gamma_i.$$

This completes the proof. □

Proof of Theorem 4. Denote by N the set of all matrices in $\mathrm{Sp}(2n, R)$ satisfying the conditions (T1) – (T3). It's easy to see that $N \leq \mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\sigma, \Gamma))$. Indeed, pick any $g \in \mathrm{Sp}(\sigma, \Gamma)$ and any $a \in N$. Then condition (T1) guarantees that

$$(aga^{-1})_{ij} = \sum_{p,q\in I} a_{ip} g_{pq} a'_{qj} \leq \sum_{p,q\in I} a_{ip}\sigma_{pq}a'_{qj} \leq \sigma_{ij}$$

for all $i, j \in I$. Now applying Proposition 1.2.2 we get

$$S_{i,-i}(aga^{-1}) \equiv \sum_{k\in I} \left(a_{ik}^2 S_{k,-k}(g) + a_{ik}^2 S_{k,-k}(a^{-1}) + \sum_{t\in I} a_{ik}^2 g_{kt}^2 S_{t,-t}(a^{-1}) \right) \mod \Gamma_i.$$

Observe that by condition (T3) it follows that $a_{ik}^2 S_{k,-k}(g^{-1}) \in a_{ik}^2 \Gamma_k \leq \Gamma_i$. Next, by condition (T2) we get $a_{ik}^2 g_{kt}^2 S_{t,-t}(a^{-1}) \in a_{ik}^2 \sigma_{kt}^{2} S_{t,-t}(a^{-1}) \leq \Gamma_i$ and $a_{ik}^2 S_{k,-k}(a^{-1}) = a_{ik}^2 \cdot 1^2 \cdot S_{k,-k}(a^{-1}) \in a_{ik}^2 \sigma_{kk}^2 S_{k,-k}(a^{-1}) \leq \Gamma_i$. Therefore, $S_{i,-i}(aga^{-1})) \in \Gamma_i$ for all i. It follows that $aga^{-1} \in \mathrm{Sp}(\sigma, \Gamma)$ and thus $a \in \mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\sigma, \Gamma))$.

The proof of the inclusion $\mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma, \Gamma), \mathrm{Sp}(\sigma, \Gamma)) \leq N$ is slightly trickier. Consider an arbitrary matrix a in $\mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma, \Gamma), \mathrm{Sp}(\sigma))$ and a short (σ, Γ)-transvection $T_{rs}(\xi)$. By definition of the transporter we get

$$\delta_{ij} + a_{ir}\xi a'_{sj} - \varepsilon(r)\varepsilon(s)a_{i,-s}\xi a'_{-r,j} = ({}^a T_{rs}(\xi))_{ij} \in \sigma_{ij}. \tag{1.17}$$

Now given two short (σ, Γ)-transvections $T_{rs}(\xi)$ and $T_{st}(\zeta)$ such that $r \neq \pm t$ we get by a straightforward computation

$$\delta_{ij} + a_{ir}\xi\zeta a'_{tj} = (aT_{rs}(\xi)T_{st}(\zeta)a^{-1})_{ij} - (aT_{rs}(\xi)a^{-1})_{ij} - (aT_{st}(\zeta)a^{-1})_{ij} + \delta_{ij}. \tag{1.18}$$

And therefore using (1.17) we get the inclusions $h_{ir}\sigma_{rs}\sigma_{sr}\zeta a'_{tj} \leq \sigma_{ij}$ for all $i, j, s, r, t \in I$ such that $s \neq \pm r, \pm t$ and $r \neq \pm t$.

Next for a long (σ, Γ)-elementary transvection $T_{s,-s}(\alpha)$ we get

$$\delta_{ij} + a_{is}\alpha a'_{-s,j} = ({}^a T_{s,-s}(\alpha))_{ij} \in \sigma_{ij}. \tag{1.19}$$

Finally for $r, s \in I$ such that $r \neq \pm s$ we get

$$\delta_{ij} + a_{ir}\xi\alpha a'_{-s} = (aT_{rs}(\xi)T_{s,-s}(\alpha)a^{-1})_{ij} - (aT_{rs}(\xi)a^{-1})_{ij} - (aT_{s,-s}(\alpha)a^{-1})_{ij} + \delta_{ij} \quad (1.20)$$

and therefore by (1.19) we get the inclusions $a_{ir}\sigma_{rs}\Gamma_{s,-s}a'_{rj} \leq \sigma_{ij}$ for all $i, j \in I$ and all $r, s \in I$ such that $s \neq \pm r$.

Now let r and t be two indices such that $r \neq \pm t$. Then either $\nu(r) = \{\pm r, \pm t\}$, or there exists an index $s \sim t$ such that $s \neq \pm r, \pm t$. In the former case using (1.20) we get

$$a_{ir}\sigma_{rt}a'_{tj} = a_{ir}R\sigma_{rt}a'_{tj} = a_{ir}\sigma_{r,-t}\Gamma_{-t,t}a'_{tj} \leq \sigma_{ij}$$

for all $i, j \in I$. In the latter case using (1.18) we get

$$a_{ir}\sigma_{rt}a'_{tj} = a_{ir}\sigma_{rt}Ra'_{tj} = a_{ir}\sigma_{rs}\sigma_{st}a'_{tj} \leq \sigma_{ij}$$

for all $i, j \in I$.

Now assume $t = r$. Then there exists an index $s \sim r$ such that $s \neq \pm r$ and using the fact that $\sum_{l \in I} a'_{sl}a_{ls} = 1$ we get

$$a_{ir}\sigma_{rr}a'_{rj} = \sum_{l \in I}(a_{ir}\sigma_{rs}a'_{sl})(a_{ls}\sigma_{sr}a'_{rj}) \leq \sum_{l \in I}\sigma_{il}\sigma_{lj} \leq \sigma_{ij}.$$

Finally if $t = -r$ then $\sigma_{r,-r} = \sum_{l \neq \pm r}\sigma_{rl}\sigma_{l,-r} + \langle\Gamma_r\rangle$. By (1.19) it follows that

$$a_{ir}\langle\Gamma_r\rangle a'_{-r,j} \leq \sigma_{ij}.$$

It's only left to notice that

$$a_{ir}\sigma_{rl}\sigma_{l,-r}a'_{rj} = \sum_k(a_{ir}\sigma_{rl}a'_{lk})(a_{kl}\sigma_{l,-r}a'_{-r,j}) \leq \sum_k \sigma_{ik}\sigma_{kj} \leq \sigma_{ij}.$$

Therefore, any matrix a in the transporter satisfies condition (T1). In particular, we can apply Proposition 1.2.2 to any such matrix a.

Pick any short (σ, Γ)-elementary transvection $T_{jk}(\xi)$. By Proposition 1.2.2 we get

$$S_{i,-i}(aT_{jk}(\xi)a^{-1}) \equiv a_{ij}^2\xi^2 S_{k,-k}(a^{-1}) + a_{i,-k}^2\xi^2 S_{-j,j}(a^{-1}) \mod \Gamma_i. \quad (1.21)$$

Now, given a long (σ, Γ)-elementary transvection $T_{j,-j}(\alpha)$ we obtain by the same proposition

$$S_{i,-i}(aT_{j,-j}(\alpha)a^{-1}) \equiv a_{ij}^2\alpha + a_{ij}^2\alpha^2 S_{-j,j}(a^{-1}) \mod \Gamma_i. \quad (1.22)$$

Given two short (σ, Γ)-elementary transvections $T_{jk}(\xi)$ and $T_{km}(\zeta)$ such that $j \neq \pm m$ we get

$$\begin{aligned} S_{i,-i}(aT_{jk}(\xi)T_{km}(\zeta)a^{-1}) \equiv\ & a_{ij}^2\xi^2 S_{k,-k}(a^{-1}) + a_{i,-k}^2\xi^2 S_{-j,j}(a^{-1}) \\ &+ a_{ik}^2\zeta^2 S_{m,-m}(a^{-1}) + a_{i,-m}^2\zeta^2 s_{-k,k}(a^{-1}) \\ &+ a_{ij}^2\xi^2\zeta^2 S_{m,-m}(a^{-1}) \mod \Gamma_i. \end{aligned} \quad (1.23)$$

Finally given a short and a long (σ,Γ)-elementary transvections $T_{jk}(\xi)$ and $T_{k,-k}(\alpha)$ we get

$$
\begin{aligned}
S_{i,-i}(aT_{jk}(\xi)T_{k,-k}(\zeta)a^{-1}) \equiv a_{ij}^2\xi^2 S_{k,-k}(a^{-1}) + a_{i,-k}^2\xi^2 S_{-j,j}(a^{-1}) \\
+ a_{ik}^2\alpha + a_{ik}^2\alpha^2 S_{-k,k}(a^{-1}) \\
+ a_{ij}^2\xi^2\alpha^2 S_{-k,k}(a^{-1}) \mod \Gamma_i.
\end{aligned}
\tag{1.24}
$$

Comparing (1.23) and (1.21) we get the inclusions

$$a_{ij}^2\sigma_{jk}^{\boxed{2}}\sigma_{km}^{\boxed{2}}S_{k,-k}(a^{-1}) \in \Gamma_i$$

for all $i,j,k,m \in I$ such that $j \neq \pm k, \pm m$ and $k \neq \pm m$. Similarly comparing (1.24) with (1.22) and (1.21) we get the inclusions

$$a_{ij}^2\sigma_{jm}^{\boxed{2}}\Gamma_m^{\boxed{2}}S_{-m,m}(a^{-1}) \in \Gamma_i$$

for all $i,j,m \in I$ such that $j \neq \pm m$.

Now let $j,m \in I$ such that $j \neq \pm m$. As $h(\nu) \geq (4,3)$ either there exists an index $k \sim m$ such that $k \neq \pm j, \pm m$ or $-m \sim m$. In the first case we get

$$a_{ij}^2\sigma_{jm}^{\boxed{2}}S_{m,-m}(a^{-1}) = a_{ij}^2\sigma_{jk}^{\boxed{2}}R^{\boxed{2}}S_{m,-m}(a^{-1}) = a_{ij}^2\sigma_{jk}^{\boxed{2}}\sigma_{km}^{\boxed{2}}S_{m,-m}(a^{-1}) \leq \Gamma_i$$

for all $i \in I$. In the second case we get similarly

$$a_{ij}^2\sigma_{jm}^{\boxed{2}}S_{m,-m}(a^{-1}) = a_{ij}^2\sigma_{j,-m}^{\boxed{2}}R^{\boxed{2}}S_{m,-m}(a^{-1}) = a_{ij}^2\sigma_{j,-m}^{\boxed{2}}\Gamma_{-m}^{\boxed{2}}S_{m,-m}(a^{-1}) \leq \Gamma_i$$

for all $i \in I$. To prove the inclusions (T2) for the matrix a it's only left to consider the cases when $m = j$ and $m = -j$. Fix an index $k \sim j$ such that $k \neq \pm j$. Observe that

$$1 = \left(\sum_{t\in I} a'_{kt}a_{tk}\right)^2 \equiv \sum_{t\in I} a'^2_{kt}a^2_{tk} \mod 2R$$

and therefore

$$
\begin{aligned}
a_{ij}^2\sigma_{jm}^{\boxed{2}}S_{m,-m}(a^{-1}) &= a_{ij}^2\sigma_{jk}^{\boxed{2}}\left(\sum_{t\in I} a'_{kt}a_{tk}\right)^2 \sigma_{km}^{\boxed{2}}S_{m,-m}(a^{-1}) \\
&\equiv \sum_t \left(a_{ij}^2\sigma_{jk}^{\boxed{2}}(a'^2_{kt})\right)\left(a_{tk}^2\sigma_{km}^{\boxed{2}}S_{m,-m}(a^{-1})\right) \\
&\leq \sum_{t\in I}\sigma_{it}^2\Gamma_t \leq \Gamma_i,
\end{aligned}
$$

where the congruence is meant modulo Γ_i.

Thus a satisfies condition (T2). Finally using (1.22) and (T2) we get the inclusions (T3) for all $i,j \in I$. Thus we have proved that

$$\mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma,\Gamma), \mathrm{Sp}(\sigma,\Gamma)) \leq N.$$

Finally, it is easy to see that Transp is contravariant in the first variable and therefore

$$\mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\sigma,\Gamma)) \leq \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma,\Gamma),\mathrm{Sp}(\sigma,\Gamma)).$$

Hence also

$$\mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\sigma,\Gamma)) = \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma,\Gamma),\mathrm{Sp}(\sigma,\Gamma)) = N.$$

□

1.3 Extraction of transvections in parabolic subgroups

The current section is devoted to the extraction of transvections in parabolic subgroups of $\mathrm{Sp}(2n,R)$. We will show that the matrices in H lying in some explicitly defined subgroups of parabolic subgroups lie in $\mathrm{Sp}(\sigma,\Gamma)$, where (σ,Γ) is the form net of ideals associated with H. The results of this section as well as the methods involved in their proofs are standard for this area of research. However, there are not many papers where similar results are stated for form nets of ideals. In this sense, the results of this sections are new.

Till the end of this section we fix a subgroup H of $\mathrm{Sp}(2n,R)$ such that $\mathrm{Ep}(\nu,R) \leq H$ and denote by (σ,Γ) the form net of ideals associated with H. We will start with the extraction of transvections from the unipotent radical of type U_1.

Lemma 1.3.1. *Assume $h(\nu) \geq (4,3)$. Let a be a matrix in H such that for some index p, $a_{ij} = \delta_{ij}$ whenever $i \neq -p$ and $j \neq p$. Additionally assume that $a_{pp} = a_{-p,-p} = 1$. Then a is contained in $\mathrm{Ep}(\sigma,\Gamma)$ and consequently in $\mathrm{Sp}(\sigma,\Gamma)$.*

Proof. A direct calculation using only the fact that a is a symplectic matrix shows that a can be decomposed in $\mathrm{Ep}(2n,R)$ as follows:

$$a = \left(\prod_{j=1}^{n} T_{-p,j}(a_{-p,j})T_{-p,-j}(a_{-p,-j})\right) T_{-p,p}(S_{-p,p}(a)). \tag{1.25}$$

Our goal is to prove, that each short elementary transvection on the right-hand side of the equation (1.25) is contained in H and thus in $\mathrm{Ep}(\sigma,\Gamma)$. Then, as a is contained in H, the last multiplier $T_{-p,p}(S_{-p,p}(a))$ is also contained in H and the proof is complete.

Fix any $j \neq \pm p$. We will show that $T_{-p,j}(a_{-p,j}) \in \mathrm{Ep}(\sigma,\Gamma)$. For $j \sim -p$ this inclusion is trivial as $\sigma_{-p,j} = R$. Suppose that $j \nsim -p$. According to Proposition 1.1.7 there exists an index $k \neq \pm p$ such that (k,j) is an A-type base pair. Consider the matrix $b = [a, T_{jk}(1)]$ in H. Using the Steinberg relations (R3) and (R4) we get that b is actually a product of two commuting short elementary transvections

$$b = T_{-p,k}(a_{-p,j})T_{-p,-j}(*),$$

and the parameter of the second one is not important. Now if $-j \sim -p$, then the second elementary transvection in the decomposition of b above automatically lies in H.

Therefore so does the first one. Finally, assuming that $\pm j \nsim -p$, either there exists an index $h \sim j \sim k$ such that $h \neq \pm j, \pm h, \pm p$ or $k \sim -k$. In the first case it follows again from the Steinberg relations that

$$T_{-p,h}(a_{-p,j}) = [b, T_{kh}(1)] \in H$$

and thus also $T_{-p,j}(a_{-p,j}) \in H$ by Proposition 1.1.6. In the last case, one gets that

$$T_{-p,j}(a_{-p,j}) = [[b, T_{k,-k}(1)], T_{-k,j}(1)] \in H.$$

This completes the proof. □

The next lemma allows us to perform the extraction of elementary transvections in the whole parabolic subgroup of type P_1.

Lemma 1.3.2. *Assume $h(\nu) \geq (4,3)$. Let (p,q) be an* A*-type base pair and a a matrix in H such that $a_{*p} = e_{*p}$. Then $a_{qj}, a_{-j,-q} \in \sigma_{qj}$ for all $j \neq -p$. Moreover if $a \in \mathrm{Sp}(\sigma)$ then also $S_{q,-q}(a) \in \Gamma_q$.*

Proof. First, as a is symplectic, we also have $a'_{*p} = e_{*p}$ and also $a_{-p,*} = a'_{-p,*} = e_{-p,*}$. Consider the matrix $b = a^{-1}T_{pq}(1)a$. It's easy to see that

$$b = e + a'_{*p}a_{q*} - \varepsilon_p\varepsilon_q a'_{*,-q}a_{-p,*} = e + e_{*p}a_{q*} - \varepsilon_p\varepsilon_q a'_{*,-q}e_{-p,*},$$

and thus b satisfies the conditions of Lemma 1.3.1. Therefore $b \in \mathrm{Ep}(\sigma,\Gamma)$. In particular, $a_{qj} = b_{pj} \in \sigma_{qj}$ for all $j \neq \pm p$ and $S_{p,-p}(b) \in \Gamma_p$. Applying all the above to the matrix a^{-1} we get the inclusions $a_{-q,-j} = \pm a'_{jq} \in \sigma_{jq}$ for all $j \neq -p$.

Now if $a \in \mathrm{Sp}(\sigma)$ then by Corollary 1.1.5 we get

$$S_{p,-p}(b) \equiv (a')^2_{pp}S_{q,-q}(a) + (a')^2_{p,-q}S_{-p,p}(a) \mod \Gamma_p.$$

By the conditions of the lemma $a'_{pp} = 1$ and $S_{-p,p}(a) = 0$, therefore $S_{q,-q}(a) \in \Gamma_q$. □

The next lemma allows us to extract elementary transvections from matrices having zeros in some specific positions.

Corollary 1.3.3. *Assume $h(\nu) \geq (4,3)$. Let a be a matrix in H. Pick two indices k and p. If there exist two more indices $h, l \sim k$ such that $k, l \neq \pm h$ (but l can be equal to k or $-k$) and $a_{-h,-h} = 1$, $a_{k,-h} = a_{kl} = a_{-h,l} = 0$, then $a_{kj} \in \sigma_{kj}$ for all $j \neq -l$. Moreover if $a \in \mathrm{Sp}(\sigma)$ then $S_{k,-k}(a) \in \Gamma_k$.*

Proof. Consider the short root element

$$b = a^{-1}T_{hk}(1)a = e + a'_{*h}a_{k*} - \varepsilon_h\varepsilon_k a'_{*,-k}a_{-h,*}.$$

As $a_{kl} = a_{-h,l} = 0$ it follows that $b_{*l} = e_{*l}$. Thus by lemma 1.3.2 we get $b_{hj} \in \sigma_{hj}$ for all $j \neq -l$. It's only left to notice that $b_{hj} = \delta_{hj} + a'_{hh}a_{kj} \pm a'_{h,-k}a_{-h,j} = \delta_{hj} + a_{kj}$.

Now if $a \in \mathrm{Sp}(\sigma)$ then also $b \in \mathrm{Sp}(\sigma)$. By Lemma 1.3.2 we get $S_{h,-h}(b) \in \Gamma_h$. Now by Corollary 1.1.5 we get

$$S_{h,-h}(b) \equiv (a')^2_{hh} S_{k,-k}(a) + (a')^2_{h,-k} S_{-h,h}(a) \mod \Gamma_h.$$

By the conditions of the lemma $a'_{hh} = 1$ and $a'_{h,-k} = 0$. Therefore

$$S_{k,-k}(a) \equiv S_{h,-h}(b) \equiv 0 \mod \Gamma_k.$$

□

The next three lemmas are direct corollaries of the last two results. The classes of matrices described below may look artificial, however they represent some subgroups of a parabolic subgroup which appear in the proof of Theorem 3.

Lemma 1.3.4. *Assume $h(\nu) \geq (4,3)$. Let (p,q) be a C-type base pair and let a be a matrix in H such that $a_{ij} = \delta_{ij}$ whenever $i \neq \pm p, -q$ and $j \neq \pm p, q$. Then $a \in \mathrm{Sp}(\sigma)$.*

Proof. It's enough to prove the inclusions $a_{kp}, a_{k,-p}, a_{kq} \in \sigma_{kp}$ for all k. If $k \sim p$ then the inclusions $a_{kp}, a_{k,-p}, a_{kq} \in \sigma_{kp}$ are trivial. Thus assume from now on that $k \nsim p$. As $h(\nu) \geq (4,3)$ the index k can either be included into an A-type base triple (k,h,l) or a C-type base pair (k,h). In the last case put $l = -k$. Then $h \neq \pm k$ and $a_{-h,-h} = 1$ and $a_{k,-h} = a_{kl} = a_{-h,l} = 0$. Using Corollary 1.3.3 we get the inclusions $a_{kp}, a_{kq}, a_{k,-p} \in \sigma_{kp}$ for ll k. □

Lemma 1.3.5. *Assume $h(\nu) \geq (4,4)$. Let (p,q) be an* A*-type base pair such that the equivalence class of p is non-self-conjugate. Let $a \in H$ be a matrix such that $a_{ij} = \delta_{ij}$ whenever $i \neq -p, -q$ and $j \neq p, q$. Then $a_{kp} \in \sigma_{kp}$ for all $k \neq -p, -q$. If additionally $a \in \mathrm{Sp}(\sigma)$ then also $S_{-p,p}(a) \in \Gamma_{-p}$.*

Proof. As $h(\nu) \geq (3,4)$ and $p \nsim -p$ it follows that p can be included into some A-type base quadruple (p,q,h,l). Consider the matrix

$$b = T_{-p,-h}(-a_{-p,-h}) T_{-q,-h}(-a_{-q,-h}) a.$$

Clearly $b \in H$ and $b_{*,-h} = e_{*,-h}$. By Lemma 1.3.2 we get $b_{ip} \in \sigma_{ip}$ for all $i \neq -h$. Therefore $a_{ip} = b_{ip} \in \sigma_{ip}$ for all $i \neq -p, -q, \pm h$, in particular $a_{-k,p} \in \sigma_{-k,p}$. Swapping the indices k and h above we get the last inclusion $a_{-h,p} \in \sigma_{-h,p}$.

Now if $a \in \mathrm{Sp}(\sigma)$ then by Lemma 1.3.2 we get $S_{-p,p}(b) \in \Gamma_{-p}$. According to Corollary 1.1.4 we get

$$S_{-p,p}(b) \equiv S_{-p,p}(a) + a^2_{-p,-k} S_{-k,k}(a) \mod \Gamma_{-p}.$$

It's only left to notice that $S_{-k,k}(a) = 0$. □

Lemma 1.3.6. *Assume $h(\nu) \geq (4,5)$. Let (p,q,t) be an* A*-type base triple such that $p \nsim -p$ and let a be an element of H such that $a_{ij} = \delta_{ij}$ whenever $i \neq -q, -t, \pm p$ and $j \neq q, t, \pm p$. Then $a_{kp}, a_{kq} \in \sigma_{kp}$ for all $k \neq -p, -q, -t$. If additionally $a \in \mathrm{Sp}(\sigma)$, then also $S_{-q,q}(a) \in \Gamma_{-q}$.*

Proof. Fix an index $k \not\sim p$ such that $k \neq -p, -q, -t$. As $h(\nu) \geq (4, 5)$, there exists an index $h \sim k$ such that $h \neq \pm k, -p, -q, -t$. If the class of k is self-conjugate, put $l = -k$. Otherwise, if $k \not\sim -p$ there exists another $l \sim k$ such that $l \neq \pm k, \pm h, -p, -q, -t$. If $k \sim -p$ set $l = -t$. In any case, $a_{-h,-h} = 1$ and $a_{k,-h} = a_{kl} = a_{-h,l} = 0$ and $l \neq -p, -q$. Therefore by Corollary 1.3.3 we get $a_{kp}, a_{kq} \in \sigma_{kp}$ for all $k \neq -p, -q, -t$.

Suppose that $a \in \mathrm{Sp}(\sigma)$. By Corollary 1.3.3 we also get $S_{k,-k}(a) \in \Gamma_k$ for all $k \not\sim p$ such that $k \neq -p, -q, -t$. Finally assuming that the $p \not\sim -p$ there exist indices h, l such that (p, q, t, h, l) is an A-type base quintuple. Consider the matrices

$$b = T_{-q,h}(-a_{-q,h})a \qquad\qquad c = T_{-q,-l}(-b_{-q,-l})b.$$

Clearly $b, c \in \mathrm{Sp}(\sigma)$. It's easy to see that $c_{hh} = 1$ and $c_{-q,h} = c_{-q,-l} = c_{h,-l} = 0$. Thus by Corollary 1.3.3 we get the inclusion $S_{-q,q}(c) \in \Gamma_q$. Now by Corollary 1.1.4 we get

$$S_{-q,q}(c) \equiv S_{-q,q}(b) + b^2_{-q,-l}S_{-l,l}(b) = S_{-q,q}(a) + a^2_{-q,h}S_{h,-h}(a) + b^2_{-q,-l}S_{-l,l}(a) \mod \Gamma_{-q}.$$

We have already proved that $S_{h,-h}(a) \in \Gamma_{-l}$. Also, note that $a_{h,-l} = 0$ and thus $b_{-q,-l} = a_{-q,-l}$. Therefore

$$S_{-q,q}(a) + a^2_{-q,-l}S_{-l,l}(a) \in \Gamma_{-q}. \tag{1.26}$$

To eliminate this last summand we have to perform one more trick. First of all after swapping the indices l and h in (1.26) we get

$$S_{-q,q}(a) + a^2_{-q,-h}S_{-h,h}(a) \in \Gamma_{-q}. \tag{1.27}$$

Consider the matrix $d = T_{-q,-h}(a_{-q,-h})a$. It's clear that $d \in \mathrm{Sp}(\sigma)$ and d itself satisfies the conditions of this lemma. Therefore we get the inclusion (1.26) for the matrix d also and with the use of Corollary 1.1.4 we get:

$$S_{-q,q}(a) + a^2_{-q,-h}S_{-h,h}(a) + (a_{-q,-l} + a_{-h,-l})^2 S_{-l,l}(a) = S_{-q,q}(d) + d^2_{-q,-l}S_{-l,l}(d) \in \Gamma_{-q}. \tag{1.28}$$

Now recall that $a_{-h,-l} = 0$ and subtract (1.27) from (1.28). We get

$$a^2_{-q,-l}S_{-l,l}(a) \in \Gamma_{-q}. \tag{1.29}$$

Combining (1.26) and (1.29) we get $S_{-q,q}(a) \in \Gamma_{-q}$. This completes the proof. □

1.4 Extraction of transvections using a long root element

Suppose $h(\nu) \geq (4, 5)$. Let H be a subgroup of $\mathrm{Sp}(2n, R)$ such that $\mathrm{Ep}(\nu, R) \leq H$ and let (σ, Γ) be the form net of ideals associated with H. In this section we will show that any long root element

$$b = aT_{s,-s}(\xi)a^{-1} = e + a_{*s}\xi a'_{-s,*},$$

where $a \in H$ and $\xi \in \Gamma_s$, is contained in $\mathrm{Sp}(\sigma, \Gamma)$.

For any $g \in \mathrm{Sp}(2n, R)$ let $\widehat{g} = g - e$. We don't require that the matrix $\widehat{g}$ belongs to $\mathrm{Sp}(2n, R)$. The hat notation will be used only to shorten the formulas. We set $\widehat{g}_{ij} = g_{ij} - \delta_{ij}$ for any $i, j \in I$ and $\widehat{g'}_{ij} = g'_{ij} - \delta_{ij}$ for any $i, j \in I$.

The following property of long root elements can be checked straightforwardly. For any $i, j \in I$,

$$\begin{aligned} \widehat{b}_{*i}\widehat{b}'_{j*} &= -\varepsilon_i\varepsilon_j\widehat{b}_{*,-j}\widehat{b}'_{-i,*}, \\ \widehat{b}_{*i}\widehat{b}_{j*} &= -\varepsilon_i\varepsilon_j\widehat{b}_{*,-j}\widehat{b}_{-i,*}. \end{aligned} \tag{1.30}$$

A slightly more sophisticated calculation shows that for the long root element b and any indices $i, j, k \in I$ we have the equality

$$\widehat{b}^2_{ij}S_{k,-k}(b^{-1}) = -\varepsilon_j\varepsilon_k\widehat{b}^2_{i,-k}S_{-j,j}(b^{-1}) + (\varepsilon_k + \varepsilon_j)\widehat{b}^2_{i,-k}\widehat{b}'_{-j,j}. \tag{1.31}$$

Indeed using (1.30) we get

$$\begin{aligned} \widehat{b}^2_{ij}S_{k,-k}(b^{-1}) &= -\varepsilon_k\left(\sum_{m>0}\left(\widehat{b}_{ij}\widehat{b}'_{km}\right)\left(\widehat{b}_{ij}\widehat{b}'_{k,-m}\right)\right) - \varepsilon_k\widehat{b}^2_{ij}\widehat{b}'_{k,-k} \\ &= -\varepsilon_k\left(\sum_{m>0}\left(\widehat{b}_{i,-k}\widehat{b}'_{-j,m}\right)\left(\widehat{b}_{i,-k}\widehat{b}'_{-j,-m}\right)\right) - \varepsilon_k\widehat{b}^2_{ij}\widehat{b}'_{k,-k} \\ &= -\varepsilon_k\varepsilon_j\widehat{b}^2_{i,-k}S_{-j,j}(b^{-1}) + \varepsilon_k\widehat{b}^2_{i,-k}\widehat{b}'_{-j,j} - \varepsilon_k\widehat{b}^2_{ij}\widehat{b}'_{k,-k} \\ &= -\varepsilon_k\varepsilon_j\widehat{b}^2_{i,-k}S_{-j,j}(b^{-1}) + (\varepsilon_k + \varepsilon_j)\widehat{b}^2_{i,-k}\widehat{b}'_{-j,j}. \end{aligned}$$

The last two formulas will be used repeatedly in the proofs of the current section.

The following lemma is the analogue of Lemma 12 of [Vav08] although with slightly different conditions on α and β and covering the case of long root elements as opposed to long root involutions in the mentioned paper. Note, that this is indeed a more general case as any long root involution is a product of some long root elements.

Lemma 1.4.1. *Assume $h(\nu) \geq (4, 5)$. Let H be a subgroup in $\mathrm{Sp}(2n, R)$ such that $\mathrm{Ep}(\nu, R) \leq H$ and let (σ, Γ) be the form net of ideals associated with H. Let a be a matrix in H and $T_{s,-s}(\xi)$ be a (σ, Γ)-elementary long transvection. Let b denote the long root element $aT_{s,-s}(\xi)a^{-1}$. If (p, h) is an* A*-type base pair then*

$$a_{ps}b_{ih} \in \sigma_{ih} \text{ for all } i \neq -p, -q. \tag{1.32}$$

If additionally $b \in \mathrm{Sp}(\sigma)$ then also

$$a^2_{ps}S_{-h,h}(b^{-1}) \in \Gamma_{-h}. \tag{1.33}$$

Proof. If the class of p is non-self-conjugate, let q be an index in I such that (p, q, h) is an A-type base triple. If the class of p is self-conjugate, set $q = -p$. If $i \sim h$ the inclusions (1.32) are trivial. Assume $i \nsim h$ and consider the matrix $c = bT_{hp}(\alpha)T_{hq}(\beta)b^{-1}$ where $\alpha, \beta \in R$ such that $\alpha a_{ps} + \beta a_{qs} = 0$. Then

$$c = e + b_{*h}(\alpha b'_{p*} + \beta b'_{q*}) - \varepsilon_h(\varepsilon_p\alpha b_{*,-p} + \varepsilon_q\beta b_{*,-q})b'_{-h,*}.$$

As $\alpha a_{ps} + \beta a_{qs} = 0$ it's easy to see that

$$\alpha \widehat{b}'_{p*} + \beta \widehat{b}_{q*} = 0 \qquad \text{and} \qquad \varepsilon_p \alpha \widehat{b}_{*,-p} + \varepsilon_q \beta \widehat{b}_{*,-q} = 0.$$

Therefore

$$c = b_{*h}(\alpha e_{p*} + \beta e_{q*}) - \varepsilon_h(\varepsilon_p \alpha e_{*,-p} + \varepsilon_q \beta e_{*,-q}) b'_{-h,*}.$$

In particular $c_{ij} = \delta_{ij}$ for all $i \neq -q, -p$ and $j \neq p, q$. Hence, the matrix c satisfies the conditions of Lemma 1.3.4 if $q = -p$ or Lemma 1.3.5 if $q \neq -p$. Therefore $c_{iq} \in \sigma_{iq}$ for all $i \neq -p, -q$. Put $\alpha = -a_{qs}, \beta = a_{ps}$ and note that $c_{iq} = \beta b_{ih}$ for all $i \neq -p, -q$. Therefore $a_{ps} b_{ih} \in \sigma_{ih}$ for all $i \neq -p, -q$.

Now suppose $b \in \mathrm{Sp}(\sigma)$. If $p \sim -p$, the inclusions (1.33) are trivial. If $p \nsim -p$, it is clear that the matrix c is also contained in $\mathrm{Sp}(\sigma)$ and by the lemma 1.3.5 we get $S_{-q,q}(c) \in \Gamma_{-q}$. By Corollary 1.1.5 we get

$$\begin{aligned} S_{-q,q}(c) \equiv\ & b^2_{-q,h} \alpha^2 S_{p,-p}(b^{-1}) + b^2_{-q,h} \beta^2 S_{q,-q}(b^{-1}) + \\ & + b^2_{-q,-p} \alpha^2 S_{-h,h}(b^{-1}) + b^2_{-q,-q} \beta^2 S_{-h,h}(b^{-1}) \end{aligned} \tag{1.34}$$

Now using (1.31) we get

$$b^2_{-q,h} S_{p,-p}(b^{-1}) = -\varepsilon_h \varepsilon_p b^2_{-q,-p} S_{-h,h}(b^{-1}) + (\varepsilon_h + \varepsilon_p) b^2_{-q,-p} b'_{-h,h} \tag{1.35}$$

$$b^2_{-q,h} S_{q,-q}(b^{-1}) = -\varepsilon_h \varepsilon_q \widehat{b}^2_{-q,-q} S_{-h,h}(b^{-1}) + (\varepsilon_h + \varepsilon_q) \widehat{b}^2_{-q,-q} b'_{-h,h}. \tag{1.36}$$

Therefore as $2S_{-h,h}(b^{-1}) \in 2\sigma_{-h,h} \leq \Gamma_{-h}$, we get combining (1.34), (1.35) and (1.36) that

$$S_{-q,q}(c) \equiv \beta^2 S_{-h,h}(b^{-1}) + \alpha^2(\varepsilon_h + \varepsilon_p) b^2_{-q,-p} b'_{-h,h} + \beta^2 (\varepsilon_h + \varepsilon_q) \widehat{b}^2_{-q,-q} b'_{-h,h}. \tag{1.37}$$

It's only left to notice that $\pm \alpha^2 b^2_{-q,-p} \pm \beta^2 \widehat{b}^2_{-q,-q} \in 2R$ and therefore we get from (1.37) that $S_{-q,q}(c) \equiv \beta^2 S_{-h,h}(b^{-1}) \mod \Gamma_{-h}$. Therefore

$$a^2_{ps} S_{-h,h}(b^{-1}) = \beta^2 S_{-h,h}(b^{-1}) \in \Gamma_{-h}.$$

This completes the proof.

□

In the next two lemmas we use short root elements to extract more transvections. It turns out to suffice for the proof of Theorem 3.

Lemma 1.4.2. *Assume $h(\nu) \geq (4,5)$. Let H be a subgroup of $\mathrm{Sp}(2n, R)$ such that $\mathrm{Ep}(\nu, R) \leq H$ and let (σ, Γ) be the form net of ideals associated with H. Let a be a matrix in H, $T_{s,-s}(\xi)$ a long (σ, Γ)-elementary transvection and b the long root element $aT_{s,-s}(\xi)a^{-1}$. Let p be an index in I such that the equivalence class of $p \in I$ is non-self-conjugate. Then for any $i \in I$ we have the inclusion*

$$b_{ip} \in \sigma_{ip}. \tag{1.38}$$

If additionally $b \in \mathrm{Sp}(\sigma)$, then $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$.

Proof. By the assumptions of the lemma we can choose an A-type base quintuple (p, q, h, t, l) containing p. The proof is made by applying Lemma 1.4.1 to the matrix b and then a modification of Lemma 1.4.1 to the short root element $c = {}^bT_{ph}(1) \in H$. The element c has the difficulty that it is not a long root element but a short one and thus does not fit Lemma 1.4.1. However it turns out that for our goals it suffices that b itself is a long root element.

Take take any α and β in R such that $\alpha(1+2\widehat{b}_{pp})+\beta 2a_{qp} = 0$ and consider the matrix

$$d = cT_{hp}(\alpha)T_{hq}(\beta)c^{-1} = e + c_{*h}(\alpha c'_{p*} + \beta c'_{q*}) - \varepsilon_h(\varepsilon_p \alpha c_{*,-p} + \varepsilon_q \beta c_{*,-q})c'_{-h,*}. \tag{1.39}$$

We will show that d satisfies the conditions of Lemma 1.3.6 and moreover that $d_{iq} = \beta b_{ih}$ for all $i \neq -p, -q, -h$.

Denote by $\widetilde{d}$ the first term $c_{*h}(\alpha c'_{p*} + \beta c'_{q*})$ of the right-hand side of the expression in (1.39). A direct calculation using the formulas (1.30) shows that

$$\begin{aligned}
\widetilde{d} &= c_{*h}(\alpha(e_{p*} - b_{pp}b'_{h*} + \varepsilon_p\varepsilon_h b_{p,-h}b'_{-p,*}) + \beta(e_{q*} - b_{qp}b'_{h*} + \varepsilon_p\varepsilon_h b_{q,-h}b'_{-p,*})) = \\
&= c_{*h}(\alpha(e_{p*} - b_{pp}b'_{h*} - \widehat{b}_{pp}b'_{h*} + \widehat{b}_{pp}e_{h*} + \varepsilon_p\varepsilon_h b_{p,-h}e_{-p,*}) + \\
&\qquad + \beta(e_{q*} - b_{qp}b'_{h*} - b_{qp}b'_{h*} + b_{qp}e_{h*} + \varepsilon_p\varepsilon_h b_{q,-h}e_{-p,*})) = \\
&= c_{*h}(-(\alpha(1+2\widehat{b}_{pp}) + \beta(2b_{qp}))b'_{h*} + \\
&\qquad + \alpha(e_{p*} + \widehat{b}_{pp}e_{h*} + \varepsilon_p\varepsilon_h b_{p,-h}e_{-p,*}) + \beta(e_{q*} + b_{qp}e_{h*} + \varepsilon_p\varepsilon_h b_{q,-h}e_{-p,*})) \\
&= c_{*h}(\alpha(e_{p*} + \widehat{b}_{pp}e_{h*} + \varepsilon_p\varepsilon_h b_{p,-h}e_{-p,*}) + \beta(e_{q*} + b_{qp}e_{h*} + \varepsilon_p\varepsilon_h b_{q,-h}e_{-p,*})).
\end{aligned}$$

This formula shows that $\widetilde{d}$ is a matrix all columns of whose except for the ones indexed by $p, -p, q$ and h are zero and $\widetilde{d}_{*q} = \beta c_{*h}$. Symmetrically the last term of d, namely $-\varepsilon_h(\varepsilon_p\alpha c_{*,-p} + \varepsilon_q\beta c_{*,-q})c'_{-h,*}$ is a matrix all the rows of whose except for the ones indexed by $p, -p, -q, -h$ are zero. Indeed using Proposition 1.1.1 one gets

$$-\varepsilon_h(\varepsilon_p\alpha c_{*,-p} + \varepsilon_q\beta c_{*,-q})c'_{-h,*} = (\mathfrak{p}c_{*h}(\alpha c'_{p*} + \beta c'_{q*})\mathfrak{p})^t = -(\mathfrak{p}\,\widetilde{d}\,\mathfrak{p}^{-1})^t,$$

where $\mathfrak{p}$ is a product of a permutation matrix swapping the indices i and $-i$ for all i and a scalar matrix with entries ± 1.

Therefore d satisfies the conditions of Lemma 1.3.6 and thus $\beta c_{ih} = d_{iq} \in \sigma_{iq}$ for all $i \neq p, -p, -q, -h$ while the inclusion corresponding to $i = p$ is trivial.

We are ready to prove the inclusions $b_{ip} \in \sigma_{ip}$. Take $i \nsim p$. Thus $i \neq p, q, h, t, l$. Suppose $i \neq -p, -q, -h$. Using (1.30) we get

$$\begin{aligned}
\sigma_{ip} \ni d_{iq} = \beta c_{ih} &= (1+2\widehat{b}_{pp})(b_{ip}b'_{hh} - \varepsilon_h\varepsilon_p b_{i,-h}b'_{-p,h}) = (1+2\widehat{b}_{pp})(b_{ip} + 2b_{ip}\widehat{b}'_{hh}) = \\
&= b_{ip} + 2b_{ip}\widehat{b}'_{hh} + 2\widehat{b}_{pp}b_{ip} + 4\widehat{b}_{pp}b_{ip}\widehat{b}'_{hh}.
\end{aligned} \tag{1.40}$$

The second and the fourth summands immediately above, namely $2b_{ip}\widehat{b}'_{hh}$ and $4\widehat{b}_{pp}b_{ip}\widehat{b}'_{hh}$, are both multiples of $a_{hs}b_{ip}$ which belongs to σ_{ip} by Lemma 1.4.1 for all $i \neq -h, -q$. Thus so does the rest of (1.40), in other words

$$b_{ip} + 2\widehat{b}_{pp}b_{ip} \in \sigma_{ip}. \tag{1.41}$$

Now recall that ξ is an arbitrary element of the additive subgroup $\Gamma_{s,-s}$ and thus we can get the inclusion (1.41) with ξ substituted by $-\xi$. Combining it with the original inclusion (1.41) we get

$$2b_{ip} \in \sigma_{ip}.$$

This inclusion in turn together with (1.41) gives us the required inclusions (1.38) for all $i \neq -p, -q, -h$, in particular $b_{-t,p} \in \sigma_{-t,p}$.

Applying the inclusions (1.38) to the matrix $f = T_{-t,-p}(1)aT_{s,-s}(\xi)a^{-1}T_{-t,-p}(-1)$ instead of to b we get

$$b_{-t,p} + b_{-p,p} = f_{-t,p} \in \sigma_{-t,p},$$

and therefore also

$$b_{-p,p} \in \sigma_{-p,p}.$$

Finally, the inclusions for $i = -h$ and $i = -q$ are obtained by swapping the indices t, q and h in all the reasoning above. Therefore the inclusions (1.38) hold for any $i \in I$.

Now assume $b \in \mathrm{Sp}(\sigma)$. Then $c, d \in \mathrm{Sp}(\sigma)$. By Corollary 1.1.5 we get

$$S_{i,-i}(c^{-1}) \equiv b_{ip}^2 S_{h,-h}(b^{-1}) + b_{i,-h}^2 S_{-p,p}(b^{-1}) \mod \Gamma_i. \tag{1.42}$$

Next by Lemma 1.4.1 we have

$$a_{-p,s}^2 S_{h,-h}(b^{-1}) \in \Gamma_h, \quad a_{hs}^2 S_{-p,p}(b^{-1}) \in \Gamma_{-p}. \tag{1.43}$$

Combining (1.42) and (1.43) we get

$$\begin{aligned} S_{q,-q}(c^{-1}) \in \Gamma_q + b_{qp}^2 S_{h,-h}(b^{-1}) + b_{q,-h}^2 S_{-p,p}(b^{-1}) = \\ \pm \xi^2 a_{qs}^2 a_{-p,s}^2 S_{h,-h}(b^{-1}) \pm \xi^2 a_{qs}^2 a_{hs}^2 S_{-p,p}(b^{-1}) \in \Gamma_q + \Gamma_{-q} \end{aligned} \tag{1.44}$$

and using also (1.31) we get

$$\begin{aligned} S_{-h,h}(c^{-1}) &\in \Gamma_{-h} + b_{-h,p}^2 S_{h,-h}(b^{-1}) + b_{-h,-h}^2 S_{-p,p}(b^{-1}) \\ &= \Gamma_{-h} + ((\widehat{b}')_{-h,-h}^2 S_{-p,p}(b^{-1}) + (\varepsilon_h + \varepsilon_p)\widehat{b}'_{-h,-h} b'_{-p,p}) + \\ &\quad (1 + 2\widehat{b}_{-h,-h} + \widehat{b}_{-h,-h})^2 S_{-p,p}(b^{-1}) = S_{-p,p}(b^{-1}) + \Gamma_{-p}. \end{aligned} \tag{1.45}$$

Next by Lemma 1.3.6 we have $S_{-q,q}(d) \in \Gamma_{-q}$ and by Corollary 1.1.5 we have

$$\begin{aligned} S_{-q,q}(d) \equiv c_{-q,h}^2 \alpha^2 S_{p,-p}(c^{-1}) + c_{-q,h}^2 \beta^2 S_{q,-q}(c^{-1}) + \\ + c_{-q,-p}^2 \alpha^2 S_{-h,h}(c^{-1}) + c_{-q,-q}^2 \beta^2 S_{-h,h}(c^{-1}) \in \Gamma_{-q}. \end{aligned} \tag{1.46}$$

Recall that $\alpha = 2a_{qp}$; therefore

$$c_{-q,h}^2 \alpha^2 S_{p,-p}(c^{-1}), c_{-q,-p}^2 \alpha^2 S_{-h,h}(c^{-1}) \leq 2\sigma_{-p,p} \leq \Gamma_{-h}. \tag{1.47}$$

Now by (1.30) we have $c_{-q,h} = b_{-q,p}(b'_{hh} + \widehat{b}'_{hh})$ and therefore by (1.44) we have

$$c_{-q,h}^2 \beta^2 S_{q,-q}(c^{-1}) = b_{-q,p}^2 (b'_{hh} + \widehat{b}'_{hh})^2 \beta^2 S_{q,-q}(c^{-1}) \in \sigma_{-q,p}^2 R^2 (\Gamma_q + \Gamma_{-q}) \leq \Gamma_{-q}. \tag{1.48}$$

Again by (1.30) we have $c_{-q,-q} = 1 + 2b_{-q,p}b'_{h,-q}$ and therefore by (1.45) we get

$$c^2_{-q,-q}\beta^2 S_{-h,h}(c^{-1}) = (1 + 2b_{-q,p}b'_{h,-q})^2(1 + 2\widehat{b}_{pp})^2 S_{-h,h}(c^{-1}) \equiv S_{-p,p}(b^{-1}) \mod \Gamma_{-p}. \tag{1.49}$$

Combining (1.49), (1.48), (1.47) and (1.46) we get the inclusion

$$S_{-p,p}(b^{-1}) \in \Gamma_{-p}.$$

This completes the proof. □

Lemma 1.4.3. *Assume* $h(\nu) \geq (4, 5)$. *Let* H *be a subgroup in* $\mathrm{Sp}(2n, R)$ *such that* $\mathrm{Ep}(\nu, R) \leq H$ *and let* (σ, Γ) *be the form net of ideals associated with* H. *Let* a *be a matrix in* H, $T_{s,-s}(\xi)$ *a long* (σ, Γ)*-elementary transvection and* b *the long root element* $aT_{s,-s}(\xi)a^{-1}$. *If* p *is an index in* I *such that the equivalence class of* p *is self-conjugate then for any index* $i \in I$ *the inclusions*

$$b_{ip} \in \sigma_{ip} \tag{1.50}$$

hold.

Proof. Fix some $h \in I$ such that (p, h) is a C-type base pair. Obviously, for $i \sim p$ the inclusions (1.50) are trivial and thus from now on we assume that $i \nsim p$, in particular $i \neq \pm p, \pm h$.

Consider the matrices $c = bT_{ph}(1)b^{-1}$ and

$$d = cT_{hp}(\alpha)T_{h,-p}(\beta)c^{-1} = e + c_{*h}(\alpha c'_{p*} + \beta c'_{-p,*}) - \varepsilon_h(\varepsilon_p c_{*,-p}\alpha + \varepsilon_h\varepsilon_{-p}c_{*p}\beta)c'_{-h,*} \in H.$$

Exactly as in Lemma 1.4.2 will show that provided $\alpha = -2b_{-p,p}$ and $\beta = 1 + 2\widehat{b}_{pp}$ the matrix d satisfies the conditions of Lemma 1.3.4. Denote by $\widetilde{d}$ the matrix $c_{*h}(\alpha c'_{p*} + \beta c'_{-p,*})$ in $M(2n, R)$. Using (1.30) we get

$$\begin{aligned}
\widetilde{d} &= c_{*h}(\alpha(e_{p*} - b_{pp}b'_{h*} + \varepsilon_h\varepsilon_p b_{p,-h}b'_{-p,*}) + \beta(e_{-p,*} - b_{-p,p}b'_{h*} + \varepsilon_p\varepsilon_h b_{-p,-h}b'_{-p,*}))\\
&= c_{*h}(\alpha(e_{p*} - (1 + \widehat{b}_{pp})b'_{h*} - \widehat{b}_{pp}\widehat{b}'_{h*} + \varepsilon_p\varepsilon_h b_{p,-h}e_{-p,*}) +\\
&\qquad + \beta(e_{-p,*} - b_{-p,p}b'_{h*} - b_{-p,p}\widehat{b}'_{h*} + \varepsilon_p\varepsilon_h b_{-p,-h}e_{-p,*}))\\
&= c_{*h}(-(\alpha(1 + 2\widehat{b}_{pp}) + \beta(2b_{-p,p})) + \alpha(e_{p*} + \widehat{b}_{pp}e_{h*} + \varepsilon_p\varepsilon_h b_{p,-h}e_{-p,*}) +\\
&\qquad + \beta(e_{-p,*} + b_{-p,p}e_{h*} + \varepsilon_p\varepsilon_h b_{-p,-h}e_{-p,*}))\\
&= c_{*h}(\alpha(e_{p*} + \widehat{b}_{pp}e_{h*} + \varepsilon_p\varepsilon_h b_{p,-h}e_{-p,*}) + \beta(e_{-p,*} + b_{-p,p}e_{h*} + \varepsilon_p\varepsilon_h b_{-p,-h}e_{-p,*})).
\end{aligned}$$

Therefore all columns of $\widetilde{d}$ except for the ones indexed by $\pm p$ and h are zero and $\widetilde{d}_{*,-p} = \beta c_{*h}$. Let $\overline{d} = -\varepsilon_h(\varepsilon_p c_{*,-p}\alpha + \varepsilon_h\varepsilon_{-p}c_{*p}\beta)c'_{-h,*}$. By Proposition 1.1.1 we get

$$\overline{d} = -(\mathfrak{p}\widetilde{d}\mathfrak{p}^{-1})^t.$$

Therefore $\overline{d}$ is a matrix with all rows zero except for the ones indexed by $\pm p, -h$. Therefore $d = e + \widetilde{d} + \overline{d}$ satisfies the conditions of Lemma 1.3.4. Thus $d_{i,-p} \in \sigma_{i,-p}$ for all i. Thus for $i \nsim p$ we get

$$\begin{aligned}\sigma_{i,-p} \ni d_{i,-p} = \beta c_{ih} = (1 + 2\widehat{b}_{pp})(b_{ip}b'_{hh} - \varepsilon_h\varepsilon_p b_{i,-h}b'_{-p,h}) = \\ = (1 + 2\widehat{b}_{pp})(b_{ip} + 2b_{ip}\widehat{b}'_{hh}) = b_{ip} + 2b_{ip}\widehat{b}'_{hh} + 2\widehat{b}_{pp}b_{ip} + 4\widehat{b}_{pp}b_{ip}\widehat{b}'_{hh}.\end{aligned} \tag{1.51}$$

The last summand

$$4\widehat{b}_{pp}b_{ip}\widehat{b}'_{hh} = -4a_{ps}\xi a'_{-s,p}a_{is}\xi a'_{-s,p}a_{hs}\xi a'_{-s,h}$$

lies in σ_{ip} by Lemma 1.4.1. Thus the rest of (1.51) also lies in this ideal, namely

$$b_{ip} + 2b_{ip}\widehat{b}'_{hh} + 2\widehat{b}_{pp}b_{ip} \in \sigma_{ip}. \tag{1.52}$$

Now recall that ξ is an arbitrary element of $\Gamma_{s,-s}$. Thus we can make the substitution $\xi \mapsto -\xi$ in (1.52), and subtract the result from (1.52) itself. This operation doubles the summand with an odd number of multipliers ξ and cancels the ones with an even number. We get

$$2b_{ip} \in \sigma_{ip}.$$

Looking back at (1.52) we also get

$$b_{ip} \in \sigma_{ip}.$$

This finishes the proof. □

We are ready to prove the main results of this chapter.

Proof of Theorem 3. Pick any matrix $a \in H$ and any long long (σ, Γ)-elementary transvection $T_{s,-s}(\xi)$. Denote by b the long root element $aT_{s,-s}(\xi)a^{-1}$. Pick an index $p \in I$. If the equivalence class of p is self-conjugate, by Lemma 1.4.3 we get the inclusions $b_{ip} \in \sigma_{ip}$ for all $i \in I$. If the equivalence class of p is non-self-conjugate, we get the inclusions $b_{ip} \in \sigma_{ip}$ for all i by Lemma 1.4.2. Therefore $b \in \mathrm{Sp}(\sigma)$.

Now if $p \sim -p$ we obviously have $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$. Assume $p \nsim -p$. By Lemma 1.4.2 we also get the inclusions $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$. Therefore $b^{-1} \in \mathrm{Sp}(\sigma, \Gamma)$ and by Corollary 1.1.3 also $b \in \mathrm{Sp}(\sigma, \Gamma)$. Due to the arbitrary choice of the elementary transvections $T_{s,-s}(\xi)$ we can conclude that $a \in \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}^L(\sigma, \Gamma), \mathrm{Sp}(\sigma, \Gamma))$. □

Proof of Theorem 5. Pick any matrix $a \in H$ and any short (σ, Γ)-elementary transvection $T_{sr}(\xi)$. Denote by b the short root element $aT_{sr}(\xi)a^{-1}$. Then $b \in H$. Since $1 \in \Gamma_{-i}$, because the equivalence class of i is self-conjugate, we get by Theorem 3 that

$$b_{-i,-i} \cdot 1 \cdot b_{ij} \in \sigma_{-i,j} = \sigma_{ij}.$$

But

$$\begin{aligned}
b_{-i,-i}b_{ij} &= (1 + a_{-i,s}\xi a'_{r,-i} - \varepsilon_s\varepsilon_r a_{-i,-r}\xi a'_{-s,-i})(a_{is}\xi a'_{rj} - \varepsilon_s\varepsilon_r a_{i,-r}\xi a'_{-s,j}) = \\
&= b_{ij} + a_{-i,s}\xi a'_{r,-i}a_{is}\xi a'_{rj} - \varepsilon_s\varepsilon_r a_{-i,s}\xi a'_{r,-i}a_{i,-r}\xi a'_{-s,j} - \\
&\quad - \varepsilon_s\varepsilon_r a_{-i,-r}\xi a'_{-s,-i}a_{is}\xi a'_{rj} + a_{-i,-r}\xi a'_{-s,-i}a_{i,-r}\xi a'_{-s,j} = \\
&= b_{ij} - \varepsilon_i\varepsilon_r(a_{i,-r}a'_{rj})\xi^2 a_{-i,s}a_{is} - \varepsilon_s\varepsilon_r(a_{-i,s}a'_{-s,j})\xi^2 a'_{r,-i}a_{i,-r} - \\
&\quad - \varepsilon_s\varepsilon_r(a_{-i,-r}a'_{rj})\xi^2 a'_{-s,-i}a_{is} + \varepsilon_i\varepsilon_s(a_{is}a'_{-s,j})\xi^2 a_{-i,-r}a_{i,-r}.
\end{aligned}$$

Finally by Theorem 1 we know that $a_{i,-r}a'_{rj}, a_{-i,s}a'_{-s,j}, a_{-i,-r}a'_{rj}, a_{is}a'_{-s,j}$ are all contained in σ_{ij}. Therefore b_{ij} is also contained in σ_{ij} and $b \in \mathrm{Sp}(\sigma) = \mathrm{Sp}(\sigma,\Gamma)$. Therefore $H \leq \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma,\Gamma), \mathrm{Sp}(\sigma,\Gamma))$.

Now we prove the uniqueness of a form net (σ,Γ) such that

$$\mathrm{Ep}(\nu,R) \leq \mathrm{Ep}(\sigma,\Gamma) \leq H \leq \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma,\Gamma), \mathrm{Sp}(\sigma,\Gamma)).$$

Assume there exists an exact form net of ideals $(\sigma',\Gamma') \geq [\nu]_R$ such that $\mathrm{Ep}(\sigma',\Gamma') \leq H$, which doesn't coincide with the net (σ,Γ) associated with H. Therefore there exists an index $s \neq \pm r$ and an element $\xi \in \sigma_{sr} \setminus \sigma'_{sr}$. Clearly, $h = T_{sr}(\xi) \in H \leq \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma',\Gamma'), \mathrm{Sp}(\sigma',\Gamma'))$ and thus we can apply Theorem 4 to the matrix h and the form net of ideals (σ',Γ') and get $\xi = h_{sr}\cdot 1\cdot h'_{rr} \in \sigma'_{sr}$. This contradicts the assumption that $\xi \notin \sigma'_{sr}$. $\square$

2 Sandwich classification in symplectic groups: general case

In this chapter we continue describing overgroups of elementary block-diagonal subgroups in classical symplectic groups. The central result of this chapter is the following theorem. For a form net of ideals (σ, Γ) denote by $\mathrm{Ep}^S(\sigma, \Gamma)$ the subgroup of $\mathrm{Ep}(2n, R)$ generated by all short (σ, Γ)-elementary transvections, i.e.

$$\mathrm{Ep}^S(\sigma, \Gamma) = \langle T_{ij}(\xi) \mid i \neq \pm j, \xi \in \sigma_{ij} \rangle .$$

Theorem 6. *Let ν be a unitary equivalence relation on the index set I such that $h(\nu) \geq (4, 5)$. Let H be a subgroup of the classical symplectic group $\mathrm{Sp}(2n, R)$ over a commutative ring R. Suppose $\mathrm{Ep}(\nu, R) \leq H$ and let (σ, Γ) denote the form net of ideals associated with H. Then*

$$H \leq \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}^S(\sigma, \Gamma), \mathrm{Sp}(\sigma, \Gamma)).$$

Theorem 6 is the analogue of Theorem 3 for short root elements instead of the long ones. However the proof of Theorem 6 is noticeably more complicated than that of Theorem 3. One of the primary technical reasons for this is that the rows (or columns) of the matrix $b - e$, where b is a long root element, are linearly dependant, unlike in the case when b is a short root element. In fact, if a is an arbitrary matrix in $\mathrm{Sp}(2n, R)$, $T_{sr}(\xi)$ a short symplectic transvection, b the short root element $aT_{sr}(\xi)a^{-1}$ and $p \neq q \in I$ then sufficient conditions for the rows $(b - e)_{p*}$ and $(b - e)_{q*}$ to be linearly dependant is that the system of equations

$$\begin{cases} \alpha a_{ps} & + & \beta a_{qs} & = 0 \\ \alpha a_{p,-r} & + & \beta a_{q,-r} & = 0 \end{cases} \tag{2.1}$$

has a nontrivial solution. It's easy to see that $\alpha b_{p*} + \beta b_{q*} = \alpha e_{p*} + \beta e_{q*}$ for any solution (α, β) of the system (2.1). Under the assumption that the system (2.1) has a nonzero determinant, one can see that a solution of (2.1) can be found in the form of a degree 2 polynomial in the variables $a_{ps}, a_{qs}, a_{p,-r}$ and $a_{q,-r}$. Unfortunately, this is not very rewarding as it turns out to be very hard to get rid of a quadratic multiplier during the extraction of transvections. However, if we assume that the entries $a_{p,-r}$ and $a_{q,-r}$ are zero then the system (2.1) becomes equivalent to the single linear equation $\alpha a_{ps} + \beta a_{qs} = 0$ which always has a nontrivial solution and thus we can use approximately the same techniques as in Chapter 1. This brings us to the problem of obtaining zeros in given positions of the matrix a. This task is particularly easy over a local ground ring. But to get from an arbitrary commutative ring to a commutative local ring requires using a localization method to prove Theorem 6. Theorem 6 together with Theorems 3 and 4 will yield the standard sandwich classification theorem for the classical symplectic group:

Theorem 7. *Let ν be a unitary equivalence relation on the index set I such that $h(\nu) \geq (4,5)$. Let H be a subgroup of $\mathrm{Sp}(2n,R)$ such that $\mathrm{Ep}(\nu,R) \leq H$. Then there exists a unique exact form net of ideals $(\sigma,\Gamma) \geq [\nu]_R$ such that*

$$\mathrm{Ep}(\sigma,\Gamma) \leq H \leq \mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\sigma,\Gamma)).$$

The rest of this chapter is organized in the following way. In Section 2.1 we establish a convenient setting for our localization proof, and call it the standard setting. Section 2.2 is devoted to extracting transvections using first an element of a small parabolic subgroup and then a short root element $aT_{sr}(\xi)a^{-1}$, $a \in H$, provided that some coefficients of a are zero. In Section 2.3 we will show that the intersection of an overgroup H of a block-diagonal subgroup and the principal congruence subgroup of level the Jacobson radical (in fact, a set of matrices slightly larger than this intersection) is contained in the form net subgroup of level the net associated with H. This result is used in Section 2.4 to prove the standard sandwich classification theorem for the classical symplectic group over a local ring. In Section 2.5 we generalize this result using localization to arbitrary commutative rings and prove Theorems 6 and 7.

2.1 Standard setting

Let R be a commutative associative unital ring, R' a subring of R and S a subset of the intersection $R' \cap R^*$, where R^* stands for the set of invertible elements of the ring R. We call the triple (R,R',S) *a standard setting* if for any $\xi \in R$ there exist an element x in S such that $x\xi \in R'$. Clearly, the canonical ring homomorphism $S^{-1}R' \to R$ is an isomorphism. Now let (σ',Γ') be an exact form net of ideals of rank $2n$ over R' such that $[\nu]_{R'} \leq (\sigma',\Gamma')$. For each $i,j \in I$ set

$$\sigma_{ij} = \{\xi \in R \mid \exists x \in S\ x\xi \in \sigma'_{ij}\}$$
$$\Gamma_i = \{\alpha \in R \mid \exists x \in S\ x^2\alpha \in \Gamma'_i\}.$$

We will call the pair (σ,Γ) *the S-closure of the form net of ideals* (σ',Γ') [*in* R]. We will show (Proposition 2.1.1) that S-closures of exact form D-nets of ideals are themselves exact form D-nets of ideals.

Fix a subgroup H of $\mathrm{Sp}(2n,R)$. We call a net (σ',Γ') over R' *S-associated with the subgroup H* if the following two conditions are fulfilled:

1. $\mathrm{Ep}(\sigma',\Gamma') \leq H$

2. For any elementary symplectic transvection $T_{sr}(\xi)$ contained in H there exists an element $x \in S$ such that $x^{(1+\delta_{r,-s})}\xi \in (\sigma',\Gamma')_{sr}$.

It is easy to see that a subgroup may have several different S-associated nets, but their S-closures in R will coincide.

We will introduce now a family of net-like objects. For an arbitrary $g \in \mathrm{Sp}(2n, R)$ set

$$\begin{aligned} \sigma^g_{ij} &= \{\xi \in R \mid \exists x \in S\ \forall \theta \in R'\ {}^gT_{ij}(x\xi\theta) \in H\}, i \neq \pm j \\ \sigma^g_{ii} &= R \\ \Gamma^g_i &= \{\alpha \in R \mid \exists x \in S\ \forall \theta \in R'\ {}^gT_{i,-i}(x^2\theta^2\alpha) \in H\} \\ \sigma^g_{i,-i} &= \sum_{j \neq \pm i} \sigma^g_{ij}\sigma^g_{j,-i} + \langle \Gamma^g_i \rangle_R\,, \end{aligned} \tag{2.2}$$

where the product $\sigma^g_{ij}\sigma^g_{j,-i}$ denotes the Minkowski product, that is the ideal generated by all products $\xi\zeta$, where $\xi \in \sigma^g_{ij}$ and $\zeta \in \sigma^g_{j,-i}$. In general there is no guarantee that the objects (σ^g, Γ^g), defined in the obvious way from the above data are form nets of ideals. We will show that in cases of interest to us the objects (σ^g, Γ^g) are form nets of ideals and coincide with the S-closure of any net which is S-associated with the subgroup H.

For the rest of this section we fix a standard setting (R, R', S), a unitary equivalence relation ν and a subgroup H of $\mathrm{Sp}(2n, R)$.

Proposition 2.1.1. *Let (σ', Γ') be an exact major form net of ideals over R' and (σ, Γ) the S-closure of (σ', Γ') in R. Then (σ, Γ) is an exact major form net of ideals over R. Further, assume that $h(\nu) \geq (4, 3)$ and that (σ', Γ') is S-associated with the subgroup H. Then the form net of ideals (σ, Γ) is coordinate-wise equal to (σ^e, Γ^e).*

Proof. Clearly $\sigma_{ij} = R$ whenever $i \sim j$ and $\Gamma_i = R$ whenever $i \sim -i$. We will show first that for all $i, j \in I$ the sets σ_{ij} and Γ_i are additive subgroups of R. Let $\xi, \zeta \in (\sigma, \Gamma)_{ij}$. By definition, there exist elements x, y in S such that $x^{(1+\delta_{j,-i})}\xi, y^{(1+\delta_{j,-i})}\zeta \in (\sigma', \Gamma')_{ij}$. As (σ', Γ') is a form net of ideals, it follows that $(xy)^{(1+\delta_{j,-i})}\xi, (xy)^{(1+\delta_{j,-i})}\zeta \in (\sigma', \Gamma')_{ij}$ and thus also $(xy)^{(1+\delta_{j,-i})}(\xi + \zeta) \in (\sigma', \Gamma')_{ij}$. Therefore $\xi + \zeta \in (\sigma, \Gamma)_{ij}$. The rest of the properties of (σ, Γ) as an exact form net of ideals can be deduced in the same way from the corresponding properties of (σ', Γ').

Assume $h(\nu) \geq (4, 3)$. It's obvious that $(\sigma^e, \Gamma^e)_{ij} \leq (\sigma, \Gamma)_{ij}$ for all possible indices i and j and thus also that $\sigma^e_{i,-i} \leq \sigma_{i,-i}$ for all $i \in I$. The reverse inclusions are obtained in the following way. Fix some $i \nsim j$ and $\xi \in (\sigma, \Gamma)_{ij}$. By definition, there exists an element $x \in S$ such that $T_{ij}(x^{(1+\delta_{i,-j})}\xi) \in H$. Assume first, $i \neq -j$. Then, as $h(\nu) \geq (4, 3)$, there exists another index $k \sim j$ such that $k \neq \pm j, \pm i$. Then $T_{jk}(\theta), T_{kj}(1) \in H$ for all $\theta \in R'$ and therefore

$$T_{ij}(x\theta\xi) = [[T_{ij}(x\xi), T_{jk}(\theta)], T_{kj}(1)] \in H.$$

Hence, $\xi \in \sigma^e_{ij}$. If $i = -j$ then there exists another index $k \sim i$ such that $k \neq \pm i$. As (σ, Γ) is an exact form net of ideals, it follows by Proposition 1.1.6 that $x^2\xi \in \Gamma'_{k,-k}$. Thus we get

$$T_{i,-i}(-\varepsilon_i\varepsilon_j x^2\theta^2\xi)T_{k,-i}(x^2\theta\xi) = [T_{k,-k}(x^2\xi), T_{-k,-i}(\theta)] \in H.$$

If $k \sim -i$, then $T_{k,-i}(x^2\theta\xi) \in H$ and therefore $T_{i,-i}(-\varepsilon_i\varepsilon_j x^2\theta^2\xi) \in H$. If $k \nsim -i$, there exists another index $l \sim k$ such that $l \neq \pm k, \pm i$. By relation (R4)

$$T_{k,-i}(-x^2\theta\xi) = [T_{kl}(1), [T_{lk}(-1), [T_{k,-k}(x^2\xi), T_{-k,-i}(\theta)]]] \in H.$$

Therefore $T_{i,-i}(-\varepsilon_i\varepsilon_j x^2\theta^2\xi) \in H$ and $\xi \in \Gamma_i^e$. Summing up, $(\sigma,\Gamma)_{ij} \leq (\sigma,\Gamma)_{ij}^e$ for all $i,j \in I$. As (σ,Γ) is exact, it follows that $\sigma_{i,-i} \leq \sigma_{i,-i}^e$. This completes the proof. □

The last proposition allows us to consider the elementary form net subgroup $\mathrm{Ep}(\sigma,\Gamma)$ of $\mathrm{Sp}(2n,R)$. The following proposition establishes certain properties of the objects (σ^g,Γ^g) which follow directly from its definition and the Steinberg relations. This will show that (σ^g,Γ^g) is "almost a form net of ideals".

Proposition 2.1.2. *Assume $h(\nu) \geq (4,3)$. Let g be an element of $\mathrm{Ep}(\sigma,\Gamma)$. If $[\nu]_R \leq (\sigma^g,\Gamma^g)$ coordinate-wise then the following inclusions hold:*

1. $\sigma_{ij}^g\sigma_{jk}^g \leq \sigma_{ik}^g$ *for all $i \neq \pm j$, $j \neq \pm k$*
2. $\Gamma_i^g\sigma_{-i,k}^g \leq \sigma_{ik}^g$ *and* $\sigma_{i,-k}^g\Gamma_{-k}^g \leq \sigma_{ik}^g$ *for all $i,k \in I$*
3. $(\sigma_{ij}^g)^{2}\Gamma_j^g \leq \Gamma_i^g$ *for all $i \neq \pm j$*
4. $2\sigma_{ij}^g\sigma_{j,-i}^g \leq \Gamma_i^g$ *for all $i \neq \pm j$,*

where products are Minkowski products.

Proof. 1. The first property follows directly from the Steinberg relation (R4). Indeed, pick any $\xi \in \sigma_{ij}$ and any $\zeta \in \sigma_{jk}$ such that $i \neq \pm j, \pm k$ and $j \neq \pm k$. Then there exist elements $x_\xi, x_\zeta \in S$ such that ${}^gT_{ij}(x_\xi\xi), {}^gT_{jk}(x_\zeta\theta\zeta) \in H$ for all $\theta \in R'$. By relation (R4) we get

$$ {}^gT_{ik}(x_\xi x_\zeta\theta\xi\zeta) = [{}^gT_{ij}(x_\xi\xi), {}^gT_{jk}(x_\zeta\theta\zeta)] \in H $$

for all $\theta \in R'$. Therefore $\xi\zeta \in \sigma_{ik}^g$. The corresponding inclusions for the cases when $i = \pm k$ trivially follow from the definition of (σ^g,Γ^g).

2. The second inclusion is trivial when $i = \pm k$ for the same reason as above. Assume $i \neq \pm k$. We will prove the inclusion $\Gamma_i^g\sigma_{-i,k}^g \leq \sigma_{ik}^g$. The other one can be treated similarly. Pick any $\alpha \in \Gamma_i$ and $\xi \in \sigma_{-i,k}$. Then there exist elements $x_\alpha, x_\xi \in S$ such that for any $\theta \in R'$ we have

$$ {}^gT_{i,-i}(x_\alpha^2\alpha), {}^gT_{-i,k}(x_\xi\theta\xi) \in H. $$

By relation (R6) it follows that

$$ {}^gT_{ik}(x_\alpha^2 x_\xi\theta\alpha\xi){}^gT_{-k,k}(x_\alpha^2 x_\xi^2\theta^2\alpha\xi^2) = [{}^gT_{i,-i}(x_\alpha^2\alpha), {}^gT_{-i,k}(x_\xi\theta\xi)] \in H. \tag{2.3} $$

If $k \sim -k$ then by the definition of Γ_{-k}^g we get ${}^gT_{-k,k}(x_\alpha^2 x_\xi^2\theta^2\alpha\xi^2) \in H$. Thus we get ${}^gT_{ik}(x_\alpha^2 x_\xi\theta\alpha\xi) \in H$ and $\alpha\xi \in \sigma_{ik}^g$. If $k \nsim -k$ then, as $h(\nu) \geq (4,3)$, there exists another index $l \sim i$ such that $l \neq \pm i, \pm k$. Then there exist elements $x_1, x_2 \in S$ such that ${}^gT_{li}(x_1), {}^gT_{il}(x_2) \in H$. By the Steinberg relations (R3) and (R4) together with (2.3) we get

$$ {}^gT_{ik}(x_1x_2x_\alpha^2x_\xi\theta\alpha\xi) = [{}^gT_{il}(x_2), [{}^gT_{li}(x_1), {}^gT_{ik}(x_\alpha^2x_\xi\theta\alpha\xi){}^gT_{-k,k}(x_\alpha^2x_\xi^2\theta^2\alpha\xi^2)]] \in H $$

for all $\theta \in R'$. It follows that $\alpha\xi \in \sigma_{ik}^g$.

3. The next series of inclusions is established similarly. Fix some indices $i \neq \pm j$, an element $\xi \in \sigma^g_{ij}$ and an element $\alpha \in \Gamma^g_j$. Then there exist elements $x_\xi, x_\alpha \in S$ such that

$$ {}^gT_{i,-j}(x_\xi x_\alpha^2 \theta\xi\alpha) \cdot T_{i,-i}(x_\xi^2 x_\alpha^2 \theta^2 \xi^2 \alpha) = [{}^gT_{ij}(x_\xi\theta\xi), {}^gT_{j,-j}(x_\alpha^2 \alpha)] \in H \qquad (2.4) $$

for all $\theta \in R'$. By assertion (2) of the current lemma, the first term of the left-hand side of (2.4) is contained in H whenever θ is a multiple of some $x_0 \in S$. Therefore the second term of (2.4) is also contained in H for the same values of parameter θ. This shows that $\xi^2\alpha \in \Gamma^g_i$.

4. Finally, fix an index $i \neq \pm j$, an element $\xi \in \sigma^g_{ij}$ and an element $\zeta \in \sigma^g_{j,-i}$. Then there exist $x_\xi, x_\zeta \in S$ such that ${}^gT_{ij}(x_\xi\theta\xi), {}^gT_{j,-i}(x_\zeta\theta\zeta) \in H$ for all $\theta \in R'$; in particular ${}^gT_{ij}(x_\xi x_\zeta\theta\xi), {}^gT_{j,-i}(x_\zeta x_\xi\theta\zeta) \in H$ for all $\theta \in R'$. By the Steinberg relation (R5) it follows that

$$ {}^gT_{i,-i}(2x_\xi^2 x_\zeta^2 \theta^2 \xi\zeta) \in H $$

for all $\theta \in R'$. Hence, $\xi\zeta \in \Gamma_i$. □

Lemma 2.1.3. *Assume $h(\nu) \geq (4,3)$. Let (σ', Γ') be an exact major form net of ideals over R', which is S-associated with H. Let (σ, Γ) denote the S-closure of (σ', Γ') in R. Then for every $g \in \mathrm{Ep}(\sigma, \Gamma)$ the coordinate-wise equality*

$$ (\sigma, \Gamma) = (\sigma^g, \Gamma^g) \qquad (2.5) $$

holds. In particular, each such (σ^g, Γ^g) is an exact major form net of ideals over R.

Proof. We will prove this lemma by induction on the word length $L(g)$ of g in terms of the generators of $\mathrm{Ep}(\sigma, \Gamma)$. Proposition 2.1.1 serves as a base of induction, namely it shows that when $L(g) = 0$ and $g = e$ we have the equality $(\sigma, \Gamma) = (\sigma^e, \Gamma^e)$.

Before proving the induction step, we will prove a slightly stronger statement. Namely, assume $g \in \mathrm{Ep}(\sigma, \Gamma)$ such that $(\sigma, \Gamma) \leq (\sigma^g, \Gamma^g)$. Fix an element $T_{pq}(\zeta) \in \mathrm{Ep}(\sigma, \Gamma)$. We will show that $(\sigma^g, \Gamma^g) \leq (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})$. Note that, as $(\sigma, \Gamma) \leq (\sigma^g, \Gamma^g)$, it follows that $\zeta \in (\sigma^g, \Gamma^g)_{pq}$. Fix any $\xi \in (\sigma^g, \Gamma^g)_{sr}$ for some indices $s \neq r$. Then there exists an element $x_\xi \in S$ such that for every $\theta \in R'$ the inclusion ${}^gT_{sr}(x_\xi^\kappa \theta^\kappa \xi) \in H$ holds, where $\kappa = 1 + \delta_{s,-r}$. For any $x \in S$ we have the equality

$$ {}^{gT_{pq}(\zeta)}T_{sr}(x^\kappa\theta^\kappa\xi) = {}^g[T_{pq}(\zeta), T_{sr}(x^\kappa\theta^\kappa\xi)] \cdot {}^gT_{sr}(x^\kappa\theta^\kappa\xi). \qquad (2.6) $$

Below we will construct an element x_0 such that after the substitution $x = x_0$ the right-hand side of (2.6) is contained in H for all $\theta \in R'$. It will follow that $\xi \in (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})_{sr}$.

Clearly the second term of the right-hand side of (2.6) is contained in H whenever x is a multiple of x_ξ. The first term, which we will denote by $h = h(\theta)$, requires a more detailed investigation. First, assume that the transvections $T_{sr}(*)$ and $T_{pq}(*)$ commute. In this case, $h = e$ and thus we can put $x_0 = x_\xi$. Assume that $h \neq e$. The following six alternatives exhaust all possibilities:

(1) $s \neq \pm r, p \neq \pm q$ and one of the following holds

(i) $s = q, r \neq \pm p$

(ii) $r = -q, s \neq \pm p$

(iii) $s = -p, r \neq \pm q$

(iv) $-r = -p, s \neq \pm q$.

Then h is a single short transvection. We will prove only the case (i). The other ones can be treated similarly. By the Steinberg relation (R4), $h = {}^gT_{pr}(x\theta\zeta\xi)$. Recall that

$$[\nu]_R \leq (\sigma, \Gamma) \leq (\sigma^g, \Gamma^g).$$

By Proposition 2.1.2 we get $\zeta\xi \in \sigma^g_{pr}$. Therefore there exists an element $x_\zeta \in S$ such that h is contained in H whenever x is a multiple of x_ζ. Put $x_0 = x_\xi x_\zeta$.

(2) $p \neq \pm q, s \neq \pm r$ and one of the following holds:

(i) $s = q, r = -p$

(ii) $s = -p, r = q$

(iii) $s = p, r = -q$

(iv) $s = -q, r = p$.

In this case we can compute h using the Steinberg relation (R5). Again, we will prove only the case (i). As $\zeta \in \sigma^g_{pq}$, there exists an element $x_\zeta \in S$ such that ${}^gT_{pq}(x_\zeta\zeta) \in H$. Then $h \in H$ for all $\theta \in R'$ and $x_0 = x_\xi x_\zeta$. Indeed,

$$h = [{}^gT_{pq}(\zeta), {}^gT_{q,-p}(x_\zeta x_\xi \theta\xi)] = {}^gT_{p,-p}(2\zeta x_\zeta x_\xi\theta\xi) = [{}^gT_{pq}(\zeta x_\zeta), {}^gT_{q,-p}(x_\xi\theta\xi)] \in H$$

for every $\theta \in R'$ due to the choice of x_ξ and x_ζ.

(3) $q = -p, s \neq \pm r$ and either $s = -p$ or $r = p$. In both cases h is a product of a long and a short symplectic elementary transvection. We will consider only the first option. By relation (EU6),

$$h = [{}^gT_{p,-p}(\zeta), {}^gT_{-p,r}(x\theta\xi)] = {}^gT_{pr}(x\theta\xi\zeta){}^gT_{-r,r}(\pm x^2\theta^2\xi^2\zeta). \tag{2.7}$$

By Proposition 2.1.2 it follows that $\xi\zeta \in \Gamma^g_p\sigma_{-p,r} \leq \sigma^g_{pr}$ and $\xi^2\zeta \in (\sigma^g_{-p,r})^{\boxed{2}}\Gamma^g_p \leq \Gamma^g_{-r}$. Therefore there exist elements $x_{\xi\zeta}, x_{\xi^2\zeta} \in S$ such that the first term of the right-hand side of (2.7) belongs to H whenever x is a multiple of $x_{\xi\zeta}$ and the second term whenever x is a multiple of $x_{\xi^2\zeta}$. Put $x_0 = x_\xi x_{\xi\zeta} x_{\xi^2\zeta}$.

(4) $p \neq \pm q, r = -s$ and either $s = q$ or $s = -p$. Then h is a product a long and a short transvection. We prove only the first option, $s = q$. By the Steinberg relations (R1) and (R6) we have

$$h = [{}^gT_{pq}(\zeta), {}^gT_{q,-q}(x^2\theta^2\xi)] = {}^gT_{p,-q}(\pm x^2\theta^2\xi\zeta){}^gT_{p,-p}(\pm x^2\theta^2\zeta^2\xi). \tag{2.8}$$

As before, by Proposition 2.1.2 we have $\xi\zeta \in \sigma^g_{pq}\Gamma^g_q \leq \sigma^g_{p,-q}$ and $\zeta^2\xi \in (\sigma^g_{pq})^{\boxed{2}}\Gamma^g_q \leq \Gamma^g_p$. Therefore there exist elements $x_{\xi\zeta}, x_{\zeta^2\xi} \in S$ such that the right-hand side of (2.8) is contained in H whenever x is a multiple of $x_{\xi\zeta}x_{\zeta^2\xi}$. Put $x_0 = x_\xi x_{\xi\zeta} x_{\zeta^2\xi}$.

(5) Either $s = q, r = p$ or $s = -p, r = -q$. Without loss of generality we can assume the former. In this case, we can't apply any of the Steinberg relations directly, but we can first decompose $T_{sr}(*)$ as a product of transvections for which we know the commutators with $T_{pq}(*)$. As $h(\nu) \geq (4,3)$ there exists either another index $h \sim p$ such that $h \neq \pm p, \pm q$, or $p \sim -p$. In the first case,

$$\begin{aligned} {}^{gT_{pq}(\zeta)}T_{qp}(x\theta\xi) &= {}^{gT_{pq}(\zeta)}[T_{qh}(y\theta\xi), T_{hp}(z)] \\ &= {}^{g}[[T_{pq}(\zeta), T_{qh}(y\theta\xi)]T_{qh}(y\theta\xi), [T_{pq}(\zeta), T_{hp}(z)]T_{hp}(z)] \\ &= [{}^{g}T_{ph}(y\theta\zeta\xi) \cdot {}^{g}T_{qh}(y\theta\xi), {}^{g}T_{hq}(-z\zeta) \cdot {}^{g}T_{hp}(z)] \end{aligned} \tag{2.9}$$

whenever $x = yz$. Observe that $\xi \in \sigma^g_{qp} \leq \sigma^g_{qp}R = \sigma^g_{qp}\sigma^g_{ph} \leq \sigma^g_{qh}$, $\xi\zeta \in \sigma^g_{pq}\sigma^g_{qh} \leq \sigma^g_{ph}$, $\zeta \in \sigma^g_{pq} \leq R\sigma^g_{pq} = \sigma^g_{hp}\sigma^g_{pq} \leq \sigma^g_{hq}$ and $1 \in \sigma^g_{hp}$. Thus we can choose y and z in S such that all four terms of the right-hand side of (2.9) are contained in H for all $\theta \in R'$. Then we can put $x_0 = yzx_\xi$.

If the equivalence class of p equals $\{\pm p, \pm q\}$ then we can decompose $T_{sr}(*)$ in a different way, using long transvections. Namely,

$$\begin{aligned} {}^{gT_{pq}(\zeta)}T_{qp}(x\theta\xi) &= {}^{gT_{pq}(\zeta)}\left([T_{q,-q}(y^2), T_{-q,p}(z\theta\xi)]T_{-p,p}(\pm z^2\theta^2\xi^2 y)\right) = \\ &= {}^{g}\left([T_{p,-q}(\zeta y^2)T_{p,-p}(\pm\zeta^2 y^2)T_{q,-q}(y^2),\right. \\ &\quad T_{-q,q}(-2\zeta z\theta\xi)T_{-q,p}(z\theta\xi)]T_{q,-p}(\pm\zeta z^2\theta^2\xi^2 y^2) \\ &\quad \left. T_{q,-q}(\pm\zeta^2 z^2\theta^2\xi^2 y^2)T_{-p,p}(\pm z^2\theta^2\xi^2 y^2)\right), \end{aligned} \tag{2.10}$$

whenever $x = y^2 z$. Using the previous cases, we can choose y and z such that the right-hand side of (2.10) is contained in H for all $\theta \in R'$. Put $x_0 = y^2 z$.

(6) $q = s = -p, r = p$. Then there exists an index $h \sim p$ such that $h \neq \pm p$. Then

$$\begin{aligned} {}^{gT_{p,-p}(\zeta)}T_{-p,p}(x^2\theta^2\xi) &= {}^{gT_{p,-p}(\zeta)}\left([T_{-h,h}(y^2\theta^2\xi), T_{h,p}(\pm z)]\times\right. \\ &\quad \left. T_{-h,p}(\pm y^2 z\theta^2\xi)\right) \\ &= {}^{g}T_{-h,-p}(y^2 z\theta^2\zeta\xi) \cdot {}^{g}T_{-h,h}(y^4 z^2\theta^2\xi^2\zeta) \cdot \\ &\quad {}^{g}T_{-h,p}(\pm y^2 z\theta^2\xi), \end{aligned} \tag{2.11}$$

whenever $x = y^2 z$. Observe that

$$\begin{aligned} \xi &\in \Gamma^g_{-p} \leq R\Gamma^g_{-p} = \sigma^g_{-h,-p}\Gamma^g_{-p} \leq \sigma^g_{-h,p} \\ \xi\zeta &\in \sigma^g_{-h,p}\Gamma^g_{p} \leq \sigma^g_{-h,-p} \\ \xi^2\zeta &\in (\sigma^g_{-h,p})^{\boxed{2}}\Gamma^g_{p} \leq \Gamma^g_{-h}. \end{aligned}$$

Hence we can choose elements y and z in S such that every term of the right-hand side of (2.11) is contained in H for all $\theta \in R'$. Put $x_0 = y^2 z$.

The alternatives above are exhaustive. Therefore $(\sigma^g, \Gamma^g)_{sr} \leq (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})_{sr}$ for all $s \neq r \in I$. The inclusions $\sigma^{gT_{pq}(\zeta)}_{ii} \leq \sigma^g_{ii}$ and $\sigma^{gT_{pq}(\zeta)}_{i,-i} \leq \sigma^g_{i,-i}$ follow easily from the definition of (σ^g, Γ^g). Therefore we have proved that $(\sigma^g, \Gamma^g) \leq (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})$ coordinate-wise, whenever $(\sigma, \Gamma) \leq (\sigma^g, \Gamma^g)$.

The induction step looks as follows. Assume that for all elements $g \in \mathrm{Ep}(\sigma, \Gamma)$ such that $L(g) \leq L_0$, the equality (2.5) holds. Let $T_{pq}(\zeta)$ be an elementary transvection in $\mathrm{Ep}(\sigma, \Gamma)$ such that $L(g \cdot T_{pq}(\zeta)) = L_0 + 1$. Then, as we have proved above, $(\sigma^g, \Gamma^g) \leq (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})$, in particular $(\sigma, \Gamma) \leq (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})$. For the same reason

$$(\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)}) \leq (\sigma^{gT_{pq}(\zeta)T_{pq}(-\zeta)}, \Gamma^{gT_{pq}(\zeta)T_{pq}(-\zeta)}) = (\sigma^g, \Gamma^g).$$

Summing up, by induction we get the required equality (2.5) for all $g \in \mathrm{Ep}(\sigma, \Gamma)$. $\square$

We will also use the lemma above in the form of the following obvious corollary. It represents the concept of a common denominator for a finite family of fractions.

Corollary 2.1.4. *Assume $h(\nu) \geq (4, 3)$. Let (σ', Γ') be an exact major form net of ideals which is S-associated with the subgroup H and let (σ, Γ) be the S-closure of (σ, Γ) in R. Then for any finite family $\{T_{s_i, r_i}(\xi_i)\}_{i \in L}$ of (σ, Γ)-elementary transvections and any finite family $\{g_i\}_{i \in K}$ of elements of $\mathrm{Ep}(\sigma, \Gamma)$ there exists an element $x \in S$ such that*

$$^{g_i}T_{s_j, r_j}((x\theta)^{(1+\delta_{s_j, -r_j})}\xi_j) \in H$$

for all $i \in K, j \in J$ and $\theta \in R'$.

2.2 Extraction of transvections

In this section we perform the extraction of transvections first using matrices in small parabolic subgroups and then using short root elements. The results of this section directly correspond to and follow the general lines of the results of Sections 1.3 and 1.4.

Throughout this section we fix a standard setting (R, R', S), a unitary equivalence relation ν, a subgroup H of $\mathrm{Sp}(2n, R)$ and an exact major form net of ideals (σ', Γ') which is S-associated with H. We let (σ, Γ) denote the S-closure of (σ', Γ') in R.

Extraction of transvections in parabolic subgroups

Lemma 2.2.1. *Let a be a matrix in $\mathrm{Sp}(2n, R)$ such that for some index $p \in I$ the following conditions hold:*

1. $a_{pp} = a_{-p,-p} = 1$
2. $a_{ij} = \delta_{ij}$ *whenever $i \neq -p$ and $j \neq p$.*

Then

$$a = \left(\prod_{1 \leq j \neq \pm p \leq n} T_{-p,j}(a_{-p,j}) T_{-p,-j}(a_{-p,-j}) \right) T_{-p,p}(S_{-p,p}(a)). \tag{2.12}$$

Further, suppose $h(\nu) \geq (4, 3)$ and there exists an element $g \in \mathrm{Ep}(\sigma, \Gamma)$ such that $^g a \in H$. Then $a \in \mathrm{Ep}(\sigma, \Gamma)$.

Proof. The decomposition (2.12) can be checked by a straightforward calculation. Let $j \neq \pm p$. If $j \sim -p$ then the inclusion $a_{-p,j} \in \sigma_{-p,j}$ is trivial as ν is major. From now on, assume $j \nsim -p$. As $h(\nu) \geq (4,3)$, we can choose an index $k \sim j$ such that $k \neq \pm j, \pm p$. Choose using Corollary 2.1.4 an element $x_1 \in S$ such that ${}^g T_{jk}(x_1) \in H$. Then

$$X = {}^g(T_{-p,k}(a_{-p,j}x_1)T_{-p,-j}(\pm a_{-p,-k}x_1)) = {}^g[a, T_{jk}(x_1)] \in H.$$

If the equivalence class of j is non-self-conjugate then either $j \sim p$ or there exists another index $h \sim j$ such that $h \neq \pm k, \pm j, \pm p$. In the former case, $\pm a_{-p,-k}x_1 \in \sigma^g_{-p,-j} = R$ and by Corollary 2.1.4 the element x_1 can be chosen such that ${}^g T_{-p,-j}(\pm a_{-p,-k}x_1)) \in H$ and thus also ${}^g T_{-p,k}(a_{-p,j}x_1) \in H$. It follows that $a_{-p,j} \in \sigma^g_{-p,k} = \sigma^g_{-p,j}$. In the latter case choose using Corollary 2.1.4 an element $x_2 \in S$ such that ${}^g T_{kh}(x_2), {}^g T_{hk}(x_2\theta) \in H$ for any $\theta \in R'$. Then for the same θ we get

$$T_{-p,j}(x_1x_2^2\theta a_{-p,j}) = [[X, T_{kh}(x_2)], T_{hk}(x_2\theta)] \in H.$$

If the equivalent class of j is self-conjugate, it contains at least the elements $\pm j, \pm k$. Pick an element $x_2 \in S$ such that ${}^g T_{k,-j}(x_2), {}^g T_{-j,k}(x_2), {}^g T_{kj}(x_2\theta) \in H$ for all $\theta \in R'$. Then

$${}^g T_{-p,j}(x_1x_2^3\theta a_{-p,j}) = [[[X, {}^g T_{k,-j}(x_2)], {}^g T_{-j,k}(x_2)], {}^g T_{kj}(x_2\theta)] \in H.$$

Therefore $a_{-p,j} \in \sigma^g_{-p,j}$ and by Lemma 2.1.3 $a_{-p,j} \in \sigma_{-p,j}$ for all $j \neq \pm p$.

In order to prove that $a \in \mathrm{EU}(\sigma, \Gamma)$ it only remains to show that $T_{-p,p}(S_{-p,p}(a)) \in \mathrm{Ep}(\sigma, \Gamma)$. If $-p \sim p$ then $\Gamma_{-p} = R$ and the inclusion $T_{-p,p}(S_{-p,p}(a)) \in \mathrm{Ep}(\sigma, \Gamma)$ is trivial. Assume $p \nsim -p$. Set

$$g_1 = g \prod_{j>0, j\neq \pm p} T_{-p,j}(a_{-p,j})T_{-p,-j}(a_{-p,-j}).$$

Then $g_1 T_{-p,p}(S_{-p,p}(a))g^{-1} = {}^g a \in H$ and $g_1, g^{-1} \in \mathrm{EU}(\sigma, \Gamma)$. As $p \nsim -p$, we can choose two more indices q and t such that (p, q, t) is an A-type base triple. Pick an element $y_1 \in S$ such that ${}^g T_{pq}(y\theta), {}^{g_1} T_{pq}(y\theta) \in H$ for all $\theta \in R'$ whenever y is a multiple of y_1. By the Steinberg relation (R6) we have

$$\begin{aligned} {}^{g_1}T_{-p,q}(y\theta S_{-p,p}(a)) \cdot {}^{g_1}T_{-q,q}(-\varepsilon_p\varepsilon_q y^2\theta^2 S_{-p,p}(a)) \cdot {}^{g_1}T_{pq}(y\theta) \\ = {}^{g_1}[T_{-p,p}(S_{-p,p}(a)), T_{pq}(y\theta)] \cdot {}^{g_1}T_{pq}(y\theta) \\ = \left(g_1 T_{-p,p}(S_{-p,p}(a))g^{-1}\right)\left(gT_{pq}(y\theta)g^{-1}\right)\left(gT_{-p,p}(-S_{-p,p}(a))g_1^{-1}\right). \end{aligned} \tag{2.13}$$

The right-hand side of (2.13) as well as the third term of the left-hand side of (2.13) is contained in H whenever y is a multiple of y_1 in S. Therefore

$${}^{g_1}T_{-p,q}(y\theta S_{-p,p}(a)) \cdot {}^{g_1}T_{-q,q}(-\varepsilon_p\varepsilon_q y^2\theta^2 S_{-p,p}(a)) \in H \tag{2.14}$$

for all $\theta \in S$ whenever y is a multiple of y_1. Pick $y_2 \in S$ such that ${}^{g_1}T_{-p,-t}(y_2)$, ${}^{g_1}T_{-t,-p}(y_2) \in H$. We get

$$\begin{aligned} {}^{g_1}T_{-p,q}(yy_2^2\theta S_{-p,p}(a)) = [{}^{g_1}T_{-p,-t}(y_2), [{}^{g_1}T_{-t,-p}(y_2), {}^{g_1}T_{-p,q}(y\theta S_{-p,p}(a)) \cdot \\ {}^{g_1}T_{-q,q}(-\varepsilon_p\varepsilon_q y^2\theta^2 S_{-p,p}(a))]] \end{aligned}$$

and thus by the choice of y_2 together with (2.14) we get that ${}^{g_1}T_{-p,q}(y\theta S_{-p,p}(a)) \in H$ for all $\theta \in R'$ whenever y is a multiple of $y_1y_2^2$. Combining this result again with (2.14) we get that ${}^{g_1}T_{-q,q}(-\varepsilon_p\varepsilon_q y^2\theta^2 S_{-p,p}(a)) \in H$ for all $\theta \in S$ whenever y is a multiple of $y_1y_2^2$. Thus, $S_{-p,p}(a) \in \Gamma_{-q}$ and by Proposition 1.1.6 $S_{-p,p}(a) \in \Gamma_{-p}$. This completes the proof. □

Lemma 2.2.2. *Assume $h(\nu) \geq (4,3)$. Let (p,q) be an A-type base pair and let a be an element of $\mathrm{Sp}(2n,R)$ such that $a_{*p} = e_{*p}$ or $a_{-p,*} = e_{-p,*}$. Assume that there exist elements $g_1, g_2 \in \mathrm{Ep}(\sigma,\Gamma)$ such that $g_1ag_2 \in H$. Then the inclusion $a_{qj} \in \sigma_{qj}$ holds for each $j \neq -p$. If additionally $a \in \mathrm{Sp}(\sigma)$ then also $S_{q,-q}(a) \in \Gamma_q$.*

Proof. As a is symplectic it's easy to see that the conditions of the lemma provide the equalities $a_{*p} = a'_{*p} = e_{*p}$ and $a_{-p,*} = a'_{-p,*} = e_{-p,*}$. Choose via Corollary 2.1.4 an element $x \in S$ such that ${}^{g_2^{-1}}T_{pq}(x) \in H$ and consider the matrix

$$\begin{aligned} b = a^{-1}T_{pq}(x)a &= e + a'_{*p}xa_{q*} - \varepsilon_p\varepsilon_q a'_{*,-q}xa_{-p,*} \\ &= e + e_{*p}xa_{q*} - \varepsilon_p\varepsilon_q a'_{*,-q}xe_{-p,*}. \end{aligned}$$

It's easy to see that $b_{ij} = \delta_{ij}$ whenever $i \neq p$ and $j \neq -p$, that $b_{pp} = b_{-p,-p} = 1$ and that

$${}^{g_1}b = (g_1ag_2)(g_2^{-1}T_{pq}(x)g_2)(g_2^{-1}a^{-1}g_1^{-1}) \in H.$$

Therefore by Lemma 2.2.1 the inclusion $b_{pj} \in \sigma_{pj}$ holds for each $j \neq \pm p$. Note that $b_{pj} = xa_{qj}$ whenever $j \neq \pm p$. Therefore $a_{qj} \in \sigma_{qj}$ for all $j \neq -p$. By Lemma 2.2.2 we also get the inclusion $S_{p,-p}(b) \in \Gamma_q$. Assume $a \in \mathrm{Sp}(\sigma)$. By the corollary 1.1.5 we have

$$S_{p,-p}(b) \equiv {a'_{pp}}^2 x^2 S_{q,-q}(a) + {a'_{p,-q}}^2 x^2 S_{-p,p}(a) \mod \Gamma_p^{\min}.$$

Recall that $a_{-p,*} = e_{-p,*}$. Thus $S_{-p,p}(a) = 0$. Further $a'_{pp} = 1$, and therefore $S_{p,-p}(b) \equiv S_{q,-q}(a) \mod \Gamma_p$. Hence, $S_{q,-q}(a) \in \Gamma_q$. □

Lemma 2.2.3. *Assume $h(\nu) \geq (4,4)$. Let (p,q) be an A-type base pair and a be an element of $\mathrm{Sp}(2n,R)$ such that $a_{ij} = \delta_{ij}$ whenever $i \neq -p,-q$ and $j \neq p,q$. Assume that there exists an element $g \in \mathrm{Ep}(\sigma,\Gamma)$ such that ${}^{g}a \in H$. Then the inclusion $a_{kp} \in \sigma_{kp}$ holds for each $k \neq -p,-q$. If additionally $a \in \mathrm{Sp}(\sigma)$ then also $S_{-p,p}(a) \in \Gamma_{-p}$.*

Proof. Fix any $k \nsim p$. As $h(\nu) \geq (4,4)$, there exists an index $h \sim k$ such that $h \neq \pm k, \pm p, \pm q$. Pick using Corollary 2.1.4 an element $x \in S$ such that ${}^{g}T_{hk}(x) \in H$ and consider the matrix

$$b = a^{-1}T_{hk}(x)a = e + a'_{*h}xa_{k*} - \varepsilon_h\varepsilon_k a'_{*,-k}xa_{-h,*}.$$

It is easy to see that $b_{hp} = xa_{kp}$ and ${}^{g}b \in H$. Further, there exists an index $l \sim k$ such that $l \neq \pm h, -p$ and $b_{*l} = e_{*l}$. Indeed, if $k \sim -k$, one can simply take $l = -k$. If the class of k is non-self-conjugate then such l exists due to the condition $h(\nu) \geq (4,4)$ (l can be equal to $-q$ if $-q \sim k$). Therefore, by Lemma 2.2.2 we get $xa_{kp} \in \sigma_{hp} = \sigma_{kp}$. Thus $a_{kp} \in \sigma_{kp}$.

Assume $a \in \mathrm{Sp}(\sigma)$. If the equivalence class of p is self-conjugate, clearly $S_{-p,p}(a) \in R = \Gamma_{-p}$. If the equivalence class of p is non-self-conjugate then, as $h(\nu) \geq (4,4)$, there exists an index $t \sim p$ such that (p,q,t) is an A-type base triple. Consider the matrix

$$c = T_{-p,-t}(-a_{-p,-t})T_{-q,-t}(a_{-q,-t})a.$$

As $t \sim p \sim q$ it follows that $c \in \mathrm{Sp}(\sigma)$. Note that $a_{-t,-t} = 1$, hence $c_{*,-t} = e_{*,-t}$ and

$$gT_{-q,-t}(a_{-q,-t})T_{-p,-t}(a_{-p,-t})cg^{-1} \in H.$$

By Lemma 2.2.2 it follows that $S_{-p,p}(c) \in \Gamma_{-p}$. Finally, by Corollary 1.1.4 we have

$$S_{-p,p}(c) \equiv S_{-p,p}(a) + a^2_{-p,-t}S_{-t,t}(a) \mod \Gamma_{-p}$$

and as $S_{-t,t}(a) = 0$ we get $S_{-p,p}(a) \in \Gamma_{-p}$. □

Lemma 2.2.4. *Assume $h(\nu) \geq (4,4)$. Let p be an index in I with self-conjugate equivalence class and let a be an element of $\mathrm{Sp}(2n,R)$ such that $a_{ij} = \delta_{ij}$ whenever $i \neq \pm p$ and $j \neq \pm p$. If there exists an element $g \in \mathrm{Ep}(\sigma,\Gamma)$ such that ${}^g a \in H$, then $a_{kp} \in \sigma_{kp}$ for all $k \in I$.*

Proof. If $k \sim p$ the inclusion $a_{kp} \in \sigma_{kp}$ is trivial. Assume $k \nsim p$, in particular $k \neq \pm p$. As $h(\nu) \geq (4,4)$, there exists another index $h \sim k$ such that $h \neq \pm k \pm p$. Pick using Corollary 2.1.4 an element $x \in S$ such that ${}^g T_{hk}(x) \in H$ and consider the matrix

$$b = a^{-1}T_{hk}(x)a = e + a'_{*h}ma_{k*} - \varepsilon_h\varepsilon_k a'_{*,-k}ma_{-h,*}.$$

We will show that it satisfies the conditions of Lemma 2.2.2. Indeed, by choice of x the inclusion ${}^g b \in H$ holds. Pick an index q such that (p,q) is a C-type base pair. It is easy to see that $q \nsim \pm k$. Clearly $b_{*q} = e_{*q}$. Applying Lemma 2.2.2 to the matrix a, the elementary transvection $T_{hk}(x)$ and the pair $(-p,q)$, we get $b_{-p,j} \in \sigma_{-p,j}$ for all $j \neq -q$. Thus $b_{-p,-h} \in \sigma_{-p,-h} = \sigma_{kp}$ and it is only left to notice that $b_{-p,-h} = -\varepsilon_h\varepsilon_k a'_{-p,-k}xa_{-h,-h} = \pm xa_{kp}$. Therefore $a_{kp} \in \sigma_{kp}$ for all $k \in I$. □

Extraction of transvections using short root elements

Lemma 2.2.5. *Assume $h(\nu) \geq (4,4)$. Let (p,q,h) be an* A*-type base triple, a an element of $\mathrm{Sp}(2n,R)$ and $T_{sr}(\xi)$ a short elementary transvection. Let b denote the short root element $aT_{sr}(\xi)a^{-1}$. Suppose that $a_{p,-r} = a_{q,-r} = 0$. Assume that there exist elements $g_1, g_2 \in \mathrm{Ep}(\sigma,\Gamma)$ such that $g_1ag_2 \in H$ and ${}^{g_2^{-1}}T_{sr}(\xi) \in H$. Then $a_{ps}b_{ih} \in \sigma_{ih}$ for all $i \neq -p,-q$. If additionally $a \in \mathrm{Sp}(\sigma)$ then also $a^2_{ps}S_{-h,h}(b^{-1}) \in \Gamma_{-h}$.*

Proof. It is easy to see that ${}^{g_1}b \in H$. Indeed

$${}^{g_1}b = (g_1ag_2)(g_2^{-1}T_{sr}(\xi)g_2)(g_2^{-1}a^{-1}g_1^{-1}) \in H.$$

Using Corollary 2.1.4 pick an element $x \in S$ such that ${}^{g_1}T_{hp}(-a_{qs}x), {}^{g_1}T_{hq}(a_{ps}x) \in H$. Set $\alpha = -a_{qs}x$ and $\beta = a_{ps}x$. Clearly, $\alpha a_{ps} + \beta a_{qs} = 0$. Consider the matrix

$$c = bT_{hp}(\alpha)T_{hq}(\beta)b^{-1} = e + b_{*h}(\alpha b'_{p*} + \beta b'_{q*}) - \varepsilon_h(\varepsilon_p \alpha b_{*,-p} + \varepsilon_q \beta b_{*,-q})b'_{-h,*}. \tag{2.15}$$

Obviously, ${}^{g_1}c \in H$ and by the conditions that $a_{p,-r} = a_{q,-r} = 0$ and $\alpha a_{ps} + \beta a_{qs} = 0$ it easily follows that $(\alpha b'_{p*} + \beta b'_{q*}) = (\alpha e_{p*} + \beta e_{q*})$ and $(\varepsilon_p \alpha b_{*,-p} + \varepsilon_q \beta b_{*,-q}) = (\varepsilon_p \alpha e_{*,-p} + \varepsilon_q \beta e_{*,-q})$. Thus we can rewrite (2.15) as follows

$$c = e + b_{*h}(\alpha e_{p*} + \beta e_{q*}) - \varepsilon_h(\varepsilon_p \alpha e_{*,-p} + \varepsilon_q \beta e_{*,-q})b'_{-h,*}.$$

In particular, $c_{ij} = \delta_{ij}$ whenever $i \neq -p, -q$ and $j \neq p, q$. Applying Lemma 2.2.3 to the matrix c we get the inclusion $c_{iq} \in \sigma_{iq}$ for all $i \neq -p, -q$. Notice that $c_{iq} = \beta b_{ih} = x a_{ps} b_{ih}$ for $i \neq -p, -q$. This completes the proof of the first part of the lemma.

Assume $a \in \mathrm{Sp}(\sigma)$. It follows that $b, c \in \mathrm{Sp}(\sigma)$. Therefore by Lemma 2.2.3 we get the inclusion $S_{-q,q}(c) \in \Gamma_{-q}$. By Corollary 1.1.5 we have

$$\begin{aligned} S_{-q,q}(c) \equiv\ & b^2_{-q,h}x^2(a^2_{qs}S_{p,-p}(b^{-1}) + a^2_{ps}S_{q,-q}(b^{-1})) \\ & + (b^2_{-q,-p}a^2_{qs} + b^2_{-q,-q}a^2_{ps})x^2 S_{-h,h}(b^{-1}) \in \Gamma_{-q}. \end{aligned} \tag{2.16}$$

As $S_{-h,h}(b^{-1}) \in \sigma_{-q,q}$ and $2\sigma_{-q,q} \leq \Gamma_i$ it follows that

$$\begin{aligned} (b^2_{-q,-p}a^2_{qs} + b^2_{-q,-q}a^2_{ps})x^2 S_{-h,h}(b^{-1}) \equiv \\ (b_{-q,-p}a_{qs} + b_{-q,-q}a_{ps})^2 x^2 S_{-h,h}(b^{-1}) \mod \Gamma_{-q}. \end{aligned} \tag{2.17}$$

A straightforward calculation shows that $b_{-q,-p}a_{qs} + b_{-q,-q}a_{ps} \in a_{ps} + 2R$, which together with (2.17) yields

$$(b^2_{-q,-p}a^2_{qs} + b^2_{-q,-q}a^2_{ps})x^2 S_{-h,h}(b^{-1}) \equiv a^2_{ps}x^2 S_{-h,h}(b^{-1}) \mod \Gamma_{-q}. \tag{2.18}$$

As $a_{p,-r} = a_{q,-r} = 0$, we have

$$\begin{aligned} b'_{pj} &= \delta_{pj} - a_{ps}\xi a'_{rj} \\ b'_{qj} &= \delta_{qj} - a_{qs}\xi a'_{rj}. \end{aligned}$$

In particular, $b'_{p,-p} = b'_{q,-q} = b'_{p,-q} = b'_{q,-p} = 0$. Therefore

$$\begin{aligned} a^2_{qs}S_{p,-p}(b^{-1}) &= -\varepsilon_p \sum_{j>0, j\neq \pm p, \pm q} a^2_{qs} b'_{pj} b'_{p,-j} = -\varepsilon_p \sum_{j>0, j\neq \pm p, \pm q} a^2_{qs} a^2_{ps} \xi^2 a'_{rj} a'_{r,-j} \\ &= \varepsilon_p \varepsilon_q a^2_{ps} S_{q,-q}(b^{-1}) \end{aligned}$$

and thus

$$b^2_{-q,h}x^2(a^2_{qs}S_{p,-p}(b^{-1}) + a^2_{ps}S_{q,-q}(b^{-1})) \in 2\sigma_{-q,q} \leq \Gamma_{-q}. \tag{2.19}$$

Combining (2.16), (2.17), (2.18) and (2.19) we get the inclusion $a^2_{ps}x^2 S_{-h,h}(b^{-1}) \in \Gamma_{-h}$. By the property ($\Gamma 2$) of a form net of ideals it follows that $a^2_{ps} S_{-h,h}(b^{-1}) \in \Gamma_{-h}$. □

Lemma 2.2.6. *Assume $h(\nu) \geq (4,4)$. Let (p,q) be a* C*-type base pair, a an element of* $\mathrm{Sp}(2n,R)$ *and $T_{s,-s}(\xi)$ a long elementary transvection. Let b denote the long root element $aT_{s,-s}(\xi)a^{-1}$. If there exist elements $g_1, g_2 \in \mathrm{Ep}(\sigma,\Gamma)$ such that $g_1ag_2 \in H$ and ${}^{g_2^{-1}}T_{s,-s}(\xi) \in H$, then $a_{ps}b_{ih} \in \sigma_{ih}$ for all $i \in I$.*

Proof. Clearly ${}^{g_1}b \in H$. Indeed

$$ {}^{g_1}b = g_1aT_{s,-s}(\xi)a^{-1}g_1^{-1} = (g_1ag_2)({}^{g_2^{-1}}T_{s,-s}(\xi))(g_2^{-1}a^{-1}g_1^{-1}) $$

and each term in brackets is contained in H. Pick $x \in S$ such that ${}^{g_1}T_{hp}(-a_{-p,s}x)$, ${}^{g_1}T_{h,-p}(a_{ps}x) \in H$. Set $\alpha = -a_{-p,s}x$ and $\beta = a_{ps}x$. Clearly $a_{ps}\alpha + a_{-p,s}\beta = 0$. It easily follows that $b'_{p*}\alpha + b'_{-p,*}\beta = e_{p*}\alpha + e_{-p,*}\beta$ and $(\varepsilon_p\alpha b_{*,-p} + \varepsilon_{-p}\beta b_{*p}) = (\varepsilon_p\alpha e_{*,-p} + \varepsilon_{-p}\beta e_{*p})$. Consider the matrix

$$ \begin{aligned} c = bT_{hp}(\alpha)T_{h,-p}(\beta)b^{-1} &= e + b_{*h}(b'_{p*}\alpha + b_{-p,*}\beta) - \varepsilon_h(\varepsilon_p\alpha b_{*,-p} + \varepsilon_{-p}\beta b_{*p})b'_{-h,*} \\ &= e + b_{*h}(e_{p*}\alpha + e_{-p,*}\beta) - \varepsilon_h(\varepsilon_p\alpha e_{*,-p} + \varepsilon_{-p}\beta e_{*p})b'_{-h,*}. \end{aligned} $$

It is easy to see that $c_{ij} = \delta_{ij}$ whenever $i \neq \pm p$ and $j \neq \pm p$. By Lemma 2.2.4 it follows that $c_{i,-p} \in \sigma_{ip}$ for all $i \in I$. It's only left to notice that $c_{i,-p} = \beta b_{ih} = xa_{ps}b_{ih}$ whenever $i \neq \pm p$. Hence $a_{ps}b_{ih} \in \sigma_{ih}$ for all $i \in I$. □

2.3 At the level of the Jacobson radical

Let $J = \mathrm{Rad}(R)$ be the Jacobson radical of the ground ring R and $\mathrm{Sp}(2n,R,J)$ be the principal congruence subgroup of $\mathrm{Sp}(2n,R)$ of level J, i.e. the subgroup $\{g \in \mathrm{Sp}(2n,R) \mid g_{ij} \equiv \delta_{ij} \mod J \text{ for all } i,j \in I\}$. In this section we will focus on the extraction of transvections using matrices having a submatrix which looks like a submatrix of an element of $\mathrm{Sp}(2n,R,J)$. By definition, every element x of the Jacobson radical J is quasi-regular, i.e. $1 + xy \in R^*$ for all $y \in R$, in particular $1 + x \in R^*$. It follows that $R^* + J \leq R^*$. Indeed, let x be invertible and $y \in J$ then, as y is quasi-regular, $1 + x^{-1}y \in R^*$. Therefore, and $x + y = x(1 + x^{-1}y)$ is a unit since it is a product of two units. We will use this property without reference.

Throughout this section we fix a standard setting (R, R', S), a unitary equivalence relation ν, a subgroup H of $\mathrm{Sp}(2n,R)$ which contains $\mathrm{Ep}(\nu,R')$ and an exact major form net of ideals (σ',Γ') which is S-associated with H. Let J denote the Jacobson radical of the ring R and let (σ,Γ) denote the S-closure in R of the form net of ideals (σ',Γ').

Lemma 2.3.1. *Assume $h(\nu) \geq (4,4)$. Let (p,q,h,t,l) be an A-type base quintuple and a an element of* $\mathrm{Sp}(2n,R)$ *such that $a_{-p,-t} = a_{-h,-t} = a_{-l,-t} = a_{pq} = a_{hq} = 0$, $a_{-p,q}a'_{t,-p}$, $a_{-h,q}a'_{t,-h} \in J$ and $a_{qq} \in R^*$. If there exist elements g_1 and g_2 in* $\mathrm{Ep}(\sigma,\Gamma)$ *such that $g_1ag_2 \in H$ then $a_{p,-t} \in \sigma_{p,-t}$.*

Proof. Pick using Corollary 2.1.4 an element $x \in S$ such that ${}^{g_2^{-1}}T_{qt}(x) \in H$ and consider the matrix $b = aT_{qt}(x)a^{-1}$. By choice of the parameter x we have

$$^{g_1}b = (g_1 a g_2)(g_2^{-1}T_{qt}(x)g_2)(g_2^{-1}a^{-1}a_1^{-1}) \in H.$$

Pick using Corollary 2.1.4 another element $y \in S$ such that ${}^{g_1}T_{-p,-h}(y) \in H$ and consider the matrix $c = bT_{-p,-h}(y)b^{-1}$. Clearly, ${}^{g_1}c \in H$ for the same reason as above. We are going to apply Lemma 2.2.5 to the matrix b, the short elementary transvection $T_{-p,-h}(y)$ and the A-type base triple $(-p,-l,-h)$. In order to do this we have to show first that $b_{-p,h} = b_{-l,h} = 0$. Indeed, by the assumptions of this lemma $a'_{th} = \varepsilon_t\varepsilon_h a_{-h,-t} = 0$ and $a_{-p,-t} = a_{-l,-t} = 0$. Thus

$$\begin{aligned} b_{-p,h} &= a_{-p,q}xa'_{th} - \varepsilon_q\varepsilon_t a_{-p,-t}xa'_{-q,h} = 0 \\ b_{-l,h} &= a_{-l,q}xa'_{th} - \varepsilon_q\varepsilon_t a_{-l,-t}xa'_{-q,h} = 0. \end{aligned}$$

As $a_{-p,q}a'_{t,-p} \in J$, it follows that

$$\begin{aligned} b_{-p,-p} &= 1 + a_{-p,q}xa'_{t,-p} - \varepsilon_q\varepsilon_t a_{-p,-t}xa'_{-q,-p} \\ &= 1 + a_{-p,q}xa'_{t,-p} \in 1 + J \leq R^*. \end{aligned}$$

Therefore by Lemma 2.2.5 we get the inclusions $b_{-p,-p}c_{i,-h} \in \sigma_{i,-h}$ and, as $b_{-p,-p}$ is invertible, also $c_{i,-h} \in \sigma_{i,-h}$ for all $i \neq p, l$. In particular,

$$b_{q,-p}yb'_{-h,-h} - \varepsilon_p\varepsilon_h b_{qh}yb'_{p,-h} = c_{q,-h} \in \sigma_{q,-h}. \tag{2.20}$$

Recall that $a_{-h,q}a'_{t,-h} \in J$ Therefore

$$b'_{-h,-h} = 1 - a_{-h,q}xa'_{t,-h} + \varepsilon_q\varepsilon_t a_{-h,-t}xa'_{-q,-h} = 1 - a_{-h,q}xa'_{t,-h} \in 1 + J \leq R^*. \tag{2.21}$$

Observe that

$$b'_{p,-h} = -a_{pq}xa'_{t,-h} + \varepsilon_q\varepsilon_t a_{p,-t}xa'_{-q,-h} = 0. \tag{2.22}$$

Substituting (2.21) and (2.22) into (2.20) we get the inclusion $b_{q,-p} \in \sigma_{q,-h} = \sigma_{q,-p}$. Finally, recall that $a_{qq} \in R^*$ and $a'_{-q,-p} = \varepsilon_q\varepsilon_p a_{pq} = 0$. Therefore

$$a_{qq}xa'_{t,-p} = a_{qq}xa'_{t,-p} - \varepsilon_q\varepsilon_t a_{q,-t}xa'_{-q,-p} = b_{q,-p} \in \sigma_{q,-p}.$$

Thus $a'_{t,-p} \in \sigma_{q,-p}$. It only remains to notice that

$$a_{p,-t} = -\varepsilon_p\varepsilon_t a'_{t,-p} \in \sigma_{q,-p} = \sigma_{p,-t}.$$

□

Lemma 2.3.2. *Assume $h(\nu) \geq (4,4)$. Let (p,q,h,t,l) be an A-type base quintuple and a an element of $\mathrm{Sp}(2n,R)$ such that at least one of the following three conditions holds:*

1. *The entries $a_{-p,-t}, a_{-h,-t}, a_{pq}, a_{hq}, a_{-p,q}a'_{t,-p}$ and $a_{-h,q}a'_{t,-h}$ are contained in the Jacobson radical and the entries a_{pp}, a_{qq} and $a_{-t,-t}$ are invertible.*

2. *The rows a_{p*}, a_{q*}, a_{h*} and the column $a_{*,-t}$ coincide modulo the Jacobson radical with the corresponding rows and columns of the identity matrix.*

3. *The rows $a_{-t,*}, a_{-h,*}, a_{-p,*}$ and columns a_{*p} and a_{*q} coincide modulo the Jacobson radical with the corresponding rows and columns of the identity matrix.*

If there exist $g_1, g_2 \in \mathrm{Ep}(\sigma, \Gamma)$ such that $g_1 a g_2 \in H$ then $a_{p,-t} \in \sigma_{p,-t}$.

Proof. Note that the first condition in the statement of this lemma trivially follows from any of the others. Consider the matrix

$$b = aT_{pq}(-a_{pp}^{-1}a_{pq}).$$

By the assumption that the entry a_{pq} is contained in the Jacobson radical $b \equiv a \mod J$. Clearly, $b_{pq} = 0$, $b_{p,-t} = a_{p,-t}$ and $b_{qq}, b_{-t,-t} \in R^*$. Further, consider the matrix

$$c = T_{hq}(-b_{hq}b_{qq}^{-1})T_{-p,-t}(-b_{-p,-t}b_{-t,-t}^{-1})T_{-h,-t}(-b_{-h,-t}b_{-t,-t}^{-1})T_{-l,-t}(-b_{-l,-t}b_{-t,-t}^{-1})b.$$

As $b \equiv a \mod J$, we have $b_{hq}, b_{-p,-t}$ and $b_{-h,-t}$ in the Jacobson radical. Therefore $c_{i*} \equiv a_{i*} \mod J$ whenever $i \neq t, -l$ (and also $c'_{*j} \equiv a'_{*j} \mod J$ whenever $j \neq l, -t$). In particular $c_{-p,q}c'_{t,-p}, c_{-h,q}c'_{t,-h} \in J$ and $c_{qq} \in R^*$. It is easy to see that $c_{pq} = c_{hq} = c_{-p,-t} = c_{-h,-t} = c_{-l,-t} = 0$. Finally, $g_3 c g_4 \in H$, where

$$\begin{aligned} g_3 = g_1 \big(&T_{hq}(-b_{hq}b_{qq}^{-1})T_{-p,-t}(-b_{-p,-t}b_{-t,-t}^{-1})\times \\ &T_{-q,-t}(-b_{-q,-t}b_{-t,-t}^{-1})T_{-l,-t}(-b_{-l,-t}b_{-t,-t}^{-1})\big)^{-1}, \\ g_4 = T_{pq}&(a_{pp}^{-1}a_{pq})g_2. \end{aligned}$$

Clearly, g_3 and g_4 are contained in $\mathrm{Ep}(\sigma, \Gamma)$. Therefore, c satisfies the conditions of Lemma 2.3.1 and it follows that $c_{p,-t} \in \sigma_{p,-t}$. It's only left to notice that $c_{p,-t} = a_{p,-t}$. □

Lemma 2.3.3. *Assume $h(\nu) \geq (4, 4)$. Let (p, q, h, t) be an A-type base quadruple and a an element of $\mathrm{Sp}(2n, R)$ such that $a_{p,-h}, a_{q,-h}, a_{t,-h} \in \sigma_{p,-p} \cap J$, $a_{-h,p} \in \sigma_{-h,p} \cap J$. Suppose $a_{qp} \in J$ and $a_{pp}, a_{-h,-h} \in R^*$ and suppose that there exists an element $g \in \mathrm{Ep}(\sigma, \Gamma)$ such that ${}^g a \in H$. Then $a_{ip} \in \sigma_{ip}$ for all $i \in I$. If additionally $a \in \mathrm{Sp}(\sigma)$ then also $S_{-h,h}(a^{-1}) \in \Gamma_{-h}$.*

Proof. Consider the matrix

$$b = T_{-h,p}(-a_{-h,p}a_{pp}^{-1})a.$$

As $a_{-h,p} \in J$ it follows that $b \equiv a \mod J$. Additionally $b_{p,-h}, b_{q,-h}$ and $b_{t,-h}$ are contained in $\sigma_{p,-p}$ and $b_{-h,p} = 0$. Consider the matrix

$$c = T_{p,-h}(-b_{p,-h}b_{-h,-h}^{-1})T_{q,-h}(-b_{q,-h}b_{-h,-h}^{-1})T_{t,-h}(-b_{t,-h}b_{-h,-h}^{-1})b.$$

Again, $c \equiv a \mod J$, in particular $c_{pp}, c_{-h,-h} \in R^*$. As c is a symplectic matrix, it also follows that $c'_{-p,h} = 0$ and $c'_{hh} \in R^*$. It's easy to see that $c_{p,-h} = c_{q,-h} = c_{t,-h} = c_{-h,p} = 0$. Finally, gg_1cg^{-1} is contained in H, where

$$\begin{aligned} g_1 = \big(&T_{p,-h}(-b_{-h,-h}^{-1}b_{p,-h})T_{q,-h}(-b_{-h,-h}^{-1}b_{q,-h})\times \\ &T_{t,-h}(-b_{-h,-h}^{-1}b_{t,-h})T_{-h,p}(-a_{pp}^{-1}a_{-h,p})\big)^{-1} \in \mathrm{Ep}(\sigma, \Gamma). \end{aligned}$$

Pick an element $x \in S$ such that ${}^gT_{ph}(x) \in H$. Applying Lemma 2.2.5 to the matrix c, the short symplectic transvection $T_{ph}(x)$ and the A-type base triple (p, q, h), we get for all $i \neq -p, -q$ the inclusion $c_{pp}(cT_{ph}(x)c^{-1})_{ih} \in \sigma_{ih}$. As c_{pp} is invertible, we also get

$$\delta_{ih} + c_{ip}xc'_{hh} - \varepsilon_p\varepsilon_h c_{i,-h}xc'_{-p,h} = (cT_{ph}(x)c^{-1})_{ih} \in \sigma_{ih} \tag{2.23}$$

for all $i \neq -p, -q$. We can apply Lemma 2.2.5 to the same matrix and transvection, but to a different A-type base triple (p, t, h) and get the inclusion (2.23) also for $i = -q$. As c'_{hh} is invertible and $c'_{-p,h} = 0$, it follows from (2.23) that $c_{ip} \in \sigma_{ih}$ for all $i \neq h, -p$. Observe that $a_{ip} = c_{ip}$ for all $i \neq p, q, t, -h$. Therefore $a_{ip} \in \sigma_{ip}$ for all $i \neq p, q, t, -h, -p$. The inclusions $a_{ip} \in \sigma_{ip}$ for $i = p, q$ and t are trivial and the corresponding inclusion for $i = -h$ is provided by the assumptions of the lemma. Therefore $a_{ip} \in \sigma_{ip}$ for all $i \neq -p$.

Pick an element $y \in S$ such that ${}^gT_{pq}(y) \in H$ and consider the matrix $d = T_{pq}(y)a$. Clearly, it satisfies all the conditions of this lemma. Indeed, $d_{p,-h} = a_{p,-h} + ya_{q,-h} \in \sigma_{p,-p} \cap J$, $d_{pp} = a_{pp} + ya_{qp} \in R^* + J \leq R^*$ and the rest of the entries of d involved in the conditions of this lemma coincide with the corresponding entries of a itself. Thus we get the inclusions $d_{ip} \in \sigma_{ip}$ for all $i \neq -p$. In particular, $d_{-q,p} \in \sigma_{-q,p}$. It's only left to notice that $d_{-q,p} = a_{-q,p} - \varepsilon_p\varepsilon_q a_{-p,p}$ and $a_{-q,p}$ is already contained in $\sigma_{-q,p}$, while y is invertible. Therefore $a_{-p,p} \in \sigma_{-p,p}$.

Assume $a \in \mathrm{Sp}(\sigma)$. By Lemma 2.2.5 we get the inclusion $S_{-h,h}(c^{-1}) \in \Gamma_{-h}$. As $a^{-1} = c^{-1}g_1^{-1}$, by Corollary 1.1.4 we get $S_{-h,h}(a^{-1}) \in \Gamma_{-h}$. □

We combine Lemmas 2.3.2 and 2.3.3 in the following corollary.

Corollary 2.3.4. *Assume $h(\nu) \geq (4,4)$. Let (p, q, h, t, l) be an A-type base quintuple and a an element of $\mathrm{Sp}(2n, R)$. Let I' denote the set $\{p, q, h, t\}$. Suppose that $a_{i*} \equiv e_{i*} \mod J$ and $a_{*,-i} \equiv e_{*,-i} \mod J$ whenever $i \in I'$. Further, suppose that there exists an element $g \in \mathrm{Ep}(\sigma, \Gamma)$ such that ${}^ga \in H$. Then $a_{ip} \in \sigma_{ip}$ for all $i \in I$. If additionally $a \in \mathrm{Sp}(\sigma)$ then also $S_{-p,p}(a^{-1}) \in \Gamma_{-p}$.*

Proof. It's easy to see that the matrix a satisfies condition (2) of Lemma 2.3.2. Thus we can conclude that the entries $a_{p,-h}, a_{q,-h}$ and $a_{t,-h}$ are contained in $\sigma_{p,-p}$. Moreover, the same entries are contained in the Jacobson radical by assumption. Since a also satisfies the condition (3) of Corollary 2.3.2, it follows that $a_{-h,p}$ is contained in $\sigma_{-h,p}$. Note that by assumption, $a_{pp}, a_{-h,-h} \in R^*$ and $a_{qp} \in J$. Summing up, a satisfies the conditions of Lemma 2.3.3. Hence $a_{ip} \in \sigma_{ip}$ for all $i \in I$. If $a \in \mathrm{Sp}(\sigma)$ then by Lemma 2.3.3 we get the inclusion $S_{-h,h}(a^{-1}) \in \Gamma_{-h}$. Switching the indices p and h in the reasoning above, we get the required inclusion $S_{-p,p}(a^{-1}) \in S_{-p,p}$. □

Lemma 2.3.5. *Assume $h(\nu) \geq (4,4)$. Let (p, h) be a C-type base pair and a an element of $\mathrm{Sp}(2n, R)$ such that $a_{pp}, a_{-p,-p} \in R^*$ and $a_{-h,-p} \in J$. If there exists an element $g \in \mathrm{Ep}(\sigma, \Gamma)$ such that ${}^ga \in H$, then $a_{ip} \in \sigma_{ip}$ for all $i \in I$.*

Proof. Consider the matrix

$$b = T_{hp}((-a_{hp} + 1)a_{pp}^{-1})a.$$

Clearly, $b_{pp} = a_{pp}$ is invertible, $b_{hp} = 1$ and

$$b_{-p,-p} = a_{-p,-p} - \varepsilon_h \varepsilon_p(-a_{hp} + 1)a_{pp}^{-1} a_{-h,-p} \in R^* + J \leq R^*.$$

Pick using Corollary 2.1.4 an element $x \in S$ such that ${}^g T_{p,-p}(x^2) \in H$. By Lemma 2.2.6 we get the inclusions

$$b_{pp}(\delta_{i,-h} + b_{ip} x^2 b'_{-p,-h}) = b_{pp}(bT_{p,-p}(x^2)b^{-1})_{i,-h} \in \sigma_{i,-h}$$

for all $i \in I$. It's only left to notice that b_{pp} and $x^2 b'_{-h,-p}$ are invertible and thus $b_{ip} \in \sigma_{i,-h}$ for all $i \in I$. Finally, $a_{ip} = b_{ip}$ whenever $i \neq h, -p$. Thus $a_{ip} \in \sigma_{ip}$. □

The following corollary is an illustration of application of Corollary 2.3.4 and Lemma 2.3.5. Suppose $R' = R$ and $S = \{1\}$. Then it is clear that $(\sigma', \Gamma') = (\sigma, \Gamma)$ is the net associated with H in $\mathrm{Sp}(2n, R)$. Corollary 2.3.4 together with Lemma 2.3.5 yield the following corollary.

Corollary 2.3.6. *Let R be a commutative associative unital ring with Jacobson radical J. Let ν be a unitary equivalence relation on the index set I such that $h(\nu) \geq (4, 5)$. Let H be a subgroup of $\mathrm{Sp}(2n, R)$ such that $\mathrm{Ep}(\nu, R) \leq H$ and let (σ, Γ) be the form net associated with H. Then*

$$H \cap \mathrm{Sp}(2n, R, J) \leq \mathrm{Sp}(\sigma, \Gamma),$$

where $\mathrm{Sp}(2n, R, J)$ denotes the principal congruence subgroup of $\mathrm{Sp}(2n, R)$ of level J.

2.4 Over a local ring

Throughout this section fix a standard setting (R, R', S), where R is a commutative local ring. Let J denote the Jacobson radical of R (which is the only maximal ideal of R). Further, fix a subgroup H of $\mathrm{Sp}(2n, R)$ and an exact major form net of ideals (σ', Γ') which is S-associated with H. Let (σ, Γ) denote the S-closure of the (σ', Γ') in R. In this section we will show that

$$H \leq \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}^S(\sigma', \Gamma'), \mathrm{Sp}(\sigma, \Gamma))$$

provided $h(\nu) \geq (4, 5)$.

Lemma 2.4.1. *Assume $h(\nu) \geq (4, 4)$. Let a be an element of H and $T_{sr}(\xi)$ a short (σ', Γ')-elementary transvection. Let b denote the short root element $aT_{sr}(\xi)a^{-1}$. If (p, h) is a C-type base pair then for all $i \in I$ the inclusion*

$$b_{ip} \in \sigma_{ip} \tag{2.24}$$

holds.

Proof. We will organize the analysis into four cases.

1. Assume that the elements $a_{-p,-r}, a_{p,-r}$ and one of the element $a_{-h,-r}$ or $a_{h,-r}$ is contained in J. Without loss of generality, we may assume that $a_{-h,-r} \in J$. Then it's easy to see that b_{pp} and $b_{-p,-p}$ are invertible and $b_{-h,-p} \in J$. Indeed,

$$\begin{aligned} b_{pp} &= 1 + a_{ps}\xi a'_{rp} - \varepsilon_s\varepsilon_r a_{p,-r}\xi a'_{-s,p} \in 1 + J \leq R^*, \\ b_{-p,-p} &= 1 + a_{-p,s}\xi a'_{r,-p} - \varepsilon_s\varepsilon_r a_{-p,-r}\xi a'_{-s,-p} \in 1 + J \leq R^*, \\ b_{-h,-p} &= a_{-h,s}\xi a'_{r,-p} - \varepsilon_s\varepsilon_r a_{-h,-r}\xi a'_{-s,p} \in J. \end{aligned}$$

Therefore, the matrix b satisfies the conditions of Lemma 2.3.5 and we get the inclusions $b_{ip} \in \sigma_{ip}$ for all $i \in I$.

2. Assume that either $a_{h,-r}$ or $a_{-h,-r}$ is invertible. Without loss of generality we may assume that $a_{h,-r}$ is invertible. Consider the matrices

$$c = T_{ph}(-a_{p,-r}a_{h,-r}^{-1})T_{-p,h}(-a_{-p,-r}a_{h,-r}^{-1}) \qquad d = T_{-h,h}(-c_{-h,-r}b_{h,-r}^{-1})d.$$

It's easy to see that $d_{p,-r} = d_{-p,-r} = d_{-h,-r} = 0$ and by definition $g_1 d = a \in H$, where $g_1 \in \mathrm{Ep}(\sigma, \Gamma)$. By the previous case we get the inclusion $f_{ip} \in \sigma_{ip}$ for all $i \in I$, where $f = dT_{sr}(\xi)d^{-1}$. It's only left to notice that $b_{ip} = f_{ip}$ whenever $i \neq -h, \pm p$ and the rest of the required inclusions are trivial.

3. Assume $a_{-p,-r}$ is invertible. Consider the matrix $c = T_{h,-p}(1)a$. Clearly, $c_{h,-r}$ is invertible and $T_{h,-p}(-1)c \in H$. By case 2 we get the inclusions $f_{ip} \in \sigma_{ip}$, where $f = cT_{sr}(\xi)c^{-1}$. Finally, $b_{ip} = f_{ip}$ whenever $i \neq h, p$.

4. Assume $a_{p,-r}$ is invertible. By the case 2, the inclusion $b_{ih} \in \sigma_{ih}$ holds for any $i \in I$. Consider the matrix $c = T_{hp}(1)a$. Then $c_{h,-r}$ is invertible and by case 2 we get the inclusions $f_{ip} \in \sigma_{ip}$ for all $i \in I$, where $f = cT_{sr}(\xi)c^{-1}$. Finally, $f_{ip} = b_{ip} + b_{ih}$ whenever $i \neq h, -p$ and thus also $b_{ip} \in \sigma_{ip}$ for all $i \in I$.

As the ring R is local, the cases above exhaust all the possibilities for the entries $a_{p,-r}, a_{-p,-r}, a_{h,-r}$ and $a_{-h,-r}$. Therefore $b_{ip} \in \sigma_{ip}$ for all $i \in I$. □

Lemma 2.4.2. *Assume $h(\nu) \geq (4,4)$, Let a be an element of H, $T_{sr}(\xi)$ a short (σ', Γ')-elementary transvection and b the short root element $aT_{sr}(\xi)a^{-1}$. Let (p,q,h,t,l) be an* A*-type base quintuple. Then the inclusion*

$$b_{ip} \in \sigma_{ip} \tag{2.25}$$

holds for any $i \in I$. If additionally $b \in \mathrm{Sp}(\sigma)$ then also

$$S_{-p,p}(b^{-1}) \in \Gamma_{-p}. \tag{2.26}$$

Proof. Denote by I' the set $\{p,q,h,t,l\}$. This proof is organized as follows.

1. We will show that if $a_{l,-r}$ is invertible then the inclusion (2.25) holds for any $i \in I$ and if additionally $b \in \mathrm{Sp}(\sigma)$ then also (2.26) holds.

2. We will show that if there exists an index $i \in I'$ such that $a_{i,-r}$ or a_{is} is invertible then the inclusion (2.25) holds for any $i \in I$ and if additionally $b \in \mathrm{Sp}(\sigma)$ then also (2.26) holds. This case can be reduced to the previous one.

3. Finally, if $a_{i,-r}, a_{is} \in J$ for all $i \in I'$ then the required inclusions (2.25) and (2.26) can be obtained using Corollary 2.3.4.

1. Suppose $a_{l,-r} \in R^*$. Let

$$g = T_{tl}(a_{t,-r}a_{l,-r}^{-1})T_{hl}(a_{h,-r}a_{l,-r}^{-1})T_{ql}(a_{q,-r}a_{l,-r}^{-1})T_{pl}(a_{p,-r}a_{l,-r}^{-1}).$$

Let $c = g^{-1}a$ and $d = cT_{sr}(\xi)c^{-1} = {}^{g^{-1}}b$. It is easy to see that $c_{p,-r} = c_{q,-r} = c_{h,-r} = c_{t,-r} = 0$ and if $b \in \mathrm{Sp}(\sigma)$ then also $d \in \mathrm{Sp}(\sigma)$. We will consider three subcases:

i. There is an index $i_1 \in I' \setminus \{p, l\}$ such that $c_{i_1,s}$ is invertible.

ii. The only $i_1 \in I' \setminus \{l\}$ such that $c_{i_1,s} \in R^*$ is $i_1 = p$.

iii. $c_{is} \in J$ for all $i \in I' \setminus \{l\}$.

For each of the cases (i)–(iii) we will prove that $d_{ip} \in \sigma_{ip}$ for all $i \in I$ and if $b \in \mathrm{Sp}(\sigma)$ then $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$. Note that $d_{ip} = b_{ip}$ for all $i \neq p, q, h, t, -l$, the inclusions $b_{ip} \in \sigma_{ip}$ for $i = p, q, h, t$ are trivial and

$$d_{-l,p} = b_{-l,p} \pm \zeta_p b_{-p,p} \pm \zeta_q b_{-q,p} \pm \zeta_h b_{-h,p} \pm \zeta_t b_{-t,p},$$

where $\zeta_i \in R$ and $b_{-i,p} \in \sigma_{-p,p}$ for $i = p, q, t, h$. Therefore $b_{-l,p} \in \sigma_{-l,p}$. Summing up, $b_{ip} \in \sigma_{ip}$ for all $i \in I$.

Assume that $b \in \mathrm{Sp}(\sigma)$. By Corollary 1.1.4 we get

$$S_{-p,p}(b^{-1}) \equiv S_{-p,p}({}^{g^{-1}}b^{-1}) = S_{-p,p}(d^{-1}) \mod \Gamma_{-p}.$$

Hence, $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$. This completes the analysis of the case 1. Now we consider the cases (1.i)–(1.iii).

1.i. Suppose there exists an index $i_1 \in I' \setminus \{p, l\}$ such that $c_{i_1,s} \in R^*$. By Lemma 2.2.5 we get $c_{i_1,s}d_{ip} \in \sigma_{ip}$ for all $i \neq -i_1$ and if $b \in \mathrm{Sp}(\sigma)$ then also $c_{i_1,s}^2 S_{-p,p}(d^{-1}) \in \Gamma_{-p}$. As $c_{i_1,s}$ is invertible, it follows that $d_{ip} \in \sigma_{ip}$ for all $i \neq -i_1$ and if $b \in \mathrm{Sp}(\sigma)$ then $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$.

Pick an index $i_2 \in I' \setminus \{p, l, i_1\}$. If $c_{i_2,s}$ is invertible in R we can replace i_1 in the reasoning above with i_2 and get the missing inclusion $d_{-i_1,p} \in \sigma_{-i_1,p}$. If $c_{i_2,s}$ is not invertible, consider the matrices $f = T_{i_1,i_2}(1)c$ and $g = fT_{sr}(\xi)f^{-1} = {}^{T_{i_1,i_2}(1)}d$. Clearly $f_{p,-r} = f_{q,-r} = f_{h,-r} = f_{t,-r} = 0$ and $f_{i_1,s}, f_{i_2,s} \in R^*$. Moreover ${}^{gT_{i_1,i_2}(-1)}b \in H$. Therefore by Lemma 2.2.5 we get $f_{i_1,s}g_{-i_2,p} \in \sigma_{-p,p}$ and thus $g_{-i_2,p} \in \sigma_{-p,p}$. Finally $g_{-i_2,p} = d_{-i_2,p} \pm d_{-i_1,p}$ and, as $d_{-i_2,p} \in \sigma_{-p,p}$, it follows also that $d_{-i_1,p} \in \sigma_{-p,p}$. Therefore $d_{ip} \in \sigma_{ip}$ for all $i \in I$ and if $b \in \mathrm{Sp}(\sigma)$ then also $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$.

1.ii. Suppose $c_{hs}, c_{qs}, c_{ts} \in J$, but $c_{ps} \in R^*$. By the case (1.i) the inclusion $d_{ih} \in \sigma_{ih}$ holds for any $i \in I$ and if $b \in \mathrm{Sp}(\sigma)$ then also $S_{-h,h}(d^{-1}) \in \Gamma_{-h}$. Consider the matrices $f = T_{hp}(1)c$ and $g = {}^{T_{hp}(1)}d$. Then ${}^{gT_{hp}(-1)}b \in H$, $f_{p,-r} = f_{q,-r} = f_{h,-r} = f_{t,-r} = 0$ and $f_{hs} \in R^*$. By case (1.i) we get $g_{ip} \in \sigma_{ip}$ for all $i \in I$ and if $b \in \mathrm{Sp}(\sigma)$ then also $S_{-p,p}(g^{-1}) \in \Gamma_{-p}$. Observe that $g_{ip} = d_{ip} + d_{ih}$ for all $i \neq h, -p$ and, as d_{ih} is already contained in $\sigma_{-p,p}$ for all $i \in I$, we get $d_{ip} \in \sigma_{ip}$ for all $i \neq -p$. Finally

$$g_{-p,p} = d_{-p,p} \pm d_{-h,p} \pm d_{-p,h} \pm d_{-h,h} \tag{2.27}$$

and, as $d_{-h,p} = \pm d^{-1}_{-p,h} = \pm d_{-p,h}$, the last three summands in (2.27) are contained in $\sigma_{-p,p}$. Thus $d_{-p,p} \in \sigma_{-p,p}$. If $b \in \mathrm{Sp}(\sigma)$ then by Corollary 1.1.4 we have

$$S_{-p,p}(g^{-1}) = S_{-p,p}({}^{T_{hp}(1)}d^{-1}) \equiv S_{-p,p}(d^{-1}) + S_{-h,h}(d^{-1}) \mod \Gamma_{-p}$$

and, as $S_{-h,h}(d^{-1}) \in \Gamma_{-p}$, we get $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$.

1.iii. Suppose $c_{is}, c_{i,-r} \in J$ for all $i \in I' \setminus \{l\}$. Then $b_{ij} \equiv \delta_{ij} \mod J$ whenever $i \in I' \setminus \{l\}$ or $-j \in I' \setminus \{l\}$. By Corollary 2.3.4 we get the required inclusions $d_{ip} \in \sigma_{ip}$ for all $i \in I$ and if $b \in \mathrm{Sp}(\sigma)$ then also $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$.

2. Suppose $a_{l,-r} \in J$, but there still exists an index $i_1 \in I'$ such that $a_{i_1,-r} \in R^*$ or $a_{i_1,s} \in R^*$. First, assume $a_{i_1,-r} \in R^*$. By case 1 we have $b_{il} \in \sigma_{il}$ for all $i \in I$ and if $b \in \mathrm{Sp}(\sigma)$ then also $S_{-l,l}(b^{-1}) \in \Gamma_{-l}$. Consider the matrix $c = T_{l,i_1}(1)a \in H$. Let d denote the matrix ${}^{T_{l,i_1}(1)}b$. Then $c_{l,-r} \in R^*$ and by case 1 we get $d_{ip} \in \sigma_{ip}$ and if $b \in \mathrm{Sp}(\sigma)$ then also $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$. Note that if $a \in \mathrm{Sp}(\sigma)$ we have

$$S_{-p,p}(d^{-1}) \equiv S_{-p,p}(T_{l,i_1}(1)b^{-1}) \equiv S_{-p,p}(b^{-1}) + \delta_{p,i_1} S_{-l,l}(b^{-1}).$$

Therefore $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$. As $d_{ip} = b_{ip} + \delta_{i_1,p} b_{il}$ for all $i \neq l, -i_1$, it follows that $b_{ip} \in \sigma_{ip}$ for all $i \neq -i_1$. Finally we have

$$d_{-i_1,p} = b_{-i_1,p} \pm b_{-l,p} + \delta_{i_1,p} b_{-i_1,l} \pm \delta_{i_1,p} b_{-l,l} \tag{2.28}$$

and, as $b_{-l,p} = \pm b_{-p,l}$, the last three summands in (2.28) are contained in $\sigma_{-p,p}$. Therefore $b_{-i_1,p} \in \sigma_{-p,p}$. Thus $b_{ip} \in \sigma_{ip}$ for all $i \in I$ and if $b \in \mathrm{Sp}(\sigma)$ then also $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$.

Finally, if $a_{i_1,s} \in R^*$ we can use the Steinberg relation (R1), namely $T_{sr}(\xi) = T_{-r,-s}(\pm\xi)$. Set $d = aT_{-r,-s}(\xi)a^{-1}$. We have already shown that in this case $d_{ip} \in \sigma_{ip}$ for all $i \in I$ and $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$. Finally $b_{ip} = \pm d_{ip}$ for all $i \neq p$ and if $b \in \mathrm{Sp}(\sigma)$ then $S_{-p,p}(d^{-1}) = S_{-p,p}(b^{-1}) \pm 2b_{-p,p}(2 - b_{-p,-p}) \equiv S_{-p,p}(b^{-1}) \mod \Gamma_{-p}$. Therefore $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$.

3. Suppose $a_{is}, a_{i,-r} \in J$ for all $i \in I'$. Exactly as in case (1.iii) we get the required inclusions by Corollary 2.3.4.

This completes the proof. □

2.5 Localization

In this section we will prove Theorems 6 and 7. Suppose Theorem 1 holds for any Noetherian ground ring. It is a well known fact that every commutative ring R is a direct limit $\varinjlim R'$ of its Noetherian subrings R'. Fix a commutative ring R, a unitary equivalence relation ν such that $h(\nu) \geq (4,5)$ and a subgroup H such that $\mathrm{Ep}(\nu, R) \leq H$. Let (σ, Γ) denote the exact major form net of ideals associated with H, cf. Lemma 1.2.1. For any Noetherian subring R' of R set $H' = H \cap \mathrm{Sp}(2n, R')$. Then $\mathrm{Ep}(\nu, R') \leq \mathrm{Ep}(\nu, R) \cap \mathrm{Sp}(2n, R') \leq H'$. By Lemma 1.2.1 there exists an exact major form net of ideals (σ', Γ') over R' associated with H'. By the construction in Lemma 1.2.1 of a form net associated with a subgroup it follows that if $R' \leq R''$ then $(\sigma', \Gamma') \leq (\sigma'', \Gamma'')$. Clearly $Sp(\sigma', \Gamma') \leq \mathrm{Sp}(\sigma'', \Gamma'')$. As any element g of $\mathrm{Sp}(2n, R)$ is contained in $\mathrm{Sp}(2n, R'')$ for some Noetherian subring R'' of R such that $R' \leq R''$, it follows that

$$\mathrm{Sp}(\sigma, \Gamma) = \varinjlim_{R' \text{ is Noetherian}} \mathrm{Sp}(\sigma', \Gamma'). \tag{2.29}$$

Pick any $a \in H$ and $T_{sr}(\xi) \in \mathrm{Ep}(\sigma, \Gamma)$. There exists a Noetherian subring R' of R such that $a, T_{sr}(\xi) \in \mathrm{Sp}(2n, R')$. Clearly, $a \in H'$ and $T_{sr}(\xi) \in \mathrm{Ep}(\sigma', \Gamma')$. By assumption, Theorem 1 holds for the ground ring R'. Therefore

$$aT_{sr}(\xi)a^{-1} \in \mathrm{Sp}(\sigma', \Gamma'). \tag{2.30}$$

Moreover the inclusion (2.30) holds for any Noetherian subring R'' such that $R' \leq R''$. Combining (2.30) with (2.29) we deduce that

$$\mathrm{Ep}(\sigma, \Gamma) \leq H \leq \mathrm{Transp}_{\mathrm{Sp}(2n,R)}(\mathrm{Ep}(\sigma, \Gamma), \mathrm{Sp}(\sigma, \Gamma)).$$

Therefore we only need to prove the existence of an exact major form net of ideals satisfying the sandwich inclusions in Theorem 6 for a Noetherian ground ring.

Proof of Theorem 6. As remarked above, it suffices to prove the theorem when R is Noetherian. Pick an element a in H, a short (σ, Γ)-elementary transvection $T_{sr}(\xi)$ and denote by b the short root element $aT_{sr}(\xi)a^{-1}$. Our goal is to show that b is contained in $\mathrm{Sp}(\sigma, \Gamma)$. For each $i, j \in I$ put

$$X_{ij} = \{\xi \in R \mid \xi b_{ij} \in \sigma_{ij}\}$$
$$Z_i = \{\xi \in R \mid \xi^2 S_{i,-i}(b) \in \Gamma_i\}.$$

We will show that the sets X_{ij} and Z_i are unimodular, i.e. generate the unit ideal R, for all $i, j \in I$. Fix a maximal ideal $\mathfrak{m}$ of R and denote by S the compliment $R \backslash \mathfrak{m}$ of $\mathfrak{m}$ in R. Let $R_\mathfrak{m}$ denote the localization $S^{-1}R$ of the ring R at the multiplicative system S and $F_\mathfrak{m}$ denote the corresponding localization morphism $R \to R_\mathfrak{m}$. Let $R'_\mathfrak{m}$ denote the image of R under $F_\mathfrak{m}$ and $S_\mathfrak{m}$ denote the image of S under $F_\mathfrak{m}$. Clearly $(R_\mathfrak{m}, R'_\mathfrak{m}, S_\mathfrak{m})$ is a standard setting, cf. Section 2.1. We show there is an element $x_0 \in S$ such that $F_\mathfrak{m}$ is injective on x_0R. Indeed, for each $x \in S$ set $Ann(x) = \{\xi \in R \mid x\xi = 0\}$. As R is Noetherian, the set of ideals $\{Ann(x) \mid x \in S\}$ contains a maximal element $Ann(x_0)$. Let $\xi x_0, \zeta x_0$ be two arbitrary elements of x_0R. Assume $F_\mathfrak{m}(\xi x_0) = F_\mathfrak{m}(\zeta x_0)$. Then there exists an element $y \in S$ such that $yx_0(\xi - \zeta) = 0$. Therefore $\xi - \zeta \in Ann(yx_0) \geq Ann(x_0)$ and by the maximality of $Ann(x_0)$ it follows that $\xi - \zeta \in Ann(x_0)$. Consequently $\xi x_0 = \zeta x_0$. Therefore the localization morphism $F_\mathfrak{m}$ is injective on x_0R.

Let $H_\mathfrak{m}$ denote the image of H under $\mathrm{M}(F_\mathfrak{m})$. Clearly $\mathrm{Ep}(\nu, R'_\mathfrak{m}) \leq H_\mathfrak{m}$. The following proposition allows lifting transvections from $H_\mathfrak{m}$ to H.

Proposition 2.5.1. *Let $\zeta \in R$ and $x \in S$. If $T_{pq}(\frac{\zeta}{x}) \in H_\mathfrak{m}$ then*

$$T_{pq}(x_0^{(1+\delta_{p,-q})} \cdot x^{\delta_{p,-q}} \cdot \zeta) \in H.$$

Proof. If $p \sim q$, the inclusion $T_{pq}(x_0^{(1+\delta_{p,-q})} \cdot x^{\delta_{p,-q}} \cdot \zeta) \in H$ is trivial. Assume $p \nsim q$ and $p \neq -q$. There exists another index $h \sim q$ such that $h \neq \pm p, \pm q$. By the Steinberg relation (R4)

$$T_{pq}(F_\mathfrak{m}(\zeta)) = [[T_{pq}(\frac{\zeta}{x}), T_{qh}(x)], T_{hq}(1)] \in H_\mathfrak{m}.$$

Pick any pre-image g of the matrix $T_{pq}(F_\mathfrak{m}(\zeta))$ contained in H. Then again by (R4)

$$T_{pq}(F_\mathfrak{m}(\zeta x_0)) = \mathrm{M}(F_\mathfrak{m})([[g, T_{qh}(x_0)], T_{hq}(1)]) \in \mathrm{M}(F_\mathfrak{m})(\mathrm{Sp}(2n, R, x_0R) \cap H).$$

As $F_\mathfrak{m}$ is injective on $\mathrm{Sp}(2n, R, x_0R)$, it follows that $T_{pq}(\zeta x_0) \in H$.

Assume $q = -p$ and $p \nsim -p$. Pick two more indices $h, t \in I$ such that (p, h, t) is an A-type base triple. By the Steinberg relations (R3), (R4) and (R6) we get

$$\begin{aligned} T_{h,-h}(-\varepsilon_p\varepsilon_h F_\mathfrak{m}(\zeta x)) = {} & [T_{p,-p}(\frac{\zeta}{x}), T_{-p,-h}(x)] \\ & \times [T_{pt}(-1), [T_{tp}(1), [T_{p,-p}(\frac{\zeta}{x}), T_{-p,-h}(x)]]] \in H_\mathfrak{m}. \end{aligned}$$

Pick any pre-image g of $T_{h,-h}(-\varepsilon_p\varepsilon_h F_\mathfrak{m}(\zeta x))$, which is contained in H. Then by the Steinberg relations (R3), (R4) and (R6)

$$\begin{aligned} T_{p,-p}(F_\mathfrak{m}(\zeta x x_0^2)) &= \mathrm{M}(F_\mathfrak{m})([g, T_{-h,-p}(x_0)][T_{ht}(-1), [T_{th}(1), [g, T_{-h,-p}(1)]]]) \\ &\in \mathrm{M}(F_\mathfrak{m})(\mathrm{Sp}(2n, R, x_0R) \cap H). \end{aligned}$$

By the injectivity of $\mathrm{M}(F_\mathfrak{m})$ on $\mathrm{Sp}(2n, R, x_0R)$ it follows that $T_{p,-p}(\zeta x x_0^2) \in H$. □

Let $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ denote the coordinate-wise image of (σ, Γ) under $F_\mathfrak{m}$. It is easy to see that $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ is an exact major form net of ideals over $R'_\mathfrak{m}$. Proposition 2.5.1 allows us to conclude that $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ $S_\mathfrak{m}$-associated, cf. Section 2.1, with $H_\mathfrak{m}$. Indeed, the inclusion $\mathrm{Ep}(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m}) \leq H_\mathfrak{m}$ is obvious. Suppose $T_{pq}(\frac{\xi}{x}) \in H_\mathfrak{m}$ for some elementary transvection $T_{pq}(\frac{\xi}{x})$ in $\mathrm{Ep}(2n, R_\mathfrak{m})$. By Proposition 2.5.1 it follows that $T_{pq}(\frac{\xi}{x}(xx_0)^{1+\delta_{p,-q}}) = T_{pq}(\xi x^{\delta_{p,-q}} x_0^{(1+\delta_{p,-q})}) \in H$. Thus $\xi x^{\delta_{p,-q}} x_0^{(1+\delta_{p,-q})} \in (\sigma, \Gamma)_{pq}$ and $F_\mathfrak{m}(\xi) F_\mathfrak{m}(x^{\delta_{p,-q}} x_0^{(1+\delta_{p,-q})}) \in (\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})_{pq}$. Therefore $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ is indeed $S_\mathfrak{m}$-associated with $H_\mathfrak{m}$.

Let $(\sigma_\mathfrak{m}, \Gamma_\mathfrak{m})$ denote the $S_\mathfrak{m}$-closure of $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ in $R_\mathfrak{m}$. By a combination of Lemmas 2.4.2 and 2.4.1 it follows that $\mathrm{M}(F_\mathfrak{m})(b) \in \mathrm{Sp}(\sigma_\mathfrak{m}, \Gamma_\mathfrak{m})$.

Now we will show that each set X_{ij} and Z_i contains an element of S and thus each is not contained in $\mathfrak{m}$. For $i \sim j$ it is easy to see that $X_{ij} = R$, therefore we may assume that $i \nsim j$. Let $i \neq -j$. As $F_\mathfrak{m}(b_{ij}) \in (\sigma_\mathfrak{m})_{ij}$, there exists an element $x \in S$ such that $T_{ij}(F_\mathfrak{m}(b_{ij}x)) \in H_\mathfrak{m}$. By Proposition 2.5.1 it follows that $b_{ij}xx_0 \in \sigma_{ij}$ which means that $xx_0 \in X_{ij}$. If $j = -i$ then there exists an index $k \sim i$ such that $i \neq \pm k$. Therefore $F_\mathfrak{m}(b_{i,-i}) \in (\sigma_\mathfrak{m})_{i,-i} = (\sigma_\mathfrak{m})_{i,-k}$. Exactly as in the previous case we get that $b_{ij}xx_0 \in \sigma_{i,-k} = \sigma_{i,-i}$ for some $x \in S$. Finally, let $i \nsim -i$. As $F_\mathfrak{m}(S_{i,-i}(b)) \in (\Gamma_m)_i$, there exists an element $x \in S$ such that $T_{i,-i}(F_\mathfrak{m}(S_{i,-i}(b)x^2)) \in H_\mathfrak{m}$. By Proposition 2.5.1 it follows that $T_{i,-i}(F_\mathfrak{m}(S_{i,-i}(b)x^2x_0^2)) \in H$, which yields that $x_0x \in Z_i$.

Fix some indices i and j. We have shown that the set X_{ij} is unimodular, therefore $b_{ij} \in \langle X_{ij} \rangle b_{ij} \in \sigma_{ij}$. Thus $b \in \mathrm{Sp}(\sigma)$. We have also shown that the set Z_i is also unimodular. Therefore there exist elements $\zeta_1, \ldots, \zeta_k \in Z_i$ and elements $\xi_1, \ldots, \xi_k \in R$ such that $\sum_{t=1}^k \xi_t \zeta_t = 1$. Thus $\sum_{t=1}^k \xi_t^2 \zeta_t^2 \equiv 1 \mod 2R$ and

$$S_{i,-i}(b) \in \sum_{t=1}^{k} \xi_t^2 (\zeta_t^2 S_{i,-i}(b)) + 2RS_{i,-i}(b) \leq \Gamma_i + 2\sigma_{i,-i} \leq \Gamma_i.$$

Summing up, $b \in \mathrm{Sp}(\sigma, \Gamma)$. This completes the proof. □

Proof of Theorem 7. Combining Theorems 6, 3 and 4 we get that there exists an exact major form net of ideals (σ, Γ), namely the form net of ideals associated with H, such that

$$\mathrm{Ep}(\sigma, \Gamma) \leq H \leq \mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\sigma, \Gamma)). \tag{2.31}$$

It only remains to prove that a form net of ideals such that (2.31) holds, is unique. Assume the contrary: let (τ, B) be an exact major form net over (R, Λ) such that

$$\mathrm{Ep}(\tau, B) \leq H \leq \mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\tau, B)),$$

but (τ, B) is not equal to (σ, Γ). As (σ, Γ) is maximal among exact form nets such that $\mathrm{Ep}(\sigma, \Gamma) \leq H$, it follows that $(\tau, b) \leq (\sigma, \Gamma)$. Pick any $\xi \in (\sigma, \Gamma)_{ij}$. Then $T_{ij}(\xi) \in H \leq \mathrm{N}_{\mathrm{Sp}(2n,R)}(\mathrm{Sp}(\tau, B))$. First, assume $i \neq -j$. By property (T1) of Theorem 4 applied to the net (τ, B) it follows, because $(T_{ij}(\xi))_{jj} = 1$, that

$$\xi = (T_{ij}(\xi))_{ij} \cdot 1 \cdot (T_{ij}(\xi))_{jj} \leq (T_{ij}(\xi))_{ij} \cdot \tau_{jj} \cdot (T_{ij}(\xi))_{jj} \leq \tau_{ij}.$$

Therefore $\tau_{ij} = \sigma_{ij}$ for all $i \neq -j$. If $i = -j$ then by property (T2) of Theorem 4

$$\xi = (T_{i,-i}(\xi))_{ii}^2 \cdot 1^2 \cdot S_{i,-i}(T_{i,-i}(\xi)) \leq (T_{i,-i}(\xi))_{ii}^2 \cdot \tau_{ii}^{\boxed{2}} \cdot S_{i,-i}(T_{i,-i}(\xi)) \leq B_i.$$

Therefore $B_i = \Gamma_i$ for all $i \in I$. Finally as both form nets (σ, Γ) and (τ, B) are exact, it follows that $(\sigma, \Gamma) = (\tau, B)$. □

3 Sandwich classification in Unitary groups

In this chapter we will prove the main results of this dissertation, Theorems 1 and 2. The general toolbox we use in this chapter mimics the one used in the case of the classical symplectic group. In the Chapter 2 we relied on the fact that the localization of a commutative ring at the compliment of a maximal ideal is a local ring and thus for any element ξ of the localization, ξ is either invertible or is contained in the Jacobson radical of the localization. In the non-commutative setting, localizing a form ring (R, Λ), which is module finite over a subring C of the center of R, at the compliment in C of a maximal ideal of C gives only a noncommutative semilocal ring. In order to overcome this obstacle we have to first factor the Jacobson radical out of the semilocal ring, which yields a semisimple ring; and then use Morita theory to reduce the semisimple case to the case of a product of division rings with involution with symmetry. Our result over semisimple rings are in the spirit of [Dyb07].

Now we explain the general flow of the proof of Theorem 1 of the Introduction and simultaneously describe the structure of the rest of the chapter. The prerequisites are gathered in Section 3.1. We start with a form ring (R, Λ) over a quasi-finite ring R, a unitary equivalence relation ν subject to a certain condition on the minimal sizes of equivalence classes and a subgroup H of $\mathrm{U}(2n, R, \Lambda)$ that contains the elementary block-diagonal group $\mathrm{EU}(\nu, R, \Lambda)$. In Section 3.2 we define form nets of ideals and form net subgroups and construct the form net of ideals associated with H, namely the maximal exact form net of ideals (σ, Γ) such that the corresponding elementary form net subgroup $\mathrm{EU}(\sigma, \Gamma)$ is contained in H. Theorem 2 of the Introduction is also proved there. Using a localization method presented in Section 3.10 we reduce the proof of Theorem 1 over a quasi-finite ring to a similar result over a semilocal ring. However, the image of the subgroup H in the localization does not have to contain the elementary block-diagonal subgroup defined by ν, and therefore we can't define the form net of ideals associated with the image of H. To get around this problem, we refine concept of the form net of ideals associated with a subgroup by a pair of nets with special properties. For this reason we introduce in Section 3.3 the concept of a standard setting and the corresponding theory of form nets of ideals. For this theory to work, it is crucial that the canonical map of the original ring to the quotient of a localization by its Jacobson radical is surjective. In Section 3.4 we present the method of extracting elementary unitary matrices, first, in small parabolic subgroups and, next, using a root element. We continue extracting in Section 3.5 elementary unitary matrices using elements close to the principal congruence subgroup of the level the Jacobson radical. This allows us to reduce the proof of Theorem 1 to the case of a semisimple ground ring. This reduction is done in Section 3.9. Section 3.6 contains important technical results on morphisms of standard settings, direct decompositions, form ring scaling and direct limits of form rings, form nets of ideals and related subgroups. In Section 3.7 we further reduce the proof of Theorem 1 to the case of a division ring or a product of two copies of a division ring interchanged by the involution with symmetry. Finally, in Section 3.8 we prove Theorem 1 in these two cases.

3.1 Preliminaries

Hermitian and quadratic forms. Our goal now is to define the even unitary group which first appeared in [Bak69]. We will mostly follow the notations of [BV00] while working in the generality of [HO89].

Let R be an associative unital ring. Equip R with an *anti-automorphism* $\overline{\cdot}$, i.e. an automorphism of the underlying additive group of R which satisfies the property $\overline{ab} = \overline{b}\cdot\overline{a}$ for all a and b in R. Let M be a right R-module. We will call a map

$$f : M \times M \longrightarrow R$$

a $\overline{\cdot}$*-sesquilinear form* or simply a $\overline{\cdot}$*-form on* M if it is additive in both variables and satisfies the $1\frac{1}{2}$- linearity property

$$f(xa, yb) = \overline{a}f(x, y)b$$

for all $x, y \in M$ and $a, b \in R$. Assume there is a unit λ in R such that the square of the automorphism $\overline{\cdot}$ is given by conjugation by λ, i.e. such that $\overline{\overline{a}} = \lambda a\lambda^{-1}$ for all $a \in R$, and in addition $\overline{\lambda} = \lambda^{-1}$. In this situation with will call λ *a symmetry* for $\overline{\cdot}$ and the pair $(\overline{\cdot}, \lambda)$ *an involution with symmetry.* We will also say that $\overline{\cdot}$ *is an involution with symmetry* λ. We will call and the triple $(R, \overline{\cdot}, \lambda)$ *a ring with involution with symmetry.* Note that an involution with symmetry is also called in the literature *an anti-structure.*

Note that in [BV00] and in many other sources the symmetry λ was assumed to be central; however this assumption is not necessary and most of the results concerning unitary groups hold without it, but computations become more tedious because one has to constantly keep track of the position of λ in any product of elements. Given two rings with involution with symmetry $(R, \overline{\cdot}, \lambda)$ and $(Q, \widehat{\cdot}, \mu)$ we call a ring morphism $\varphi : R \longrightarrow Q$ *a morphism of rings involution with symmetry* if φ preserves the involution with symmetry, i.e. $\widehat{\cdot}\circ\varphi = \varphi\circ\overline{\cdot}$ and $\varphi(\lambda) = \mu$. An *isomorphism of rings involution with symmetry* is by definition a morphism of rings with involution with symmetry, which is invertible as a ring morphism and its inverse morphism is also a morphism of rings with involution with symmetry. Clearly an image of a morphism of rings with involution with symmetry is a ring with involution with symmetry. If $(R, \overline{\cdot}, \lambda)$ is a ring with involution with symmetry and R' is a subring of R we call $(R', \overline{\cdot}, \lambda)$ a *subring of* R *with involution with symmetry* if λ is contained in R' and R' is stable under $\overline{\cdot}$.

A $\overline{\cdot}$-form h on M is called λ*-hermitian* if

$$h(x, y) = \overline{h(y, x)}\lambda$$

for all $x, y \in M$. This is an example where the position of λ is important. Given a $\overline{\cdot}$-form f we can always construct a λ-hermitian form by setting $h(x,y) = f(x,y) + \overline{f(y,x)}\lambda$.

A choice of an involution with symmetry fixes two additive subgroups of the ring R

$$\Lambda^{\min} = \Lambda^{\min}(R) = \{\alpha - \overline{\alpha}\lambda \mid \alpha \in R\},$$
$$\Lambda^{\max} = \Lambda^{\max}(R) = \{\alpha \in R \mid \overline{\alpha}\lambda = -\alpha\}.$$

We will call an additive subgroup Λ of R *a form parameter for* R in the sense of [Bak81], if

(Λ1) $\Lambda^{\min} \leq \Lambda \leq \Lambda^{\max}$

(Λ2) $\overline{\alpha}\Lambda\alpha \leq \Lambda$ for any $\alpha \in R$.

For each $i \in I$ set

$$\Lambda_i = \Lambda\lambda^{(-1-\varepsilon(i))/2}.$$

Clearly both subgroups $\Lambda^{\min}$ and $\Lambda^{\max}$ are form parameters for R. They are called *the minimum and maximum form parameters* respectively. The pair (R, Λ) is called a *form ring* [*over* R]. By R in the notation of a form ring we will always mean ring equipped with involution with symmetry. In case it is important which particular involution with symmetry on R we have in mind we will write $((R, \overline{\cdot}, \lambda), \Lambda)$ in place of (R, Λ). Clearly the form parameters of a given ring with involution with symmetry form a lattice with respect to inclusions. For a ring with involution with symmetry $(R, \overline{\cdot}, \Lambda)$ we will denote the lattice of form parameters for $(R, \overline{\cdot}, \lambda)$ by $\mathrm{FP}(R, \overline{\cdot}, \Lambda)$.

The notion of a form subring is defined in the natural way. Namely, *a form subring* (R', Λ') *of* (R, Λ) consists of a subring R' of R such that $\lambda \in R'$, the involution with symmetry on R induces the involution with symmetry on R', and $\Lambda' \subset \Lambda$. Clearly, $(R', \Lambda \cap R')$ is a form subring of (R, Λ).

let $((R, \overline{\cdot}, \lambda), \Lambda)$ be a form ring and $(R', \overline{\cdot}, \lambda)$ be a subring of R with involution with symmetry. Let Λ' be a form parameter for R' such that $\Lambda' \subseteq \Lambda$. Then the form ring (R', Λ') is called *a form subring of the form ring* (R, Λ).

Given two form rings $((R_1, \overline{\cdot}, \lambda_1), \Lambda_1)$ and $((R_2, \widehat{\cdot}, \lambda_2), \Lambda_2)$ we will call a morphism of rings with involution with symmetry $\varphi : (R_1, \overline{\cdot}, \lambda_1) \to (R_2, \widehat{\cdot}, \lambda_2)$ *a form ring morphism* if $\varphi(\Lambda_1) \leq \Lambda_2$. We will call a form ring morphism *a form ring isomorphism* if it is an isomorphism of rings with involution with symmetry and $\varphi(\Lambda_1) = \Lambda_2$. The following obvious proposition provides an example of a morphism of form rings.

Lemma 3.1.1. *Let $((R_1, \overline{\cdot}, \lambda_1), \Lambda_1)$ be a form ring and $(R_2, \widehat{\cdot}, \lambda_2)$ be a ring with involution with symmetry. Let $\varphi : (R_1, \overline{\cdot}, \lambda_1) \to (R_2, \widehat{\cdot}, \lambda_2)$ be a morphism of rings with involution with symmetry. Then $\Lambda_2 = \varphi(\Lambda_1)$ is a form parameter for $(\varphi(R_2), \widehat{\cdot}, \lambda_2)$ and φ defines a morphism $(R_1, \overline{\cdot}, \lambda_1) \to (\varphi(R_2), \widehat{\cdot}, \lambda_2)$ of form rings. If φ is an isomorphism of rings with involution with symmetry then φ defines an isomorphism $(R_1, \overline{\cdot}, \lambda_1) \to (\varphi(R_2), \widehat{\cdot}, \lambda_2)$ of form rings.*

Proof. Note that $\varphi(R_1)$ is closed under the action of $\widehat{\cdot}$ and $\lambda_2 = \varphi(\lambda_1) \in \varphi(R_1)$. Therefore $(\varphi(R_1), \widehat{\cdot}, \lambda_2)$ is a ring with involution with symmetry. It is easy to see that $\Lambda^{\min}(\varphi(R_1)) = \varphi(\Lambda^{\min}(R_1))$ and $\Lambda^{\max}(\varphi(R_1)) = \varphi(\Lambda^{\max}(R_1))$. Therefore the property $(\Lambda 1)$ above is fulfilled for Λ_2. Pick any $\zeta \in \varphi(R_1)$ and any pre-image ξ of ζ under φ. Then

$$\widehat{\zeta}\Lambda_2\zeta = \varphi(\overline{\xi}\Lambda_1\xi) \leq \varphi(\Lambda_1) = \Lambda_2,$$

therefore Λ_2 is a form parameter for $(\varphi(R_1), \widehat{\cdot}, \lambda_2)$. The second conclusion of this lemma is obvious. □

Fix a right R-module M and a $\overline{\cdot}$-form f on M. Define the maps

$$h : M \times M \longrightarrow R \qquad \text{and} \qquad q : M \longrightarrow R/\Lambda$$

by setting

$$h(x,y) = f(x,y) + \overline{f(y,x)}\lambda \qquad \text{and} \qquad q(x) = f(x,x) + \Lambda,$$

for all $x, y \in M$. We call the pair (h,q) *a* Λ*-quadratic form on* M, and the form f in this case is said to *define* (h,q). As we have already mentioned, the form h constructed in this way is always λ-hermitian. A non-zero vector $x \in M$ is called *isotropic* if $q(x) = 0$ in R/Λ and *anisotropic* otherwise.

A *quadratic module over* (R,Λ) is a triple (M,h,q), where M is an R-module and (h,q) is a Λ-quadratic form on M. (M,h,q) is called *free hyperbolic* if the module M has an ordered basis $\mathfrak{X} = \{x_1, \dots, x_{2n}\}$ consisting of isotropic vectors such that the Gram matrix $(h(x_i,x_j))$ of h in this basis equals

$$\begin{pmatrix} & e_n \\ \lambda e_n & \end{pmatrix},$$

where e_n is an identity matrix of size $n \times n$. Any such basis $\mathfrak{X}$ is called a *hyperbolic basis of the module* M. Following [BV00] we will use a different basis for quadratic modules as it allows to shorten some computations. We fix an ordered index set $I = \{1, \dots, n, -n, \dots, -1\}$ and define the ordered basis $\mathfrak{E} = \{e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\}$ by putting

$$e_i = \begin{cases} x_i & \text{if } i > 0 \\ x_{n-i} & \text{if } i < 0. \end{cases}$$

In this basis the form f defining the Λ-quadratic form (h,q) has the Gram matrix $(h(e_i,e_j))$ equal to $\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$, where $p = \operatorname{sdiag}(1,\dots,1)$ is an $n \times n$ matrix which has 1's along the second (skew) diagonal and zeros elsewhere, i.e.

$$f(x,y) = \overline{x}^t \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} y = \overline{x}_1 y_{-1} + \dots + \overline{x}_n y_{-n}.$$

Thus

$$h(x,y) = f(x,y) + \overline{f(y,x)}\lambda = \overline{x}_1 y_{-1} + \dots + \overline{x}_n y_{-n} + \overline{x}_{-1}\lambda y_1 + \dots + \overline{x}_{-n}\lambda y_n$$

and

$$q(x) = f(x,x) + \Lambda = \overline{x}_1 x_{-1} + \dots + \overline{x}_n x_{-n} + \Lambda.$$

Unitary groups. We define *the unitary group* $\mathrm{U}(M)$ *of a quadratic module* (M, h, q) as the group of isometries of the quadratic module M, i.e.

$$\mathrm{U}(M) = \{g \in \mathrm{GL}(M) \mid h(gx, gy) = h(x, y) \text{ and } q(gx) = q(x) \text{ for all } x, y \in M\}.$$

Define *the even (or hyperbolic) unitary group* $\mathrm{U}(2n, R, \Lambda)$ *of rank* $2n$ *over the form ring* (R, Λ) to be the unitary group of the free quadratic module (R^{2n}, h, q), where the Λ-quadratic form (h, q) is defined by the form f with Gram matrix $\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$.

It follows from the defintion of an even unitary group that if (R', Λ') is a form subring of the form ring (R, Λ) then $\mathrm{U}(2n, R', \Lambda')$ is a subgroup of $\mathrm{U}(2n, R, \Lambda)$. The following lemma provides a characterisation of $\mathrm{U}(2n, R, \Lambda)$ in terms of matrix elements. It's a direct analogue of Lemma 2.3 of [BV00] (or lemma 3.4 of [Bak81], in the case of the ordered basis $x_1, \ldots, x_2 n$ above) and can be proven exactly in the same way.

Lemma 3.1.2. *A necessary and sufficient condition for a matrix* $g \in \mathrm{GL}(2n, R)$ *to belong to* $\mathrm{U}(2n, R, \Lambda)$ *is that the following two properties are satisfied*

(U1) $g'_{ij} = \lambda^{-(\varepsilon(i)+1)/2} \overline{g}_{-j,-i} \lambda^{(\varepsilon(j)+1)/2}$ *for all* $i, j \in I$

(U2) $S_{i,-i}(g) = \sum_{j>0} g_{ij} g'_{j,-i} \in \Lambda_i = \Lambda \lambda^{(-\varepsilon(i)-1)/2}$ *for all* $i \in I$.

We call the sum $S_{i,-i}(g)$ *the length of i'-th row of the matrix g*. Note that condition (2) above can be replaced by a similar condition on columns rather than rows. Further, in the case of a maximal form parameter Λ, condition (1) infers (2), cf. [Bak81, Th. 1.1].

Elementary subgroup. We introduce two special types of elements of $\mathrm{U}(2n, R, \Lambda)$ called *elementary unitary matrices*. For any pair of indices $i, j \in I$ such that $i \neq \pm j$ and an element $\xi \in R$ set

$$T_{ij}(\xi) = e + \xi e_{ij} - \lambda^{(\varepsilon(j)-1)/2} \overline{\xi} \lambda^{(1-\varepsilon(i))/2} e_{-j,-i}.$$

Such a matrix is called *a short elementary unitary matrix*. For any $i \in I$ and $\alpha \in \Lambda_i = \Lambda \lambda^{-(\varepsilon(i)+1)/2}$ set

$$T_{i,-i}(\alpha) = e + \alpha e_{i,-i}.$$

Such a matrix is called *long elementary unitary matrices.*

Another important class of unitary matrices is the class of *elementary diagonal unitary matrices* which are defined for every $i \in I$ and every invertible θ in R as follows:

$$D_i(\theta) = e + (\theta - 1) e_{ii} + \lambda^{(\varepsilon(i)-1)/2} (\overline{\theta^{-1}} - 1) \lambda^{(1-\varepsilon(i))/2} e_{-i,-i}.$$

It is an easy exercise to check that the short and long elementary unitary matrices as well as the elementary diagonal unitary matrices actually belong to $\mathrm{U}(2n, R, \Lambda)$. Denote by $\mathrm{EU}(2n, R, \Lambda)$ the subgroup of $\mathrm{U}(2n, R, \Lambda)$ generated by all long and short elementary unitary matrices. We call the subgroup $\mathrm{EU}(2n, R, \Lambda)$ *the elementary unitary* *subgroup.* Denote by $\Delta(2n, R)$ the subgroup of $\mathrm{U}(2n, R, \Lambda)$ generated by all elementary diagonal unitary matrices. The subgroup $\Delta(2n, R)$ is called *the diagonal subgroup of* $\mathrm{U}(2n, R, \Lambda)$. Note that $\Delta(2n, R)$ does not depend on the choice of a form parameter. But an elementary diagonal unitary matrix is not necessarily contained in the elementary unitary subgroup.

Elementary unitary matrices satisfy the following set of relations known as *elementary relations* which can be checked by straightforward calculation.

(EU1) $T_{ij}(\xi) = T_{-j,-i}(-\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(i))/2})$ for all $i \neq j$ and additionally $\xi \in \Lambda_i$ if $j = -i$

(EU2) $T_{ij}(\xi)T_{ij}(\zeta) = T_{ij}(\xi + \zeta)$ for all $i \neq j$ and $\xi, \zeta \in R$

(EU3) $[T_{ij}(\xi), T_{kl}(\zeta)] = e$, whenever $i \neq l, -k, j$, $j \neq k, -l$ and $k \neq l$

(EU4) $[T_{ij}(\xi), T_{jk}(\zeta)] = T_{ik}(\xi\zeta)$ for all $i \neq \pm j, \pm k$, $j \neq \pm k$

(EU5) $[T_{ij}(\xi), T_{j,-i}(\zeta)] = T_{i,-i}(\xi\zeta - \lambda^{(-1-\varepsilon(i))/2}\overline{(\xi\zeta)}\lambda^{(1-\varepsilon(i))/2})$

(EU6) $[T_{i,-i}(\alpha), T_{-i,j}(\xi)] = T_{ij}(\alpha\xi)T_{-j,j}(-\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi)$ for all $i \neq \pm j$, $\alpha \in \Lambda_i$ and $\xi \in R$.

The last relation we are going to consider is a combination of (EU1) and (EU6) and thus is not independent. However it is useful in this form.

(EU6′) $[T_{ij}(\xi), T_{j,-j}(\alpha)] = T_{i,-j}(\xi\alpha)T_{i,-i}(\xi\alpha\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(i))/2})$ for all $i \neq \pm j$, $\xi \in R$ and $\alpha \in \Lambda_j$.

Adopting the terminology of Chevalley groups we will call any conjugate of a long or short elementary unitary matrix *a long or short root element* respectively.

Form ideals. As we have mentioned in the introduction to this dissertation, in the context of even unitary groups over rings with 2 not invertible, the concept of an ideal in a ring is not fine enough and has to be replaced by the concept of a form ideal, cf. [Bak81]. Let J be an involution-invariant (i.e. $\overline{J} = J$) ideal of the ring R. Set

$$\Omega^{\min} = \Omega^{\min}(J) = \{\xi - \overline{\xi}\lambda \mid \xi \in J\} + \langle \overline{\xi}\alpha\xi \mid \xi \in J, \alpha \in \Lambda \rangle$$

and

$$\Omega^{\max} = \Omega^{\max}(J) = J \cap \Lambda,$$

where angular brackets stand for the additive subgroup generated by the argument. We call an additive subgroup Ω in R *a relative form parameter of level* J if

(1) $\Omega^{\min} \leq \Omega \leq \Omega^{\max}$

(2) $\overline{\xi}\Omega\xi \leq \Omega$ for all $\xi \in R$.

In this case the pair (J, Ω) is called *a form ideal of the form ring* (R, Λ). If $J = R$ then $(J, \Omega) = (R, \Lambda)$. Define

$$\Omega_i = \Omega\lambda^{(-1-\varepsilon(i))/2}.$$

3.2 Unitary form nets of ideals and related objects

Form nets of ideals. Fix a form ring (R, Λ). We call a square array $\sigma = (\sigma_{ij})_{i,j\in I}$ of $(2n)^2$ two-sided ideals of R *a unitary net of ideals* over R if the following two conditions are fulfilled:

(Σ1) $\sigma_{ij} = \overline{\sigma}_{-j,-i}$ for all $i, j \in I$

(Σ2) $\sigma_{ij}\sigma_{jk} \le \sigma_{ik}$ for all $i, j, k \in I$.

A choice of a unitary net of ideals fixes for each $i \in I$ two additive subgroups of $\sigma_{i,-i}$

$$\begin{aligned}\Gamma_i^{\min} &= \{\alpha - \lambda^{(-1-\varepsilon(i))/2}\overline{\alpha}\lambda^{(1-\varepsilon(i))/2} \mid \alpha \in \sigma_{i,-i}\}\\ \Gamma_i^{\max} &= \sigma_{i,-i} \cap \Lambda_i,\end{aligned}$$

where

$$\Lambda_i = \Lambda\lambda^{(-1-\varepsilon(i))/2}.$$

We will call a column $\Gamma = (\Gamma_i)_{i\in I}$ of $2n$ additive subgroups of $\sigma_{i,-i}$ such that

(Γ1) $\Gamma_i^{\min} \le \Gamma_i \le \Gamma_i^{\max}$ for each $i \in I$

(Γ2) $\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\Gamma_i\xi \le \Gamma_{-j}$ for all $i, j \in I$ and $\xi \in \delta_{-i,j} + \sigma_{-i,j}$

a column of form parameters of level σ. Note that the condition (Γ2) above can be equivalently stated as follows

(Γ2′) $\xi\Gamma_j\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(i))/2} \le \Gamma_i$ for all $i, j \in I$ and $\xi \in \delta_{ij} + \sigma_{ij}$.

We call a pair (σ, Γ), where σ is a unitary net of ideals and Γ a column of form parameters of level σ, *a form net of ideals* [*of rank* $2n$] *over* (R, Λ). We call a form net of ideals (σ, Γ) *a form* D*-net of ideals* if $\sigma_{ii} = R$ for all $i \in I$; further we will call (σ, Γ) *exact* if for any index $i \in I$ the equality

$$\sigma_{i,-i} = \sum_{j\neq\pm i} \sigma_{ij}\sigma_{j,-i} + \langle\Gamma_i\rangle$$

holds. The next proposition formalizes the following idea. In the case of the classical symplectic group the minimal form parameter $\Gamma_i^{\min}$ was always equal to $2\sigma_{i,-i}$. This allowed us in many situations in which form parameters play a role to perform computations as if we were working over a ring of characteristic 2. It turns out that a similar effect takes place in the general situation.

Proposition 3.2.1. *Let* (σ, Γ) *be a form net of ideals over* (R, Λ)*, which is not necessarily exact or a D-net. Let* $i \in I$*. Then for each* $\alpha \in \Gamma_i^{\max}$ *we have*

$$\overline{\alpha} = -\lambda^{(\varepsilon(i)+1)/2}\alpha\lambda^{(\varepsilon(i)-1)/2}.$$

Thus

$$2\Gamma_i^{\max} \le \Gamma_i^{\min}.$$

Proof. Recall that by definition $\Gamma_i^{\max} = \sigma_{i,-i} \cap \Lambda_i$. Pick any $\alpha \in \Gamma_i^{\max} = \sigma_{i,-i} \cap \Lambda_i \leq \sigma_{i,-i} \cap \Lambda^{\max}\lambda^{(-1-\varepsilon(i))/2}$. As $\alpha \in \sigma_{i,-i}$, by definition of $\Gamma_i^{\min}$ we have

$$\alpha \equiv \lambda^{(-1-\varepsilon(i))/2}\overline{\alpha}\lambda^{(1-\varepsilon(i))/2} \mod \Gamma_i^{\min}. \tag{3.1}$$

Take an element $\beta \in \Lambda \leq \Lambda^{\max}$ such that $\alpha = \beta\lambda^{(-1-\varepsilon(i))/2}$. By definition of $\Lambda^{\max}$ we get $\overline{\beta} = -\beta\overline{\lambda}$. Therefore

$$\overline{\alpha} = \lambda^{(1+\varepsilon(i))/2}\overline{\beta} = -\lambda^{(1+\varepsilon(i))/2}\beta\overline{\lambda} = -\lambda^{(1+\varepsilon(i))/2}\alpha\lambda^{(\varepsilon(i)-1)/2}. \tag{3.2}$$

This is precisely the first formula in the statement of this proposition. Substituting (3.2) into (3.1) we get $\alpha \equiv -\alpha \mod \Gamma_i^{\min}$. Clearly, as α was chosen arbitrarily this amounts to saying that $2\Gamma_i^{\max} \leq \Gamma_i^{\min}$. □

Another simple but important proposition shows that the left-hand side of (Γ2) is additive with respect to ξ.

Proposition 3.2.2. *Let (σ, Γ) be a form* D*-net of ideals over (R, Λ). Let $\alpha \in \Gamma_i^{\max}$ and $\xi, \zeta \in \sigma_{-i,j}$. Then*

$$\begin{aligned}\lambda^{(\varepsilon(j)-1)/2}\overline{(\xi+\zeta)}\lambda^{(1+\varepsilon(i))/2}\alpha(\xi+\zeta) \equiv \lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi \\ + \lambda^{(\varepsilon(j)-1)/2}\overline{\zeta}\lambda^{(1+\varepsilon(i))/2}\alpha\zeta \mod \Gamma_{-j}^{\min}.\end{aligned}$$

Proof. Consider

$$\begin{aligned}&\lambda^{(\varepsilon(j)-1)/2}\overline{(\xi+\zeta)}\lambda^{(1+\varepsilon(i))/2}\Gamma_i(\xi+\zeta) \\ &\quad - \lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi - \lambda^{(\varepsilon(j)-1)/2}\overline{\zeta}\lambda^{(1+\varepsilon(i))/2}\alpha\zeta \\ &\quad = \lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\zeta + \lambda^{(\varepsilon(j)-1)/2}\overline{\zeta}\lambda^{(1+\varepsilon(i))/2}\alpha\xi \mod \Gamma_{-j}^{\min}.\end{aligned} \tag{3.3}$$

Using the second part of Proposition 3.2.1 we get

$$\lambda^{(\varepsilon(j)-1)/2}\overline{(\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\zeta)}\lambda^{(1+\varepsilon(j))/2} = \lambda^{(\varepsilon(j)-1)/2}\overline{\zeta}\lambda^{(1+\varepsilon(i))/2}\alpha\xi$$

As $\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\zeta \in \sigma_{-j,j}$, it follows that

$$\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\zeta + \lambda^{(\varepsilon(j)-1)/2}\overline{\zeta}\lambda^{(1+\varepsilon(i))/2}\alpha\xi \in \Gamma_{-j}^{\min}. \tag{3.4}$$

Comparing (3.3) and (3.4) we get the required inclusions. □

Clearly, a similar proposition can be stated for the left-hand side of ($\Gamma 2'$).

An important example of a form net of ideals is a *constant* form net of ideals. Pick a form ideal (J, Ω) and define $\sigma_{ij} = J$ whenever $i \neq j$ and $\Gamma_i = \Omega\lambda^{(-\varepsilon(i)-1)/2}$. Then (σ, Γ) is an exact form net of ideals over (R, Λ). If we additionally set $\sigma_{ii} = R$ then we get an exact form D-net. Another important example of a form net of ideals is provided by the following proposition.

Proposition 3.2.3. *Let (σ, Γ) be an exact form* D*-net of ideals and (J, Ω) be a form ideal with maximal relative form parameter, i.e. $\Omega = J \cap \Lambda$. For each $i, j \in I$ set*

$$(\sigma + J)_{ij} = \sigma_{ij} + J \qquad \text{and} \qquad (\Gamma + \Omega)_i = \Gamma_i + \Omega\lambda^{(-\varepsilon(i)-1)/2} = \Gamma_i + \Omega_i.$$

Then $(\sigma + J, \Gamma + \Omega)$ is an exact form D*-net of ideals.*

Proof. It's easy to see that $(\sigma + J)$ is a unitary net of ideals. Considering the corresponding minimal form parameter it's easy to see that

$$\begin{aligned}\Gamma_i^{\min}(\sigma + J) &= \{\alpha - \lambda^{(-\varepsilon(i)-1)/2}\overline{\alpha}\lambda^{(1-\varepsilon(i))/2} \mid \alpha \in \sigma_{i,-i} + J\} \\ &= \Gamma_i^{\min}(\sigma) + \{\alpha - \overline{\alpha}\lambda \mid \alpha \in J\}\lambda^{(-\varepsilon(i)-1)/2} \le \Gamma_i^{\min}(\sigma) + \Omega^{\min}\lambda^{(-\varepsilon(i)-1)/2}.\end{aligned}$$

Similarly we get

$$\Gamma_i^{\max}(\sigma + J) = (\sigma_{i,-i} + J) \cap \Lambda_i = \Gamma_i^{\max}(\sigma) + \Omega^{\max}\lambda^{(-\varepsilon(i)-1)/2}.$$

It's also easy to see that $(\Gamma + \Omega)_i$ is an additive subgroup for any $i \in I$. Therefore condition $(\Gamma1)$ is fulfilled for $\Gamma + \Omega$. Now we will check the condition $(\Gamma2')$. Pick any $\xi \in \delta_{ij} + (\sigma + J)_{ij}$ and any $\alpha \in (\Gamma + \Omega)_j$. Then there exist elements ξ_1 in $\delta_{ij} + \sigma_{ij}$, ξ_2 in J, α_1 in Γ_j and $\alpha_2 \in \Omega\lambda^{(-\varepsilon(i)-1)/2}$ such that $\xi = \xi_1 + \xi_2$ and $\alpha = \alpha_1 + \alpha_2$.

Note that for any $\beta \in \Lambda_j^{\max}$ the following formula holds:

$$\overline{\beta} = -\lambda^{(\varepsilon(j)+1)/2}\beta\lambda^{(\varepsilon(j)-1)/2}, \tag{3.5}$$

in particular (3.5) holds for α_1 and α_2. As we have mentioned prior to the proposition, any form ideal defines to a constant form D-net of ideals, in particular the relation $(\Gamma2')$ holds for the constant D-net of ideals corresponding to the form ideal (R, Λ). Therefore we get the inclusion

$$\zeta\Lambda_j\lambda^{(\varepsilon(j)-1)/2}\overline{\zeta}\lambda^{(1-\varepsilon(j))/2} \le \Lambda_j$$

for any $\xi \in R$ and as $\Lambda_i = \Lambda_j\lambda^{(\varepsilon(j)-\varepsilon(i))/2}$ it follows that

$$\zeta\Lambda_j\lambda^{(\varepsilon(j)-1)/2}\overline{\zeta}\lambda^{(1-\varepsilon(i))/2} \le \Lambda_i. \tag{3.6}$$

Consider the equality

$$\xi\alpha\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(i))/2} = (\xi_1 + \xi_2)(\alpha_1 + \alpha_2)\lambda^{(\varepsilon(j)-1)/2}(\overline{\xi_1} + \overline{\xi_2})\lambda^{(1-\varepsilon(i))/2}. \tag{3.7}$$

Expanding the brackets in (3.7) we get eight terms of the form

$$\xi_p\alpha_q\lambda^{(\varepsilon(j)-1)/2}\overline{\xi_h}\lambda^{(1-\varepsilon(i))/2}, \tag{3.8}$$

where $p, q, h \in \{1, 2\}$. The term (3.8) corresponding to $p = q = h = 1$ is contained in $\Gamma_i \le (\Gamma + \Omega)_i$ by the relation $(\Gamma2')$ for the form net (σ, Γ). If $p = h$ and at least one of the indices p, q or h is equal to 2, the corresponding term (3.8) is contained in $\Lambda_i \cap I \le (\Gamma + \Omega)_i$ by the relation (3.6).

Finally if $p \neq h$, by (3.5) we can see that

$$\begin{aligned}
&\xi_p \alpha_q \lambda^{(\varepsilon(j)-1)/2} \overline{\xi_h} \lambda^{(1-\varepsilon(i))/2} + \xi_h \alpha_q \lambda^{(\varepsilon(j)-1)/2} \overline{\xi_p} \lambda^{(1-\varepsilon(i))/2} = \\
&\quad \xi_p \alpha_q \lambda^{(\varepsilon(j)-1)/2} \overline{\xi_h} \lambda^{(1-\varepsilon(i))/2} - \lambda^{(-\varepsilon(i)-1)/2} \xi_p \alpha_q \lambda^{(\varepsilon(j)-1)/2} \overline{\xi_h} \lambda^{(1-\varepsilon(i))/2} \lambda^{(1-\varepsilon(i))/2} \\
&\quad \in \Gamma_i^{\min}(J) \leq \Omega_i,
\end{aligned}$$

where $\Gamma_i^{\min}(J)$ stands for the minimal i'th form parameter for the constant form net with all off-diagonal ideals equal to J. Therefore the term (3.5) is contained in $(\Gamma + \Omega)_i$ whenever $p \neq h$. Summing up, (3.7) is contained in $(\Gamma + \Omega)_i$ and thus $(\sigma + I, \Gamma + \Omega)$ is indeed a form D-net of ideals over (R, Λ). It's easy to check that it is exact. □

The last important example of a form net of ideals which we would like to mention is the form net of ideals corresponding to an equivalence relation. Fix a unitary equivalence relation ν and set

$$\sigma_{ij} = \begin{cases} R & \text{, if } i \sim j \\ 0 & \text{, otherwise} \end{cases} \qquad \Gamma_i = \begin{cases} \Lambda_i & \text{, if } i \sim -i \\ 0 & \text{otherwise} \end{cases}.$$

Then the pair (σ, Γ) is a form net of ideals over (R, Λ). We will denote this net by

$$[\nu]_{(R,\Lambda)}.$$

Introduce a partial ordering on the set of all form nets of ideals of a given rank over (R, Λ) by setting $(\sigma', \Gamma') \leq (\sigma'', \Gamma'')$ if and only if for all $i, j \in I$ the inclusions $\sigma'_{ij} \leq \sigma''_{ij}$ and $\Gamma'_i \leq \Gamma''_i$ hold. Call a form net of ideals (σ, Γ) *major with respect to an equivalence relation* ν if $[\nu]_{(R,\Lambda)} \leq (\sigma, \Gamma)$. If (σ, Γ) is a major form net of ideals and and $h(\nu) \geq (4, 0)$ then $\sigma_{i,-i} = R$ and $\Gamma_i = \Lambda_i$ for all $i \in I$. For convenience, given an element $\xi \in R$ and indices s and r we will write

$$\xi \in (\sigma, \Gamma)_{sr}$$

if $\xi \in \sigma_{sr}$ when $r \neq -s$ and $\xi \in \Gamma_s$ when $r = -s$. Call an elementary unitary matrix $T_{sr}(\xi)$ (σ, Γ)-*elementary* if $\xi \in (\sigma, \Gamma)_{sr}$.

We finish this subsection with a proposition which can informally be stated as follows: if a form net of ideals is major with respect to an equivalence relation with sufficiently big blocks, then it is partitioned into rectangular blocks in which all ideals are equal and all form parameters are equal up to a multiplication by a power of λ. The proof is left as an exercise using the elementary relations.

Proposition 3.2.4. *Assume $h(\nu) \geq (4, 3)$. Let (σ, Γ) be an exact major form net of ideals over (R, Λ). Then for all $i \sim j$ and any $k \in I$ the equalities*

1. $\sigma_{ik} = \sigma_{jk}$ *and* $\sigma_{ki} = \sigma_{kj}$
2. $\Gamma_i \lambda^{(\varepsilon(i)-\varepsilon(j))/2} = \lambda^{(\varepsilon(i)-\varepsilon(j))/2} \Gamma_i = \Gamma_j$.

hold.

Form net subgroups. Given a form net of ideals (σ, Γ) we can define the following two subgroups of $\mathrm{U}(2n, R, \Lambda)$. Call

$$\mathrm{U}(\sigma,\Gamma) = \{g \in \mathrm{U}(2n,R,\Lambda) \mid g_{ij} \equiv \delta_{ij} \mod \sigma_{ij}, S_{i,-i}(g) \in \Gamma_i \text{ for all } i,j \in I\}.$$

the form net subgroup of level (σ, Γ) and

$$\mathrm{EU}(\sigma,\Gamma) = \langle T_{ij}(\xi), T_{i,-i}(\alpha) \mid i \neq \pm j, \xi \in \sigma_{ij}, \alpha \in \Gamma_i \rangle\,.$$

the elementary form net subgroup of level (σ, Γ).

If $(\sigma, \Gamma) = [\nu]_{(R,\Lambda)}$ for some equivalence relation ν, we will denote the corresponding elementary form net subgroup $\mathrm{EU}(\sigma, \Gamma)$ by

$$\mathrm{EU}(\nu, R, \Lambda)$$

and call it *the elementary block-diagonal subgroup of type* ν.

We still have to prove that $\mathrm{U}(\sigma, \Gamma)$ is a group. It's clear that

$$\mathrm{U}(\sigma) = \mathrm{U}(2n,R,\Lambda) \cap \mathrm{GL}(\sigma) = \{g \in \mathrm{U}(2n,R,\Lambda) \mid g_{ij} \equiv \delta_{ij} \mod \sigma_{ij} \text{ for all } i,j \in I\}$$

is a group because $\mathrm{U}(2n, R, \Lambda)$ and $\mathrm{GL}(\sigma)$ are subgroups of $\mathrm{GL}(2n, R)$. Therefore it only remains to prove that for any $a, b \in \mathrm{U}(\sigma, \Gamma)$ the inclusions $S_{i,-i}(ab) \in \Gamma_i$ and $S_{i,-i}(a^{-1}) \in \Gamma_i$ hold for any $i \in I$. The next proposition allows us to compute the length of a row of a product of two matrices. This is a standard computation and it was done in a slightly weaker form by Dybkova in [Dyb04]. In fact, Lemma 1 therein is precisely the second conclusion of Proposition 3.2.5 below.

Proposition 3.2.5. *Let a and b be elements of* $\mathrm{U}(2n, R, \Lambda)$*. Then for any $i \in I$ the equality*

$$\begin{aligned}
S_{i,-i}(ab) = S_{i,-i}(a) &+ \sum_k a_{ik} S_{k,-k}(b) a'_{-k,-i} \\
&- \sum_{j,k,l>0} (a_{ik} b_{k,-j} b'_{-j,l} a'_{l,-i} - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{ik} b_{k,-j} b'_{-j,l} a'_{l,-i})} \lambda^{(1-\varepsilon(i))/2}) \\
&- \sum_{k,j>0;l>k} ((a_{i,-k} b_{-k,-j} b'_{-j,l} a'_{l,-i} - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{i,-k} b_{k,-j} b'_{-j,l} a'_{l,-i})} \lambda^{(1-\varepsilon(i))/2}) \\
&+ (a_{ik} b_{k,-j} b'_{-j,-l} a'_{-l,-i} - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{ik} b_{k,-j} b'_{-j,-l} a'_{-l,-i})} \lambda^{(1-\varepsilon(i))/2})).
\end{aligned}$$

holds. In particular, if (σ, Γ) is a form net of ideals which is not necessarily exact of a D*-net over (R, Λ) and $a, b \in \mathrm{U}(\sigma)$, then*

$$S_{i,-i}(ab) \equiv S_{i,-i}(a) + \sum_k a_{ik} S_{k,-k}(b) a'_{-k,-i} \mod \Gamma_i^{\min}. \tag{3.9}$$

Finally, if $a, b \in \mathrm{U}(\sigma, \Gamma)$, then $ab, a^{-1} \in \mathrm{U}(\sigma, \Gamma)$.

Proof. We start from the following simple but useful observation. Let $i, j, k, l \in I$. Then

$$a_{ik} b_{kj} b'_{jl} a'_{l,-i} = \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{i,-l} b_{-l,-j} b'_{-j,-k} a'_{-k,-i})} \lambda^{(1-\varepsilon(i))/2}. \tag{3.10}$$

This equality directly follows from condition (1) in Lemma 3.1.2 and the definition of an involution with symmetry. By definition of the length of a row we get

$$S_{i,-i}(ab) = \sum_{j>0} (ab)_{ij} (ab)'_{j,-i} = \sum_{j>0} \sum_{k,l} a_{ik} b_{kj} b'_{jl} a'_{l,-i}.$$

Collect the summands corresponding to the four different combinations of the signs of the indices k and l, rewrite them in a way such that all the sums are taken over positive indices and group them in the following way:

$$\begin{aligned} S_{i,-i}(ab) = &\sum_{j,k,l>0} (a_{ik} b_{kj} b'_{jl} a'_{l,-i} + a_{i,-k} b_{-k,j} b'_{j,-l} a'_{-l,-i}) + \\ &+ \sum_{j,k,l>0} (a_{i,-k} b_{-k,j} b'_{jl} a'_{l,-i} + a_{ik} b_{kj} b'_{j,-l} a'_{-l,-i}). \end{aligned} \tag{3.11}$$

Denote the first sum in (3.11) by X and the second by Y. We will first simplify X. Recall that $\sum_j b_{kj} b'_{jl} = \delta_{kl}$. Using 3.10 we get

$$\begin{aligned} X = &\sum_{k,l>0} \sum_{j \in I} a_{ik} b_{kj} b'_{jl} a'_{l,-i} \\ &- \sum_{j,k,l>0} (a_{ik} b_{k,-j} b'_{-j,l} a'_{l,-i} - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{ik} b_{k,-j} b'_{-j,l} a'_{l,-i})} \lambda^{(1-\varepsilon_i)/2}) \\ = &\ S_{i,-i}(a) \\ &- \sum_{j,k,l>0} (a_{ik} b_{k,-j} b'_{-j,l} a'_{l,-i} - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{ik} b_{k,-j} b'_{-j,l} a'_{l,-i})} \lambda^{(1-\varepsilon_i)/2}). \end{aligned} \tag{3.12}$$

Now consider the summand Y. We group together the summands of Y such that $l > k$, $l = k$ and $l < k$ to obtain three summands. In the last summand we swap the indices l and k. Finally we join the first and last summands together. Summing up,

$$\begin{aligned} Y = &\sum_{j,k>0} \sum_{l>k} ((a_{i,-k} b_{-k,j} b'_{jl} a'_{l,-i} + a_{i,-l} b_{-l,j} b'_{jk} a'_{k,-i}) \\ &+ (a_{ik} b_{kj} b'_{j,-l} a'_{-l,-i} + a_{il} b_{lj} b'_{j,-k} a'_{-k,-i})) \\ &+ \sum_{j,k>0} (a_{i,-k} b_{-k,j} b'_{jk} a'_{k,-i} + a_{ik} b_{kj} b'_{j,-k} a'_{-k,-i}). \end{aligned} \tag{3.13}$$

Denote the first sum in (3.13) by Z and the second by W. Clearly

$$W = \sum_k a_{ik} \left(\sum_{j>0} b_{kj} b'_{j,-k} \right) a'_{-k,-i} = \sum_k a_{ik} S_{k,-k}(b) a'_{-k,-i}. \tag{3.14}$$

Simplify Z as follows.

$$\begin{aligned}
Z &= \sum_{j,k>0} \sum_{l>k} ((a_{i,-k} b_{-k,j} b'_{jl} a'_{l,-i} + a_{i,-l} b_{-l,j} b'_{jk} a'_{k,-i}) \\
&\quad + (a_{ik} b_{kj} b'_{j,-l} a'_{-l,-i} + a_{il} b_{lj} b'_{j,-k} a'_{-k,-i})) \\
&= (\text{by } (3.10)) \\
&\quad \sum_{k>0} \sum_{j\in I} \sum_{l>k} (a_{i,-k} b_{-k,j} b'_{jl} a'_{l,-i} + a_{ik} b_{kj} b'_{j,-l} a'_{-l,-i}) \\
&\quad - \sum_{k,j>0} \sum_{l>k} ((a_{i,-k} b_{-k,-j} b'_{-j,l} a'_{l,-i} - \\
&\quad - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{i,-k} b_{k,-j} b'_{-j,l} a'_{l,-i})} \lambda^{(1-\varepsilon(i))/2}) \\
&\quad + (a_{ik} b_{k,-j} b'_{-j,-l} a'_{-l,-i} - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{ik} b_{k,-j} b'_{-j,-l} a'_{-l,-i})} \lambda^{(1-\varepsilon(i))/2})) \\
&= (\text{because } \sum_{j\in I} b_{kj} b'_{jl} = \delta_{kl}) \\
&\quad - \sum_{k,j>0} \sum_{l>k} ((a_{i,-k} b_{-k,-j} b'_{-j,l} a'_{l,-i} - \\
&\quad - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{i,-k} b_{k,-j} b'_{-j,l} a'_{l,-i})} \lambda^{(1-\varepsilon(i))/2}) \\
&\quad + (a_{ik} b_{k,-j} b'_{-j,-l} a'_{-l,-i} - \lambda^{-(\varepsilon(i)+1)/2} \overline{(a_{ik} b_{k,-j} b'_{-j,-l} a'_{-l,-i})} \lambda^{(1-\varepsilon(i))/2})).
\end{aligned} \tag{3.15}$$

Combining the equalities (3.11), (3.12), (3.13), (3.15) and (3.14) we get the first part of the lemma. The second part follows from the definition of $\Gamma_i^{\min}$ and the fact that $a, b \in \mathrm{U}(\sigma)$. It follows form (3.9) that if $a, b \in \mathrm{U}(\sigma, \Gamma)$ then their product ab is also in $\mathrm{U}(\sigma, \Gamma)$. Finally it's obvious that $e \in \mathrm{U}(\sigma, \Gamma)$. Therefore if $a \in \mathrm{U}(\sigma, \Gamma)$ then we get

$$0 = S_{i,-i}(e) = S_{i,-i}(a^{-1} \cdot a) \equiv S_{i,-i}(a^{-1}) + \sum_k a'_{ik} S_{k,-k}(a) a_{-k,-i} \equiv S_{i,-i}(a^{-1}) \mod \Gamma_i.$$

Therefore $S_{i,-i}(a^{-1}) \in \Gamma_i$ for all $i \in I$. □

We will often be interested in the following corollary of the last lemma.

Corollary 3.2.6. *Let (σ, Γ) be a form net of ideals over R, that is not necessarily exact or a* D*-net. Let a be a matrix in $\mathrm{U}(\sigma)$ and $T_{pq}(\xi)$ a short (σ, Γ)-elementary unitary matrix. Then the following congruences hold modulo $\Gamma_i^{\min}$:*

1. $S_{i,-i}(T_{pq}(\xi)a) \equiv \begin{cases} S_{i,-i}(a), & \text{if } i \neq p, -q \\ S_{p,-p}(a) + \xi S_{q,-q}(a) \lambda^{(\varepsilon(q)-1)/2} \overline{\xi} \lambda^{(1-\varepsilon(p))/2}, & \text{if } i = p \\ S_{-q,q}(a) + \lambda^{(\varepsilon(q)-1)/2} \overline{\xi} \lambda^{(1-\varepsilon(p))/2} S_{-p,p}(a) \xi & \text{if } i = -q. \end{cases}$

2. $S_{i,-i}(aT_{pq}(\xi)) \equiv S_{i,-i}(a)$ *for all* $i \in I$.

3. $\begin{aligned} S_{i,-i}(aT_{pq}(\xi)a^{-1}) &\equiv a_{ip} \xi S_{q,-q}(a^{-1}) \lambda^{(\varepsilon(q)-1)/2} \overline{\xi} \lambda^{(1-\varepsilon(p))/2} a'_{-p,-i} \\ &\quad + a_{i,-q} \lambda^{(\varepsilon(q)-1)/2} \overline{\xi} \lambda^{(1-\varepsilon(p))/2} S_{-p,p}(a^{-1}) \xi a'_{q,-i}. \end{aligned}$

Given a morphism $\varphi : ((R_1, \bar{\cdot}, \lambda_1), \Lambda_1) \to ((R_2, \hat{\cdot}, \lambda_2), \Lambda_2)$ of form rings and a form net of ideals (σ, Γ) we define *the image under φ of the form net of ideals* (σ, Γ) to be a pair (τ, B), where τ is a $2n \times 2n$ array with entries

$$\tau_{ij} = \varphi(\sigma_{ij})$$

and B is a $2n$ column with entries

$$B_i = \varphi(\Gamma_i).$$

It is easy to see that an image of a form net of ideals is a form net of ideals. The following proposition shows how form net subgroups are transformed by unitary group morphisms induced by form ring morphisms. We leave it without a proof.

Proposition 3.2.7. *Let $\varphi : ((R_1, \bar{\cdot}, \lambda_1), \Lambda_1) \to ((R_2, \hat{\cdot}, \lambda_2), \Lambda_2)$ be a morphism of form rings. Let (σ, Γ) be a form net of ideals over (R_1, Λ_1). Let (τ, B) denote the form net of ideals over (R_1, Λ_2) defined by applying φ to (σ, Γ). Then*

1. *(τ, B) is a form net of ideals over $(\varphi(R_1), \varphi(\Lambda_1))$*
2. *if (σ, Γ) is an exact or* D- *form net, then (τ, B) is an exact or* D- *form net, respectively.*
3. $\mathrm{M}(\varphi)(\mathrm{U}(\sigma, \Gamma)) = \mathrm{U}(\tau, B)$
4. $\mathrm{M}(\varphi)(\mathrm{EU}(\sigma, \Gamma)) = \mathrm{EU}(\tau, B)$.

Description of the transporter In this paragraph we will prove Theorem 2. The following proposition will allow computing lengths of root elements corresponding to matrices which satisfy property (T1).

Proposition 3.2.8. *Let (σ, Γ) be an exact form* D*-net of ideals over (R, Λ) and a an element of* $\mathrm{U}(2n, R, \Lambda)$ *that satisfies property* (T1) *of Theorem 2, namely:*

$$a_{ij}\sigma_{jk}a'_{kl} \leq \sigma_{il}$$

for all $i, j, k, l \in I$. Then for any matrix $g \in \mathrm{U}(\sigma, \Gamma)$ and any index $i \in I$ the following congruence holds modulo $\Gamma_i^{\min}$:

$$S_{i,-i}(aga^{-1}) \equiv \sum_{k \in I} a_{ik} \left(S_{k,-k}(g) + S_{k,-k}(a^{-1}) + \sum_{t \in I} g_{kt} S_{t,-t}(a^{-1}) g'_{-t,-k} \right) a'_{-k,-i}.$$

Proof. By Proposition 3.2.5 we have

$$
\begin{aligned}
S_{i,-i}(aga^{-1}) = S_{i,-i}(a) &+ \sum_{k\in I} a_{ik} S_{k,-k}(ga^{-1}) a'_{-k,-i} \\
&- \sum_{j,k,l>0} \Big(a_{ik}(ga^{-1})_{k,-j}(ga^{-1})'_{-j,l} a'_{l,-i} \\
&\quad -\lambda^{(-\varepsilon(i)-1)/2} \overline{(a_{ik}(ga^{-1})_{k,-j}(ga^{-1})'_{-j,l} a'_{l,-i})} \lambda^{(1-\varepsilon(i))/2} \Big) \\
&- \sum_{j,k>0} \sum_{l>k} \Big(\Big(a_{i,-k}(ga^{-1})_{-k,-j}(ga^{-1})'_{-j,l} a'_{l,-i} \\
&\quad - \lambda^{(-\varepsilon(i)-1)/2} \overline{(a_{i,-k}(ga^{-1})_{-k,-j}(ga^{-1})'_{-j,l} a'_{l,-i})} \lambda^{(1-\varepsilon(i))/2} \Big) \\
&\quad + \Big(a_{ik}(ga^{-1})_{k,-j}(ga^{-1})'_{-j,-l} a'_{-l,-i} \\
&\quad - \lambda^{(-\varepsilon(i)-1)/2} \overline{(a_{ik}(ga^{-1})_{k,-j}(ga^{-1})'_{-j,-l} a'_{-l,-i})} \lambda^{(1-\varepsilon(i))/2} \Big) \Big) .
\end{aligned} \tag{3.16}
$$

Consider the expression $a_{ik}(ga^{-1})_{k,-j}(ga^{-1})'_{-j,l} a'_{l.-i}$. By property (T1) we get

$$
a_{ik}(ga^{-1})_{k,-j}(ga^{-1})'_{-j,l} a'_{l.-i} = \sum_{p,q\in I} (a_{ik} g_{kp} a'_{p,-j})(a_{-j,q} g'_{ql} a'_{l,-i}) \in \sigma_{i,-j}\sigma_{-j,-i} \le \sigma_{i,-i}.
$$

Therefore the whole term

$$
a_{ik}(ga^{-1})_{k,-j}(ga^{-1})'_{-j,l} a'_{l.-i} - \lambda^{(-\varepsilon(i)-1)/2} \overline{(a_{ik}(ga^{-1})_{k,-j}(ga^{-1})'_{-j,l} a'_{l.-i})} \lambda^{(1-\varepsilon(i))/2}
$$

of the second sum in (3.16) is contained in $\Gamma_i^{\min}$. For the same reason, the third sum in (3.16) is also contained in $\Gamma_i^{\min}$ and (3.16) can be rewritten as follows:

$$
S_{i,-i}(aga^{-1}) \equiv S_{i,-i}(a) + \sum_{k\in I} a_{ik} S_{k,-k}(ga^{-1}) a'_{-k,-i} \mod \Gamma_i^{\min}.
$$

By Proposition 3.2.5 we obtain

$$
\begin{aligned}
S_{i,-i}(aga^{-1}) \equiv S_{i,-i}(a) + \sum_{k\in I} \Bigg(& a_{ik} S_{k,-k}(g) a'_{-k,-i} \\
&+ \sum_{t\in I} a_{ik} g_{kt} S_{t,-t}(a^{-1}) g'_{-t,-k} a'_{-k,-i} - \sum_{j,t,l>0} a_{ik} \Big(g_{kt} a'_{t,-j} a_{-j,l} g'_{l,-k} \\
&\quad -\lambda^{(-\varepsilon(k)-1)/2} \overline{(g_{kt} a'_{t,-j} a_{-j,l} g'_{l,-k})} \lambda^{(1-\varepsilon(k))/2} \Big) a'_{-k,-i} \\
&- \sum_{t,j>0} \sum_{l>t} a_{ik} \Big(\Big(g_{k,-t} a'_{-t,-j} a_{-j,l} g'_{l,-k} \\
&\quad -\lambda^{(-\varepsilon(k)-1)/2} \overline{(g_{k,-t} a'_{-t,-j} a_{-j,l} g'_{l,-k})} \lambda^{(1-\varepsilon(k))/2} \Big) \\
&\quad + \Big(g_{kt} a'_{t,-j} a_{-j,-l} g'_{-l,-k} \\
&\quad -\lambda^{(-\varepsilon(k)-1)/2} \overline{(g_{kt} a'_{t,-j} a_{-j,-l} g'_{-l,-k})} \lambda^{(1-\varepsilon(k))/2} \Big) \Big) a'_{-k,-i} \Bigg) .
\end{aligned} \tag{3.17}
$$

Note that

$$(a_{ik}g_{kt}a'_{t,-j})(a_{-j,l}g'_{l,-k}a'_{-k,-i}) \leq \sigma_{i,-i}$$

and

$$a_{ik}\lambda^{(-\varepsilon(k)-1)/2}\overline{(g_{kt}a'_{t,-j}a_{-j,l}g'_{l,-k})}\lambda^{(1-\varepsilon(k))/2}a'_{-k,-i} = \\ \lambda^{(-\varepsilon(i)-1)/2}\overline{(a_{ik}g_{kt}a'_{t,-j}a_{-j,l}g'_{l,-k}a'_{-k,-i})}\lambda^{(1-\varepsilon(i))/2}.$$

Therefore the whole term

$$a_{ik}(g_{kt}a'_{t,-j}a_{-j,l}g'_{l,-k} - \lambda^{(-\varepsilon(k)-1)/2}\overline{g_{kt}a'_{t,-j}a_{-j,l}g'_{l,-k}})\lambda^{(1-\varepsilon(k))/2})a'_{-k,-i}$$

of (3.17) is contained in $\Gamma_i^{\min}$. For the same reason the terms of the third sum in (3.17) are also contained in $\Gamma_i^{\min}$. Thus we can rewrite (3.17) as follows

$$\begin{aligned} S_{i,-i}(aga^{-1}) \equiv S_{i,-i}(a) + \sum_{k\in I} a_{ik}S_{k,-k}(g)a'_{-k,-i} \\ + \sum_{k,t\in I} a_{ik}g_{kt}S_{t,-t}(a^{-1})g'_{-t,-k}a'_{-k,-i} \mod \Gamma_i^{\min}. \end{aligned} \tag{3.18}$$

Apply (3.18) to the special case when $g = e$:

$$0 = S_{i,-i}(a \cdot a^{-1}) \equiv S_{i,-i}(a) + \sum_{k\in I} a_{ik}S_{k,-k}(a^{-1})a'_{-k,-i} \mod \Gamma_i^{\min}. \tag{3.19}$$

Note that

$$S_{i,-i}(a) = \sum_{k\in I} a_{ij}a'_{j,-i} \in \sum_{k\in I} a_{ij}\sigma_{jj}a'_{j,-i} \leq \sigma_{i,-i}$$

and, as $a \in \mathrm{U}(2n, R, \Lambda)$, it follows that $S_{i,-i}(a) \in \Lambda_i^{\max}$. Therefore $S_{i,-i}(a) \in \Gamma_i^{\max}$. By Proposition 3.2.1 it follows that $2S_{i,-i}(a) \in \Gamma_i^{\min}$. Hence, we can rewrite (3.19) as

$$S_{i,-i}(a) \equiv \sum_{k\in I} a_{ik}S_{k,-k}(a^{-1})a'_{-k,-i} \mod \Gamma_i^{\min}. \tag{3.20}$$

Substituting (3.20) into (3.18) we get the required congruence

$$S_{i,-i}(aga^{-1}) \equiv \sum_{k\in I} a_{ik}\left(S_{k,-k}(g) + S_{k,-k}(a^{-1}) + \sum_{t\in I} g_{kt}S_{t,-t}(a^{-1})g'_{-t,-k}\right)a'_{-k,-i}$$

for all possible indices $i, k, t \in I$. □

Proof of Theorem 2. Let N denote the set of all matrices in $\mathrm{U}(2n, R, \Lambda)$ satisfying conditions (T1)–(T3). It's easy to see that $N \leq \mathrm{N}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{U}(\sigma, \Gamma))$. Indeed, pick any $g \in \mathrm{U}(\sigma, \Gamma)$ and $a \in N$. By condition (T1) it follows that

$$(aga^{-1})_{ij} = \sum_{p,q\in I} a_{ip}g_{pq}a'_{qj} \leq \sum_{p,q\in I} a_{ip}\sigma_{pq}a'_{qj} \leq \sigma_{ij} \tag{3.21}$$

for all indices i and j. Applying Proposition 3.2.8 to the matrix a and an element g of $\mathrm{U}(\sigma,\Gamma)$ we get the following congruence modulo Γ_i:

$$
\begin{aligned}
S_{i,-i}(aga^{-1}) \equiv \sum_{k\in I} a_{ik}\Bigg(S_{k,-k}(g) + S_{k,-k}(a^{-1}) \\
+ \sum_{t\in I} g_{kt} S_{t,-t}(a^{-1}) g'_{-t,-k}\Bigg) a'_{-k,-i}.
\end{aligned}
\tag{3.22}
$$

By property (T3) it follows that

$$
a_{ik} S_{k,-k}(g) a'_{-k,-i} \le a_{ik}\Gamma_k a'_{-k,-i} \le \Gamma_i. \tag{3.23}
$$

By property (T2) it follows that

$$
\begin{aligned}
a_{ik} g_{kt} S_{t,-t}(a^{-1}) g'_{-t,-k} a'_{-k,-i} = \\
a_{ik} g_{kt} S_{t,-t}(a^{-1}) \lambda^{(\varepsilon(t)-1)/2}\overline{g_{kt}}\lambda^{(1-\varepsilon(k))/2} a'_{-k,-i} \in \Gamma_i.
\end{aligned}
\tag{3.24}
$$

As $1\in\sigma_{kk}$, by property (T2) we get

$$
a_{ik} S_{k,-k}(a^{-1}) = a_{ik}\cdot 1\cdot S_{k,-k}(a^{-1})\lambda^{(\varepsilon(k)-1)/2}\overline{1}\lambda^{(1-\varepsilon(k))/2} a'_{-k,-i} \in \Gamma_i. \tag{3.25}
$$

Combining (3.22), (3.23), (3.24) and (3.25) we get the inclusion $S_{i,-i}(aga^{-1}) \in \Gamma_i$ for all $i\in I$. This together with (3.21) allows us to conclude that $aga^{-1}\in \mathrm{U}(\sigma,\Gamma)$ whenever $a\in N$ and $g\in\mathrm{U}(\sigma,\Gamma)$ Therefore $N \le \mathrm{N}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{U}(\sigma,\Gamma))$.

Now we will show that $\mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma,\Gamma),\mathrm{U}(\sigma,\Gamma)) \le N$. Fix an element a of $\mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma,\Gamma),\mathrm{U}(\sigma,\Gamma))$ and a short (σ,Γ)-elementary unitary matrix $T_{sr}(\xi)$. By definition of the transporter

$$
\delta_{ij} + a_{is}\xi a'_{rj} - a_{i,-r}\lambda^{(\varepsilon(r)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(s))/2}a'_{-s,j} = (aT_{sr}(\xi)a^{-1})_{ij} \in \sigma_{ij} \tag{3.26}
$$

for all $i,j\in I$. If $T_{-r,r}(\alpha)$ is a long (σ,Γ)-elementary unitary matrix then

$$
\delta_{ij} + a_{i,-r}\alpha a'_{-r,j} = (aT_{-r,r}(\alpha)a^{-1})_{ij} \in \sigma_{ij} \tag{3.27}
$$

If $T_{rt}(\zeta)$ is another short (σ,Γ)-elementary unitary matrix then

$$
\begin{aligned}
(\delta_{ij} + a_{is}\xi a'_{rj} - a_{i,-r}\lambda^{(\varepsilon(r)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(s))/2}a'_{-s,j}) \\
+(\delta_{ij} + a_{ir}\zeta a'_{tj} - a_{i,-t}\lambda^{(\varepsilon(t)-1)/2}\overline{\zeta}\lambda^{(1-\varepsilon(r))/2}a'_{-r,j}) + a_{is}\xi\zeta a'_{tj} - \delta_{ij} \\
= (aT_{sr}(\xi)T_{rt}(\zeta)a^{-1})_{ij} \in \sigma_{ij}
\end{aligned}
\tag{3.28}
$$

for all $i,j\in I$. Similarly we have

$$
\begin{aligned}
(\delta_{ij} + a_{i,-r}\alpha a'_{-r,j}) + ((\delta_{ij} + a_{ir}\zeta a'_{tj} - a_{i,-t}\lambda^{(\varepsilon(t)-1)/2}\overline{\zeta}\lambda^{(1-\varepsilon(r))/2}a'_{-r,j}) \\
+ a_{i,-r}\alpha\zeta a'_{tj} - \delta_{ij} \\
= (aT_{-r,r}(\alpha)T_{rt}(\zeta)a^{-1})_{ij} \in \sigma_{ij}
\end{aligned}
\tag{3.29}
$$

for all $i, j \in I$. Comparing (3.28) with (3.26) we conclude that

$$a_{is}\xi\zeta a'_{tj} = (aT_{sr}(\xi)T_{rt}(\zeta)a^{-1})_{ij} - (aT_{sr}(\xi)a^{-1})_{ij} - (aT_{rt}(\zeta)a^{-1})_{ij} - \delta_{ij} \in \sigma_{ij}.$$

Therefore due to the arbitrary choice of ξ and ζ,

$$a_{is}\sigma_{sr}\sigma_{rt}a'_{tj} \leq \sigma_{ij} \tag{3.30}$$

for all $i, j \in I$ and all $s, r, t \in I$ such that $s \neq \pm r, \pm t$ and $r \neq \pm t$. Similarly comparing (3.29) with (3.27) and (3.26) we get

$$a_{i,-r}\alpha\zeta a'_{tj} = (aT_{-r,r}(\alpha)T_{rt}(\zeta)a^{-1})_{ij} - (aT_{-r,r}(\alpha)a^{-1})_{ij} - (aT_{rt}(\zeta)a^{-1})_{ij} - \delta_{ij} \in \sigma_{ij}$$

for all $i, j \in I$ and varying ζ and α we get the inclusion

$$a_{i,-r}\Gamma_{-r}\sigma_{rt}a'_{tj} \in \sigma_{ij} \tag{3.31}$$

for all $i, j \in I$ and all $r, t \in I$ such that $r \neq \pm t$. Exactly in the same way we get the inclusions

$$a_{is}\Gamma_s\sigma_{-s,t}a'_{tj} \in \sigma_{ij} \tag{3.32}$$

for the same indices as in (3.31) We are ready to show that the condition (T1) is fulfilled for the matrix a. First, let $s \neq \pm t$. Assume, there exists another index $r \sim t$ such that $r \neq \pm s, \pm t$. In this case $\sigma_{sr} = \sigma_{sr}R = \sigma_{st}R = \sigma_{st}\sigma_{tr}$ By (3.30) it follows that

$$a_{is}\sigma_{st}a'_{tj} = a_{is}\sigma_{sr}\sigma_{rt}a'_{tj} \leq \sigma_{ij}$$

for all $i, j \in I$. The only situation when we fail to find an index r as described above is when the equivalence class of t is precisely $\{\pm s, \pm t\}$. In this case by assumption of the theorem we have $R\Lambda + \Lambda R = R$. It's easy to see that the last condition is equivalent to $R\Lambda_{-t} + \Lambda_s R = R$ Therefore $\sigma_{st} = R = R\Lambda_{-t} + \Lambda_s R = R = \sigma_{s,-t}\Gamma_{-t} + \Gamma_s\sigma_{-s,t}$. Combining (3.31) with (3.32) we get

$$a_{is}\sigma_{st}a'_{tj} \leq a_{is}\sigma_{s,-t}\Gamma_{-t}a'_{tj} + a_{is}\Gamma_s\sigma_{-s,t}a'_{tj} \leq \sigma_{ij}$$

for all $i, j \in I$. Therefore $a_{is}\sigma_{st}a'_{tj} \leq \sigma_{ij}$ whenever $s \neq \pm t$. If $s = \pm t$ then there exists an index $r \sim t$ such that $r \neq \pm s$ and we get

$$a_{is}\sigma_{st}a'_{tj} = a_{is}\sigma_{sr} \cdot 1 \cdot \sigma_{rt}a'_{tj} = \sum_{p \in I}(a_{is}\sigma_{sr}a'_{rp})(a_{pr}\sigma_{rt}a'_{tj} \leq \sigma_{ij}$$

for all i and j in I. Summing up, the matrix a satisfies property (T1).

Pick a short (σ, Γ)-elementary unitary matrix $T_{jk}(\xi)$. By Proposition 3.2.8 the following congruence modulo $\Gamma_i^{\min}$ takes place:

$$\begin{aligned} S_{i,-i}(aT_{jk}(\xi)a^{-1}) \equiv{}& a_{ij}\xi S_{k,-k}(a^{-1})\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \\ &+ a_{i,-k}\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}S_{-j,j}(a^{-1})\xi a'_{k,-i}. \end{aligned} \tag{3.33}$$

Pick a long (σ,Γ)-elementary unitary matrix $T_{k,-k}(\alpha)$. By Proposition 3.2.8 the congruence

$$S_{i,-i}(aT_{k,-k}(\alpha)a^{-1}) \equiv a_{ik}\alpha a'_{-k,-i}+ \\ a_{ik}\alpha S_{-k,k}(a^{-1})\lambda^{(-\varepsilon(k)-1)/2}\overline{\alpha}\lambda^{(1-\varepsilon(k))/2}a'_{-k,-i} \mod \Gamma_i^{\min} \tag{3.34}$$

holds. Fix yet another short (σ,Γ)-elementary unitary matrix $T_{kl}(\zeta)$. By Proposition 3.2.8 the following congruence modulo $\Gamma_i^{\min}$ holds:

$$\begin{aligned} S_{i,-i}(aT_{jk}(\xi)T_{kl}(\zeta)a^{-1}) \equiv\ & a_{ij}\xi S_{k,-k}(a^{-1})\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \\ & + a_{i,-k}\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}S_{-j,j}(a^{-1})\xi a'_{k,-i} \\ & + a_{ik}\xi S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(k))/2}a'_{-k,-i} \\ & + a_{i,-l}\lambda^{(\varepsilon(l)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(k))/2}S_{-k,k}(a^{-1})\xi a'_{l,-i} \\ & + a_{ij}\xi\zeta S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi\zeta}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i}. \end{aligned} \tag{3.35}$$

Similarly considering a product of a short and a long elementary unitary matrix we get by Proposition 3.2.8 the following congruence modulo $\Gamma_i^{\min}$:

$$\begin{aligned} S_{i,-i}(aT_{jk}(\xi)T_{k,-k}(\alpha)a^{-1}) \equiv\ & a_{ij}\xi S_{k,-k}(a^{-1})\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \\ & + a_{i,-k}\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}S_{-j,j}(a^{-1})\xi a'_{k,-i} \\ & + a_{ij}\alpha a'_{-k,-i} + a_{ik}\alpha S_{-k,k}(a^{-1})\lambda^{(-\varepsilon(k)-1)/2}\overline{\alpha}\lambda^{(1-\varepsilon(k))/2}a'_{-k,-i} \\ & + a_{ij}\xi\alpha S_{-k,k}(a^{-1})\lambda^{(-\varepsilon(k)-1)/2}\overline{\xi\alpha}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i}. \end{aligned} \tag{3.36}$$

Denote the left-hand side of (T2) by $X(\xi)$. By Proposition 3.2.2 we have $X(\xi+\zeta) \equiv X(\xi)+X(\zeta) \mod \Gamma_i^{\min}$. Therefore it is enough to check property (T2) for some set of additive generators of the ideal σ_{jk}.

Recall that $a \in \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma,\Gamma),\mathrm{U}(\sigma,\Gamma))$ Comparing (3.35) with (3.33) we get the inclusion

$$a_{ij}\xi\zeta S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi\zeta}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \in \Gamma_i \tag{3.37}$$

for all $i \in I$, all $j,k,l \in I$ such that $j \neq \pm k, \pm l$, $k \neq \pm l$ and all $\xi \in \sigma_{jk}$ and $\zeta \in \sigma_{kl}$. Similarly comparing (3.36) with (3.33) and (3.34) we get the inclusion

$$a_{ij}\xi\alpha S_{-k,k}(a^{-1})\lambda^{(-\varepsilon(k)-1)/2}\overline{\xi\alpha}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \in \Gamma_i \tag{3.38}$$

for all $i \in I$ and all $j,k \in I$ such that $j \neq \pm k$ and all $\xi \in \sigma_{jk}$ and $\alpha \in \Gamma_k$. Similarly we get the inclusion

$$a_{ij}\alpha\xi S_{-k,k}(a^{-1})\lambda^{(-\varepsilon(k)-1)/2}\overline{\alpha\xi}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \in \Gamma_i \tag{3.39}$$

for all $i \in I$ and all $j,k \in I$ such that $j \neq \pm k$ and all $\alpha \in \Gamma_j$ and $\xi \in \sigma_{-j,-k}$.

We are ready to check property (T2) for the matrix a. Fix two indices $j,l \in I$ such that $j \neq \pm l$. If there exists an index $k \sim l$ such that $k \neq \pm jm \pm l$ then $\sigma_{jl} = \sigma_{jk}\sigma_{kl}$ and furthermore the inclusion (T2)

$$a_{ij}\xi S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \in \Gamma_i$$

for any i and any $\xi \in \sigma_{jl}$ is given by (3.37) together with Proposition 3.2.2. If we fail to find the index k as above then it follows that the equivalence class of l consists precisely of four elements $\{\pm j, \pm l\}$. Therefore $R\Lambda + \Lambda R = R$. Hence $\sigma_{jl} = \sigma_{j,-l}\Gamma_{-l} + \Gamma_j \sigma_{-j,l}$ and the inclusion (T2)

$$a_{ij}\xi S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \in \Gamma_i$$

for all $i \in I$ and $\xi \in \sigma_{jl}$ is obtained as a combination of (3.38) and (3.39).

If $j = \pm l$ then there exists an index $k \sim j$ such that $k \neq \pm j$ and thus $k \neq \pm l$. Then $\sigma_{jl} = \sigma_{jk}\sigma_{il}$. Pick any $\xi_1 \in \sigma_{jk}$ and any $\xi_2 \in \sigma_{kl}$. Consider

$$\begin{aligned} & a_{ij}\xi_1\xi_2 S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi_1\xi_2}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \\ &= a_{ij}\xi_1 \left(\sum_{p\in I} a'_{kp}a_{pk}\right) \xi_2 S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi_2}\lambda^{(1-\varepsilon(k))/2} \\ &\quad \times \left(\sum_{q\in I} a'_{-kq}a_{q,-k}\right) \lambda^{(\varepsilon(k)-1)/2}\overline{\xi_1}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} \\ &= \sum_{p,q\in I} (a_{ij}\xi_1 a'_{kp})a_{pk}\xi_2 S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi_2}\lambda^{(1-\varepsilon(k))/2} \\ &\quad \times a'_{-kq}(a_{q,-k}\lambda^{(\varepsilon(k)-1)/2}\overline{\xi_1}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i}). \end{aligned} \tag{3.40}$$

Denote the summand of the right-hand side of (3.40) by $Y(p,q)$. It's easy to see that $Y(p,-p) \in \Gamma_i$ for all $p \in I$. Indeed, we have already proved that

$$a_{pk}\xi_2 S_{l,-l}(a^{-1})\lambda^{(\varepsilon(l)-1)/2}\overline{\xi_2}\lambda^{(1-\varepsilon(k))/2}a'_{-k,-p} \in \Gamma_p$$

for all p. Note that $a_{ij}\xi_1 a'_{kp} \in \sigma_{ip}$ and

$$a_{-p,-k}\lambda^{(\varepsilon(k)-1)/2}\overline{\xi_1}\lambda^{(1-\varepsilon(j))/2}a'_{-j,-i} = \lambda^{(\varepsilon(p)-1)/2}\overline{(a_{ij}\xi_1 a'_{kp})}\lambda^{(1-\varepsilon(i))/2}.$$

Therefore by relation ($\Gamma 2'$) it follows that $Y(p,-p)$ is indeed contained in Γ_i. It's clear that $Y(p,q) \in \sigma_{i,-i}$ for all $p,q \in I$. It's easy to see that $Y(p,q) + Y(-q,-p) \in \Gamma_i^{\min}$ for all $p,q \in I$. Therefore

$$\sum_{p,q\in I} Y(p,q) = \sum_{p} Y(p,-p) + \sum_{p>-q} (Y(p,q) + Y(-q,-p)) \in \Gamma_i.$$

Therefore property (T2) holds for the matrix a for all possible indices. Combining (T2) with (3.34) we get property (T3) for all $i,j \in I$. Therefore

$$\mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma,\Gamma), \mathrm{U}(\sigma,\Gamma)) \leq N.$$

It's only left to notice that, as Transp is contravariant in the first variable, it follows that

$$\begin{aligned} \mathrm{N}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{U}(\sigma,\Gamma)) &= \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{U}(\sigma,\Gamma), \mathrm{U}(\sigma,\Gamma)) \\ &\leq \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma,\Gamma), \mathrm{U}(\sigma,\Gamma)) \leq N. \end{aligned}$$

□

Form net associated with a subgroup Fix a subgroup H of $\mathrm{U}(2n, R, \Lambda)$. An exact form net of ideals (σ, Γ) is called *the form net of ideals associated with* H if $\mathrm{Ep}(\sigma, \Gamma) \le H$ and if for any exact form net of ideals (σ', Γ') such that $\mathrm{Ep}(\sigma', \Gamma') \le H$ it follows that $(\sigma', \Gamma') \le (\sigma, \Gamma)$. It is obvious, that if a form net of ideals associated with H exists then it is unique. The following lemma shows that it exists whenever H contains $\mathrm{EU}(\nu, R, \Lambda)$ and $h(\nu) \ge (4, 3)$.

Lemma 3.2.9. *Assume* $h(\nu) \ge (4, 3)$. *Let* H *be a subgroup of* $\mathrm{U}(2n, R, \Lambda)$ *such that* $\mathrm{EU}(\nu, R, \Lambda) \le H$. *Set*

$$\begin{aligned} \sigma_{ij} &= \{\xi \in R \mid T_{ij}(\xi) \in H\} \text{ for all } i \neq \pm j \\ \Gamma_i &= \{\alpha \in \Lambda_i \mid T_{i,-i}(\alpha) \in H\} \text{ for all } i \in I \\ \sigma_{ii} &= R \text{ and} \\ \sigma_{i,-i} &= \sum_{j \neq \pm i} \sigma_{ij}\sigma_{j,-i} + \langle \Gamma_i \rangle \text{ for all } i \in I. \end{aligned} \tag{3.41}$$

Then (σ, Γ) *is the form net of ideals associated with* H.

Proof. By the elementary relation (EU2) it follows that σ_{ij} and Γ_i are additive subgroups for all $i, j \in I$. We will show that σ_{ij} is a two-sided ideal for all $i, j \in I$. Indeed, take three indices $i, j, k \in I$ such that $i \neq \pm j, \pm k$ and $j \neq \pm k$ and any $\xi \in \sigma_{ij}, \zeta \in \sigma_{jk}$. Then by the elementary relation (EU4) we get

$$T_{ik}(\xi\zeta) = [T_{ij}(\xi), T_{jk}(\zeta)] \in H.$$

In other words

$$\sigma_{ij}\sigma_{jk} \le \sigma_{ik}, \quad \text{whenever } i \neq \pm j, \pm k \text{ and } j \neq \pm k. \tag{3.42}$$

If $i \sim j$, then $\sigma_{ij} = R$ is obviously a two-sided ideal. By definition, $\sigma_{i,-i}$ is also a two-sided ideal. Assume $i \nsim j$ and $i \neq \pm j$. As $h(\nu) \ge (4, 3)$, there exists another index $k \sim j$ such that $k \neq \pm i, \pm j$. Then $\sigma_{jk} = \sigma_{kj} = R$ and by (3.42) we get

$$\sigma_{ij}R = \sigma_{ij}\sigma_{jk} \le \sigma_{ik} \le \sigma_{ik}R = \sigma_{ik}\sigma_{kj} \le \sigma_{ij}.$$

Therefore σ_{ij} is a right ideal. Similarly, σ_{ij} is a left ideal and, consequently, a two-sided ideal.

Now we will prove that

$$\Gamma_i\sigma_{-i,j} \le \sigma_{ij} \tag{3.43}$$

for all $i, j \in I$. If $i \sim j$ this inclusion (3.43) is trivial. The inclusion (3.43) for $i = -j$ follows from the following obvious observation:

$$\Gamma_i\sigma_{-i,j} = \Gamma_i R \le \sigma_{i,-i}.$$

Assume $i \nsim j$ and $i \neq -j$. Pick any $\alpha \in \Gamma_i$ and $\xi \in \sigma_{-i,j}$. According to the elementary relation (EU6) we have

$$x = T_{ij}(\alpha\xi)T_{-j,j}(\beta) = [T_{i,-i}(\alpha), T_{-i,j}(\xi)] \le H,$$

where $\beta = -\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi$ is an element of Λ_{-j}. If $j \sim -j$ then $T_{-j,j}(\beta) \in H$ for all $\beta \in \Gamma_{-j}$. Therefore $T_{ij}(\alpha\xi) \in H$. If $j \nsim -j$ then there exists an index $k \sim i$ such that $k \neq \pm i, \pm j$. In this case by the elementary relation (EU4) we have

$$T_{ij}(\alpha\xi) = [T_{ik}(1), [T_{ki}(1), x]] \in H.$$

Therefore $\alpha\xi \in \sigma_{ij}$. This completes the proof of the inclusion (3.43) for all $i, j \in I$.

Assume $i \neq \pm j$. Combing (EU6) and (3.43) we have

$$\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\Gamma_i\xi \leq \Gamma_{-j} \tag{3.44}$$

for all $\xi \in \sigma_{-i,j}$. Indeed, in this situation by (EU6) we get

$$T_{ij}(\alpha\xi)T_{-j,j}(\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi) = [T_{i,-i}(\alpha), T_{-i,j}(\xi)] \in H \tag{3.45}$$

whenever $\alpha \in \Gamma_i$. By (3.43) the first term of the left-hand side of (3.45) is also in H.

The inclusion $\Gamma_i^{\min} \leq \Gamma_i$ follows from the relation (EU5). Indeed, fix an index $i \in I$. As Γ_i is an additive subgroup, it's enough to prove that $\alpha - \lambda^{-(1+\varepsilon(i))/2}\overline{\alpha}\lambda^{(1-\varepsilon(i))/2} \in \Gamma_i$ only for α in some set generating $\sigma_{i,-i}$ as an additive subgroup. First, take α of the form $\xi\zeta$, where $\xi \in \sigma_{ij}$ and $\zeta \in \sigma_{j,-i}$ for some $j \neq \pm i$. By (EU5) we get

$$T_{i,-i}(\alpha - \lambda^{-(1+\varepsilon(i))/2}\overline{\alpha}\lambda^{(1-\varepsilon(i))/2}) = [T_{ij}(\xi), T_{j,-i}(\zeta)] \in H.$$

Now let α be an additive generator of $\langle\Gamma_i\rangle_R$, i.e. $\alpha = \xi_1\beta\xi_2$ for some $\beta \in \Gamma_i$ and $\xi_1, \xi_2 \in R$. Pick any index $j \sim i$ such that $j \neq \pm i$. Then $1 \in \sigma_{-i,-j}$ and by (3.44) we get $\lambda^{(\varepsilon(i)-\varepsilon(j))/2}\beta \in \lambda^{(\varepsilon(i)-\varepsilon(j))/2}\Gamma_i \leq \Gamma_j$. As $\xi_2 \in \sigma_{-j,-i}$, by (3.43) we get $\lambda^{(\varepsilon(i)-\varepsilon(j))/2}\beta\xi_2 \leq \Gamma_j\sigma_{-j,-i} \leq \sigma_{j,-i}$. Finally, as $\xi_1\lambda^{(\varepsilon(j)-\varepsilon(i))/2} \in R = \sigma_{ij}$, it follows by (EU5) that

$$T_{i,-i}(\alpha - \lambda^{-(1+\varepsilon(i))/2}\overline{\alpha}\lambda^{(1-\varepsilon(i))/2}) = [T_{ij}(\xi_1\lambda^{(\varepsilon(j)-\varepsilon(i))/2}), T_{j,-i}(\lambda^{(\varepsilon(i)-\varepsilon(j))/2}\beta\xi_2)] \in H.$$

Therefore all additive generators of $\Gamma_i^{\min}$ are contained in Γ_i. Thus $\Gamma_i^{\min} \leq \Gamma_i$ for all $i \in I$.

Now we will prove that $\sigma_{ij}\sigma_{jk} \leq \sigma_{ik}$ for all $i, j, k \in I$. We have already proved (see (3.42)) these inclusions whenever $i \neq \pm j, \pm k$ and $j \neq \pm k$. If $i \sim k$ then $\sigma_{ik} = R$ and the inclusion $\sigma_{ij}\sigma_{jk} \leq \sigma_{ik}$ is trivial. The inclusion corresponding to the case when $k = -i$ follows from the definition of $\sigma_{i,-i}$. Assume $k \nsim i$, $k \neq \pm i$. If $j = k$ or $i = j$ then the corresponding inclusion $\sigma_{ij}\sigma_{jk} \leq \sigma_{ik}$ is equivalent to the fact that each σ_{ij} is a two-sided ideal. Therefore it only remains to consider two possibilities. The first one is that $i = -j$ and $k \neq \pm i$. The second one is that $j = -k$ and $i \neq \pm k$. We will only consider the first one. The second one can be treated similarly. Observe that

$$\sigma_{i,-i}\sigma_{-i,k} = \sum_{l \neq \pm i} \sigma_{il}\sigma_{l,-i}\sigma_{-i,k} + \langle\Gamma_i\rangle\,\sigma_{-i,k}.$$

By (3.43) we know that $\langle\Gamma_i\rangle\,\sigma_{-i,k} \leq \sigma_{ik}$. Moreover for each $l \neq \pm i, \pm k$ we already know that $\sigma_{il}\sigma_{l,-i}\sigma_{-i,k} \leq \sigma_{ik}$. Finally, for $l = k$ $\sigma_{ik}\sigma_{k,-i}\sigma_{-i,k} \leq \sigma_{ik}$ as σ_{ik} is an ideal. We are only left to prove that $\sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,k} \leq \sigma_{ik}$. As $k \nsim i$ there exists another index $t \sim k$ such that $t \neq \pm k, \pm i$. Then

$$\sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,k} \leq \sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,k}\sigma_{kt} \leq \sigma_{i,-k}\sigma_{-k,-i}\sigma_{-i,t} \leq \sigma_{i,-k}\sigma_{-k,t} \leq \sigma_{it}.$$

Therefore $\sigma_{i,-i}\sigma_{-i,k} \leq \sigma_{it} \leq \sigma_{it}\sigma tk \leq \sigma_{ik}$. Thus we have proven the inclusion

$$\sigma_{ij}\sigma_{jk} \leq \sigma_{ik}$$

for all $i, j, k \in I$.

Now we will prove the inclusion ($\Gamma 2$). We have already checked (see (3.44)) these inclusions for all $i \neq \pm j$. Let $j \in \{\pm i\}$. Fix some $\xi \in \sigma_{-i,j}$. There exists an index $k \sim i$ such that $k \neq \pm i$. Then $\xi \in \sigma_{-i,k}$ and by (3.44) we get

$$\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(i))/2}\Gamma_i\xi \leq \Gamma_{-k}. \tag{3.46}$$

As $1 \in \sigma_{kj}$, by (3.44) we get

$$\lambda^{(\varepsilon(j)-\varepsilon(k))/2}\Gamma_{-k} \leq \Gamma_{-j}. \tag{3.47}$$

Combining (3.46) and (3.47) we get

$$\lambda^{(\varepsilon(j)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(i))/2}\Gamma_i = \lambda^{(\varepsilon(j)-\varepsilon(k))/2}\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(i))/2}\Gamma_i\xi \leq \Gamma_{-j}.$$

Therefore, the inclusions ($\Gamma 2$) hold for all $i, j \in I$.

Notice that the inclusion $\Gamma_i \leq \Gamma_i^{\max}$ is obvious by (3.41). The condition $\sigma_{ij} = \overline{\sigma}_{-j,-i}$ follows obviously from (EU1) whenever $j \neq \pm i$ and is trivial when $i = j$. Finally, if $j = -i$ there exists another index $k \sim i$ such that $k \neq \pm i$. Then

$$\sigma_{i,-i} \leq \sigma_{ki}\sigma_{i,-i} \leq \sigma_{k,-i} \leq \sigma_{ik}\sigma_{k,-i} \leq \sigma_{i,-i}$$

and thus $\sigma_{i,-i} = \sigma_{k,-i}$. Similarly $\sigma_{i,-k} = \sigma_{i,-i}$. Thus

$$\sigma_{i,-i} = \sigma_{k,-i} = \overline{\sigma}_{i,-k} = \overline{\sigma}_{i,-i}.$$

Therefore (σ, Γ) is a form net of ideals. It is clear that (σ, Γ) is exact and major. By definition, $\mathrm{EU}(\sigma, \Gamma) \leq H$. Finally, it is easy to see that for any exact form net of ideals (σ', Γ') such that $\mathrm{EU}(\sigma', \Gamma') \leq H$ it follows that $(\sigma', \Gamma') \leq (\sigma, \Gamma)$. This completes the proof. □

3.3 Standard setting

In this section we are going to introduce the concept of a standard setting. Let (R, Λ) be an associative [unital] form ring and (R', Λ') be a form subring of (R, Λ). Let S be a multiplicative set in R' (i.e. a multiplicatively closed set containing the identity) such that $S \subseteq R^* \cap \mathrm{Center}(R) \cap \{\xi \in R \mid \overline{\xi} = \xi\}$. Assume that for every $\xi \in R$ there exists an element $x \in S$ such that $x\xi \in R'$ and for every $\alpha \in \Lambda$ there exists an elements $x \in S$ such that $x^2\alpha \in \Lambda'$. Then we call the triple $((R,\Lambda),(R',\Lambda'),S)$ *a standard setting.* In this setting we will associate with a subgroup $H \le \mathrm{U}(2n,R,\Lambda)$ such that $\mathrm{EU}(\nu,R,\Lambda) \le H$, a form net of ideals (σ',Γ') over (R',Λ') and a form net of ideals (σ,Γ) over (R,Λ) such that $\mathrm{EU}(\sigma,\Gamma) \le H$ and (σ,Γ) is S-related (in appropriate sense) with (σ',Γ').

Let (σ',Γ') be an exact form net of ideals of size $2n$ over (R',Λ'). For all indices $i,j \in I$ define the following subsets in R:

$$\begin{aligned}\sigma_{ij} &= \{\xi \in R \mid \exists x \in S,\ x\xi \in \sigma'_{ij}\}\\ \Gamma_i &= \{\alpha \in \Lambda_i \mid \exists x \in S,\ x^2\alpha \in \Gamma'_i\}.\end{aligned}$$

The pair (σ,Γ) is called *the S-closure of the form net (σ',Γ') in the form ring (R,Λ).* We will show that (σ,Γ) is an exact form D-net over (R,Λ).

Fix a subgroup H of $\mathrm{U}(2n,R,\Lambda)$. We say that a form net (σ',Γ') over (R',Λ) is *S-associated with the subgroup H* if the following two conditions are fulfilled:

1. $\mathrm{EU}(\sigma',\Gamma') \le H$
2. For any elementary unitary matrix $T_{sr}(\xi)$ contained in H there exists an element $x \in S$ such that the inclusion $x^{(1+\delta_{r,-s})}\xi \in (\sigma',\Gamma')_{sr}$ holds.

Next we introduce a family of net-like objects defined by H. Under certain conditions they become form nets of ideals over (R,Λ). For any $g \in \mathrm{U}(2n,R,\Lambda)$ and any $i \ne \pm j$ set

$$\begin{aligned}\sigma^g_{ij} &= \{\xi \in R \mid \exists x \in S,\ \forall\theta, \in S\ {}^gT_{ij}(x\theta\xi) \in H\}\\ \sigma^g_{ii} &= R\\ \Gamma^g_i &= \{\alpha \in \Lambda_i \mid \exists x \in S,\ \forall\theta \in S,\ {}^gT_{i,-i}(x^2\theta^2\alpha) \in H\}\\ \sigma^g_{i,-i} &= \sum_{k\ne\pm i} \sigma^g_{ik}\sigma^g_{k,-i} + \langle\Gamma^g_i\rangle\,,\end{aligned} \tag{3.48}$$

where the product $\sigma^g_{ik}\sigma^g_{k,-i}$ above is the additive subgroup generated by all products $\xi\zeta$, where $\xi \in \sigma^g_{ij}$ and $\zeta \in \sigma^g_{j,-i}$.

The rest of this section directly follows the analogous results for the symplectic groups presented in Section 2.1. A lot of these results translate to the current setting literally. Freely speaking, all the computations involving only short elementary unitary matrices remain unchanged.

For the rest of this section we fix a standard setting $((R,\Lambda),(R',\Lambda'),S)$, a unitary equivalence relation ν on our index set I and a subgroup $H \le \mathrm{U}(2n,R,\Lambda)$.

Proposition 3.3.1. *Let (σ',Γ') be an exact major form net of ideals over (R',Λ'). Let (σ,Γ) denote the S-closure of (σ',Γ') in (R,Λ). Then (σ,Γ) is an exact major form net of ideals over (R,Λ). Further, suppose $h(\nu) \ge (4,3)$ and (σ',Γ') is S-associated with H. Then the form net of ideals (σ,Γ) is coordinate-wise equal to (σ^e,Γ^e).*

Proof. It's clear that $\sigma_{ij} = R$ whenever $i \sim j$ and $\Gamma_i = \Lambda_i$ whenever $i \sim -i$. We will show that for all $i,j \in I$ the sets σ_{ij} and Γ_i are additive subgroups of R. Indeed, let $\xi,\zeta \in$

$(\sigma,\Gamma)_{ij}$. Then there exist elements x,y in S such that $x^{(1+\delta_{j,-i})}\xi, y^{(1+\delta_{j,-i})}\zeta \in (\sigma',\Gamma')_{ij}$. As (σ',Γ') is a form net of ideals, it follows that $(xy)^{(1+\delta_{j,-i})}\xi, (xy)^{(1+\delta_{j,-i})}\zeta \in (\sigma',\Gamma')_{ij}$. Thus $(xy)^{(1+\delta_{j,-i})}(\xi+\zeta) \in (\sigma',\Gamma')_{ij}$. Therefore $\xi+\zeta \in (\sigma,\Gamma)_{ij}$. The remaining properties of (σ,Γ) as an exact form net of ideals can be deduced in the same way from the corresponding properties of (σ',Γ').

Suppose $h(\nu) \geq (4,3)$ and (σ',Γ') is S-associated with the subgroup H. It's obvious that $(\sigma^e,\Gamma^e)_{ij} \leq (\sigma,\Gamma)_{ij}$ for all possible indices i and j. Thus also $\sigma^e_{i,-i} \leq \sigma_{i,-i}$ for all i. The reverse inclusions are obtained in the following way. Fix some $i \not\sim j$ and $\xi \in (\sigma,\Gamma)_{ij}$. By definition, it means that there exists an element $x \in S$ such that $T_{ij}(x^{(1+\delta_{i,-j})}\xi) \in H$. If $i \neq -j$ then, as $h(\nu) \geq (4,3)$, there exists another index $k \sim j$ such that $k \neq \pm j, \pm i$. Then $T_{jk}(\theta), T_{kj}(1) \in H$ for all $\theta \in S$. Therefore

$$T_{ij}(x\theta\xi) = [[T_{ij}(x\xi), T_{jk}(\theta)], T_{kj}(1)] \in H.$$

Hence, $\xi \in \sigma^e_{ij}$. If $i = -j$ then there exists another index $k \sim i$ such that $k \neq \pm i$. As (σ',Γ') is an exact form net of ideals, it follows by Proposition 3.2.4 that $\lambda^{(\varepsilon(i)-\varepsilon(k))/2}x^2\xi \in \Gamma'_{k,-k}$. By relation (EU6) we have

$$T_{k,-i}(-\lambda^{(\varepsilon(i)-\varepsilon(k))/2}x^2\theta\xi)T_{i,-i}(x^2\theta^2\xi) = [T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}x^2\xi), T_{-k,-i}(-\theta)] \in H$$

for all $\theta \in S$. If $k \sim -i$, then $T_{k,-i}(-\lambda^{(\varepsilon(i)-\varepsilon(k))/2}x^2\theta\xi) \in H$ and therefore also $T_{i,-i}(x^2\theta^2\xi) \in H$. If $k \not\sim -i$ then there exists another index $l \sim k$ such that $l \neq \pm k, \pm i$. By (EU4)

$$T_{k,-i}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}x^2\theta\xi) = [T_{kl}(1), [T_{lk}(1), [T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}x^2\xi), T_{-k,-i}(\theta)]]] \in H.$$

Therefore $T_{i,-i}(x^2\theta^2\xi) \in H$. Hence, $\xi \in \Gamma^e_i$. Summing up, $(\sigma,\Gamma)_{ij} \leq (\sigma,\Gamma)^e_{ij}$ for all $i,j \in I$. Clearly $\sigma_{i,-i} \leq \sigma^e_{i,-i}$ by definition. This completes the proof. □

The following proposition shows that (σ^g,Γ^g) has at least those properties of exact form nets of ideals which follow directly from the elementary relations. Given two subsets $V_1, V_2 \subseteq R$ we will call the additive subgroup $\langle \xi\zeta \mid \xi \in V_1, \zeta \in V_2\rangle$ of R *the Minkowsi product of V_1 and V_2*.

Proposition 3.3.2. *Assume $h(\nu) \geq (4,3)$. Let H be a subgroup of $\mathrm{U}(2n,R,\Lambda)$, g an element of $\mathrm{U}(2n,R,\Lambda)$ and (σ^g,Γ^g) as in (3.48). If $[\nu]_{(R,\Lambda)} \leq (\sigma^g,\Gamma^g)$ then the following inclusions hold:*

(1) $\sigma^g_{ij}\sigma^g_{jk} \leq \sigma^g_{ik}$ *for all $i \neq \pm j$, $j \neq \pm k$.*

(2) $\Gamma^g_i\sigma^g_{-i,k} \leq \sigma^g_{ik}$ *and* $\sigma^g_{i,-k}\Gamma^g_{-k} \leq \sigma^g_{ik}$ *for all $i,k \in I$*

(3) $\xi\zeta - \lambda^{(-1-\varepsilon(i))/2}\overline{\xi\zeta}\lambda^{(1-\varepsilon(i))/2} \in \Gamma^g_i$ *for all $i \neq \pm j$, $\xi \in \sigma^g_{ij}$ and $\zeta \in \sigma^g_{j,-i}$*

(4) $\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\Gamma^g_i\xi \leq \Gamma^g_{-k}$ *for all $i \neq \pm k$ and $\xi \in \sigma^g_{-i,k}$*

(4′) $\xi\Gamma^g_i\lambda^{(\varepsilon(i)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(k))/2} \leq \Gamma^g_k$ *for all $i \neq \pm k$ and $\xi \in \sigma^g_{ki}$,* *where the products are Minkowski products.*

Proof. By the elementary relation (EU2), each σ^g_{ij} and Γ^g_{ij} is an additive subgroup of R.

(1). The inclusion (1) for $i = \pm k$ is trivial by definition of σ^g_{ik}. Assume $i \neq \pm k$. Fix an element $\xi \in \sigma^g_{ij}$ and $\zeta \in \sigma^g_{jk}$. By definition of (σ^g, Γ^g) there exist elements $x_\xi, x_\zeta \in S$ such that ${}^gT_{ij}(x_\xi\theta\xi), {}^gT_{jk}(x_\zeta\zeta) \in H$ for all $\theta \in S$. By relation (EU4) we get

$$ {}^gT_{ik}(x_\xi x_\zeta\theta\xi\zeta) = [{}^gT_{ij}(x_\xi\theta\xi), {}^gT_{jk}(x_\zeta\zeta)] \in H $$

for all $\theta \in S$. Therefore $\xi\zeta \in \sigma^g_{ik}$.

(2) and (4). For $i = \pm k$ the inclusion (2) is trivial. Assume $i \neq \pm k$. We will show that $\Gamma^g_i\sigma^g_{-i,k} \leq \sigma^g_{ik}$ as well as (4). The second inclusion in (2) and (4′) are treated similarly. Pick any $\alpha \in \Gamma^g_i$ and $\xi \in \sigma^g_{-i,k}$. Then there exist elements $x_\alpha, x_\xi \in S$ such that ${}^gT_{i,-i}(x^2_\alpha\alpha), {}^gT_{-i,k}(x_\xi\theta\xi) \in H$ for all $\theta \in S$. By relation (EU6) we have

$$ \begin{aligned} {}^gT_{ik}(x^2_\alpha x_\xi\theta\alpha\xi){}^gT_{-k,k}(-x^2_\xi x^2_\alpha\theta^2\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi) \\ = [{}^gT_{i,-i}(x^2_\alpha\alpha), {}^gT_{-i,k}(x_\xi\theta\xi)] \in H. \end{aligned} \tag{3.49} $$

As $h(\nu) \geq (4,3)$, either there exist another index $l \sim i$ such that $l \neq \pm i, \pm k$, or $\nu(i) = \{\pm i, \pm k\}$. Assume the former. As $1 \in R = \sigma^g_{il} = \sigma^g_{li}$, there exist elements $x_1, x_2 \in S$ such that ${}^gT_{il}(x_1), {}^gT_{li}(x_2) \in H$. By (EU3), (EU4) and (3.49) combined we get

$$ \begin{aligned} {}^gT_{ik}(x^2_\alpha x_\xi x_1x_2\theta\alpha\xi) = [{}^gT_{il}(x_1), [{}^gT_{li}(x_2), {}^gT_{ik}(x^2_\alpha x_\xi\theta\alpha\xi)\times \\ {}^gT_{-k,k}(-x^2_\xi x^2_\alpha\theta^2\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi)]] \in H \end{aligned} $$

for all $\theta \in S$. This shows that $\alpha\xi \in \sigma^g_{ik}$. We can reformulate the last statement as follows: ${}^gT_{ik}(x^2_\alpha x_\xi\theta\alpha\xi) \in H$ whenever θ is a multiple of x_1x_2. Combined with (3.49), this yields the inclusion

$$ {}^gT_{-k,k}(-x^2_\xi x^2_\alpha x_1x_2\theta^2\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi) \in H $$

for all $\theta \in S$. Thus $-\lambda^{(\varepsilon(k)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(i))/2}\alpha\xi \in \Gamma_{-k}$. It's only left to recall that Γ_{-k} is an additive group.

Finally, if $\nu(i) = \{\pm i, \pm k\}$ then $\sigma^g_{ik} = R$ and the corresponding inclusions (2) and (4) are trivial.

(3). Fix indices $i \neq \pm j$, an element $\xi \in \sigma^g_{ij}$ and an element $\zeta \in \sigma^g_{j,-i}$. By definition of (σ^g, Γ^g), there exist elements $x_\xi, x_\zeta \in S$ such that for all $\theta \in S$ the subgroup H contains the elements ${}^gT_{ij}(x^2_\xi\theta\xi)$ and ${}^gT_{j,-i}(x^2_\zeta\theta\zeta)$. By relation (EU5) we get

$$ {}^gT_{i,-i}\big(x^2_\xi x^2_\zeta\theta^2(\xi\zeta - \lambda^{(-1-\varepsilon(i))/2}\overline{\xi\zeta}\lambda^{(1+\varepsilon(i))/2})\big) = [{}^gT_{ij}(x^2_\xi\theta\xi), {}^gT_{j,-i}(x^2_\zeta\theta\zeta)] \in H $$

for all $\theta \in S$. Therefore $\xi\zeta - \lambda^{(-1-\varepsilon(i))/2}\overline{\xi\zeta}\lambda^{(1+\varepsilon(i))/2} \in \Gamma^g_i$. This proves the inclusions (3). $\square$

Lemma 3.3.3. *Assume $h(\nu) \geq (4,3)$ and either $h(\nu) \geq (6,3)$ or $R\Lambda + \Lambda R = R$. Let H be a subgroup of $\mathrm{U}(2n, R, \Lambda)$ and (σ, Γ) the S-closure of an major form net of ideals (σ', Γ') which is S-associated with H. Let g be an element of $\mathrm{EU}(\sigma, \Gamma)$ and (σ^g, Γ^g) as in (3.48). Then*

$$(\sigma^g, \Gamma^g) = (\sigma, \Gamma). \tag{3.50}$$

In particular, (σ^g, Γ^g) is an exact major form net of ideals over (R, Λ).

Proof. We will prove this lemma via induction on the word length $L(g)$ of g in terms of the generators of $\mathrm{EU}(\sigma, \Gamma)$. Proposition 3.3.1 serves as a base of induction, namely it shows that when $L(g) = 0$, i.e. $g = e$, we get the equality $(\sigma, \Gamma) = (\sigma^e, \Gamma^e)$.

Before proving the induction step, we will prove a slightly stronger statement. Suppose that $g \in \mathrm{EU}(\sigma, \Gamma)$ is such that $(\sigma, \Gamma) \leq (\sigma^g, \Gamma^g)$. Fix an element $T_{pq}(\zeta)$ of $\mathrm{EU}(\sigma, \Gamma)$. We will show in this case that $(\sigma^g, \Gamma^g) \leq (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})$. Note that, as $(\sigma, \Gamma) \leq (\sigma^g, \Gamma^g)$, it follows that $\zeta \in (\sigma^g, \Gamma^g)_{pq}$. Pick indices $s \neq r \in I$ and $\xi \in (\sigma^g, \Gamma^g)_{sr}$. There exists an element $x_\xi \in S$ such that the inclusion ${}^gT_{sr}(x_\xi^\kappa \theta^\kappa \xi) \in H$ holds for every $\theta \in R'$, where $\kappa = 1 + \delta_{s,-r}$. For any $x \in S$ we have the following equality

$${}^{gT_{pq}(\zeta)}T_{sr}(x^\kappa \theta^\kappa \xi) = {}^g[T_{pq}(\zeta), T_{sr}(x^\kappa \theta^\kappa \xi)] \cdot {}^gT_{sr}(x^\kappa \theta^\kappa \xi). \tag{3.51}$$

Below we will construct an element x_0 such that after the substituting $x = x_0$ the right-hand side of (3.51) is contained in H for all $\theta \in S$. This will infer that $\xi \in (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})_{sr}$.

Clearly the second term of the right-hand side of (3.51) is contained in H whenever x is a multiple of x_ξ. The first term, which we will denote by $h = h(\theta)$, requires a more detailed investigation. Consider the following exhaustive list of alternatives:

1. $p \neq r, -s$ and $q \neq s, -r$. According to the relation (EU3) we get $h = e \in H$.

2. $p \neq \pm q, s \neq \pm r$ and one of the following options holds:
   (i) $s = q, r \neq \pm p$
   (ii) $r = -q, s \neq \pm p$
   (iii) $s = -p, r \neq \pm q$
   (iv) $-r = -p, s \neq \pm q$.

   In any of these cases, h is equal to a single short elementary unitary matrix. We will consider only the first case. The other cases can be reduced to it using relations (EU1) and (EU2). By relation (EU4) we have

   $$h = T_{pr}(x\theta\zeta\xi).$$

   By Proposition 3.3.2 we get $\zeta\xi \in \sigma^g_{pq}\sigma^g_{qr} \leq \sigma^g_{pr}$. Therefore there exists an element $x_{\zeta\xi} \in S$ such that h is in H whenever x is a multiple of $x_{\zeta\xi}$. Put $x_0 = x_\xi x_{\zeta\xi}$.

3. $p \neq \pm q, s \neq \pm r$ and one of the following options hold:

(i) $s = q, r = -p$

(ii) $s = -p, r = q$

(iii) $s = p, r = -q$

(iv) $s = -q, r = p$.

In any of these four cases h is a single long elementary unitary matrix. We will consider only the first case. By relation (EU5) we get

$$\begin{aligned} h &= T_{p,-p}(\zeta x\theta\xi - \lambda^{(-1-\varepsilon(p))/2}\overline{\zeta x\theta\xi}\lambda^{(1-\varepsilon(p))/2}) \\ &= T_{p,-p}\left(x\theta(\zeta\xi - \lambda^{(-1-\varepsilon(p))/2}\overline{\zeta\xi}\lambda^{(1-\varepsilon(p))/2})\right) \\ &= [T_{pq}(y\zeta), T_{sr}(z\theta\xi)], \end{aligned} \tag{3.52}$$

whenever $x = yz$. As $\zeta \in \sigma^g_{pq}$, there exists an element $x_\zeta \in S$ such that ${}^gT_{pq}(y\zeta) \in H$ whenever y is a multiple of x_ζ in S. Put $x_0 = x_\xi x_\zeta$, $y = x_\zeta$, and $z = x_\xi$. Then the equality (3.52) shows that $h(\theta) \in H$ for all $\theta \in S$.

4. $p \neq \pm q, r = -s$ and either $s = q$ or $s = -p$. Then h is a product of a long and a short elementary unitary matrix. We will consider only the first option, $s = q$. By relation (EU6′) we get

$$h = {}^gT_{p,-q}(x\theta\zeta\xi){}^gT_{p,-p}(x\theta\zeta\xi\lambda^{(\varepsilon(q)-1)/2}\overline{\zeta}\lambda^{(1-\varepsilon(p))/2}). \tag{3.53}$$

By Proposition 3.3.2 it follows that $\zeta\xi \in \sigma^g_{pq}\Gamma^g_q \leq \sigma^g_{p,-q}$. Therefore there exists an element $x_{\zeta\xi} \in S$ such that the first term of the right-hand side of (3.53) is contained in H for all $\theta \in S$ whenever x is a multiple of $x_{\zeta\xi}$. As $\zeta \in \sigma^g_{pq}$ and $\xi \in \Gamma^g_q$, by Proposition 3.3.2 we have

$$\zeta\xi\lambda^{(\varepsilon(q)-1)/2}\overline{\zeta}\lambda^{(1-\varepsilon(p))/2} \in \Gamma^g_p.$$

Therefore there exists an element $x_{\zeta\xi\zeta} \in S$ such that the second term of the right-hand side of (3.53) is contained in H for all $\theta \in S$ whenever x is a multiple of $x_{\zeta\xi\zeta}$. Put $x_0 = x_\xi x_{\zeta\xi} x_{\zeta\xi\zeta}$.

5. $q = -p, s \neq \pm r$ and either $s = -p$ or $r = p$. In both cases h is a product of a long and a short elementary unitary matrix. We will consider only the first option. By relation (EU6) we get

$$\begin{aligned} h &= {}^g[T_{p,-p}(\zeta), T_{-p,r}(x\theta\xi)] = \\ &= {}^gT_{pr}(x\theta\zeta\xi){}^gT_{-r,r}\left(x^2\theta^2(-\lambda^{(\varepsilon(r)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(p))/2}\zeta\xi)\right). \end{aligned} \tag{3.54}$$

By Proposition 3.3.2 we have $\zeta\xi \in \Gamma^g_p\sigma^g_{-p,r} \leq \sigma^g_{pr}$ and

$$-\lambda^{(\varepsilon(r)-1)/2}\overline{\xi}\lambda^{(1+\varepsilon(p))/2}\zeta\xi \in \Gamma^g_{-r}.$$

Therefore there exist elements $x_{\zeta\xi}, x_{\xi\zeta\xi} \in S$ such that the first term of the right-hand side of (3.54) is contained in H for all $\theta \in S$ whenever x is a multiple of $x_{\zeta\xi}$ and the second term whenever x is a multiple of $x_{\xi\zeta\xi}$. Put $x_0 = x_\xi x_{\zeta\xi} x_{\xi\zeta\xi}$.

6. $p \neq \pm q, s \neq \pm r$ and either $s = q, r = p$ or $s = -p, r = q$. As usual, we will only consider the first case. In this situation we can't directly apply any of the relations (EU3)–(EU6). Therefore we have to decompose the elementary unitary matrix $T_{qp}(x\theta\xi)$ into a product of other elementary unitary matrices, for which we can compute the commutator with $T_{pq}(\zeta)$. This is precisely the place in our proof where the condition, that either $h(\nu) \geq (6,3)$ or $R\Lambda + \Lambda R = R$ is used.

   Suppose there exists an index $l \sim p$ such that $l \neq \pm p, \pm q$. Then for any elements $x, y, z \in S$ such that $x = yz$ and any $\theta \in S$ we have by relation (EU4)

$$\begin{aligned} {}^{gT_{pq}(\zeta)}T_{qp}(x\theta\xi) &= {}^{g}[{}^{T_{pq}(\zeta)}T_{ql}(y\theta\xi), {}^{T_{pq}(\zeta)}T_{lp}(z)] \\ &= {}^{g}[[T_{pq}(\zeta), T_{ql}(y\theta\xi)]T_{ql}(y\theta\xi), [T_{pq}(\zeta), T_{lp}(z)]T_{lp}(z)] \\ &= [{}^{g}T_{pl}(y\theta\zeta\xi) \cdot {}^{g}T_{ql}(y\theta\xi), {}^{g}T_{lq}(-z\zeta) \cdot {}^{g}T_{lp}(z)]. \end{aligned} \tag{3.55}$$

   Recall that $\zeta\xi \in R = \sigma^g_{pl}$, $1 \in R = \sigma^g_{lp}$. By Proposition 3.3.2, $\xi \in \sigma^g_{qp} \leq \sigma^g_{qp} R = \sigma^g_{qp}\sigma^g_{pl} \leq \sigma^g_{ql}$ and $\zeta \in \sigma^g_{pq} \leq R\sigma^g_{pq} = \sigma^g_{lp}\sigma^g_{pq} \leq \sigma^g_{lq}$. Therefore there exist elements $y_0, z_0 \in S$ such that the right-hand side of (3.55) is contained in H for all $\theta \in S$ whenever y is a multiple of y_0 and z is a multiple of z_0. In this case we can put $x_0 = y_0 z_0$.

   The only case when we fail to choose an index l as above is when the equivalence class of p is precisely $\{\pm p, \pm q\}$. By assumption, we have $R\Lambda + \Lambda R = R$. As λ is invertible, it follows that $R\Lambda_{-p} + \Lambda_q R = R$. In particular, there exists a natural number m and elements $\eta_i, \eta'_i \in R$ and $\alpha_i \in \Lambda_{-p}, \alpha'_i \in \Lambda_q$ for all $1 \leq i \leq m$ such that $\xi = \sum_{i=1}^m (\eta_i\alpha_i + \alpha'_i\eta'_i)$. For the sake of clarity, we will assume $m = 1$, in other words $\xi = \eta\alpha + \alpha'\eta'$ for some $\alpha \in \Lambda_{-p}$, $\alpha' \in \Lambda_q$ and $\eta, \eta' \in R$. Assuming $x = yz$, by relations (EU2), (EU6) and (EU6′) we have

$$\begin{aligned} {}^{gT_{pq}(\zeta)}T_{qp}(x\theta\xi) &= {}^{gT_{pq}(\zeta)}T_{qp}(x\theta\eta\alpha) \cdot {}^{gT_{pq}(\zeta)}T_{qp}(x\theta\alpha'\eta') \\ &= {}^{gT_{pq}(\zeta)}\big([T_{q,-p}(y\theta\eta), T_{-p,p}(z\alpha)]\times \\ &\qquad T_{q,-q}(-y^2 z\theta^2\eta\alpha\lambda^{(-1-\varepsilon(p))/2}\overline{\eta}\lambda^{(1-\varepsilon(q))/2})\times \\ &\qquad [T_{q,-q}(y\alpha'), T_{-q,p}(z\theta\eta')]\times \\ &\qquad T_{-p,p}(z^2 y\theta^2\lambda^{(\varepsilon(p)-1)/2}\overline{\eta'}\lambda^{(1+\varepsilon(q))/2}\alpha'\eta')\big). \end{aligned} \tag{3.56}$$

   Observe that by cases (3) and (4) we have $\eta \in R = \sigma^g_{q,-p} \leq \sigma^{gT_{pq}(\zeta)}_{q,-p}$, $\eta' \in R = \sigma^g_{-q,p} = \sigma^{gT_{pq}(\zeta)}_{-q,p}$ $\alpha \in \Lambda_{-p} = \Gamma^g_{-p} \leq \Gamma^{gT_{pq}(\zeta)}_{-p}$ and $\alpha' \in \Lambda_q = \Gamma^g_q \leq \Gamma^{gT_{pq}(\zeta)}_q$. Finally, by Proposition 3.3.2 we get

$$\begin{aligned} \lambda^{(\varepsilon(p)-1)/2}\overline{\eta'}\lambda^{(1+\varepsilon(q))/2}\alpha'\eta' &\in \Gamma^g_{-p} \leq \Gamma^{gT_{pq}(\zeta)}_{-p} \\ \eta\alpha\lambda^{(-1-\varepsilon(p))/2}\overline{\eta}\lambda^{(1-\varepsilon(q))/2} &\in \Gamma^g_q \leq \Gamma^{gT_{pq}(\zeta)}_q. \end{aligned}$$

   Summing up, there exist elements $y_0, z_0 \in S$ such that the right-hand side of (3.56) is contained in H for all $\theta \in S$ whenever y is a multiple of y_0 and z is a multiple of z_0. Put $x = y_0 z_0$. The case that $m > 1$ is treated similarly with the additional help of relation (EU2).

7. The last possible option is that $q = s = -p$, and $r = p$. In this case we again have to first decompose $T_{-p,p}(x\theta\xi)$. As $h(\nu) \geq (4,3)$, there exists another index $l \sim p$ such that $l \neq \pm p$. By relation (EU6) we get

$$\begin{aligned} {}^{gT_{p,-p}(\zeta)}T_{-p,p}(x\theta\xi) = {}^{gT_{p,-p}(\zeta)}[T_{-l,l}(-y\theta\lambda^{(\varepsilon(l)-\varepsilon(p))/2}\xi), T_{lp}(z)] \times \\ {}^{gT_{p,-p}(\zeta)}T_{-l,p}(yz\theta\lambda^{(\varepsilon(l)-\varepsilon(p))/2}\xi), \end{aligned} \tag{3.57}$$

whenever $x = yz^2$. As $\xi \in \Gamma^g_{-p}$ and $1 \in \sigma^g_{pl}$, by Proposition 3.3.2 together with cases (1) and (5) above we get that $\lambda^{(\varepsilon(l)-\varepsilon(p))/2}\xi \in \Gamma^g_{-l} \leq \Gamma^{gT_{p,-p}(\zeta)}_{-p}$, $1 \in \sigma^g_{lp} \leq \sigma^{gT_{p,-p}(\zeta)}$ and $\lambda^{(\varepsilon(l)-\varepsilon(p))/2}\xi \in \Gamma^g_{-l}\sigma^g_{lp} \leq \sigma^g_{-l,p} \leq \sigma^{gT_{p,-p}(\zeta)}_{-l,p}$. Therefore we can choose elements y and z in S such that for all $\theta \in S$ the right-hand side of (3.57) is contained in H. Put $x = yz^2$.

The induction step looks as follows. Assume that for all elements $g \in \mathrm{EU}(\sigma,\Gamma)$ such that $L(g) \leq L_0$, the equality (2.5) holds. Let $T_{pq}(\zeta)$ be an elementary unitary matrix in $\mathrm{EU}(\sigma,\Gamma)$ such that $L(g \cdot T_{pq}(\zeta)) = L_0 + 1$. Then, as we have proved above, $(\sigma^g, \Gamma^g) \leq (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})$, in particular $(\sigma,\Gamma) \leq (\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)})$. For the same reason

$$(\sigma^{gT_{pq}(\zeta)}, \Gamma^{gT_{pq}(\zeta)}) \leq (\sigma^{gT_{pq}(\zeta)T_{pq}(-\zeta)}, \Gamma^{gT_{pq}(\zeta)T_{pq}(-\zeta)}) = (\sigma^g, \Gamma^g).$$

Summing up, by induction we get the required equality (3.50) for all $g \in \mathrm{EU}(\sigma,\Gamma)$. □

Corollary 3.3.4. *Assume $h(\nu) \geq (4,3)$ and either $h(\nu) \geq (6,3)$ or $R\Lambda + \Lambda R = R$. Let H be a subgroup of $\mathrm{U}(2n,R,\Lambda)$ and (σ,Γ) the S-closure of an exact major form net of ideals (σ',Γ') which is S-associated with H. Then for any finite family $\{T_{s_i,r_i}(\xi_i)\}_{i\in L}$ of (σ,Γ)-elementary unitary matrices and any finite family $\{g_i\}_{i\in K}$ of elements of $\mathrm{EU}(\sigma,\Gamma)$ there exists an element $x \in S$ such that*

$${}^{g_i}T_{s_j,r_j}((x\theta)^{(1+\delta_{s_j,-r_j})}\xi_j) \in H$$

for all $i \in K, j \in J$ and $\theta \in S$.

Remark 3.3.5. Assume $h(\nu) \geq (4,3)$, $R = R'$ and $\Lambda = \Lambda'$. Fix a subgroup $H \geq \mathrm{EU}(\nu,R,\Lambda)$ of $\mathrm{U}(2n,R,\Lambda)$. Let (σ,Γ) denote the form net of ideals associated with H (cf. Lemma 3.2.9). It is clear that any form net of ideals S-associated with the subgroup H coincides with (σ,Γ). Further the S-closure of any form net of ideals coincides with the original form net of ideals. Finally, for any $g \in \mathrm{EU}(\sigma,\Gamma) \leq H$ the net (σ^g,Γ^g) by definition coincides with (σ,Γ). In this case Propositions 3.3.1 and 3.3.2, Lemma 3.3.3 and Corollary 3.3.4 are redundant.

3.4 Extraction of elementary matrices

In this section we extract elementary unitary matrices first using elements of small parabolic subgroups and then using certain root elements. We generally follow the scheme of proof of the analogous results for the classical symplectic group in Section

2.2. Throughout this section we fix a standard setting $((R,\Lambda),(R',\Lambda'),S)$, a unitary equivalence relation ν on the index set I of size $2n$, a subgroup H of $\mathrm{U}(2n,R,\Lambda)$ and an exact major form net (σ',Γ') which is S-associated with H. Let (σ,Γ) denote the S-closure of (σ',Γ') in (R,Λ).

The results of this section rely on the conclusion of Lemma 3.3.3. We can ensure that the conclusion of Lemma 3.3.3 holds either by satisfying the hypotheses of Lemma 3.3.3 itself hold or using Remark 3.3.5. We will say that *ν is good for the standard setting above* if at least one of the following three options holds:

1. $h(\nu) \geq (6,3)$
2. $h(\nu) \geq (4,3)$ and $R\Lambda + \Lambda R = R$
3. $h(\nu) \geq (4,3)$ and $(R,\Lambda) = (R',\Lambda')$.

Lemma 3.4.1. *Let a be an element of $\mathrm{U}(2n,R,\Lambda)$ such that for some index $p \in I$ the following conditions are satisfied*

1. $a_{pp} = a_{-p,-p} = 1$
2. $a_{ij} = \delta_{ij}$ *whenever $i \neq -p$ and $j \neq p$.*

Then

$$a = \left(\prod_{j>0, j\neq \pm p} T_{-p,j}(a_{-p,j}) T_{-p,-j}(a_{-p,-j}) \right) T_{-p,p}(S_{-p,p}(a)). \tag{3.58}$$

Further, suppose ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$ and there exists an element $g \in \mathrm{EU}(\sigma,\Gamma)$ such that ${}^g a \in H$. Then $a \in \mathrm{EU}(\sigma,\Gamma)$.

Proof. As $a \in \mathrm{EU}(2n,R,\Lambda)$, it follows by Lemma 3.1.2 that $S_{-p,p}(a) \in \Lambda_i$. Therefore all the elementary unitary matrices on the right-hand side of (3.58) are defined. The equality (3.58) can be verified by a direct matrix calculation.

Assume ${}^g a \in H$ for some $g \in \mathrm{EU}(\sigma,\Gamma)$ and ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Either by Lemma 3.3.3 or by Remark 3.3.5 we have $(\sigma^g,\Gamma^g) = (\sigma,\Gamma) \geq [\nu]_{(R,\Lambda)}$.

Fix some $j \neq \pm p$. We will show that $T_{-p,j}(a_{-p,j}) \in \sigma_{-p,j}$. If $j \sim -p$ then, as $[\nu]_{(R,\Lambda)} \leq (\sigma,\Gamma)$, the inclusion $T_{-p,j}(a_{-p,j}) \in \sigma_{-p,j}$ is trivial. Assume, $j \nsim -p$. Pick any index $k \sim j$ such that $k \neq \pm j, \pm p$. By Corollary 3.3.4 there exists an element $x_1 \in S$ such that ${}^g T_{jk}(x_1\theta) \in H$ for all $\theta \in H$. By the elementary relations (EU1), (EU3) and (EU4) we have

$${}^g T_{-p,k}(x\theta a_{-p,j}) {}^g T_{-p,-j}(-\lambda^{(\varepsilon(j)-\varepsilon(k))/2} x a_{-p,-k}) = {}^g[a, T_{jk}(x\theta)] \in H, \tag{3.59}$$

whenever x is a multiple of x_1.

Let $b(\theta)$ denote the left-hand side of (3.59). By choice of x_1, $b(\theta) \in H$ for all $\theta \in H$ whenever x is a multiple of x_1. If $j \sim p$ then there exists an element $x_2 \in S$ such that the second term of $b(\theta)$ is contained in H whenever x is a multiple of x_2. Thus, the first term thereof is also contained in H for all $\theta \in S$ whenever x is a multiple of x_1x_2. In this case we get $a_{-p,j} \in \sigma_{-p,j}$.

Assume $j \not\sim \pm p$. If the equivalence class of j is non-self-conjugate then there exists yet another index $l \sim j \sim k$ such that $l \neq \pm j, \pm k, \pm p$. By Corollary 3.3.4 we can choose an element $x_3 \in S$ such that ${}^gT_{kl}(x_3), {}^gT_{lj}(x_3) \in H$. Then substituting $x = x_1$ into $b(\theta)$ by (EU4) we get

$$T_{-p,j}(x_1x_3^2\theta a_{-p,j}) = [[b(\theta), {}^gT_{kl}(x_3), {}^gT_{lj}(x_3)] \in H$$

for all $\theta \in S$. Therefore $a_{-p,j} \in \sigma_{-p,j}$. Finally, if the equivalence class of j is self-conjugate then there exists an element $x_4 \in S$ such that ${}^gT_{k,-j}(x_4), {}^gT_{-j,k}(x_4)$, and ${}^gT_{kj}(x_4) \in H$. Again, substituting $x = x_1$ into $b(\theta)$ and using (EU4) we get

$$T_{-p,j}(x_1x_4^3\theta a_{-p,j}) = [[[b(\theta), {}^gT_{k,-j}(x_4)], {}^gT_{-j,k}(x_4)], {}^gT_{kj}(x_4)] \in H$$

for all $\theta \in S$. Summing up, we've proved that $a_{-p,j} \in \sigma_{-p,j}$ for all $j \neq \pm p$. To prove that $a \in \mathrm{EU}(\sigma, \Gamma)$ it only remains to show that $T_{-p,p}(S_{-p,p}(a)) \in \mathrm{EU}(\sigma, \Gamma)$. If $-p \sim p$ then $\Gamma_{-p} = \Lambda_{-p}$ and the required inclusion is trivial. Assume, $p \not\sim -p$. Set

$$g_1 = g \prod_{j>0, j\neq \pm p} T_{-p,j}(a_{-p,j})T_{-p,-j}(a_{-p,-j}).$$

Then $g_1T_{-p,p}(S_{-p,p}(a))g^{-1} = {}^ga \in H$ and $g_1, g^{-1} \in \mathrm{EU}(\sigma, \Gamma)$. As $p \not\sim -p$, we can choose two more indices q and t such that (p, q, t) is an A-type base triple. Pick an element $y_1 \in S$ such that ${}^gT_{pq}(y_1\theta), {}^{g_1}T_{pq}(y_1\theta) \in H$ for all $\theta \in S$. Therefore

$$\begin{aligned} {}^{g_1}T_{-p,q}(y\theta S_{-p,p}(a)) \cdot {}^{g_1}T_{-q,q}(-y^2\theta^2\lambda^{(\varepsilon(q)-\varepsilon(p))/2}S_{-p,p}(a)) \cdot {}^{g_1}T_{pq}(y\theta) \\ = {}^{g_1}[T_{-p,p}(S_{-p,p}(a)), T_{pq}(y\theta)] \cdot {}^{g_1}T_{pq}(y\theta) \\ = \left(g_1T_{-p,p}(S_{-p,p}(a))g^{-1}\right)\left(gT_{pq}(y\theta)g^{-1}\right)\left(gT_{-p,p}(-S_{-p,p}(a))g_1^{-1}\right). \end{aligned} \tag{3.60}$$

The right-hand side and the third term of the left-hand side of (3.60) are contained in H whenever y is a multiple of y_1 in S. Therefore

$${}^{g_1}T_{-p,q}(y\theta S_{-p,p}(a)) \cdot {}^{g_1}T_{-q,q}(-y^2\theta^2\lambda^{(\varepsilon(q)-\varepsilon(p))/2}S_{-p,p}(a)) \in H \tag{3.61}$$

for all $\theta \in S$ whenever y is a multiple of y_1. Pick $y_2 \in S$ such that ${}^{g_1}T_{-p,-t}(y_2)$ and ${}^{g_1}T_{-t,-p}(y_2) \in H$. By relations (EU3) and (EU4) we get

$$\begin{aligned} {}^{g_1}T_{-p,q}(yy_2^2\theta S_{-p,p}(a)) = [{}^{g_1}T_{-p,-t}(y_2), [{}^{g_1}T_{-t,-p}(y_2), {}^{g_1}T_{-p,q}(y\theta S_{-p,p}(a))\cdot \\ {}^{g_1}T_{-q,q}(-y^2\theta^2\lambda^{(\varepsilon(q)-\varepsilon(p))/2}S_{-p,p}(a))]]. \end{aligned} \tag{3.62}$$

By the choice of y_2 together with (3.62) we get that ${}^{g_1}T_{-p,q}(y\theta S_{-p,p}(a)) \in H$ for all $\theta \in S$ whenever y is a multiple of $y_1y_2^2$. Combining this result with (3.61) we get that ${}^{g_1}T_{-q,q}(-y^2\theta^2\lambda^{(\varepsilon(q)-\varepsilon(p))/2}S_{-p,p}(a)) \in H$ for all $\theta \in S$ whenever y is a multiple of $y_1y_2^2$. Thus, $-\lambda^{(\varepsilon(q)-\varepsilon(p))/2}S_{-p,p}(a) \in \Gamma_{-q}$. Finally, by Proposition 3.2.4 we have $S_{-p,p}(a) \in \Gamma_{-p}$. This completes the proof. □

Lemma 3.4.2. *Assume that ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Let (p,q) be an A-type base pair and a an element in $\mathrm{U}(2n,R,\Lambda)$ such that either $a_{*,-p}=e_{*,-p}$ or $a_{p*}=e_{p*}$. Suppose, there exist elements $g_1,g_2\in \mathrm{EU}(\sigma,\Gamma)$ such that $g_1ag_2\in H$. Then $a_{-q,j}\in\sigma_{-q,j}$ for all $j\neq p$. If additionally $a\in\mathrm{U}(\sigma)$ then also $S_{-q,q}(a)\in\Gamma_{-q}$.*

Proof. Note that $a_{*,-p}=a'_{*,-p}=e_{*,-p}$ and $a_{p*}=a'_{p*}=e_{p*}$. Choose using Corollary 3.3.4 an element $x\in S$ such that ${}^{g_2}T_{-p,-q}(x)\in H$ and consider the matrix

$$\begin{aligned}b=a^{-1}T_{-p,-q}(x)a&=e+a'_{*,-p}xa_{-q,*}-a'_{*q}\lambda^{(\varepsilon(p)-\varepsilon(q))/2}xa_{p*}\\&=e+e_{*,-p}xa_{-q,*}-a'_{*q}\lambda^{(\varepsilon(p)-\varepsilon(q))/2}xe_{p*}.\end{aligned}$$

It's easy to see that $b_{pp}=b_{-p,-p}=1$, $b_{ij}=\delta_{ij}$ whenever $i\neq -p$ and $j\neq p$ and $b_{-p,j}=xa_{-q,j}$ whenever $j\neq\pm p$. Clearly

$${}^{g_2^{-1}}b=\left(g_2^{-1}a^{-1}g_1^{-1}\right)\left(g_1T_{-p,-q}(x)g_1^{-1}\right)(g_1ag_2)\in H.$$

By Lemma 3.4.1 we get $a_{-q,j}=x^{-1}b_{-p,j}\in\sigma_{-p,j}=\sigma_{-q,j}$ for any $j\neq\pm p$ and $S_{-p,p}(b)\in\Gamma_{-p}$.

Assume that $a\in\mathrm{U}(\sigma)$. Note that $a_{pp}=a'_{-p,-p}=1$ and $S_{p,-p}(a)=0$. By Corollary 3.2.6 we have

$$\begin{aligned}S_{-p,p}(b)&\equiv a'_{-p,-p}xS_{-q,q}(a)\lambda^{(\varepsilon(-q)-1)/2}\overline{x}\lambda^{(1-\varepsilon(-p))/2}a_{pp}\\&\quad+a'_{-p,q}\lambda^{(\varepsilon(-q)-1)/2}\overline{x}\lambda^{(1-\varepsilon(-p))/2}S_{p,-p}(a)xa_{-q,p}\\&=xS_{-q,q}(a)\lambda^{(\varepsilon(-q)-1)/2}\overline{x}\lambda^{(1-\varepsilon(-p))/2}\mod\Gamma_{-p}.\end{aligned}$$

Therefore, $xS_{-q,q}(a)\lambda^{(\varepsilon(-q)-1)/2}\overline{x}\lambda^{(1-\varepsilon(-p))/2}\in\Gamma_{-p}$. As x is invertible, by property $(\Gamma2')$ we have also $S_{-q,q}(a)\in\Gamma_{-q}$. □

Lemma 3.4.3. *Assume that $h(\nu)\geq(4,4)$ and that ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Let (p,q) be an* A*-type base pair and a an element in $\mathrm{U}(2n,R,\Lambda)$ such that $a_{ij}=\delta_{ij}$ whenever $i\neq -p,-q$ and $j\neq p,q$. Suppose there exists an element $g\in\mathrm{EU}(\sigma,\Gamma)$ such that ${}^ga\in H$. Then $a_{kp}\in\sigma_{kp}$ for all $k\neq -p,-q$. If additionally $a\in\mathrm{U}(\sigma)$ then also $S_{-p,p}(a)\in\Gamma_{-p}$.*

Proof. First, we will prove the inclusion $a_{kp}\in\sigma_{kp}$ for all $k\neq -p,-q$. If $k\sim p$ then the inclusion $a_{kp}\in\sigma_{kp}$ is trivial. Assume $k\nsim p$. There exists another index $h\sim k$ such that $h\neq\pm k,\pm p,\pm q$. Using Corollary 3.3.4 pick an element $x\in S$ such that ${}^gT_{hk}(x)\in H$ and consider the matrix

$$b=a^{-1}T_{hk}(x)a=e+a'_{*h}xa_{k*}-a'_{*,-k}\lambda^{(\varepsilon(k)-1)/2}\overline{x}\lambda^{(1-\varepsilon(h))/2}a_{-h,*}.$$

By choice of x it's clear that ${}^gb\in H$. We will show that there exists an index $l\sim k$ such that $l\neq -p,\pm h$ and $b_{*l}=e_{*l}$. In this case by Lemma 3.4.2 applied to the matrix b, the short elementary unitary matrix $T_{hk}(x)$ and an A-type base pair (l,h), the inclusion $a_{kp}=b_{hp}\in\sigma_{kp}$ holds. Now we will show the existence of an index l as above. If the equivalence class of h is self-conjugate then we can simply set $l = -k$. If the equivalence class of h is non-self-conjugate then there are two more possibilities. If $h \sim -p$ then we can set $l = -q$. Finally, if $h \nsim -p$ then the equivalence class of h contains as least 4 elements and doesn't intersect with $\pm\nu(p)$. Therefore we may take for l any index in $\nu(k) \setminus \{k, h\}$. In any of the above cases the index l clearly satisfies the required properties. Therefore, the first part of this lemma is proved.

Assume that $a \in \mathrm{U}(\sigma)$. If $p \sim -p$ then the inclusion $S_{-p,p}(a) \in \Gamma_{-p}$ is trivial. Assume, $p \nsim -p$. There exists another index $t \in I$ such that (p, q, t) is an A-type base triple. Let g_1 denote the matrix $T_{-p,-t}(-a_{-p,-t})T_{-q,-t}(-a_{-q,-t})$. Clearly, $g_1 \in \mathrm{EU}(\sigma, \Gamma)$. Consider the matrix $c = g_1 a$. It's clear that $c_{*,-t} = e_{*,-t}$, $c \in \mathrm{U}(\sigma)$ and $gg_1^{-1}cg^{-1} \in H$. By Lemma 3.4.2 we get the inclusion $S_{-p,p}(c) \in \Gamma_{-p}$. Finally, by Corollary 3.2.6 we get

$$S_{-p,p}(c) \equiv S_{-p,p}(a) + a_{-p,-t}S_{-t,t}(a)\lambda^{(\varepsilon(-t)-1)/2}\overline{a_{-p,-t}}\lambda^{(1-\varepsilon(-p))/2} \mod \Gamma_{-p}. \tag{3.63}$$

It is clear that $S_{-t,t}(a) = 0$. By (3.63) it follows that $S_{-p,p}(a) \in \Gamma_{-p}$. □

Lemma 3.4.4. *Assume that $h(\nu) \geq (4, 4)$ and that ν is good for the standard setting $((R, \Lambda), (R', \Lambda'), S)$. Let p be an index in I such that $p \sim -p$ and let a be a matrix in $\mathrm{U}(2n, R, \Lambda)$ such that $a_{ij} = \delta_{ij}$ whenever $i \neq \pm p$ and $j \neq \pm p$. Suppose, there exists an element $g \in \mathrm{EU}(\sigma, \Gamma)$ such that ${}^g a \in H$. Then $a_{kp} \in \sigma_{kp}$ for all $k \in I$.*

Proof. If $k \sim p$ then the inclusion $a_{kp} \in \sigma_{kp}$ is trivial. Assume $k \nsim p$. There exists an index $h \sim k$ such that $h \neq \pm k, \pm p$. Using Corollary 3.3.4 pick an element $x \in S$ such that ${}^g T_{hk}(x) \in H$ and consider the matrix

$$b = a^{-1}T_{hk}(x)a = e + a'_{*h}xa_{k*} - a'_{*,-k}\lambda^{(\varepsilon(k)-1)/2}\overline{x}\lambda^{(1-\varepsilon(h))/2}a_{-h,*}.$$

Clearly ${}^g b \in H$. Pick an index $q \in I$ such that $(-q, p)$ is an A-type base pair. It's easy to see that $a_{kq} = a_{-h,q} = 0$. Therefore $b_{*q} = e_{*q}$. By Lemma 3.4.2 applied to the matrix b and the A-type base pair $(-q, p)$ we get $b_{-p,j} \in \sigma_{-p,j}$ for all $j \neq -q$. In particular, $b_{-p,-h} \in \sigma_{-p,-h} = \overline{\sigma_{kp}}$. Observe that

$$b_{-p,-h} = -a'_{-p,-k}\lambda^{(\varepsilon(k)-1)/2}x\lambda^{(1-\varepsilon(h))/2}a_{-h,-h}$$

and $a_{-h,-h} = 1$. By Lemma 3.1.2, $a'_{-p,-k} = \lambda^{(\varepsilon(p)-1)/2}\overline{a_{kp}}\lambda^{(1-\varepsilon(k))/2}$. Summing up, we get

$$-\lambda^{(\varepsilon(p)-1)/2}\overline{a_{kp}}x\lambda^{(1-\varepsilon(h))/2} = b_{-p,-h} \in \overline{\sigma_{kp}}.$$

Clearly, this yields the inclusion $a_{kp} \in \sigma_{kp}$. □

Lemma 3.4.5. *Assume that $h(\nu) \geq (4, 4)$ and that ν is good for the standard setting $((R, \Lambda), (R', \Lambda'), S)$. Let p, q and h be three indices in I such that either (p, q, h) is an A-type base triple or (p, h) is a C-type base pair and $q = -p$. Let a be an element of $\mathrm{U}(2n, R, \Lambda)$ such that $g_1 a g_2 \in H$ for some elements $g_1, g_2 \in \mathrm{EU}(\sigma, \Gamma)$. Let $T_{sr}(\xi)$ be an elementary unitary matrix (long or short) such that ${}^{g_2^{-1}}T_{sr}(\xi) \in H$ and let $b = aT_{sr}(\xi)a^{-1}$. If $s \neq \pm r$, assume additionally that $a_{p,-r} = a_{q,-r} = 0$. Let $\alpha, \beta \in R$ be a* *solution of the equation* $\alpha a_{ps}+\beta a_{qs}=0$*. Then* $b_{ih}\alpha, b_{ih}\beta\in\sigma_{ih}$ *for all* $i\neq -p,-q$*. If additionally* $b\in \mathrm{U}(\sigma)$ *then also*

$$\begin{aligned}&\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})\alpha\in\Gamma_{-p}\\&\lambda^{(\varepsilon(q)-1)/2}\overline{\beta}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})\beta\in\Gamma_{-q}.\end{aligned}\tag{3.64}$$

Proof. In the case that the elementary unitary $T_{sr}(\xi)$ is short, the additional condition $a_{p,-r}=a_{q,-r}=0$ guarantees the following property:

$$\begin{aligned}b'_{p*}&=e_{p*}-a_{ps}\xi a'_{r*}\\b'_{q*}&=e_{q*}-a_{qs}\xi a'_{r*}\\b_{*,-p}&=e_{*,-p}-a_{*,-r}\lambda^{(\varepsilon(r)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(s))/2}a'_{-s,-p}\\b_{*,-q}&=e_{*,-q}-a_{*,-r}\lambda^{(\varepsilon(r)-1)/2}\overline{\xi}\lambda^{(1-\varepsilon(s))/2}a'_{-s,-q}.\end{aligned}\tag{3.65}$$

Using Lemma 3.1.2 we can rewrite the last two equalities in (3.65) as follows

$$\begin{aligned}b_{*,-p}&=e_{*,-p}-a_{*,-r}\lambda^{(\varepsilon(r)-1)/2}\overline{\xi}\overline{a_{ps}}\lambda^{(1-\varepsilon(p))/2}\\b_{*,-q}&=e_{*,-q}-a_{*,-r}\lambda^{(\varepsilon(r)-1)/2}\overline{\xi}\overline{a_{qs}}\lambda^{(1-\varepsilon(q))/2}.\end{aligned}\tag{3.66}$$

Combining the condition $\alpha a_{ps}+\beta a_{qs}=0$ and the first two equalities in (3.65), we get the following property

$$\alpha b'_{p*}+\beta b'_{q*}=\alpha e_{p*}+\beta e_{q*}.\tag{3.67}$$

Similarly (3.66) yields

$$b_{*,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}+b_{*,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta}=e_{*,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}+e_{*,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta}.\tag{3.68}$$

Pick using Corollary 3.3.4 an element $x\in S$ such that ${}^{g_1}T_{hp}(x\alpha), {}^{g_1}T_{hq}(x\beta)\in H$. Consider the matrix

$$c=bT_{hp}(x\alpha)T_{hq}(x\beta)b^{-1}.$$

Note that

$${}^{g_1}b=(g_1ag_2)(g_2^{-1}T_{sr}(\xi)g_2)(g_2^{-1}a^{-1}g_1^{-1})\in H.$$

Therefore ${}^{g_1}c\in H$. It's easy to see that c has the shape described in Lemma 3.4.3, namely $c_{ij}=\delta_{ij}$ whenever $i\neq -p,-q$ and $j\neq p,q$. The first equality below is straightforward and the second one is due to (3.68) and (3.67):

$$\begin{aligned}c&=e+b_{*h}x\alpha b'_{p*}-b_{*,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}x\lambda^{(1-\varepsilon(h))/2}b'_{-h,*}\\&\quad+b_{*h}x\beta b'_{q*}-b_{*,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta}x\lambda^{(1-\varepsilon(h))/2}b'_{-h,*}\\&=e+b_{*h}x\alpha e_{p*}-e_{*,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}x\lambda^{(1-\varepsilon(h))/2}b'_{-h,*}\\&\quad+b_{*h}x\beta e_{q*}-e_{*,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta}x\lambda^{(1-\varepsilon(h))/2}b'_{-h,*}.\end{aligned}$$

By Lemma 3.4.3 applied to the matrix c we get the inclusions $c_{ip}, c_{iq}\in\sigma_{ip}$ for all $i\neq -p,-q$. Note that $c_{ip}=b_{ih}\alpha, c_{iq}=b_{ih}\beta$ whenever $i\neq -p,-q$. This proves the first conclusion of the current lemma.

If $p \sim -p$ then inclusions (3.64) are trivial. Assume $p \nsim -p$ and $b \in \mathrm{U}(\sigma)$. Then $c \in \mathrm{U}(\sigma)$. By Lemma 3.4.3 we get the inclusions $S_{-p,p}(c) \in \Gamma_{-p}, S_{-q,q}(c) \in \Gamma_{-q}$. We will show that these inclusions yield the inclusions (3.64). Expand $S_{-p,p}(c)$ using Corollary 3.2.6 in the following way

$$\Gamma_{-p} \ni S_{-p,p}(c) = X + Y, \tag{3.69}$$

where

$$\begin{aligned} X = b_{-p,h}(x\alpha)S_{p,-p}(b^{-1})\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}b'_{-h,p} + \\ b_{-p,h}(x\beta)S_{q,-q}(b^{-1})\lambda^{(\varepsilon(q)-1)/2}\overline{(x\beta)}\lambda^{(1-\varepsilon(h))/2}b'_{-h,p} \end{aligned} \tag{3.70}$$

and

$$\begin{aligned} Y = b_{-p,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})(x\alpha)b'_{pp} + \\ b_{-p,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{(x\beta)}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})(x\beta)b'_{qp}. \end{aligned} \tag{3.71}$$

We will show that the expression X is trivial modulo Γ_{-p} and the expression Y is congruent to

$$\lambda^{(\varepsilon(p)-1)/2}\overline{x\alpha}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})(x\alpha)$$

modulo Γ_{-p}. Assuming this for the moment and keeping in mind that $\overline{x} = x$ and that x is central we get the required inclusion

$$\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})\alpha \in \Gamma_{-p}.$$

The second inclusion in (3.64) can be treated similarly.

We start by considering the expression X. In the following computation the first equality is by definition of $S_{q,-q}(\cdot)$, the second by (3.68) and (3.67) and the third again by definition of $S_{p,-p}(\cdot)$:

$$\begin{aligned} \beta S_{q,-q}(b^{-1})\lambda^{(\varepsilon(q)-1)/2}\overline{\beta} = \\ = \sum_{j>0} \beta(b'_{qj} - \delta_{qj})(b_{j,-q} - \delta_{j,-q})\lambda^{(\varepsilon(q)-1)/2}\overline{\beta} \\ + (-1)^{(\varepsilon(q)-1)/2}\beta b_{q,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta} \\ = \sum_{j>0} \alpha(b'_{pj} - \delta_{pj})(b_{j,-p} - \delta_{j,-p})\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha} \\ + (-1)^{(\varepsilon(q)-1)/2}\beta b_{q,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta} \\ = \alpha S_{p,-p}(b^{-1})\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha} - (-1)^{(\varepsilon(p)-1)/2}\alpha b_{p,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha} \\ + (-1)^{(\varepsilon(q)-1)/2}\beta b_{q,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta}. \end{aligned} \tag{3.72}$$

If $s \neq \pm r$ then the last two summands on the right-hand side of (3.72) are equal to zero, as $b_{p,-p} = b_{q,-q} = 0$. If $r = -s$ then it's easy to see the following:

$$\begin{aligned} (-1)^{(\varepsilon(q)-1)/2}\beta b_{q,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta} &= (-1)^{(\varepsilon(q)-1)/2}\beta a_{qs}\xi a'_{-s,-q}\lambda^{(\varepsilon(q)-1)/2}\overline{\beta} \\ &= (-1)^{(\varepsilon(q)-1)/2}\beta a_{qs}\xi\lambda^{(\varepsilon(s)-1)/2}\overline{a_{qs}}\overline{\beta} \\ &= (-1)^{(\varepsilon(q)-1)/2}\alpha a_{ps}\xi\lambda^{(\varepsilon(s)-1)/2}\overline{a_{ps}\alpha} \\ &= (-1)^{(\varepsilon(q)-1)/2}\alpha a_{ps}\xi a'_{-s,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha} \\ &= (-1)^{(\varepsilon(q)-1)/2}\alpha b_{p,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}, \end{aligned} \tag{3.73}$$

where the first equality is straightforward, the second is by Lemma 3.1.2, the third is by the condition $\alpha a_{ps} + \beta a_{qs} = 0$, the penultimate is again by Lemma 3.1.2 and the last is straightforward. Using (3.73) we can rewrite (3.72) as follows:

$$\begin{aligned}\beta S_{q,-q}(b^{-1})\lambda^{(\varepsilon(q)-1)/2}\overline{\beta} &= \alpha S_{p,-p}(b^{-1})\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha} \\ &+ \left((-1)^{(\varepsilon(q)-1)/2} - (-1)^{(\varepsilon(p)-1)/2}\right)\alpha b_{p,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}.\end{aligned} \tag{3.74}$$

Note that the last formula works regardless whether or not $s = -r$. Using (3.74) we can rewrite (3.70) as

$$\begin{aligned}X &= 2b_{-p,h}(x\alpha)S_{p,-p}(b^{-1})\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}b'_{-h,p}+ \\ &\quad \left((-1)^{(\varepsilon(q)-1)/2} - (-1)^{(\varepsilon(p)-1)/2}\right) b_{-p,h}(x\alpha)b_{p,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}b'_{-h,p} \\ &= 2Z + \left((-1)^{(\varepsilon(q)-1)/2} - (-1)^{(\varepsilon(p)-1)/2}\right) W,\end{aligned}$$

where

$$Z = b_{-p,h}(x\alpha)S_{p,-p}(b^{-1})\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}b'_{-h,p} \tag{3.75}$$

and

$$W = b_{-p,h}(x\alpha)b_{p,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}b'_{-h,p}. \tag{3.76}$$

We will show that $Z, W \in \Gamma^{\max}_{-p}$. Then, as the coefficient $\left((-1)^{(\varepsilon(q)-1)/2} - (-1)^{(\varepsilon(p)-1)/2}\right)$ takes only even values (namely, $-2, 0$ and 2), Proposition 3.2.1 allows us to conclude that

$$X = 2Z + \left((-1)^{(\varepsilon(q)-1)/2} - (-1)^{(\varepsilon(p)-1)/2}\right) W \in 2\Gamma^{\max}_{-p} \leq \Gamma^{\min}_{-p}. \tag{3.77}$$

Indeed, as $S_{p,-p}(b^{-1}) \in \Lambda_p$, we get by relation $(\Gamma 2')$ applied to the constant net generated by the ideal (R, Λ) that

$$(x\alpha)S_{p,-p}(b^{-1})\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2} \in \Lambda_h.$$

By combining property $(\Gamma 2')$ and Lemma 3.1.2 we have

$$b_{-p,h}(x\alpha)S_{p,-p}(b^{-1})\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}b'_{-h,p} \in \Lambda_{-p}. \tag{3.78}$$

Comparing (3.75) with (3.78) we get

$$Z \in \Lambda_{-p}. \tag{3.79}$$

If $s \neq \pm r$, then $b_{p,-p} = 0$ and thus $W = 0 \in \Gamma^{\max}_{-p}$. Assume $r = -s$. As $\xi \in \Gamma_s \leq \Lambda_s$, we get by Lemma 3.1.2 and property $(\Gamma 2')$ applied to the constant net (R, Λ)

$$b_{p,-p} = a_{ps}\xi a'_{-s,-p} = a_{ps}\xi\lambda^{(\varepsilon(s)-1)/2}\overline{a_{ps}}\lambda^{(1-\varepsilon(p))/2} \in \Lambda_p.$$

By property $(\Gamma 2')$ applied to the constant net (R, Λ) we get

$$(x\alpha)b_{p,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2} \in \Lambda_p.$$

By combining property ($\Gamma 2'$) and Lemma 3.1.2, we get

$$b_{-p,h}(x\alpha)b_{p,-p}\lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}b'_{-h,p} \in \Lambda_{-p}. \tag{3.80}$$

Comparing (3.80) with (3.76) we get

$$W \in \Lambda_{-p}. \tag{3.81}$$

It's easy to see that, as $b \in \mathrm{U}(\sigma)$ and $x\alpha \in R = \sigma_{hp}$, we have $Z, W \in \sigma_{-p,p}$. By (3.79) and (3.81) we have $Z, W \in \Lambda_{-p}^{\max}$. As we noticed before, this yields (3.77).

Now we will treat Y in a similar way. Using Lemma 3.1.2 we can rewrite Y as follows:

$$\begin{aligned} Y = \lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha b'_{pp})}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})(x\alpha b'_{pp})+ \\ \lambda^{(\varepsilon(p)-1)/2}\overline{(x\beta b'_{qp})}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})(x\beta b'_{qp}). \end{aligned}$$

Note that $S_{-h,h}(b^{-1}) \in \Gamma_{-h}^{\max}$ and $x\alpha b'_{pp}, x\beta b'_{qp} \in \sigma_{hp}$. By Proposition 3.2.2 it follows that

$$Y \equiv \lambda^{(\varepsilon(p)-1)/2}\overline{(x(\alpha b'_{pp}+\beta b'_{qp}))}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})(x(\alpha b'_{pp}+\beta b'_{qp})) \mod \Gamma_{-p}^{\min}. \tag{3.82}$$

Using (3.67) we can rewrite (3.82) as

$$Y \equiv \lambda^{(\varepsilon(p)-1)/2}\overline{(x\alpha)}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})(x\alpha) \mod \Gamma_{-p}^{\min}. \tag{3.83}$$

As we noticed before, the combination of (3.69) (3.77) and (3.83) suffices to deduce that

$$\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})\alpha \in \Gamma_{-p}.$$

This completes the proof. □

Corollary 3.4.6. *Assume that $h(\nu) \geq (4,4)$ and that ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Let p, q and h be three indices in I such that either (p,q,h) is an* A*-type base triple or (p,h) is a* C*-type base pair and $q = -p$. Let a be a matrix in $\mathrm{U}(2n,R,\Lambda)$ such that $g_1 a g_2 \in H$ for some element $g_1, g_2 \in \mathrm{EU}(\sigma,\Gamma)$. Let $T_{sr}(\xi)$ be an elementary unitary matrix (long or short) such that ${}^{g_2^{-1}}T_{sr}(\xi) \in H$ and let $b = aT_{sr}(\xi)a^{-1}$. If $s \neq \pm r$, assume additionally $a_{p,-r} = a_{q,-r} = 0$. If one of the following conditions hold*

1. *a_{ps} or a_{qs} is invertible in R*
2. *a_{ps} or $a_{qs} = 0$*
3. *R is either a division ring or a product of two copies of a division ring*

then $b_{ih} \in \sigma_{ih}$ for all $i \neq -p, -q$. If additionally $b \in \mathrm{U}(\sigma)$ then also $S_{-h,h}(b^{-1}) \in \Gamma_{-h}$.

Proof. Suppose we can find a solution of the equation $\alpha a_{ps} + \beta a_{qs} = 0$ such that α, β or $\alpha + \beta$ is invertible. First, assume α is invertible. By Lemma 3.4.5 the inclusion $b_{ih}\alpha \in \sigma_{ih}$ holds for all $i \neq -p, -q$. Thus $b_{ih} \in \sigma_{ih}$ for all $i \neq -p, -q$. If $b \in \mathrm{U}(\sigma)$ then by Lemma 3.4.5 we also get the inclusion

$$\lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})\alpha \in \Gamma_{-p}.$$

As $\alpha^{-1} \in \sigma_{ph}$, it follows by property (Γ2) that $S_{-h,h}(b^{-1}) \in \Gamma_{-h}$.

The case that β is invertible can be treated similarly. Assume $\alpha + \beta$ is invertible. By Lemma 3.4.5 we get the inclusions $b_{ih}\alpha, b_{ih}\beta \in \sigma_{ih}$ for all $i \neq -p, -q$. Therefore also $b_{ih}(\alpha + \beta) \in \sigma_{ih}$ and $b_{ih} \in \sigma_{ih}$. If $b \in \mathrm{U}(\sigma)$ then we get by Lemma 3.4.5 the inclusions

$$\begin{aligned} \lambda^{(\varepsilon(p)-1)/2}\overline{\alpha}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})\alpha &\in \Gamma_{-p} \\ \lambda^{(\varepsilon(q)-1)/2}\overline{\beta}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})\beta &\in \Gamma_{-q}. \end{aligned} \tag{3.84}$$

Rewrite the second inclusion using Proposition 3.2.4 as follows:

$$\lambda^{(\varepsilon(p)-1)/2}\overline{\beta}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})\beta \in \Gamma_{-p}. \tag{3.85}$$

Clearly, $S_{-h,h}(b^{-1}) \in \Gamma_{-h}^{\max}$. Applying Proposition 3.2.2 to the first inclusion in (3.84) and to (3.85) we get the inclusion

$$\lambda^{(\varepsilon(p)-1)/2}\overline{(\alpha+\beta)}\lambda^{(1-\varepsilon(h))/2}S_{-h,h}(b^{-1})(\alpha+\beta) \in \Gamma_{-p}. \tag{3.86}$$

As $(\alpha+\beta)^{-1} \in \sigma_{ph}$, by combining property (Γ2) and (3.86) we get the required inclusion $S_{-h,h}(b^{-1}) \in \Gamma_{-h}$.

It's only left to show that we can always find α and β as above. First, assume that a_{ps} is invertible. Then we can put $\alpha = -a_{qs}a_{ps}^{-1}$ and $\beta = 1$. If $a_{ps} = 0$ put $\alpha = 1$ and $\beta = 0$. The case when a_{qs} is invertible or equals zero is treated similarly. Finally, assume the ground ring R is either a division ring or a direct product of two division rings. In the first case the element a_{ps} is always either zero or invertible and thus we can use one of the cases above. Assume R is a product of two division rings both the elements a_{ps} and a_{qs} are neither invertible nor zero. Assume $a_{ps} = (x, 0)$ and $a_{qs} = (y, 0)$, where $a_1, b_1 \neq 0$. Then we can put $\alpha = (-yx^{-1}, 1)$ and $\beta = (1, 1)$. The case when $a_{ps} = (0, x)$ and $a_{qs} = (0, y)$ is treated in the same way. Finally, assume $a_{ps} = (x, 0)$ and $a_{qs} = (0, y)$. Then we can set $\alpha = (0, 1)$, $\beta = (1, 0)$ and thus $\alpha + \beta = 1$ is invertible. The last case when $a_{ps} = (0, x)$ and $a_{qs} = (y, 0)$ is treated similarly. This completes the proof. □

3.5 At the level of the Jacobson radical

Fix a standard setting $((R, \Lambda), (R', \Lambda'), S)$, a unitary equivalence relation ν on our index set I, a subgroup H of $\mathrm{U}(2n, R, \Lambda)$ and an exact major form net of ideals (σ', Γ') which is S-associated with H. Let (σ, Γ) denote the S-closure of (σ', Γ') in (R, Λ). Let J denote the Jacobson radical of the ring R. In this section we continue extracting elementary unitary matrices, this time, using elements "close to" the principal congruence subgroup

$$\mathrm{U}(J) = \mathrm{U}(2n, (R, \Lambda), (J, \Omega^{\max}(J))) = \mathrm{U}(2n, R, \Lambda) \cap \mathrm{GL}(2n, R, J)$$

of level J. The general idea is as follows. Pick any matrix $a \in \mathrm{U}(2n, R)$ such that $g_1 a g_2 \in H$ for some elements $g_1, g_2 \in \mathrm{EU}(\sigma, \Gamma)$. Corollary 3.4.6 shows that it is possible to extract transvections using short root elements once some zero entries are present. It is also clear that if we have a unit on the diagonal, say, $a_{pp} \in R^*$ for some $p \in I$ and know that $a_{hp} \in \sigma_{hp}$ for some $h \neq \pm p$ then the matrix $T_{hp}(-a_{hp}a_{pp}^{-1})a$ has zero in position (h, p) and $T_{hp}(-a_{hp}a_{pp}^{-1}) \in \mathrm{EU}(\sigma, \Gamma)$. Thus, we can create zeros in positions with equivalent coordinates, provided that some of the diagonal elements are invertible. This allows us to extract entries on the block skew-diagonal (Corollary 3.5.2) and then, using zeros on the block skew-diagonal, we can extract any other entry.

For the rest of this section we will be assuming that ν is good for our standard setting, cf. Section 3.4.

Lemma 3.5.1. *Assume that $h(\nu) \geq (4, 4)$ and that ν is good for the standard setting $((R, \Lambda), (R', \Lambda'), S)$. Let (p, q, h, t, l) be an A-type base quintuple and a an element of $\mathrm{U}(2n, R, \Lambda)$ such that $a_{-p,-t} = a_{-h,-t} = a_{-l,-t} = a_{pq} = a_{hq} = 0$, $a_{-p,q}a'_{t,-p}, a_{-h,q}a'_{t,-h} \in J$ and $a_{qq} \in R^*$. If there exist elements g_1 and g_2 in $\mathrm{EU}(\sigma, \Gamma)$ such that $g_1 a g_2 \in H$ then $a_{p,-t} \in \sigma_{p,-t}$.*

Proof. Pick using Lemma 3.3.3 an element $x \in S$ such that ${}^{g_2^{-1}}T_{qt}(x) \in H$ and consider the element $b = aT_{qt}(x)a^{-1}$ of $\mathrm{U}(2n, R, \Lambda)$. By choice of the parameter x we have

$$^{g_1}b = (g_1 a g_2)(g_2^{-1}T_{qt}(x)g_2)(g_2^{-1}a^{-1}a_1^{-1}) \in H.$$

Pick again using Lemma 3.3.3 an element $y \in S$ such that ${}^{g_1}T_{-p,-h}(y) \in H$ and consider yet another matrix $c = bT_{-p,-h}(y)b^{-1}$. Clearly, ${}^{g_1}c \in H$ for the same reason as above. We are going to apply Corollary 3.4.6 to the element b, the elementary unitary matrix $T_{qt}(x)$ and the A-type base triple $(-p, -l, -h)$. In order to do this we have to show first that $b_{-p,h} = b_{-l,h} = 0$ and $b_{-p,-p}$ is invertible. Indeed, by assumption of this lemma, $a'_{th} = \lambda^{(-\varepsilon(t)-1)/2}\overline{a_{-h,-t}}\lambda^{(\varepsilon(h)+1)/2} = 0$ as well as $a_{-p,-t} = a_{-l,-t} = 0$. Thus

$$\begin{aligned} b_{-p,h} &= a_{-p,q}xa'_{th} - a_{-p,-t}\lambda^{(\varepsilon(t)-1)/2}\overline{x}\lambda^{(1-\varepsilon(q))/2}a'_{-q,h} = 0 \\ b_{-l,h} &= a_{-l,q}xa'_{th} - a_{-l,-t}\lambda^{(\varepsilon(t)-1)/2}\overline{x}\lambda^{(1-\varepsilon(q))/2}a'_{-q,h} = 0. \end{aligned}$$

As $a_{-p,q}a'_{t,-p} \in J$, it follows that

$$\begin{aligned} b_{-p,-p} &= 1 + a_{-p,q}xa'_{t,-p} - a_{-p,-t}\lambda^{(\varepsilon(t)-1)/2}\overline{x}\lambda^{(1-\varepsilon(q))/2}a'_{-q,-p} \\ &= 1 + a_{-p,q}xa'_{t,-p} \in 1 + J \leq R^*. \end{aligned}$$

By Corollary 3.4.6 applied to the matrix b, the short elementary unitary matrix $T_{-p,-h}(y)$ and the A-type base triple $(-p, -l, -h)$, we get the inclusion $c_{i,-h} \in \sigma_{i,-h}$ for all $i \neq p, l$. In particular,

$$b_{q,-p}yb'_{-h,-h} - b_{qh}\lambda^{(\varepsilon(p)-1)/2}\overline{y}\lambda^{(1-\varepsilon(h))/2}b'_{p,-h} = c_{q,-h} \in \sigma_{q,-h}. \tag{3.87}$$

Recall that $a_{-h,q}a'_{t,-h} \in J$. Thus,

$$\begin{aligned} b'_{-h,-h} &= 1 - a_{-h,q}xa'_{t,-h} + a_{-h,-t}\lambda^{(\varepsilon(t)-1)/2}\overline{x}\lambda^{(1-\varepsilon(q))/2}a'_{-q,-h} \\ &= 1 - a_{-h,q}xa'_{t,-h} \in 1 + J \leq R^* \end{aligned} \tag{3.88}$$

and also

$$b'_{p,-h} = -a_{pq}xa'_{t,-h} + a_{p,-t}\lambda^{(\varepsilon(t)-1)/2}\overline{x}\lambda^{(1-\varepsilon(q))/2}a'_{-q,-h} = 0. \quad (3.89)$$

Substituting (3.88) and (3.89) into (3.87) we get the inclusion $b_{q,-p} \in \sigma_{q,-h} = \sigma_{q,-p}$. Recall that by assumption of the lemma we have $a_{qq} \in R^*$ and

$$a'_{-q,-p} = \lambda^{(\varepsilon(q)-1)/2}\overline{a_{pq}}\lambda^{(1-\varepsilon(p))/2} = 0.$$

Then

$$a_{qq}xa'_{t,-p} = a_{qq}xa'_{t,-p} - a_{q,-t}\lambda^{(\varepsilon(t)-1)/2}\overline{x}\lambda^{(1-\varepsilon(q))/2}a'_{-q,-p} = b_{q,-p} \in \sigma_{q,-p}.$$

Therefore $a'_{t,-p} \in \sigma_{q,-p}$. It only remains to notice that

$$a_{p,-t} = \lambda^{(-\varepsilon(p)-1)/2}\overline{a'_{t,-p}}\lambda^{(1-\varepsilon(t))/2} \in \overline{\sigma_{q,-p}} = \sigma_{p,-q} = \sigma_{p,-t}.$$

□

Corollary 3.5.2. *Assume that $h(\nu) \geq (4,4)$ and that ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Let (p,q,h,t,l) be an A-type base quintuple and a an element of $\mathrm{U}(2n,R,\Lambda)$ such that at least one of the following three conditions holds:*

1. *The elements $a_{-p,-t}, a_{-h,-t}, a_{pq}, a_{hq}, a_{-p,q}a'_{t,-p}$ and $a_{-h,q}a'_{t,-h}$ are contained in the Jacobson radical and the elements a_{pp}, a_{qq} and $a_{-t,-t}$ are invertible.*
2. *The rows a_{p*}, a_{q*}, a_{h*} and the column $a_{*,-t}$ coincide modulo the Jacobson radical with the corresponding rows and columns of the identity matrix.*
3. *The rows $a_{-t,*}, a_{-h,*}, a_{-p,*}$ and the columns a_{*p} and a_{*q} coincide modulo the Jacobson radical with the corresponding rows and columns of the identity matrix.*

If there exist elements $g_1, g_2 \in \mathrm{EU}(\sigma,\Gamma)$ such that $g_1ag_2 \in H$ then $a_{p,-t} \in \sigma_{p,-t}$.

Proof. Note that the first option in the statement of the corollary trivially follows from any of the others. Consider the matrix

$$b = aT_{pq}(-a_{pp}^{-1}a_{pq}).$$

By the assumption that the entry a_{pq} is contained in the Jacobson radical, $b \equiv a \mod J$. Clearly, $b_{pq} = 0$, $b_{p,-t} = a_{p,-t}$ and $b_{qq}, b_{-t,-t} \in R^*$. Further, consider the matrix

$$c = T_{hq}(-b_{hq}b_{qq}^{-1})T_{-p,-t}(-b_{-p,-t}b_{-t,-t}^{-1})T_{-h,-t}(-b_{-h,-t}b_{-t,-t}^{-1})T_{-l,-t}(-b_{-l,-t}b_{-t,-t}^{-1})b.$$

As $b \equiv a \mod J$, we have $b_{hq}, b_{-p,-t}$ and $b_{-h,-t}$ are in the Jacobson radical. Therefore $c_{i*} \equiv a_{i*} \mod J$ whenever $i \neq t, -l$ (and also $c'_{*j} \equiv a'_{*j} \mod J$ whenever $j \neq l, -t$). In particular $c_{-p,q}c'_{t,-p}, c_{-h,q}c'_{t,-h} \in J$ and $c_{qq} \in R^*$. It is easy to see that $c_{pq} = c_{hq} = c_{-p,-t} = c_{-h,-t} = c_{-l,-t} = 0$. Finally, $g_3cg_4 \in H$, where

$$\begin{aligned} g_3 &= g_1\big(T_{hq}(-b_{hq}b_{qq}^{-1})T_{-p,-t}(-b_{-p,-t}b_{-t,-t}^{-1})\times \\ &\qquad T_{-q,-t}(-b_{-q,-t}b_{-t,-t}^{-1})T_{-l,-t}(-b_{-l,-t}b_{-t,-t}^{-1})\big)^{-1}, \\ g_4 &= T_{pq}(a_{pp}^{-1}a_{pq})g_2. \end{aligned}$$

Clearly, g_3 and g_4 are contained in $\mathrm{EU}(\sigma,\Gamma)$. Therefore, c satisfies the conditions of Lemma 3.5.1 and it follows that $c_{p,-t} \in \sigma_{p,-t}$. It's only left to notice that $c_{p,-t} = a_{p,-t}$. □

Lemma 3.5.3. *Assume that $h(\nu) \geq (4,4)$ and that ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Let (p,q,h,t) be an A-type base quadruple and a an element of $\mathrm{U}(2n,R,\Lambda)$ such that $a_{p,-h}, a_{q,-h}, a_{t,-h} \in \sigma_{p,-p} \cap J$, $a_{-h,p} \in \sigma_{-h,p} \cap J$. Suppose $a_{qp} \in J$ and a_{pp} and $a_{-h,-h}$ are units and suppose there exists an element $g \in \mathrm{EU}(\sigma,\Gamma)$ such that ${}^g a \in H$. Then $a_{ip} \in \sigma_{ip}$ for all $i \in I$. If additionally $a \in \mathrm{U}(\sigma)$, then also $S_{-h,h}(a^{-1}) \in \Gamma_{-h}$.*

Proof. Consider the matrix
$$b = T_{-h,p}(-a_{-h,p}a_{pp}^{-1})a.$$
As $a_{-h,p} \in J$ it follows that $b \equiv a \mod J$. Additionally $b_{p,-h}, b_{q,-h}$ and $b_{t,-h}$ are contained in $\sigma_{p,-p}$ and $b_{-h,p} = 0$. Consider the matrix

$$c = T_{p,-h}(-b_{p,-h}b_{-h,-h}^{-1})T_{q,-h}(-b_{q,-h}b_{-h,-h}^{-1})T_{t,-h}(-b_{t,-h}b_{-h,-h}^{-1})b.$$

Again, $c \equiv a \mod J$, in particular $c_{pp}, c_{-h,-h} \in R^*$. By Lemma 3.1.2 it also follows that $c'_{-p,h} = 0$ and $c'_{hh} \in R^*$. It's easy to see that $c_{p,-h} = c_{q,-h} = c_{t,-h} = c_{-h,p} = 0$. Finally, gg_1cg^{-1} is contained in H, where

$$g_1 = \big(T_{p,-h}(-b_{-h,-h}^{-1}b_{p,-h})T_{q,-h}(-b_{-h,-h}^{-1}b_{q,-h})\times \\ T_{t,-h}(-b_{-h,-h}^{-1}b_{t,-h})T_{-h,p}(-a_{pp}^{-1}a_{-h,p})\big)^{-1} \in \mathrm{EU}(\sigma,\Gamma).$$

Pick an element $x \in S$ such that ${}^gT_{ph}(x) \in H$. Applying Corollary 3.4.6 to the matrix c, the short elementary unitary matrix $T_{ph}(x)$ and the A-type base triple (p,q,h) we get the inclusions

$$\delta_{ih} + c_{ip}xc'_{hh} - c_{i,-h}\lambda^{(-\varepsilon(p)-1)/2}\overline{x}\lambda^{(\varepsilon(h)+1)/2}c'_{-p,h} \in \sigma_{ih} \tag{3.90}$$

for all $i \neq -p, -q$. We can apply Corollary 3.4.6 to the same matrix and an elementary matrix, but to a different A-type base triple (p,t,h) and get the inclusion (3.90) also for $i = -q$. As c'_{hh} is invertible and $c'_{-p,h}$ is equal to zero, it follows from (3.90) that $c_{ip} \in \sigma_{ih}$ for all $i \neq h, -p$. Observe that $a_{ip} = c_{ip}$ for all $i \neq p,q,t,-h$. Thus $a_{ip} \in \sigma_{ip}$ for all $i \neq p,q,t,-h,-p$. The inclusion $a_{ip} \in \sigma_{ip}$ for $i = p,q,t$ is trivial and the inclusion $a_{-h,p} \in \sigma_{-h,p}$ is provided by the assumption of the lemma. Therefore $a_{ip} \in \sigma_{ip}$ for all $i \neq -p$.

Pick an element $y \in S$ such that ${}^gT_{pq}(y) \in H$ and consider the matrix $d = T_{pq}(y)a$. Clearly, it satisfies all the conditions of this lemma. Indeed, $d_{p,-h} = a_{p,-h} + ya_{q,-h} \in \sigma_{p,-p} \cap J$, $d_{pp} = a_{pp} + ya_{qp} \in R^* + J \leq R^*$ and the rest of the entries involved in the conditions of this lemma coincide with the corresponding entries of a itself. Thus we get the inclusions $d_{ip} \in \sigma_{ip}$ for all $i \neq -p$. In particular, $d_{-q,p} \in \sigma_{-q,p}$. It's only left to notice that $d_{-q,p} = a_{-q,p} - \lambda^{(-\varepsilon(p)-1)/2}\overline{y}\lambda^{(\varepsilon(q)+1)/2}a_{-p,p}$ and $a_{-q,p}$ is already contained in $\sigma_{-q,p}$, while $\lambda^{(-\varepsilon(p)-1)/2}\overline{y}\lambda^{(\varepsilon(q)+1)/2}$ is invertible. Therefore $a_{-p,p} \in \sigma_{-p,p}$.

If $a \in \mathrm{U}(\sigma)$ then by Corollary 3.4.6 we get the inclusion $S_{-h,h}(c^{-1}) \in \Gamma_{-h}$. As $a^{-1} = c^{-1}g_1^{-1}$, we get by Corollary 3.2.6 that $S_{-h,h}(a^{-1}) \in \Gamma_{-h}$. □

Corollary 3.5.4. *Assume that $h(\nu) \geq (4,4)$ and that ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Let (p,q,h,t,l) be an A-type base quintuple and a an element of $\mathrm{U}(2n,R,\Lambda)$. Let I' denote the set $\{p,q,h,t\}$. Suppose $a_{i*} \equiv \alpha_i e_{i*} \mod J$ and $a_{*,-i} \equiv \alpha_{-i} e_{*,-i} \mod J$ whenever $i \in I'$, where α_i is some invertible element of R for each $i \in I'$. Further, suppose there exists an element $g \in \mathrm{EU}(\sigma,\Gamma)$ such that ${}^g a \in H$. Then $a_{ip} \in \sigma_{ip}$ for all $i \in I$. If additionally $a \in \mathrm{U}(\sigma)$ then also $S_{-p,p}(a^{-1}) \in \Gamma_{-p}$.*

Proof. It's easy to see that the matrix a satisfies condition (2) of Corollary 3.5.2. Thus we can conclude that the entries $a_{p,-h}, a_{q,-h}$ and $a_{t,-h}$ are contained in $\sigma_{p,-p}$. Moreover, the same entries are contained in the Jacobson radical by assumption of this corollary. Next, a also satisfies condition (3) of Corollary 3.5.2. Therefore, $a_{-h,p}$ is contained in $\sigma_{-h,p}$. By assumption, $a_{pp}, a_{-h,-h} \in R^*$ and $a_{qp} \in J$. Summing up, a satisfies the conditions of Lemma 3.5.3. Therefore $a_{ip} \in \sigma_{ip}$ for all $i \in I$. If $a \in \mathrm{U}(\sigma)$ then by Lemma 3.5.3 we get the inclusion $S_{-h,h}(a^{-1}) \in \Gamma_{-h}$. Switching the indices p and h in the reasoning above, we get the required inclusion $S_{-p,p}(a^{-1}) \in S_{-p,p}$. □

We will state another version of the last corollary. It is proved exactly in the same way.

Corollary 3.5.5. *Assume that $h(\nu) \geq (4,4)$ and that the equivalence relation ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Let (p,q,h,t,l) be an A-type base quintuple and a an element of $\mathrm{U}(2n,R,\Lambda)$. Let I' denote the set $\{p,q,h,t\}$. Suppose $a_{-i*} \equiv \alpha_{-i} e_{-i,*} \mod J$ and $a_{*i} \equiv \alpha_i e_{*i} \mod J$ whenever $i \in I'$, where α_i is some invertible element of R for each $i \in I'$. Further, suppose that $a_{*,-h} \equiv e_{*,-h} \mod J$ and that there exists an element $g \in \mathrm{EU}(\sigma,\Gamma)$ such that ${}^g a \in H$. Then $a_{ip} \in \sigma_{ip}$ for all $i \in I$. If additionally $a \in \mathrm{U}(\sigma)$ then also $S_{-h,h}(a^{-1}) \in \Gamma_{-h}$.*

Proof. As in Corollary 3.5.4 it follows from Corollary 3.5.2 that $a_{ij} \in \sigma_{ij}$ whenever $i \in I' \cup (-I')$ and $j \sim -i$. By assumption, $a_{pq}, a_{-h,p}, a_{p,-h}, a_{q,-h}, a_{t,-h} \in J$ and $a_{pp}, a_{-h,-h} \in R$. By Lemma 3.5.3 it follows that $a_{ip} \in \sigma_{ip}$ for all $i \in I$. If $a \in \mathrm{U}(\sigma)$ then by Lemma 3.5.3 we get the inclusion $S_{-h,h}(a^{-1}) \in \Gamma_{-h}$. □

Lemma 3.5.6. *Assume that $h(\nu) \geq (4,4)$ and that ν is good for the standard setting $((R,\Lambda),(R',\Lambda'),S)$. Let (p,h) be a C-type base pair and a an element in $\mathrm{U}(2n,R,\Lambda)$ such that $a_{p,-h} \cap J$ and $a_{pp}, a_{-h,-h} \in R^*$. If there exist an element $g \in \mathrm{EU}(\sigma,\Gamma)$ such that ${}^g a \in H$, then $a_{ip} \in \sigma_{ip}$ for all $i \in I$.*

Proof. Consider the matrix
$$b = T_{-h,p}(-a_{-h,p}a_{pp}^{-1})a.$$
Clearly, $b_{-h,p} = 0$ and $b_{p*} = a_{p*}$, in particular b_{pp} is invertible and $b_{p,-h} \in J$. As $a_{p,-h} \in J$, it follows that
$$b_{-h,-h} = a_{-h,-h} - a_{-h,p}a_{pp}^{-1}a_{p,-h} \in R^* + J \leq R^*.$$
Consider the matrix
$$c = T_{p,-h}(-b_{p,-h}b_{-h,-h}^{-1})T_{-p,-h}(-b_{-p,-h}b_{-h,-h}^{-1})b.$$

Clearly, $c_{p,-h} = c_{-p,-h} = 0$, $c_{-h,p} = b_{-h,p} = 0$, $c_{-h,-h} = b_{-h,-h}$ is invertible. As $b_{-h,p} = 0$, it follows that

$$c_{pp} = b_{pp} - b_{p,-h}b_{-h,-h}^{-1}b_{-h,p} = b_{pp} \in R^*.$$

By Lemma 3.1.2 this means that c'_{hh} is invertible and $c'_{-p,h} = 0$. Finally, $gg_1cg^{-1} \in H$, where

$$g_1 = \left(T_{p,-h}(-b_{p,-h}b_{-h,-h}^{-1})T_{-p,-h}(-b_{-p,-h}b_{-h,-h}^{-1})T_{-h,p}(-a_{-h,p}a_{pp}^{-1})\right)^{-1} \in \mathrm{EU}(\sigma, \Gamma).$$

Pick an element $x \in S$ such that ${}^gT_{ph}(x) \in H$. Applying Corollary 3.4.6 to the matrix c, the short elementary unitary matrix $T_{ph}(x)$ and the C-type base pair (p, h) we get that

$$\delta_{ih} + c_{ip}xc'_{hh} - c_{i,-h}\lambda^{(-\varepsilon(p)-1)/2}\overline{x}\lambda^{(\varepsilon(h)+1)/2}c'_{-p,h} = (cT_{ph}(x)c^{-1})_{ih} \in \sigma_{ih} \qquad (3.91)$$

for all $i \in I$. Recall that $c'_{-h,p} = 0$ and $c'_{hh} \in R^*$. Therefore the inclusions (3.91) can be rewritten as $c_{ip} \in \sigma_{ih} = \sigma_{ip}$ for all $i \in I$. Recall that for all $i \neq \pm p, \pm h$, $a_{ip} = c_{ip}$ and therefore $a_{ip} \in \sigma_{ip}$ for all $i \neq \pm p, \pm h$. The missing four inclusions are trivial. □

Corollary 3.5.7. *Assume that $h(\nu) \geq (4, 4)$ and that ν is good for the standard setting $((R, \Lambda), (R', \Lambda'), S)$. Let (p, h) be a C-type base pair and a be an element in $\mathrm{U}(2n, R, \Lambda)$ such that either $a_{i*} \equiv \alpha_i e_{i*} \mod J$ for $i \in \{p, -h\}$ or $a_{*i} \equiv \alpha_i e_{*i} \mod J$ for $i \in \{p, -h\}$, where $\alpha_i \in R^*$ for $i \in \{p, -h\}$. Suppose there exists an element $g \in \mathrm{EU}(\sigma, \Gamma)$ such that ${}^ga \in H$. Then $a_{ip} \in \sigma_{ip}$ for all $i \in I$.*

The last corollary allows us to perform radical reduction whenever the minimal size of self-conjugate components is at least 6. However, it's also possible to do this when this bound is lowered to 4. In this case we utilize the condition $R\Lambda + \Lambda R = R$.

Lemma 3.5.8. *Assume that $h(\nu) \geq (4, 4)$ and that ν is good for the standard setting $((R, \Lambda), (R', \Lambda'), S)$. Let (p, h) be a C-type base pair and a be a matrix in $\mathrm{U}(2n, R, \Lambda)$ such that $a_{pp}, a_{-p,-p} \in R^*$ and $a_{-h,-p} \in J$. Suppose there exists an element $g \in \mathrm{EU}(\sigma, \Gamma)$ such that ${}^ga \in H$. Then $a_{ip}\Lambda \in \sigma_{ip}$ for all $i \in I$.*

Proof. Consider the matrix

$$b = T_{hp}((-a_{hp} + 1)a_{pp}^{-1})a.$$

Clearly, $b_{pp} = a_{pp}$ is invertible, $b_{hp} = 1$ and

$$b_{-p,-p} = a_{-p,-p} - \lambda^{(\varepsilon(p)-1)/2}\overline{(-a_{hp} + 1)a_{pp}^{-1}}\lambda^{(1-\varepsilon(h))/2}a_{-h,-p} \in R^*.$$

Pick any $\alpha \in \Lambda_p$ and an element $x \in S$ such that ${}^gT_{p,-p}(x^2\alpha) \in H$. As b_{pp} is invertible, we get by Corollary 3.4.6

$$\delta_{i,-h} + b_{ip}x^2\alpha b'_{-p,-h} = (bT_{p,-p}(x^2\alpha)b^{-1})_{i,-h} \in \sigma_{i,-h}$$

for all $i \in I$. As $xb'_{-h,-p}$ is invertible, $b_{ip}\alpha \in \sigma_{i,-h}$ for all $i \in I$ and $\alpha \in \Lambda_p$. Finally, $a_{ip} = b_{ip}$ whenever $i \neq h, -p$. Thus $a_{ip}\Lambda \leq \sigma_{ip}$. □

3.6 Scaling, form ring morphisms, direct products and direct limits

In this section we will show that all the conditions used in the statement of our main result, Theorem 1, are compatible with form ring direct decompositions, direct limits, form ring scaling and isomorphisms of even unitary groups induced by isomorphisms of form rings. Therefore in addressing the problem of describing the overgroups of elementary block-diagonal subgroups of even unitary groups over quasi-finite rings we may restrict ourselves to proving Theorem 1 for a certain class of rings provided that all finite rings can be obtained from this class of rings using the four operations mentioned above.

Let $((R_1,\Lambda_1),(R_1',\Lambda_1'),S_1)$ and $((R_2,\Lambda_2),(R_2',\Lambda_2'),S_2)$ be standard settings and let $\varphi:\ (R_1,\Lambda_1)\longrightarrow(R_2,\Lambda_2)$ be a morphism of form rings. We will call φ *a morphism of standard settings* if $\varphi(R_1')\leq R_2'$, $\varphi(\Lambda_1')\leq\Lambda_2'$ and $\varphi(S_1)\subseteq S_2$.

Lemma 3.6.1. *Let $((R_i,\Lambda_i),(R_i',\Lambda_i'),S_i)$ be a standard setting for each $i\in\{1,2\}$ and let*

$$\varphi:((R_1,\Lambda_1),(R_1',\Lambda_1'),S_1)\longrightarrow((R_2,\Lambda_2),(R_2',\Lambda_2'),S_2)$$

be a morphism of standard settings. Then:

1. *$((\varphi(R_1),\varphi(\Lambda_1)),(\varphi(R_1'),\varphi(\Lambda_1')),\varphi(S_1))$ is a standard setting.*
2. *$\mathrm{M}(\varphi)(U(2n,R_1',\Lambda_1'))\leq\mathrm{U}(2n,R_2',\Lambda_2')$.*

Fix a subgroup H_1 of $\mathrm{U}(2n,R_1,\Lambda_1)$ and let (σ_1',Γ_1') be an exact form D*-net of ideals over (R_1,Λ_1), which is S-associated with H_1. Denote by (σ_1,Γ_1) the S-closure of (σ_1',Γ_1') in (R_1,Λ_1). Denote by (σ_2',Γ_2') and (σ_2,Γ_2) the images of (σ_1',Γ_1') and (σ_1,Γ_2) under φ respectively. Then:*

3. *Suppose H_1 contains the kernel of $\mathrm{M}(\varphi)$. Then (σ_2',Γ_2') is an exact form* D*-net, which is $\varphi(S_1)$-associated with H_2 as a subgroup of $\mathrm{U}(2n,\varphi(R_1),\varphi(\Lambda_1))$, and (σ_2,Γ_2) is the $\varphi(S_1)$-closure of (σ_2',Γ_2') in $(\varphi(R_1),\varphi(\Lambda_1))$.*
4. *The following equalities hold:*

$$\mathrm{M}(\varphi)(\mathrm{U}(\sigma_1',\Gamma_1'))=\mathrm{U}(\sigma_2',\Gamma_2'),\quad \mathrm{M}(\varphi)(\mathrm{EU}(\sigma_1',\Gamma_1'))=\mathrm{EU}(\sigma_2',\Gamma_2'),$$
$$\mathrm{M}(\varphi)(\mathrm{U}(\sigma_1,\Gamma_1))=\mathrm{U}(\sigma_2,\Gamma_2),\quad \mathrm{M}(\varphi)(\mathrm{EU}(\sigma_1,\Gamma_1))=\mathrm{EU}(\sigma_2,\Gamma_2).$$

Proof. As $\varphi(\Lambda_1')\subseteq\Lambda_2'$, it follows by Lemma 3.1.1 that $(\varphi(R_1'),\varphi(\Lambda_1'))$ is a form subring of $(\varphi(R_1),\varphi(\Lambda_1))$ and a form subring of subring of (R_2',Λ_2'). Now the conclusion (1) is straightforward.

Applying Proposition 3.2.7 to the constant form net of ideals defined by the unit form ideal (R_1',Λ_1') of the form ring (R_1',Λ_1') we get the assertion (2). By Proposition 3.2.7 we also get (4).

By Proposition 3.2.7, (σ'_2, Γ'_2) is an exact form D-net of ideals over $(\varphi(R'_1), \varphi(\Lambda'_1))$. It is clear that $\mathrm{EU}(\sigma'_2, \Gamma'_2) = \varphi(\mathrm{EU}(\sigma'_1, \Gamma'_1)) \leq \varphi(H_1) = H_2$. Let $T_{sr}(\xi) \in H_2$ for some $\xi = \varphi(\zeta) \in \varphi(R_1)$. As H_1 contains the kernel of $\mathrm{M}(\varphi)$, it follows that $T_{sr}(\zeta) \in H_1$. Thus there exists an element $x \in S_1$ such that $x^{(1+\delta_{s,-r})}\zeta \in (\sigma'_1, \Gamma'_1)_{sr}$. Therefore $\varphi(x)^{(1+\delta_{s,-r})}\xi \in (\sigma'_2, \Gamma'_2)_{sr}$. Summing up, (σ', Γ') is an exact form D-net of ideals, which is $\varphi(S_1)$-associated with H_2 as a subgroup of $\mathrm{U}(2n, \varphi(R_1), \varphi(\Lambda_1))$. The fact that (σ_2, Γ_2) is the $\varphi(S_1)$-closure of (σ'_2, Γ'_2) in $(\varphi(R_1), \varphi(\Lambda_1))$ is straightforward. $\square$

The following proposition plays a crucial role in the localization method used in Section 3.10 and allows to lift certain elementary unitary matrices along not necessarily injective morphisms.

Proposition 3.6.2. *Let* $\varphi : ((R_1, \Lambda_1), (R'_1, \Lambda'_1), S_1) \to ((R_2, \Lambda_2), (R'_2, \Lambda'_2), S_2)$ *be a morphism of standard settings such that* φ *is injective as a ring morphism on the principal ideal* x_0R*, where* $x_0 \in R'_1 \cap \mathrm{Center}(R_1) \cap \{\xi \in R \mid \overline{\xi} = \xi\}$*. Let* H *be a subgroup of* $\mathrm{U}(2n, R_1, \Lambda_1)$ *such that* $\mathrm{EU}(\nu, R'_1, \Lambda'_1) \leq H$*, where* ν *is a unitary equivalence relation on the index set* I *such that* $h(\nu) \geq (4, 3)$*. Suppose that the elementary unitary matrix* $T_{sr}(\xi) \in \mathrm{EU}(2n, R_2, \Lambda_2)$ *is contained in* $\varphi(H)$*. Further, suppose there exists an element* $\xi' \in R'_1$ *and an element* $x \in R'_1 \cap \mathrm{Center}(R_1) \cap \{\xi \in R \mid \overline{\xi} = \xi\}$ *such that* $\varphi(\xi') = \varphi(x)^{(1+\delta_{s,-r})}\xi$ *and that if* $r = -s$ *then* $\xi' \in (\Lambda'_1)_s$*. Then* $T_{sr}(x_0^{(1+\delta_{s,-r})}\xi') \in H$*.*

Proof. First, assume $s \sim r$. Note that $x_0^{(1+\delta s,-r)}\xi' \in R'_1$ and if $r = -s$ then also $x_0^{(1+\delta s,-r)}\xi' \in (\Lambda'_1)_s$. Therefore $T_{sr}(x_0^{(1+\delta s,-r)}\xi') \in \mathrm{EU}(\nu, R'_1, \Lambda'_1) \leq H$. From now on assume $s \not\sim -r$. Fix a pre-image h of $T_{sr}(\xi)$ which is contained in H. If $s \neq -r$ then there exists another index $t \sim r$ such that $t \neq \pm r, \pm s$. Then

$$[[h, T_{rt}(x)], T_{tr}(x_0)] \in H \cap \mathrm{GL}(2n, R, x_0R).$$

Therefore by the relation (EU4)

$$\begin{aligned} T_{sr}(\varphi(x_0\xi')) &= T_{sr}(\varphi(x_0)\varphi(x)\xi) = [[T_{sr}(\xi), T_{rt}(\varphi(x))], T_{tr}(\varphi(x_0))] \\ &= \mathrm{M}(\varphi)([[h, T_{rt}(x)], T_{tr}(x_0)]) \in \mathrm{M}(\varphi)(H \cap \mathrm{GL}(2n, R, x_0R)). \end{aligned} \tag{3.92}$$

Clearly, the matrix $T_{sr}(x_0\xi')$ is a pre-image of $T_{sr}(\varphi(x_0\xi'))$ and is contained in $\mathrm{M}(2n, R, x_0R)$. As $\mathrm{M}(\varphi)$ is injective on $\mathrm{M}(2n, R, x_0R)$, it follows that $T_{sr}(x_0\xi')$ is the only pre-image of $T_{sr}(\varphi(x_0\xi'))$ in $\mathrm{M}(2n, R, x_0R)$. Thus by (3.92) it follows that $T_{sr}(x_0\xi') \in H$.

Assume $r = -s$ and $s \not\sim -s$. There exists two more indices l, t in I such that (s, t, l) is an A-type base triple. Then

$$[h, T_{-s,-t}(x_0x)] \cdot [T_{sl}(-1), [T_{ls}(1), [h, T_{-s,-t}(x_0x)]]] \in H \cap \mathrm{M}(2n, R, x_0R).$$

By the elementary relations (EU3), (EU4) and (EU6) we get

$$\begin{aligned}
T_{t,-t}(\varphi(-\lambda^{(\varepsilon(s)-\varepsilon(t))/2}x_0^2\xi')) &= T_{t,-t}(-\lambda^{(\varepsilon(s)-\varepsilon(t))/2}\varphi(x_0)^2\varphi(x)^2\xi)\\
&= T_{s,-t}(\xi\varphi(x_0)x)T_{t,-t}(-\lambda^{(\varepsilon(s)-\varepsilon(t))/2}\varphi(x_0)^2\varphi(x)^2\xi)T_{s,-t}(-\xi\varphi(x_0)x)\\
&= T_{s,-t}(\xi\varphi(x_0)x)T_{t,-t}(-\lambda^{(\varepsilon(s)-\varepsilon(t))/2}\varphi(x_0)^2\varphi(x)^2\xi)\\
&\quad\times [T_{sl}(\varphi(-1)),[T_{ls}(\varphi(1)),T_{s,-t}(\xi\varphi(x_0)x)\\
&\quad\times T_{t,-t}(-\lambda^{(\varepsilon(s)-\varepsilon(t))/2}\varphi(x_0)^2\varphi(x)^2\xi)]]\\
&= [T_{s,-s}(\xi),T_{-s,-t}(\varphi(x_0x))]\\
&\quad\times [T_{sl}(\varphi(-1)),[T_{ls}(\varphi(1)),[h,T_{-s,-t}(\varphi(x_0x))]]]\\
&= \mathrm{M}(\varphi)([h,T_{-s,-t}(x_0x)]\cdot[T_{sl}(-1),[T_{ls}(1),[h,T_{-s,-t}(x_0x)]]])\\
&\in \mathrm{M}(\varphi)(H\cap \mathrm{M}(2n,R,x_0R)).
\end{aligned}\tag{3.93}$$

For the same reason as in the previous case, (3.93) yields the inclusion

$$X = T_{t,-t}(-\lambda^{(\varepsilon(s)-\varepsilon(t))/2}x_0^2\xi') \in H.$$

Finally, by the elementary relations (EU3), (EU4) and (EU6) we get

$$T_{s,-s}(x_0^2\xi') = [X,T_{-t,-s}(1)]\cdot[T_{tl}(-1),[T_{lt}(1),[X,T_{-t,-s}(1)]]] \in H.$$

□

Note that the assumption that x_0 is central in not essential and was imposed just to make the computations easier. A similar result can be stated without this assumption.

Direct decompositions. Let $(R_1,\bar{\cdot},\lambda_1)$ and $(R_2,\hat{\cdot},\lambda_2)$ be two rings with involution with symmetry. Then we can naturally consider the ring $(R_1\times R_2)$ with a componentwise involution with symmetry. We will call the resulting ring $(R_1\times R_2,(\bar{\cdot},\hat{\cdot}),(\lambda_1,\lambda_2))$ *the direct product of the rings with involution with symmetry* $(R_1,\bar{\cdot},\lambda)$ *and* $(R_2,\hat{\cdot},\mu)$. In this case the projections

$$\begin{aligned}
\mathrm{pr}_1 &: (R_1\times R_2,(\bar{\cdot},\hat{\cdot}),(\lambda_1,\lambda_2)) \longrightarrow (R_1,\bar{\cdot},\lambda_1),\\
\mathrm{pr}_2 &: (R_1\times R_2,(\bar{\cdot},\hat{\cdot}),(\lambda_1,\lambda_2)) \longrightarrow (R_2,\hat{\cdot},\mu)
\end{aligned}$$

are morphisms of rings with involution with symmetry. Let $((R_i,\Lambda_i),(R_i',\Lambda_i'),S_i)$ be standard settings with respect to involutions with symmetry (K_i,λ_i) for $i\in\{1,2\}$. Then it's easy to see that

$$((R_1\times R_2,\Lambda_1\times\Lambda_2),(R_1'\times R_2',\Lambda_1'\times\Lambda_2'),S_1\times S_2)$$

is also a standard setting. We will call such a standard setting *the direct product of the standard settings* $((R_1,\Lambda_1),(R_1',\Lambda_1'),S_1)$ *and* $((R_2,\Lambda_2),(R_2',\Lambda_2'),S_2)$. The following lemma shows that the notion of a standard setting is compatible with direct products.

Lemma 3.6.3. *Let a standard setting $((R,\Lambda),(R',\Lambda'),S)$ be equal to a direct product of two standard settings*

$$((R,\Lambda),(R',\Lambda'),S) = ((R_1,\Lambda_1),(R'_1,\Lambda'_1),S_1) \times ((R_2,\Lambda_2),(R'_2,\Lambda'_2),S_2).$$

Then

1. $\mathrm{U}(2n,R,\Lambda) = \mathrm{U}(2n,R_1,\Lambda_1) \times \mathrm{U}(2n,R_2,\Lambda_2)$ *and* $\mathrm{U}(2n,R',\Lambda') = \mathrm{U}(2n,R'_1,\Lambda'_1) \times \mathrm{U}(2n,R'_2,\Lambda'_2)$*, where the direct products are meant as direct products of abstract groups.*

Further, let H be a subgroup of $\mathrm{U}(2n,R,\Lambda)$*, (σ',Γ') and (σ,Γ) be exact form nets of ideals over (R',Λ') and (R,Λ), respectively. Denote by (σ_i,Γ_i) and (σ'_i,Γ'_i) the respective images of (σ,Γ) and (σ',Γ') under* pr_i *for $i\in\{1,2\}$. Denote by H_i the image of H under* pr_i *for $i\in\{1,2\}$.*

2. *Let ν be a unitary equivalence relation on the index set I such that $h(\nu)\geq(4,3)$. Then* $\mathrm{EU}(\nu,R',\Lambda')\leq H$ *if and only if* $\mathrm{EU}(\nu,R'_i,\Lambda'_i)\leq H_i$ *for $i\in\{1,2\}$. Further, (σ',Γ') is a major form net of ideals which is S-associated with H if and only if (σ'_i,Γ'_i) is a major form net of ideals which is S_i-associated with H_i for $i\in\{1,2\}$. Finally, (σ,Γ) is the S-closure of (σ',Γ') in (R,Λ) if and only if (σ_i,Γ_i) is the S_i-closure of (σ'_i,Γ'_i) in (R_i,Λ_i) for $i\in\{1,2\}$.*

3. *The following equalities hold*

$$\begin{aligned}\mathrm{EU}(\sigma,\Gamma) &= \mathrm{EU}(\sigma_1,\Gamma_1)\times\mathrm{EU}(\sigma_2,\Gamma_2)\\ \mathrm{U}(\sigma,\Gamma) &= \mathrm{U}(\sigma_1,\Gamma_1)\times\mathrm{U}(\sigma_2,\Gamma_2)\\ \mathrm{EU}(\sigma',\Gamma') &= \mathrm{EU}(\sigma'_1,\Gamma'_1)\times\mathrm{EU}(\sigma'_2,\Gamma'_2)\\ \mathrm{U}(\sigma',\Gamma') &= \mathrm{U}(\sigma'_1,\Gamma'_1)\times\mathrm{U}(\sigma'_2,\Gamma'_2).\end{aligned}$$

4. *The condition $R\Lambda+\Lambda R=R$ is equivalent to the condition $R_1\Lambda_1+\Lambda_1R_1=R_1$ and $R_2\Lambda_2+\Lambda_2R_2=R_2$.*

Proof. 1. By assumption the unitary groups $\mathrm{U}(2n,R_i,\Lambda_i)$ and $\mathrm{U}(2n,R'_i,\Lambda'_i)$ are defined. The fact that $\mathrm{U}(2n,R,\Lambda)$ is a direct product of $\mathrm{U}(2n,R_1,\Lambda_1)$ and $\mathrm{U}(2n,R_2,\Lambda)$ is obvious because $\mathrm{M}(\mathrm{pr}_i)(a)_{jk}=\mathrm{pr}_i(a_{jk})$ for all $i\in\{1,2\}$ and $j,k\in I$. The same goes for the second equality in (1).

2. It is clear that $\mathrm{EU}(\nu,R',\Lambda')\leq H$ if and only if $\mathrm{EU}(\nu,R'_i,\Lambda'_i)\leq H_i$ for $i\in\{1,2\}$. Assume, (σ',Γ') is a major form net of ideals which is S-associated with H and (σ,Γ) is the S-closure of (σ',Γ') in (R,Λ). We will show that (σ'_i,Γ'_i) is a major form net of ideals which is S_i-associated with H_i and (σ_i,Γ_i) is the S_i-closure of (σ'_i,Γ'_i) in (R_i,Λ_i) for $i\in\{1,2\}$. The converse is straightforward.

Recall that pr_i is a surjective morphism of forms rings. By Proposition 3.2.7 it follows that (σ'_i,Γ'_i) is an exact form D-net of ideals over (R_i,Λ_i) for each $i\in\{1,2\}$. We will show now that (σ'_i,Γ'_i) is S_i-associated with H_i. Note that pr_1 is injective on $(1,0)R=R_1$. Let

$T_{sr}(\xi) \in H_1$. Pick an element $x \in S$ such that $x(\xi,0) \in R'$ if $s \neq -r$ and $x^2(\xi,0) \in (\Lambda')_s$ if $s = -r$. Set $\xi' = x^{(1+\delta_{s,-r})}(\xi,0)$. Then $\mathrm{pr}_1(\xi') = \mathrm{pr}_1(x)^{(1+\delta_{s,-r})}\xi$. By Proposition 3.6.2 it follows that $T_{sr}((1,0)x^{(1+\delta_{s,-r})}(\xi,0)) \in H$. Therefore there exists an element $y \in S$ such that $(xy)^{(1+\delta_{s,-r})}(\xi,0) \in (\sigma',\Gamma')_{sr}$. It follows that $\mathrm{pr}_1(xy)^{(1+\delta_{s,-r})}\xi \in (\sigma_1',\Gamma_1')_{sr}$. Summing up, we have proved that (σ_1',Γ_1') is a net S_1-associated with H_1. Similarly, (σ_2',Γ_2') is a net S_2-associated with H_2.

3. Now we will show that (σ_i,Γ_i) is the S_i-closure of (σ_i',Γ_i') in (R_i,Λ_i). Denote the S_i-closure of (σ_i',Γ_i') in (R_i,Λ_i) by (τ_i,B_i). It's easy to see that (σ_i,Γ_i) is contained in (τ_i,B_i). Let $\xi \in (\tau_i,B_i)_{sr}$. Then there exists an element $x_i \in S_i$ such that $T_{sr}(x_i^{(1+\delta_{s,-r})}\xi) \in H_i$. Pick an element $y \in S$ such that $(yx_i)^{(1+\delta_{s,-r})}(\xi,0) \in R'$. Clearly, if $s = -s$ then $\xi \in (\Lambda_i)_s$, therefore y can be chosen so that $(yx_i)^{(1+\delta_{s,-r})}(\xi,0) \in \Lambda_s$. Finally, $\mathrm{pr}_i((yx_i)^{(1+\delta_{s,-r})}(\xi,0) = \mathrm{pr}_i((1,1))\xi$. By Proposition 3.6.2 it follows that

$$T_{sr}((yx_i)^{(1+\delta_{s,-r})}(\xi,0)) \in H.$$

Therefore there exists an element $z \in S$ such that $(zyx_i)^{(1+\delta_{s,-r})}(\xi,0) \in (\sigma',\Gamma')_{sr}$. Thus $(\xi,0) \in (\sigma,\Gamma)_{sr}$. It only remains to notice that in this case $\xi \in (\sigma_i,\Gamma_i)_{sr}$. Therefore $(\tau_i,B_i) \leq (\sigma_i,\Gamma_i)$ and thus (σ_i,Γ_i) is indeed the S_i-closure of (σ_i',Γ_i') in (R_i,Λ_i).

4. The conclusion 4 is obvious. □

We will use Lemma 3.6.3 in combination with the following proposition which shows that for a standard setting $((R,\Lambda),(R',\Lambda'),S)$ a direct decomposition of R as a ring with involution with symmetry induces a direct decomposition of $((R,\Lambda),(R',\Lambda'),S)$ as a standard setting.

Proposition 3.6.4. *Let $((R,\Lambda),(R',\Lambda'),S)$ be a standard setting with respect to an involution with symmetry $(\bar{\cdot},\lambda)$ and let $(R,\bar{\cdot},\lambda) = (R_1,K_1,\lambda_1)\times(R_2,K_2,\lambda_2)$ be a direct product of rings with involution with symmetry. Denote the corresponding projections by pr_1 and pr_2. Let $R_i' = \mathrm{pr}_i(R')$, $\Lambda_i' = \mathrm{pr}_i(\Lambda')$ and $S_i' = \mathrm{pr}_i(S)$ for $i \in \{1,2\}$. Then $((R_i,\Lambda_i),(R_i',\Lambda_i'),S_i)$ is a standard setting for each $i \in \{1,2\}$ and*

$$((R,\Lambda),(R',\Lambda'),S) = ((R_1,\Lambda_1),(R_1',\Lambda_1'),S_1)\times((R_2,\Lambda_2),(R_2',\Lambda_2'),S_2).$$

Proof. Straightforward. □

Scaling. The next topic we are going to discuss is scaling of form rings as presented in [Bak81, §9] or [HO89, Ch. 5, Sec. 1C, p. 191]. Let $(R,\bar{\cdot},\lambda)$ be a ring with involution with symmetry. Given an invertible element β in R we can introduce a new involution with symmetry on R by setting $\widehat{\xi} = \beta\bar{\xi}\beta^{-1}$ and $\mu = \beta\overline{\beta^{-1}}\lambda$. It's easy to see that $(\widehat{\cdot},\mu)$ is indeed an involution with symmetry on R. We say that the ring with involution with symmetry $(R,\widehat{\cdot},\mu)$ is obtained from the ring with involution with symmetry $(R,\widehat{\cdot},\lambda)$ by *scaling by* β. If $((R,\bar{\cdot},\lambda),\Lambda)$ is a form ring, then $((R,\widehat{\cdot},\mu),\beta\Lambda)$ is a form ring. We will say that the form ring $(R,\beta\Lambda)$ is obtained from the form ring (R,Λ) by *scaling by* β. Note that scaling is not a morphism of form rings, however it several nice properties. First of all, the corresponding unitary groups are conjugate in $\mathrm{GL}(2n,R)$, namely

$$\mathrm{U}(2n,R,\beta\Lambda) = B\,\mathrm{U}(2n,R,\Lambda)B^{-1},$$

where $B = \operatorname{diag}(1, \dots 1, \beta, \dots, \beta)$. Moreover this correspondence preserves all the important classes of matrices. First,

$$BD_i(\theta)B^{-1} = D_i^{\beta}(\theta),$$

where $D_i^{\beta}(\theta)$ denotes the elementary diagonal unitary matrix in $\mathrm{U}(2n, R, \beta\Lambda)$, cf. Section 3.1. Further, as mentioned in [Dyb07], if (σ, Γ) is a form net of ideals over (R, Λ) and $\Gamma_i^{\beta} = \beta^{\frac{1-\varepsilon(i)}{2}}\Gamma_i\beta^{\frac{-1-\varepsilon(i)}{2}}$, then (σ, Γ^{β}) is a form net of ideals over $(R, \beta\Lambda)$ and

$$\mathrm{U}(\sigma, \Gamma^{\beta}) = B\,\mathrm{U}(\sigma, \Gamma)B^{-1} \qquad\qquad \mathrm{EU}(\sigma, \Gamma^{\beta}) = B\,\mathrm{EU}(\sigma, \Gamma)B^{-1}.$$

More precisely,

$$BT_{ij}(\xi)B^{-1} = T_{ij}^{\beta}(\xi),$$

where $T_{ij}^{\beta}(\xi)$ stands for a short elementary unitary matrix in $\mathrm{EU}(2n, R, \beta\Lambda)$, and

$$BT_{i,-i}(\alpha)B^{-1} = T_{i,-i}^{\beta}(\beta^{\frac{1-\varepsilon(i)}{2}}\alpha\beta^{\frac{-1-\varepsilon(i)}{2}}),$$

where $T_{i,-i}^{\beta}(\cdot)$ stands for a long elementary unitary matrix in $\mathrm{EU}(2n, R, \beta\Lambda)$.

Finally, it's easy to see that if H is a subgroup of $\mathrm{U}(2n, R, \Lambda)$ such that $\mathrm{EU}(\nu, R, \Lambda) \leq H$ and (σ, Γ) is the form net of ideals associated with H, then (σ, Γ^{β}) is the form net of ideals associated with BHB^{-1} as a subgroup of $\mathrm{U}(2n, R, \beta\Lambda)$. In the present paper we will not discuss scaling of standard settings. The following proposition clearly follows from the reasoning above.

Proposition 3.6.5. *Let (R, Λ) be a form ring and $(R, \beta\Lambda)$ be the form ring obtained from (R, Λ) by scaling by β. Assume Theorem 1 holds for the ground form ring (R, Λ). Then it holds also for the ground form ring $(R, \beta\Lambda)$.*

Direct limits.

Proposition 3.6.6. *Let $((R, \bar{\cdot}, \lambda), \Lambda)$ be a form ring such that R is a direct limit $\varinjlim R_i$ of some involution invariant subrings of R that contain the symmetry λ. Denote by Λ_i the intersection $R_i \cap \Lambda$ for all i. If Theorem 1 holds for each of the rings (R_i, Λ_i) then it also holds for the ring (R, Λ).*

Proof. Let (σ, Γ) denote the exact major form net of ideals associated with H, which constructed in Lemma 3.2.9. For any ring R_i in the directed system, set $H_i = H \cap \mathrm{U}(2n, R_i, \Lambda_i)$. Then $\mathrm{EU}(\nu, R_i, \Lambda_i) \leq \mathrm{EU}(\nu, R, \Lambda) \cap \mathrm{U}(2n, R_i, \Lambda_i) \leq H_i$. By Lemma 3.2.9 there exists an exact major form net of ideals (σ_i, Γ_i) associated with H_i. By the construction in Lemma 3.2.9 of a form net of ideals associated with a subgroup, it follows that if $(R_i, \Lambda_i) \leq (R_j, \Lambda_j)$ then $(\sigma_i, \Gamma_i) \leq (\sigma_j, \Gamma_j)$. Clearly $\mathrm{U}(\sigma_i, \Gamma_i) \leq \mathrm{U}(\sigma_j, \Gamma_j)$ because the net subgroup is defined in terms of inclusions. As any element of g of $\mathrm{U}(2n, R, \Lambda)$ is contained in $\mathrm{U}(2n, R_i, \Lambda_i)$ for some R_i in the directed system, it follows that

$$\mathrm{U}(\sigma, \Gamma) = \varinjlim \mathrm{U}(\sigma_i, \Gamma_i). \tag{3.94}$$

Pick any $a \in H$ and $T_{sr}(\xi) \in \mathrm{EU}(\sigma, \Gamma)$. There exists a subring R_i of R such that $a, T_{sr}(\xi) \in \mathrm{U}(2n, R_i, \Lambda_i)$. Clearly, $a \in H_i$ and $T_{sr}(\xi) \in \mathrm{EU}(\sigma_i, \Gamma_i)$. By assumption, Theorem 1 holds for the ground form ring (R_i, Λ_i). Therefore

$$aT_{sr}(\xi)a^{-1} \in \mathrm{U}(\sigma_i, \Gamma_i). \tag{3.95}$$

Moreover the inclusion (3.95) holds for any subring R_j in the directed system such that $R_i \leq R_j$. Combining (3.95) with (3.94) we deduce that

$$\mathrm{EU}(\sigma, \Gamma) \leq H \leq \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma, \Gamma), \mathrm{U}(\sigma, \Gamma)). \tag{3.96}$$

The proof of the uniqueness of (σ, Γ) in (3.96) can be deduced easily from Theorem 2. This will be done in the end of Section 3.10. □

3.7 Unitary groups over semisimple Artinian rings

In this section we reduce the study of even dimensional unitary groups over semisimple Artinian rings to the case of unitary groups over products of division rings. Fix a semisimple Artinian ring R together with an involution with symmetry $(\bar{\cdot}, \lambda)$. The famous Artin-Wedderburn theorem establishes a ring isomorphism of R and a finite direct product of full matrix rings over division rings

$$R \cong \mathrm{M}(m_1, D_1) \times \cdots \times \mathrm{M}(m_N, D_N), \tag{3.97}$$

where the dimensions m_i are defined up to permutation of indices and the divisions rings D_i up to permutation of indices and isomorphisms. We will show that every even dimensional unitary group over R is isomorphic to a direct product of unitary groups, where each group in the product is either an even dimensional unitary group over D_i or an even dimensional unitary group over $D_i \times D_j$ for some i and j such that the involution interchanges D_i and D_j. This result follows form a more general Morita theory of qudratic modules, cf. [Bak81, §9]. Results of this section regarding form nets of ideals mimic those of [Dyb07] and we utilize numerous results thereof.

Our immediate goal is to classify the possible involutions with symmetry on R in terms of direct decomposition of rings with involution with symmetry, cf. Section 3.6. The direct decomposition (3.97) of R provides us with N central primitive (i.e. generating a minimal two-sided ideal) idempotents $f_1, \ldots, f_N$, where f_i is the N-tuple having the identity of $\mathrm{M}(m_i, D_i)$ as it's i-th component and zeros elsewhere. Clearly, $f_1 + \cdots + f_N = 1$ and $f_i R \cong M(m_i, D_i)$. It's easy to see that $\overline{f_1}, \ldots, \overline{f_N}$ is also a system of central primitive idempotents such that $\overline{f_1} + \cdots + \overline{f_N} = 1$. By the uniqueness condition in the Artin-Wedderburn theorem, there exists a permutation π of the indices $1, \ldots, N$ such that $\overline{f_i}R \cong \mathrm{M}(m_{\pi(i)}, D_{\pi(i)})$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$. If $\pi(i) = i$ then the involution $\bar{\cdot}$ leaves $\mathrm{M}(m_i, D_i)$ invariant and induces an involution J_i with symmetry λ_i on $\mathrm{M}(m_i, D_i)$. If $\pi(i) \neq i$, then $\pi^2(i) = i$ because $\overline{\overline{\xi}} = \lambda \xi \lambda^{-1}$ for $\xi \in R$. In this case the involution leaves $\mathrm{M}(m_i, D_i) \times \mathrm{M}(m_{\pi(i)}, D_{\pi(i)})$ invariant and induces on $\mathrm{M}(m_i, D_i) \times \mathrm{M}(m_{\pi(i)}, D_{\pi(i)})$ an involution J_i with symmetry $(\lambda_i, \lambda_{\pi(i)})$. It follows that $m_i = m_{\pi(i)}$ and $D_i \cong D_{\pi(i)}$. The action of the involution J_i on each of the components induces two anti-automorphisms $\cdot^*$ and $\cdot^\perp$ of $\mathrm{M}(m_i, D_i)$ such that J_i sends each (ξ, ζ) to $(\zeta^*, \xi^\perp)$. Further, the compositions $\cdot^{\perp *}$ and $\cdot^{*\perp}$ are given by conjugation by λ_i and $\lambda_{\pi(i)}$, respectively. Finally $\lambda_i^{-1} = \lambda_{\pi(i)}^*$ and $\lambda_{\pi(i)}^{-1} = \lambda_i^\perp$. Summing up, we have proved the following two propositions.

Proposition 3.7.1. *Let R be a semisimple Artinian ring with involution with symmetry $(\bar{\cdot}, \lambda)$. Then R is isomorphic as a ring with involution with symmetry to a direct product of rings with involution with symmetry (R_i, J_i, λ_i), where each R_i is either a simple Artinian ring or a product of two copies of a simple Artinian ring, the symmetries λ_i are images λ under the projections $\mathrm{pr}_i : R \to R_i$ and the involutions J_i with symmetry λ_i are obtained from $\bar{\cdot}$ by taking compositions with the projections pr_i. Each of the components (R_i, J_i, λ_i) can't be further decomposed as direct products of simple Artinian rings with involution with symmetry.*

Proposition 3.7.2. *Let $(R, \bar{\cdot}, \lambda)$ be a ring with involution with symmetry such that R is a direct product of two copies of a simple Artinian ring Q. Then either $(R, \bar{\cdot}, \lambda)$ can either be decomposed as a direct product of two rings with involution with symmetry over Q, or there exist two anti-automorphisms $\cdot^*$ and $\cdot^\perp$ of the ring Q such that $\overline{(a,b)} = (b^*, a^\perp)$ for all a in b in Q. Moreover, if $\lambda = (\lambda_1, \lambda_2)$ then the compositions $\cdot^{\perp *}$ and $\cdot^{*\perp}$ are given by conjugation by λ_1 and λ_2 respectively. Finally, $\lambda_1^{-1} = \lambda_2^*$ and $\lambda_2^{-1} = \lambda_1^\perp$.*

It is well known that the anti-automorphisms of $\mathrm{M}(m, D)$ where D is a division ring are described in terms of the anti-automorphisms of D. Check for example [Jac43, ch. 2, §5, theorem 8] for the proof.

Proposition 3.7.3. *Let $R = \mathrm{M}(m, D)$ be a simple Artinian ring, where D is a division ring. Then for any anti-automorphism $\widehat{\cdot}$ of the ring R there exists an anti-isomorphism $\bar{\cdot}$ of the division ring D and a matrix $b \in \mathrm{GL}(m, D)$ such that for any $a \in \mathrm{M}(m, D)$ the equality*

$$\widehat{a} = b \cdot \overline{a}^t \cdot b^{-1}$$

holds, where $\overline{a}^t$ is the hermitian transpose of a with respect to the involution $\bar{\cdot}$, i.e. $(\overline{a}^t)_{ij} = \overline{a}_{ji}$ for all $i, j \in I$.

In other words, all the anti-automorphisms of $\mathrm{M}(m, D)$ are defined by anti-automorphisms of D up to an inner automorphism of $\mathrm{M}(m, D)$.

Let Q be any associative unital ring. Following [Dyb07] we will pay special attention to the anti-automorphisms of $\mathrm{M}(m, Q)$ obtained from an anti-automorphism of Q via the following procedure. Let $\bar{\cdot}$ be an anti-automorphism of Q. Consider an automorphism $\cdot^*$ sending each $a \in \mathrm{M}(m, Q)$ to $a^* = p\overline{a}^t p$, where $\overline{a}^t$ is the hermitian transpose of a with respect to $\bar{\cdot}$ and $p = \mathrm{sdiag}(1, \ldots, 1)$ is the monomial matrix having ones on the skew-diagonal and zeros elsewhere. For each i in $\{1, \ldots, m\}$, set

$$i^\times = m + 1 - i.$$

Then

$$(a^*)_{ij} = \overline{a}_{j^\times, i^\times},$$

i.e. a^* is obtained from a by first applying the anti-automorphism $\overline{\cdot}$ to each of its entries and then reflecting it with respect to the skew-diagonal. We will call the anti-automorphism $\cdot^*$ *the standard anti-automorphism of* $\mathrm{M}(m, Q)$ *corresponding to* $\overline{\cdot}$. It's an obvious corollary of Proposition 3.7.3 that any anti-automorphism of $R = M(m, D)$, where D is a division ring, is a composition of a standard anti-automorphism and an inner automorphism of R.

Clearly $\mathrm{M}(m, D) \times \mathrm{M}(m, D)$ is isomorphic as a ring to $\mathrm{M}(m, D \times D)$ via the map which sends a pair of matrices $(a, b) \in \mathrm{M}(m, D) \times \mathrm{M}(m, D)$ to the matrix $A \in \mathrm{M}(m, D \times D)$ such that $A_{ij} = (a_{ij}, b_{ij})$ for all $1 \leq i, j \leq m$. Denote this isomorphism by φ. Given an involution $\widehat{\cdot}$ on $\mathrm{M}(m, D) \times \mathrm{M}(m, D)$ with symmetry λ we obtain an involution $\widetilde{\cdot} = \varphi \circ \widehat{\cdot} \circ \varphi^{-1}$ on $\mathrm{M}(m, D \times D)$ with symmetry $\varphi(\lambda)$. Therefore any ring with involution with symmetry over $\mathrm{M}(m, D) \times \mathrm{M}(m, D)$ is isomorphic to a ring with involution with symmetry over $\mathrm{M}(m, D \times D)$. By Lemma 3.6.1 form nets and the related subgroups over isomorphic rings with involution with symmetry biject onto one another. In future we won't distinguish the elements of $\mathrm{M}(m, D) \times \mathrm{M}(m, D)$ and $\mathrm{M}(m, D \times D)$, neither the involutions $\widehat{\cdot}$ and $\widetilde{\cdot}$.

The next proposition shows that all the anti-automorphisms of a product of two copies of a matrix ring over a division ring are standard up to an inner automorphism.

Proposition 3.7.4. *Let D be a division ring and $\widehat{\cdot}$ an anti-automorphism of the matrix ring* $\mathrm{M}(m, D \times D)$*. Then there exists an anti-automorphism $\overline{\cdot}$ of $D \times D$ and a matrix $b \in \mathrm{GL}(m, D \times D)$ such that for any $a \in \mathrm{M}(m, D \times D)$*

$$\widehat{a} = b \cdot a^* \cdot b^{-1},$$

where $\cdot^*$ is the standard anti-automorphism on $\mathrm{M}(m, D \times D)$ *corresponding to the anti-automorphism $\overline{\cdot}$ of $D \times D$.*

Proof. Consider the image X of $\mathrm{M}(m, D \times \{0\}) = \mathrm{M}(m, D) \times \{0\}$ under $\widehat{\cdot}$. By the Artin-Wedderburn theorem either $X = \mathrm{M}(m, D \times \{0\})$ or $X = \mathrm{M}(m, \{0\} \times D)$. In the first case, it's easy to see that $\overline{\cdot}$ can be decomposed as a direct product of two anti-automorphisms of $\mathrm{M}(m, D)$ and the conclusion of the proposition can be easily deduced from Proposition 3.7.3. If $X = \mathrm{M}(m, \{0\} \times D)$ then we get two anti-automorphisms $\cdot^J$ and $\cdot^K$ of $\mathrm{M}(m, D)$ such that for any $(x, y) \in \mathrm{M}(m, D) \times \mathrm{M}(m, D)$ the equality

$$\widehat{(x, y)} = (y^J, x^K)$$

holds. By Proposition 3.7.3 there exist anti-automorphisms J_1, K_1 of D and elements $c, d \in \mathrm{GL}(m, D)$ such that

$$x^K = d \cdot x^{K_2} \cdot d^{-1} \qquad\qquad y^J = c \cdot x^{J_2} \cdot c^{-1},$$

where K_2 and J_2 stand for standard anti-automorphisms of $\mathrm{M}(m, D)$ corresponding to K_1 and J_1 respectively. Set $\overline{(\xi, \zeta)} = (\zeta^{J_1}, \xi^{K_1})$ for any (ξ, ζ) in $D \times D$. Further, set $b = (c, d)$. Clearly, $\overline{\cdot}$ is an anti-automorphism of $D \times D$, b is an element of $\mathrm{GL}(m, D) \times \mathrm{GL}(m, D) = \mathrm{GL}(m, D \times D)$ and $\widehat{a} = b \cdot a^* \cdot b^{-1}$ for any $a \in \mathrm{M}(m, D \times D)$. □

Given an associative unital ring Q such that all the anti-automorphism of $\mathrm{M}(m, Q)$ are standard up to an inner automorphism a reasonable question is, whether the involutions with symmetries of Q define involutions with symmetries on $\mathrm{M}(m, Q)$ and vice versa? The simple answer is, yes. The following proposition gives a formalization of this fact. In the particular case of Q being a division ring it is precisely Proposition 3 of [Dyb07].

Proposition 3.7.5. *Let Q be an associative unital ring and $\widehat{\cdot}$ an anti-automorphism of the ring $R = M(m, Q)$. Suppose there is an anti-automorphism $\overline{\cdot}$ of Q and a matrix b in $\mathrm{GL}(m, Q)$ such that $\widehat{a} = ba^*b^{-1}$ for all $a \in R$, where $\cdot^*$ is the standard anti-automorphism of R corresponding to $\overline{\cdot}$. Let $\mathrm{diag} : Q \to R$ be the diagonal embedding of Q into R. Then the following holds:*

1. *If $\widehat{\cdot}$ is an involution on R with symmetry l then there exists an element $\lambda \in Q$ such that $\mathrm{diag}(\lambda) = b^*b^{-1}l$ and $\overline{\cdot}$ is an involution on Q with symmetry λ.*
2. *If $\overline{\cdot}$ is an involution on Q with symmetry λ then $\widehat{\cdot}$ is an involution on R with symmetry $l = b(b^*)^{-1}\,\mathrm{diag}(\lambda)$.*

Proof. 1. By definition of an involution with symmetry, for any $a \in \mathrm{M}(m, Q)$ we have

$$lal^{-1} = \widehat{\widehat{a}} = \widehat{ba^*b^{-1}} = b(b^{-1})^*a^{**}b^*b^{-1}. \tag{3.98}$$

Therefore

$$a^{**}(b^*b^{-1}l) = (b^*b^{-1}l)a \tag{3.99}$$

for any $a \in \mathrm{M}(m, Q)$. It's easy to see how standard anti-automorphisms act on matrix units, namely for any $i, j \in \{1, \dots, m\}$, $e^*_{ij} = e_{j^\times, i^\times}$. Thus $e^{**}_{ij} = e_{ij}$ for all i, j. This, together with (3.99), yields that the invertible matrix $(b^*b^{-1}l)$ commutes with all matrix units. Therefore $(b^*b^{-1}l)$ is scalar. Denote by λ the pre-image of $(b^*b^{-1}l)$ under diag. By (3.98) it follows that $\overline{\overline{\xi}} = \lambda\xi\lambda^{-1}$ for any $\xi \in Q$. Finally, the condition $\lambda\overline{\lambda} = 1$ is equivalent to the condition $\mathrm{diag}(\lambda)\,\mathrm{diag}(\lambda)^* = e$ and thus easily follows from the condition $l \cdot \widehat{l} = 1$.

2. The second conclusion of the lemma can be checked via a straightforward computation. □

Summing up, any involution with symmetry on a matrix ring over either a division ring or a product of two copies of a division ring is standard up to scaling of form rings. By Proposition 3.6.5 it follows that all the form nets and the corresponding subgroups respect scaling, therefore we may in future limit ourselves to considering only standard involutions with symmetries on matrix rings.

The following proposition is a well know recipe for building a form parameter of a matrix ring from a form parameter of the ground ring. It has appeared in the literature multiple times, e.g. [HO89, Ex. 4, sec. 5.1C, p. 191] or [Dyb07, Prop. 4]. We include a sketch of the proof, whose full details are easy to recover.

Proposition 3.7.6. *Let Q be an associative unital ring with involution with symmetry $(\overline{\cdot}, l)$, R the matrix ring $\mathrm{M}(m, Q)$ of rank $m \geq 1$ over Q and $\cdot^*$ the standard involution* *on R with symmetry $\lambda = \mathrm{diag}(l)$ corresponding to the involution $\bar{\cdot}$ with symmetry l on Q. Then the map Ψ_{fp}, sending a form parameter Λ of the ring R to the set*

$$\Psi_{\mathrm{fp}}(\Lambda) = \{\alpha \in Q \mid \alpha e_{i,i^\times} \in \Lambda \textit{ for some index } 1 \le i \le m\}$$

is an isomorphism of the lattice $\mathrm{FP}(R, \cdot^*, \lambda)$ of form parameters for $(R, \cdot^*, \lambda)$ onto the lattice $\mathrm{FP}(Q, \bar{\cdot}, l)$ of form parameters for $(Q, \bar{\cdot}, l)$. The inverse of Ψ_{fp} is given by the following formula

$$\begin{aligned}\Psi_{\mathrm{fp}}^{-1}(L) &= \{a \in R \mid a_{ij} = -\overline{a_{j^\times,i^\times}}l \textit{ for all } i,j \textit{ and } a_{i,i^\times} \in L \textit{ for all } i\} \\ &= \{a \in R \mid a_{i,i^\times} \in L \textit{ for all } i\} \cap \Lambda^{\max}.\end{aligned}$$

Proof. Let $\Lambda^{\max}$ and $\Lambda^{\min}$ denote the maximal and minimal, respectively, form parameters for $(R, \cdot^*, \lambda)$ and $L^{\max}$ and $L^{\min}$ the maximal and minimal, respectively, form parameters for $(Q, \bar{\cdot}, l)$.

First, we will show that Ψ_{fp} and Ψ_{fp}^{-1} preserve inclusions and then show they send form parameters to form parameters. Applying property (Λ2) of the form parameter Λ to matrix units it's easy to conclude that $\alpha e_{i,i^\times} \in \Lambda$ if and only if $\alpha p \in \Lambda$, where $p = \mathrm{sdiag}(1, \dots, 1)$. It immediately follows that Ψ_{fp} preserves inclusions as well as additive subgroups, $\Psi_{\mathrm{fp}}(\Lambda^{\min}) = L^{\min}$ and $\Psi_{\mathrm{fp}}(\Lambda^{\max}) = L^{\max}$. The property ($\Lambda$2) for $\Psi_{\mathrm{fp}}(\Lambda)$ follows from the corresponding property for Λ applied to scalar matrices. Therefore Ψ_{fp} is a well defined morphism from $\mathrm{FP}(R, \cdot^*, \lambda)$ to $\mathrm{FP}(Q, \bar{\cdot}, l)$.

Denote by φ the map which sends a form parameter L for $(Q, \bar{\cdot}, l)$ to the set

$$\{a \in R \mid a_{ij} = -\overline{a_{j^\times,i^\times}}l \text{ for all } i,j \text{ and } a_{i,i^\times} \in L \text{ for all } i\}.$$

Write the definition of the maximal and minimal form parameters for R in terms of matrix entries:

$$\begin{aligned}\Lambda^{\max} &= \{a \in R \mid a_{ij} = -\overline{a_{j^\times,i^\times}}l \text{ for all } i,j\}, \\ \Lambda^{\min} &= \{a \in R \mid \forall i,j \in \{1,\dots,m\}\ a_{ij} = -\overline{a_{j^\times,i^\times}}l; \forall i\ \exists \alpha_i \in Q : a_{i,i^\times} = \alpha_i - \overline{\alpha_i}l\} \\ &= \{a \in R \mid a_{i,i^\times} \in L^{\min}\} \cap \Lambda^{\max}.\end{aligned}$$

The definition of φ can be rewritten as follows

$$\varphi(L) = \{a \in R \mid a_{i,i^\times} \in L \text{ for all } i\} \cap \Lambda^{\max}. \tag{3.100}$$

It's clear that $\varphi(L^{\min}) = \Lambda^{\min}$, $\varphi(L^{\max}) = \Lambda^{\max}$ and φ preserves inclusions. The condition (Λ2) for $\varphi(L)$ also follows from the corresponding condition for L in a view of (3.100).

It's easy to see that $\Psi_{\mathrm{fp}} \circ \varphi = \mathrm{id}$. Finally, one can easily check that $a - \sum_{i=1}^m a_{i,i^\times} e_{i,i^\times}$ is contained in $\Lambda^{\min}$ whenever a is contained in $\Lambda^{\max}$. Thus

$$a \equiv \sum_{i=1}^m a_{i,i^\times} e_{i,i^\times} \mod \Lambda^{\min}$$

for all $a \in \Lambda^{\max}$. Therefore, Ψ_{fp} is strictly monotone. Let Λ denote a form parameter for $(R, \cdot^*, \lambda)$ and $a \in \Lambda$. Then

$$a_{i,i^\times} e_{i,i^\times} = e^*_{i^\times,i^\times} a e_{i^\times,i^\times} \in \Lambda$$

for all $1 \leq i \leq m$. Thus $\Lambda \leq \varphi(\Psi_{\mathrm{fp}}(\Lambda))$. Finally, we have to prove that $\varphi(\Psi_{\mathrm{fp}}(\Lambda)) \leq \Lambda$. Assume for some Λ the form parameter $\varphi(\Psi_{\mathrm{fp}}(\Lambda))$ is strictly greater than Λ. We have proved, that Ψ_{fp} is strictly monotone. Recall that $\Psi_{\mathrm{fp}} \circ \varphi = \mathrm{id}$. Thus $\Psi_{\mathrm{fp}}(\Lambda) = \Psi_{\mathrm{fp}}(\varphi(\Psi_{\mathrm{fp}}(\Lambda)))$ is strictly greater than $\Psi_{\mathrm{fp}}(\Lambda)$, which is obviously false. It follows that $\varphi(\Psi_{\mathrm{fp}}(\Lambda)) \leq \Lambda$ which finishes the proof that $\varphi \circ \Psi_{\mathrm{fp}} = \mathrm{id}$. □

For the proposition we need to introduce the following notation. Define a map $\mathrm{ind} : I_{2n} \to I_{2mn}$ by putting

$$\mathrm{ind}(j) = \begin{cases} (j-1)m & , j > 0 \\ jm - 1 & , j < 0. \end{cases}$$

Given a matrix A in $\mathrm{GL}(2n, R)$, where $R = \mathrm{M}(m, Q)$, we want to rewrite it as a matrix in $\mathrm{GL}(2mn, Q)$. We do this by letting a denote the $mn \times mn$ matrix over Q whose Q-coefficient $a_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l}$ in the $(\mathrm{ind}(i)+k, \mathrm{ind}(j)+l)$'th position, where $i, j \in I_{2n}$ and $k, l \in \{1, \ldots, m\}$, is given by

$$(A_{ij})_{kl} = a_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l}.$$

$(\mathrm{ind}(i), \mathrm{ind}(j))$ stands for the coordinates of the (i, j)'th $m \times m$ block in a. From now on, whenever we write $a_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l}$ for a matrix a in $\mathrm{GL}(2mn, Q)$ we mean that i, j, m, l are uniquely determined indices such that i, j are in I_{2n} and m, l are in $\{1, \ldots, m\}$. The following relation can be checked via a straightforward calculation:

$$-(\mathrm{ind}(i) + k) = \mathrm{ind}(-i) + k^\times.$$

Proposition 3.7.7. *Let $(Q, \bar{\cdot}, l)$ be an associative ring with involution with symmetry, R the matrix ring $\mathrm{M}(m, Q)$ of rank $m \geq 1$ over Q and $\cdot^*$ the standard involution on R with symmetry $\lambda = \mathrm{diag}(l)$ corresponding to the involution with symmetry $(\bar{\cdot}, l)$ on Q. Let L be a form parameter for Q and $\Lambda = \Psi_{\mathrm{fp}}^{-1}(L)$ be the corresponding form parameter for R. Then a matrix $g \in \mathrm{GL}(2n, R)$ is contained in $\mathrm{U}(2n, R, \Lambda)$ if and only if as a matrix in $\mathrm{GL}(2mn, Q)$, g is contained in $\mathrm{U}(2mn, Q, L)$.*

Proof. The proof is a routine check of the conditions of Lemma 3.1.2. Let A denote a matrix in $\mathrm{GL}(2n, R)$ and let a denote the same matrix as an element of $\mathrm{GL}(2mn, Q)$. Write condition (U1) of Lemma 3.1.2 for the matrix A in terms of matrix entries:

$$\begin{aligned} a_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l} &= (A'_{ij})_{kl} = \left(\lambda^{(-1-\varepsilon(i))/2} \left(A_{-j,-i}\right)^* \lambda^{(\varepsilon(j)+1)/2}\right)_{kl} \\ &= l^{(-1-\varepsilon(i))/2} \overline{(A_{-j,-i})_{l^\times,k^\times}} l^{(\varepsilon(j)+1)/2} \\ &= l^{(-1-\varepsilon(i))/2} \overline{a_{\mathrm{ind}(-j)+l^\times,\mathrm{ind}(-i)+k^\times}} l^{(\varepsilon(j)+1)/2} \\ &= l^{(-1-\varepsilon(\mathrm{ind}(i)+k))/2} \overline{a_{-(\mathrm{ind}(j)+l),-(\mathrm{ind}(i)+k)}} l^{(\varepsilon(\mathrm{ind}(j)+l)+1)/2}. \end{aligned}$$

Then we get precisely condition (U1) of Lemma 3.1.2 for the matrix a. As we have mentioned after Lemma 3.1.2, condition (U1) thereof yields condition (U2) for the maximal form parameter.

Assume both A and a satisfy condition (U1). According to Proposition 3.7.6 we have

$$\Lambda = \{g \in R \mid g_{i,i^\times} \in L \text{ for all } i\} \cap \Lambda^{\max}.$$

As $S_{i,-i}(A) \in \Lambda_i^{\max}$ for all $i \in I$, condition (2) for A is equivalent to the inclusions $(S_{i,-i}(A))_{k,k^\times} \in L_k$ for all k in $\{1,\dots,m\}$. It's only left to notice that

$$\begin{aligned}(S_{i,-i}(A))_{k,k^\times} &= \sum_{j>0}\sum_{l=1}^{m}(A_{ij})_{kl}(A'_{j,-i})_{l,k^\times}\\ &= \sum_{j>0}\sum_{l=1}^{m} a_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l}\,a'_{\mathrm{ind}(j)+l,\mathrm{ind}(-i)+k^\times}\\ &= \sum_{\mathrm{ind}(j)+l>0} a_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l}\,a'_{\mathrm{ind}(j)+l,-(\mathrm{ind}(i)+k)}\\ &= S_{\mathrm{ind}(i)+k,-(\mathrm{ind}(i)+k)}(a).\end{aligned}$$

Thus conditions (U2) for the matrices A and a are equivalent. □

An important question is if the isomorphism of unitary groups described in the last proposition preserves elementary subgroups (full, block-diagonal or form net). The informal answer is, it preserves sufficiently large elementary subgroups. To make a distinction, we will denote the short and long elementary unitary matrices in $\mathrm{EU}(2mn, Q, L)$ by the small letter t and the ones in $\mathrm{EU}(2n, R, \Lambda)$ by the capital letter T. Then it's easy to see that

$$T_{ij}(a) = \prod_{k,l=1}^{m} t_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l}(a_{kl}), \tag{3.101}$$

whenever $i \neq \pm j$ and

$$T_{i,-i}(a) = \prod_{k>l^\times} t_{\mathrm{ind}(i)+k,\mathrm{ind}(-i)+l}(a_{kl}) \cdot \prod_{k=1}^{m} t_{\mathrm{ind}(i)+k,-(\mathrm{ind}(i)+k)}(a_{kk^\times}). \tag{3.102}$$

It follows that $\mathrm{EU}(2n, R, \Lambda) \leq \mathrm{EU}(2mn, Q, L)$. Let $t_{\mathrm{ind}(i)+k,-(\mathrm{ind}(i)+k)}(\alpha)$ be a long elementary unitary matrix in $\mathrm{EU}(2mn, Q, L)$. By (3.102) it follows that

$$t_{\mathrm{ind}(i)+k,-(\mathrm{ind}(i)+k)}(\alpha) = T_{i,-i}(\alpha e_{k,k^\times}) \in \mathrm{EU}(2n, R, \Lambda).$$

Let $t_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l}(\xi)$ be a short elementary unitary matrix in $\mathrm{EU}(2mn, Q, L)$. If $i \neq j$ then by (3.101) we have

$$t_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l}(\xi) = T_{ij}(\xi e_{kl}) \in \mathrm{EU}(2n, R, \Lambda).$$

However, if $i = j$ then

$$t_{\mathrm{ind}(i)+k,\mathrm{ind}(i)+l}(\xi) = D_i(e + \xi e_{kl}),$$

where D_i is defined as right after Lemma 3.1.2. We know that such an element can, but does not have to be contained in the elementary subgroup $\mathrm{EU}(2n, R, \Lambda)$. We will address this issue in Proposition 3.7.9.

We will establish now a correspondence between form nets of ideals over the form rings (R, Λ) and (Q, L). The following recipe in a slightly weaker form (namely, only for the case when R is a simple Artinian ring) is described in [Dyb07]. We will call a form net (σ, Γ) of order $2mn$ over (Q, L) an *m-block form net* if $\sigma_{\mathrm{ind}(i_1)+k_1,\mathrm{ind}(j_1)+l_1} = \sigma_{\mathrm{ind}(i_2)+k_2,\mathrm{ind}(j_2)+l_2}$ whenever $i_1 = i_2$ and $j_1 = j_2$ and $\Gamma_{ind(i_1)+k_1} = \Gamma_{\mathrm{ind}(i_2)+k_2}$ whenever $i_1 = i_2$, i.e. σ consists of $4n^2$ "blocks" of equal ideals of size $m \times m$ and Γ consists of $2n$ "blocks" of m equal form parameters each. We will show that form nets of ideals of order $2n$ over (R, Λ) can be identified with m-block form nets of order $2mn$ over (Q, L). For a form net of ideals (σ, Γ) of order $2n$ over (R, Λ) we will construct an m-block form net of ideals (τ, B) of order $2mn$ over (Q, L) by setting

$$\tau_{\mathrm{ind}(i)+k,\mathrm{ind}(j)+l} = \{\xi \in Q \mid \xi e \in \sigma_{ij}\} \qquad \text{and} \qquad B_{\mathrm{ind}(i)+k} = \Psi_{\mathrm{fp}}(\Gamma_i)$$

for all $i, j \in I_{2n}$ and $k, l \in \{1, \dots, m\}$. We will denote the form net of ideals (τ, B) constructed above by $\Psi_{\mathrm{fn}}(\sigma, \Gamma)$.

Proposition 3.7.8. *Let $(Q, \bar{\cdot}, l)$ be an associative unital ring with involution with symmetry, R denote the matrix ring $\mathrm{M}(m, Q)$ of rank $m \geq 1$ over Q and $\cdot^*$ be the standard involution on R with symmetry $\lambda = \mathrm{diag}(l)$ corresponding to the involution with symmetry $(\bar{\cdot}, l)$ on Q. Let L be a form parameter for Q and $\Lambda = \Psi_{\mathrm{fp}}^{-1}(L)$ the corresponding form parameter for R. Then the map Ψ_{fn} defined immediately before is a bijection of the set of all exact form* D*-nets of order $2n$ over the form ring (R, Λ) onto the set of all m-block exact form* D*-nets of order $2mn$ over the form ring (Q, L).*

Proof. Fix an exact form D-net (σ, Γ) of rank $2n$ over (R, Λ) Let (τ, B) denote the image $\Psi_{\mathrm{fn}}(\sigma, \Gamma)$ of (σ, Γ). We will first show that Ψ_{fn} is an exact form D-net of ideals over (Q, L). It's clear that τ is a unitary D-net of ideals over Q. As in the proof of Proposition 3.7.6, condition ($\Gamma 2$) in the definition of a form net allows us to conclude that

$$\Psi_{\mathrm{fp}}(\Gamma_i) = \{\xi \in Q \mid \xi p \in \Gamma_i\},$$

where p denotes the matrix $\mathrm{sdiag}(1, \dots, 1)$. It immediately follows from this observation that Ψ_{fp} preserves inclusions, additive subgroups and maps $\Gamma_i^{\min}(\sigma)$ and $\Gamma_i^{\max}(\sigma)$ to $\Gamma_i^{\min}(\tau)$ and $\Gamma_i^{\max}(\tau)$ respectively. Condition ($\Gamma 2$) for B also follows in an obvious way from the corresponding condition for Γ. Finally, the exactness condition for the net (τ, B) can be checked straightforwardly.

The injectivity of Ψ_{fn} is clear and it's easy to see that for any m-block form D-net (τ, B) of rank $2mn$ over (Q, L), $\Psi_{\mathrm{fn}}^{-1}(\tau, B)$ is a net (σ, Γ) such that $\sigma_{ij} = \mathrm{M}(m, \tau_{\mathrm{ind}(i)+1,\mathrm{ind}(j)+1})$ and $\Gamma_i = \Psi_{\mathrm{fp}}^{-1}(B_{\mathrm{ind}(i)+1})$. The proof of this fact precisely follows from the corresponding part the proof of Proposition 3.7.6. □

Given an equivalence relation ν on $I_{2n} = \{1, \dots, n, -n, \dots, -1\}$, define an "$m$-block" equivalence relation $\Psi_{\text{er}}(\nu)$ on I_{2mn} be setting $\text{ind}(i) + k \sim^{\Psi_{\text{er}}(\nu)} \text{ind}(j) + l$ if and only if $i \sim^{\nu} j$.

Proposition 3.7.9. *Let ν be a unitary equivalence relation on the index set I_{2n} such that $h(\nu) \geq (4, 2)$. Let $(Q, \bar{\cdot}, l)$ be an associative unital ring with involution with symmetry, R denote the matrix ring $\mathrm{M}(m, Q)$ of rank $m \geq 1$ over Q and $\cdot^*$ be the standard involution on R with symmetry $\lambda = \text{diag}(l)$ corresponding to the involution with symmetry $(\bar{\cdot}, l)$ on Q. Let L be a form parameter for Q and $\Lambda = \Psi_{\text{fp}}^{-1}(L)$ the corresponding form parameter for R. Let (σ, Γ) be an exact major form net of ideals of rank $2n$ over (R, Λ). Let $(\tau, B) = \Psi_{\text{fn}}(\sigma, \Gamma)$ denote the corresponding exact form net of rank $2mn$ over (Q, L). Then $\mathrm{EU}(\sigma, \Gamma) = \mathrm{EU}(\tau, B)$ and $\mathrm{U}(\sigma, \Gamma) = \mathrm{U}(\tau, B)$, where the equalities are understood modulo the isomorphism of groups $\mathrm{U}(2n, R, \Lambda)$ and $\mathrm{U}(2mn, Q, L)$. In particular, $\mathrm{EU}(2n, R, \Lambda) = \mathrm{EU}(2mn, Q, L)$ whenever $n \geq 2$. Further, suppose $h(\nu) \geq (4, 3)$ and (σ, Γ) is the form net of ideals associated with a subgroup $H \geq \mathrm{EU}(\nu, R, \Lambda)$. Then (τ, B) is the form net of ideals associated with H as a subgroup of $\mathrm{U}(2mn, Q, L)$.*

Proof. We will first show the inclusion $\mathrm{EU}(\sigma, \Gamma) \leq \mathrm{EU}(\tau, B)$. Pick any short elementary unitary matrix $T_{ij}(a) \in \mathrm{EU}(\sigma, \Gamma)$. Then for any possible indices k and l we have

$$a_{kl} e = \sum_{h=1}^{m} e_{hk} a e_{lh} \in \sigma_{ij}. \tag{3.103}$$

Therefore $a_{kl} \in \tau_{\text{ind}(i)+k, \text{ind}(j)+l}$ for all k and l. Thus by (3.101) we conclude that $T_{ij}(a) \in \mathrm{EU}(\tau, B)$. Pick any $T_{i,-i}(a) \in \mathrm{EU}(\sigma, \Gamma)$. Then $a_{kl} \in \tau_{\text{ind}(i)+k, \text{ind}(j)+l}$ for all $k > l^{\times}$. By relation (Γ2) we get

$$a_{k,k^{\times}} e_{k,k^{\times}} = e^{*}_{k^{\times},k^{\times}} a e_{k^{\times},k^{\times}} \in \Gamma_i \tag{3.104}$$

for all k. Thus $a_{k,k^{\times}} \in B_{\text{ind}(i)+k}$ for all k. Hence, by 3.102 we get the inclusion $T_{i,-i}(a) \in \mathrm{EU}(\tau, b)$. Therefore $\mathrm{EU}(\sigma, \Gamma) \leq \mathrm{EU}(\tau, B)$.

It's easy to see that $t_{\text{ind}(i)+k, \text{ind}(j)+l}(\xi) \in \mathrm{EU}(\sigma, \Gamma)$ whenever $i \neq j$. If $i = j$ then, as $h(\nu) \geq (4, 2)$, there exists an index $i' \sim i$ such that $i' \neq \pm i$. Therefore,

$$t_{\text{ind}(i)+k, \text{ind}(i)+l}(\xi) = [t_{\text{ind}(i)+k, \text{ind}(i')+k}(1), t_{\text{ind}(i')+k, \text{ind}(i)+l}(\xi)] \in \mathrm{EU}(\sigma, \Gamma). \tag{3.105}$$

Thus $\mathrm{EU}(\tau, B) \leq \mathrm{EU}(\sigma, \Gamma)$. The equality $\mathrm{U}(\tau, B) = \mathrm{U}(\sigma, \Gamma)$ follows straightforwardly from (3.103) and (3.104).

Assume $h(\nu) \geq (4, 3)$ and (σ, Γ) is the form net of ideals associated with a subgroup $H \geq \mathrm{EU}(\nu, R, \Lambda)$. It's easy to see that $\Psi_{\text{fn}}([\nu]_{(R,\Lambda)}) = [\nu]_{(Q,L)}$. By Lemma 3.2.9 the net (τ', B') associated with H as a subgroup of $\mathrm{U}(2mn, Q, L)$ is defined. Let $t_{\text{ind}(i)+k, \text{ind}(j)+l}(\xi)$ be an elementary unitary matrix in $\mathrm{EU}(2mn, Q, L)$ contained in H. If $i \neq j$ the inclusion $\xi \in (\tau, B)_{\text{ind}(i)+k, \text{ind}(j)+l}$ easily follows from the formulas (3.101) and (3.102). If $i = j$ then $\tau_{ij} = Q = \tau'_{ij}$. Therefore (τ, B) is the form net of ideals associated with H as a subgroup of $\mathrm{U}(2mn, Q, L)$. □

3.8 Sandwich classification over division rings

In this section we will prove Theorem 1 provided that the ground ring is either a division ring or a product of two copies of a division ring. We will show that form net subgroups over such rings are quite close to the corresponding elementary form net subgroup. This fact will be an important ingredient for the radical reduction described in the next section.

Throughout this section we fix a form ring $((R, \bar{\cdot}, \lambda), \Lambda)$, a unitary equivalence relation ν on the index set I and a subgroup H of $\mathrm{U}(2n, R, \Lambda)$ such that $\mathrm{EU}(\nu, R, \Lambda) \leq H$. We will refer to the results of Section 3.4 in the special case of the trivial standard setting $((R, \Lambda), (R, \Lambda), \{1\})$. The equivalence relation ν is automatically good for the standard setting above whenever $h(\nu) \geq (4, 3)$.

Proposition 3.8.1. *Assume $h(\nu) \geq (4, 3)$. Let (σ, Γ) be an exact major form net of ideals over (R, Λ), b a matrix in $\mathrm{U}(2n, R, \Lambda)$ and p some index in I. Let q and h be two indices in the equivalence class of p such that $q \neq \pm h$ (however p can be contained in $\{\pm q, \pm h\}$). Let c denote the conjugate ${}^{T_{qh}(\xi)}b$ of b by $T_{qh}(\xi)$ for some $\xi \in R$. Then:*

1. *Suppose $b_{ip} \in \sigma_{ip}$ for all $i \in I$. If $p = h$, suppose that $b_{iq} \in \sigma_{ip}$. If $p = -q$, suppose that $b_{i,-h} \in \sigma_{ip}$. Then $c_{ip} \in \sigma_{ip}$ for all $i \in I$.*

2. *Suppose $b \in \mathrm{U}(\sigma)$ and $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$. If $p = h$, suppose that $S_{-q,q}(b^{-1}) \in \Gamma_{-q}$. If $p = -q$, suppose that $S_{h,-h}(b^{-1}) \in \Gamma_h$. Then $S_{-p,p}(c^{-1}) \in \Gamma_{-p}$.*

Proof. Consider the matrix $d = T_{qh}(\xi)b$. Clearly, $c = dT_{qh}(-\xi)$. A direct calculation shows that

$$d_{ip} = b_{ip} + \delta_{iq}\xi b_{hp} - \delta_{i,-h}\lambda^{(\varepsilon(h)-1)/2}\bar{\xi}\lambda^{(1-\varepsilon(q))/2}b_{-q,p} \tag{3.106}$$

$$c_{ip} = d_{ip} - \delta_{ph}d_{iq}\xi + \delta_{p,-q}d_{i,-h}\lambda^{(\varepsilon(h)-1)/2}\bar{\xi}\lambda^{(1-\varepsilon(q))/2}. \tag{3.107}$$

Clearly, $d_{ip} = b_{ip} \in \sigma_{ip}$ whenever $i \neq q, -h$. The inclusion $d_{qp} \in \sigma_{qp} = R$ is trivial. It follows by (3.106) that $d_{-h,p} = b_{-h,p} - \lambda^{(\varepsilon(h)-1)/2}\bar{\xi}\lambda^{(1-\varepsilon(q))/2}b_{-q,p}$ and both $b_{-h,q}$ and $b_{-q,p}$ are contained in $\sigma_{-h,q}$. Thus $d_{-h,p}$ is contained in $\sigma_{-h,p}$. Therefore $d_{ip} \in \sigma_{ip}$ for all $i \in I$. Similarly, if $p = h$, then $d_{iq} \in \sigma_{ip}$ for all $i \in I$ and if $p = -q$ then $d_{i,-h} \in \sigma_{ip}$ for all $i \in I$. It follows by 3.107 that $c_{ip} \in \sigma_{ip}$ for all $i \in I$.

Assume $b \in \mathrm{U}(\sigma)$. Clearly $T_{qh}(\xi) \in \mathrm{U}(\sigma)$. By Corollary 3.2.6 we get

$$\begin{aligned} S_{-p,p}(c^{-1}) &= S_{-p,p}(T_{qh}(\xi)b^{-1}T_{qh}(-\xi)) \equiv S_{-p,p}(T_{qh}(\xi)b^{-1}) \\ &\equiv S_{-p,p}(b^{-1}) + \delta_{-p,q}\xi S_{h,-h}(b^{-1})\lambda^{(\varepsilon(h)-1)/2}\bar{\xi}\lambda^{(1-\varepsilon(q))/2} \\ &\quad + \delta_{-p,-h}\lambda^{(\varepsilon(h)-1)/2}\bar{\xi}\lambda^{(1-\varepsilon(q))/2}S_{-q,q}\xi \mod \Gamma_{-p}^{\min}. \end{aligned} \tag{3.108}$$

Relations $(\Gamma 2)$ and $(\Gamma 2')$ together with (3.108) give us the required inclusion $S_{-p,p}(c^{-1})$. □

The following lemma on generating ideals in products of division rings is trivial and is stated without a proof. Note that the angular brackets here stand for a two-sided ideal generated by the argument.

Lemma 3.8.2. *Let D be a division ring an let R be either equal to D or to $D \times D$. Let u and v be two elements of R. Then there exist an element ξ in $\{0,1\}$ if $R = D$ or $\{(0,0),(1,0),(0,1),(1,1)\}$ if $R = D \times D$ such that $\langle u, v\rangle \leq \langle u + \xi v\rangle$.*

Lemma 3.8.3. *Assume that $h(\nu) \geq (4,4)$ and that R is either a division ring D or a direct product $D \times D$ of two copies of a division ring. Let (σ, Γ) be the form net of ideals associated with H, a an element of H, $T_{sr}(\xi)$ a short (σ,Γ)-elementary unitary matrix and (p,q,h,l) an* A*-type base quadruple. Denote by b the short root element $aT_{sr}(\xi)a^{-1}$. Then $b_{ip} \in \sigma_{ip}$ for all $i \in I$. If additionally $b \in \mathrm{U}(\sigma)$ then also $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$.*

Proof. If $r = -s$ then the conclusion of this proposition follows directly from Corollary 3.4.6. Assume $r \neq \pm s$.

1. Assume that $a_{p,-r}, a_{q,-r}, a_{h,-r} \in \langle a_{l,-r}\rangle_R$. Then there exist elements ξ_p, ξ_q and ξ_h in R such that $a_{i,-r} = -\xi_i a_{l,-r}$ for $i = p,q,h$. Consider the matrix

$$c = T_{pl}(\xi_p)T_{ql}(\xi_q)T_{hl}(\xi_h)a.$$

Clearly, $c_{p,-r} = c_{q,-r} = c_{h,-r} = 0$ and $g = ca^{-1} \in \mathrm{EU}(\sigma,\Gamma)$. Denote by d the matrix $cT_{sr}(\xi)c^{-1} = {}^{g}b$. It's clear that ${}^{g^{-1}}d \in H$. By Corollary 3.4.6 we get the inclusions $d_{ip} \in \sigma_{ip}$ for all $i \in I$. By Proposition 3.8.1 we get the inclusions $b_{ip} \in \sigma_{ip}$ for all $i \in I$. Assume that $b \in \mathrm{U}(\sigma)$. Then $d \in \mathrm{U}(\sigma)$. By Corollary 3.4.6 we get the inclusion $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$. Finally, by Proposition 3.8.1 we also get the inclusion $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$.

2. Assume $a_{p,-r} \in \langle a_{l,-r}\rangle_R$. By Lemma 3.8.2 we may choose elements ζ_q, ζ_h and ζ_t in R such that $a_{p,-r}, a_{q,-r}, a_{h,-r} \in \langle c_{l,-r}\rangle_R$, where

$$c = T_{lq}(\zeta_q)T_{lh}(\zeta_h)a.$$

Let $f = cT_{sr}(\xi)c^{-1}$. By case 1 we have $d_{ip} \in \sigma_{ip}$. By Proposition 3.8.1 we get $b_{ip} \in \sigma_{ip}$. Assume $b \in \mathrm{U}(\sigma)$. Then $d \in \mathrm{U}(\sigma)$. By case 1 the inclusion $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$ holds. By Proposition 3.8.1 we get $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$.

3. Assume $a_{l,-r} \in \langle a_{p,-r}\rangle_R$. By case 2 we have the inclusions $b_{il} \in \sigma_{il}$ and $S_{-l,l}(b^{-1}) \in \Gamma_{-l}$. By Lemma 3.8.2 there exists an element $\zeta \in R$ such that $a_{p,-r} \in \langle a_{l,-r} + \zeta a_{p,-r}\rangle_R$. Consider the matrices $c = T_{lp}(\zeta)a$ and $d = cT_{sr}(\xi)c^{-1} = {}^{T_{lp}(\zeta)}b$. By case 2 we obtain $d_{ip} \in \sigma_{ip}$ for all $i \in I$. By Proposition 3.8.1 we get $b_{ip} \in \sigma_{ip}$. Assume $b \in \mathrm{U}(\sigma)$. Then $d \in \mathrm{U}(\sigma)$. By case 2 it follows that $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$. By Proposition 3.8.1 we get $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$.

If R is a division ring then the options 2 and 3 are exhaustive. Let $R = D \times D$ be a product of two copies of a division ring. Assume that neither of options 2 and 3 holds. Then the unordered pair of ideals $\langle a_{p,-r}\rangle_R$ and $\langle a_{l,-r}\rangle_R$ is $\{0\} \times D$ and $D \times \{0\}$. If $a_{h,-r} \in \langle a_{p,-r}\rangle_R$, or $a_{p,-r} \in \langle a_{h,-r}\rangle_R$ then by case 3 or 2, respectively, we get the required inclusions. In the opposite case, it's clear that $\langle a_{h,-r}\rangle_R = \langle a_{l,-r}\rangle_R$. Then there exists an element ζ such that $\zeta a_{h,-r} = a_{l,-r}$. Consider the matrices $c = T_{lh}(-\zeta)a$ and $d = {}^{T_{lh}(-\zeta)}b$. Clearly, $c_{l,-r} = 0$. Therefore $c_{l,-r} \in \langle c_{p,-r}\rangle_R$. By case 3 it follows that $d_{ip} \in \sigma_{ip}$ for all $i \in I$. By Proposition 3.8.1 it follows that $b_{ip} \in \sigma_{ip}$ for all $i \in I$. Assume $b \in \mathrm{U}(\sigma)$. Then $d \in \mathrm{U}(\sigma)$. It follows by case 3 that $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$. This together with Proposition 3.8.1 yields the inclusion $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$. □

Lemma 3.8.4. *Assume $h(\nu) \geq (4,4)$ and R is either a division ring D or a direct product $D \times D$ of two copies of a division ring. Let (σ, Γ) be the exact major form net of ideals associated with H, a an element of H, $T_{sr}(\xi)$ a short (σ, Γ)- elementary unitary matrix and (p, h) a* C*-type base pair. Denote by b the short root element $aT_{sr}(\xi)a^{-1}$. Then $b_{ip} \in \sigma_{ip}$ for all $i \in I$.*

Proof. If $r = -s$ then the conclusion of this proposition follows directly from Corollary 3.4.6.

Assume $r \neq \pm s$. Consider the following cases.

1. Assume $a_{h,-r}, a_{-h,-r} \in \langle a_{-p,-r} \rangle$. Pick elements ξ_h, ξ_{-h} such that $a_{h,-r} = -\xi_h a_{-p,-r}$ and $a_{-h,-r} = -\xi_{-h} a_{-p,-r}$. Let

$$c = T_{h,-p}(\xi_h) T_{-h,-p}(\xi_{-h}) a.$$

Clearly, $g = ca^{-1} \in \mathrm{EU}(\sigma, \Gamma)$. Let d denote the root element $cT_{sr}(\xi)c^{-1} = {}^{g^{-1}}b$. By construction, $c_{h,-r} = c_{-h,-r} = 0$. By Corollary 3.4.6 it follows that $d_{ip} \in \sigma_{ip}$ for all $i \in I$. By Proposition 3.8.1 we obtain $b_{ip} \in \sigma_{ip}$ for all $i \in I$.

2. Assume $a_{h,-r} \in R^*$. By case 1 we have $b_{i,-h} \in \sigma_{i,-h}$ for all $i \in I$. Let

$$c = T_{-p,h}((1 - a_{-p,-r}) a_{h,-r}^{-1}) a \qquad\qquad d = cT_{sr}(\xi)c^{-1}.$$

Then $c_{-p,-r} = 1$. By case 1 it follows that $d_{ip} \in \sigma_{ip}$ for all $i \in I$. By Proposition 3.8.1 we also obtain the inclusion $b_{ip} \in \sigma_{ip}$ for all $i \in I$.

2'. Assume $a_{-h,-r} \in R^*$. This case can be treated in the same way as case 2.

3. Assume that $a_{h,-r} = a_{-h,-r} = 0$. By Corollary 3.4.6 we obtain the required inclusion $b_{ip} \in \sigma_{ip}$ for all $i \in I$.

If $R = D$ then the cases 2, 2' and 3 are exhaustive. Assume $R = D \times D$. By Proposition 3.7.2 it follows that either R can be decomposed as a direct product of rings, each with involution with symmetry, or $\overline{D \times 0} = 0 \times D$. In the former case, the conclusion of this lemma follows from the case when $R = D$, Lemma 3.6.3 and Proposition 3.6.4. If $\overline{D \times 0} = 0 \times D$ then there exists an invertible element in $\Lambda^{\min}$. Indeed, $\overline{(1,0)} = (0, \alpha)$, where $\alpha \neq 0$, therefore

$$\alpha_0 = (1, -\alpha\lambda_2) = (1,0) - \overline{(1,0)}\lambda \in \Lambda^{\min} \cap R^*.$$

We will continue with a case by case analysis.

4. Assume $\langle a_{h,-r}, a_{-h,-r} \rangle = R$, but neither $a_{h,-r}$ nor $a_{-h,-r}$ is invertible. In this case $a_{h,-r}$ and $a_{-h,-r}$ generate two maximal ideals $D \times 0$ and $0 \times D$ of R. It's easy to see that $a_{h,-r} + \alpha_0 \lambda^{(-1-\varepsilon(h))/2} a_{-h,-r}$ is invertible. Let

$$c = T_{h,-h}(\alpha_0 \lambda^{(-1-\varepsilon(h))/2}) a \qquad\qquad d = cT_{sr}(\xi)c^{-1}.$$

Therefore $c_{h,-r}$ is invertible. By case 2 we get the inclusion $d_{ip} \in \sigma_{ip}$ for all $i \in I$. It's only left to notice that $b_{ip} = d_{ip}$ for all $i \neq h$ and that the inclusion $b_{hp} \in \sigma_{hp}$ is trivial.

5. Assume none of the cases 1, 2', 2, 3 or 4 hold. Then each of the entries $a_{h,-r}$ and $a_{-h,-r}$ generate the same maximal ideal of R. Without loss of generality we may assume

that this ideal is $D \times 0$. Then, as case 1 doesn't hold, it follows that $a_{-p,-r} \in 0 \times D$. We will split this case in two subcases.

5a. Assume $D \times 0 \leq \langle a_{p,-r} \rangle$. By case 1 we get the inclusion $b_{i,-p} \in \sigma_{ip}$ for all $i \in I$. It's easy to see that $D \times 0 \leq \langle \alpha_0 \lambda^{(\varepsilon(p)-1)/2} a_{p,-r} \rangle$. Consider the matrices

$$c = T_{-p,p}(\alpha_0 \lambda^{(\varepsilon(p)-1)/2}) a \qquad\qquad d = cT_{sr}(\xi)c^{-1}.$$

Clearly, $c_{h,-r} = a_{h,-r}$, $c_{-h,-r} = a_{-h,-r}$ and $D \times 0 \leq \langle c_{-p,-r} \rangle$. By case 2 it follows that $d_{ip} \in \sigma_{ip}$ for all $i \in I$. It's only left to notice that $b_{ip} = d_{ip} - d_{i,-p}\alpha_0 \lambda^{(\varepsilon(p)-1)/2}$ for all $i \neq -p$. Therefore $b_{ip} \in \sigma_{ip}$ for all $i \in I$.

5b. Assume $a_{p,-r} \leq 0 \times D$. Consider the matrices

$$c = T_{ph}(1) a \qquad\qquad d = cT_{sr}(\xi)c^{-1}.$$

It's easy to see that $D \times 0 \leq \langle c_{p,-r} \rangle$, $c_{h,-r} = a_{h,-r}$ and $c_{-p,-r} = a_{-p,-r}$. If $a_{-p,-r} = 0$ then $c_{-h,-r} = a_{-h,-r}$. By case 5a it follows that $d_{ip} \in \sigma_{ip}$ for all $i \in I$. If $a_{p,-r}$ generates the ideal $0 \times D$ then $c_{-h,-r}$ is invertible as a sum of the generator $a_{-h,-r}$ of the ideal $D \times 0$ and the generator $-\lambda^{(\varepsilon(h)-\varepsilon p)/2} a_{-p,-r}$ of the ideal $0 \times D$. By case 2' it follows that $d_{ip} \in \sigma_{ip}$ for all $i \in I$. As $b_{ip} = d_{ip}$ for all $i \neq p, -h$ it follows that $b_{ip} \in \sigma_{ip}$ for all $i \in I$. □

Combining the last two lemmas we get the following important result.

Corollary 3.8.5. *Assume that $h(\nu) \geq (4,4)$ and that R is either a division ring D or a direct product $D \times D$ of two copies of a division ring. Let $H \geq \mathrm{EU}(\nu, R, \Lambda)$ be a subgroup of $\mathrm{U}(2n, R, \Lambda)$ and (σ, Γ) the form net of ideals associated with H. Then*

$$H \leq \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma, \Gamma), \mathrm{U}(\sigma, \Gamma)).$$

Further, combining Corollary 3.8.5, Propositions 3.7.9, 3.6.5, 3.7.1, 3.6.4 and Lemma 3.6.3 we get the following corollary.

Corollary 3.8.6. *Assume that $h(\nu) \geq (4,4)$ and that R is a semisimple Artinian ring. Let H be a subgroup of $\mathrm{U}(2n, R, \Lambda)$ which contains $\mathrm{EU}(\nu, R, \Lambda)$ and let (σ, Γ) be the form net of ideals associated with H. Then*

$$H \leq \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma, \Gamma), \mathrm{U}(\sigma, \Gamma)).$$

We finish this section by describing the form net subgroup $\mathrm{U}(\sigma, \Gamma)$ over a division ring or a product of two copies of a division ring. This description is required for the radical reduction.

Lemma 3.8.7. *Let (σ, Γ) be a form* D*-net of ideals over an arbitrary form ring (R, Λ). Then $\Delta(2n, R)$ normalizes $\mathrm{EU}(\sigma, \Gamma)$, where $\Delta(2n, R)$ is defined as after Lemma 3.1.2.*

Proof. It's enough to check that for any generators $D_i(\theta)$ of $\Delta(2n, R)$ and $T_{jk}(\xi)$ of $\mathrm{EU}(\sigma, \Gamma)$ the conjugate $h = {}^{D_i(\theta)}T_{jk}(\xi)$ is contained in $\mathrm{EU}(\sigma, \Gamma)$. Clearly, $h = e$ whenever $i \neq j, -k$. Assume $i = j$.

If $k = -j = -i$ and $\xi \in \Gamma_i$ a direct matrix calculation shows that

$$h = T_{i,-i}(\theta\xi\lambda^{(\varepsilon(i)-1)/2}\overline{\theta}\lambda^{(1-\varepsilon(i))/2}).$$

By ($\Gamma 2'$) we get $h \in \mathrm{EU}(\sigma, \Gamma)$. If $k \neq \pm i$ and $\xi \in \sigma_{ik}$ then it's easy to check that

$$h = T_{ik}(\theta\xi)$$

and thus $h \in \mathrm{EU}(\sigma, \Gamma)$.

If $k = -i$, then by the relation (EU1) we get

$$T_{j,-i}(\xi) = T_{i,-j}(-\lambda^{(\varepsilon(-i)-1)/2)}\overline{\xi}\lambda^{(1-\varepsilon(j))/2})$$

and we already know that in this situation $h \in \mathrm{EU}(\sigma, \Gamma)$. □

Let $C_1, \dots C_K$ denote the self-conjugate equivalence classes of ν and $C_{K+1}, -C_{K+1}, \dots, C_N, -C_N$ the non-self-conjugate classes of ν. For each i from 1 to N fix a representative p_i of the class C_i. Denote by I' the set of all $\pm p_i$ for $i = 1, \dots, N$. Clearly, I' contains exactly one representative of each non-self-conjugate equivalence class and exactly two representatives $\pm p_i$, which are opposite by sign, of each self-conjugate class. Clearly $I' = -I'$. For each set I' constructed via the procedure above we define a subgroup $F_{I'}(R, \Lambda)$ of $\mathrm{U}(2n, R, \Lambda)$ as follows

$$F_{I'}(R, \Lambda) = \{a \in \mathrm{U}(2n, R) \mid a_{ij} = \delta_{ij} \text{ whenever } i \notin I' \text{ or } j \notin I'\}.$$

Lemma 3.8.8. *Assume R is a division ring or a product of two copies of a division ring. Let (σ, Γ) be an exact major form net of ideals over (R, Λ) and let I' be chosen as described immediately before this lemma. Then*

$$\mathrm{U}(\sigma, \Gamma) \leq \mathrm{EU}(\sigma, \Gamma) \cdot \Delta(2n, R) \cdot F_{I'}(R, \Lambda).$$

Proof. We will start with an element of $\mathrm{U}(\sigma, \Gamma)$ and multiply it on the left consecutively by elements of $\mathrm{EU}(\sigma, \Gamma) \cdot \Delta(2n, R)$ until we reach an element of $F_{I'}(R, \Lambda)$. Note, that by Lemma 3.8.7 the subgroups $\mathrm{EU}(\sigma, \Gamma)$ and $\Delta(2n, R)$ commute.

Pick a matrix a in $\mathrm{U}(\sigma, \Gamma)$. Let I'' denote the minimal subset of I such that $I' \subseteq I''$, $I'' = -I''$ and such that $a_{pq} = \delta_{pq}$ whenever $p \in I \setminus I''$ and $q \in I \setminus I''$. If $I'' = I'$ then $a \in F_{I'}(R, \Lambda)$. If $I'' \supsetneq I'$ then we will show that there exists an index $i \in I'' \setminus I'$ and an element of $d \in \mathrm{EU}(\sigma, \Gamma) \cdot \Delta(2n, R) \cdot a$ such that $d_{pq} = \delta_{pq}$ whenever $p \in I \setminus (I'' \setminus \{\pm i\})$ or $q \in I \setminus (I'' \setminus \{\pm i\})$. Clearly $d \in \mathrm{U}(\sigma, \Gamma)$, because (σ, Γ) is a major form net of ideals, thus $\mathrm{U}(\sigma, \Gamma)$ contains $\Delta(2n, R)$. Repeating this procedure sufficient number of times, we will end up with a matrix in $\mathrm{EU}(\sigma, \Gamma) \cdot \Delta(2n, R) \cdot F_{I'}(R, \Lambda)$.

Step 1. Let (i, k) be an A-type base pair such that $i \in I'' \setminus I'$ and $k \in I''$. In this step we will construct a matrix b in $\mathrm{EU}(\sigma, \Gamma) \cdot \Delta(2n, R) \cdot a$ such that $b_{ii} \in R^*$ and $b_{pq} = \delta_{pq}$

whenever $p \in I \setminus I''$ or $q \in I \setminus I''$. If a_{ii} is already invertible, simply put $b = a$. Suppose a_{ii} is not invertible. First, assume that R is a division ring. Then $a_{ii} = 0$. Observe, that there exists an index $j \neq i$ such that $a_{ji} \neq 0$ and $\sigma_{ij} = R$, otherwise

$$1 = \sum_{j \in I} a'_{ij} a_{ji} \leq \sum_{j \in I \setminus \{i\}} \sigma_{ij} a_{ji} = 0,$$

which is impossible. Suppose such j can be chosen in $I'' \setminus \{\pm i\}$ and consider the matrix $b = T_{ij}(1)a$. Clearly, $b_{ii} = a_{ji}$ is invertible. As both i and j are contained in I'', the condition that $b_{pq} = \delta_{pq}$ whenever $p, q \in I \setminus I''$ is preserved. If the only such j is equal to $-i$, by the choice of j we have $a_{ki} = 0$ (as $\sigma_{ik} = \sigma_{ii} = R$) and $a_{-k,i} = 0$ (as $\sigma_{i,-k} = \sigma_{i,-i} = R$). Then we can consider the matrix $b = T_{i,-k}(1)T_{-k,-i}(1)a$. Clearly, $b_{ii} = a_{i,-i}$ is invertible. Again, $\pm i, \pm k \in I''$ and thus the condition that $b_{pq} = \delta_{pq}$ whenever $p \in I \setminus I''$ or $q \in I \setminus I''$ is preserved.

Assume R is the product $D \times D$ of two copies of a division ring D. R has precisely two maximal ideals $D \times 0$ and $0 \times D$. Assume, a_{ii} is contained in $D \times 0$. We will construct a matrix c in $\mathrm{EU}(\sigma, \Gamma) \cdot \Delta(2n, R) \cdot a$ such that $\langle (c_{ii})_1 \rangle_D = \langle (a_{ii})_1 \rangle_D$ and $(c_{ii})_2 \neq 0$, where $c_{ii} = ((c_{ii})_1, (c_{ii})_2)$ and $a_{ii} = ((a_{ii})_1, (a_{ii})_2)$. If $(a_{ii})_1 \neq 0$ then we can assume that $(a_{ki})_1 = 0$. Otherwise, we can replace a with $T_{ki}(-a_{ki}(((a_{ii})_1)^{-1}, 0))a$. Additionally, we will ensure that $c_{pq} = \delta_{pq}$ whenever $p \in I \setminus I''$ or $q \in I \setminus I''$. Observe, that there exists an index $j \in I''$ such that $a_{ji} \notin D \times 0$ and $0 \times D \leq \sigma_{ij}$. Indeed, assume the contrary, then $\sigma_{ji} \leq D \times 0$ and

$$1 = \sum_{j \in I} a'_{ij} a_{ji} \leq \sum_{j \in I \setminus \{i\}} \sigma_{ij} a_{ji} \leq D \times 0,$$

which is impossible. If such j can be chosen not equal to $\pm i$, put $c = T_{ij}((0,1))a$. If the only such j can be chosen equal to $-i$, put $c = T_{i,-k}((0,1))T_{-k,-i}((0,1))a$. Clearly, in any of the two cases above the matrix c satisfies all the required conditions. Similarly we can construct a matrix b in $\mathrm{EU}(\sigma, \Gamma) \cdot \Delta(2n, R) \cdot c$ such that $(b_{ii})_2 = (c_{ii})_2 \neq 0$ and $(b_{ii})_1 \neq 0$, where $c_{ii} = ((c_{ii})_1, (c_{ii})_2)$ and $b_{ii} = ((b_{ii})_1, (b_{ii})_2)$. Therefore $b_{ii} \in R^*$. It is easy to see that the operation above preserves the condition $b_{pq} = \delta_{pq}$ whenever $p \in I \setminus I''$ or $q \in I \setminus I''$.

Step 2. In this step we will replace the matrix b with a matrix d in $\mathrm{EU}(\sigma, \Gamma) \cdot \Delta(2n, R) \cdot b$ such that $d_{pq} = \delta_{pq}$ whenever $p \in I \setminus (I'' \setminus \{\pm i\})$ or $q \in I \setminus (I'' \setminus \{\pm i\})$. Without loss of generality we may assume that $b_{ii} = 1$. Indeed, we can always replace b with with the matrix $D_i(b_{ii}^{-1})b$. Consider the matrix

$$c = \prod_{\varepsilon(j) = -\varepsilon(i), j \neq -i} T_{ji}(-b_{ji}) T_{-j,i}(-b_{-j,i}) b.$$

It's clear that $c_{ji} = 0$ whenever $j \neq \pm i$ and $c_{ii} = 1$. Finally,

$$\begin{aligned} c_{-i,i} &= b_{-i,i} + \sum_{\varepsilon(j) = -\varepsilon(i), j \neq -i} \lambda^{(\varepsilon(i)-1)/2} \overline{b_{-j,i}} \lambda^{(1+\varepsilon(j))/2} b_{ji} \\ &= 1 \cdot b_{-i,i} \sum_{\varepsilon(j) = -\varepsilon(i), j \neq -i} b'_{-ij} b_{ji} \sum_{\varepsilon(j) = -\varepsilon(i)} b'_{-ij} b_{ji} = \pm S_{-i,i}(b^{-1}) \in \Gamma_{-i}. \end{aligned}$$

Thus, $T_{-i,i}(-c_{-i,i}) \in \mathrm{EU}(\sigma,\Gamma)$. Put $g = T_{-i,i}(-c_{-i,i})c$. Clearly, $g_{*i} = e_{*i}$ and consequently $g_{-i,*} = e_{-i,*}$. In particular, $g_{-i,-i} = 1$. Consider the matrix

$$f = \prod_{\varepsilon(j)=\varepsilon(i), j\neq i} T_{j,-i}(-g_{j,-i})T_{-j,-i}(-g_{-j,-i})g.$$

Notice that $f_{*i} = e_{*i}$, $f_{-i,*} = e_{-i,*}$ and $f_{j,-i} = 0$ whenever $j \neq \pm i$. As before, one can easily check that $f_{i,-i} = \pm S_{i,-i}(g^{-1}) \in \Gamma_i$. Finally, put $d = T_{i,-i}(-f_{i,-i})f$. By construction, $d \in \mathrm{EU}(\sigma,\Gamma)\cdot\Delta(2n,R)\cdot a$. It's easy to see that $d_{pq} = \delta_{pq}$ whenever $p \in I \setminus (I'' \setminus \{\pm i\})$ or $q \in I \setminus (I'' \setminus \{\pm i\})$. □

Let (Q,L) be a form ring. Consider the form ring (R,Λ), where $R = \mathrm{M}(m,Q)$ and $\Lambda = \Psi_{\mathrm{fp}}^{-1}(L)$. Fix a unitary equivalence relation ν on I_{2n} and let I' be chosen as before Lemma 3.8.8. Consider the subset $I'' = \{\pm(\mathrm{ind}(i)+1) \mid i \in I', i > 0\}$ of I_{2mn}. Using Proposition 3.7.7 it's easy to see that $F_{I''}(Q,L) \leq F_{I'}(R,\Lambda)$. Further, $\Delta(Q,L) \leq \Delta(R,\Lambda)$. This combined with Propositions 3.7.9, 3.6.4 and Lemma 3.6.3 allows us to deduce the following corollary.

Corollary 3.8.9. *Assume $h(\nu) \geq (4,3)$ and R is a semi-simple Artinian ring. Let (σ,Γ) be an exact major form net of ideals over (R,Λ) and let I' be chosen as described before Lemma 3.8.8. Then*

$$\mathrm{U}(\sigma,\Gamma) \leq \mathrm{EU}(\sigma,\Gamma)\cdot\Delta(2n,R)\cdot F_{I'}(R,\Lambda).$$

3.9 Radical reduction

Throughout this section we fix a standard setting $((R,\Lambda),(R',\Lambda'),S)$, a unitary equivalence ν on I and a subgroup H of $\mathrm{U}(2n,R,\Lambda)$. We will study the factor of a standard setting $((R,\Lambda),(R',\Lambda'),S)$ by the Jacobson radical J of the ring R. First of all, it's easy to see that the Jacobson radical is invariant under the action of the involution with symmetry $\overline{\cdot}$. Indeed, an element ξ in R belongs to J if and only if $1-\xi\zeta$ is left invertible for all $\zeta \in R$. As $\overline{\cdot}$ is an anti-automorphism, this is equivalent to saying that $\overline{1-\xi\zeta} = 1-\overline{\zeta}\cdot\overline{\xi}$ is right-invertible for all $\zeta \in R$. As $\overline{R} = R$, it follows that $\overline{\xi}$ is also in the Jacobson radical. Therefore we may consider the factor form ring $(R/J,\Lambda/\Omega_J)$, where $\Omega_J = J \cap \Lambda$ is the maximal relative form parameter of level J. Next, $J' = R' \cap J$ is obviously a two-sided ideal in R'. It's also clear, that

$$\overline{J'} = \overline{R' \cap J} = \overline{R'} \cap \overline{J} = R' \cap J = J'.$$

Therefore we may consider the factor form ring $(R'/J',\Lambda'/\Omega_{J'})$, where $\Omega_{J'} = \Lambda' \cap J'$ is the maximal relative form parameter in R' of level J'. Notice that, as $\Lambda' \leq \Lambda$, it follows that

$$\Omega_{J'} = \Lambda' \cap J' = R' \cap \Lambda \cap \Lambda' \cap J = (\Lambda \cap J) \cap (R' \cap \Lambda') = \Omega_J \cap \Lambda'.$$

Therefore $(R'/J', \Lambda'/\Omega_{J'})$ can be viewed as a form subring of $(R/J, \Lambda/\Omega_J)$. Clearly, the subset $S/J = \{x + J' \mid x \in S\}$ in R'/J' is a multiplicative set which is contained in $(R/J)^* \cap \mathrm{Center}(R/J) \cap \{\xi \in R/J \mid \overline{\xi} = \xi\}$. Summing up,

$$((R/J, \Lambda/\Omega_J), (R'/J', \Lambda'/\Omega_{J'}), S/J)$$

is a standard setting and the reduction morphism $\rho : R \to R/J$ is a morphism of standard settings. Denote by $\mathrm{U}(J)$ the principal congruence subgroup $\mathrm{U}(2n, (R, \Lambda), (J, \Omega_J)) = \mathrm{U}(2n, R, \Lambda) \cap \mathrm{GL}(2n, R, J)$ of $\mathrm{U}(2n, R, \Lambda)$ of level (J, Ω_J). Note that $\mathrm{U}(J)$ is a normal subgroup in $\mathrm{U}(2n, R, \Lambda)$.

Lemma 3.9.1. *Assume that $h(\nu) \geq (4, 5)$ and that either $h(\nu) \geq (6, 5)$ or $R\Lambda + \Lambda R = R$. Further, assume $\Lambda' = \Lambda \cap R'$. Let H be a subgroup of $\mathrm{U}(2n, R, \Lambda)$ and (σ', Γ') an exact major form net of ideals which is S-associated with H. Let (σ, Γ) denote the S-closure of (σ', Γ') in (R, Λ). Then $(\sigma' + J', \Gamma' + \Omega_{J'})$ (cf. Lemma 3.2.3) is an exact major form net of ideals which is S-associated with the subgroup $\mathrm{U}(J) \cdot H$ and the form net of ideals $(\sigma + J, \Gamma + \Omega_J)$ is the S-closure of $(\sigma' + J', \Gamma' + \Omega_{J'})$.*

Proof. The inclusion $\mathrm{EU}(\sigma' + J', \Gamma' + \Omega_{J'}) \leq \mathrm{U}(J) \cdot H$ is straightforward. Pick any $T_{sr}(\xi) \in \mathrm{U}(J) \cdot H$, where $\xi \in R$. If $s \sim r$ the existence of an element $x \in S$ such that $x\xi \in (\sigma', \Gamma')_{sr} \leq (\sigma' + J', \Gamma' + \Omega_{J'})_{sr}$ is guaranteed by definition of a standard setting. Assume that $s \nsim r$. Pick any element $g \in \mathrm{U}(J)$ such that $h = gT_{sr}(\xi) \in H$. Our immediate goal is to show that $h \in \mathrm{U}(\sigma)$. Clearly, $h_{i*} \equiv e_{i*} \mod J$ whenever $i \neq s, -r$ and $h_{*j} \equiv e_{*j} \mod J$ whenever $j \neq r, -s$.

Pick any index $p \in I$. Assume, the equivalence class of p is self-conjugate. Then we can always choose an index $l \sim p$ such that $l \neq \pm p$ and either $h_{i*} \equiv e_{i*} \mod J$ for $i \in \{p, -l\}$ or $h_{*i} \equiv e_{*i} \mod J$ for $i \in \{p, -l\}$. By Corollary 3.5.7 it follows that $h_{ip} \in \sigma_{ip}$ for all $i \in I$. Next, assume that the equivalence class of p is non-self-conjugate and $p \neq s, -r$. If the equivalence class of p contains not more than one element of the set $\{s, -r\}$, clearly there exists an A-type base quadruple (p, q, t, l) such that $a_{i*} \equiv e_{i*} \mod J$ and $a_{*,-i} \equiv e_{*,-i} \mod J$ whenever $i \in \{p, q, t, l\}$. By Corollary 3.5.4 it follows that $a_{ip} \in \sigma_{ip}$ for all $i \in I$. Assume the equivalence class of p contains both s and $-r$. As $s \nsim r$, it follows that $p \neq \pm s, \pm r$. Recall that $h(\nu) \geq (4, 5)$. Therefore there exist at least one more index $l \sim p$ such that $l \neq \pm p, \pm s, \pm r$. Then $h_{-i,*} \equiv e_{-i,*} \mod J$ and $h_{*i} \equiv a_{*i} \mod J$ whenever $i \in \{p, l, s, -r\}$. Additionally, $h_{*l} \equiv e_{*l}$. By Corollary 3.5.5 we get $h_{ip} \in \sigma_{ip}$ for all $i \in I$. Now we have to consider the cases when $p \in \{s, -r\}$. Assume, $p = s$. Recall that $s \nsim r$. Therefore there exists at least three more indices $q, l, t \in I$ such that (s, q, l, t) is an A-type base quadruple and $r \neq \pm q, \pm l, \pm t$. Consider the matrix $f = T_{st}(-1)hT_{st}(1)$. Clearly, $f_{-i,*} \equiv e_{-i,*}$ and $f_{*i} \equiv e_{*i}$ whenever $i \in \{s, q, h, t\}$. Moreover $f_{*l} \equiv e_{*l}$. By Corollary 3.5.5 it follows that $f_{it} \in \sigma_{it}$. It's only left to notice that $f_{it} = h_{it} + h_{is}$ whenever $i \neq s, -t$. We we have already shown that $h_{it} \in \sigma_{is}$, therefore $h_{is} \in \sigma_{is}$ for $i \neq -t$. Finally,

$$f_{-t,t} = h_{-t,t} + h_{-t,s} - h_{-s,t} - h_{-s,s} \equiv h_{-t,s} \mod \sigma_{-t,s}.$$

Therefore $h_{is} \in \sigma_{is}$ for all $i \in I$. The case when $p = -r$ can be treated in the same way. Therefore, $h \in \mathrm{U}(\sigma)$.

Assume $s \neq -r$. Then $g_{sr} + g_{ss}\xi = h_{sr} \in \sigma_{sr}$. Recall that $g_{sr} \in J$ and g_{ss} is invertible, therefore $\xi \in \sigma_{sr} + J$. It's obvious that there exists an element $x \in S$ such that $x\xi \in \sigma'_{sr} + J'$. Let $r = -s$. Note that by assumption, $s \not\sim r = -s$. Thus the equivalence class of s is non-self-conjugate. Therefore there exist three more indices $q, t, l \in I$ such that (s, q, t, l) is an A-type base quadruple. It's easy to see that $h_{i*} \equiv e_{i*}$ and $h_{*,-i} \equiv e_{*,-i}$ whenever $i \in \{-s, -q, -h, -l\}$. By Corollary 3.5.4 it follows that $S_{s,-s}(h^{-1}) \in \Gamma_s$. By Proposition 3.2.5 we have

$$\begin{aligned}
\Gamma_s \ni S_{s,-s}(h^{-1}) = S_{s,-s}(T_{s,-s}(-\xi)g^{-1}) &= -\xi + \sum_k a_{sk} S_{k,-k}(g^{-1}) a'_{-k,-s} \\
&- \sum_{j,k,l>0} (a_{sk} g'_{k,-j} g_{-j,l} a'_{l,-s} \\
&\quad - \lambda^{-(\varepsilon(s)+1)/2} \overline{(a_{sk} g'_{k,-j} g_{-j,l} a'_{l,-s})} \lambda^{(1-\varepsilon_s)/2}) \\
&- \sum_{k,j>0;l>k} ((a_{s,-k} g'_{-k,-j} g_{-j,l} a'_{l,-s} \\
&\quad - \lambda^{-(\varepsilon(s)+1)/2} \overline{(a_{s,-k} g'_{k,-j} g_{-j,l} a'_{l,-s})} \lambda^{(1-\varepsilon(s))/2}) \\
&+ (a_{sk} g'_{k,-j} g_{-j,-l} a'_{-l,-s} \\
&\quad - \lambda^{-(\varepsilon(s)+1)/2} \overline{(a_{sk} g'_{k,-j} g_{-j,-l} a'_{-l,-s})} \lambda^{(1-\varepsilon(s))/2})),
\end{aligned} \tag{3.109}$$

where a stands for $T_{s,-s}(-\xi)$. It's easy to see that every term of the last two big sums in 3.109 contains a factor $g'_{k,-j}$, $\overline{g'_{k,-j}}$, $g_{-j,l}$ or $\overline{g_{-j,l}}$, which is contained in J. Moreover, these two big sums are obviously contained in in $\Lambda_s^{\min}$ and therefore in $(\Omega_J)_s$. Every length $S_{k,-k}(g^{-1})$ is contained in $(\Omega_J)_k$ for the same reason. Thus

$$a_{sk} S_{k,-k}(g^{-1}) a'_{-k,-s} = a_{sk} S_{k,-k}(g^{-1}) \lambda^{(\varepsilon(k)-1)/2} \overline{a_{sk}} \lambda^{(1-\varepsilon(s))/2} \in (\Omega_J)_s,$$

where the equality is due to Lemma 3.1.2 and the inclusion is due to property ($\Gamma 2'$). This combined with (3.109) yields the inclusion $-\xi \in \Gamma_s + (\Omega_J)_s$. Again, it's clear that there exists an element $x \in S$ such that $x^2\xi \in \Gamma'_s + (\Omega_{J'})_s$. Thus we have proved that $(\sigma' + J', \Gamma' + \Omega_{J'})$ is a form net of ideals which is S-associated with H. The fact that the net $(\sigma + J, \Gamma + \Omega_J)$ is the S-closure in R of the net $(\sigma' + J', \Gamma' + \Omega_{J'})$ is straightforward. □

Corollary 3.9.2. *Assume that $h(\nu) \geq (4, 5)$ and that either $h(\nu) \geq (6, 5)$ or $R\Lambda + \Lambda R = R$. Let $\rho : R \to R/J$ denote the reduction morphism and assume that the restriction $\rho|_{R'} : R' \to R/J$ of ρ is surjective. Let H be a subgroup of $\mathrm{U}(2n, R, \Lambda)$, (σ', Γ') an exact major form net of ideals which is S-associated with H, and (σ, Γ) the S-closure of (σ', Γ') in (R, Λ). Then $(R'/J', \Lambda'/\Omega_{J'}) = (R/J, \Lambda/\Omega_J)$ and the images (σ'_J, Γ'_J) and (σ_J, Γ_J) under ρ of the form nets of ideals (σ', Γ') and (σ, Γ), respectively, coincide and each is the net associated with $H_J = \mathrm{M}(\rho)(H)$.*

Proof. The surjectivity of $\rho|_{R'}$ immediately yields the equality of the form rings $(R'/J', \Lambda'/\Omega_{J'})$ and $(R/J, \Lambda/\Omega_J)$. Clearly, $H_J = \mathrm{M}(\rho)(\mathrm{U}(J) \cdot H)$ and (σ'_J, Γ'_J) coincides with the image of $(\sigma' + J', \Gamma' + \Omega_{J'})$. By Lemma 3.9.1 we know that $(\sigma' + J', \Gamma' + \Omega_{J'})$ is an exact form net of ideals which is S-associated with $\mathrm{U}(J)\cdot H$ and that $(\sigma+J,\Gamma+\Omega_J)$ is the S-closure of $(\sigma'+J',\Gamma'+\Omega_{J'})$. As $\mathrm{U}(J)\cdot H$ contains the kernel $\mathrm{U}(J)$ of the reduction morphism ρ, by Lemma 3.6.1 it follows that (σ'_J,Γ'_J) is a net S/J-associated with H_J and that (σ_J,Γ_J) is the S-closure of (σ'_J,Γ'_J). By Remark 3.3.5 it follows that the form net of ideals (σ'_J,Γ'_J) is the net associated with H_J and coincides with (σ_J,Γ_J). □

Corollary 3.9.2 shows, in particular, that the full pre-image of $\mathrm{EU}(\sigma_J,\Gamma_J)$ under ρ is contained in $\mathrm{EU}(\sigma',\Gamma')\cdot\mathrm{U}(J)$. Now we perform the radical reduction.

Lemma 3.9.3. *Assume that $h(\nu)\geq(4,5)$ and that either $h(\nu)\geq(6,5)$ or $R\Lambda+\Lambda R=R$. Let $((R,\Lambda),(R',\Lambda'),S)$ be a standard setting such that R is a semilocal ring and the canonical morphism $R'\to R/J$ is surjective. Let H be a subgroup of $\mathrm{U}(2n,R,\Lambda)$, (σ',Γ') an exact major form net which is S-associated with H, and (σ,Γ) the S-closure of (σ',Γ') in (R,Λ). Then*

$$H\leq\mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\sigma',\Gamma'),\mathrm{U}(\sigma,\Gamma)).$$

Proof. Let $T_{sr}(\xi)$ be a (σ',Γ')-elementary unitary matrix and a an element in H. Consider the matrix $b=aT_{sr}(\xi)a^{-1}$. We will show that $b\in\mathrm{U}(\sigma,\Gamma)$. Denote by ρ the reduction morphism $R\to R/J$. Combining Corollary 3.8.6 with Corollary 3.9.2 we obtain the inclusion $\rho(b)\in\mathrm{U}(\sigma_J,\Gamma_J)$, where (σ_J,Γ_J) is the image of (σ',Γ') under ρ. By Corollary 3.8.9 the inclusion

$$\mathrm{U}(\sigma_J,\Gamma_J)\leq\mathrm{EU}(\sigma_J,\Gamma_J)\cdot\Delta(2n,R/J)\cdot F_{I'}(R/J,\Lambda/(J\cap\Lambda))$$

holds for any choice of I'. As we have mentioned after Corollary 3.9.2, the pre-image of $\mathrm{EU}(\sigma_J,\Gamma_J)$ under ρ is contained in $\mathrm{EU}(\sigma',\Gamma')\cdot\mathrm{U}(J)$. Further, the pre-image of $\Delta(2n,R/J)\cdot F_{I'}(R/J,\Lambda/(J\cap\Lambda))$ under ρ is contained in the group $FJ_{I'}(R,\Lambda)$ defined as follows:

$$\begin{aligned}FJ_{I'}(R,\Lambda)=\{a\in\mathrm{U}(2n,R,\Lambda)\mid{}& a_{ii}\in R^*\text{ whenever }i\notin I',\\ & a_{ij}\in J\text{ whenever }i\neq j\text{ and either }i\notin I'\text{ or }j\notin I'\}.\end{aligned}$$

Note, that $FJ_{I'}(R,\Lambda)$ contains $\mathrm{U}(J)$. Summing up, we get the inclusion

$$b\in\mathrm{EU}(\sigma',\Gamma')\cdot FJ_{I'}(R,\Lambda)$$

for any choice of I'. Fix any decomposition $b=c\cdot d$, where $c\in\mathrm{EU}(\sigma',\Gamma')$ and $d\in FJ_{I'}(R,\Lambda)$. Clearly $d=c^{-1}b$ is also contained in H. Thus $d\in H\cap FJ_{I'}(R,\Lambda)$.

Fix an index $p\notin I'$. Assume that the equivalence class of p is non-self-conjugate, then we can chose an A-type base quintuple (p,q,h,t,l) such that $p,q,h,t\notin I'$. By Corollary 3.5.4 we get $d_{ip}\in\sigma_{ip}$ for all $i\in I$ and, as $c\in\mathrm{U}(\sigma,\Gamma)$, also $b_{ip}\in\sigma_{ip}$ for all $i\in I$.

Assume p is self-conjugate. If $h(\nu)\geq(6,5)$ then we can choose a C-type base pair (p,h) such that h is also not contained in I'. By Corollary 3.5.7 we get the inclusions $d_{ip}\in\sigma_{ip}$ for all $i\in I$. Thus $b_{ip}\in\sigma_{ip}$ for all $i\in I$. Finally, if $R\Lambda+\Lambda R$ then by Lemma 3.5.8 we get $d_{ip}\Lambda\leq\sigma_{ip}$ and thus also $b_{ip}\Lambda\leq\sigma_{ip}$ for all $i\in I$.

Applying the results above to all possible sets of representatives I', we get the inclusions $b_{ip} \in \sigma_{ip}$ for all $i \in I$ and all p such that $p \nsim -p$. If $h(\nu) \geq (6,5)$ we also have the inclusions $b_{ip} \in \sigma_{ip}$ for all $i \in I$ and all p such that $p \sim -p$. Finally, if $R\Lambda + \Lambda R = R$ then we have the inclusions $b_{ip}\Lambda \leq \sigma_{ip}$ for all $i \in I$ and all $p \sim -p$. Therefore if $h(\nu) \geq (6,5)$ we already have $b \in \mathrm{U}(\sigma)$. Assume $R\Lambda + \Lambda R = R$. Fix some C-type base pair (p,h) and some $\eta \in R$. Consider the matrix $f = T_{ph}(\eta)a$ and $g = fT_{sr}(\xi)f^{-1}$. Applying the results above to the matrix g we get the inclusions $g_{ih}\Lambda \leq \sigma_{ih}$ for all $i \in I$. Note that $g_{ih} = b_{ih} + b_{ip}\eta$ for all $i \neq p, -h$ and thus $b_{ip}\eta\Lambda \leq \sigma_{ip}$ for all $i \in I$. By arbitrariness of choice of η, it follows that $b_{ip}R\Lambda \leq \sigma_{ip}$ for all $i \in I$. Thus $b_{ip} \in b_{ip}R = b_{ip}R\Lambda + b_{ip}\Lambda R \leq \sigma_{ip}$ for all $i \in I$. We conclude that $b \in \mathrm{U}(\sigma)$.

For any $p \nsim -p$ we can choose I' in such a way that $p \notin I'$. By Corollary 3.5.4 we get $S_{-p,p}(d^{-1}) \in \Gamma_{-p}$ for any choice of p. Note, that $d^{-1} = b^{-1}c$ and both b and c are contained in $\mathrm{U}(\sigma,\Gamma)$. By Proposition 3.2.5 the following congruence holds:

$$S_{-p,p}(d^{-1}) = S_{-p,p}(b^{-1}c) \equiv S_{-p,p}(b^{-1}) + \sum_{k\in I} b'_{-p,k}S_{k,-k}(c)b_{-k,p} \mod \Gamma^{\min}_{-p}. \tag{3.110}$$

As $c \in \mathrm{EU}(\sigma,\Gamma)$, it follows that $S_{k,-k}(c) \in \Gamma_k$ for all $k \in I$. Finally, combining property (U1) of Lemma 3.1.2 with property (Γ2) of a form net of ideals we get

$$b'_{-p,k}S_{k,-k}(c)b_{-k,p} = \lambda^{(\varepsilon(p)-1)/2}\overline{b_{-k,p}}\lambda^{(1+\varepsilon(k))/2}S_{k,-k}(c)b_{-k,p} \in \Gamma_{-p} \tag{3.111}$$

for all $k \in I$. Substituting (3.111) into (3.110) we conclude that $S_{-p,p}(b^{-1}) \in \Gamma_{-p}$ As p was chosen arbitrarily it follows that $b^{-1} \in \mathrm{U}(\sigma,\Gamma)$ and thus $b \in \mathrm{U}(\sigma,\Gamma)$. □

3.10 Localization

Proposition 3.10.1. *Let $(R, \overline{\cdot}, \lambda)$ be a ring with involution with symmetry, where R is finitely generated as a module over its center. Then R is the direct limit of a directed system $\{(R_i, \overline{\cdot}, \lambda)\}_\Theta$ of subrings with involution with symmetry of $(R, \overline{\cdot}, \lambda)$ such that for each index $i \in \Theta$ there exists an involution invariant finitely generated Noetherian subring C_i in the center of R_i such that R_i is a finitely generated C_i-module, $\overline{c} = c$ for any $c \in C_i$ and any form parameter Λ_i for R_i is a C_i-module.*

Proof. Let L denote the center of R. Clearly the ring L is involution invariant, i.e. $\overline{L} = L$. By assumption,

$$R = x_1L + \cdots + x_NL$$

for some $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in R$. For each i from 1 to N the product x_ix_j can be expressed as a sum $\sum_{k=1}^N x_ka_{ijk}$, where $a_{ijk} \in L$. For each i from 1 to N the element $\overline{x_i}$ can be expressed as a sum $\sum_{k=1}^N x_kb_{ik}$, where $a_{ik} \in L$. Finally, $\lambda = \sum_{k=1}^N x_kc_{ik}$, where $c_{ik} \in L$. Consider the ring

$$K = \mathbb{Z}[a_{ijk}, \overline{a_{ijk}}, b_{ik}, \overline{b_{ik}}, c_{ik}, \overline{c_{ik}} \mid 1 \leq i,j,k \leq N].$$

It is clear, that K is an involution-invariant, finitely generated, commutative (and thus Noetherian) ring that contains λ. Clearly, L is a K-algebra. Moreover L is the limit of the directed set $\{L_i\}_{i\in\Theta}$ of all involution invariant K-subalgebras L_i of L which are finitely generated over K (and thus also Noetherian).

Fix an index $i \in \Theta$ set $R_i = x_1 L_i + \dots x_N L_i$. It's easy to see that R_i is an involution invariant subring of R such that $\lambda \in R_i$ for each $i \in \Theta$. It is also clear that $R = \varinjlim R_i$. For each $i \in \Theta$, let C_i be the ring spanned by all elements $c\overline{c}$, where $c \in L_i$. Our goal is to show that R_i is is finitely generated as a C_i-module. We represent here the proof by Hazrat ([Haz02, Lemma 3.7]). Note that C_i also contains the elements of the shape $c + \overline{c}$ for any $c \in L_i$. Indeed

$$c + \overline{c} = (c+1)\overline{(c+1)} - c\overline{c} - 1.$$

Pick elements $a_1, \dots, a_k \in L_i$ such that they generate L_i as a $\mathbb{Z}$-algebra (i.e. any element of L_i is a polynomial in variables $a_1, \dots, a_k$ and coefficients in $\mathbb{Z}$). Then L is also a finitely generated ring over C_i (with the same set of generators). Next, L_i is an integral extension of C_i. Indeed, any of it's $\mathbb{Z}$-generators a_i satisfies a monic polynomial

$$X^2 - (a_i + \overline{a}_i)X + a_i\overline{a}_i.$$

Therefore L_i is an integral extension of C_i and a finitely generated ring over C_i. It's well known that in this case L_i is a finitely generated module over C_i (see for example [Kap70, p.11, Theorem 17]). It also follows that if Λ_i is a form parameter for R_i, then Λ_i is a C_i-module, i.e. $c\Lambda_i \leq \Lambda_i$ for all $c \in C_i$. □

Proof of Theorem 1. Note that if (R, Λ) is a quasi-finite form ring we can always choose a directed system $\{R_i\}_{i\in\Theta}$ such that $R = \varinjlim R_i$ and each ring R_i is an involution invariant ring module finite over its center that contains λ. Combining this observation with Propositions 3.10.1 and 3.6.6 we can from the very beginning assume that R is a finitely generated module over a finitely generated subring C in the center of R such that $\overline{c} = c$ whenever $c \in C$ and Λ is a C-module. Let (σ, Γ) be the net associated with H. Fix an element $a \in H$ and an elementary unitary matrix $T_{sr}(\xi)$ in $\mathrm{EU}(\sigma, \Gamma)$. Let b denote the conjugate $aT_{sr}(\xi)a^{-1}$. We will show that $b \in \mathrm{U}(\sigma, \Gamma)$.

Pick a maximal ideal $\mathfrak{m}$ of C. Let S denote the compliment of $\mathfrak{m}$ in C. Consider the localisation $R_\mathfrak{m} = S^{-1}R$ of the ring R at the multiplicative set S together with the localization morphism $F_\mathfrak{m} : R \to R_\mathfrak{m}$. It is well known that $R_\mathfrak{m}$ is semilocal. Let $\Lambda_\mathfrak{m}$ denote the form parameter $S^{-1}\Lambda$ for $R_\mathfrak{m}$, R' denote the image $F_\mathfrak{m}(R)$ of R in $R_\mathfrak{m}$, $\Lambda'_\mathfrak{m}$ denote the form parameter $F_\mathfrak{m}(\Lambda)$ for $R'_\mathfrak{m}$ and $S_\mathfrak{m}$ denote the multiplicative set $F_\mathfrak{m}(S)$. It is easy to see that $((R_m, \Lambda_\mathfrak{m}), (R'_\mathfrak{m}, \Lambda'_\mathfrak{m}), S_\mathfrak{m})$ is a standard setting. Clearly we can consider the form ring (R, Λ) as a trivial standard setting $((R, \Lambda), (R, \Lambda), \{1\})$. It's easy to see that $F_\mathfrak{m}$ is a morphism of standard settings $((R, \Lambda), (R, \Lambda), \{1\})$ and $((R_\mathfrak{m}, \Lambda_\mathfrak{m}), (R'_\mathfrak{m}, \Lambda'_\mathfrak{m}), S_\mathfrak{m})$ We will show that there exists an element $x_0 \in S$ such that $F_\mathfrak{m}$ is injective on x_0R. For each $s \in S$ set $Ann(s) = \{\xi \in R \mid s\xi = 0\}$. Note that $Ann(s)$ is an ideal of R for any $s \in S$. As R is Noetherian, it follows by Zorn's lemma that there are maximal elements in the set $\{Ann(s) \mid s \in S\}$. Let $x_0 \in S$ be such that $Ann(x_0)$ is maximal. Then $F_\mathfrak{m}$ is injective on x_0R. Indeed, let $F_\mathfrak{m}(x_0\xi) = F_\mathfrak{m}(x_0\zeta)$. Then there exists an element $s \in S$ such that $sx_0(\xi - \zeta) = 0$, therefore $(\xi - \zeta) \in Ann(sx_0) \supseteq Ann(x_0)$. As $Ann(x_0)$ is maximal it follows that $(\xi - \zeta) \in Ann(x_0)$ and $x_0\xi = x_0\zeta$.

We will show that the image $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ under $F_\mathfrak{m}$ of the form net of ideals (σ, Γ) is $S_\mathfrak{m}$-associated with the image $H_\mathfrak{m} = \mathrm{M}(F_\mathfrak{m})(H)$ of H in $\mathrm{U}(2n, R'_\mathfrak{m}, \Lambda'_\mathfrak{m})$. By Proposition 3.2.7 it follows that $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ is an exact form D-net of ideals over $(R'_\mathfrak{m}, \Lambda'_\mathfrak{m})$ and $\mathrm{M}(F_\mathfrak{m})(\mathrm{U}(\sigma, \Gamma)) = \mathrm{U}(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$, in particular, $\mathrm{U}(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m}) \leq H_\mathfrak{m}$. Let $T_{sr}(\xi)$ be an element of $\mathrm{EU}(2n, R_\mathfrak{m}, \Lambda_\mathfrak{m})$ contained in $H_\mathfrak{m}$. Then $\xi = \frac{\zeta}{x}$, where $\zeta \in R$ and $x \in S$, thus $F_\mathfrak{m}(\zeta x^{\delta_{s,-r}}) = F_\mathfrak{m}(x)^{(1+\delta_{s,-r})}\xi$. Clearly, if $s = -r$ that $\zeta x^{\delta_{s,-r}} \in \Lambda_s$. By Proposition 3.6.2 it follows that $T_{sr}(x_0^{(1+\delta_{s,-r})}\zeta x^{\delta_{s,-r}}) \in H$. Therefore $x_0^{(1+\delta_{s,-r})}\zeta x^{\delta_{s,-r}} \in (\sigma, \Gamma)_{sr}$. Thus $(xx_0)^{(1+\delta_{s,-r})}\xi \in (\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$. Hence, $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ is indeed a from net of ideals which is $S_\mathfrak{m}$-associated with $H_\mathfrak{m}$.

Let $J_\mathfrak{m}$ denote the Jacobson radical of the ring $R_\mathfrak{m}$. Then the canonical morphism $R'_\mathfrak{m} \to R_\mathfrak{m}/J_\mathfrak{m}$ is surjective. Indeed, as the ring C is commutative it follows that $C/\mathfrak{m}$ is a field, therefore $S+\mathfrak{m}$ is invertible in $C/\mathfrak{m}$. Thus, the canonical morphism $R'_\mathfrak{m}/F_\mathfrak{m}(\mathfrak{m}) \to R_\mathfrak{m}/(S^{-1}\mathfrak{m}\cdot R_\mathfrak{m})$ is surjective. Further, the ring $R_\mathfrak{m}$ is module finite over the commutative local ring $S^{-1}C$ with the Jacobson radical $S^{-1}\mathfrak{m}$. According to [Bas68, P. I, Ch. III, §2] the inclusion $R_\mathfrak{m}\cdot S^{-1}\mathfrak{m} \leq J_\mathfrak{m}$ holds. Therefore the canonical morphism $R/(S^{-1}\mathfrak{m}\cdot R_\mathfrak{m}) \to R_\mathfrak{m}/J_\mathfrak{m}$ is surjective. Summing up, the canonical morphism
$R'_\mathfrak{m} \to R_\mathfrak{m}/J_\mathfrak{m}$ is surjecive as a composition of surjective morphisms

$$R'_\mathfrak{m} \longrightarrow R'_\mathfrak{m}/F_\mathfrak{m}(\mathfrak{m}) \longrightarrow R_\mathfrak{m}/(S^{-1}\mathfrak{m} \cdot R_\mathfrak{m}) \longrightarrow R_\mathfrak{m}/J_\mathfrak{m}.$$

By Lemma 3.9.3 it follows that

$$H_\mathfrak{m} \leq \mathrm{Transp}(\mathrm{EU}(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m}), \mathrm{U}(\sigma_\mathfrak{m}, \Gamma_\mathfrak{m})), \tag{3.112}$$

where $(\sigma_\mathfrak{m}, \Gamma_\mathfrak{m})$ is the $S_\mathfrak{m}$-closure of the $(\sigma'_\mathfrak{m}, \Gamma'_\mathfrak{m})$ in $(R_\mathfrak{m}, \Lambda_\mathfrak{m})$. One can show that $(\sigma_\mathfrak{m}, \Gamma_\mathfrak{m})$ is the localization $(S^{-1}\sigma, S^{-1}\Gamma)$ of the form net of ideals (σ, Γ), although it is not important for this proof. Denote by X_{ij} the set of all elements $x \in C$ such that $xb_{ij} \in \sigma_{ij}$. We will show that X_{ij} contains an element of S for all $i \neq j$. Assume $i \neq \pm j$. By (3.112) it follows that $F_\mathfrak{m}(b_{ij}) \in (\sigma_\mathfrak{m})_{ij}$. Thus there exists an element $x \in S$ such that $T_{ij}(F_\mathfrak{m}(x)F_\mathfrak{m}(b_{ij})) \in H$. By Proposition 3.6.2 it follows that $T_{ij}(x_0xb_{ij}) \in H$. Therefore $x_0xb_{ij} \in \sigma_{ij}$ and $x_0x \in X_{ij} \cap S$. If $i = -j$ then there exists an index $k \in I$ such that (i, k) is an A-type base pair. Then $F_\mathfrak{m}(b_{i,-i}) \in (\sigma_\mathfrak{m})_{i,-i} = (\sigma_\mathfrak{m})_{k,-i}$. In the same way as above, this yields that $x_0x \in X_{i,-i} \cap S$ for some $x \in S$.

Similarly, for any $i \in I$ let $Z_{i,-i}$ be the set of all elements $x \in C$ such that $xS_{i,-i}(b) \in \Gamma_i$. We will also show that $Z_{i,-i}$ contains an element of S. By (3.112) it follows that $S_{i,-i}(F_\mathfrak{m}(b)) \in (\Gamma_\mathfrak{m})_i$. Thus there exists an element $x \in S$ such that

$$T_{i,-i}(F_\mathfrak{m}(x)^2F_\mathfrak{m}(S_{i,-i}(b))) \in H.$$

By Proposition 3.6.2 it follows that $T_{i,-i}(x_0^2x^2S_{i,-i}(b)) \in H$. Therefore $x_0^2x^2S_{i,-i}(b) \in \Gamma_i$ and $x_0^2x^2 \in Z_{i,-i}$.

Assume $i \neq j$. Because the maximal ideal $\mathfrak{m}$ of C is arbitrary it follows that the set X_{ij} generates the whole ring C as an ideal, therefore $b_{ij} \in \langle X_{ij}\rangle_C \cdot b_{ij} \leq \sigma_{ij}$. Similarly,

$Z_{i,-i}$ also generates the ring C as an ideal. As Γ_i is a C-module, it follows that $S_{i,-i}(b) \leq \langle Z_{i,-i}\rangle_C \cdot S_{i,-i}(b) \leq \Gamma_i$. Summing up, $b \in \mathrm{U}(\sigma,\Gamma)$. Therefore,

$$\mathrm{EU}(\sigma,\Gamma) \leq H \leq \mathrm{Transp}(\mathrm{EU}(\sigma,\Gamma),\mathrm{U}(\sigma,\Gamma)). \qquad (3.113)$$

The uniqueness of an exact form D-net of ideals (σ,Γ) such that (3.113) holds follows from Theorem 2. Indeed, let (τ, B) be any such exact form D-net not coinciding with (σ,Γ). Clearly, $(\tau,B) \leq (\sigma,\Gamma)$. Fix some $i \neq \pm j$ and pick any $\xi \in \sigma_{ij}$. Then $T_{ij}(\xi) \in H \leq \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\tau,B),\mathrm{U}(\tau,B))$. By Theorem 2 it follows that $\xi = \xi \cdot 1 \cdot 1 \in (T_{ij}(\xi))_{ij}\tau_{jj}(T_{ij}(\xi))'_{jj} \leq \tau_{ij}$. Therefore $\tau_{ij} = \sigma_{ij}$ for all $i \neq -j$. Let $\alpha \in \Gamma_i$ for some $i \in I$. Then $T_{i,-i}(-\alpha) \in H \leq \mathrm{Transp}_{\mathrm{U}(2n,R,\Lambda)}(\mathrm{EU}(\tau,B),\mathrm{U}(\tau,B))$ and $1 \in \tau_{ii}$. Therefore by Theorem 2 (property (T2)) it follows that

$$\alpha = (T_{i,-i}(-\alpha))_{ii} \cdot 1 \cdot S_{i,-i}(T_{i,-i}(\alpha))\lambda^{(\varepsilon(i)-1)/2}\overline{1}\lambda^{(1-\varepsilon(i))/2} \cdot (T_{i,-i}(\alpha))_{-i,-i} \in B_{i,-i}.$$

Hence, $\Gamma_i = B_i$ for all $i \in I$. Finally, as both form nets of ideals (σ,Γ) and (τ,B) are exact, it follows that $\sigma_{i,-i} = \tau_{i,-i}$ for all $i \in I$. Thus $(\sigma,\Gamma) = (\tau,B)$. $\square$

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