

CRITICAL WELL-POSEDNESS RESULTS FOR NONLINEAR SCHRÖDINGER EQUATIONS ON COMPACT MANIFOLDS

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Introduction

Physical relevance

Typically, the situation in physics is quite complex and one tries to approximate a physical behavior by partial differential equations. For instance, certain phenomena in electromagnetics, optics, mechanics, general relativity, and fluid mechanics can be approximately described by nonlinear waves. There is a huge number of nonlinear wave equations. In this thesis, we consider the *nonlinear Schrödinger equation* (NLS)

$$i\partial_t u + \Delta u = F(u).$$

The linear Schrödinger equation is one of the fundamental equations in quantum mechanics. It provides a description of a particle in a non-relativistic setting. The nonlinear Schrödinger equation is a prototypical dispersive nonlinear partial differential equation (see [Section 1.5.1](#)) and has a much more complicated structure as well as many applications in physics. Some relevant fields of application are nonlinear optics, propagation of the electric field in optical fibers, self-focusing and collapse of Langmuir waves in plasma physics and the behavior of deep water waves in the ocean. Moreover, various phenomena arising in Heisenberg ferromagnets and magnons, self-channeling of a high-power ultra-short laser in matter, condensed matter theory, dissipative quantum mechanics, and electromagnetic fields may be described by the NLS. [[APT04](#), [BOR15](#), [SS99](#)]

The nonlinear Schrödinger equation may also be derived from quantum dynamics of many-body systems, see [[ESY07](#)]. The fundamental principle of quantum mechanics states that a quantum system of N particles is described by a wave function of N variables satisfying a Schrödinger equation. In realistic systems, N is so large that a direct solution of the Schrödinger equation for interacting systems is clearly an impossible task. Thus, many-body systems are usually approximated by simpler dynamics where only the time evolution of a few cumulative degrees of freedom is monitored. In the simplest case only the one-particle marginal densities are considered. This means that the many-body pair interaction is replaced by an effective nonlinear mean-field potential and higher order quantum correlations are neglected. The cubic nonlinear Schrödinger equation then appears in the context of Bose–Einstein condensation with short range interactions in suitable scaling limits. The Bose–Einstein condensation is a state of matter consisting of dilute bosonic particles which are cooled to a temperature close to absolute zero. At this temperature, these particles tend to occupy the lowest quantum state, which can be expressed mathematically as the ground state of an energy functional related to the NLS. This phenomenon was proposed by Bose [[Bos24](#)] and Einstein [[Ein24](#), [Ein25](#)] in 1924–1925. Not so long ago, two groups, one led by Cornell–Wiemann [[AEM+95](#)] and the other by Ketterle [[DMA+95](#)], were awarded the Physics Nobel Prize in 2001 for (independently) verifying the Bose–Einstein condensation by experiments. Recently, the nonlinear Schrödinger equation on the tori \mathbb{T}^2 and \mathbb{T}^3 have been derived from many-body quantum systems as well. [[ESY07](#), [KSS11](#), [Soh14](#)]

The nonlinear Schrödinger equation on the Euclidean space

The nonlinear Schrödinger equation has been studied intensively within the last decades. We refer to [SS99, Caz03, Tao06, LP15] for some nice reviews. In this thesis, we mainly consider the NLS with a quintic nonlinearity, that is

$$i\partial_t u + \Delta u = \pm |u|^4 u. \quad (0.1)$$

The equation is called *defocusing* if the right-hand side has a plus and *focusing* if the right-hand side has a minus. The quintic NLS posed on \mathbb{R}^3 with initial data in $H^1(\mathbb{R}^3)$ is called *energy-critical* since if u is a solution to (0.1), then the scaled solution $(t, x) \mapsto \lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda x)$ solves (0.1) and leaves the homogeneous Sobolev norm $\dot{H}^1(\mathbb{R}^3)$ and the energy

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 dx \pm \frac{1}{6} \int_{\mathbb{R}^3} |u(t, x)|^6 dx$$

invariant. For sub-quintic nonlinearities, the corresponding Cauchy problem on \mathbb{R}^3 is called *sub-critical*. Given a nonlinearity with super-quintic powers, the corresponding equation on \mathbb{R}^3 is called *super-critical*. As explained in Section 1.5.2, studying the energy-critical equation is more challenging than studying the sub-critical case and hence, of a particular interest.

We say that a Cauchy problem is *locally well-posed* in H^s if for any choice of initial data $\phi \in H^s$, there exists a positive time T that may depend on the initial data such that a solution to the initial value problem exists on the time interval $[0, T)$, is unique, and the solution map depends Lipschitz continuously on the initial data ϕ . In sub-critical results the time of existence usually depends only on the norm of the initial data. If T can be chosen arbitrarily large, we call the Cauchy problem *globally well-posed*. Local and global well-posedness of the nonlinear Schrödinger equation posed on \mathbb{R}^n have been studied extensively. Various sub-critical and critical results have been obtained, cf. [SS99, Section 3.2] and [Caz03, Chapter 4].

Local and small data global well-posedness of both the focusing and the defocusing energy-critical NLS on \mathbb{R}^3 have been proved by Cazenave–Weissler [CW89] in 1989. It took many years until Colliander–Keel–Staffilani–Takaoka–Tao [CKS⁺08] finally showed that the defocusing NLS is also globally well-posed for arbitrarily large initial data in $H^1(\mathbb{R}^3)$. On the other hand, Christ–Colliander–Tao [CCT03, Theorem 1] showed that the quintic focusing and defocusing NLS on \mathbb{R}^3 fail to be well-posed in $H^s(\mathbb{R}^3)$ for $s < 1$. In addition, they demonstrated that the focusing and defocusing energy-super-critical NLS on \mathbb{R}^3 are ill-posed in $H^1(\mathbb{R}^3)$.

One of the fundamental tools in the aforementioned well-posedness results is the dispersive estimate,

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C|t|^{-\frac{n}{2}} \|u(0)\|_{L^1(\mathbb{R}^n)},$$

where u is a solution to the free Schrödinger equation $i\partial_t u + \Delta u = 0$. This shows that if the initial datum $u(0)$ has suitable integrability in space, then the solution has a decay in time. In many situations, the initial data do not have good integrability properties as one often assumes the initial data to lie in a Sobolev space $H^s(\mathbb{R}^n)$. However, from the dispersive estimate one can derive a useful set of estimates, known as *Strichartz estimates*, which can handle this type of initial data, see Section 1.5.2 for more details.

The nonlinear Schrödinger equation on compact manifolds

In the following, we consider the NLS on boundaryless, compact, smooth Riemannian manifolds. The behavior of solutions on such domains changes completely. For instance, the dispersive estimate fails to hold true. This becomes obvious by considering the flat standard torus. Since solutions on this manifold are periodic in time, dispersion in the classical sense can not be present here.

Moreover, the mathematical tools at our disposal change. An important tool one misses when moving to the setting of compact manifolds (except of tori) is the Fourier transform. However, the spectral resolution of the Laplace–Beltrami operator Δ_g compensates this loss near-completely. Frequency localization projectors that have been used on \mathbb{R}^n (and can be used on tori) can be replaced by spectral localization projectors. They are given as spectral multipliers instead of Fourier multipliers.

Another difference is the following: Solutions to the free Schrödinger equation on \mathbb{R}^n have the structure of oscillatory integrals

$$u(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi - t|\xi|^2)} \widehat{u_0}(\xi) d\xi, \quad (0.2)$$

where u_0 denotes the initial datum. The behavior of oscillatory integrals has been studied in great detail, see e.g. [Ste93, Chapters VIII–IX]. On compact manifolds, free solutions are given as exponential sums such as

$$u(t, x) = \sum_{k \in \mathbb{N}_0} e^{-it\lambda_k} (h_k u_0)(x),$$

where λ_k , $k \in \mathbb{N}_0$, denote the eigenvalues of the Laplace–Beltrami operator and h_k the projection on the corresponding eigenspace, see Section 1.4. The connection to (0.2) becomes particularly apparent on the standard torus, in which case free solutions are given by

$$u(t, x) = \sum_{\xi \in \mathbb{Z}^n} e^{i(x \cdot \xi - t|\xi|^2)} \widehat{u_0}(\xi).$$

Some ideas that have been used to obtain estimates for oscillatory integrals, such as integration by parts, do not work for exponential sums and hence, we need a different approach. In analytic number theory there is a classical theory about exponential sums, which may be found in [Vau97, Kor92]. The main contributions to relevant results regarding exponential sums appearing in this context, however, are due to Bourgain [Bou89, Bou93a]. Some of these estimates require sophisticated arguments. In this thesis, we want to point out that the presented well-posedness results rely on exponential sum estimates, whose proofs do not require complicated arguments. To demonstrate this, we provide detailed proofs for all exponential sum estimates we shall use in Section 1.3.2 and Section 1.3.3. Corollary 1.39 below, for instance, was often cited to be a special case of the more general estimate given in [Bou89, formula (4.1)], see also [Her13, Lemma 3.1]. Here, we show how to get Corollary 1.39 from a variant of the classical Hardy–Littlewood circle method.

Apart from the technical difficulties described above, the essential argument used in the Euclidean setting fails, cf. [HTT11, pages 329–330]. On \mathbb{R}^3 , the Strichartz estimate [KT98, Corollary 1.4]

$$\|u\|_{L_t^\infty H_x^1} + \|u\|_{L_t^2 W_x^{1,6}} \lesssim \|u(0)\|_{H^1} + \|(i\partial_t + \Delta)u\|_{L_t^2 W_x^{1,6/5}} \quad (0.3)$$

plays an important role to establish local and small data global well-posedness. Applied to the quintic NLS (0.1), using Hölder’s estimate, and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ yields

$$\|u\|_{L_t^\infty H_x^1} + \|u\|_{L_t^2 W_x^{1,6}} \lesssim \|u(0)\|_{H^1} + \|u\|_{L_t^2 W_x^{1,6}} \|u\|_{L_t^2 H_x^1}^4.$$

As Cazenave–Weissler [CW90, Section 4] showed by applying the Picard iteration scheme, this implies local and small data global well-posedness in $H^1(\mathbb{R}^3)$. This approach breaks down in the case of compact manifolds since inequality (0.3) fails. Indeed, on the torus it follows from adapting the one-dimensional counterexample of [Bou93a, Section 2, Remark 2] to the three-dimensional situation, and for \mathbb{S}^3 it was shown in [BGT04, Section 4.2].

More differences and details that are a bit more technical are postponed to Section 1.5.2.

Related results and main results of this thesis

Let (M, g) be a three-dimensional, smooth, compact Riemannian manifold without boundary. One major part of the present thesis is to study large data local and small data global well-posedness of the energy-critical nonlinear Schrödinger equation, that is

$$\begin{cases} i\partial_t u + \Delta_g u = \pm |u|^4 u & \text{in } [0, T) \times M \\ u(0, \cdot) = \phi & \text{on } M \end{cases} \quad (0.4)$$

with $\phi \in H^1(M)$. We call a Cauchy problem posed on a compact n -manifold energy-critical if the corresponding problem posed on \mathbb{R}^n is energy-critical. The terms sub-critical and super-critical are defined analogously.

This line of research was initiated by Bourgain [Bou93a], who proved that the energy-sub-critical NLS (i.e. nonlinearity with power less than 5) is globally well-posed for sufficiently small H^1 -data. In 2007, Bourgain extended his sub-critical result to the class of rectangular tori

$$\mathbb{T}_\theta^3 := \mathbb{R}^3 / (2\pi\theta_1^{-1/2}\mathbb{Z} \times 2\pi\theta_2^{-1/2}\mathbb{Z} \times 2\pi\theta_3^{-1/2}\mathbb{Z}), \quad (\theta_1, \theta_2, \theta_3) \in (0, \infty)^3.$$

The approach used by Bourgain relies heavily on the particular structure of the torus. In a series of papers, Burq–Gérard–Tzvetkov [BGT04, BGT05a, BGT05b, BGT07] developed a theory to prove sub-critical global well-posedness of (0.4) on $M = \mathbb{S}^3$ and $M = \mathbb{S} \times \mathbb{S}_\rho^2$, where \mathbb{S}_ρ^2 is the embedded sphere of radius ρ in \mathbb{R}^3 .¹ One of their main, newly developed tools is a set of multilinear spectral cluster estimates, which hold on any compact manifold. If one considers single eigenfunctions, these estimates seem to be only relevant for “sphere like manifolds” as they are far from being optimal for eigenfunctions on the torus.

In 2011, Herr–Tataru–Tzvetkov [HTT11] were the first to prove a local and small data global well-posedness for the energy-critical NLS on a compact manifold, namely the flat torus \mathbb{T}^3 . Parts of their proof rely deeply on the given structure of the spectrum of Δ_g . However, by simple geometric considerations, it is possible to extend this result to rectangular tori with rational ratios². In 2013, Herr [Her13] was able to extend this result to Zoll manifolds, which are manifolds whose geodesics are simple and closed with a common minimal period such as \mathbb{S}^3 . Herr used in an essential way that the eigenvalues of the Laplace–Beltrami operator are clustered around square numbers.

¹More generally, Burq–Gérard–Tzvetkov proved well-posedness for three-dimensional Zoll manifold and $\mathbb{S} \times M$, where M is a two-dimensional Zoll manifold.

²In this case, there exists $k \in \mathbb{N}$ such that the scaled torus $k\mathbb{T}^3$ can be viewed as a disjoint union of parallel translates of the original rational torus \mathbb{T}_θ^3 , see [GOW14, pages 977–978].

In the present thesis, we shall consider the energy-critical NLS (0.4) on general rectangular tori (with possibly irrational ratios) and on products of spheres, i.e. $\mathbb{S} \times \mathbb{S}_\rho^2$. In Chapter 2, we prove local and small data global well-posedness. The well-posedness result on rectangular 3-tori has been published by the present author in [Str14] and extends the results in [HTT11, GOW14]. Moreover, we present a proof of a multilinear Strichartz estimate, which implies scaling-critical local well-posedness of the NLS with nonlinearity $\pm|u|^{2k+1}u$, $k \geq 3$, on two-dimensional rectangular tori. This result is part of [Str14] and extends an earlier result of Guo–Oh–Wang [GOW14] who proved the same result for $k \geq 6$. In this thesis, we also give the first proof of local and small data global well-posedness of the energy-critical NLS on $\mathbb{S} \times \mathbb{S}_\rho^2$. It extends a previous result of Herr and the author [HS15], in which the special case $\mathbb{S} \times \mathbb{S}^2$, i.e. $\rho = 1$, was treated. In the joint work [HS15], the essential contributions of Sebastian Herr were Sections 1, 3, and 4; the present author’s contribution is essentially Section 2. As in the Euclidean setting, it is known that the energy-super-critical focusing and defocusing NLS on an analytic manifold fail to be well-posed in H^1 [Tho08]. In the same work, it was also proven that both the focusing and the defocusing quintic NLS are ill-posed in H^s for $s < 1$. Hence, our study completes the analysis of local well-posedness in H^1 on rectangular tori and products of spheres in three dimensions.

The domain $\mathbb{S} \times \mathbb{S}_\rho^2$ is particularly interesting as it can be considered as an intermediate case between the torus \mathbb{T}^3 and the sphere \mathbb{S}^3 . To see this, let us first compare their spectra of the Laplace–Beltrami operator $\sigma(-\Delta_g)$:

M	$\sigma(-\Delta_g)$
\mathbb{T}^3	$\ell^2 + m^2 + n^2, \quad \ell, m, n \in \mathbb{Z}$
$\mathbb{S} \times \mathbb{S}_\rho^2$	$m^2 + \rho^{-2}(n^2 + n), \quad m \in \mathbb{Z}, n \in \mathbb{N}_0$
\mathbb{S}^3	$n^2 + 2n, \quad n \in \mathbb{N}_0$

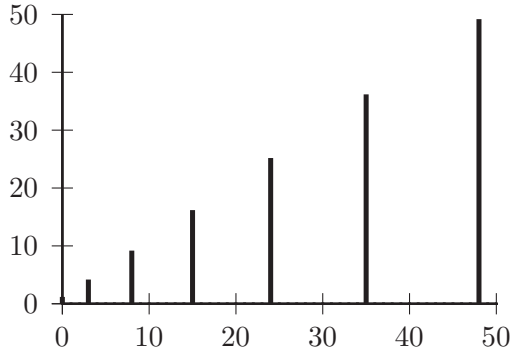
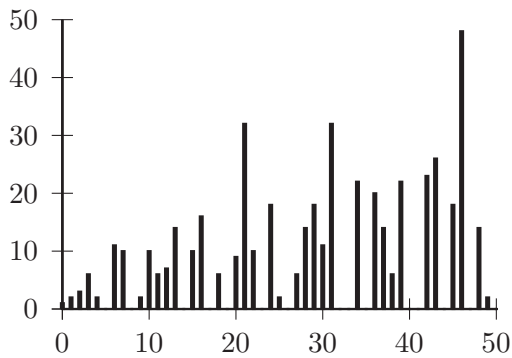
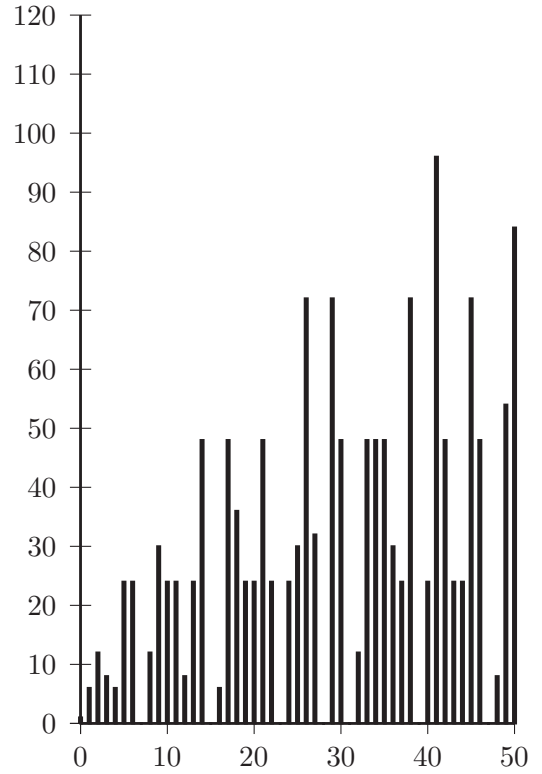
The spectrum $\sigma(-\Delta_g)$ on the torus is—as the sum of three square numbers—badly localized, whereas the eigenvalues of the Laplace–Beltrami operator on the three-dimensional sphere are essentially square numbers and hence, well localized. The spectrum of $-\Delta_g$ on $\mathbb{S} \times \mathbb{S}_\rho^2$ is mainly given as the sum of two square numbers and thus, in a certain sense, it is intermediate between the two. A similar picture emerges regarding the multiplicities of the eigenvalues. On \mathbb{S}^3 and $\mathbb{S} \times \mathbb{S}_\rho^2$, the multiplicities behave well-tempered. On the torus, though, the multiplicities of the eigenvalues vary heavily and increase fast. These facts are illustrated in Figure 0.1–Figure 0.3 below.

On the contrary, the eigenfunctions on the torus have very good algebraic properties since the product of two eigenfunctions equals an eigenfunction again. This is not the case for the eigenfunctions on \mathbb{S}^3 , the so called spherical harmonics. Though, the product of two spherical harmonics of degree m and ℓ can be expanded in terms of spherical harmonics of degree less or equal to $m + \ell$.

Another argument why \mathbb{T}^3 and \mathbb{S}^3 may be considered as extreme cases is due to the L^p -bounds of their eigenfunctions. While the L^p -norms of eigenfunctions on the torus are bounded, the L^p -norms of spherical harmonics present a bad concentration.

The study of the nonlinear Schrödinger equation on $\mathbb{S} \times \mathbb{S}_\rho^2$ is also interesting since one has to combine the different approaches used on the torus and the sphere, which could be a first step in understanding better how more general classes of manifolds can be treated. It seems that one has to find a way to balance the concentration of eigenfunctions and the repartition of the

spectrum. However, our knowledge about the spectrum and the eigenfunctions of the Laplace–Beltrami operator on arbitrary manifolds is poor, which makes it hard to obtain results for arbitrary manifolds. Since the NLS is locally well-posed in the two extreme cases, \mathbb{T}^3 and \mathbb{S}^3 , Burq–Gérard–Tzvetkov [BGT05b, page 257] conjectured that a similar local well-posedness result might hold true on any boundaryless, smooth, compact Riemannian 3-manifold.

Figure 0.1: $\sigma(-\Delta_g)$ on \mathbb{S}^3 Figure 0.2: $\sigma(-\Delta_g)$ on $\mathbb{S} \times \mathbb{S}^2$ Figure 0.3: $\sigma(-\Delta_g)$ on \mathbb{T}^3

In [HS15], Herr discovered that a certain trilinear Strichartz estimate based on L^2 -spaces, see [Assumption 2.1](#), is sufficient to conclude energy-critical local well-posedness and small data global well-posedness on any smooth, compact Riemannian 3-manifold without boundary. The proof of this conditional result given in [HS15] relies on earlier works and hence, we take the opportunity to review the whole argument in [Section 2.2](#).

Another goal of this thesis was to find a common approach to prove local and small data global well-posedness results in this setting. The first big step was the conditional result by Herr that reduces the study to proving a trilinear Strichartz estimate. In the present work, we verify this trilinear estimate for rectangular tori and products of spheres. So far, we were able to carve out the following general strategy:

- (i) Exploit almost orthogonality in space and time to restrict the spectrum of the high-frequency term to a smaller set whose size can be expressed involving a negative power of the largest frequency. See [Section 2.3.4](#) for rectangular tori, [Section 2.5.5](#) for product of spheres, and part b) in the proof of [Her13, Proposition 3.6] for Zoll manifolds.
- (ii) Prove scale invariant $L_t^p L_x^q$ -bounds on exponential sums arising from the linear evolution formula. Of course, the aim is to choose p and q as small as possible. For these bounds, it is usually hard to make use of the additional spectral localization introduced in (i). Hence, the additional restriction of the spectrum of the high-frequency term is usually

neglected. In the case of tori, these exponential sums are given by the linear evolution, see [Lemma 2.10](#). On products of spheres and Zoll manifolds, the exponential sums are not the respective linear evolutions but they are strongly related, cf. [Lemma 2.19](#) and [[Her13](#), Lemma 3.1].

- (iii) So far, the additional localization of the spectrum of the high-frequency term has not been used. However, this is easy in the $L_{t,x}^\infty$ -estimate since it leads essentially to a lattice point counting problem. Interpolating this with the estimates obtained in (ii) provides $L_t^p L_x^q$ -bounds that take the additional restriction of the spectrum in (i) into account. See [Corollary 2.11](#) (tori) and [Corollary 2.20](#) (products of spheres). On Zoll manifolds, the interpolation argument is not needed since the $L_{t,x}^p$ -bounds in [[Her13](#), Lemma 3.1] already take the spectral restriction in (i) into account.
- (iv) Finally, one considers the trilinear estimates, applies the almost orthogonality property in (i) and the estimates obtained in (ii) and (iii) to conclude the desired inequality.

Once a good local theory is obtained, one may ask for global well-posedness of the defocusing NLS (0.4) even for arbitrarily large initial data in $H^1(M)$. Ionescu–Pausader [[IP12b](#)] developed a method that allows to answer this question on the standard torus³. Shortly after, Pausader–Tzvetkov–Wang carried over the idea to \mathbb{S}^3 [[PTW14](#)]. Recently, the present author extended the global well-posedness result given in [[IP12b](#)] to the class of rectangular tori [[Str15](#)]. In this thesis, we provide a slightly modified proof that requires only Strichartz estimates in a smaller range instead of using Killip–Viřan’s result in [[KV14](#), Theorem 1.1]. The Strichartz estimates we apply follow essentially from the exponential sum estimates that are proved in [Section 1.3](#). Since small data global well-posedness on $\mathbb{S} \times \mathbb{S}_\rho^2$ is studied here, one might ask for large data global well-posedness on this domain. The difficulties arising are briefly discussed in [Section 3.7](#).

Unlike on \mathbb{R}^n , global control on compact manifolds can not come from dispersive decay. Hence, one can only hope for a local-in-time control instead of a global-in-time control. This local-in-time control has to be uniform over all small time intervals and has to handle nonzero contributions on each time interval. Presumably, solutions with large frequencies lead to complicated dynamics even in short time. Due to the non-dispersive nature of the geometry, this effect could be amplified and lead to even stronger nonlinear interactions producing even larger frequencies, cf. [[IP12b](#), page 1582]. On \mathbb{R}^n , this effect is compensated by dispersion.

The approach developed by Ionescu–Pausader relies strongly on the corresponding global well-posedness result on \mathbb{R}^3 [[CKS⁺08](#)]: It is proved that concentration in a certain critical norm can only happen around a point in space-time. This must occur in a way which can be compared to Euclidean solutions within a small time interval. However, these Euclidean-like solutions are controlled by the Euclidean well-posedness theory.

³Builds on their earlier article [[IP12a](#)] and a joint work with Staffilani [[IPS12](#)].

1 Basics

The first chapter of this thesis is devoted to introduce notation, function spaces, and to collect some basic propositions. Most parts of this chapter are a review of well-known material and cited from various sources. [Section 1.3.2](#) and [Section 1.3.3](#) contain some exponential sum estimates that are known and have been used before but either without a detailed proof or as a special case of more general statements, which require sophisticated arguments to prove. We aim to show that the exponential sum estimates used in this thesis may be obtained using rather simple arguments.

1.1 Notation

Before we start with the actual content of this thesis, we fix some notation that is used throughout this work.

The set of positive integers shall be denoted by $\mathbb{N} := \{1, 2, 3, \dots\}$, and we define the set of all non-negative integers by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We write $A \lesssim B$ if there exists a harmless constant $C > 0$ such that $A \leq CB$. Analogously, we denote $A \gtrsim B$ if $B \lesssim A$. If $A \lesssim B$ and $A \gtrsim B$, then we write $A \approx B$. If we want to emphasize the dependence of the constant, then we write $A \lesssim_s B$ for $A \leq C(s)B$, where the constant $C(s)$ depends on s . The terms $A \gtrsim_s B$ and $A \approx_s B$ are defined accordingly. We write $A \ll B$ if for a large constant $C > 1$ we have $CA \leq B$. Correspondingly, $A \gg B$ means that $B \ll A$.

For a multi-index $\alpha \in \mathbb{N}_0^n$ we denote as usual $|\alpha| := \alpha_1 + \dots + \alpha_n$, $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$, and $\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$.

The indicator function of a subset A of a set X shall be denoted by $\mathbf{1}_A: X \rightarrow \{0, 1\}$.

The Euclidean norm on \mathbb{R}^n is denoted by $|\cdot|$ and the standard inner product is written as

$$x \cdot y = \sum_{j=1}^n x_j y_j, \quad x, y \in \mathbb{R}^n.$$

Function that are k -times continuously differentiable are denoted by C^k , and C^∞ denotes the set of all functions that are differentiable for all degrees of differentiation. The space C_0^∞ is the subspace of all functions C^∞ with compact support.

We use the convention that sums over capital letters denote a dyadic summation. For instance, we write for $c \geq 1$,

$$\sum_{N \geq c} a_N := \sum_{j \in \mathbb{N}_0: 2^j \geq c} a_{2^j} \quad \text{and} \quad \sum_{N \leq c} a_N := \sum_{j \in \mathbb{N}_0: 2^j \leq c} a_{2^j}.$$

1.2 Function spaces and the Fourier transform

This section is devoted to briefly review function spaces and some of their basic properties that play a crucial role in the present thesis. Moreover, the crucial Fourier transform is introduced. We start with introducing the well-known L^p -spaces in [Section 1.2.1](#). Beyond defining those spaces, we are going to cite some results which will be used in the sequel. In [Section 1.2.2](#), the Fourier transform of Schwartz functions and tempered distributions are defined. This allows us to define Sobolev spaces of fractional order. Lesser-known are the U^p - and V^p -spaces that have become increasingly popular in the theory of dispersive partial differential equations. These spaces may be viewed as a powerful replacement for Bourgain's Fourier restriction spaces $X^{s,b}$. The U^p - and V^p -spaces are introduced in [Section 1.2.3](#).

1.2.1 L^p -spaces and Sobolev spaces

This subsection essentially follows [[Gra08](#), Chapter 1] and [[LL97](#)].

Let Ω be a measure space with a positive measure μ . We begin by defining the spaces of all μ -measurable functions on Ω whose modulus to the p th power is μ -summable.

Definition 1.1 (L^p -spaces). For $1 \leq p \leq \infty$ we define the space $L^p(\Omega, \mu)$ to be the following class of measurable functions:

$$L^p(\Omega, \mu) := \{f: \Omega \rightarrow \mathbb{C} : f \text{ is } \mu\text{-measurable and } \|f\|_{L^p(\Omega, \mu)} < \infty\},$$

where

$$\|f\|_{L^p(\Omega, \mu)} := \left(\int_{\Omega} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}, \quad \text{if } 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\Omega, \mu)} := \operatorname{ess\,sup}_{x \in \Omega} |f(x)| = \inf \{ \lambda \geq 0 : \mu(\{x \in \Omega : |f(x)| > \lambda\}) = 0 \}.$$

Remark.

- (i) To simplify notation, we write $L^p(\Omega)$ or L^p instead of $L^p(\Omega, \mu)$ if confusions are impossible. If μ is the Lebesgue measure, then we simply denote $d\mu(x)$ by dx .
- (ii) $\|\cdot\|_{L^p(\Omega, \mu)}$ does not distinguish all different measurable functions. For instance, from $\|f - g\|_{L^p(\Omega, \mu)} = 0$ we can only conclude that $f(x) = g(x)$ μ -almost everywhere. For this reason, we identify two functions that differ only on a μ -null set. To make that precise, we consider equivalence classes $[f]$ of measurable functions defined via the equivalence relation $f \sim g$ if $f = g$ μ -a.e. on Ω . If $L^p(\Omega, \mu)$ is defined so that its elements are not functions but the equivalence classes $[f]$, then $\|\cdot\|_{L^p(\Omega, \mu)}$ defines a norm.
- (iii) The space $L^2(\Omega, \mu)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{L^2(\Omega, \mu)} := \int_{\Omega} f(x) \overline{g(x)} d\mu(x).$$

- (iv) In this thesis, we use L^p -spaces with mixed norms. We refer to [[BP61](#)] for more details.

◇

There is another useful description of the L^p -norm via the *distribution function*

$$d_f(\lambda) := \mu(\{x \in \Omega : |f(x)| > \lambda\}).$$

This quantity provides information about the size of f but not about the behavior of f near a given point. Translations of a function on \mathbb{R}^n , for instance, does not change d_f . However, the provided information is sufficient to write the L^p -norm in terms of the distribution function.

Lemma 1.2 ([Gra08, Proposition 1.1.4]). *For $f \in L^p(\Omega, \mu)$ and $1 \leq p < \infty$ we have*

$$\|f\|_{L^p(\Omega, \mu)}^p = p \int_0^\infty \lambda^{p-1} d_f(\lambda) d\lambda.$$

We recall some well-known statements about L^p -spaces. The first inequality is named after the German mathematician Otto Hölder (1859–1937). The formulation of the lemma is taken from [LL97, Theorem 2.3].

Lemma 1.3 (Hölder's inequality). *Let $1 \leq p \leq \infty$ and p' its conjugate Hölder exponent, i.e. $1 = \frac{1}{p} + \frac{1}{p'}$ with the convention that $\frac{1}{\infty} := 0$. Moreover, let $f \in L^p(\Omega, \mu)$ and $g \in L^{p'}(\Omega, \mu)$. Then the pointwise product, given by $(fg)(x) = f(x)g(x)$, is in $L^1(\Omega, \mu)$ and*

$$\left| \int_{\Omega} fg d\mu \right| \leq \|f\|_{L^p(\Omega, \mu)} \|g\|_{L^{p'}(\Omega, \mu)}.$$

Remark. The special case $p = p' = 2$ coincides with the *Cauchy–Schwarz inequality*

$$\left| \int_{\Omega} fg d\mu \right|^2 \leq \int_{\Omega} |f|^2 d\mu \int_{\Omega} |g|^2 d\mu. \quad \diamond$$

The next inequality got its name from Hermann Minkowski (1864–1909), a German mathematician and physicist. A special case of Minkowski's inequality is the triangle inequality for the $L^p(\Omega, \mu)$ -norm, in this case ν is the counting measure and $q = 1$. A proof for $q = 1$ may be found in [LL97, Theorem 2.4]. A simple modification of this proof yields the result for $q > 1$.

Lemma 1.4 (Minkowski's inequality). *Suppose that Ω and Γ are any two spaces with σ -finite measures μ and ν , respectively. Let $f: \Omega \times \Gamma \rightarrow \mathbb{C}$ be a $\mu \times \nu$ -measurable function and $1 \leq q \leq p \leq \infty$. Then,*

$$\left(\int_{\Omega} \left(\int_{\Gamma} |f(x, y)|^q d\nu(y) \right)^{\frac{p}{q}} d\mu(x) \right)^{\frac{1}{p}} \leq \left(\int_{\Gamma} \left(\int_{\Omega} |f(x, y)|^p d\mu(x) \right)^{\frac{q}{p}} d\nu(y) \right)^{\frac{1}{q}}$$

with the obvious modifications for $q < p = \infty$ and $q = p = \infty$.

Now, we come to the identification of $L^p(\Omega, \mu)^*$, the dual of $L^p(\Omega, \mu)$, for $1 \leq p < \infty$, see e.g. [LL97, Theorem 2.14].

Lemma 1.5 (The dual of $L^p(\Omega, \mu)$). *When $1 \leq p < \infty$ the dual of $L^p(\Omega, \mu)$ is $L^{p'}(\Omega, \mu)$, where p' is conjugate Hölder exponent, in the sense that every $L \in L^p(\Omega, \mu)^*$ has the form*

$$L(g) = \int_{\Omega} v(x)g(x)d\mu(x)$$

for some unique $v \in L^{p'}(\Omega, \mu)$. In all cases, even $p = \infty$, L given as above is in $L^p(\Omega, \mu)^*$ and its norm

$$\|L\| := \sup\{|L(f)| : \|f\|_{L^p(\Omega, \mu)} \leq 1\} = \|v\|_{L^{p'}(\Omega, \mu)}.$$

A special kind of product of two functions on \mathbb{R}^n is the convolution. To keep the definition as general as possible, we do not require any restrictions on those two functions and accept that the right-hand side in the following definition might be undefined.

Definition 1.6 (Convolution). For $f, g: \mathbb{R}^n \rightarrow \mathbb{C}$ we define the *convolution of f and g* to be the function $f * g: \mathbb{R}^n \rightarrow \mathbb{C}$ given by

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y) dy.$$

Remark.

- (i) By a change of variables, one immediately sees commutativity, i.e. $f * g = g * f$.
- (ii) One has to make sure that the integral on the right-hand side is well-defined. Hölder's inequality, for instance, implies that this is the case whenever $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$. Young's inequality (see [Lemma 1.7](#) below), named after the English mathematician William Henry Young (1863–1942), shows that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1 \leq \frac{1}{p} + \frac{1}{q}$, then the integral is finite almost everywhere and defines a function that is in $L^r(\mathbb{R}^n)$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. \diamond

Lemma 1.7 (Young's inequality for convolutions). *Let $1 \leq p, q, r \leq \infty$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ as well as $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then $f * g \in L^r(\mathbb{R}^n)$ and*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq C_{p,q,r,n} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Remark.

- (i) Minkowski's inequality is a special case since it implies for $r \geq 1$,

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

- (ii) There is a more general version of Young's inequality which may be found in [\[LL97, Theorem 4.2\]](#).
- (iii) Convolutions may be defined on locally compact groups and Young's inequality also holds in this setting, see [\[Gra08, Section 1.2.2\]](#). In [Lemma 1.34](#) below, we state it in the case where the locally compact group is given by \mathbb{T}^n . \diamond

Convolutions may be applied to show that smooth functions with compact support are dense in L^p , see e.g. [\[AF03, Corollary 2.30\]](#) and [\[LL97, Lemma 2.19\]](#).

Lemma 1.8 (Density). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

The next result is known as Schur's lemma and provides sufficient conditions for linear operators to be bounded on L^p . We cite Schur's lemma from [\[Gra08, Appendix I.1\]](#).

Lemma 1.9 (Schur's lemma). *Let (X, μ) and (Y, ν) be two σ -finite measure spaces and $K: (X, \mu) \times (Y, \nu) \rightarrow \mathbb{C}$. Furthermore, let T be a linear operator given by*

$$T(f)(x) = \int_Y K(x, y) f(y) d\nu(y),$$

where f is bounded and compactly supported. If K satisfies

$$\begin{aligned} \sup_{x \in X} \int_Y |K(x, y)| d\nu(y) &= A < \infty, \\ \sup_{y \in Y} \int_X |K(x, y)| d\mu(x) &= B < \infty, \end{aligned}$$

then the operator T extends to a bounded operator from $L^p(Y)$ to $L^p(X)$ with norm $A^{1-\frac{1}{p}} B^{\frac{1}{p}}$ for $1 \leq p \leq \infty$.

Remark.

- (i) The result is named after the German mathematician Issai Schur (1875–1941). In 1911, Schur [Sch11] proved a matrix version of the lemma for $p = 2$. More about the history of Schur’s lemma can be found in [Gra08, page 461].
- (ii) For positive operators, i.e. K is a non-negative measurable function on $X \times Y$, the version of Schur’s lemma in [Gra08, Appendix I.2] provides necessary and sufficient conditions for the L^p boundedness. \diamond

We end this subsection with a useful interpolation statement between L^p -spaces, see e.g. [Gra08, Theorem 1.3.4]. Let $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ be the space of all functions $f: \mathbb{R}^n \rightarrow \mathbb{C}$ such that there exists $f_1 \in L^{p_0}(X, \mu)$ and $f_2 \in L^{p_1}(X, \mu)$ with $f = f_1 + f_2$. Note that $L^p(X, \mu) \subseteq L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ for $p_0 \leq p \leq p_1$.

Proposition 1.10 (Riesz–Thorin interpolation). *Let (X, μ) and (Y, ν) be two measure spaces. Let T be a linear operator defined on $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ and taking values in the set of ν -measurable functions on Y . Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that*

$$\begin{aligned} \|T(f)\|_{L^{q_0}(Y, \nu)} &\leq M_0 \|f\|_{L^{p_0}(X, \mu)} \quad \text{for all } f \in L^{p_0}(X, \mu), \\ \|T(f)\|_{L^{q_1}(Y, \nu)} &\leq M_1 \|f\|_{L^{p_1}(X, \mu)} \quad \text{for all } f \in L^{p_1}(X, \mu). \end{aligned}$$

Then for all $0 < \vartheta < 1$ and $f \in L^p(X, \mu)$ we have

$$\|T(f)\|_{L^q(Y, \nu)} \leq M_0^{1-\vartheta} M_1^\vartheta \|f\|_{L^p(X, \mu)},$$

where

$$\frac{1}{q} = \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1} \quad \text{and} \quad \frac{1}{p} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}.$$

1.2.2 The Schwartz class and the Fourier transform

This subsection is devoted to introduce one of the most important tools in harmonic analysis: the Fourier transform. From the definition of the Fourier transform (see Definition 1.12 below) it is obvious that it may be defined for functions $f \in L^1(\mathbb{R}^n)$. However, we are going to define the Fourier transform on a smaller class of functions, the space of Schwartz functions that is denoted by $\mathcal{S}(\mathbb{R}^n)$. The reason is that the space turns out to be a natural environment, for instance, since the Fourier transform defines a homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself and the Fourier inversion formula holds in it. On the contrary, if the Fourier transform would be defined as an operator on $L^1(\mathbb{R}^n)$, then the Fourier inversion formula requires the additional assumption that the Fourier transform is in $L^1(\mathbb{R}^n)$.

The whole subsection is pretty close to the nice introduction given in [Gra08, Section 2.2].

The Schwartz space

Schwartz functions are—roughly speaking—smooth functions for which the function and all of its derivatives decay faster than the reciprocal of any polynomial at infinity. The space is named after the French mathematician and Fields medalist Laurent Schwartz (1915–2002).

Definition 1.11 (Schwartz functions).

- (i) A complex-valued function $f \in C^\infty(\mathbb{R}^n)$ is called *Schwartz function* if for every pair of multi-indices α and β there exists a positive constant $C_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta} < \infty.$$

The quantities $\rho_{\alpha,\beta}(f)$ are called the *Schwartz seminorms* of f . The set of all Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^n)$.

- (ii) A sequence $(f_k)_{k \in \mathbb{N}_0}$ in $\mathcal{S}(\mathbb{R}^n)$ is said to be *convergent to* $f \in \mathcal{S}(\mathbb{R}^n)$ if for all multi-indices α and β it holds that

$$\rho_{\alpha,\beta}(f_k - f) = \sup_{x \in \mathbb{R}^n} |x^\alpha (\partial^\beta (f_k - f))(x)| \rightarrow 0$$

as $k \rightarrow \infty$.

Remark.

- (i) There is an alternative characterization of Schwartz functions which is very useful. A smooth function $f: \mathbb{R}^n \rightarrow \mathbb{C}$ is in $\mathcal{S}(\mathbb{R}^n)$ if and only if for all positive integers N and all multi-indices α there exists a positive constant $C_{\alpha,N}$ such that

$$|(\partial^\alpha f)(x)| \leq C_{\alpha,N} (1 + |x|)^{-N}$$

for all $x \in \mathbb{R}^n$, see [Gra08, Remark 2.2.4].

- (ii) If ρ_j is an enumeration of the Schwartz seminorms $\rho_{\alpha,\beta}$, then

$$d(f, g) := \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f - g)}{1 + \rho_j(f - g)}$$

defines a metric on $\mathcal{S}(\mathbb{R}^n)$. It is easy to check that $\mathcal{S}(\mathbb{R}^n)$ is complete with respect to d . Hence, $\mathcal{S}(\mathbb{R}^n)$ is a *Fréchet space*, i.e. it is a complete metrizable locally convex space. [Gra08, pages 96–97]

- (iii) Obviously, $C_0^\infty(\mathbb{R}^n)$ is contained in $\mathcal{S}(\mathbb{R}^n)$ and convergence in $C_0^\infty(\mathbb{R}^n)$ implies convergence in $\mathcal{S}(\mathbb{R}^n)$. The function $x \mapsto e^{-|x|^2}$ is a Schwartz function but not in $C_0^\infty(\mathbb{R}^n)$.
- (iv) Convergence in $\mathcal{S}(\mathbb{R}^n)$ is stronger than convergence in all $L^p(\mathbb{R}^n)$. [Gra08, Proposition 2.2.6] \diamond

The Fourier transform of Schwartz functions

We define the Fourier transform as an operator acting on $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform got its name from the French mathematician and physicist (1768–1830).

Definition 1.12 (Fourier transform on $\mathcal{S}(\mathbb{R}^n)$). Let $f \in \mathcal{S}(\mathbb{R}^n)$.

(i) We define the *Fourier transform of f* as

$$\mathcal{F}_{\mathbb{R}^n}(f)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

Sometimes we also write $\widehat{f} := \mathcal{F}_{\mathbb{R}^n}(f)$.

(ii) The *inverse Fourier transform of f* is defined as

$$\mathcal{F}_{\mathbb{R}^n}^{-1}(f)(x) := \mathcal{F}_{\mathbb{R}^n}(f)(-x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

We sometimes also write $\check{f} := \mathcal{F}_{\mathbb{R}^n}^{-1}(f)$.

Remark. Note that the notation $\widehat{\cdot}$ and $\check{\cdot}$ clashes with the notation in [Definition 1.26](#) below. Whenever it is clear from the context, we use $\widehat{\cdot}$ and $\check{\cdot}$ without mentioning whether it is meant in the sense of [Definition 1.12](#) or [Definition 1.26](#). \diamond

Now, we collect some important properties of the Fourier transform that may be found in e.g. [\[Gra08, Proposition 2.2.11\]](#). Let us first introduce some notation: For a measurable function f on \mathbb{R}^n , $x, y \in \mathbb{R}^n$, and $a > 0$ we define the *translation* and *dilation* of f by

$$\tau^y(f)(x) := f(x - y) \quad \text{and} \quad \delta^a(f)(x) := f(ax),$$

respectively.

Lemma 1.13 (Properties of $\mathcal{F}_{\mathbb{R}^n}$). *Given two functions $f, g \in \mathcal{S}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$, α a multi-index, and $a > 0$, we have the following:*

(i) $\|f\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)},$

(ii) $\widehat{f + g} = \widehat{f} + \widehat{g},$

(iii) $\widehat{\lambda f} = \lambda \widehat{f},$

(iv) $\widehat{\tau^y(f)}(\xi) = e^{-iy \cdot \xi} \widehat{f}(\xi),$

(v) $(e^{ix \cdot y} f(x))^\wedge(\xi) = \tau^y(\widehat{f})(\xi),$

(vi) $(\delta^a(f))^\wedge(\xi) = a^{-n} \delta^{a^{-1}}(\widehat{f})(\xi),$

(vii) $(\partial^\alpha f)^\wedge(\xi) = (i\xi)^\alpha \widehat{f}(\xi),$

(viii) $(\partial^\alpha \widehat{f})(\xi) = ((-ix)^\alpha f(x))^\wedge(\xi),$

(ix) $\widehat{f} \in \mathcal{S}(\mathbb{R}^n),$

(x) $\widehat{f * g} = \widehat{f} \widehat{g},$

(xi) $\widehat{f \circ A}(\xi) = \widehat{f}(A\xi),$ where A is an orthogonal matrix and ξ is a column vector.

Remark. It is not hard to prove that analogue statements hold true for the inverse Fourier transform. \diamond

The following lemma investigates the relation between the Fourier transform and the inverse Fourier transform.

Proposition 1.14 ([Gra08, Theorem 2.2.14]). *Given $f, g, h \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$(i) \int_{\mathbb{R}^n} f(x)\widehat{g}(x) dx = \int_{\mathbb{R}^n} \widehat{f}(x)g(x) dx,$$

$$(ii) \text{ Fourier inversion: } (\widehat{f})^\vee = f = (\check{f})^\wedge,$$

$$(iii) \text{ Parseval's relation: } \int_{\mathbb{R}^n} f(x)\overline{h(x)} dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{h}(\xi)} d\xi,$$

$$(iv) \text{ Plancherel's identity: } \|f\|_{L^2(\mathbb{R}^n)} = \|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|\check{f}\|_{L^2(\mathbb{R}^n)},$$

$$(v) \int_{\mathbb{R}^n} f(x)g(x) dx = \int_{\mathbb{R}^n} \widehat{f}(x)\check{g}(x) dx.$$

On the one hand, the Fourier transform may easily be extended to the space $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ since the integrability ensures that the integrals in Definition 1.12 are convergent and most of the results in Lemma 1.13 hold true for those functions.¹ On the other hand, for $L^2(\mathbb{R}^n)$ functions the integrals in Definition 1.12 do not converge absolutely. However, since $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, there is a unique bounded extension of the Fourier transform and the inverse Fourier transform on $L^2(\mathbb{R}^n)$. This extension is also an isometry on $L^2(\mathbb{R}^n)$. The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ and its extension share most of its properties, see [Gra08, Section 2.2.4], and hence, we do not distinguish them notationally.

From a simple interpolation of Plancherel's identity and Lemma 1.13 (i), we can extend the Fourier transform on $L^p(\mathbb{R}^n)$ for $1 < p < 2$, see e.g. [Gra08, Proposition 2.2.16].

Lemma 1.15 (Hausdorff–Young inequality). *Let $1 \leq p \leq 2$. For every function $f \in L^p(\mathbb{R}^n)$ we have the estimate*

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}.$$

The Fourier transform of tempered distributions

It is also possible to give a meaning to the Fourier transform on the space of tempered distributions. The following definitions and results as well as more details may be found e.g. in [Gra08, Section 2.3].

Definition 1.16 (Tempered distribution). The space of *tempered distributions* is defined as

$$\mathcal{S}'(\mathbb{R}^n) := \{u: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C} : u \text{ is linear and continuous}\}.$$

Remark.

(i) It is common to denote the evaluation of $u \in \mathcal{S}'(\mathbb{R}^n)$ at $f \in \mathcal{S}(\mathbb{R}^n)$ as

$$\langle u, f \rangle = u(f).$$

(ii) Functions g that do not increase too quickly can be thought of as tempered distributions via the identification $g \mapsto L_g$, where L_g is the functional

$$L_g(f) := \int_{\mathbb{R}^n} g(x)f(x) dx, \quad f \in \mathcal{S}. \quad \diamond$$

¹To be precise, (i)–(vi) as well as (x) and (xi). [Gra08, Section 2.2.4]

It is obvious that the following definitions are well-defined and coincide with the previous definitions whenever they apply to u .

Definition 1.17. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be tempered distribution and $f \in \mathcal{S}(\mathbb{R}^n)$.

(i) Let α be a multi-index, then

$$\langle \partial^\alpha u, f \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha f \rangle.$$

(ii) The Fourier transform \hat{u} and the inverse Fourier transform \check{u} are defined by

$$\langle \hat{u}, f \rangle := \langle u, \hat{f} \rangle \quad \text{resp.} \quad \langle \check{u}, f \rangle := \langle u, \check{f} \rangle.$$

Remark. Hölder's inequality shows that every $L^p(\mathbb{R}^n)$ function is a tempered distribution. Hence, the Fourier transform defined in [Definition 1.17](#) is indeed defined on a larger set compared to the extension to $L^2(\mathbb{R}^n)$ of Fourier transform defined in [Definition 1.12](#). Let $u \in L^2(\mathbb{R}^n)$. On the one hand, it follows that $\hat{u} \in L^2(\mathbb{R}^n)$ and hence,

$$\langle \hat{u}, f \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \hat{u}(x) f(x) dx = \int_{\mathbb{R}^n} u(x) \hat{f}(x) dx$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ by using [Proposition 1.14 \(i\)](#). On the other hand, if we consider u as a tempered distribution, then we have, by definition,

$$\langle \hat{u}, f \rangle = \langle u, \hat{f} \rangle = \int_{\mathbb{R}^n} u(x) \hat{f}(x) dx$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Hence, the extension to $L^2(\mathbb{R}^n)$ of [Definition 1.12](#) and [Definition 1.17](#) indeed coincide. \diamond

We refer to [[Gra08](#), Proposition 2.3.22] for a list of properties of the (inverse) Fourier transform of a tempered distribution.

Sobolev spaces

Next, we use the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ to define Sobolev spaces and study some of their properties. Compared to $L^p(\mathbb{R}^n)$, these spaces give more precise information about the regularity of a function. We follow the nice introduction in [[Caz03](#), Section 1.4].

Definition 1.18 (Sobolev spaces). Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ be given.

(i) We define the *inhomogeneous Sobolev space*

$$H^{s,p}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}_{\mathbb{R}^n}^{-1}((1 + |\xi|^2)^{\frac{s}{2}} \hat{u}) \in L^p(\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{H^{s,p}} := \|\mathcal{F}_{\mathbb{R}^n}^{-1}((1 + |\xi|^2)^{\frac{s}{2}} \hat{u})\|_{L^p(\mathbb{R}^n)}.$$

(ii) The *homogeneous Sobolev space* is defined as

$$\dot{H}^{s,p}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}_{\mathbb{R}^n}^{-1}(|\xi|^s \hat{u}) \in L^p(\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{\dot{H}^{s,p}} := \|\mathcal{F}_{\mathbb{R}^n}^{-1}(|\xi|^s \hat{u})\|_{L^p(\mathbb{R}^n)}.$$

(iii) We denote $H^s(\mathbb{R}^n) := H^{s,2}(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n) := \dot{H}^{s,2}(\mathbb{R}^n)$ for brevity.

Remark.

- (i) The space $H^s(\mathbb{R}^n)$ is a Hilbert space and $H^{s,p}(\mathbb{R}^n)$ is a Banach space. One trivially sees that $H^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.
- (ii) $H^{s_1,p}(\mathbb{R}^n) \hookrightarrow H^{s_2,p}(\mathbb{R}^n)$ if $s_1 \geq s_2$.
- (iii) If $p < \infty$, then $(H^{s,p}(\mathbb{R}^n))^* = H^{-s,p'}(\mathbb{R}^n)$ [BL76, Corollary 6.2.8].
- (iv) For $m \in \mathbb{N}_0$ and $1 < p < \infty$ it follows that

$$H^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n) : \partial^\alpha u \in L^p(\mathbb{R}^n) \text{ for } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq m\},$$

where $\partial^\alpha u$ has to be understood in the sense of tempered distributions. The norm

$$\|u\|_{W^{m,p}(\mathbb{R}^n)} := \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ 0 \leq |\alpha| \leq m}} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)}$$

is equivalent to $\|\cdot\|_{H^{m,p}(\mathbb{R}^n)}$. [BL76, Theorem 6.2.3] ◇

Proposition 1.19 (Sobolev embedding theorem). *Let $s \in \mathbb{R}$.*

- (i) *If $1 < p \leq q < \infty$, $r \in \mathbb{R}$ with $s - \frac{n}{p} = r - \frac{n}{q}$, then*

$$H^{s,p}(\mathbb{R}^n) \hookrightarrow H^{r,q}(\mathbb{R}^n).$$

- (ii) *If $1 \leq p < \infty$ and $0 < s < \frac{n}{p}$, then*

$$H^{s,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{pn}{n-sp}}(\mathbb{R}^n).$$

- (iii) *If $1 \leq p < \infty$, $k \in \mathbb{N}_0$ and $[s] > k + \frac{n}{p}$, then every element of $H^{s,p}(\mathbb{R}^n)$ can be modified on a set of measure zero so that the resulting function is bounded and is k -times continuously differentiable.*

The first statement may be found in e.g. [BL76, Theorem 6.5.1] and the second embedding is an immediate consequence of the first statement. The third embedding is a simple consequence of

$$H^{s,p}(\mathbb{R}^n) \hookrightarrow H^{[s],p}(\mathbb{R}^n) = W^{[s],p}(\mathbb{R}^n)$$

and the Sobolev embedding theorem for the latter spaces, cf. [LL97, Theorem 8.8].

1.2.3 The spaces U^p and V^p

This subsection introduces the spaces U^p and V^p . We closely follow some parts of [KTV14, Chapter I.4] and refer the reader to [HHK09, HTT11, Her13, HTT14] for more details.

In 1924, Norbert Wiener [Wie24] studied functions of bounded p -variation. These spaces were used in several contexts such as Riemann–Stieltjes integrals [You36] and rough paths [Lyo94, Lyo98]. In 2005, Koch–Tataru [KT05] were the first who realized that the spaces of bounded p -variation and its “dual” U^p -spaces may be used to sharpen Bourgain’s technique of $X^{s,b}$ -spaces that have often been applied to achieve well-posedness results for dispersive equations. Indeed, for the well-posedness result for the Kadomtsev–Petviashvili II equation obtained by Hadac–Herr–Koch [HHK09] the $X^{s,b}$ -spaces seem to be insufficient. The theory of the U^p - and V^p -spaces including some basic properties were worked out in [HHK09] for the first time.

Ever since, these spaces have repeatedly been applied to dispersive equations. Herr–Tataru–Tzvetkov were the first who successfully applied these spaces to gain energy-critical small data global well-posedness of the NLS on the three-dimensional flat torus. Later, Ionescu–Pausader used these spaces to extend this global well-posedness for initial data with arbitrary large H^1 -norm. Recently, the first book reviewing these spaces was published [KTV14].

These spaces shall only be briefly introduced in this thesis. Aside from the definitions and some basic properties, we cite an important interpolation and a duality result.

For the remainder of this subsection, $(X, \|\cdot\|_X)$ shall denote a Banach space with the norm $\|\cdot\|_X$.

In the following chapters, we rely on Proposition 1.23 (v), Lemma 1.24, and Lemma 1.25 below. These results do not hold for functions in the space V^p as it is defined in [KTV14] but for the subspace of right-continuous functions in V^p that is called V_{rc}^p in [KTV14]. For this reason, we only define V_{rc}^p as in [KTV14, pages 44–45] but call it V^p for brevity.

Definition 1.20 (V^p -space). Let $1 \leq p < \infty$. The space $V^p = V^p(X)$ is the space of right-continuous functions $v: \mathbb{R} \rightarrow X$ such that

$$\|v\|_{V^p}^p := \sup_{-\infty < t_0 < \dots < t_K \leq +\infty} \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_X^p < +\infty$$

with the convention $v(+\infty) := 0$, and in addition, we require $\lim_{t \rightarrow -\infty} v(t) = 0$.

We collect some properties of these spaces that may be found in [KTV14, page 45].

Proposition 1.21 (Properties of V^p).

- (i) The space V^p is a Banach space.
- (ii) We have $\|\cdot\|_{\text{sup}} \leq \|\cdot\|_{V^p}$ for all $1 \leq p < \infty$.
- (iii) If $1 \leq p \leq q < \infty$, then $V^p \hookrightarrow V^q$ and for all $v \in V^p$,

$$\|v\|_{V^q} \leq \|v\|_{V^p}.$$

The following definition of U^p is given in [KTV14, Definition I.4.10].

Definition 1.22 (U^p -space). Let $1 \leq p < \infty$. A right-continuous step function $a: \mathbb{R} \rightarrow X$ is called a U^p -atom if

$$a(t) = \sum_{k=1}^K \mathbf{1}_{[t_{k-1}, t_k)}(t) \phi_k, \quad \sum_{k=1}^K \|\phi_k\|_X^p = 1$$

for a partition $-\infty < t_0 < \dots < t_K \leq \infty$. Let $(a_j)_{j \in \mathbb{N}}$ be a sequence of atoms and let $(\lambda_j)_{j \in \mathbb{N}}$ be a summable sequence. Then

$$u := \sum_{j=1}^{\infty} \lambda_j a_j$$

is a U^p -function. We define the space $U^p = U^p(X)$ as the set of functions having such a representation and endow it with the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$

We state some properties of U^p -spaces given in [KTV14, pages 46–49].

Proposition 1.23 (Properties of U^p).

(i) If a is a U^p -atom, then $\|a\|_{U^p} \leq 1$. The norm of an U^p -atom may be less than 1. Determining the norm of an atom is a difficult task.

(ii) Functions in U^p are continuous from the right and the limit as $t \rightarrow -\infty$ vanishes.

(iii) The expression $\|\cdot\|_{U^p}$ defines a norm on U^p , and U^p is closed with respect to this norm. Moreover, $\|\cdot\|_{\text{sup}} \leq \|\cdot\|_{U^p}$.

(iv) If $1 \leq p \leq q < \infty$, then $U^p \hookrightarrow U^q$ and for all $u \in U^p$,

$$\|u\|_{U^q} \leq \|u\|_{U^p}.$$

(v) If $1 \leq p < \infty$, then $U^p \hookrightarrow V^p \hookrightarrow L^\infty(\mathbb{R}, X)$ and for all $u \in U^p$,

$$\|u\|_{V^p} \lesssim \|u\|_{U^p}.$$

(vi) If $1 < p < q < \infty$, then $V^p \hookrightarrow U^q$.

Later, we apply the following interpolation type property of the U^p - and V^p -spaces. The statement may be found in [HTT11, Lemma 2.4] and [HHK09, Proposition 2.20]. It is worth mentioning that there is a more general interpolation statement in [KTV14, Lemma I.4.12].

Lemma 1.24 (Interpolation). Let $q_1, q_2, q_3 > 2$, $(X, \|\cdot\|_X)$ be a Banach space, and let

$$T: U^{q_1} \times U^{q_2} \times U^{q_3} \rightarrow X$$

be a bounded, trilinear operator with $\|T(u_1, u_2, u_3)\|_X \leq C_1 \prod_{j=1}^3 \|u_j\|_{U^{q_j}}$. In addition, assume that there exists $C_2 \in (0, C_1]$ such that the estimate $\|T(u_1, u_2, u_3)\|_X \leq C_2 \prod_{j=1}^3 \|u_j\|_{U^2}$ holds true. Then, T satisfies the estimate

$$\|T(u_1, u_2, u_3)\|_X \lesssim C_2 \left(\ln \frac{C_1}{C_2} + 1 \right)^3 \prod_{j=1}^3 \|u_j\|_{V^2}$$

for $u_1, u_2, u_3 \in V^2$.

The following duality statement plays a crucial role in our analysis, too. The statement is taken from [KTV14, Corollary I.4.24].

Lemma 1.25 (Duality). Let $1 < p < \infty$ and H be a Hilbert space with complex inner product $\langle \cdot, \cdot \rangle$ and dual space H^* . Assume $u \in U^p(H)$ with $\partial_t u \in L^1_{\text{loc}}(\mathbb{R}, H)$ and $v \in V^p(H)$, then the following duality statements hold true:

$$\begin{aligned} \|u\|_{U^p(H)} &= \sup \left\{ \int_{\mathbb{R}} \langle \partial_t u(t), \overline{v(t)} \rangle dt : v \in C_0^\infty(\mathbb{R}, H^*), \|v\|_{V^{p'}(H^*)} = 1 \right\}, \\ \|v\|_{V^p(H)} &= \sup \left\{ \int_{\mathbb{R}} \langle \partial_t u(t), \overline{v(t)} \rangle dt : u \in C_0^\infty(\mathbb{R}, H^*), \|u\|_{U^{p'}(H^*)} = 1 \right\}, \end{aligned}$$

where $1 = \frac{1}{p} + \frac{1}{p'}$.

1.3 Fourier series and exponential sums

Some basic facts about the Fourier analysis on the torus $\mathbb{T}^n := \mathbb{R}^n / (2\pi\mathbb{Z})^n$ shall be discussed in this section. Related exponential sums play an important role in the study of the nonlinear Schrödinger equation on boundaryless compact manifolds. Estimates for exponential sums are addressed in this section, too. A variant of the Hausdorff–Young inequality for non-periodic functions is studied in [Section 1.3.2](#). L^p -estimates of exponential sums are discussed in [Section 1.3.3](#). Most of the results in the latter two subsections have been applied before but either without giving a thorough proof or as a special case of a more general statement. By giving detailed proofs for all exponential sum estimates we rely on, we would like to demonstrate that verifying these estimates do not require sophisticated arguments.

1.3.1 Fourier series

In this subsection, we adapt parts of the introduction in [[Gra08](#), Section 3.1] to $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$ instead of $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

Functions on \mathbb{T}^n can be considered as functions on \mathbb{R}^n with the property that $f(2\pi\xi + x) = f(x)$ for all $\xi \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. Those functions are called 2π -periodic in every coordinate. The measure on \mathbb{T}^n is given by the restriction of the n -dimensional Lebesgue measure to $\mathbb{T}^n = [0, 2\pi]^n$. It is a simple consequence of Hölder’s inequality that the $L^p(\mathbb{T}^n)$ -spaces are nested and $L^1(\mathbb{T}^n)$ contains all $L^p(\mathbb{T}^n)$ -spaces for $p \geq 1$.

Definition 1.26.

- (i) For a complex-valued function $f \in L^1(\mathbb{T}^n)$ and $\xi \in \mathbb{Z}^n$ we define the ξ th *Fourier coefficient* of f by

$$\mathcal{F}(f)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{T}^n} f(x) e^{-ix \cdot \xi} dx.$$

Sometimes we also write $\hat{f} := \mathcal{F}(f)$.

- (ii) The *Fourier series* of f at $x \in \mathbb{T}^n$ is given by

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{\xi \in \mathbb{Z}^n} \hat{f}(\xi) e^{ix \cdot \xi}.$$

So far it is not clear in which sense and for which $x \in \mathbb{T}^n$ the Fourier series converges. However, the following lemma holds, see e.g. [[Gra08](#), Proposition 3.1.14].

Lemma 1.27 (Fourier inversion). *If $f \in L^1(\mathbb{T}^n)$ with $\sum_{\xi \in \mathbb{Z}^n} |\hat{f}(\xi)| < \infty$, then*

$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{\xi \in \mathbb{Z}^n} \hat{f}(\xi) e^{ix \cdot \xi}$$

almost everywhere. As a consequence f equals almost everywhere a continuous function.

Remark. In light of the previous lemma, for a function $f: \mathbb{Z}^n \rightarrow \mathbb{C}$ with $\sum_{\xi \in \mathbb{Z}^n} |f(\xi)| < \infty$ and $x \in \mathbb{T}^n$ we write,

$$\mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{\xi \in \mathbb{Z}^n} f(\xi) e^{ix \cdot \xi}$$

and sometimes also $\check{f} = \mathcal{F}^{-1}(f)$. ◇

We collect some properties:

Lemma 1.28 (Properties of \mathcal{F}). *Given two functions $f, g \in L^1(\mathbb{T}^n)$, $y \in \mathbb{T}^n$, $\lambda \in \mathbb{C}$, $\xi, \eta \in \mathbb{Z}^n$, and α a multi-index, we have*

- (i) $\sup_{\xi \in \mathbb{Z}^n} |\widehat{f}(\xi)| \leq \|f\|_{L^1(\mathbb{T}^n)}$,
- (ii) $\widehat{f+g} = \widehat{f} + \widehat{g}$,
- (iii) $\widehat{\lambda f} = \lambda \widehat{f}$,
- (iv) $\widehat{\tau^y(f)}(\xi) = e^{-iy \cdot \xi} \widehat{f}(\xi)$,
- (v) $\widehat{(e^{ix \cdot \eta} f(x))}(\xi) = \widehat{f}(\xi - \eta)$,
- (vi) $\widehat{f}(0) = \int_{\mathbb{T}^n} f(x) dx$,
- (vii) $\widehat{f * g} = \widehat{f} \widehat{g}$,
- (viii) $\widehat{\partial^\alpha f}(\xi) = (i\xi)^\alpha \widehat{f}(\xi)$.

A useful result that connects the Fourier analysis on the torus with the Fourier analysis on \mathbb{R}^n is the Poisson summation formula, named after French mathematician and physicist Siméon Denis Poisson (1781–1840).

Proposition 1.29 (Poisson summation formula). *Suppose that $f, \widehat{f} \in L^1(\mathbb{R}^n)$ satisfy*

$$|f(x)| + |\widehat{f}(x)| \leq C(1 + |x|)^{-n-\delta}$$

for some $C, \delta > 0$. Then f and \widehat{f} are both continuous, and for all $x \in \mathbb{R}^n$ we have

$$\sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) e^{ix \cdot \xi} = \sum_{\xi \in \mathbb{Z}^n} f(x + \xi).$$

As a consequence of Hilbert space theory, we may define the Fourier transform even for functions $f \in L^2(\mathbb{T}^n)$.

Lemma 1.30 ([Gra08, Proposition 3.1.15]). *Let H be a separable Hilbert space with complex inner product $\langle \cdot, \cdot \rangle$ and let $\{\varphi_k\}_{k \in \mathbb{Z}}$ be an orthonormal system in H . Then the following are equivalent:*

(i) $\{\varphi_k\}_{k \in \mathbb{Z}}$ is a complete orthonormal system.

(ii) For every $f \in H$ we have

$$\|f\|_H^2 = \sum_{k \in \mathbb{Z}} |\langle f, \varphi_k \rangle|^2.$$

(iii) For every $f \in H$ we have

$$f = \lim_{N \rightarrow \infty} \sum_{|k| \leq N} \langle f, \varphi_k \rangle \varphi_k,$$

where the series converges in H , i.e.

$$\lim_{N \rightarrow \infty} \left\| f - \sum_{|k| \leq N} \langle f, \varphi_k \rangle \varphi_k \right\|_H = 0.$$

Consider the Hilbert space $L^2(\mathbb{T}^n)$ with complex inner product

$$\langle f, g \rangle := \int_{\mathbb{T}^n} f(x) \overline{g(x)} dx.$$

We choose φ_ξ to be the sequence of functions $x \mapsto e^{ix \cdot \xi}$ indexed by $\xi \in \mathbb{Z}^n$. It is easy to see that $\{\varphi_\xi\}_{\xi \in \mathbb{Z}^n}$ are indeed orthonormal:

$$\int_{[0, 2\pi]^n} e^{ix \cdot \xi} \overline{e^{-ix \cdot \eta}} dx = \begin{cases} 1 & \text{when } \xi = \eta, \\ 0 & \text{when } \xi \neq \eta. \end{cases}$$

In order to show the completeness of the orthonormal system $\{\varphi_\xi\}_{\xi \in \mathbb{Z}^n}$, we cite the next result that answers the question whether the Fourier coefficients uniquely determine the function.

Lemma 1.31 ([Gra08, Proposition 3.1.13]). *If $f, g \in L^1(\mathbb{T}^n)$ satisfy $\widehat{f}(\xi) = \widehat{g}(\xi)$ for all $\xi \in \mathbb{Z}^n$, then $f = g$ almost everywhere.*

The completeness is now obvious since $\langle f, \varphi_\xi \rangle = \widehat{f}(\xi)$ for all $f \in L^2(\mathbb{T}^n)$. The previous lemma now implies that if $\langle f, \varphi_\xi \rangle = 0$ for all $\xi \in \mathbb{Z}^n$, then $f = 0$ almost everywhere.

The next result is a consequence of Lemma 1.30.

Proposition 1.32 ([Gra08, Proposition 3.1.16]). *The following are valid for $f, g \in L^2(\mathbb{T}^n)$:*

(i) *Plancherel's identity:* $\|f\|_{L^2(\mathbb{T}^n)}^2 = (2\pi)^n \sum_{\xi \in \mathbb{Z}^n} |\widehat{f}(\xi)|^2.$

(ii) *The function $f(x)$ is almost everywhere equal to the $L^2(\mathbb{T}^n)$ limit of the sequence*

$$\lim_{N \rightarrow \infty} \sum_{\xi \in \mathbb{Z}^n: |\xi| \leq N} \widehat{f}(\xi) e^{ix \cdot \xi}$$

(iii) *Parseval's relation:* $\int_{\mathbb{T}^n} f(x) \overline{g(x)} dx = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) \overline{\widehat{g}(\xi)}.$

(iv) *The map $f \mapsto \{\widehat{f}(\xi)\}_{\xi \in \mathbb{Z}^n}$ is an isometry from $L^2(\mathbb{T}^n)$ to $\ell^2(\mathbb{Z}^n)$.*

We already mentioned in Section 1.2.1 that the convolution may be defined on \mathbb{T}^n and that there is a version of Young's inequality on that domain. We refer to [Gra08, Section 1.2], where the convolution is defined more generally on a locally compact group and furthermore, some convolution inequalities such as Young's inequality are proved.

Definition 1.33 (Convolution on \mathbb{T}^n). Let $f, g \in L^1(\mathbb{T}^n)$. Define the *convolution* $f * g$ by

$$(f * g)(x) := \int_{\mathbb{T}^n} f(x - y) g(y) dy.$$

Young's inequality stays exactly the same as in Lemma 1.7:

Lemma 1.34 (Young's inequality). *Let $1 \leq p, q, r \leq \infty$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ as well as $f \in L^p(\mathbb{T}^n)$ and $g \in L^q(\mathbb{T}^n)$. Then $f * g \in L^r(\mathbb{T}^n)$ and*

$$\|f * g\|_{L^r(\mathbb{T}^n)} \leq C_{p,q,r,n} \|f\|_{L^p(\mathbb{T}^n)} \|g\|_{L^q(\mathbb{T}^n)}.$$

1.3.2 Hausdorff–Young inequalities

The classical Hausdorff–Young inequality for periodic functions provides for $2 \leq p \leq \infty$ a bound for the ℓ^p -norm of the Fourier coefficients by the $L^{p'}$ -norm of the function, where p' denotes—as usual—the conjugate Hölder exponent $1 = \frac{1}{p} + \frac{1}{p'}$. For the same range of p , the L^p -norm of a periodic function may also be estimated by the $\ell^{p'}$ -norm of the Fourier coefficients, which may be seen as a “dual” estimate of the first one. In 1913, William Henry Young [You13] proved this estimate for even p . Ten years later, Felix Hausdorff [Hau23] proved the result in general. We cite the Hausdorff–Young inequality from [Kat68, Theorems IV.2.1 & IV.2.2], where it is given in one dimension. To emphasize the analogy to the proof of Proposition 1.36, we also provide a sketch of the proof of the classical Hausdorff–Young inequality.

Proposition 1.35 (Hausdorff–Young inequality). *Let $2 \leq p \leq \infty$ and p' denote the conjugate Hölder exponent.*

(i) *If $f \in L^p([0, 2\pi]^n)$, then*

$$\|\widehat{f}(\xi)\|_{\ell_\xi^p(\mathbb{Z}^n)} \leq (2\pi)^{\frac{n}{p} - \frac{n}{2}} \|f\|_{L^{p'}([0, 2\pi]^n)}.$$

(ii) *If $a \in \ell^{p'}(\mathbb{Z}^n)$, then there exists a function $f \in L^p([0, 2\pi]^n)$ such that $a_\xi = \widehat{f}(\xi)$ for all $\xi \in \mathbb{Z}^n$ and moreover,*

$$\|f\|_{L^p([0, 2\pi]^n)} \leq (2\pi)^{\frac{n}{p} - \frac{n}{2}} \|\widehat{f}(\xi)\|_{\ell_\xi^{p'}(\mathbb{Z}^n)}.$$

Sketch of the proof. The proof is taken from [Kat68, Theorems IV.2.1 & IV.2.2].

Note that for $p = 2$ the first inequality matches Plancherel’s identity. Interpolating this with the trivial estimate for $p = \infty$,

$$\sup_{\xi \in \mathbb{Z}^n} |\widehat{f}(\xi)| = \left| \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{[0, 2\pi]^n} e^{-ix \cdot \xi} f(x) dx \right| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_{L^1([0, 2\pi]^n)},$$

yields (i).

The idea of the proof of second estimate is similar. If $a \in \ell^1(\mathbb{Z}^n)$, then $f(x) := \sum_{\xi \in \mathbb{Z}^n} a_\xi e^{ix \cdot \xi}$ is continuous on $[0, 2\pi]^n$ and $\widehat{f}(\xi) = (2\pi)^{n/2} a_\xi$ for every $\xi \in \mathbb{Z}^n$. Furthermore,

$$\|f\|_{L^\infty([0, 2\pi]^n)} = \sup_{x \in [0, 2\pi]^n} \left| \sum_{\xi \in \mathbb{Z}^n} a_\xi e^{ix \cdot \xi} \right| \leq \sum_{\xi \in \mathbb{Z}^n} |a_\xi| = \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{\xi \in \mathbb{Z}^n} |\widehat{f}(\xi)|,$$

and the result follows from interpolation with Plancherel’s identity. \square

In Chapter 2, we have to deal with functions of the form $t \mapsto \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t}$, where Λ is a countable set of real numbers. Depending on Λ , this function might not be periodic. For this reason, we need a replacement for the classical Hausdorff–Young inequality. The first inequality of the following lemma has been applied before, e.g. in [Bou07, formula (1.1.9)], but yet without a rigorous proof. The second statement appeared in [BGT05b, Lemma 5.2] for $p = 2$. In this case, it can be seen as non-periodic variant of Plancherel’s identity. Similarly as for the classical Hausdorff–Young inequality, we gain the full range of p by interpolating with the trivial case $p = \infty$.

Proposition 1.36 (Non-periodic Hausdorff–Young inequality). *Assume that $2 \leq p \leq \infty$ and Λ is a countable set of real numbers. Furthermore, let p' denote the conjugate Hölder exponent.*

(i) *Then there exists $C > 0$ such that for every non-negative sequence $(a_\lambda)_{\lambda \in \Lambda}$,*

$$\left\| \sum_{\lambda \in \Lambda: |\lambda - k| \leq \frac{1}{2}} a_\lambda \right\|_{\ell_k^p} \leq C \left\| \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \right\|_{L^{p'}([0, 2\pi])}$$

holds true.

(ii) *For every compact interval $I \subset \mathbb{R}$ there exists $C_I > 0$ such that for every sequence $(a_\lambda)_{\lambda \in \Lambda}$ the “dual” estimate of (i) holds:*

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \right\|_{L^p(I)} \leq C_I \left\| \sum_{\lambda \in \Lambda: |\lambda - k| \leq \frac{1}{2}} |a_\lambda| \right\|_{\ell_k^{p'}}.$$

Proof. In this proof, $\hat{\cdot}$ shall denote the Fourier transform on \mathbb{R} .

First, we prove (i). Let $\eta: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function supported on $[-\pi, \pi]$ with $\hat{\eta}(\tau) \geq 0$ for all $\tau \in \mathbb{R}$ and $\hat{\eta}(\tau) \geq 1$ for all $\tau \in [-\frac{1}{2}, \frac{1}{2}]$. For instance, if $c > 0$ large enough, then

$$\eta(t) = c \chi_{[-\frac{1}{2}, \frac{1}{2}]} * \chi_{[-\frac{1}{2}, \frac{1}{2}]}(t)$$

fulfills this assumptions since the Fourier transform is given by $\hat{\eta}(\tau) = \tilde{c}(\sin(\tau/2)/\tau)^2$. Define $\psi: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\psi(t) := \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \eta(t).$$

Then,

$$\left\| \sum_{\lambda \in \Lambda: |\lambda - k| \leq \frac{1}{2}} a_\lambda \right\|_{\ell_k^p} \leq \left\| \sum_{\lambda \in \Lambda: |\lambda - k| \leq \frac{1}{2}} a_\lambda \hat{\eta}(k - \lambda) \right\|_{\ell_k^p} = \|\hat{\psi}(k)\|_{\ell_k^p}$$

and thus, it suffices to prove

$$\|\hat{\psi}(k)\|_{\ell_k^p} \lesssim \left\| \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \right\|_{L^{p'}([0, 2\pi])}. \quad (1.1)$$

Due to the assumption on the support of η , $\hat{\psi}(k)$ coincides with the k th Fourier coefficient of the periodic continuation of $\psi|_{[-\pi, \pi]}$. Hence, (1.1) follows from the (classical) Hausdorff–Young inequality for periodic functions.

We now turn to the proof of the “dual” estimate (ii). The estimate for $p = \infty$ is immediate since the sets $\{\lambda \in \Lambda : |\lambda - k| \leq \frac{1}{2}\}$ are essentially disjoint for different $k \in \mathbb{Z}$. By interpolation, we are left to prove the estimate in the case $p = 2$ for which we closely follow the argument of Burq–Gérard–Tzvetkov [BGT05b, Lemma 5.2].

We choose a function $\eta \in C_0^\infty(\mathbb{R})$ with the property $\eta(t) = 1$ for $t \in I$. If we define $f: \mathbb{R} \rightarrow \mathbb{C}$ as

$$f(t) := \sum_{\lambda \in \Lambda} \eta(t) a_\lambda e^{i\lambda t},$$

then

$$\hat{f}(\tau) = \sum_{\lambda \in \Lambda} \hat{\eta}(\tau - \lambda) a_\lambda,$$

which reduces the claim to

$$\|\widehat{f}\|_{L^2(\mathbb{R})} \leq C_I \left(\sum_{k \in \mathbb{Z}} \left(\sum_{\lambda \in \Lambda: |\lambda-k| \leq \frac{1}{2}} |a_\lambda| \right)^2 \right)^{\frac{1}{2}}.$$

For $\tau \in \mathbb{R}$ we estimate,

$$|\widehat{f}(\tau)| \leq \sum_{k \in \mathbb{Z}} \sum_{\lambda \in \Lambda: |\lambda-k| \leq \frac{1}{2}} |\widehat{\eta}(\tau - \lambda)| |a_\lambda| \leq \sum_{k \in \mathbb{Z}} K(\tau, k) h(k),$$

where $K: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ are defined as

$$K(\tau, k) := \sup_{\lambda \in \Lambda: |\lambda-k| \leq \frac{1}{2}} |\widehat{\eta}(\tau - \lambda)| \quad \text{and} \quad h(k) := \sum_{\lambda \in \Lambda: |\lambda-k| \leq \frac{1}{2}} |a_\lambda|,$$

respectively. A simple argument shows that the inequality $|\lambda - k| \leq \frac{1}{2}$ implies

$$\frac{1}{1 + |\tau - \lambda|} \leq \frac{C}{1 + |\tau - k|}.$$

Since $\eta \in C_0^\infty(\mathbb{R})$, we see that for every $N \in \mathbb{N}$ there exists $C_{I,N}$ such that

$$|K(\tau, k)| \leq \frac{C_{I,N}}{(1 + |\tau - k|)^N}.$$

Now, we may apply Schur's lemma to conclude the asserted estimate. \square

Next, we apply the previous Hausdorff–Young inequality to address lattice point counting. The result is not new and was used by several authors before, e.g. [Bou07, CW10, GOW14, Str14], and a rigorous proof may be found in [Str14, Lemma 3.1]. Nonetheless, we give a new proof to highlight the close relation to [Proposition 1.36](#).

Corollary 1.37. *Let $2 \leq p \leq \infty$, $r \geq \frac{1}{2}$, and Λ be a countable set. There exists $C_r > 0$ such that for all $\varphi: \Lambda \rightarrow \mathbb{R}$ and for \mathfrak{S}_k defined as*

$$\mathfrak{S}_k := \left\{ \lambda \in \Lambda : |\varphi(\lambda) - k| \leq r \right\}, \quad k \in \mathbb{Z},$$

the following estimate holds true:

$$\|\mathfrak{S}_k\|_{\ell_k^p} \leq C_r \left\| \sum_{\lambda \in \Lambda} e^{i\varphi(\lambda)t} \right\|_{L_t^{p'}([0, 2\pi])}.$$

Here, p' denotes the conjugate Hölder exponent.

Proof. First, we reduce it to

$$\|\mathfrak{S}_k\|_{\ell_k^p} \leq \left\| \sum_{\ell=-r}^{r-1} \sum_{\substack{\lambda \in \Lambda: \\ \ell \leq \varphi(\lambda) - k \leq \ell+1}} 1 \right\|_{\ell_k^p} \leq \sum_{\ell=-r}^{r-1} \left\| \sum_{\substack{\lambda \in \Lambda: \\ |\varphi(\lambda) - \ell - \frac{1}{2} - k| \leq \frac{1}{2}}} 1 \right\|_{\ell_k^p}.$$

Then, an application of [Proposition 1.36](#) yields the desired estimate:

$$\sum_{\ell=-r}^{r-1} \left\| \sum_{\substack{\lambda \in \Lambda: \\ |\varphi(\lambda) - \ell - \frac{1}{2} - k| \leq \frac{1}{2}}} 1 \right\|_{\ell_k^p} \lesssim \sum_{\ell=-r}^{r-1} \left\| \sum_{\lambda \in \Lambda} e^{i(\varphi(\lambda) - \ell - \frac{1}{2})t} \right\|_{L^{p'}([0, 2\pi])} \lesssim_r \left\| \sum_{\lambda \in \Lambda} e^{i\varphi(\lambda)t} \right\|_{L^{p'}([0, 2\pi])}$$

\square

1.3.3 L^p -estimates of exponential sums

L^p -estimates of exponential sums turned out to be substantial for studying the nonlinear Schrödinger equation on compact manifolds. Such estimates have been addressed in recent years, cf. [Bou89, Section 4], [Bou93a, Proposition 3.114], [Bou07, formula (1.1.10)], and [BGT07, Lemma 5.5.3]. In all of these works, the L^p -norm of sums like

$$\sum_{n \in J \cap \mathbb{Z}} a_n e^{in^2 t}$$

have been considered, where $J = [-N, N]$. This, however, is not sufficient for our analysis. In fact, we have to show that the L_t^p -bound does only depend on the size of J rather than on the actual position. Herr [Her13, Lemma 3.1] observed that this is the case for the exponential sum above by modifying the arguments in [Bou89, Section 4] slightly. Corollary 1.39 can be viewed as a special case of [Her13, Lemma 3.1] with $a_n = 1$ or as an extension of [Bou07, formula (1.1.10)] to intervals J which are not centered around zero. In this thesis, we want to take the opportunity to give a rigorous proof of the exponential sum estimates we shall rely on. In fact, due to technical reasons, we prove a slightly more general statement in Lemma 1.38 and conclude the required estimate in Corollary 1.39.

Lemma 1.38. *Let $I \subseteq \mathbb{R}$ be a compact interval and $4 < p \leq \infty$. There exists a constant $C > 0$ such that for any $N \geq 1$, $b \in \mathbb{Z}$, and any multiplier $(\sigma_n)_n$ satisfying*

- (i) *for all $n \in \mathbb{Z}$: $0 \leq \sigma_n \leq 1$; for all $n \in [-N, N]$: $\sigma_n = 1$; for all $n \notin [-2N, 2N]$: $\sigma_n = 0$,*
- (ii) *the sequence $(\sigma_{n+1} - \sigma_n)_n$ is bounded by $\frac{2}{N}$ and has bounded variation by $\frac{2}{N}$,*

the estimate

$$\left\| \sum_{n \in \mathbb{Z}} \sigma_n e^{i(n+b)^2 t} \right\|_{L_t^p(I)} \leq CN^{1-\frac{2}{p}},$$

holds true. The constant C depends only on p and $|I|$.

As a consequence, we get the same statement without the coefficient sequence that smoothens out the cut-off.

Corollary 1.39. *Let $I \subseteq \mathbb{R}$ be a compact interval and $4 < p \leq \infty$. Then, there exists a constant $C > 0$ such that for any $M \geq 1$ and $J = [b, b+M] \cap \mathbb{Z}$ with $b \in \mathbb{Z}$, we have the estimate*

$$\left\| \sum_{n \in J} e^{in^2 t} \right\|_{L_t^p(I)} \leq CM^{1-\frac{2}{p}},$$

where the constant C does only depend on p and $|I|$.

Proof. First, we note that the inequality is trivial if $p = \infty$. Hence, we may assume $p < \infty$ from now on.

By possibly increasing M by one, we may assume M to be even. Set $N := \frac{M}{2}$ and $b' := b + N$. Let σ_n be a sequence as given in Lemma 1.38 (with respect to N). Define $\hat{\psi}: \mathbb{Z} \rightarrow \mathbb{R}$ as $\hat{\psi}((n+b')^2) := \sigma_n$ for all $n \in \mathbb{Z}$ and $\hat{\psi}$ equals 0 otherwise. Note that $\psi \in L^1([0, 2\pi])$. We write

$$\begin{aligned} \left\| \sum_{n \in J} e^{in^2 t} \right\|_{L^p([0, 2\pi])} &= \left\| \sum_{m \in \mathbb{Z}} \mathbf{1}_{[b^2, (b+M)^2]}(m) \hat{\psi}(m) e^{imt} \right\|_{L^p([0, 2\pi])} \\ &= \left\| \mathcal{F}^{-1}(\mathbf{1}_{[b^2, (b+M)^2]} \hat{\psi}) \right\|_{L^p([0, 2\pi])}. \end{aligned}$$

Since $\mathbf{1}_{[b^2, (b+M)^2]}$ is a multiplier on $L^p(\mathbb{R})$ for $1 < p < \infty$ with norm independent of the size of the interval [Duo01, Proposition 3.6], the transference of multipliers to \mathbb{T}^n [Gra08, Theorem 3.6.7] and Lemma 1.38 yield

$$\|\mathcal{F}^{-1}(\mathbf{1}_{[b^2, (b+M)^2]}\widehat{\psi})\|_{L^p([0, 2\pi])} \lesssim \left\| \sum_{n \in \mathbb{Z}} \sigma_n e^{i(n+b')^2 t} \right\|_{L^p([0, 2\pi])} \lesssim N^{1-\frac{2}{p}} \lesssim M^{1-\frac{2}{p}}. \quad \square$$

The remainder of this subsection is devoted to prove Lemma 1.38.

The proof which is presented here is a variant of the Hardy–Littlewood circle method in which one splits the integration over one period in two parts which are, due to historical reasons, called major and minor arcs. The main contribution to the L^p -norm comes from the major arcs, which contain those t that are close to a reduced fraction $\frac{a}{q}$ with $1 \leq a \leq q \leq N^{1/100}$. This is easy to see if one chooses $a = q = 1$. Then, t is close to 1 and the modulus of the sum is approximately N . Lemma 1.41 below provides a more precise estimate with some additional decay in t . Lemma 1.43 below shows that we can bound the modulus of the sum by $CN^{1-1/200}$ whenever t is in a minor arc. We would like to refer the reader to [Vau97, Chapter 2] for more details and a nice introduction to this method. In fact, for $J = [1, N] \cap \mathbb{Z}$ the estimate follows essentially from the Hardy–Littlewood circle method in the way it is presented there.

In order to prove the lemma above, we start with some basic definitions and notation. Given $a, q \in \mathbb{Z}$ with either $a \neq 0$ or $q \neq 0$, we denote by $\gcd(a, q)$ the greatest common divisor of a and q . We set $\nu := \frac{1}{100}$ throughout this subsection. Furthermore, $\|x\|_{\mathbb{Z}} := \min_{n \in \mathbb{Z}} |x - n|$ denotes the distance of x to the closest integer. The following definition of major and minor arcs is standard, see [Vau97, Section 2.1].

Definition 1.40 (Major & minor arcs). Let $N > 1$ be the N given in Lemma 1.38. We define the *major arcs* \mathfrak{M} to be the disjoint union of

$$\mathfrak{M}(q, a) := \left\{ t \in [0, 1] : \left| t - \frac{a}{q} \right| \leq N^{\nu-2} \right\}$$

for all $0 \leq a \leq q \leq N^\nu$ with $\gcd(a, q) = 1$. The *minor arcs* shall be defined as $\mathfrak{m} := [0, 1] \setminus \mathfrak{M}$.

Remark. The union of the $\mathfrak{M}(q, a)$ is indeed disjoint. If $\frac{a}{q} \neq \frac{a'}{q'}$ and $q, q' \leq N^\nu$, one estimates

$$\left| t - \frac{a}{q} \right| + \left| t - \frac{a'}{q'} \right| \geq \left| \frac{a}{q} - \frac{a'}{q'} \right| \geq \frac{1}{qq'} \geq N^{-2\nu} > 2N^{\nu-2}$$

for $N > 1$. Hence, either $t \notin \mathfrak{M}(q, a)$ or $t \notin \mathfrak{M}(q', a')$. \diamond

The following Weyl type lemma is due to Bourgain [Bou93a, Lemma 3.18]. The cases $a = 0$, $a = q = 1$, and $q = N$ were not included in the original statement. We repeat the proof and add a couple of details to point out that Bourgain’s proof covers these cases as well. The main improvement over the classical Weyl inequality

$$\left| \sum_{n=1}^N e^{2\pi i(nx+n^2t)} \right| \lesssim \frac{N}{\sqrt{q}} + \sqrt{N \ln q} + \sqrt{q \ln q},$$

see e.g. [Mon94, Chapter 3, Theorem 1], is the additional decay in t . Bourgain observed that this allows to treat both major and minor arcs with this Weyl type lemma. Originally, the major arcs were treated by approximating the exponential sum by a product of two functions, either of which may be estimated, cf. [Vau97, Section 2.4]. The coefficient sequence avoids logarithmic factors on N and plays only a technical role, see the remark after the proof.

Lemma 1.41 (Weyl inequality). *Let $N \geq 1$ and $(\sigma_n)_n$ be a multiplier satisfying (i) and (ii) in Lemma 1.38. If $0 \leq a \leq q \leq N$ with $\gcd(a, q) = 1$ and $\|t - \frac{a}{q}\|_{\mathbb{Z}} < \frac{1}{qN}$, then*

$$\left| \sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i(nx + n^2 t)} \right| \leq C \frac{N}{\sqrt{q} (1 + N \|t - \frac{a}{q}\|_{\mathbb{Z}}^{1/2})}.$$

Proof. We follow mainly Bourgain's argument in [Bou93a, Lemma 3.18] but provide more details. To do so, we also adapt some ideas that have been used in [PTW14, Lemma A.1] for proving a related result.

The proof is trivial for $N = 1$, hence, we may assume $N \geq 2$ in the sequel. Note also that the case $a = 0$ can be reduced to $a = q = 1$ since the exponential sum is 1-periodic with respect to t and $\|t\|_{\mathbb{Z}} = \|t - 1\|_{\mathbb{Z}}$. Therefore, we assume $a \geq 1$ for the remainder of the proof.

We consider the square modulus

$$\left| \sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i(nx + n^2 t)} \right|^2 = \sum_{n_1, n_2 \in \mathbb{Z}} \sigma_{n_1} \sigma_{n_2} e^{2\pi i[(n_1 - n_2)x + (n_1 - n_2)(n_1 + n_2)t]}.$$

By setting $k := n_1 - n_2$ and $\ell := n_1 + n_2$, we clearly have

$$\left| \sum_{n \in \mathbb{Z}} \sigma_n e^{i(nx + n^2 t)} \right|^2 \leq \sum_{\substack{\ell \in \mathbb{Z}: \\ |\ell| \leq 4N}} \left| \sum_{\substack{k \in \mathbb{Z}: \\ k \equiv \ell \pmod{2}}} \sigma_{\frac{k+\ell}{2}} \sigma_{\frac{\ell-k}{2}} e^{2\pi i k(x + \ell t)} \right|.$$

Let $\ell \in \mathbb{Z}$ be fixed now. If ℓ is even, we write $k = 2k_1$, otherwise, we write $k = 2k_1 + 1$. In any case, $\sigma_{\frac{k+\ell}{2}} \sigma_{\frac{\ell-k}{2}} = \sigma_{[\frac{\ell+1}{2}] + k_1} \sigma_{[\frac{\ell}{2}] - k_1} =: \tau_{k_1}$. We now claim

$$\left| \sum_{k_1 \in \mathbb{Z}} \tau_{k_1} e^{2\pi i k_1(2x + 2\ell t)} \right| \lesssim \min \left\{ N, \frac{1}{N \|2x + 2\ell t\|_{\mathbb{Z}}^2} \right\} \lesssim \frac{1}{N (\|2x + 2\ell t\|_{\mathbb{Z}} + \frac{1}{N})^2}. \quad (1.2)$$

The second inequality follows from a simple case-by-case analysis. That the sum on the left-hand side is bounded by CN is also obvious. Thus, we are left to show

$$\left| \sum_{k \in \mathbb{Z}} \tau_k e^{2\pi i k y} \right| \lesssim \frac{1}{N y^2}, \quad (1.3)$$

for $\frac{1}{N} \leq |y| \leq \frac{1}{2}$.

For the purpose of proving (1.3), we first replace the multiplier τ_k by a real-valued function that coincides with τ_k for every $k \in \mathbb{Z}$. Let $\phi_N: \mathbb{R} \rightarrow [0, 1]$ be a smooth, compactly supported function with $\phi_N(n) = \sigma_n$ for all $n \in \mathbb{Z}$ as well as $|\phi'_N(y)| \leq \frac{4}{N}$ and $|\phi''_N(y)| \leq \frac{4}{N^2}$ for all $y \in \mathbb{R}$. Define

$$\psi_N: \mathbb{R} \rightarrow [0, 1], \quad \psi_N(y) := \phi_N\left(\left[\frac{\ell+1}{2}\right] + y\right) \phi_N\left(\left[\frac{\ell}{2}\right] - y\right)$$

for all $y \in \mathbb{R}$, and observe that $|\psi'_N(y)| \leq \frac{8}{N}$ and $|\psi''_N(y)| \leq \frac{16}{N^2}$ for any $y \in \mathbb{R}$. Also, we see that $\tau_k = \psi_N(k)$ for all $k \in \mathbb{Z}$. We denote by $\mathcal{F}_{2\pi}^{-1}$ the inverse Fourier transform given by

$$\mathcal{F}_{2\pi}^{-1}(f)(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(\xi) d\xi$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. By the Poisson summation formula, cf. [Gra08, Theorem 3.1.17],

$$\left| \sum_{k \in \mathbb{Z}} \psi_N(k) e^{2\pi i k y} \right| \leq \sum_{k \in \mathbb{Z}} |\mathcal{F}_{2\pi}^{-1}(\psi_N)(y + k)|.$$

Note that

$$|\mathcal{F}_{2\pi}^{-1}(\psi_N)(y)| = \frac{1}{(2\pi y)^2} \left| \int_{\mathbb{R}} \left(\frac{d^2}{d\xi^2} e^{2\pi i y \xi} \right) \psi_N(\xi) d\xi \right| \leq \frac{1}{(2\pi y)^2} \|\psi_N''\|_{L^1(\mathbb{R})} \lesssim \frac{1}{Ny^2}, \quad y \neq 0.$$

Therefore, for $\frac{1}{N} \leq |y| \leq \frac{1}{2}$,

$$\sum_{k \in \mathbb{Z}} |\mathcal{F}_{2\pi}^{-1}(\psi_N)(y + k)| \lesssim \sum_{k \in \mathbb{Z}} \frac{1}{N|y + k|^2} \lesssim \frac{1}{Ny^2} + \sum_{k \in \mathbb{N}} \frac{1}{Nk^2} \lesssim \frac{1}{Ny^2}.$$

This completes the proof of (1.2), i.e.

$$\left| \sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i(n x + n^2 t)} \right|^2 \lesssim \sum_{|\ell| \leq 4N} \frac{1}{N(\|2x + 2\ell t\|_{\mathbb{Z}} + \frac{1}{N})^2}. \quad (1.4)$$

To estimate this further, we write $t = \frac{a}{q} + m + \tau$ for some $|\tau| \leq \frac{1}{qN}$ and $m \in \mathbb{Z}$. Hence,

$$2\ell t = 2\ell \frac{a}{q} + 2\ell m + 2\ell \tau.$$

For any $k \in \mathbb{Z}$, we define $b(k) := ak \pmod{q}$, $b(k) \in \mathbb{Z}_q := \{0, 1, \dots, q-1\}$. Since a and q are coprime, a is invertible in \mathbb{Z}_q and the mapping $k \mapsto b(k)$ is a bijection $\mathbb{Z}_q \rightarrow \mathbb{Z}_q$. Hence, for each $k \in \mathbb{Z}_q$ there are at most $\lceil \frac{8N}{q} \rceil$ different values $\ell \in \{\ell \in \mathbb{Z} : |\ell| \leq 4N\}$ such that $b(\ell) = b(k)$. Moreover, for each $r \in \{0, \dots, [q/2]\}$ there exist at most four different $b \in \mathbb{Z}_q$ such that

$$\frac{r}{q} \leq \left\| 2x + 2\frac{b}{q} \right\|_{\mathbb{Z}} \leq \frac{r+1}{q}.$$

We conclude that for each $r \in \{0, \dots, [q/2]\}$ and

$$\mathcal{N}_r := \left\{ \ell \in \mathbb{Z} : |\ell| \leq 4N, \frac{r}{q} \leq \left\| 2x + 2\frac{b(\ell)}{q} \right\|_{\mathbb{Z}} < \frac{r+1}{q} \right\}$$

we have $|\mathcal{N}_r| \leq C \frac{N}{q}$. Define $\mathcal{R} := \bigcup_{r=0}^{10} \mathcal{N}_r$. We distinguish two cases: the resonant case $\ell \in \mathcal{R}$ and the non-resonant case $\ell \notin \mathcal{R}$. The latter does only exist if $q > 20$.

We consider the non-resonant case first, i.e. $\ell \in \mathcal{N}_r$ for some $r > 10$. Since $|\ell| \leq 4N$, we see that

$$\|2x + 2\ell t\|_{\mathbb{Z}} = \left\| 2x + 2\frac{b(\ell)}{q} + 2\ell \tau \right\|_{\mathbb{Z}} \geq \left\| 2x + 2\frac{b(\ell)}{q} \right\|_{\mathbb{Z}} - 2\ell |\tau| \geq \frac{r}{q} - \frac{8}{q} \geq \frac{r}{5q}.$$

We may estimate the corresponding contribution to (1.4) by

$$\sum_{\substack{\ell \in \mathbb{Z} \setminus \mathcal{R}: \\ |\ell| \leq 4N}} \frac{1}{N(\|2x + 2\ell t\|_{\mathbb{Z}} + \frac{1}{N})^2} \lesssim \frac{1}{N} \sum_{r=11}^{q/2} \sum_{\ell \in \mathcal{N}_r} \frac{q^2}{r^2} \lesssim q \sum_{r=11}^{q/2} \frac{1}{r^2} \lesssim q. \quad (1.5)$$

We are left with the resonant case. Fortunately, there are only $|\mathcal{R}| \lesssim \frac{N}{q}$ of them. Hence, it is easy to see that the contribution from the resonant case is bounded by $C \frac{N}{q} \cdot N$. We

can improve this bound further provided $|\tau| \geq \frac{1}{N^2}$. Indeed, let $b \in \mathbb{Z}_q$ be fixed now and set $\mathcal{M}_b := \{\ell \in \mathcal{R} : b(\ell) = b\}$. Note that the gap between two consecutive elements in \mathcal{M}_b is q and that for $\ell \in \mathcal{M}_b$,

$$\|2x + 2\ell t\|_{\mathbb{Z}} = \left\| 2x + 2\frac{b}{q} + 2\ell\tau \right\|_{\mathbb{Z}}.$$

Since \mathcal{R} is the union of at most 44 sets \mathcal{M}_b , there exists $C_0 > 0$ such that

$$\{\|2x + 2\ell t\|_{\mathbb{Z}} : \ell \in \mathcal{R}\}$$

is contained in at most C_0 arithmetic sequences with increment $2q|\tau|$. Thus, we may estimate the contribution from the resonant case by

$$\begin{aligned} \sum_{\ell \in \mathcal{R}} \frac{1}{N(\|2x + 2\ell t\|_{\mathbb{Z}} + \frac{1}{N})^2} &\leq C_0 \frac{1}{N} \sum_{j \in \mathbb{N}_0} \frac{1}{(2jq|\tau| + \frac{1}{N})^2} \\ &\lesssim \sum_{\substack{j \in \mathbb{N}_0: \\ 2jq|\tau| \leq 1/N}} N + \frac{1}{N} \sum_{\substack{j \in \mathbb{N}_0: \\ 2jq|\tau| > 1/N}} \frac{1}{(2jq|\tau|)^2} \lesssim \frac{1}{q|\tau|} \end{aligned}$$

provided $|\tau| \geq \frac{1}{N^2}$. In any case, we proved

$$\sum_{\ell \in \mathcal{R}} \frac{1}{N(\|2x + 2\ell t\|_{\mathbb{Z}} + \frac{1}{N})^2} \lesssim \min\left\{\frac{N^2}{q}, \frac{1}{q|\tau|}\right\}. \quad (1.6)$$

Recall that $\tau = t - \frac{a}{q} - m$. Then the conclusion follows from (1.5) and (1.6) since

$$\left| \sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i(n^2 t + nx)} \right|^2 \lesssim q + \min\left\{\frac{N^2}{q}, \frac{1}{q|\tau|}\right\} \lesssim \frac{N^2}{q(1 + N\|t - \frac{a}{q}\|_{\mathbb{Z}}^{1/2})^2}. \quad \square$$

Remark. Guo–Oh–Wang [GOW14, page 991] discussed the role of the coefficient sequence: Consider the Weyl sum without the coefficient sequence

$$W_N(t, x) := \sum_{|n| \leq N} e^{2\pi i(n^2 t + nx)}.$$

Choosing $(\sigma_n)_n$ to increase respectively decay like $\frac{1}{N}$ in $[-2N, -N]$ respectively $[N, 2N]$, we may write

$$\sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i(n^2 t + nx)} = \frac{1}{N} \sum_{k=N}^{2N-1} W_k(t, x).$$

Hence, the regularizing effect of $(\sigma_n)_n$ may be compared to the regularizing effect of the Fejér kernel over the Dirichlet kernel, cf. [SS03, Section 5.2]. \diamond

For estimating the contribution from the minor arcs, we use the next three lemmas. The first result is due to Peter Gustav Lejeune Dirichlet (1805–1859). The statement is taken from [Vau97, Lemma 2.1], where a proof is provided as well.

Lemma 1.42 (Dirichlet's lemma). *Let t denote a real number. Then, for each real number $N \geq 1$ there exists a rational number $\frac{a}{q}$ with $\gcd(a, q) = 1$, $1 \leq q \leq N$, and*

$$\left| t - \frac{a}{q} \right| \leq \frac{1}{qN}.$$

For brevity we define the following function which equals the exponential sum in [Lemma 1.38](#) except of a dilation of 2π in the argument of the exponential function. Let $(\sigma_n)_n$ be a sequence as given in [Lemma 1.38](#), then we define

$$F_b(t) := \sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i(n+b)^2 t}. \quad (1.7)$$

The next lemma shows that a better point-wise estimate than the trivial bound $|F_b(t)| \leq 4N$ can be obtained whenever t is in the minor arcs.

Lemma 1.43. *Let $N > 1$ and F_b as in (1.7). There exists $C > 0$ such that for all $t \in \mathfrak{m}$,*

$$|F_b(t)| \leq CN^{1-\frac{\nu}{2}},$$

where C does not depend on b .

Proof. Let $t \in \mathfrak{m}$. By Dirichlet's lemma, there exists a reduced fraction $\frac{a}{q}$ with $1 \leq q \leq N$ and $|t - \frac{a}{q}| \leq \frac{1}{qN}$. Since $t \in (N^{\nu-2}, 1 - N^{\nu-2})$, it follows that $0 \leq a \leq q \leq N$.

On the one hand, if $1 \leq q \leq N^\nu$, then $\|t - \frac{a}{q}\|_{\mathbb{Z}} = |t - \frac{a}{q}| > N^{\nu-2}$ because otherwise t would be in the major arcs. Applying [Lemma 1.41](#) yields

$$|F_b(t)| \leq C \frac{N}{\sqrt{q}(1 + N\|t - \frac{a}{q}\|_{\mathbb{Z}}^{1/2})} \leq C \left\| t - \frac{a}{q} \right\|_{\mathbb{Z}}^{-\frac{1}{2}} \leq CN^{1-\frac{\nu}{2}}.$$

If, on the other hand, $N^\nu < q \leq N$, [Lemma 1.41](#) implies

$$|F_b(t)| \leq C \frac{N}{\sqrt{q}(1 + N\|t - \frac{a}{q}\|_{\mathbb{Z}}^{1/2})} \leq \frac{N}{\sqrt{q}} \leq CN^{1-\frac{\nu}{2}}. \quad \square$$

Remark. Note that the previous proof corrects the proof of [[Her13](#), formula (33)]. \diamond

We also rely on a Hua type lemma. See [[Vau97](#), Lemma 2.5] for a more general version. This is the endpoint case of [Lemma 1.38](#), which has an additional loss of ε . This loss, however, can be compensated in the minor arcs. We shall provide a proof of this well-known result for the sake of completeness.

Lemma 1.44 (Hua's lemma). *For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $N \geq 1$ and F_b as in (1.7), the estimate*

$$\|F_b\|_{L^4([0, 2\pi])} \leq C_\varepsilon N^{\frac{1}{2} + \varepsilon}$$

holds true.

Proof. The proof follows the idea of [[Bou89](#), formulas (1.3)–(1.6)] for the reduction to the number of lattice points estimate and [[Her13](#), Appendix A, b)] for the bound on the lattice points.

We apply the Parseval identity with respect to t and obtain

$$\|F_b\|_{L^4([0, 2\pi])}^4 = \left\| \sum_{m, n = -2N}^{2N} \sigma_m \sigma_n e^{2\pi i[(m+b)^2 - (n+b)^2]t} \right\|_{L^2([0, 2\pi])}^2 \lesssim \sum_{k \in \mathcal{N}} \left| \sum_{\substack{(m-n)(m+n+2b)=k \\ 1 \leq m, n \leq N}} 1 \right|^2,$$

where $\mathcal{N} := \{k \in \mathbb{Z} : \exists m, n \in [-2N, 2N] \cap \mathbb{Z} \text{ s.t. } k = (m+b)^2 - (n+b)^2\}$. Obviously, $|\mathcal{N}| \leq 16N^2$. Hence, it suffices to show that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $N \geq 1$,

$$\sup_{k \in \mathbb{N}, b \in \mathbb{N}_0} |\{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \leq N, n_1(n_2 + b) = k\}| \leq C_\varepsilon N^\varepsilon. \quad (1.8)$$

If $0 \leq b \leq 10N^2$ this is a consequence of the number of divisors estimate. We refer the reader to [HW79, Theorem 315] for more details.

If $b > 10N^2$, then the set contains at most one element. Indeed, the imposed restriction is equivalent to

$$n_1 = \frac{k}{b} - \frac{n_1 n_2}{b}.$$

For fixed $k \in \mathbb{N}$ and $10N^2 < b \in \mathbb{N}_0$ the set $\bigcup_{1 \leq n_1, n_2 \leq N} \{\frac{k}{b} - \frac{n_1 n_2}{b}\}$ is contained in an interval of size less than one. Thus, there is at most one possible n_1 . \square

To treat the major arcs, we first prove a distributional inequality. This shall be used after writing the L^p -norm in terms of the distribution function.

Lemma 1.45. *Let $\lambda > 0$. For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $N > 1$ and F_b as in (1.7),*

$$|\{t \in \mathfrak{M} : |F_b(t)| > \lambda\}| \leq C_\varepsilon N^{2+\varepsilon} \lambda^{-4-\varepsilon},$$

where the constant C_ε is independent of b .

Proof. Let $0 \leq a \leq q \leq N^\nu$ with $\gcd(a, q) = 1$ be fixed. In view of Lemma 1.41, we trivially see that

$$|\{t \in \mathfrak{M}(q, a) : |F_b(t)| > \lambda\}| \leq \left| \left\{ t \in \mathfrak{M}(q, a) : \frac{N}{\sqrt{q}(1 + N\|t - \frac{a}{q}\|_{\mathbb{Z}}^{1/2})} > \lambda \right\} \right|. \quad (1.9)$$

Suppose $t \in \mathfrak{M}(q, a)$ is such that $\|t - \frac{a}{q}\|_{\mathbb{Z}} \geq q^{-1} \lambda^{-2}$, then

$$\frac{N}{\sqrt{q}(1 + N\|t - \frac{a}{q}\|_{\mathbb{Z}}^{1/2})} \leq \lambda.$$

Hence, the set in the right-hand side of (1.9) contains only those $t \in \mathfrak{M}(q, a)$ that fulfill $\|t - \frac{a}{q}\|_{\mathbb{Z}} < q^{-1} \lambda^{-2}$. From the imposed condition of the set on the right-hand side of (1.9), we get that $\lambda < \frac{N}{\sqrt{q}}$. These two observations lead to

$$|\{t \in \mathfrak{M}(q, a) : |F_b(t)| > \lambda\}| \lesssim q^{-1} \lambda^{-2} \lesssim q^{-2-\frac{\varepsilon}{2}} N^{2+\varepsilon} \lambda^{-4-\varepsilon} \quad (1.10)$$

for any $\varepsilon > 0$. Since the $\mathfrak{M}(q, a)$ are disjoint, we get

$$|\{t \in \mathfrak{M} : |F_b(t)| > \lambda\}| = \sum_{q=1}^{N^\nu} \sum_{\substack{1 \leq a \leq q: \\ \gcd(a, q)=1}} |\{t \in \mathfrak{M}(q, a) : |F_b(t)| > \lambda\}|.$$

We then use (1.10) and estimate the sum over a by q to get

$$|\{t \in \mathfrak{M} : |F_b(t)| > \lambda\}| \leq c_\varepsilon \sum_{q=1}^{N^\nu} q^{-1-\frac{\varepsilon}{2}} N^{2+\varepsilon} \lambda^{-4-\varepsilon} \leq C_\varepsilon N^{2+\varepsilon} \lambda^{-4-\varepsilon}$$

as asserted. \square

Now, we have all the ingredients, which we shall use to prove [Lemma 1.38](#).

Proof of Lemma 1.38. The estimate is trivial for $p = \infty$ and for $N = 1$. Hence, we may assume $4 < p < \infty$ and $N > 1$.

From the 2π -periodicity of the exponential sum, we notice that we may assume $I = [0, 2\pi]$ if the constant is adjusted depending on $|I|$. By a change of variable, it suffices to show

$$\int_0^1 \left| \sum_{n \in \mathbb{Z}} \sigma_n e^{2\pi i(n+b)^2 t} \right|^p dt = \int_0^1 |F_b(t)|^p dt \lesssim N^{p-2}.$$

We split the integration over $[0, 1]$ into \mathfrak{M} and \mathfrak{m} . We consider the minor arcs first:

$$\int_{\mathfrak{m}} |F_b(t)|^p dt \leq \left(\sup_{t \in \mathfrak{m}} |F_b(t)| \right)^{p-4} \int_0^1 |F_b(t)|^4 dt \lesssim (N^{1-\frac{\nu}{2}})^{p-4} N^{2+\varepsilon} \lesssim N^{p-2}$$

provided $0 < \varepsilon \leq \frac{\nu}{2}(p-4)$. Here, we used [Lemma 1.43](#) and [Lemma 1.44](#).

For the major arcs we write the L^p -norm in terms of the distribution function and apply [Lemma 1.45](#):

$$\int_{\mathfrak{M}} |F_b(t)|^p dt \leq p \int_0^{4N} \lambda^{p-1} |\{t \in \mathfrak{M} : |F_b(t)| > \lambda\}| d\lambda \lesssim N^{2+\varepsilon} \int_0^{4N} \lambda^{p-5-\varepsilon} d\lambda \lesssim N^{p-2}. \quad \square$$

1.4 Riemannian manifolds

A brief introduction to Riemannian manifolds and some statements that are needed later are provided in this section. From [Definition 1.46](#) to [Definition 1.50](#) we follow (sometimes verbatim) Chapter 1 and Chapter 3 of the book [[Jos11](#)].

Definition 1.46 (Manifold). A *manifold* M of dimension n is a connected paracompact Hausdorff space for which every point has a neighborhood U that is homeomorphic to an open subset Ω of \mathbb{R}^n . Such a homeomorphism

$$x: U \rightarrow \Omega$$

is called a (*coordinate*) *chart*. An *atlas* is a family $\{U_\alpha, x_\alpha\}_\alpha$ of charts for which the U_α constitute an open covering of M . A *compact manifold* is a manifold which is compact as a topological space.

Remark.

- (i) A point $p \in U_\alpha$ is determined by $x_\alpha(p)$. Often the index α is omitted, and the components of $x(p) \in \mathbb{R}^n$ are called *local coordinates* of p . It is customary to write the Euclidean coordinates of \mathbb{R}^n as

$$x = (x^1, \dots, x^n),$$

and these are considered as local coordinates on M when $x: U \rightarrow \Omega$ is a chart.

- (ii) A compact manifold has a finite atlas $\{U_\alpha, x_\alpha\}_{\alpha=1, \dots, K}$. ◇

Definition 1.47 (Differentiable manifold). An atlas $\{U_\alpha, x_\alpha\}$ on a manifold is called *differentiable* or *smooth* if all chart transitions

$$x_\beta \circ x_\alpha^{-1}: x_\alpha(U_\alpha \cap U_\beta) \rightarrow x_\beta(U_\alpha \cap U_\beta)$$

are differentiable of class C^∞ .

Remark. If M and N are smooth manifolds, the Cartesian product $M \times N$ also naturally carries the structure of a differentiable manifold. If $\{U_\alpha, x_\alpha\}_{\alpha \in A}$ and $\{V_\beta, y_\beta\}_{\beta \in B}$ are atlases for M and N , respectively, then $\{U_\alpha \times V_\beta, (x_\alpha, y_\beta)\}_{(\alpha, \beta) \in A \times B}$ is a differentiable atlas for $M \times N$. \diamond

Definition 1.48 (Tangent space & derivative).

- (i) Let $x = (x^1, \dots, x^n)$ be Euclidean coordinates of \mathbb{R}^n , $\Omega \subset \mathbb{R}^n$ open, $x_0 \in \Omega$. The *tangent space* of Ω at the point x_0 ,

$$T_{x_0}\Omega,$$

is the space $\{x_0\} \times E$, where E is the n -dimensional vector space spanned by the basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$.

- (ii) If $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ are open, and $f: \Omega \rightarrow \Omega'$ is differentiable, we define the *derivative* $df(x_0)$ for $x_0 \in \Omega$ as the induced linear map between the tangent spaces

$$df(x_0): T_{x_0}\Omega \rightarrow T_{f(x_0)}\Omega', \quad v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \mapsto \sum_{i=1}^n \sum_{j=1}^m v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}.$$

Definition 1.49 (Riemannian manifold). A *Riemannian metric* on a differentiable manifold M is given by a scalar product on each tangent space $T_p M$ which depends smoothly on the base point $p \in M$. A (*smooth*) *Riemannian manifold* is a differentiable manifold, equipped with a Riemannian metric.

Remark. In local coordinates $x = (x^1, \dots, x^n)$, a metric is represented by a positive definite, symmetric matrix

$$(g_{ij}(x))_{i,j=1,\dots,n},$$

where the coefficients depend smoothly on x . Since the smoothness does not depend on the choice of coordinates, smooth dependence on the base point p as required in [Definition 1.49](#) can be expressed in local coordinates. [[Jos11](#), pages 13–14] \diamond

The product of two tangent vectors $v, w \in T_p M$ with coordinate representations (v^1, \dots, v^n) and (w^1, \dots, w^n) , i.e. $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$ and $w = \sum_{j=1}^n w^j \frac{\partial}{\partial x^j}$, then is

$$\langle v, w \rangle := \sum_{i,j=1}^n g_{ij}(x(p)) v^i w^j.$$

In particular, $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = g_{ij}$. Similarly, the length of v is given by

$$\|v\| := \langle v, v \rangle^{\frac{1}{2}}.$$

The integration of a smooth, compact Riemannian manifold with boundary can now be easily understood. The *volume factor*

$$\sqrt{g} := \sqrt{\det(g_{ij})}$$

is used for the integration of functions $F: M \rightarrow \mathbb{C}$,

$$\int_M F(x) \sqrt{g(x)} dx^1 \dots dx^n.$$

The integral is independent of the choice of the coordinate representation, see [Jos11, page 14]. The space $L^p(M)$ is defined as all functions $f: M \rightarrow \mathbb{C}$ for which the following expression exists and is finite

$$\|f\|_{L^p(M)} := \left(\int_M |f(x)|^p \sqrt{g(x)} dx^1 \dots dx^n \right)^{\frac{1}{p}}.$$

It is natural to define the $L^2(M)$ -product

$$\langle f, h \rangle_{L^2(M)} := \int_M f(x) \overline{h(x)} \sqrt{g(x)} dx^1 \dots dx^n$$

for $f, h \in L^2(M)$ such that

$$\|f\|_{L^2(M)} = \langle f, f \rangle_{L^2(M)}^{\frac{1}{2}}.$$

We now extend the Euclidean Laplace operator $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ to Riemannian manifolds. Let M be a Riemannian manifold of dimension n with metric tensor g_{ij} in some local coordinates (x^1, \dots, x^n) . Let $f: M \rightarrow \mathbb{C}$ be a function on M . The *gradient* is defined as

$$\nabla_g f := \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j},$$

where $(g^{ij})_{i,j=1,\dots,n} := (g_{ij})_{i,j=1,\dots,n}^{-1}$. One easily checks $|\nabla_g f| = |df|$. Furthermore, the *divergence* of a vector field $Z = \sum_{i=1}^n Z^i \frac{\partial}{\partial x^i}$ is

$$\operatorname{div}_g Z := \frac{1}{\sqrt{g}} \sum_{j=1}^n \frac{\partial}{\partial x^j} (\sqrt{g} Z^j) = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \left\langle Z, \frac{\partial}{\partial x^i} \right\rangle \right).$$

Definition 1.50 (Laplace–Beltrami operator). The *Laplace–Beltrami operator* of a smooth function $f: M \rightarrow \mathbb{C}$ is defined as

$$\Delta_g f := -\operatorname{div}_g \nabla_g f = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \right).$$

In this thesis, we do not work with the definition of Δ_g given in Definition 1.50 but with the properties of its spectrum and eigenfunctions: If M is assumed to be compact, the spectrum $\sigma(-\Delta_g) = \{\lambda_k\}_{k \in \mathbb{N}_0}$ is discrete and positive, i.e. $\lambda_k \geq 0$ for any $k \in \mathbb{N}_0$. By reordering the λ_k , we may assume $\lambda_k \leq \lambda_{k+1}$ for every $k \in \mathbb{N}_0$. Furthermore, $\lambda_0 = 0$ and $\lambda_k \rightarrow +\infty$ as $k \rightarrow +\infty$. There exist corresponding eigenfunctions $\{\varphi_k\}_k$ which define a complete orthonormal system in $L^2(M)$. Hence, if E_k denotes the eigenspace corresponding to the eigenvalue λ_k for $k \in \mathbb{N}_0$, then

$$L^2(M) = \bigoplus_{k=0}^{\infty} E_k,$$

i.e. for $f \in L^2(M)$ we have

$$f(x) = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle_{L^2(M)} \varphi_k(x),$$

where the series converges in $L^2(M)$. For more details on this, we refer to [Shu01, Section 8.3], [Tay11, Chapter 8], and [Jos11, Section 3.2]. Furthermore, let $h_k: L^2(M) \rightarrow L^2(M)$, $h_k(f) = \langle f, \varphi_k \rangle_{L^2(M)} \varphi_k$ denote the spectral projector onto the eigenspace E_k . For later usage, we define the projector

$$p_n := \sum_{\substack{k \in \mathbb{N}: \\ |\sqrt{\lambda_k}| \in [n-1, n)}} h_k$$

for $n \in \mathbb{N}$. We fix a smooth, non-negative, even function $\eta: \mathbb{R} \rightarrow [0, 1]$ with $\eta(y) = 1$ for $|y| \leq 1$ and $\text{supp } \eta \subseteq (-2, 2)$ to define a partition of unity. For a dyadic number $N > 1$, we set

$$\eta_N(y) := \eta\left(\frac{|y|}{N}\right) - \eta\left(\frac{2|y|}{N}\right) \quad \text{and} \quad \eta_1(y) := \eta(|y|) \quad (1.11)$$

for $y \in \mathbb{R}$. Note that $\text{supp } \eta_N \subseteq (-2N, -N/2) \cup (N/2, 2N)$. For dyadic $N \geq 1$ we define the smooth projectors of dyadic scale as

$$P_N := \sum_{k \in \mathbb{N}_0} \eta_N(\sqrt{\lambda_k}) h_k \quad \text{and} \quad P_{\leq N} := \sum_{M \leq N} P_M. \quad (1.12)$$

Remark. The smooth projectors P_N are bounded operators from $L^p(M)$ to $L^p(M)$ for any $1 < p < \infty$ [Tay74, Theorem 2.2]. See also [SS89, Xu07] for more general results. \diamond

Example 1.51.

- (i) If $M = \mathbb{T}^n$, then the set of eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}_0}$ is given by $\{|\xi|^2 : \xi \in \mathbb{Z}^n\}$. The eigenfunctions are given as $\{x \mapsto e^{ix \cdot \xi}\}_{\xi \in \mathbb{Z}^n}$. [Zel08, Section 2.3]
- (ii) If $M = \mathbb{S}^n$ equipped with the standard metric, then $\lambda_k = k^2 + (n-1)k$, $k \in \mathbb{N}_0$, and the multiplicity of the eigenvalue λ_k equals $\frac{2k+n-1}{n-1} \binom{k+n-2}{k}$. The eigenfunctions to the eigenvalue λ_k are the n -dimensional spherical harmonics of degree k . See [Tay11, Chapter 8, Corollary 4.3] and [Zel08, Section 2.3]. \diamond

The Sobolev space $H^s(M)$ can be defined now.

Definition 1.52 (Sobolev space $H^s(M)$). Let $s \geq 0$. The Sobolev space $H^s(M)$ shall be defined as $H^s(M) := (1 - \Delta_g)^{-\frac{s}{2}} L^2(M)$ endowed with the norm

$$\|f\|_{H^s(M)} := \left(\sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|h_k(f)\|_{L^2(M)}^2 \right)^{\frac{1}{2}},$$

where $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$.

Remark.

- (i) Due to the L^2 -orthogonality of the spectral projectors, we have

$$\|f\|_{H^s(M)} \approx \left(\sum_{N \geq 1} N^{2s} \|P_N f\|_{L^2(M)}^2 \right)^{\frac{1}{2}}.$$

- (ii) Apparently the first comprehensive study of Sobolev spaces on Riemannian manifolds is due to Aubin [Aub76, Aub82]. The idea is to replace partial derivatives in \mathbb{R}^n by covariant derivatives in order to define Sobolev spaces of integer order. Let ∇_α with

$\alpha = 1, \dots, n$ be the covariant derivative with respect to a given local chart. For a complex-valued smooth function and $k \in \mathbb{N}_0$ we define

$$|\nabla^k f|^2 := \sum_{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k=1}^n g^{\alpha_1 \beta_1} \dots g^{\alpha_k \beta_k} \nabla_{\alpha_1} \dots \nabla_{\alpha_k} f \cdot \nabla_{\beta_1} \dots \nabla_{\beta_k} \bar{f}$$

In particular, $|\nabla^0 f| = |f|$ and $|\nabla^1 f|^2 = |\nabla f|^2$. Then, for $1 \leq p < \infty$ and $k \in \mathbb{N}_0$ one may define the Sobolev space $W^{k,p}(M)$ as the completion of $\{h \in C^\infty(M) : \|h\|_{W^{k,p}(M)} < \infty\}$ with respect to the norm

$$\|f\|_{W^{k,p}(M)} := \sum_{j=0}^k \|\nabla^j f\|_{L^p(M)}.$$

In 1983, Strichartz [Str83] (mainly in Section 4) introduced fractional Sobolev spaces as $H^{s,p}(M) := (1 - \Delta_g)^{-\frac{s}{2}} L^p(M)$ for $1 < p < \infty$ and $s \geq 0$. For $k \in \mathbb{N}_0$ these spaces coincide with $W^{k,p}(M)$, cf. [Tri92, Section 7.4.5]. We refer the reader to [Tri92, Chapter 7], [Aub98, Chapter 2], and [Heb99, Chapter 2–3] for more details.

- (iii) Sobolev embeddings for $W^{k,p}(M)$ may be found in [Aub98, Theorem 2.20]. In the case $M = \mathbb{T}^n$, Sobolev embeddings for $H^s(\mathbb{T}^n)$ were studied in [ST87, Section 3.5.5]. \diamond

We rely on the following linear spectral cluster estimate in the L^∞ -norm due to Sogge [Sog88, Proposition 2.1] and some other immediate consequences. Due to the obvious relation to Bernstein's inequality, we sometimes refer to this as Bernstein's inequality in the sequel.

Lemma 1.53. *Let M be a smooth, connected, compact manifold without boundary of dimension $n \geq 2$.*

- (i) *There exists $C > 0$ such that for all $f \in L^2(M)$ and any $k \in \mathbb{N}$,*

$$\|p_k f\|_{L^\infty(M)} \leq C k^{\frac{n-1}{2}} \|f\|_{L^2(M)}.$$

- (ii) *Let $2 \leq p \leq \infty$. There exists $C > 0$ such that for all $f \in L^2(M)$ and any dyadic $N \geq 1$,*

$$\|P_N f\|_{L^p(M)} \leq C N^{n(\frac{1}{2} - \frac{1}{p})} \|P_N f\|_{L^2(M)}.$$

- (iii) *Let $2 \leq q \leq p \leq \infty$. There exists $C > 0$ such that for all $f \in L^q(M)$ and any dyadic $N \geq 1$,*

$$\|P_N f\|_{L^p(M)} \leq C N^{n(\frac{1}{q} - \frac{1}{p})} \|P_N f\|_{L^q(M)}.$$

Proof/Reference. The first inequality was proved in [Sog88, Proposition 2.1].

In order to prove the second estimate, we first deduce from applying (i) and the Cauchy–Schwarz inequality that (cf. [Her13, Lemma 3.4]):

$$\|P_N f\|_{L^\infty(M)} \leq \sum_{k=N/2}^{2N} \|p_k(P_N f)\|_{L^\infty(M)} \lesssim \sum_{k=N/2}^{2N} k^{\frac{n-1}{2}} \|P_N f\|_{L^2(M)} \lesssim N^{\frac{n}{2}} \|P_N f\|_{L^2(M)}.$$

Now, an interpolation type argument yields the claim: Let $\frac{1}{p} = \frac{\vartheta}{2}$, then

$$\|P_N f\|_{L^p(M)} \leq \|P_N f\|_{L^2(M)}^\vartheta \|P_N f\|_{L^\infty(M)}^{1-\vartheta} \lesssim N^{n(\frac{1}{2} - \frac{1}{p})} \|P_N f\|_{L^2(M)}.$$

We are not aware of any proof of statement (iii), hence, we prove it here in detail. It suffices to prove the dual estimate

$$\|P_N g\|_{L^{q'}(M)} \leq CN^{n(\frac{1}{q}-\frac{1}{p})} \|P_N g\|_{L^p(M)} \quad (1.13)$$

for all $g \in L^p(M)$ and any $N \geq 1$. Indeed, assuming this, we define $\tilde{P}_N := P_{N/2} + P_N + P_{2N}$ for $N > 1$ and $\tilde{P}_1 := P_1 + P_2$ and deduce

$$\begin{aligned} \|P_N f\|_{L^p(M)} &= \sup_{\substack{g \in L^{p'}(M): \\ \|g\|_{L^{p'}(M)} \leq 1}} \left| \int_M P_N f(x) \tilde{P}_N g(x) dx \right| \leq \sup_{\substack{g \in L^{p'}(M): \\ \|g\|_{L^{p'}(M)} \leq 1}} \|P_N f\|_{L^q(M)} \|\tilde{P}_N g\|_{L^{q'}(M)} \\ &\leq CN^{n(\frac{1}{q}-\frac{1}{p})} \sup_{\substack{g \in L^{p'}(M): \\ \|g\|_{L^{p'}(M)} \leq 1}} \|P_N f\|_{L^q(M)} \|g\|_{L^{p'}(M)} \leq CN^{n(\frac{1}{q}-\frac{1}{p})} \|P_N f\|_{L^q(M)}. \end{aligned}$$

The dual estimate (1.13) is a consequence of the dual estimate of (ii),

$$\|P_N g\|_{L^2(M)} \leq CN^{n(\frac{1}{2}-\frac{1}{p})} \|P_N g\|_{L^p(M)},$$

and an interpolation type argument. Indeed, choose $0 \leq \vartheta \leq 1$ such that $\frac{1}{q'} = \frac{\vartheta}{2} + \frac{1-\vartheta}{p'}$, then

$$\|P_N g\|_{L^{q'}(M)} = \|P_N g\|_{L^2(M)}^\vartheta \|P_N g\|_{L^{p'}(M)}^{1-\vartheta} \leq CN^{\vartheta n(\frac{1}{2}-\frac{1}{p})} \|P_N g\|_{L^{p'}(M)}.$$

Noting that $\vartheta n(\frac{1}{2} - \frac{1}{p}) = n(\frac{1}{q} - \frac{1}{p})$ gives the desired result. \square

Burq–Gérard–Tzvetkov [BGT05a, Lemma 2.6] proved the following smallness of the product of four eigenfunctions on M , where one of the corresponding eigenvalues is much bigger than the three others. We refer the reader to [Han12, Theorem 4.2] for a more general result.

Lemma 1.54. *There exists $K \geq 1$ such that for any $\gamma \geq 1$ there exists $C_\gamma > 0$ such that for any $f_j \in L^2(M)$ and eigenvalues $\lambda_{k_j} \in \sigma(-\Delta_g)$, $j = 0, 1, 2, 3$, with $K\lambda_{k_j} \leq \lambda_{k_0}$, $j = 1, 2, 3$,*

$$\left| \int_M h_{k_0}(f_0)(x) h_{k_1}(f_1)(x) h_{k_2}(f_2)(x) h_{k_3}(f_3)(x) dx \right| \leq C_\gamma \langle \lambda_{k_0} \rangle^{-\gamma} \prod_{j=0}^3 \|f_j\|_{L^2(M)}.$$

Using this result, we may prove the following two crude Sobolev multiplication type inequalities for the fractional Sobolev spaces introduced in Definition 1.52. To our knowledge, the following lemma has not been stated anywhere else in the literature. Hence, we give the proof.

Lemma 1.55. *Let $n = 3$, $s > 0$, and $\sigma > \frac{3}{2}$. Then there exists $C > 0$ such that the following inequality holds true for all $f, g \in H^s(M) \cap H^\sigma(M)$,*

$$\|fg\|_{H^s(M)} \leq C(\|f\|_{H^s(M)} \|g\|_{H^\sigma(M)} + \|f\|_{H^\sigma(M)} \|g\|_{H^s(M)}).$$

Proof. Let $K \geq 1$ be the constant given in Lemma 1.54. Then, for $\lambda \geq 0$ and a function $f \in L^2(M)$ we define

$$f_{\ll \lambda} := \sum_{\substack{\ell \in \mathbb{N}_0: \\ K\lambda_\ell \leq \lambda}} h_\ell(f), \quad f_{\gg \lambda} := \sum_{\substack{\ell \in \mathbb{N}_0: \\ K\lambda \leq \lambda_\ell}} h_\ell(f), \quad \text{and} \quad f_{\sim \lambda} := f - f_{\ll \lambda} - f_{\gg \lambda}.$$

Let us recall the definition of the H^s -norm

$$\|f\|_{H^s(M)} = \left(\sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|h_k f\|_{L^2(M)}^2 \right)^{\frac{1}{2}}.$$

Obviously, given $k \in \mathbb{N}_0$ we can decompose the product

$$h_k(fg) = f_{\ll \lambda_k} g_{\ll \lambda_k} + f_{\sim \lambda_k} g + fg_{\sim \lambda_k} - f_{\sim \lambda_k} g_{\sim \lambda_k} + f_{\ll \lambda_k} g_{\gg \lambda_k} + f_{\gg \lambda_k} g_{\ll \lambda_k} + f_{\gg \lambda_k} g_{\gg \lambda_k}.$$

Thanks to [Lemma 1.54](#), we may estimate the terms $f_{\ll \lambda_k} g_{\ll \lambda_k}$, $f_{\ll \lambda_k} g_{\gg \lambda_k}$, and $f_{\gg \lambda_k} g_{\ll \lambda_k}$ easily. Indeed, the first term, for instance, can be treated in the following way:

$$\sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|h_k(f_{\ll \lambda_k} g_{\ll \lambda_k})\|_{L^2(M)}^2 \leq \sum_{k \geq 0} \langle \sqrt{\lambda_k} \rangle^{2s} \left(\sum_{\substack{\ell \in \mathbb{N}_0: K\lambda_\ell \leq \lambda_k, \\ m \in \mathbb{N}_0: K\lambda_m \leq \lambda_k}} \|h_k(h_\ell(f)h_m(g))\|_{L^2(M)} \right)^2.$$

By duality, we may write

$$\|h_k(h_\ell(f)h_m(g))\|_{L^2(M)} = \sup_{\substack{v \in L^2(M): \\ \|v\|_{L^2(M)} \leq 1}} \left| \int_M h_\ell(f)(x)h_m(g)(x)h_k(v)(x) dx \right|.$$

Applying [Lemma 1.54](#) with $f_3 = 1$, which is the eigenfunction corresponding to the eigenvalue 0, we get

$$\|h_k(h_\ell(f)h_m(g))\|_{L^2(M)} \lesssim_\gamma \langle \lambda_k \rangle^{-s-\gamma} \|f\|_{L^2(M)} \|g\|_{L^2(M)}.$$

By the Weyl asymptotic, see e.g. [\[GS94, Chapter 12\]](#), summing over ℓ , m , and k yields

$$\left(\sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|h_k(f_{\ll \lambda_k} g_{\ll \lambda_k})\|_{L^2(M)}^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2(M)} \|g\|_{L^2(M)}$$

provided γ is sufficiently large. The terms $f_{\ll \lambda_k} g_{\gg \lambda_k}$ and $f_{\gg \lambda_k} g_{\ll \lambda_k}$ can be handled similarly.

To estimate the contribution coming from $fg_{\sim \lambda_k}$, we proceed as follows:

$$\begin{aligned} \sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|h_k(fg_{\sim \lambda_k})\|_{L^2(M)}^2 &\leq \sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|fg_{\sim \lambda_k}\|_{L^2(M)}^2 \leq \sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|f\|_{L^\infty}^2 \|g_{\sim \lambda_k}\|_{L^2}^2 \\ &\lesssim \|f\|_{H^\sigma(M)}^2 \|g\|_{H^s(M)}^2. \end{aligned} \tag{1.14}$$

The same argument yields

$$\sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|h_k(f_{\sim \lambda_k} g)\|_{L^2(M)}^2 \lesssim \|f\|_{H^s(M)}^2 \|g\|_{H^\sigma(M)}^2.$$

Using $\ell^1 \subset \ell^2$ and Cauchy–Schwarz, we also estimate

$$\begin{aligned} \sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|h_k(f_{\sim \lambda_k} g_{\sim \lambda_k})\|_{L^2(M)}^2 &\leq \sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|f_{\sim \lambda_k}\|_{L^2(M)}^2 \|g_{\sim \lambda_k}\|_{L^\infty(M)}^2 \\ &\lesssim \|f\|_{H^s(M)}^2 \sum_{k=0}^{\infty} \|g_{\sim \lambda_k}\|_{L^\infty(M)}^2 \lesssim \|f\|_{H^s(M)}^2 \|g\|_{H^\sigma(M)}^2, \end{aligned}$$

where in the last step we used

$$\sum_{k=0}^{\infty} \|g_{\sim\lambda_k}\|_{L^\infty(M)}^2 \lesssim \sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^3 \|g_{\sim\lambda_k}\|_{L^2(M)}^2 \lesssim \|g\|_{H^\sigma(M)}^2.$$

Now, it remains to estimate the term $f_{\gg\lambda_k} g_{\gg\lambda_k}$. First, we note that it suffices to consider $\sum_{\ell \in \mathbb{N}_0: K\lambda_k \leq \lambda_\ell} h_\ell(f) g_{\sim\lambda_\ell}$ since the other contributions are bounded by $C\|f\|_{L^2(M)}\|g\|_{L^2(M)}$, which can be proved using [Lemma 1.54](#). We find that

$$\sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \left\| h_k \left(\sum_{\substack{\ell \in \mathbb{N}_0: \\ K\lambda_k \leq \lambda_\ell}} h_\ell(f) g_{\sim\lambda_\ell} \right) \right\|_{L^2(M)}^2 \leq \sum_{k=0}^{\infty} \left(\sum_{\substack{\ell \in \mathbb{N}_0: \\ K\lambda_k \leq \lambda_\ell}} \langle \sqrt{\lambda_\ell} \rangle^s \|h_k(h_\ell(f) g_{\sim\lambda_\ell})\|_{L^2(M)} \right)^2$$

using the triangle inequality and $\lambda_k \leq \lambda_\ell$. Hence,

$$\begin{aligned} \left(\sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \left\| h_k \left(\sum_{\substack{\ell \in \mathbb{N}_0: \\ K\lambda_k \leq \lambda_\ell}} h_\ell(f) g_{\sim\lambda_\ell} \right) \right\|_{L^2(M)}^2 \right)^{\frac{1}{2}} &\leq \sum_{\ell=0}^{\infty} \langle \sqrt{\lambda_\ell} \rangle^s \left(\sum_{k=0}^{\infty} \|h_k(h_\ell(f) g_{\sim\lambda_\ell})\|_{L^2(M)}^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{\ell=0}^{\infty} \langle \sqrt{\lambda_\ell} \rangle^s \|h_\ell(f) g_{\sim\lambda_\ell}\|_{L^2(M)}. \end{aligned}$$

Hölder's estimate, Bernstein's inequality, and Cauchy–Schwarz yield

$$\sum_{\ell=0}^{\infty} \langle \sqrt{\lambda_\ell} \rangle^s \|h_\ell(f) g_{\sim\lambda_\ell}\|_{L^2(M)} \lesssim \sum_{\ell=0}^{\infty} \langle \sqrt{\lambda_\ell} \rangle^{s+\frac{3}{2}} \|h_\ell(f)\|_{L^2(M)} \|g_{\sim\lambda_\ell}\|_{L^2(M)} \lesssim \|f\|_{H^s(M)} \|g\|_{H^\sigma(M)},$$

which finishes the proof. \square

Remark. On \mathbb{R}^n [Lemma 1.55](#) is known to hold if one replaces H^σ by L^∞ , see e.g. [[Tay00](#), page 104, formula (0.22)]. \diamond

A similar result holds if we assume that one function is more regular than the other.

Lemma 1.56. *Let $n = 3$, $s > 0$, and $\sigma > \frac{3}{2}$. Then there exists $C > 0$ such that the following inequality holds true for all $f \in H^s(M)$ and $g \in H^{s+\sigma}(M)$,*

$$\|fg\|_{H^s(M)} \leq C(\|f\|_{H^s(M)}\|g\|_{H^\sigma(M)} + \|f\|_{L^2(M)}\|g\|_{H^{s+\sigma}(M)}).$$

Proof. We highlight the differences to the proof of [Lemma 1.55](#).

All estimates in the previous proof are sufficient except of (1.14). This inequality may be replaced by

$$\sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|h_k(f g_{\sim\lambda_k})\|_{L^2(M)}^2 \leq \sum_{k=0}^{\infty} \langle \sqrt{\lambda_k} \rangle^{2s} \|f\|_{L^2(M)}^2 \|g_{\sim\lambda_k}\|_{L^\infty(M)}^2 \leq \|f\|_{L^2(M)}^2 \|g\|_{H^{s+\sigma}(M)}^2,$$

where we used Bernstein's inequality. \square

1.5 Dispersion

The Schrödinger equation is one of the most studied dispersive equations. We provide a brief introduction to dispersive equations in this section. A short introduction to the NLS on the Euclidean space and on compact manifolds is given and we continue the discussion about differences in the study on those domains that was stated in the [Introduction](#).

1.5.1 Dispersive equations

We follow the nice introduction given by Nataša Pavlović and Nikolaos Tzirakis at the MSRI Summer Graduate School “Dispersive Partial Differential Equations” in 2014 [PT14]. Consider a partial differential equation on \mathbb{R}^n without boundary conditions. Informally speaking, this partial differential equation is said to be dispersive if its solutions spread out in space as they evolve in time. We give another characterization after the following example that can be found in [PT14, pages 1–2].

Example 1.57. The linear homogeneous Schrödinger equation on the real line is given by

$$i\partial_t u + \partial_{xx} u = 0,$$

where $u: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ is a function. We are looking for a simple wave solution, i.e. for u of the form

$$u(t, x) = C e^{i(kx - \omega t)}.$$

By plugging this into our equation, we see that u satisfies the equation if and only if $\omega = k^2$. Hence, the frequency is a real-valued function of the wave number k . If we denote the phase velocity by $v(k) := \frac{\omega}{k}$, then

$$u(t, x) = C e^{ik(x - v(k)t)}.$$

From this, we see that the wave travels with velocity k and that waves corresponding to large wave numbers k propagate faster than waves belonging to small wave numbers.

If we choose the same wave solution ansatz for the heat equation,

$$\partial_t u + \partial_{xx} u = 0,$$

then we obtain $i\omega = k^2$. Therefore, the solutions decay exponentially in time. Using this ansatz, one can also see that the transport equation $\partial_t u - \partial_x u = 0$ and the one-dimensional wave equation $\partial_{tt} u - \partial_{xx} u = 0$ have traveling waves with constant velocity. \diamond

Dispersive equations may also be characterized by the support of the space-time Fourier transform of their solutions. If the space-time Fourier transform is supported on hypersurfaces that have non-vanishing Gaussian curvature, we call the partial differential equation dispersive. The following example can be found in [PT14, page 2].

Example 1.58. Consider the linear homogeneous Schrödinger equation on \mathbb{R}^n

$$i\partial_t u + \Delta u = 0.$$

The space-time Fourier transform fulfills

$$\tau \hat{u}(\tau, \xi) - |\xi|^2 \hat{u}(\tau, \xi) = (\tau - |\xi|^2) \hat{u}(\tau, \xi) = 0.$$

Hence, \hat{u} is supported on the paraboloid

$$\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\xi|^2\},$$

which has non-vanishing Gaussian curvature.

The linear wave equation $\partial_{tt} u - \Delta u = 0$ on \mathbb{R}^n , on the contrary, is supported on the cone $\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\xi|\}$, which has one direction in which the principal curvature vanishes. \diamond

Adding some nonlinear effects to a dispersive equation, like

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0$$

for some $p > 1$, makes the analysis of this equation much harder. If u is very small, then solutions to this equation behave almost like linear solutions since the linear term dominates the nonlinear term. However, if u is large, then the nonlinear term dominates and may cause concentration or blow-up in finite time. In general, one expects a competition between the dispersion and the nonlinearity.

Some popular examples of nonlinear dispersive equations are

- the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u - |u|^{p-1}u = 0, \quad u: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}, \quad p > 1,$$

- the Korteweg–de Vries equation

$$\partial_t u + \partial_{xxx} u + u\partial_x u = 0, \quad u: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R},$$

- the nonlinear Klein–Gordon equation

$$\partial_{tt} u - \Delta u + u + |u|^{p-1}u = 0, \quad u: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}.$$

The first two equations illustrate well in which different ways the nonlinearities and the dispersion interact. On the one hand, the global energy solution to the NLS as stated above satisfies a certain decay in time, see e.g. [CKS⁺08, Theorem 1.1] on \mathbb{R}^3 , that weakens the influence of the nonlinear term. Hence, for large times the dynamics of the NLS may be compared with the linear problem (scattering property) [SS99, Theorem 3.21]. On the other hand, this is not possible for the Korteweg–de Vries equation. Indeed, the dispersion and the nonlinearity are balanced in such a way that there are solitary waves solutions. These are waves that keep its form and size and just translate as time evolves [LP15, formula (7.6)]. Hence, a similar scattering effect cannot be present for solutions to this equation.

A partial differential equation which is posed on some compact Riemannian n -manifold without boundary is called dispersive if the corresponding equation on \mathbb{R}^n is dispersive. In this setting, we expect a different behavior. The reason is that due to the compactness of the domain, the dispersion is limited. How this can be understood is addressed in the next subsection.

1.5.2 The Schrödinger equation

Some basic facts about the linear and the nonlinear Schrödinger equation on \mathbb{R}^n are briefly introduced and the terms (energy-)critical and (energy-)sub-critical are defined. Then, both the linear and the nonlinear Schrödinger equation on compact manifolds are considered and related to the respective equation on Euclidean domains.

Euclidean domains

First, we consider the linear equation. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ the function

$$e^{it\Delta}\phi(x) := \mathcal{F}_{\mathbb{R}^n}^{-1}(e^{-i|\cdot|^2 t} \widehat{\phi})(x) = \frac{1}{(4\pi it)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} \phi(y) dy, \quad t \neq 0,$$

solves the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n \\ u(0, \cdot) = \phi & \text{on } \mathbb{R}^n. \end{cases}$$

We refer the reader to [Caz03, Lemma 2.2.4] for more details. From the definition of the solution formula, it is obvious that the L^2 -norm of the solution is conserved in time, i.e. $\|e^{it\Delta}\phi\|_{L^2(\mathbb{R}^n)} = \|\phi\|_{L^2(\mathbb{R}^n)}$ for any $t \in \mathbb{R}$. Moreover, the solution satisfies the so called *dispersive estimate*:

$$\|e^{it\Delta}\phi\|_{L^\infty(\mathbb{R}^n)} \leq (4\pi|t|)^{-\frac{n}{2}} \|\phi\|_{L^1(\mathbb{R}^n)}.$$

These two observations imply, by interpolation, the well-known estimate [Caz03, Proposition 2.2.3]

$$\|e^{it\Delta}\phi\|_{L^q(\mathbb{R}^n)} \leq (4\pi|t|)^{-n(\frac{1}{2}-\frac{1}{q})} \|\phi\|_{L^{q'}(\mathbb{R}^n)}, \quad \phi \in L^{q'}(\mathbb{R}^n), \quad (1.15)$$

where q' is the conjugated Hölder exponent of $2 \leq q \leq \infty$. From this, it is not hard to see that the Schrödinger flow does not preserve any $L^p(\mathbb{R}^n)$ -norm other than the $L^2(\mathbb{R}^n)$ -norm. The estimate (1.15) is the fundamental ingredient to the important *Strichartz estimates* [Caz03, Theorem 2.3.3]

$$\|e^{it\Delta}\phi\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \leq C \|\phi\|_{L^2(\mathbb{R}^n)}, \quad \phi \in L^2(\mathbb{R}^n), \quad (1.16)$$

which hold for every *Schrödinger admissible pair* (p, q) . These are pairs (p, q) that fulfill $\frac{2}{p} = n(\frac{1}{2} - \frac{1}{q})$ with $2 \leq p, q \leq \infty$ and $(p, q, n) \neq (2, \infty, 2)$. These estimates are named after Robert Stephen Strichartz (born 1943) who proved the inequality in the case $p = q$ [Str77]. Further contributions came from [GV84, Yaj87, KT98]. On the other hand, for functions $f \in L^2(\mathbb{R}^n)$ with $\text{supp } \widehat{f} \subseteq [-N, N]^n$, Bernstein's inequality, see e.g. [Tao06, formula (A.6)], implies

$$\|e^{it\Delta}f\|_{L^\infty(\mathbb{R} \times \mathbb{R}^n)} \lesssim N^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}.$$

By interpolation with the $L^p_{t,x}$ -estimate for $p = \frac{2(n+2)}{n}$ given by (1.16), one obtains for $f \in L^2(\mathbb{R}^n)$ with $\text{supp } \widehat{f} \subseteq [-N, N]^n$,

$$\|e^{it\Delta}f\|_{L^p(\mathbb{R} \times \mathbb{R}^n)} \lesssim N^{\frac{n}{2} - \frac{n+2}{p}} \|f\|_{L^2(\mathbb{R}^n)} \quad (1.17)$$

for $\frac{2(n+2)}{n} \leq p \leq \infty$.

Equipped with these Strichartz estimates, one may study well-posedness results for the non-linear equation

$$\begin{cases} i\partial_t u + \Delta u = \alpha|u|^{p-1}u & \text{in } \mathbb{R} \times \mathbb{R}^n \\ u(0, \cdot) = \phi & \text{on } \mathbb{R}^n, \end{cases} \quad (1.18)$$

where $\phi \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. If $\alpha = 1$ the equation is called *defocusing* and if $\alpha = -1$ it is called *focusing*. One major question in the well-posedness theory is: for which $s \in \mathbb{R}$ can one expect reasonable solutions? The *scaling symmetry* of (1.18) is important for answering this question. If $\lambda > 0$ and u a solution to (1.18), then

$$u_\lambda(t, x) := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$$

is also a solution to the same equation with initial data $\phi_\lambda(x) := \lambda^{\frac{2}{p-1}}\phi(\lambda x)$. It is easy to compute that

$$\|\phi_\lambda\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{s-s_c}\|\phi\|_{\dot{H}^s(\mathbb{R}^n)}, \quad s_c := \frac{n}{2} - \frac{2}{p-1}. \quad (1.19)$$

Hence, for $n = 3$ and $p = 5$ the Sobolev space $\dot{H}^1(\mathbb{R}^3)$ is scaling invariant. For fixed $s \in \mathbb{R}$ we now study the behavior if $\lambda \rightarrow 0$ [PT14, Section 2.1]:

- If $s > s_c$, then the \dot{H}^s -norm of the initial data ϕ_λ decreases as $\lambda \rightarrow 0$. At the same time, the time interval on which the solution u_λ is defined increases. For well-posedness results this is the best scenario. Whenever s is in this range, then the equation is called *(scaling-)sub-critical* or $H^s(\mathbb{R}^n)$ -*sub-critical*.
- If $s = s_c$, then the \dot{H}^s -norm of ϕ_λ does not change as λ tends to zero, but the time interval still magnifies for increasing λ . In many cases, local or even global well-posedness results hold true but in most cases one has to work harder than in the sub-critical case. This case is called *(scaling-)critical* or $H^s(\mathbb{R}^n)$ -*critical*.
- If $s < s_c$, the \dot{H}^s -norm of the rescaled initial data grows while the time interval magnifies as $\lambda \rightarrow 0$. This is the worst case scenario, and we can not expect even locally defined strong solutions. For those s , the equation is called *(scaling-)super-critical* or $H^s(\mathbb{R}^n)$ -*super-critical*.

Another important invariance is the *Galilean invariance*: If u is a solution to (1.18) and $v \in \mathbb{R}^n$ is a vector, then

$$u_v(t, x) := e^{-i(x \cdot v + t|v|^2)}u(t, x + 2vt)$$

is a solution to this equation with initial data $\phi_v(x) := e^{-ix \cdot v}\phi(x)$ [SS99, formula (2.3.14)–(2.3.16)].

The L^2 -mass and the energy of the solution are defined as

$$M_{\mathbb{R}^n}(u)(t) := \int_{\mathbb{R}^n} |u(t, x)|^2 dx$$

and

$$E_{\mathbb{R}^n}(u)(t) := \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(t, x)|^2 dx + \frac{\alpha}{p+1} \int_{\mathbb{R}^n} |u(t, x)|^{p+1} dx,$$

respectively. By multiplying the equation by \bar{u} , integrating over \mathbb{R}^n , and taking the imaginary part, one formally computes conservation of the L^2 -mass, i.e.

$$\frac{d}{dt}M_{\mathbb{R}^n}(u)(t) = 0.$$

Similarly, multiplying (1.18) by $\overline{\partial_t u}$, integrating, and taking the real part (formally) shows that the energy is conserved as well, i.e.

$$\frac{d}{dt}E_{\mathbb{R}^n}(u)(t) = 0.$$

There are many other invariances and conserved quantities such as invariance in time and space translation, the Gauge invariance, the pseudo-conformal invariance, and the conservation of the linear momentum to name just a few. For details about these and more invariances, we refer to [SS99, Section 2.3].

For $p = 5$ and $n = 3$ one easily calculates that for any $\lambda > 0$,

$$E_{\mathbb{R}^3}(\phi_\lambda) = E_{\mathbb{R}^3}(u_\lambda)(0) = E_{\mathbb{R}^3}(u)(0) = E_{\mathbb{R}^3}(\phi).$$

Together with (1.19), we observe that the energy and the $\dot{H}^1(\mathbb{R}^3)$ -norm scale equally. For this reason, the $H^1(\mathbb{R}^3)$ -critical NLS is also called *energy-critical*.

Compact manifolds as a domain

The nonlinear Schrödinger equation posed on a boundaryless compact manifold behaves differently. After discussing a few of those differences in the introduction, we want to continue this now. For that purpose, let (M, g) be a smooth, boundaryless, compact Riemannian manifold of dimension $n \geq 1$ with metric g . We shall use the notation introduced in Section 1.4.

As above, we first consider the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_g u = 0 & \text{in } I \times M \\ u(0, \cdot) = \phi & \text{on } M, \end{cases}$$

where $\phi \in H^s(M)$ for some $s \geq 0$ and I is an interval around zero. The unique solution is given by

$$e^{it\Delta_g} \phi(x) := \sum_{k=0}^{\infty} e^{-it\lambda_k} (h_k \phi)(x), \quad (1.20)$$

where the series converges in the $L^2(M)$ -sense. From the orthogonality of $\{h_k \phi\}_{k \in \mathbb{N}_0}$, we immediately infer the conservation of the L^2 -norm:

$$\|e^{it\Delta_g} \phi\|_{L^2(M)}^2 = \sum_{k=0}^{\infty} \|h_k \phi\|_{L^2(M)}^2 = \|\phi\|_{L^2(M)}^2.$$

Now, it is natural to ask whether a dispersive estimate like (1.15) can hold. However, due to the non-dispersive nature of the geometry, this is not the case. It is easy to construct a contradiction if we assume

$$\|e^{it\Delta_g} \phi\|_{L^q(M)} \leq \frac{C}{|t|^\nu} \|\phi\|_{L^{q'}(M)}$$

to hold for some $\nu > 0$, $q > 2$, and its conjugated Hölder exponent $q' < 2$. Indeed,

$$\|e^{it\Delta_g} \phi\|_{L^2(M)} \leq |M|^{\frac{1}{2} - \frac{1}{q}} \|e^{it\Delta_g} \phi\|_{L^q(M)} \leq \frac{C}{|t|^\nu} |M|^{\frac{1}{2} - \frac{1}{q}} \|\phi\|_{L^{q'}(M)} \leq \frac{C}{|t|^\nu} |M|^{1 - \frac{2}{q}} \|\phi\|_{L^2(M)}$$

since M is compact. For large $|t|$ this obviously contradicts the conservation of the L^2 -norm. This raises the question how Strichartz estimates look like. Burq–Gérard–Tzvetkov [BGT04, Theorem 1] proved for a Strichartz admissible pair (p, q) with $p \geq 2$ and $q < \infty$ that for any finite time interval I ,

$$\|e^{it\Delta_g} \phi\|_{L^p(I, L^q(M))} \lesssim_I \|\phi\|_{H^{\frac{1}{p}}(M)}. \quad (1.21)$$

Compared to (1.16) there is a loss of $\frac{1}{p}$ derivatives, but corresponds to half the loss of the trivial estimate given by Sobolev's embedding $H^{2/p}(M) = H^{n/2 - n/q}(M) \hookrightarrow L^q(M)$. Inequality (1.21) is not scale invariant and therefore not sufficient for proving critical results. Scale invariant improvements of this Strichartz estimate are known on a few manifolds. On the three-dimensional sphere, for instance, the scale invariant Strichartz estimate

$$\|P_N e^{it\Delta_g} \phi\|_{L^p(I \times \mathbb{S}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|\phi\|_{L^2(\mathbb{S}^3)}$$

is known to be true for $p > 4$, see [BGT07, Proposition 5.5.1] and [Her13, Lemma 3.5]². This corresponds to inequality (1.17) with exactly the same power of N . Similar estimates are also

²More generally, the aforementioned authors proved the scale invariant Strichartz estimate to hold true for an arbitrary Zoll manifold.

known for rectangular tori and for products of spheres, which are addressed in [Section 2.3](#) and [Section 2.5](#), respectively.

Now, we turn to the nonlinear equation

$$\begin{cases} i\partial_t u + \Delta_g u = \alpha |u|^{p-1} u & \text{in } I \times M \\ u(0, \cdot) = \phi & \text{on } M, \end{cases} \quad (1.22)$$

where $\phi \in H^s(M)$ and $s \geq 0$. Another concept that does not work on compact manifolds without boundary is scaling. Therefore, we have to define the meaning of sub-critical, critical, and super-critical. We call (1.22) critical if the corresponding equation posed on \mathbb{R}^n is critical. The terms sub-critical, super-critical, and energy-critical are defined analogously.

By similar arguments as in the previous subsection, one may show that the L^2 -mass

$$M(u)(t) := \int_M |u(t, x)|^2 dx \quad (1.23)$$

and the energy

$$E(u)(t) := \frac{1}{2} \int_M |\nabla_g u(t, x)|^2 dx + \frac{\alpha}{p+1} \int_M |u(t, x)|^{p+1} dx \quad (1.24)$$

are conserved.

Remark. Note that due to the compactness of M , the spaces $L^p(M)$ are nested in each other. Hence, every sufficiently smooth solution u to the defocusing equation permits the following a priori bound:

$$\|u(t)\|_{H^1(M)}^2 = \|u(t)\|_{L^2(M)}^2 + \|\nabla u(t)\|_{L^2(M)}^2 \lesssim \|u(t)\|_{L^6(M)}^2 + E(u(t)) \lesssim E(u(t))^{\frac{1}{3}} + E(u(t)). \quad \diamond$$

Depending on the manifold, there might be other invariances. The Galilean invariance, for instance, holds on rectangular tori, see [Lemma 2.10](#) for an application, but not on general compact manifolds without boundary.

As mentioned in the introduction, the lack of dispersion and important mathematical tools, such as the Fourier transform, require new ideas for studying well-posedness results. In the next two sections we present methods to overcome these difficulties on specific manifolds, namely rectangular tori and products of spheres.

2 Local and small data global well-posedness

After a few preliminary remarks in [Section 2.1](#), we prove a conditional local and small data global well-posedness result for the energy-critical nonlinear Schrödinger equation posed on a three-dimensional, compact, connected, smooth Riemannian manifold without boundary in [Section 2.2](#). This conditional result was developed in recent years without a noteworthy contribution of the present author.

The remainder of this section is devoted to results that are due to the author. The assumption for the conditional well-posedness result is verified in the case of rectangular tori ([Section 2.3](#)) and on products of spheres ([Section 2.5](#)). Moreover, [Section 2.4](#) provides a multilinear Strichartz estimate, which implies a scaling-critical local well-posedness result for the NLS on two-dimensional tori. The aforementioned results on tori have been published in [\[Str14\]](#), the result on products of spheres extends a previously published result of the author and Sebastian Herr [\[HS15\]](#).

2.1 Preliminary remarks

Well-posedness of the nonlinear Schrödinger equation on \mathbb{R}^n has been studied extensively. We give an overview of some important results on the Euclidean space to be able to put the results on compact manifolds into context.

2.1.1 Relevant results on the Euclidean space

We give a brief review over important results for the NLS on \mathbb{R}^3 with initial data in $H^1(\mathbb{R}^3)$. Several sub-critical well-posedness results have been obtained amongst other by Ginibre–Velo [\[GV79, GV85\]](#), Kato [\[Kat87\]](#), Cazenave–Weissler [\[CW88\]](#). In 1989, Cazenave–Weissler [\[CW89\]](#) considered the energy-critical case and proved that both the focusing and defocusing quintic NLS are locally well-posed for any initial data in $H^1(\mathbb{R}^3)$. If the energy of the initial data are small, then the solution is known to exist even globally in time. However, since the time of existence given by the local theory depends on the profile of the data, the argument in [\[CW89\]](#) does not extend to yield global well-posedness for large initial data.

Studying large data well-posedness for the energy-critical defocusing nonlinear Schrödinger equation posed on the Euclidean space \mathbb{R}^3 is delicate. Bourgain [\[Bou99\]](#) was the first who proved global well-posedness, though, under the additional assumption that the initial data are radial. Shortly after, Grillakis [\[Gri00\]](#) gave a different proof under the same spherical symmetry assumption. In 2008, Colliander–Keel–Staffilani–Takaoka–Tao [\[CKS⁺08\]](#) finally removed the spherical symmetry assumption and proved that the defocusing quintic NLS is even globally well-posed for arbitrarily large initial data in $H^1(\mathbb{R}^3)$. In 2003, Christ–Colliander–Tao [\[CCT03, Theorem 1\]](#) proved that the quintic NLS fails to be well-posed in $H^s(\mathbb{R}^3)$ for any $s < 1$. Moreover, they proved that the energy-super-critical focusing and defocusing NLS fails to be well-posed in $H^1(\mathbb{R}^3)$. Therefore, the well-posedness theory of quintic NLS in \mathbb{R}^3 is complete.

More details may be found in the monographs [SS99, Caz03, Tao06, LP15].

2.1.2 Selected results on compact manifolds

The study of well-posedness results on manifolds is quite new and started with a fundamental work on the domain \mathbb{T}^n by Bourgain [Bou93a] in 1993. Before we turn to the study of specific manifolds, we collect a few results that are known to hold on every compact Riemannian manifold. In the following we assume any manifold (M, g) to be a compact, connected, smooth Riemannian manifold without boundary. Due to the different behavior of the eigenfunctions and the eigenvalues (see [Introduction](#)), it is hard to establish results that hold true on large classes of manifolds or even on any manifold.

Early work has been done by Sogge [Sog88] who proved bounds on the L^p -norm of spectral clusters for second-order elliptic operators on compact manifolds. One of his results that is relevant for our study is the following sharp Bernstein type inequality [Sog88, Theorem 2.2]¹: Let $2 \leq p \leq \infty$, then for every $f \in L^2(M)$ and any $k \in \mathbb{N}$,

$$\|p_k f\|_{L^p(M)} \lesssim k^{n(\frac{1}{2} - \frac{1}{p})} \|p_k f\|_{L^2(M)}$$

holds true.

As mentioned earlier Burq–Gérard–Tzvetkov [BGT04, Theorem 1] proved the Strichartz estimate (1.21). Because of the loss of $\frac{1}{p}$ derivatives compared to the scale invariant version, one can not conclude critical well-posedness results from this estimate. However, this estimate is strong enough to gain global well-posedness in H^1 of the three-dimensional cubic defocusing nonlinear Schrödinger equation on any manifold with the properties above, see [BGT04, Theorem 3]. They even established a similar result for any two-dimensional manifold, cf. [BGT04, Theorem 2]. The two-dimensional result was later extended by Hani [Han12] who proved that the defocusing cubic NLS on two-dimensional manifolds M is globally well-posed in $H^s(M)$ for $s > \frac{2}{3}$.

Bilinear and trilinear generalizations of Sogge’s spectral cluster estimate have been obtained by Burq–Gérard–Tzvetkov [BGT05a, BGT05b]. Although these estimates hold true on every manifold M , they only led to good results on manifolds that are spectrally close to spheres [BGT05a, BGT05b, Her13, HS15].

General four-dimensional manifolds with Hartree-type nonlinearities has been studied by Gérard–Pierfelice [GP10, Theorem 1].

We want to emphasize that apart from [Her13, HS15] none of the above results are scaling-critical. Due to the precise knowledge of the spectrum and eigenfunctions, much more results (even critical) have been accomplished on specific manifolds. These are summarized in [Section 2.3](#), [Section 2.5](#), and [Section 2.6](#).

Laurent Thomann [Tho08, Theorem 1.4] established an analogue of Christ–Colliander–Tao’s ill-posedness result on general analytic manifolds. He proved that there is a sequence of times $t_n \rightarrow 0$ and a sequence of smooth Cauchy data with decreasing support and decreasing H^s -norm for $s < 1$ such that the solution to both the focusing and defocusing quintic NLS at time t_n blows up in the H^s -norm as n tends to infinity. Moreover, he showed that the focusing and defocusing super-quintic NLS fails to be well-posed in H^1 . Hence, obtaining energy-critical well-posedness results is of particular interest.

¹Actually, Sogge proved the dual estimate.

2.2 A conditional local and small data global well-posedness result

A conditional local and small data global well-posedness result is addressed in this section. It is shown that the energy-critical NLS on any three-dimensional, compact, connected, smooth Riemannian manifold without boundary is well-posed provided the trilinear Strichartz estimate given in [Assumption 2.1](#) holds. Moreover, the necessity of [Assumption 2.1](#) is discussed.

2.2.1 Sufficiency of the condition

The result discussed in this subsection was proved in [[HS15](#), Section 3] building on earlier results [[HTT11](#), [Her13](#)] and the standard contraction mapping principle. We would like to point out that this was essentially a contribution of Sebastian Herr and not of the author of the present thesis. We take the opportunity to review the complete argument and to expand it to a complete proof.

Let (M, g) be a three-dimensional, compact, connected, smooth Riemannian manifold without boundary. The Cauchy problem

$$\begin{cases} i\partial_t u + \Delta_g u = \pm |u|^4 u \\ u(0, \cdot) = \phi \end{cases} \quad (2.1)$$

with initial data in $\phi \in H^s(M)$ for $s \geq 1$ is studied. The aim of this subsection is to prove existence and uniqueness of a solution u in a suitable function space and its Lipschitz continuous dependence on the initial data provided a certain trilinear Strichartz estimate holds true.

In the sequel, we use the notation introduced in [Section 1.4](#) and [Section 1.5.2](#), and we assume:

Assumption 2.1. *There exist an interval $\tau_0 \supseteq [0, 1)$ and $\delta > 0$ such that for all $\phi_1, \phi_2, \phi_3 \in L^2(M)$ and dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ the following estimate holds true:*

$$\left\| \prod_{j=1}^3 P_{N_j} e^{it\Delta_g} \phi_j \right\|_{L^2(\tau_0 \times M)} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2} \right)^\delta N_2 N_3 \prod_{j=1}^3 \|P_{N_j} \phi_j\|_{L^2(M)}. \quad (2.2)$$

This inequality has been proved for the flat standard torus by Herr–Tataru–Tzvetkov [[HTT11](#), formula (26)] and for arbitrary rectangular tori by the author of this thesis [[Str14](#), Proposition 4.1]. Furthermore, Herr [[Her13](#)] verified [Assumption 2.1](#) on Zoll manifolds. The verification of this trilinear Strichartz estimate for $\mathbb{S} \times \mathbb{S}^2$ in [[HS15](#), Proposition 2.6] was essentially a contribution of the present author.² We review the author’s proof of [Assumption 2.1](#) for rectangular tori in [Section 2.3](#). Moreover, in [Section 2.5](#), we give the first proof of (2.2) for $M = \mathbb{S} \times \mathbb{S}_\rho^2$, where \mathbb{S}_ρ^2 is the embedded sphere of radius $\rho > 0$ in \mathbb{R}^3 , which extends the result given in [[HS15](#)].

²Note that in the case of Zoll manifolds and $\mathbb{S} \times \mathbb{S}^2$ spectral projectors with sharp cut-offs have been used, say $P_N^\#$, and hence, (2.2) holds only for those projectors. However, from the L^2 -boundedness of these sharp projectors and from the identity $P_N^\# = P_{N/2} + P_N + P_{2N}$, it is easy to see that this implies (2.2) with smooth cut-off projectors as stated in [Assumption 2.1](#).

Suitable function spaces based on U^p and V^p , which are crucial in the study of critical well-posedness problems on compact manifolds, have first been constructed by Herr–Tataru–Tzvetkov [HTT11, Definitions 2.6–2.7]. They defined similar function spaces as X^s and Y^s below but with unit scales instead of dyadic scales. In [Her13, Definition 2.3], Herr introduced resolution spaces with dyadic scales (such as X^s and Y^s below) and used them to establish well-posedness on three-dimensional Zoll manifolds. In [HS15, Section 3], Sebastian Herr finally observed that given Assumption 2.1, local and small data global well-posedness can be proved for every compact, connected, smooth, boundaryless, three-dimensional Riemannian manifold using the same dyadic scale resolution spaces X^s and Y^s on every manifold. This unifies the well-posedness results in [HTT11, Her13, Str14, HS15].

Following [HS15, Section 3], we work with the following resolution spaces.

Definition 2.2 (Resolution spaces). Let $s \in \mathbb{R}$.

- (i) The space X^s is defined as the space of all $u: \mathbb{R} \rightarrow H^s(M)$ such that $e^{-it\Delta_g} P_N u \in U^2$ for all dyadic $N \geq 1$ and

$$\|u\|_{X^s} := \left(\sum_{N \geq 1} N^{2s} \|e^{-it\Delta_g} P_N u\|_{U_t^2}^2 \right)^{\frac{1}{2}} < +\infty.$$

- (ii) The space Y^s is defined as the space of all $u: \mathbb{R} \rightarrow H^s(M)$ such that $e^{-it\Delta_g} P_N u \in V^2$ for all dyadic $N \geq 1$ and

$$\|u\|_{Y^s} := \left(\sum_{N \geq 1} N^{2s} \|e^{-it\Delta_g} P_N u\|_{V_t^2}^2 \right)^{\frac{1}{2}} < +\infty.$$

- (iii) For an interval $\tau \subset \mathbb{R}$ we denote by $X^s(\tau)$ and $Y^s(\tau)$ the restriction spaces

$$X^s(\tau) := \left\{ u: \tau \rightarrow H^s(M) : \|u\|_{X^s(\tau)} := \inf_{\substack{v \in X^s: \\ v \cdot \mathbf{1}_\tau = u \cdot \mathbf{1}_\tau}} \|v\|_{X^s} < +\infty \right\}$$

respectively

$$Y^s(\tau) := \left\{ u: \tau \rightarrow H^s(M) : \|u\|_{Y^s(\tau)} := \inf_{\substack{v \in Y^s: \\ v \cdot \mathbf{1}_\tau = u \cdot \mathbf{1}_\tau}} \|v\|_{Y^s} < +\infty \right\}.$$

Remark.

- (i) Obviously, given a function $u: \mathbb{R} \rightarrow H^s(M)$, $u \in X^s(\tau)$ should be understood as $u|_\tau \in X^s(\tau)$ and $\|u\|_{X^s(\tau)} = \|u|_\tau\|_{X^s(\tau)}$. The same should apply to $Y^s(\tau)$.
- (ii) Note that in contrast to [HS15], the spaces are defined using smooth cut-off projectors. This requires an additional argument in the proof of Lemma 2.5. \diamond

The aim of this subsection is the verification of the subsequent theorem, whose formulation is taken from [Her13, Theorem 4.1]. In the following, for $\phi_* \in H^1(M)$ and $\varepsilon > 0$ we denote by $B_\varepsilon(\phi_*)$ the open ball in $H^1(M)$ with center ϕ_* and radius ε , i.e.

$$B_\varepsilon(\phi_*) := \{\phi \in H^1(M) : \|\phi - \phi_*\|_{H^1(M)} < \varepsilon\}.$$

Theorem 2.3. *Let (M, g) be a three-dimensional, compact, connected, smooth Riemannian manifold without boundary and let $s \geq 1$. Furthermore, assume that Assumption 2.1 holds true. Then:*

Local well-posedness. For every $\phi_* \in H^1(M)$ there exist $\varepsilon > 0$ and $T = T(\phi_*) > 0$ such that the following holds true:

(i) For all initial data $\phi \in B_\varepsilon(\phi_*) \cap H^s(M)$ the Cauchy problem (2.1) has a unique solution

$$u =: \Phi(\phi) \in C([0, T], H^s(M)) \cap X^s([0, T]).$$

(ii) The solution constructed in (i) obeys the conservation laws (1.23) and (1.24), and the flow map

$$\Phi: B_\varepsilon(\phi_*) \cap H^s(M) \rightarrow C([0, T], H^s(M)) \cap X^s([0, T])$$

is Lipschitz continuous.

Small data global well-posedness. With $\phi_* = 0$ there exists $\varepsilon_0 > 0$ such that for all $T > 0$ the assertions (i) and (ii) above hold true.

First, we state some well-known results about the function spaces X^s and Y^s .

Proposition 2.4 (Properties of X^s and Y^s). Let $\tau = [a, b) \subset \mathbb{R}$ be a bounded time interval.

(i) For $s \in \mathbb{R}$ it holds that

$$X^s \hookrightarrow Y^s \hookrightarrow L^\infty(\mathbb{R}, H^s(M))$$

and

$$X^s(\tau) \hookrightarrow Y^s(\tau) \hookrightarrow L^\infty(\tau, H^s(M)).$$

(ii) In addition, assume that $0 \in \tau$. Let $s \geq 0$ and $\phi \in H^s(M)$, then we have that $e^{it\Delta_g}\phi \in X^s(\tau)$ and

$$\|e^{it\Delta_g}\phi\|_{X^s(\tau)} \lesssim \|\phi\|_{H^s(M)}.$$

(iii) Suppose $u \in Y^s$ for some $s \in \mathbb{R}$. Then,

$$\left(\sum_{N \geq 1} N^{2s} \|P_N u\|_{Y^0}^2 \right)^{\frac{1}{2}} \lesssim \|u\|_{Y^s}.$$

The corresponding statement also holds for $Y^s(\tau)$.

Proof. The embeddings given in (i) follow immediately from the embeddings

$$U^p \hookrightarrow V^p \hookrightarrow L^\infty(\mathbb{R}, L^2(M))$$

in Proposition 1.23 (v). Note that $U^p \hookrightarrow L^\infty(\mathbb{R}, L^2(M))$ and $V^p \hookrightarrow L^\infty(\mathbb{R}, L^2(M))$ hold with constant one, cf. Proposition 1.21 (iii) and Proposition 1.23 (iii).

Claim (ii) follows immediately from the definition of X^s : Indeed,

$$\|e^{it\Delta_g}\phi\|_{X^s(\tau)} \leq \|e^{it\Delta_g}\phi\|_{X^s}.$$

We then deduce that

$$\|e^{it\Delta_g}\phi\|_{X^s}^2 = \sum_{N \geq 1} N^{2s} \|e^{-it\Delta_g} P_N e^{it\Delta_g} \phi\|_{U_t^2}^2 \leq \sum_{N \geq 1} N^{2s} \|P_N \phi\|_{L^2(M)}^2 \approx \|\phi\|_{H^s(M)}^2.$$

To prove the last statement, we recall that the V^2 -norm is based on the L^2 -norm and compute

$$\sum_{N \geq 1} N^{2s} \|P_N u\|_{Y^0}^2 = \sum_{N \geq 1} N^{2s} \sum_{M \geq 1} \|e^{-it\Delta_g} P_M P_N u\|_{V^2}^2 \lesssim \sum_{N \geq 1} N^{2s} \|e^{-it\Delta_g} P_N u\|_{V^2}^2.$$

Obviously, a similar argument holds for $Y^s(\tau)$, too. \square

Let $\tau = [a, b)$ and $f \in L^1(\tau, L^2(M))$. Then, we define the Duhamel term as

$$\mathcal{I}(f)(t) := \int_a^t e^{i(t-s)\Delta_g} f(s) ds \quad (2.3)$$

for $t \in \tau$, $\mathcal{I}(f)(t) := 0$ for $t < a$, and $\mathcal{I}(f)(t) := \mathcal{I}(f)(b)$ for $t \geq b$.

The following estimate of the Duhamel term is well-known in this context. The proof of this estimate can be found for the standard torus, $\tau = [0, T)$, and spaces X^s and Y^s with unit scale in [HTT11, Proposition 2.11]. In [Her13, Lemma 2.5 (ii)], a similar result was stated with sharp spectral projectors but without a proof. The novelty here is that we show this estimate to hold true also for smooth spectral projectors. Note that in [Her13] the following, less restrictive condition was required:

$$\sup_{\substack{v \in Y^{-s}(\tau): \\ \|v\|_{Y^{-s}(\tau)}=1}} \left| \sum_{N \geq 1} \int_{\tau} \int_M P_N f(t, x) \overline{v(t, x)} dx dt \right| < \infty.$$

Lemma 2.5. *Let $s \in \mathbb{R}$ and $\tau = [a, b) \subset \mathbb{R}$. Furthermore, let $P_N f \in L^1(\tau, L^2(M))$ for all $N \geq 1$. Then, $\sum_{N \geq 1} \mathcal{I}(P_N f) =: \mathcal{I}(f)$ converges in $X^s(\tau)$ and*

$$\|\mathcal{I}(f)\|_{X^s(\tau)} \lesssim \sup_{\substack{v \in Y^{-s}(\tau): \\ \|v\|_{Y^{-s}(\tau)}=1}} \sum_{N \geq 1} \left| \int_{\tau} \int_M P_N f(t, x) \overline{v(t, x)} dx dt \right| \quad (2.4)$$

provided that the right-hand side is finite. In particular, if $f \in L^1(\tau, H^s(M))$, then

$$\|\mathcal{I}(f)\|_{X^s(\tau)} \lesssim \|f\|_{L^1(\tau, H^s(M))}. \quad (2.5)$$

Proof. For the proof of (2.7) below we adapt the idea presented in [HTT11, Proposition 2.11].

For dyadic $N > 1$ let the projectors with *sharp* cut-offs be defined as

$$P_N^\# := \sum_{\substack{k \in \mathbb{N}_0: \\ N \leq |\sqrt{\lambda_k}| < 2N}} h_k \quad \text{and} \quad P_1^\# := \sum_{\substack{k \in \mathbb{N}_0: \\ 0 \leq |\sqrt{\lambda_k}| < 2}} h_k.$$

First, we remark that it suffices to consider $P_N^\#$ instead of the smooth projectors P_N . Indeed, let $\tilde{P}_N^\# := P_{N/2}^\# + P_N^\#$ for $N > 1$ and $\tilde{P}_1^\# := P_1^\#$. We prove that for any $\tilde{P}_N^\# u \in U^2$ we have

$$\|P_N u\|_{U^2} \leq \|\tilde{P}_N^\# u\|_{U^2}. \quad (2.6)$$

Since $\tilde{P}_N^\# P_N = P_N$, this immediately implies

$$\sum_{N \geq 1} N^{2s} \|P_N u\|_{U^2}^2 \leq 2(1 + 2^{2s}) \sum_{N \geq 1} N^{2s} \|\tilde{P}_N^\# u\|_{U^2}^2.$$

In order to verify (2.6), it suffices to consider an U^2 -atom $\tilde{P}_N^\# a \neq 0$ with representation $\tilde{P}_N^\# a(t) = \sum_{k=1}^K \mathbf{1}_{[t_{k-1}, t_k)}(t) \phi_k$ with $\sum_{k=1}^K \|\phi_k\|_{L^2(M)}^2 = 1$ and a partition $(t_k)_k$. Note that $\phi_k = \tilde{P}_N^\# \phi_k$. Define $A := \sum_{k=1}^K \|P_N \phi_k\|_{L^2(M)}^2$, and observe from the boundedness of P_N in $L^2(M)$ that $0 < A \leq 1$. We may write $P_N a(t) = A \sum_{j=1}^K \mathbf{1}_{[t_{j-1}, t_j)}(t) \frac{P_N \phi_j}{A}$ which implies $\|P_N a\|_{U^2} \leq A$ and hence, (2.6) follows.

For $L \geq 1$ we prove the estimate

$$\|\mathcal{I}(P_{\leq L}f)\|_{X^s} \lesssim \sup_{\substack{v \in Y^{-s}: \\ \|v\|_{Y^{-s}}=1}} \sum_{N \geq 1} \left| \int_{\tau} \int_M P_N f(t, x) \overline{v(t, x)} dx dt \right| \quad (2.7)$$

uniformly in L . Since $t \mapsto e^{it\lambda_k} h_k(P_{\leq L}\mathcal{I}(f))(t)$ is for every $k \geq 0$ absolutely continuous and of bounded variation, we conclude that $t \mapsto e^{-it\Delta_g} P_N \mathcal{I}(P_{\leq L}f)(t)$ and $t \mapsto e^{-it\Delta_g} \tilde{P}_N^{\#} \mathcal{I}(P_{\leq L}f)(t)$ are in U^2 . Then, we see from the definition of X^s that

$$\begin{aligned} \|\mathcal{I}(P_{\leq L}f)\|_{X^s}^2 &\lesssim \sum_{1 \leq N \leq L} N^{2s} \|e^{-it\Delta_g} P_N^{\#} \mathcal{I}(P_{\leq L}f)\|_{U_t^2}^2 \\ &= \sum_{1 \leq N \leq L} N^{2s} \left\| \int_a^t e^{-is\Delta_g} P_N^{\#}(P_{\leq L}f)(s) \cdot \mathbf{1}_{\tau}(s) ds \right\|_{U_t^2}^2. \end{aligned}$$

By duality, $\|a\|_{\ell^2} = \sup_{\|b\|_{\ell^2}=1} \|ab\|_{\ell^1}$. Thus, for every $\varepsilon > 0$ we may choose a positive sequence $b \in \ell^2(2^{\mathbb{N}_0})$ with $\|b\|_{\ell^2(2^{\mathbb{N}_0})} = 1$ such that

$$\|\mathcal{I}(P_{\leq L}f)\|_{X^s} \lesssim \sum_{1 \leq N \leq L} b_N N^s \left\| \int_a^t e^{-is\Delta_g} P_N^{\#}(P_{\leq L}f)(s) \cdot \mathbf{1}_{\tau}(s) ds \right\|_{U_t^2} + \varepsilon.$$

By duality (see Lemma 1.25), for any dyadic $1 \leq N \leq L$, there is a V^2 -function $v_N \in C_0^\infty(\mathbb{R}, L^2(M))$ with $\|v_N\|_{V^2} = 1$ such that

$$\left\| \int_a^t e^{-is\Delta_g} P_N^{\#}(P_{\leq L}f)(s) \cdot \mathbf{1}_{\tau}(s) ds \right\|_{U_t^2} \leq \left| \int_{\tau} \int_M P_N^{\#}(P_{\leq L}f)(t, x) \overline{e^{it\Delta_g} v_N(t, x)} dx dt \right| + \frac{\varepsilon}{N}, \quad (2.8)$$

where—after a rotation of v_N —we may assume the integral to be positive. We now define the function $v: \tau \times M \rightarrow \mathbb{C}$,

$$v(t, x) := (1 + 2^{-s})^{-1} \sum_{1 \leq M \leq L} b_M M^s e^{it\Delta_g} P_M^{\#}(v_M)(t, x),$$

and notice that

$$P_N^{\#} v(t) = (1 + 2^{-s})^{-1} b_N N^s e^{it\Delta_g} P_N^{\#}(v_N)(t).$$

One easily verifies $v \in Y^{-s}$ and $\|v\|_{Y^{-s}} \leq 1$. Furthermore, since $\sum_{1 \leq N \leq L} P_N^{\#} P_{\leq L} = P_{\leq L}$,

$$\begin{aligned} \|\mathcal{I}(P_{\leq L}f)\|_{X^s} &\lesssim \left| \sum_{1 \leq N \leq L} \int_{\tau} \int_M P_N^{\#}(P_{\leq L}f)(t, x) \overline{v(t, x)} dx dt \right| + C\varepsilon \\ &\lesssim \sum_{N \geq 1} \left| \int_{\tau} \int_M P_N f(t, x) \overline{v(t, x)} dx dt \right| + C\varepsilon. \end{aligned}$$

Inequality (2.7) follows since $\varepsilon > 0$ was arbitrary.

We conclude (2.4) now. Since the left-hand side of the following estimate is smaller than $\|\mathcal{I}(P_{\leq L}f)\|_{X^s}$, inequality (2.7) implies

$$\left(\sum_{1 \leq N < L} N^{2s} \|e^{-it\Delta_g} P_N \mathcal{I}(f)\|_{U_t^2}^2 \right)^{\frac{1}{2}} \lesssim \sup_{\substack{v \in Y^{-s}: \\ \|v\|_{Y^{-s}}=1}} \sum_{N \geq 1} \left| \int_{\tau} \int_M P_N f(t, x) \overline{v(t, x)} dx dt \right| < \infty$$

uniformly in $L \geq 1$. Hence,

$$\left(\sum_{N \geq 1} N^{2s} \|e^{-it\Delta_g} P_N \mathcal{I}(f)\|_{U_t^2}^2 \right)^{\frac{1}{2}} < \infty,$$

which implies that $\mathcal{I}(f) \in X^s$. Thus, $\mathcal{I}(f) \in X^s(\tau)$ and the estimate (2.4) holds since the supremum in (2.4) is taken over a larger set.

The bound (2.5) follows essentially from (2.4) in conjunction with the embedding $Y^{-s}(\tau) \hookrightarrow L^\infty(\tau, H^{-s}(M))$:

$$\begin{aligned} \|\mathcal{I}(f)\|_{X^s(\tau)} &\lesssim \sup_{\substack{v \in Y^{-s}(\tau): \\ \|v\|_{Y^{-s}(\tau)} \leq 1}} \left| \int_\tau \int_M f(t, x) \overline{v(t, x)} dx dt \right| \\ &\lesssim \sup_{\substack{v \in Y^{-s}(\tau): \\ \|v\|_{Y^{-s}(\tau)} \leq 1}} \|v\|_{L^\infty(\tau, H^{-s}(M))} \|f\|_{L^1(\tau, H^s(M))} \\ &\lesssim \|f\|_{L^1(\tau, H^s(M))}. \end{aligned} \quad \square$$

Even though the estimate in the next lemma is not scale invariant, it turns out to be useful in the sequel. The estimate was proved in [Her13, Lemma 3.4].

Lemma 2.6. *Let $\tau \subset \mathbb{R}$ be a bounded interval. For all functions $u_1, u_2, u_3 \in L^\infty(\tau, L^2(M))$ and dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ the estimate*

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim |\tau|^{\frac{1}{2}} (N_2 N_3)^{\frac{3}{2}} \prod_{j=1}^3 \|P_{N_j} u_j\|_{L^\infty(\tau, L^2(M))}$$

holds true.

Proof. Hölder's estimate yields

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \leq |\tau|^{\frac{1}{2}} \|P_{N_1} u_1\|_{L_t^\infty L_x^2} \|P_{N_2} u_2\|_{L_{t,x}^\infty} \|P_{N_3} u_3\|_{L_{t,x}^\infty}.$$

We fix $t \in \tau$ and apply Bernstein's inequality, see Lemma 1.53 (ii), for $j = 2, 3$, to obtain

$$\|P_{N_j} u_j(t)\|_{L^\infty(M)} \lesssim N_j^{\frac{3}{2}} \|P_{N_j} u_j(t)\|_{L^2(M)}.$$

By taking the supremum in $t \in \tau$, we get

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim |\tau|^{\frac{1}{2}} (N_2 N_3)^{\frac{3}{2}} \prod_{j=1}^3 \|P_{N_j} u_j\|_{L^\infty(\tau, L^2(M))}. \quad \square$$

The result in Lemma 1.54 may be extended from single eigenfunctions to the spectral localization operators P_N . Hence, we get a bound for the product of four spectrally localized functions on M , where the spectrum of one function is much bigger than the spectrum of all the others. Herr [Her13, Lemma 3.3] proved that Lemma 1.54 together with the Weyl asymptotic yields this result.

Corollary 2.7. *There exists $C \geq 1$ such that if N_0, \dots, N_3 are dyadic with $C^{-1}N_0 \geq N_1, N_2, N_3 \geq 1$, then for every $\gamma \geq 1$ there exists $C_\gamma > 0$ such that for any $P_{N_j} f_j \in L^2(M)$, $j = 0, 1, 2, 3$,*

$$\left| \int_M P_{N_0} f_0(x) P_{N_1} f_1(x) P_{N_2} f_2(x) P_{N_3} f_3(x) dx \right| \leq C_\gamma N_0^{-\gamma} \prod_{j=0}^3 \|P_{N_j} f_j\|_{L^2(M)}.$$

So far, we have not used [Assumption 2.1](#). We now show that the assumption implies an analogue bound for functions in Y^0 . We follow the proof of [[HS15](#), Proposition 3.3] and add some details.

Lemma 2.8. *Let $\tau \subseteq \tau_0$ be any interval. There exists $\delta > 0$ such that for all dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ and $P_{N_j} u_j \in Y^0$, $j = 1, 2, 3$, the following holds true*

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2} \right)^\delta N_2 N_3 \prod_{j=1}^3 \|P_{N_j} u_j\|_{Y^0}.$$

Proof. In this proof, we write $C_\delta(N_1, N_2, N_3) := \left(\frac{N_3}{N_1} + \frac{1}{N_2} \right)^\delta N_2 N_3$.

The proof is split into three parts. In the first two steps, we prove the estimate with the U^2 -norm respectively the U^6 -norm on the right-hand side. Then, we interpolate those estimates in the third step to get the V^2 -norm on the right-hand side.

Step 1. We claim that if $e^{-it\Delta_g} P_{N_j} u_j \in U^2$, $j = 1, 2, 3$, then

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim C_\delta(N_1, N_2, N_3) \prod_{j=1}^3 \|e^{-it\Delta_g} P_{N_j} u_j\|_{U^2}, \quad (2.9)$$

where $\delta > 0$ is the δ as given in [Assumption 2.1](#).

It suffices to prove (2.9) for U^2 -atoms. Indeed, first, note that if $P_{N_1} u_1, P_{N_2} u_2, P_{N_3} u_3 \in U^2$ with representation $P_{N_j} u_j = \sum_{\ell=1}^{\infty} \lambda_{j,\ell} P_{N_j} a_{j,\ell}$, $j = 1, 2, 3$, then

$$\|P_{N_1} e^{it\Delta_g} u_1 P_{N_2} e^{it\Delta_g} u_2 P_{N_3} e^{it\Delta_g} u_3\|_{L^2(\tau \times M)} = \left\| \prod_{j=1}^3 \sum_{\ell=1}^{\infty} \lambda_{j,\ell} e^{it\Delta_g} P_{N_j} a_{j,\ell} \right\|_{L^2(\tau \times M)}$$

since the L^6 -Strichartz estimate implies for any $\ell_0 \geq 1$ and $j = 1, 2, 3$,

$$\left\| e^{it\Delta_g} P_{N_j} \left(\sum_{\ell=\ell_0}^{\infty} \lambda_{j,\ell} a_{j,\ell}(t) \right) - \sum_{\ell=\ell_0}^{\infty} \lambda_{j,\ell} e^{it\Delta_g} P_{N_j} a_{j,\ell}(t) \right\|_{L^6(\tau \times M)} \lesssim 2 \sum_{\ell=\ell_0}^{\infty} |\lambda_{j,\ell}|.$$

Now, let $\varepsilon > 0$ and $e^{-it\Delta_g} P_{N_j} u_j \in U^2$ with $e^{-it\Delta_g} P_{N_j} u_j = \sum_{\ell=1}^{\infty} \lambda_{j,\ell} P_{N_j} a_{j,\ell}$ and $\sum_{\ell=1}^{\infty} |\lambda_{j,\ell}| \leq \|e^{-it\Delta_g} P_{N_j} u_j\|_{U^2} + \varepsilon$ for $j = 1, 2, 3$. Note that $e^{it\Delta_g} P_{N_j} a_{j,\ell}$ are U^2 -atoms and assume that (2.9) holds for U^2 -atoms, then

$$\begin{aligned} \|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} &\leq \sum_{\ell_1, \ell_2, \ell_3=1}^{\infty} |\lambda_{1,\ell_1} \lambda_{2,\ell_2} \lambda_{3,\ell_3}| \left\| \prod_{j=1}^3 e^{it\Delta_g} P_{N_j} a_{j,\ell_j} \right\|_{L^2(\tau \times M)} \\ &\lesssim C_\delta(N_1, N_2, N_3) \sum_{\ell_1, \ell_2, \ell_3=1}^{\infty} |\lambda_{1,\ell_1} \lambda_{2,\ell_2} \lambda_{3,\ell_3}| \\ &\lesssim C_\delta(N_1, N_2, N_3) \prod_{j=1}^3 (\|e^{-it\Delta_g} P_{N_j} u_j\|_{U^2} + \varepsilon). \end{aligned}$$

Hence, the desired estimate (2.9) follows once we proved it for U^2 -atoms.

Let a_1, a_2 , and a_3 be U^2 -atoms given as

$$P_{N_j} a_j(t) = \sum_{k=1}^{K_j} \mathbf{1}_{I_{j,k}}(t) e^{it\Delta_g} P_{N_j} \phi_{j,k}, \quad \sum_{k=1}^{K_j} \|\phi_{j,k}\|_{L^2(M)}^2 = 1,$$

with pairwise disjoint right-open intervals $I_{j,1}, I_{j,2}, \dots, I_{j,K_j}$ for $j = 1, 2, 3$. The disjointedness of the intervals implies

$$\|P_{N_1} a_1 P_{N_2} a_2 P_{N_3} a_3\|_{L^2(\tau \times M)}^2 \leq \sum_{k_1, k_2, k_3} \|e^{it\Delta_g} P_{N_1} \phi_{1,k_1} e^{it\Delta_g} P_{N_2} \phi_{2,k_2} e^{it\Delta_g} P_{N_3} \phi_{3,k_3}\|_{L^2(\tau \times M)}^2,$$

where we sum over $k_j = 1, \dots, K_j$, $j = 1, 2, 3$. Assumption 2.1 yields

$$\|P_{N_1} a_1 P_{N_2} a_2 P_{N_3} a_3\|_{L^2(\tau \times M)} \lesssim C_\delta(N_1, N_2, N_3) \left(\prod_{j=1}^3 \sum_{k=1}^{K_j} \|\phi_{j,k}\|_{L^2(M)}^2 \right)^{\frac{1}{2}} \lesssim C_\delta(N_1, N_2, N_3).$$

This finally proves (2.9).

Step 2. By choosing $N = N_1 = N_2 = N_3$ and $\phi = \phi_1 = \phi_2 = \phi_3$ in Assumption 2.1, we see that the following L^6 -estimate is implied

$$\|P_N e^{it\Delta_g} \phi\|_{L^6(\tau \times M)} \lesssim N^{\frac{2}{3}} \|P_N \phi\|_{L^2(M)}.$$

By the same argument as in the first step, this estimate carries over to U^6 -atoms. Thus

$$\|P_N u\|_{L^6(\tau \times M)} \lesssim N^{\frac{2}{3}} \|e^{-it\Delta_g} P_N u\|_{U^6}$$

for $e^{-it\Delta_g} P_N u \in U^6$. Now, we deduce for $N_j \geq 1$ and $e^{-it\Delta_g} P_{N_j} u_j \in U^6$, $j = 1, 2, 3$, from Hölder's inequality that

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim (N_1 N_2 N_3)^{\frac{2}{3}} \prod_{j=1}^3 \|e^{-it\Delta_g} P_{N_j} u_j\|_{U^6}. \quad (2.10)$$

Let $N_1 \geq N_2 \geq N_3 \geq 1$ and $p \geq 1$. Another estimate, which is not scale invariant but does not depend on N_1 , follows immediately from Lemma 2.6 and $U^p \hookrightarrow L^\infty(\tau, L^2(M))$:

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim |\tau|^{\frac{1}{2}} (N_2 N_3)^{\frac{3}{2}} \prod_{j=1}^3 \|e^{-it\Delta_g} P_{N_j} u_j\|_{U^p}. \quad (2.11)$$

Step 3. In this step, we interpolate the estimates given in the first two steps. For that purpose, we distinguish two cases.

Case 1. Assume $N_2 N_3 > N_1$. Applying the interpolation statement in Lemma 1.24 to (2.9) and (2.10) yields

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim A_\delta \prod_{j=1}^3 \|e^{-it\Delta_g} P_{N_j} u_j\|_{V^2},$$

where

$$\begin{aligned} A_\delta &:= C_\delta(N_1, N_2, N_3) \left(\ln \frac{(N_1 N_2 N_3)^{\frac{2}{3}}}{C_\delta(N_1, N_2, N_3)} + 1 \right)^3 \lesssim C_\delta(N_1, N_2, N_3) \left(\ln \frac{N_1}{N_3} + 1 \right)^3 \\ &\lesssim C_{\delta'}(N_1, N_2, N_3) \end{aligned}$$

for any $0 < \delta' < \delta$. This implies the claim in this case.

Case 2. Assume $N_2 N_3 \leq N_1$. In this case, we interpolate (see [Lemma 1.24](#)) the inequalities (2.9) and (2.11) (for some $p > 2$) and get

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2(\tau \times M)} \lesssim B_\delta \prod_{j=1}^3 \|e^{-it\Delta_g} P_{N_j} u_j\|_{V^2},$$

where

$$\begin{aligned} B_\delta &:= C_\delta(N_1, N_2, N_3) \left(\ln \frac{(N_2 N_3)^{\frac{3}{2}}}{C_\delta(N_1, N_2, N_3)} + 1 \right)^3 \lesssim C_\delta(N_1, N_2, N_3) (\ln N_2 + 1)^3 \\ &\lesssim C_{\delta'}(N_1, N_2, N_3) \end{aligned}$$

for any $0 < \delta' < \delta$. This finishes the proof. \square

We prove [Theorem 2.3](#) by the contraction mapping principle in a small closed ball in the space $C([0, T], H^s(M)) \cap X^s([0, T])$. In order to do so, we solve the following integral equation for a given $\phi \in H^s(M)$,

$$u(t) = e^{it\Delta_g} \phi \mp i\mathcal{I}(|u|^4 u)(t), \quad (2.12)$$

where $\mathcal{I}(f)$ is defined as in (2.3) with $\tau = [0, T]$.

Now, we provide an estimate for the Duhamel term measured in the restriction space $X^s(\tau)$. The proof is a combination of the arguments in [[HTT11](#), Proposition 4.1] and [[Her13](#), Proposition 4.2].

Lemma 2.9. *Let $s \geq 1$ be fixed and $T > 0$ such that $[0, T) \subseteq \tau_0$. Then, for any $u_j \in X^s([0, T])$, $j = 1, \dots, 5$, the estimate*

$$\left\| \mathcal{I} \left(\prod_{j=1}^5 \tilde{u}_j \right) \right\|_{X^s([0, T])} \lesssim \sum_{k=1}^5 \|u_k\|_{X^s([0, T])} \prod_{\substack{j=1 \\ j \neq k}}^5 \|u_j\|_{X^1([0, T])}$$

holds true, where \tilde{u}_j denotes either u_j or its complex conjugate $\overline{u_j}$.

Proof. We define $\tau := [0, T)$ for brevity. From [Lemma 2.5](#), we conclude that $\mathcal{I}(\prod_{j=1}^5 \tilde{u}_j) \in X^s(\tau)$ and

$$\left\| \mathcal{I} \left(\prod_{j=1}^5 \tilde{u}_j \right) \right\|_{X^s(\tau)} \lesssim \sup_{\substack{u_0 \in Y^{-s}(\tau): \\ \|u_0\|_{Y^{-s}(\tau)} = 1}} \sum_{N_0 \geq 1} \left| \int_\tau \int_M P_{N_0} \left(\prod_{j=1}^5 \tilde{u}_j(t, x) \right) \overline{u_0(t, x)} dx dt \right|.$$

In order to get rid of the time restriction on the spaces, we consider extensions to \mathbb{R} of u_j , $j = 0, \dots, 5$, without changing the notation. Hence, it suffices to prove

$$\sum_{N_0 \geq 1} \left| \int_\tau \int_M P_{N_0} \tilde{u}_0(t, x) \prod_{j=1}^5 \tilde{u}_j(t, x) dx dt \right| \lesssim \|u_0\|_{Y^{-s}} \sum_{k=1}^5 \|u_k\|_{X^s} \prod_{\substack{j=1 \\ j \neq k}}^5 \|u_j\|_{X^1}. \quad (2.13)$$

We dyadically decompose every function into

$$\tilde{u}_j = \sum_{N_j \geq 1} P_{N_j} \tilde{u}_j, \quad j = 1, \dots, 5.$$

Since the expression is symmetric in u_1, \dots, u_5 , it suffices to replace the left-hand side of (2.13) by

$$\Sigma := \sum_{(N_0, \dots, N_5) \in \mathcal{N}} \left| \int_{\tau} \int_M \prod_{j=0}^5 P_{N_j} \tilde{u}_j(t, x) dx dt \right|,$$

where \mathcal{N} is the set of all sextuples (N_0, N_1, \dots, N_5) of dyadic numbers such that

$$N_0 \geq 1 \quad \text{and} \quad N_1 \geq N_2 \geq \dots \geq N_5 \geq 1.$$

We split Σ into $\Sigma = \Sigma_1 + \Sigma_2$, where

$$\Sigma_1 + \Sigma_2 := \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ \max\{N_0, N_2\} \approx N_1}} \left| \int_{\tau} \int_M \prod_{j=0}^5 P_{N_j} \tilde{u}_j dx dt \right| + \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ \max\{N_0, N_2\} \not\approx N_1}} \left| \int_{\tau} \int_M \prod_{j=0}^5 P_{N_j} \tilde{u}_j dx dt \right|.$$

First, we estimate the contribution from Σ_1 . By Cauchy–Schwarz, it suffices to prove

$$\begin{aligned} \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ \max\{N_0, N_2\} \approx N_1}} \|P_{N_1} u_1 P_{N_3} u_3 P_{N_5} u_5\|_{L^2(\tau \times M)} \|P_{N_0} u_0 P_{N_2} u_2 P_{N_4} u_4\|_{L^2(\tau \times M)} \\ \lesssim \|u_0\|_{Y^{-s}} \|u_1\|_{X^s} \prod_{j=2}^5 \|u_j\|_{X^1}. \end{aligned}$$

We split the sum into two parts $\Sigma_{1,1}$ and $\Sigma_{1,2}$, where $\Sigma_{1,1}$ is defined by the constraint $N_2 \leq N_0 \approx N_1$. Consequently, $\Sigma_{1,2}$ is defined by the constraint $N_0 < N_2 \approx N_1$.

Part $\Sigma_{1,1}$. Applying Lemma 2.8 twice, we obtain

$$\Sigma_{1,1} \lesssim \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ N_2 \leq N_0 \approx N_1}} N_2 N_3 N_4 N_5 \left(\frac{N_5}{N_1} + \frac{1}{N_3} \right)^\delta \left(\frac{N_4}{N_0} + \frac{1}{N_2} \right)^\delta \prod_{j=0}^5 \|P_{N_j} u_j\|_{Y^0}$$

for some $\delta > 0$. Using Cauchy–Schwarz with respect to N_5 , N_4 , N_3 , and N_2 as well as Proposition 2.4 (iii), we estimate

$$\Sigma_{1,1} \lesssim \sum_{\substack{N_0, N_1 \geq 1: \\ N_0 \approx N_1}} \|P_{N_0} u_0\|_{Y^0} \|P_{N_1} u_1\|_{Y^0} \prod_{j=2}^5 \|u_j\|_{Y^1}.$$

Since $N_0 \approx N_1$, we conclude from Cauchy–Schwarz

$$\Sigma_{1,1} \lesssim \|u_0\|_{Y^{-s}} \|u_1\|_{Y^s} \prod_{j=2}^5 \|u_j\|_{Y^1}.$$

Part $\Sigma_{1,2}$. We apply Lemma 2.8 twice and deduce

$$\Sigma_{1,2} \lesssim \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ N_0 < N_2 \approx N_1}} N_0 N_3 N_4 N_5 \left(\frac{N_5}{N_1} + \frac{1}{N_3} \right)^\delta \prod_{j=0}^5 \|P_{N_j} u_j\|_{Y^0}$$

for some $\delta > 0$. Since $N_0 \lesssim N_1$, we have $N_0 \lesssim N_1^{1+s-\nu} N_0^{\nu-s}$ for some small $0 < \nu < s$. Cauchy–Schwarz with respect to N_5, N_4, N_3 , and N_0 as well as [Proposition 2.4 \(iii\)](#) yield

$$\Sigma_{1,2} \lesssim \sum_{\substack{N_1 \geq N_2 \geq 1: \\ N_1 \approx N_2}} N_1^{1+s} \|P_{N_1} u_1\|_{Y^0} \|P_{N_2} u_2\|_{Y^0} \|u_0\|_{Y^{-s}} \prod_{j=3}^5 \|u_j\|_{Y^1}.$$

Another application of Cauchy–Schwarz and using $N_1 \approx N_2$ leads to

$$\Sigma_{1,2} \lesssim \|u_0\|_{Y^{-s}} \|u_1\|_{Y^s} \prod_{j=2}^5 \|u_j\|_{Y^1}$$

as asserted.

We now estimate the contribution from Σ_2 and split the sum into two parts $\Sigma_2 = \Sigma_{2,1} + \Sigma_{2,2}$, where $\Sigma_{2,1}$ is defined by the constraint $\max\{N_0, N_2\} \ll N_1$ and $\Sigma_{2,2}$ is defined by the constraint $N_0 \gg N_1$.

Part $\Sigma_{2,1}$. We decompose

$$\Sigma_{2,1} \leq \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ N_0, N_2 \ll N_1}} \sum_{L \geq 1} \int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt,$$

where

$$I(N_0, \dots, N_5, L)(t) := \int_M P_L \left(\prod_{j=0}^2 P_{N_j} \tilde{u}_j \right) (t, x) \prod_{j=3}^5 P_{N_j} \tilde{u}_j(t, x) dx.$$

On the one hand, if $L \gtrsim N_1$, then we have $N_3, N_4, N_5 \ll L$. We can apply [Corollary 2.7](#) to conclude for every $t \in \tau$,

$$|I(N_0, \dots, N_5, L)(t)| \lesssim L^{-5} \|P_L(P_{N_0} \tilde{u}_0 P_{N_1} \tilde{u}_1 P_{N_2} \tilde{u}_2)(t)\|_{L^2(M)} \prod_{j=3}^5 \|P_{N_j} u_j(t)\|_{L^2(M)}.$$

Now, we apply Hölder’s inequality with respect to t and [Lemma 2.6](#) to bound

$$\int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt \lesssim L^{-5} N_0^{\frac{3}{2}} N_2^{\frac{3}{2}} \prod_{j=0}^5 \|P_{N_j} u_j\|_{L^\infty(\tau, L^2(M))},$$

which in turn implies

$$\sum_{L \gtrsim N_1} \int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt \lesssim N_1^{-2} \prod_{j=0}^5 \|P_{N_j} u_j\|_{Y^0}. \quad (2.14)$$

On the other hand, if $L \ll N_1$, then $L, N_0, N_2 \ll N_1$, and we use [Corollary 2.7](#) to get

$$|I(N_0, \dots, N_5, L)(t)| \lesssim N_1^{-5} \prod_{j=0}^2 \|P_{N_j} u_j(t)\|_{L^2(M)} \|P_L(P_{N_3} \tilde{u}_3 P_{N_4} \tilde{u}_4 P_{N_5} \tilde{u}_5)(t)\|_{L^2(M)}.$$

Again, from an application of Hölder’s inequality with respect to t and [Lemma 2.6](#), we infer

$$\int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt \lesssim N_1^{-5} N_4^{\frac{3}{2}} N_5^{\frac{3}{2}} \prod_{j=0}^5 \|P_{N_j} u_j\|_{L^\infty(\tau, L^2(M))}.$$

This in conjunction with (2.14) gives

$$\sum_{L \geq 1} \int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt \lesssim N_1^{-1} \prod_{j=0}^5 \|P_{N_j} u_j\|_{Y^0},$$

and hence,

$$\Sigma_{2,1} \lesssim \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ N_0, N_2 \ll N_1}} N_1^{-1} \prod_{j=0}^5 \|P_{N_j} u_j\|_{Y^0}.$$

Using Cauchy–Schwarz with respect to N_5 , N_4 , N_3 , and N_2 yields

$$\Sigma_{2,1} \lesssim N_1^{-1} \|P_{N_0} u_0\|_{Y^0} \|P_{N_1} u_1\|_{Y^0} \prod_{j=2}^5 \|u_j\|_{Y^1}.$$

Multiplying $(\frac{N_1}{N_0})^{s-\nu}$ for some small $0 < \nu < 1$ and applying Cauchy–Schwarz with respect to N_0 and N_1 leads to

$$\Sigma_{2,1} \lesssim \|u_0\|_{Y^{-s}} \|u_1\|_{X^s} \prod_{j=2}^5 \|u_j\|_{X^1}.$$

Part $\Sigma_{2,2}$. This case may be treated similarly as term $\Sigma_{2,1}$ by switching the roles of N_0 and N_1 . By writing

$$\Sigma_{2,2} \leq \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ N_0 \gg N_1}} \sum_{L \geq 1} \int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt$$

as above, we obtain the two estimates

$$\int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt \lesssim L^{-5} N_1^{\frac{3}{2}} N_2^{\frac{3}{2}} \prod_{j=0}^5 \|P_{N_j} u_j\|_{L^\infty(\tau, L^2(M))}$$

provided $L \gtrsim N_0$ and

$$\int_{\tau} |I(N_0, \dots, N_5, L)(t)| dt \lesssim N_0^{-5} N_4^{\frac{3}{2}} N_5^{\frac{3}{2}} \prod_{j=0}^5 \|P_{N_j} u_j\|_{L^\infty(\tau, L^2(M))}$$

provided $L \ll N_0$. This implies

$$\Sigma_{2,2} \lesssim \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ N_0 \gg N_1}} N_0^{-1} \prod_{j=0}^5 \|P_{N_j} u_j\|_{Y^0}.$$

Multiplying $(\frac{N_0}{N_1})^{s-\nu}$ for some small $0 < \nu < 1$ and arguing as above, we see that

$$\Sigma_{2,2} \lesssim \|u_0\|_{Y^{-s}} \|u_1\|_{X^s} \prod_{j=2}^5 \|u_j\|_{X^1},$$

which finishes the proof. \square

Remark. If $M = \mathbb{T}^3$, orthogonality implies that there is no contribution from Σ_2 . Similarly, if $M = \mathbb{S}^3$, then $\Sigma_2 = 0$ since the product of five spherical harmonics of maximal degree k can be developed into a series of spherical harmonics of maximal degree $5k$. \diamond

Finally, we have all the ingredients to prove [Theorem 2.3](#). The strategy is close to the arguments on Euclidean spaces, see e.g. [[CW90](#), [Tao07](#)], and was first applied to obtain energy-critical well-posedness for the NLS equation posed on a compact, boundaryless manifold by Herr–Tataru–Tzvetkov [[HTT11](#), Theorems 1.1 & 1.2]. We closely follow their arguments and add the treatment for $s > 1$.

Proof of [Theorem 2.3](#). Let $s \geq 1$.

Step 1 (Small data). Due to the polynomial structure of the nonlinearity, [Lemma 2.9](#) shows that there exists $C_{s,1} \geq 1$ such that

$$\|\mathcal{I}(|u|^4u - |v|^4v)\|_{X^s([0,T])} \leq C_{s,1}(\|u\|_{X^s([0,T])}^4 + \|v\|_{X^s([0,T])}^4)\|u - v\|_{X^s([0,T])}$$

holds true for all $T > 0$ and $u, v \in X^s([0, T])$.

Given two parameters $\varepsilon_s > 0$ and $\delta_s > 0$, we define the sets

$$\begin{aligned} B_{\varepsilon_s}^s &:= \{\phi \in H^s(M) : \|\phi\|_{H^s(M)} \leq \varepsilon_s\}, \\ D_{\delta_s}^s &:= \{u \in C([0, 1], H^s(M)) \cap X^s([0, 1]) : \|u\|_{X^s([0,1])} \leq \delta_s\}. \end{aligned}$$

Note that $D_{\delta_s}^s$ is closed in $X^s([0, 1])$, which in turn implies that $D_{\delta_s}^s$ is a complete space.

For $\phi \in B_{\varepsilon_s}^s$ we intend to solve the equation

$$u = e^{it\Delta_g}\phi \mp i\mathcal{I}(|u|^4u) =: L(\phi) + NL(u),$$

by the contraction mapping principle in $D_{\delta_s}^s$. Choose

$$\delta_s := (4C_{s,1})^{-\frac{1}{4}} \quad \text{and} \quad \varepsilon_s := \frac{\delta_s}{2C_{s,0}}, \tag{2.15}$$

where $C_{s,0}$ is the implicit constant in [Proposition 2.4 \(ii\)](#). Let $\phi \in B_{\varepsilon_s}^s$, then for every $u \in D_{\delta_s}^s$ we obtain

$$\|L(\phi) + NL(u)\|_{X^s([0,1])} \leq C_{s,0}\varepsilon_s + C_{s,1}\delta_s^5 \leq \delta_s.$$

For all $u, v \in D_{\delta_s}^s$ we also deduce

$$\|NL(u) - NL(v)\|_{X^s([0,1])} \leq \frac{1}{2}\|u - v\|_{X^s([0,1])}.$$

This implies that for any $\phi \in B_{\varepsilon_s}^s$ the nonlinear map $u \mapsto L(\phi) + NL(u)$ is a contraction on $D_{\delta_s}^s$. The Banach fixed-point theorem now proves that $u \mapsto L(\phi) + NL(u)$ has a unique fixed point in $D_{\delta_s}^s$. The uniqueness in the full space is discussed in the third step. Furthermore, for two functions $\phi, \psi \in B_{\varepsilon_s}^s$ and their corresponding fixed points $u, v \in D_{\delta_s}^s$, we have

$$\|u - v\|_{X^s([0,1])} \leq C_{s,0}\|\phi - \psi\|_{H^s(M)} + \frac{1}{2}\|u - v\|_{X^s([0,1])}.$$

This proves the Lipschitz continuity of $\phi \mapsto u$ with constant $2C_{s,0}$.

Step 2 (Large data). Let $r > 0$ and $N \geq 1$ be given. For some parameters ε_s , δ_s , R_s , and T_s with the properties $0 < \varepsilon_s \leq r$ and $0 < \delta_s \leq R_s$, we define

$$\begin{aligned} B_{\varepsilon_s, r}^s &:= \{\phi \in H^s(M) : \|\phi_{>N}\|_{H^s(M)} \leq \varepsilon_s, \|\phi\|_{H^s(M)} \leq r\}, \\ D_{\delta_s, R_s, T_s}^s &:= \{u \in C([0, T_s], H^s(M)) \cap X^s([0, T_s]) : \|u_{>N}\|_{X^s([0, T_s])} \leq \delta_s, \|u\|_{X^s([0, T_s])} \leq R_s\}, \end{aligned}$$

where $f_{>N} := (\text{Id} - P_{\leq N})f$. For any $\phi \in B_{\varepsilon_s, r}^s$, one easily sees that

$$\|(L(\phi) + NL(u))_{>N}\|_{X^s([0, T_s])} \leq C_{s,0}\varepsilon_s + \|NL(u)_{>N}\|_{X^s([0, T_s])}.$$

We split $NL(u)$ into two parts,

$$NL(u) = NL_1(u_{\leq N}, u_{>N}) + NL_2(u_{\leq N}, u_{>N}),$$

such that NL_1 is at least quadratic in $u_{>N}$ and NL_2 is at least quartic in $u_{\leq N}$. Then, thanks to Lemma 2.9, we deduce for $u \in D_{\delta_s, R_s, T_s}^s$

$$\|NL_1(u_{\leq N}, u_{>N})\|_{X^s([0, T_s])} \leq C_{s,1}\delta_s^2 R_s^3. \quad (2.16)$$

Analogously, for $u, v \in D_{\delta_s, R_s, T_s}^s$,

$$\|NL_1(u_{\leq N}, u_{>N}) - NL_1(v_{\leq N}, v_{>N})\|_{X^s([0, T_s])} \leq C_{s,2}\delta_s R_s^3 \|u - v\|_{X^s([0, T_s])}.$$

For estimating $NL_2(u_{\leq N}, u_{>N})$, we use Lemma 2.5 to argue that it suffices to bound the non-linearity in $L^1([0, T_s], H^s(M))$. Hence, by Lemma 1.55, Lemma 1.56, and Hölder's inequality, one easily checks³

$$\|NL_2(u_{\leq N}, u_{>N})\|_{X^s([0, T_s])} \leq C_{s,3}N^4 \|u\|_{L^\infty([0, T_s], H^s(M))} \|u\|_{L^4([0, T_s], H^1(M))}^4 \leq C_{s,3}N^4 T_s R_s^5. \quad (2.17)$$

A similar argument gives

$$\|NL_2(u_{\leq N}, u_{>N}) - NL_2(v_{\leq N}, v_{>N})\|_{X^s([0, T_s])} \leq C_{s,4}N^4 T_s R_s^4 \|u - v\|_{X^s([0, T_s])}.$$

Set $C_s := \max\{C_{1,0}, \dots, C_{1,4}, C_{s,0}, \dots, C_{s,4}\}$, where $C_{1,j}$, $j = 0, \dots, 4$, are the corresponding constants in the case $s = 1$, and choose

$$R_s := 4C_s r, \quad \delta_s := \frac{1}{8C_s R_s^3}, \quad \varepsilon_s := \frac{\delta_s}{2C_s}, \quad \text{and} \quad T_s := \frac{\delta_s}{8C_s R_s^5 N^4}. \quad (2.18)$$

Hence, for $\phi \in B_{\varepsilon_s, r}^s$ the map

$$L(\phi) + NL: D_{\delta_s, R_s, T_s}^s \rightarrow D_{\delta_s, R_s, T_s}^s$$

is a strict contraction and therefore, has a unique fixed point u , and $\phi \mapsto u$ is Lipschitz continuous with constant $2C_s$.

Step 3 (Uniqueness). By the translation invariance in time, it suffices to consider

$$u, v \in C([0, T], H^s(M)) \cap X^s([0, T])$$

with $u(0) = v(0)$ in order to prove uniqueness. That $u = v$ for arbitrarily small $T > 0$ follows from the uniqueness of the fixed point in Step 2.

Step 4 (Time of existence). Let $\phi_* \in H^1(M)$, define $r := 2\|\phi_*\|_{H^1(M)}$, and choose $N \geq 1$ large enough such that $\|(\phi_*)_{>N}\|_{H^1(M)} \leq \frac{\varepsilon_1}{2}$, where $0 < \varepsilon_1 \leq r$ is defined by (2.18). Let $\phi \in B_{\varepsilon_1/2}(\phi_*) \cap H^s(M)$, then $\phi \in B_{\varepsilon_1, r}^1 \cap H^s(M)$. We conclude from Step 2 that there is $T_1 = T_1(r, N) > 0$ given by (2.18) and a unique solution

$$u \in C([0, T_1], H^1(M)) \cap X^1([0, T_1]),$$

³For notational convenience, we choose $\sigma = 2$ instead of $\sigma = \frac{3}{2}+$ in the application of Lemma 1.55 and Lemma 1.56, accepting that the power of N is not the best we can achieve.

which depends Lipschitz continuously on the initial data ϕ . Note that the time of existence is determined only by ϕ_* .

For $s > 1$ we now prove that this solution is even more regular on the same time interval. Let $T_{s,\max}$ be the supremum over all T_s such that

$$u \in C([0, T_s], H^s(M)) \cap X^s([0, T_s))$$

is the unique solution with initial data ϕ . Step 2 guarantees that $T_{s,\max} > 0$. Assume that $T_{s,\max} < T_1$, then consider $0 < T_s < T_{s,\max}$ and let R_1 , δ_1 , ε_1 , and T_1 be defined as in (2.18). Note that the parameters depend only on ϕ_* .

From Step 2 with $s = 1$, we get that $\|u\|_{X^1([0, T_s])} \leq R_1$ and $\|u_{>N}\|_{X^1([0, T_s])} \leq \delta_1$. Reconsidering (2.16) and applying the bounds on u in X^1 , we infer

$$\|NL_1(u_{\leq N}, u_{>N})\|_{X^s([0, T_s])} \lesssim \|u_{>N}\|_{X^1([0, T_s])} \|u\|_{X^1([0, T_s])}^3 \|u\|_{X^s([0, T_s])} \leq C_s \delta_1 R_1^3 \|u\|_{X^s([0, T_s])},$$

where C_s is defined as in Step 2. We may also improve (2.17) to

$$\|NL_2(u_{\leq N}, u_{>N})\|_{X^s([0, T_s])} \leq C_s N^4 T_s R_1^4 \|u\|_{X^s([0, T_s])}.$$

Hence,

$$\|u\|_{X^s([0, T_s])} \leq C_s \|\phi\|_{H^s(M)} + C_s (\delta_1 R_1^3 + N^4 T_s R_1^4) \|u\|_{X^s([0, T_s])}$$

for any $T_s < T_{s,\max}$, and we conclude from the embedding $X^s([0, T_s]) \hookrightarrow L^\infty([0, T_s], H^s(M))$ that

$$\sup_{t \in [0, T_{s,\max})} \|u(t)\|_{H^s(M)} \leq 2C_s \|\phi\|_{H^s(M)}.$$

Consequently, for every sequence $(t_n)_n$ with $t_n \in [0, T_{s,\max})$ and $t_n \rightarrow T_{s,\max}$ as $n \rightarrow \infty$ we have $u(t_n) \in H^s(M)$ for any $n \in \mathbb{N}$. Thus, there exist a subsequence $(t_{n_k})_k$ and $v \in H^s(M)$ with $u(t_{n_k}) \rightharpoonup v$ in $H^s(M)$ as $k \rightarrow \infty$. By the Rellich–Kondrachov embedding theorem, we see that $u(t_{n_k}) \rightarrow v$ in $H^1(M)$ as $k \rightarrow \infty$. Since $T_{s,\max} < T_1$ we know that $u(t_{n_k}) \rightarrow u(T_{s,\max})$ in $H^1(M)$ as $k \rightarrow \infty$. Therefore, we deduce $v = u(T_{s,\max}) \in H^s(M)$. Solving the equation (2.1) with initial data $u(T_{s,\max})$ forward and backward in time, which is possible by Step 2, we see that the solution u can be uniquely extended in $H^s(M)$. This contradicts the definition of $T_{s,\max}$ and hence, $T_{s,\max} \geq T_1$.

The Lipschitz continuity for $s > 1$ follows since r , N , ε_1 , δ_1 , R_1 , and T_1 depend only on ϕ_* and

$$\|u - v\|_{X^s([0, T_1])} \leq C_{s,0} \|\phi - \psi\|_{H^s(M)} + 2C_{s,1} R_1^4 \|u - v\|_{X^s([0, T_1])}.$$

Step 5 (Global well-posedness, defocusing case). We first consider $s = 1$. Because of the first step, we only have to prove a suitable a priori bound on solutions in $H^1(M)$.

The conservation laws (1.23) and (1.24) and the Sobolev embedding $H^1(M) \hookrightarrow L^6(M)$ imply that there exists some $d > 0$ such that for every t ,

$$\|u(t)\|_{H^1(M)}^2 \leq 2E(u(0)) + 2M(u(0)) \leq \|u(0)\|_{H^1(M)}^2 + d^2 \|u(0)\|_{H^1(M)}^6. \quad (2.19)$$

If $\|u(0)\|_{H^1(M)}$ is sufficiently small, then it follows that for ε_1 as in (2.15) the solution satisfies $\|u(t)\|_{H^1(M)} \leq \varepsilon_1$ for any interval of existence. Hence, we can iterate the argument in the first step indefinitely and extend the local well-posedness result to global well-posedness.

For $s > 1$ we proceed as follows. Let $\phi \in H^s(M)$ with H^1 -norm small enough such that the solution exists globally in H^1 and Step 1 with $s = 1$ is applicable. Let $T_{s,\max}$ be the supremum over all T_s such that

$$u \in C([0, T_s], H^s(M)) \cap X^s([0, T_s))$$

is the unique solution with initial data ϕ . Define $v := u(\cdot + T_{s,\max} - \frac{1}{2})$ and $T'_{s,\max} := \frac{1}{2}$ if $T_{s,\max} \geq 1$ and $v := u$ and $T'_{s,\max} := T_{s,\max}$ otherwise. From Step 1 with $s = 1$ and (2.19) we deduce that

$$\|v\|_{X^1([0,1])} \leq 2C_{1,0}\|v(0)\|_{H^1(M)} \leq 2C_{1,0}(\|\phi\|_{H^1(M)}^2 + d^2\|\phi\|_{H^1(M)}^6)^{\frac{1}{2}}$$

and therefore, we gain the following a priori estimate

$$\|v\|_{X^s([0,T_s])} \leq C_{s,0}\|v(0)\|_{H^s(M)} + \frac{1}{2}\|v\|_{X^s([0,T_s])}$$

for any $0 < T_s < T'_{s,\max}$ provided $\|\phi\|_{H^1(M)}$ is sufficiently small. By similar arguments as above, this yields

$$\sup_{t \in [0, T'_{s,\max})} \|v(t)\|_{H^s(M)} \leq 2C_{s,0}\|v(0)\|_{H^s(M)},$$

and we conclude $u(T_{s,\max}) \in H^s(M)$. Solving (2.1) forward and backward in time with initial data $u(T_{s,\max})$ contradicts the choice of $T_{s,\max}$.

Step 6 (Global well-posedness, focusing case). In this case, the argument is a bit different. For $u \in X^1([0,1])$ we have

$$\begin{aligned} \|u(t)\|_{H^1(M)}^2 &\leq 2E(u(0)) + 2M(u(0)) + \frac{1}{3}\|u(t)\|_{L^6(M)}^6 \\ &\leq \|u(0)\|_{H^1(M)}^2 + d^2\|u(0)\|_{H^1(M)}^6 + d^2\|u(t)\|_{H^1(M)}^6 \end{aligned} \quad (2.20)$$

Consider the function $f: [0, \infty) \rightarrow \mathbb{R}$ given by $f(x) := x - d^2x^3$. The function f increases from 0 to its maximum value $2/(3\sqrt{3}d)$ in $x = 1/(\sqrt{3}d)$. Moreover, $f(x) \geq (2/3)x$ on the interval $I := [0, (\sqrt{3}d)^{-1}]$. In (2.20), we have proved that $f(\|u(t)\|_{H^1(M)}^2) < \varepsilon_0^2$ for all $t \in [0, 1)$ and all initial data satisfying

$$\|u(0)\|_{H^1(M)}^2 + d^2\|u(0)\|_{H^1(M)}^6 < \varepsilon_0^2.$$

If we choose $\varepsilon_0^2 = \min\{2/(3\sqrt{3}d), (2/3)\varepsilon_1^2\}$, where ε_1 is given as in (2.15), then we see by the continuity of $t \mapsto \|u(t)\|_{H^1(M)}^2$ that $\|u(t)\|_{H^1(M)}^2 \in I$ for every $t \in [0, 1)$. Thus, $\|u(t)\|_{H^1(M)}^2 \leq (3/2)\varepsilon_0^2 \leq \varepsilon_1^2$ for all $t \in [0, 1)$, from which we infer that the small data local well-posedness argument may be iterated.

The conclusion for $s > 1$ works exactly as in the previous step. \square

2.2.2 On the necessity of the condition

After the discussion of the sufficiency of Assumption 2.1 the question rises whether the trilinear Strichartz estimate is also necessary. In [Gér06, Theorem 5.7 i)] (take $s = 1$), Gérard answered this question by stating that Assumption 2.1 with $\delta = 0$ is necessary to obtain Theorem 2.3. In a joint paper with the present author, Herr [HS15, Section 4] provided a proof of this by adapting the arguments of Burq–Gérard–Tzvetkov in [BGT05a, Remark 2.12]. We want to point out that there was no noteworthy contribution of the author to this discussion. In the remainder of this subsection, we repeat the argument in [HS15, Section 4] almost verbatim.

Fix $T > 0$ and consider the map

$$F: H^1(M) \rightarrow H^1(M), \quad F(\phi) = u(T),$$

where u is a solution of (2.1) with initial data $u(0) = \phi$. The fifth order differential of F at the origin is given by

$$D^5 F(0)(h) = \mp 12i \int_0^T e^{i(T-\tau)\Delta_g} \sum_{\sigma \in \Sigma_5} H_{\sigma(1)}(\tau) \overline{H_{\sigma(2)}(\tau)} H_{\sigma(3)}(\tau) \overline{H_{\sigma(4)}(\tau)} H_{\sigma(5)}(\tau) d\tau,$$

where $h := (h_1, \dots, h_5)$, $H_j(\tau) := e^{i\tau\Delta_g} h_j$, and we sum over the $10 = \binom{5}{2}$ of the $5! = 120$ permutations $\sigma \in \Sigma_5$, which give rise to different pairs $(\sigma(2), \sigma(4))$. Indeed, from (2.12) it follows that $DF(0)(h) = e^{iT\Delta_g} h$, $D^j F(0) = 0$ for $2 \leq j \leq 4$, and we obtain the above formula. If we specify to $h_2 = h_3 = h_4 = h_5$, we obtain two contributions

$$\sum_{\sigma \in \Sigma_5} H_{\sigma(1)} \overline{H_{\sigma(2)}} H_{\sigma(3)} \overline{H_{\sigma(4)}} H_{\sigma(5)} = 6H_1 |H_2|^4 + 4\overline{H_1} H_2^3 \overline{H_2}.$$

Now, let us assume that $D^5 F(0): (H^1(M))^5 \rightarrow H^1(M)$ is bounded. Then, we infer

$$\left| \int_M D^5 F(0)(h_1, h_2, \dots, h_2) \overline{H_1}(T) dx \right| \lesssim \|h_1\|_{H^1(M)} \|h_1\|_{H^{-1}(M)} \|h_2\|_{H^1(M)}^4.$$

Because of

$$\operatorname{Re} \left(6|H_1|^2 |H_2|^4 + 4\overline{H_1}^2 H_2^3 \overline{H_2} \right) \geq 2|H_1|^2 |H_2|^4,$$

we conclude that

$$\int_0^T \int_M |H_1|^2 |H_2|^4 dx dt \lesssim \|h_1\|_{H^1(M)} \|h_1\|_{H^{-1}(M)} \|h_2\|_{H^1(M)}^4.$$

We set $h_1 := P_{N_1} \phi_1$, and for $\phi_2, \phi_3 \in H^1(M)$ we write

$$e^{it\Delta_g} \phi_2 e^{it\Delta_g} \phi_3 = \frac{1}{4} \left((e^{it\Delta_g} \phi_2 + e^{it\Delta_g} \phi_3)^2 - (e^{it\Delta_g} \phi_2 - e^{it\Delta_g} \phi_3)^2 \right)$$

to obtain the bound

$$\|e^{it\Delta_g} P_{N_1} \phi_1 e^{it\Delta_g} \phi_2 e^{it\Delta_g} \phi_3\|_{L^2([0,T] \times M)} \lesssim \|P_{N_1} \phi_1\|_{L^2(M)} \|\phi_2\|_{H^1(M)} \|\phi_3\|_{H^1(M)},$$

which implies the estimate in Assumption 2.1 but only with $\delta = 0$.

2.3 Rectangular tori in three dimensions

This section is devoted to verify Assumption 2.1 on flat rectangular 3-tori, which means that the energy-critical NLS is locally well-posedness and globally well-posedness for small initial data. We start with an overview of some related results and set up the framework. We shall then prove the trilinear estimate in three steps. We first provide linear Strichartz estimates, then exploit almost orthogonality, and finally conclude the desired trilinear estimate. This proof is due to the author and has already been published in [Str14].

2.3.1 Selected results

The nonlinear Schrödinger equation on flat tori has been the most investigated among all compact manifolds. Aside from the precise knowledge of the spectrum and the eigenfunctions, one main reason might be that due to the periodicity of functions on \mathbb{T}^n , one has access to

the theory of Fourier series, which is often applied in this context. At first sight, the theory of Fourier series with a common period seems not to be applicable if one considers general rectangular tori. Indeed, if one of the ratios of the periods is irrational, there is no common period. However, by a simple change of coordinates, one can always choose \mathbb{T}^n as the base space, which leads to a modified Laplace–Beltrami operator, see (2.21)–(2.23). This simple change of coordinates allows to use the theory of Fourier series also in this setting.

First, we sum up related results on the flat torus \mathbb{T}^3 . In 1993, Bourgain [Bou93a] started this line of research with a fundamental work. He established Strichartz estimates [Bou93a, Proposition 3.114] in several dimensions and deduced well-posedness in certain sub-critical regimes from it, see [Bou93a, Theorems 1–4]. Out of this results we just pick those that are most relevant for our study. For $p > 4$ the scale invariant Strichartz estimate

$$\|e^{it\Delta_g} f\|_{L^p(I \times \mathbb{T}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|f\|_{L^2(\mathbb{T}^3)}$$

holds true for all $f \in L^2(\mathbb{T}^3)$ with $\text{supp } \hat{f} \subseteq [-N, N]^3$, cf. [Bou93a, formula (3.117)]. Using this inequality, Bourgain was able to show that the focusing NLS equation with nonlinearity $|u|^{\alpha-1}u$ and initial data $\phi \in H^1(\mathbb{T}^3)$ which has sufficiently small H^1 -norm is globally well-posed for $3 \leq \alpha < 5$. Since $\alpha < 5$, Bourgain did not reach the energy-critical case. Local and small data global well-posedness for both the focusing and the defocusing equation in the energy-critical case ($\alpha = 5$) was achieved by Herr–Tataru–Tzvetkov [HTT11, Theorems 1.1 and 1.2] in 2011. One of their crucial observations is the existence of almost orthogonality in time, which is exploited in the proof of [HTT11, Proposition 3.5]. Just one year later, Ionescu–Pausader [IP12b, Theorem 1.1] showed that the energy-critical defocusing NLS equation on \mathbb{T}^3 is globally well-posed even for arbitrarily large H^1 -data. Global well-posedness is addressed in Chapter 3.

It was again Bourgain [Bou07] who initiated the study of the nonlinear Schrödinger equation on three-dimensional *rectangular* tori. He proved Strichartz estimates for free solutions on this domain for a smaller range of $L_t^p L_x^q$ -norms compared to \mathbb{T}^3 [Bou07, Proposition 1.1]. From this, he deduced that the energy-sub-critical defocusing NLS equation on rectangular 3-tori is locally and globally well-posed in H^1 [Bou07, Proposition 1.2]. The first scaling-critical results on rectangular tori were established by Guo–Oh–Wang [GOW14, Theorem 1.5]. They proved critical local well-posedness on this set of manifolds for nonlinearities $|u|^{\alpha-1}u$ with odd $\alpha \geq 7$ and initial data in the corresponding scale invariant space H^s . Furthermore, they considered the energy critical case $\alpha = 5$ on 3-dimensional rectangular tori, where two of the periods are the same [GOW14, Appendix B]. In the following, we prove a trilinear Strichartz estimate, which, by Section 2.2.1, implies that the energy-critical NLS on any 3-dimensional rectangular torus is locally well-posed and in addition, globally well-posed provided the initial data have small H^1 -norm. The author already published this result in [Str14, Proposition 4.1]. This result is highly significant for the study of large data global well-posedness on this domain, which is pursued in Chapter 3.

More authors contributed to today’s knowledge about the nonlinear Schrödinger equation on tori. We are not aiming to give a full list but we want to mention some important results. Building on an earlier work of Bourgain–Demeter [BD15], Killip–Viřan [KV14, Theorem 1.1] extended Bourgain’s above-mentioned scaling invariant Strichartz estimate for free solutions to the NLS on rectangular tori in any dimension $n \geq 1$ to a larger range of $L_{t,x}^p$ -norms. They were able to bound free solutions in L^p for $p > \frac{2(n+2)}{n}$. These results are optimal in the sense that the Strichartz estimates are known to fail for $p = \frac{2(n+2)}{n}$ [Bou93a, Section 2, Remark 2].

Comparing it to the range of Strichartz estimates on \mathbb{R}^3 , see (1.17), one notices that the estimates on rectangular tori cover the same range except of the endpoint. Further contributions to linear Strichartz estimate came from [Bou07, CW10, Dem13, Bou13, GOW14].

Sub-critical well-posedness on tori in several dimensions has been addressed e.g. in [Bou93a, Bou93b, Bou04, DPST07, Bou07, CKS⁺10, CW10, Han12, Dem13, GOW14].

The nonlinear Schrödinger equation on tori in various critical regimes have been studied in [Wan13b, HTT11, HTT14, GOW14, Str14, KV14]. The NLS on rectangular tori with nonlinearity $\pm|u|^{2k+1}u$ is known to be locally well-posed in the scaling space in the following situations:

- $n = 2$ and $k \geq 3$ [Str14], see also [GOW14] for $k \geq 6$,
- $n = 3$ and $k \geq 2$ [Str14], see also [GOW14] for $k \geq 3$,
- $n \geq 4$ and $k \geq 2$ [GOW14].

The case $n = 2$ and $k \geq 3$ is pursued in Section 2.4. Using Bourgain's Strichartz estimate in [Bou13], Herr–Tataru–Tzvetkov [HTT14] proved that the energy-critical NLS equation on \mathbb{T}^4 is globally well-posed for small initial data. This is remarkable as it is the only energy-critical well-posedness result on a 4-dimensional compact manifold known yet.

It is also worth to mention that the non-elliptic nonlinear Schrödinger equation on \mathbb{T}^2 has been considered in [GT12, Wan13a]. Moreover, rough potentials [BBZ13] and the fractional Schrödinger equation [DET13] have been studied.

2.3.2 Set-up

We start with some basic definitions and notation. \mathbb{T}^n shall denote the flat standard torus $\mathbb{T}^n := \mathbb{R}^n / (2\pi\mathbb{Z})^n$. Recall from Definition 1.26 that we use the following convention for the Fourier transform on \mathbb{T}^n

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{T}^n} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{Z}^n,$$

so that we have the Fourier inversion formula

$$f(x) = \frac{1}{(2\pi)^{n/2}} \sum_{\xi \in \mathbb{Z}^n} \hat{f}(\xi) e^{ix \cdot \xi}, \quad x \in \mathbb{T}^n.$$

Let the spectral projectors $P_N: L^2(\mathbb{T}^n) \rightarrow L^2(\mathbb{T}^n)$ be defined as in (1.12). More generally, given a set $\mathcal{S} \subseteq \mathbb{Z}^n$, we define $P_{\mathcal{S}}$ to be the Fourier multiplier operator with symbol $\mathbf{1}_{\mathcal{S}}$, where $\mathbf{1}_{\mathcal{S}}$ denotes the characteristic function of \mathcal{S} .

Given any $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in (0, \infty)^n$, we define the flat rectangular torus by

$$\mathbb{T}_{\boldsymbol{\theta}}^n := \mathbb{R}^n / (2\pi\theta_1^{-1/2}\mathbb{Z} \times \dots \times 2\pi\theta_n^{-1/2}\mathbb{Z}).$$

We shall use the standard torus $\mathbb{T}^n = \mathbb{T}_{(1, \dots, 1)}^n$ as base space. Let $\tilde{\phi} \in H^{sc}(\mathbb{T}_{\boldsymbol{\theta}}^n)$, and suppose $v: (-T, T) \times \mathbb{T}_{\boldsymbol{\theta}}^n \rightarrow \mathbb{C}$ solves the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t v + \Delta_g v = \pm|v|^{2k+1}v \\ v(0, \cdot) = \tilde{\phi}, \end{cases} \quad (2.21)$$

where $k \in \mathbb{N}$. Recall from (1.19) that the scaling-critical Sobolev index is given by

$$s_c = \frac{n}{2} - \frac{1}{k}. \quad (2.22)$$

Let $u: (-T, T) \times \mathbb{T}^n \rightarrow \mathbb{C}$ and $\phi \in H^{s_c}(\mathbb{T}^n)$ be defined as

$$u(t, x) := v(t, (\theta_1^{-1/2}x_1, \dots, \theta_n^{-1/2}x_n))$$

and $\phi(x_1, \dots, x_n) := \tilde{\phi}(\theta_1^{-1/2}x_1, \dots, \theta_n^{-1/2}x_n)$, respectively. By a change of spatial variables, one easily verifies that u is a solution to

$$\begin{cases} i\partial_t u + \Delta_{\theta} u = \pm |u|^{2k+1} u \\ u(0, \cdot) = \phi. \end{cases} \quad (2.23)$$

Here, the modified Laplace–Beltrami operator Δ_{θ} is defined via $\Delta_{\theta} := \theta_1 \partial_{x_1}^2 + \dots + \theta_n \partial_{x_n}^2$. On the Fourier side this corresponds to

$$\mathcal{F}(\Delta_{\theta} f)(\xi) := -Q(\xi) \hat{f}(\xi), \quad Q(\xi) := \theta_1 \xi_1^2 + \dots + \theta_n \xi_n^2, \quad (2.24)$$

for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n$. Using this notation, the free solution to (2.23) is given by

$$(e^{it\Delta_{\theta}} \phi)(x) = \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(\xi) e^{i(\xi \cdot x - Q(\xi)t)}. \quad (2.25)$$

By a change of variable in time, without loss of generality we may assume $\theta_1 = 1$. This turns out to be useful in the proof of Lemma 3.21 below. From now on, we study (2.23).

The mass and the energy,

$$\begin{aligned} M(u)(t) &= \frac{1}{2} \int_{\mathbb{T}^n} |u(t, x)|^2 dx, \\ E(u)(t) &= \frac{1}{2} \int_{\mathbb{T}^n} |\nabla_{\theta} u(t, x)|^2 dx \pm \frac{1}{2k+2} \int_{\mathbb{T}^n} |u(t, x)|^{2k+2} dx, \end{aligned} \quad (2.26)$$

are conserved in time, whenever $u: (-T, T) \times \mathbb{T}^n \rightarrow \mathbb{C}$ is a strong solution of (2.23). Here,

$$\nabla_{\theta} := (\theta_1^{1/2} \partial_{x_1}, \dots, \theta_n^{1/2} \partial_{x_n}).$$

For $N, M \geq 1$ we define the collection of rectangular sets

$$\begin{aligned} \mathcal{R}_{N,M}^n &:= \{ \mathcal{R} \subseteq \mathbb{R}^n : \exists z \in \mathbb{Z}^n, O \text{ orthogonal } n \times n\text{-matrix s.t.} \\ &\quad O\mathcal{R} + z \subseteq [-N, N]^{n-1} \times [-M, M] \}. \end{aligned}$$

Moreover, we set $\mathcal{C}_N^n := \mathcal{R}_{N,N}^n$.

We consider the three-dimensional quintic, i.e. $k = 2$, NLS in the present section. In Section 2.4, (2.23) in two dimensions with $k \geq 3$ is studied.

2.3.3 Linear Strichartz estimates

The following linear Strichartz estimate for free solutions on rectangular tori was verified by Bourgain [Bou07, Proposition 1.1]. Besides almost orthogonality, this is the main ingredient for the trilinear Strichartz estimate in Proposition 2.13.

Lemma 2.10. *Let $p > \frac{16}{3}$ and $\tau_0 \subset \mathbb{R}$ be a bounded interval. For every $N \geq 1$, $\mathcal{C} \in \mathcal{C}_N^3$, and $\phi \in L^2(\mathbb{T}^3)$ we have*

$$\|P_{\mathcal{C}}e^{it\Delta_{\theta}}\phi\|_{L^p(\tau_0, L^4(\mathbb{T}^3))} \lesssim N^{\frac{3}{4}-\frac{2}{p}}\|P_{\mathcal{C}}\phi\|_{L^2(\mathbb{T}^3)}.$$

Proof/Reference. Using essentially the exponential sum estimates given in Section 1.3, Bourgain [Bou07, Proposition 1.1] proved

$$\|P_Ne^{it\Delta_{\theta}}\phi\|_{L^p(\tau_0, L^4(\mathbb{T}^3))} \lesssim N^{\frac{3}{4}-\frac{2}{p}}\|P_N\phi\|_{L^2(\mathbb{T}^3)}. \quad (2.27)$$

Moreover, he remarked that the inequality holds true also for the projector $P_{\mathcal{C}}$. Below we show that this may be accomplished from a Galilean transformation.

We modify the arguments from [HTT11, Proposition 3.1] to treat Δ_{θ} , see also [Bou93a, formulas (5.7)–(5.8)]. Denote $\xi \cdot_{\theta} \zeta := \xi_1\zeta_1 + \theta_2\xi_2\zeta_2 + \theta_3\xi_3\zeta_3$, and let ξ_0 be the center of \mathcal{C} . Applying the adapted Galilean transformation

$$x \cdot_{\theta} \xi + tQ(\xi) = x \cdot_{\theta} \xi_0 + tQ(\xi_0) + (x + 2t\xi_0) \cdot_{\theta} (\xi - \xi_0) + tQ(\xi - \xi_0),$$

which can be easily verified, allows to shift the center of the cube \mathcal{C} to the origin, i.e. to $\mathcal{C}_0 := \mathcal{C} - \xi_0$. Define $\phi_0 := e^{-ix \cdot_{\theta} \xi_0} \phi(x)$, and note that $\widehat{\phi_0}(\xi) = \widehat{\phi}(\xi + \xi_0)$ implies $\|P_{\mathcal{C}}\phi\|_{L^2(\mathbb{T}^3)} = \|P_{\mathcal{C}_0}\phi_0\|_{L^2(\mathbb{T}^3)}$. We also observe that

$$P_{\mathcal{C}_0}e^{it\Delta_{\theta}}\phi_0(t, x) = \sum_{\xi \in \mathcal{C}_0} e^{i(x \cdot_{\theta} \xi - tQ(\xi))} \widehat{\phi_0}(\xi) = \sum_{\xi \in \mathcal{C}} e^{i(x \cdot_{\theta} (\xi - \xi_0) - tQ(\xi - \xi_0))} \widehat{\phi}(\xi).$$

Set $\Theta := \text{diag}(1, \theta_2, \theta_3)$, and observe that $x \cdot_{\theta} \xi = (\Theta^{-1}x) \cdot_{\theta} \xi$. Rewriting the phase as

$$x \cdot_{\theta} (\xi - \xi_0) - tQ(\xi - \xi_0) = (\Theta^{-1}x + 2t\xi_0) \cdot_{\theta} \xi - tQ(\xi) - x \cdot_{\theta} \xi_0 - tQ(\xi_0)$$

leads to

$$P_{\mathcal{C}_0}e^{it\Delta_{\theta}}\phi_0(t, x) = e^{-i(x \cdot_{\theta} \xi_0 + tQ(\xi_0))} P_{\mathcal{C}}e^{it\Delta_{\theta}}\phi(t, \Theta^{-1}x + 2t\xi_0).$$

Therefore, $\|P_{\mathcal{C}}e^{it\Delta_{\theta}}\phi\|_{L^p(\tau_0 \times \mathbb{T}^3)} = \|P_{\mathcal{C}_0}e^{it\Delta_{\theta}}\phi_0\|_{L^p(\tau_0 \times \mathbb{T}^3)}$. With this, Lemma 2.10 follows immediately from (2.27). \square

Remark. Building on an earlier work of Bourgain–Demeter [BD15], Killip–Viřan [KV14, Theorem 1.1] proved this Strichartz estimate to hold true for free solutions measured in $L^p(\tau_0 \times \mathbb{T}^3)$ with $p > \frac{10}{3}$. In this thesis, we want to point out that the Strichartz estimate gained from the exponential sum estimate in Corollary 1.39 is sufficient to obtain the local and small data global well-posedness result. As it will be seen in Chapter 3, it is even strong enough for proving global well-posedness for arbitrary large initial data in $H^1(\mathbb{T}^3)$. \diamond

Corollary 2.11. *Let $p > \frac{16}{3}$ and $4 \leq q < \frac{3p}{4}$. Then, for all $N, M \geq 1$ with $N \geq M$, $\mathcal{R} \in \mathcal{R}_{N, M}^3$, and all $\phi \in L^2(\mathbb{T}^3)$ it holds*

$$\|P_{\mathcal{R}}e^{it\Delta_{\theta}}\phi\|_{L^p(\tau_0, L^q(\mathbb{T}^3))} \lesssim N^{1-\frac{2}{p}-\frac{1}{q}}M^{\frac{1}{2}-\frac{2}{q}}\|P_{\mathcal{R}}\phi\|_{L^2(\mathbb{T}^3)}.$$

Proof. The statement is implied by Lemma 2.10, the estimate

$$\|P_{\mathcal{R}}e^{it\Delta_{\theta}}\phi\|_{L^{\infty}(\tau_0\times\mathbb{T}^3)} \leq |\mathcal{R} \cap \mathbb{Z}^3|^{\frac{1}{2}} \|P_{\mathcal{R}}\phi\|_{L^2(\mathbb{T}^3)} \lesssim NM^{\frac{1}{2}} \|P_{\mathcal{R}}\phi\|_{L^2(\mathbb{T}^3)},$$

which follows from Cauchy–Schwarz in the Fourier space, and Hölder’s estimate. The conclusion works as follows: Set $f(t, x) := |P_{\mathcal{R}}e^{it\Delta_{\theta}}\phi(x)|$, $\varepsilon := \frac{4p}{q} - \frac{16}{3} > 0$, and $\vartheta := \frac{4}{q} \leq 1$. Then,

$$\|P_{\mathcal{R}}e^{it\Delta_{\theta}}\phi\|_{L_t^p L_x^q} = \|f^{\vartheta} f^{1-\vartheta}\|_{L_t^p L_x^q} \leq \|f\|_{L_t^{\frac{16}{3+\varepsilon}} L_x^4}^{\vartheta} \|f\|_{L_{t,x}^{\infty}}^{1-\vartheta} \lesssim N^{1-\frac{2}{p}-\frac{1}{q}} M^{\frac{1}{2}-\frac{2}{q}} \|P_{\mathcal{R}}\phi\|_{L^2(\mathbb{T}^2)}. \quad \square$$

2.3.4 Almost orthogonality

In several applications it turned out to be beneficial to use almost orthogonality in time. This was first observed by Herr–Tataru–Tzvetkov [HTT11, Proof of Proposition 3.5] for the standard torus \mathbb{T}^3 and later also applied for Zoll manifolds such as \mathbb{S}^3 [Her13, Proof of Proposition 3.6] and $\mathbb{S} \times \mathbb{S}^2$ [HS15, Proof of Proposition 2.6]. Since free solutions on rectangular tori are in general not periodic in time, we can not expect almost orthogonality in the same way as in the aforementioned articles. However, the next lemma states that in the non-periodic setting one gets an additional term with arbitrarily high polynomial decay on the second highest frequency. In view of Assumption 2.1, this term is negligible. The following result by the author of this thesis can be found in [Str14, Lemma 3.2].

Lemma 2.12. *Let $\nu > 0$, $k \in \mathbb{N}$, and $\tau_0 \subset \mathbb{R}$ be a bounded time interval. Furthermore, let $\tau_1 \supset \overline{\tau_0}$ be an open interval. Then, for all $\phi_1, \dots, \phi_{2k+1} \in L^2(\mathbb{T}^n)$ and dyadic numbers $N_1 \geq \dots \geq N_{2k+1} \geq 1$ there exist finitely many rectangles $\mathcal{R}_{\ell} \in \mathcal{R}_{N_2, M}^n$, where $M := \max\{\frac{N_2^2}{N_1}, 1\}$, with the properties that $P_{N_1} = \sum_{\ell \in \mathbb{Z}} P_{\mathcal{R}_{\ell}} P_{N_1}$ and*

$$\begin{aligned} \left\| \prod_{j=1}^{2k+1} P_{N_j} e^{it\Delta_{\theta}} \phi_j \right\|_{L^2(\tau_0 \times \mathbb{T}^n)}^2 &\lesssim \sum_{\ell \in \mathbb{Z}} \left\| P_{\mathcal{R}_{\ell}} P_{N_1} e^{it\Delta_{\theta}} \phi_1 \prod_{j=2}^{2k+1} P_{N_j} e^{it\Delta_{\theta}} \phi_j \right\|_{L^2(\tau_1 \times \mathbb{T}^n)}^2 \\ &\quad + N_2^{-\nu} \prod_{j=1}^{2k+1} \|P_{N_j} \phi_j\|_{L^2(\mathbb{T}^n)}^2. \end{aligned}$$

Proof. Note that we may assume $N_1 \gg N_2$.

Step 1. We show that due to spatial almost orthogonality, it suffices to prove the desired estimate in the case

$$P_{\mathcal{C}} P_{N_1} e^{it\Delta_{\theta}} \phi_1 = P_{N_1} e^{it\Delta_{\theta}} \phi_1, \quad (2.28)$$

where $\mathcal{C} \in \mathcal{C}_{N_2}^n$. To prove this, we consider a partition of \mathbb{Z}^n into countably many, disjoint cubes in \mathbb{Z}^n of size N_2 :

$$\mathbb{Z}^n = \bigcup_{\ell \in \mathbb{Z}} \mathcal{C}_{\ell}, \quad \mathcal{C}_{\ell} \in \mathcal{C}_{N_2}^n.$$

We claim that for fixed $t \in \tau_0$,

$$\left\| \prod_{j=1}^{2k+1} P_{N_j} e^{it\Delta_{\theta}} \phi_j \right\|_{L^2(\mathbb{T}^n)}^2 \approx \sum_{\ell \in \mathbb{Z}} \left\| P_{\mathcal{C}_{\ell}} P_{N_1} e^{it\Delta_{\theta}} \phi_1 \prod_{j=2}^{2k+1} P_{N_j} e^{it\Delta_{\theta}} \phi_j \right\|_{L^2(\mathbb{T}^n)}^2.$$

Indeed, fix any $t \in \tau_0$. For a given $\ell \in \mathbb{Z}$ there are only finitely many (independent of N_j , $j = 1, \dots, 2k+1$) $\tilde{\ell} \in \mathbb{Z}$ such that

$$\left\langle P_{C_\ell} P_{N_1} e^{it\Delta_\theta} \phi_1 \prod_{j=2}^{2k+1} P_{N_j} e^{it\Delta_\theta} \phi_j, P_{C_{\tilde{\ell}}} P_{N_1} e^{it\Delta_\theta} \phi_1 \prod_{j=2}^{2k+1} P_{N_j} e^{it\Delta_\theta} \phi_j \right\rangle_{L^2(\mathbb{T}^n)} \neq 0. \quad (2.29)$$

We consider lattice points $\xi_1 \in \text{supp } \mathcal{F}(P_{C_\ell} P_{N_1} e^{it\Delta_\theta} \phi_1)$, $\tilde{\xi}_1 \in \text{supp } \mathcal{F}(P_{C_{\tilde{\ell}}} P_{N_1} e^{it\Delta_\theta} \phi_1)$, and $\xi_j, \tilde{\xi}_j \in \mathcal{N}_j := \text{supp } \mathcal{F}(P_{N_j} e^{it\Delta_\theta} \phi_j)$, $j = 2, \dots, 2k+1$. Then, (2.29) follows from the fact that

$$\int_{\mathbb{T}^n} e^{ix \cdot [\xi_1 + \dots + \xi_{2k+1} - (\tilde{\xi}_1 + \dots + \tilde{\xi}_{2k+1})]} dx = 0$$

whenever the distance of C_ℓ and $C_{\tilde{\ell}}$ is larger than $4kN_2$. Therefore, we may assume (2.28).

Step 2. As in the proof of [HTT11, Proposition 3.5], we define the following partition: Let ξ_0 be the center of \mathcal{C} and define disjoint strips of width $M := \max\{\frac{N_2}{N_1}, 1\}$ which are orthogonal to ξ_0 :

$$\mathcal{R}_\ell := \{\xi \in \mathcal{C} : \xi \cdot \xi_0 \in [|\xi_0|M\ell, |\xi_0|M(\ell+1))\} \in \mathcal{R}_{N_2, M}^n.$$

We conclude from the construction that $\angle(\xi, \xi_0) \lesssim \frac{N_2}{N_1}$ for all $\xi \in \mathcal{R}_\ell$. Since $N_1 \gg N_2$, we have $\angle(\xi, \xi_0) \leq \frac{1}{2}$. Therefore,

$$\xi \cdot \xi_0 = |\xi||\xi_0| \cos \angle(\xi, \xi_0) \approx N_1^2,$$

which implies that $\ell \geq 0$ and $\ell \approx \frac{N_1}{M}$ because $|\xi_0| \approx N_1$. Since $\mathcal{C} = \dot{\bigcup}_{\ell \in \mathbb{Z}} \mathcal{R}_\ell$, we clearly have $P_{\mathcal{C}} P_{N_1} e^{it\Delta_\theta} \phi_1 = \sum_{\ell \in \mathbb{Z}} P_{\mathcal{R}_\ell} P_{N_1} e^{it\Delta_\theta} \phi_1$.

Let $\chi \in C_0^\infty(\mathbb{R})$ be a non-negative cut-off function satisfying $\chi(t) = 1$ for all $t \in \tau_0$ and $\chi(t) = 0$ for all $t \in \mathbb{R} \setminus \tau_1$. Obviously,

$$\left\| \prod_{j=1}^{2k+1} P_{N_j} e^{it\Delta_\theta} \phi_j \right\|_{L^2(\tau_0 \times \mathbb{T}^n)}^2 \leq \left\| \sqrt{\chi}(t) \prod_{j=1}^{2k+1} P_{N_j} e^{it\Delta_\theta} \phi_j \right\|_{L^2(\tau_1 \times \mathbb{T}^n)}^2 \lesssim I_1 + I_2,$$

where

$$I_1 := \sum_{\ell \approx N_1/M} \left\| P_{\mathcal{R}_\ell} P_{N_1} e^{it\Delta_\theta} \phi_1 \prod_{j=2}^{2k+1} P_{N_j} e^{it\Delta_\theta} \phi_j \right\|_{L^2(\tau_1 \times \mathbb{T}^n)}^2,$$

and I_2 is defined as

$$\sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \left\langle \chi(t) P_{\mathcal{R}_\ell} P_{N_1} e^{it\Delta_\theta} \phi_1 \prod_{j=2}^{2k+1} P_{N_j} e^{it\Delta_\theta} \phi_j, P_{\mathcal{R}_{\tilde{\ell}}} P_{N_1} e^{it\Delta_\theta} \phi_1 \prod_{j=2}^{2k+1} P_{N_j} e^{it\Delta_\theta} \phi_j \right\rangle_{L^2(\mathbb{R} \times \mathbb{T}^n)}.$$

We are left to show that

$$|I_2| \lesssim N_2^{-\nu} \|P_{\mathcal{C}} P_{N_1} \phi_1\|_{L^2(\mathbb{T}^n)}^2 \prod_{j=2}^{2k+1} \|P_{N_j} \phi_j\|_{L^2(\mathbb{T}^n)}^2.$$

Since we extended the integration with respect to t to \mathbb{R} , we may interpret this integration as Fourier transform on \mathbb{R} . Then, taking the absolute value, we end up with

$$|I_2| \lesssim \sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \sum_{\substack{n_1 \in \mathcal{R}_\ell, \tilde{n}_1 \in \mathcal{R}_{\tilde{\ell}}, \\ n_j, \tilde{n}_j \in \mathcal{N}_j, j=2, \dots, 2k+1}} |\mathcal{F}_{\mathbb{R}^n}(\chi)| \left(\sum_{j=1}^{2k+1} (Q(n_j) - Q(\tilde{n}_j)) \right) \prod_{j=1}^{2k+1} |\hat{\phi}_j(n_j)| |\overline{\hat{\phi}_j(\tilde{n}_j)}|. \quad (2.30)$$

Similarly to the proof of [HTT11, Proposition 3.5], we get

$$\left| \sum_{j=1}^{2k+1} (Q(n_j) - Q(\tilde{n}_j)) \right| = M^2 |\ell - \tilde{\ell}| (\ell + \tilde{\ell}) + \mathcal{O}(M^2 \ell) + \mathcal{O}(M^2 \tilde{\ell}) \gtrsim N_2^2 \langle \ell - \tilde{\ell} \rangle$$

since $\ell, \tilde{\ell} \approx \frac{N_1}{M}$ and $|\ell - \tilde{\ell}| \gg 1$. Thus, for any $\mu > 0$ we may estimate

$$|\mathcal{F}_{\mathbb{R}^n}(\chi)| \left(\sum_{j=1}^{2k+1} (Q(n_j) - Q(\tilde{n}_j)) \right) \lesssim_{\mu} N_2^{-2\mu} \langle \ell - \tilde{\ell} \rangle^{-\mu}.$$

Using Cauchy–Schwarz with respect to $n_j, \tilde{n}_j, j = 1, \dots, 2k+1$, on the right-hand side of (2.30) yields

$$|I_2| \lesssim N_2^{-\nu} \sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \langle \ell - \tilde{\ell} \rangle^{-\mu} \|P_{\mathcal{R}_\ell} \phi_1\|_{L^2(\mathbb{T}^n)} \|P_{\mathcal{R}_{\tilde{\ell}}} \phi_1\|_{L^2(\mathbb{T}^n)} \prod_{j=2}^{2k+1} \|P_{N_j} \phi_j\|_{L^2(\mathbb{T}^n)}^2$$

provided $\nu \leq 2\mu - (2k+1)n$. Finally, Schur’s lemma implies

$$\sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \langle \ell - \tilde{\ell} \rangle^{-\mu} \|P_{\mathcal{R}_\ell} \phi_1\|_{L^2(\mathbb{T}^n)} \|P_{\mathcal{R}_{\tilde{\ell}}} \phi_1\|_{L^2(\mathbb{T}^n)} \lesssim \|P_{\mathcal{C}} P_{N_1} \phi_1\|_{L^2(\mathbb{T}^n)}^2$$

provided $\mu > 1$. This finishes the proof. \square

2.3.5 The trilinear Strichartz estimate

The linear Strichartz estimates in Lemma 2.10 and Corollary 2.11 as well as the almost orthogonality in Lemma 2.12 allow us to prove the desired trilinear L^2 -estimate.

Proposition 2.13. *Let $\tau \subset \mathbb{R}$ be a bounded time interval. There exists $\delta > 0$ such that for all $\phi_1, \phi_2, \phi_3 \in L^2(\mathbb{T}^3)$ and dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ the following estimate holds true:*

$$\left\| \prod_{j=1}^3 P_{N_j} e^{it\Delta_{\theta}} \phi_j \right\|_{L^2(\tau_0 \times \mathbb{T}^3)} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2} \right)^{\delta} N_2 N_3 \prod_{j=1}^3 \|P_{N_j} \phi_j\|_{L^2(\mathbb{T}^3)}.$$

Proof. From Lemma 2.12, we see that we may replace the projector P_{N_1} by $P_{\mathcal{R}} P_{N_1}$ with $\mathcal{R} \in \mathcal{R}_{N_2, M}^3$ and $M := \max\{N_2^2/N_1, 1\}$ provided we magnify the time interval τ_0 to an open interval $\tau_1 \supset \overline{\tau_0}$.

Let $p_1 > \frac{16}{3}$ and $4 < q_1 < \frac{3p_1}{4}$. Furthermore, let p_2 and q_2 be defined via the relations $\frac{1}{2} = \frac{2}{p_1} + \frac{1}{p_2}$ and $\frac{1}{2} = \frac{2}{q_1} + \frac{1}{q_2}$, respectively. Hölder’s estimate yields

$$\begin{aligned} & \|P_{\mathcal{R}} P_{N_1} e^{it\Delta_{\theta}} \phi_1 P_{N_2} e^{it\Delta_{\theta}} \phi_2 P_{N_3} e^{it\Delta_{\theta}} \phi_3\|_{L^2(\tau_1 \times \mathbb{T}^3)} \\ & \leq \|P_{\mathcal{R}} P_{N_1} e^{it\Delta_{\theta}} \phi_1\|_{L_t^{p_1} L_x^{q_1}} \|P_{N_2} e^{it\Delta_{\theta}} \phi_2\|_{L_t^{p_2} L_x^{q_2}} \|P_{N_3} e^{it\Delta_{\theta}} \phi_3\|_{L_t^{p_2} L_x^{q_2}}. \end{aligned} \quad (2.31)$$

Applying Lemma 2.10, Corollary 2.11, and Bernstein’s inequality, we infer

$$\begin{aligned} (2.31) & \lesssim M^{\frac{1}{2} - \frac{2}{q_1}} N_2^{\frac{5}{2} - \frac{4}{p_1} - \frac{4}{q_1}} N_3^{\frac{4}{p_1} + \frac{6}{q_1} - 1} \|P_{\mathcal{R}} P_{N_1} \phi_1\|_{L_x^2} \prod_{j=2}^3 \|P_{N_j} \phi_j\|_{L^2(\mathbb{T}^3)} \\ & \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2} \right)^{\frac{1}{2} - \frac{2}{q_1}} N_2^{\frac{7}{2} - \frac{4}{p_1} - \frac{8}{q_1}} N_3^{\frac{4}{p_1} + \frac{8}{q_1} - \frac{3}{2}} \|P_{\mathcal{R}} P_{N_1} \phi_1\|_{L_x^2} \prod_{j=2}^3 \|P_{N_j} \phi_j\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

Then, the claim follows for p_1 sufficiently close to $\frac{16}{3}$ and q_1 sufficiently close to 4. \square

2.4 Rectangular tori in two dimensions

After studying rectangular 3-tori in the previous section, we briefly discuss scaling-critical well-posedness on two-dimensional rectangular tori. A multilinear Strichartz estimate is proved, which implies scaling-critical local well-posedness results by similar arguments as in Section 2.2, cf. also [GOW14, Section 5] and the references therein: Define appropriate iteration spaces that in which one may control the Duhamel term, cf. Lemma 2.9. Then, a fixed-point argument similar to the proof of Theorem 2.3 proves local well-posedness. Hence, Proposition 2.17 leads to:

Theorem 2.14. *Let s_c be defined by (2.22) and let $3 \leq k \in \mathbb{N}$. Then, for all $s \geq s_c$ the initial value problem (2.21) is locally well-posed in $H^s(\mathbb{T}_\theta^2)$.*

We refer to Theorem 2.3 for a precise formulation of this theorem. Scaling-critical small data global well-posedness can not be concluded as in three dimensions since the energy and H^{s_c} -norm scale differently. Hence, the conservation of energy can not be exploited as in the proof of Theorem 2.3.

Theorem 2.14 extends previous results of Guo–Oh–Wang [GOW14, Theorem 1.5] who proved the same result for $k \geq 6$. This is accomplished by using a new trilinear Strichartz estimate which serves as an improved replacement for applying Hölder’s inequality and linear Strichartz estimates. The result is already published in [Str14, Section 3].

We use the notation introduced in Section 2.3.2 and consider (2.23) on \mathbb{T}^2 instead, with the modified Laplace–Beltrami operator Δ_θ given by (2.24).

First, the following trilinear Strichartz estimate is proved by using ideas of [Bou07]. This improves [GOW14, Lemma 5.9] (for $d = 2$). The main point here is that we do not get any factor of the highest frequency.

Lemma 2.15. *Let $2 < p \leq 4$. Then, for any $N, M \geq 1$ with $N \geq M$, $\mathcal{C}_1 \in \mathcal{C}_N^2$, $\mathcal{C}_2, \mathcal{C}_3 \in \mathcal{C}_M^2$, and $\phi_1, \phi_2, \phi_3 \in L^2(\mathbb{T}^2)$ we have*

$$\left\| \prod_{j=1}^3 P_{\mathcal{C}_j} e^{it\Delta_\theta} \phi_j \right\|_{L^p(\tau_0, L^2(\mathbb{T}^2))} \lesssim M^{2-\frac{2}{p}} \prod_{j=1}^3 \|P_{\mathcal{C}_j} \phi_j\|_{L^2(\mathbb{T}^2)}.$$

Proof. This proof is a trilinear variant of the proof of [Bou07, Proposition 1.1]. Hence, we omit details and refer the reader also to the proof of Lemma 2.19 below, in which a similar argument is applied. For brevity we write $L_t^p L_x^q := L^p(\tau_0, L^q(\mathbb{T}^2))$ and $L_t^p := L^p(\tau_0)$.

The left-hand side may be estimated by

$$\left[\sum_{a \in \mathbb{Z}^2} \left\| \sum_{\substack{n \in \mathcal{C}_2, \\ m \in \mathcal{C}_3}} \widehat{\phi}_1(a-n-m) \widehat{\phi}_2(n) \widehat{\phi}_3(m) e^{2\pi i(Q(a-n-m)+Q(n)+Q(m))t} \right\|_{L_t^p}^2 \right]^{\frac{1}{2}}$$

using Plancherel’s identity with respect to x and Minkowski’s inequality. Now, applying Hausdorff–Young (Proposition 1.36 (ii)) and setting $c_{j,n} := |\widehat{\phi}_j(n)|$ yields

$$\|\cdots\|_{L_t^p} \lesssim \left[\sum_{k \in \mathbb{Z}} \sum_{|Q(a-n-m)+Q(n)+Q(m)-k| \leq \frac{1}{2}} c_{1,a-n-m} c_{2,n} c_{3,m} \right]^{\frac{p-1}{p-1}} \left[\right]^{\frac{p-1}{p}}.$$

One easily verifies that $|Q(a - n - m) + Q(n) + Q(m) - k| \leq \frac{1}{2}$ may be written as

$$|Q(3\tilde{n} - 2a) + 3Q(\tilde{m}) + 2Q(a) - 6k| \leq 3,$$

where $\tilde{n} := n + m$ and $\tilde{m} := n - m$. Hence,

$$|\{(n, m) \in \mathcal{C}_2 \times \mathcal{C}_3 : |Q(a - n - m) + Q(n) + Q(m) - k| \leq \frac{1}{2}\}| \lesssim |\mathfrak{S}_\ell|,$$

where

$$\mathfrak{S}_\ell := \{(\tilde{n}, \tilde{m}) \in \tilde{\mathcal{C}}_2 \times \tilde{\mathcal{C}}_3 : |Q(\tilde{n}) + 3Q(\tilde{m}) - \ell| \leq 4\},$$

$\ell := [6k - 2Q(a)] \in \mathbb{Z}$, and cubes

$$\tilde{\mathcal{C}}_2 = ([b_1, b_1 + 10M] \times [b_2, b_2 + 10M]) \cap \mathbb{Z}^2 \quad \text{and} \quad \tilde{\mathcal{C}}_3 = ([b_3, b_3 + 10M] \times [b_4, b_4 + 10M]) \cap \mathbb{Z}^2$$

for some $b_1, \dots, b_4 \in \mathbb{Z}$. This observation and applying Hölder's inequality twice yield

$$\|\cdots\|_{L_t^p} \lesssim \left(\sum_{\ell \in \mathbb{Z}} |\mathfrak{S}_\ell|^{\frac{p}{p-2}} \right)^{\frac{p-2}{2p}} \left(\sum_{n \in \mathcal{C}_2, m \in \mathcal{C}_3} c_{1, a-n-m}^2 c_{2, n}^2 c_{3, m}^2 \right)^{\frac{1}{2}},$$

which in turn implies

$$\left\| \prod_{j=1}^3 P_{\mathcal{C}_j} e^{it\Delta_\theta} \phi_j \right\|_{L^p(\tau_0, L^2(\mathbb{T}^2))} \lesssim \left(\sum_{\ell \in \mathbb{Z}} |\mathfrak{S}_\ell|^{\frac{p}{p-2}} \right)^{\frac{p-2}{2p}} \prod_{j=1}^3 \|P_{\mathcal{C}_j} \phi_j\|_{L^2(\mathbb{T}^2)}.$$

The assumption $p \leq 4$ ensures that $\frac{p}{p-2} \geq 2$. Thus, by [Corollary 1.37](#) and [Corollary 1.39](#), we may estimate

$$\left(\sum_{\ell \in \mathbb{Z}} |\mathfrak{S}_\ell|^{\frac{p}{p-2}} \right)^{\frac{p-2}{p}} \lesssim \left\| \prod_{j=1}^2 \sum_{(\tilde{m}, \tilde{n}) \in \mathcal{C}_j} e^{i\theta_j(\tilde{n}_j^2 + 3\tilde{m}_j^2)t} \right\|_{L_t^{\frac{p}{2}}(I)} \lesssim M^{4(1-\frac{1}{p})}$$

for some compact interval $I \subseteq \mathbb{R}$ provided $p > 2$. This implies the desired estimate. \square

Corollary 2.16. *Let $p > 6$.*

(i) *For every $N \geq 1$, $\mathcal{C} \in \mathcal{C}_N^2$, and $\phi \in L^2(\mathbb{T}^2)$ we have*

$$\|P_{\mathcal{C}} e^{it\Delta_\theta} \phi\|_{L^p(\tau_0, L^6(\mathbb{T}^2))} \lesssim N^{\frac{2}{3} - \frac{2}{p}} \|P_{\mathcal{C}} \phi\|_{L^2(\mathbb{T}^2)}.$$

(ii) *Let $6 \leq q < p$. Then, for all $N, M \geq 1$ with $N \geq M$, $\mathcal{R} \in \mathcal{R}_{N, M}^2$, and $\phi \in L^2(\mathbb{T}^2)$ it holds that*

$$\|P_{\mathcal{R}} e^{it\Delta_\theta} \phi\|_{L^p(\tau_0, L^q(\mathbb{T}^2))} \lesssim N^{\frac{1}{2} + \frac{1}{q} - \frac{2}{p}} M^{\frac{1}{2} - \frac{3}{q}} \|P_{\mathcal{R}} \phi\|_{L^2(\mathbb{T}^2)}.$$

Proof. The first estimate is a direct consequence of [Lemma 2.15](#) provided $p \leq 12$. The estimate

$$\|P_{\mathcal{C}} e^{it\Delta_\theta} \phi\|_{L^\infty(\tau_0 \times \mathbb{T}^2)} \lesssim N \|P_{\mathcal{C}} \phi\|_{L^2(\mathbb{T}^2)}$$

is trivial from Cauchy–Schwarz. For $12 < p < \infty$, the desired estimate follows from Hölder's inequality and the estimates for $p = 12$ and $p = \infty$.

The second statement follows from (i), the estimate

$$\|P_{\mathcal{R}} e^{it\Delta_\theta} \phi\|_{L^\infty(\tau_0 \times \mathbb{T}^2)} \leq |\mathcal{R} \cap \mathbb{Z}^2|^{\frac{1}{2}} \|P_{\mathcal{R}} \phi\|_{L^2(\mathbb{T}^2)} \lesssim (NM)^{\frac{1}{2}} \|P_{\mathcal{R}} \phi\|_{L^2(\mathbb{T}^2)},$$

which may easily be obtained by applying Cauchy–Schwarz in Fourier space, and Hölder's inequality. The conclusion works as follows: Set $f(t, x) := |P_{\mathcal{R}} e^{it\Delta_\theta} \phi(x)|$, $\varepsilon := \frac{6p}{q} - 6 > 0$, and $\vartheta := \frac{6}{q} \leq 1$. Then,

$$\|P_{\mathcal{R}} e^{it\Delta_\theta} \phi\|_{L_t^p L_x^q} = \|f^\vartheta f^{1-\vartheta}\|_{L_t^p L_x^q} \leq \|f\|_{L_t^{\vartheta+\varepsilon} L_x^6}^\vartheta \|f\|_{L_t^\infty}^{1-\vartheta} \lesssim N^{\frac{1}{2} + \frac{1}{q} - \frac{2}{p}} M^{\frac{1}{2} - \frac{3}{q}} \|P_{\mathcal{R}} \phi\|_{L^2(\mathbb{T}^2)}. \quad \square$$

Now, we prove the multilinear estimate from which the claimed well-posedness result follows by standard arguments.

Proposition 2.17. *Let $k \geq 3$. There exists $\delta > 0$ such that for all $\phi_1, \dots, \phi_{k+1} \in L^2(\mathbb{T}^2)$ and dyadic numbers $N_1 \geq \dots \geq N_{k+1} \geq 1$ the following estimate holds true*

$$\left\| \prod_{j=1}^{k+1} P_{N_j} e^{it\Delta_\theta} \phi_j \right\|_{L^2(\tau_0 \times \mathbb{T}^2)} \lesssim \left(\frac{N_{k+1}}{N_1} + \frac{1}{N_2} \right)^\delta \|P_{N_1} \phi_1\|_{L^2(\mathbb{T}^2)} \prod_{j=2}^{k+1} N_j^{s_c} \|P_{N_j} \phi_j\|_{L^2(\mathbb{T}^2)}.$$

Proof. Thanks to the almost orthogonality argument in Lemma 2.12, it suffices to replace $P_{N_1} e^{it\Delta_\theta} \phi_1$ by $P_{\mathcal{R}} P_{N_1} e^{it\Delta_\theta} \phi_1$, where $\mathcal{R} \in \mathcal{R}_{N_2, M}^2$ with $M = \max\{N_2^2/N_1, 1\}$ provided we magnify the time interval to an open interval $\tau_1 \supset \overline{\tau_0}$.

Let $6 < p_1, q_1 < 8$ and $3 < p_2 \leq \frac{24}{5}$. Furthermore, let p_3 and q_2 be defined via the relations $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{k-2}{p_3}$ and $\frac{1}{2} = \frac{1}{q_1} + \frac{1}{3} + \frac{k-2}{q_2}$, respectively. By Hölder's estimate the following holds true:

$$\begin{aligned} \left\| P_{\mathcal{R}} P_{N_1} e^{it\Delta_\theta} \phi_1 \prod_{j=2}^{k+1} P_{N_j} e^{it\Delta_\theta} \phi_j \right\|_{L_{t,x}^2} &\leq \|P_{\mathcal{R}} P_{N_1} e^{it\Delta_\theta} \phi_1\|_{L_t^{p_1} L_x^{q_1}} \|P_{N_2} e^{it\Delta_\theta} \phi_2 P_{N_3} e^{it\Delta_\theta} \phi_3\|_{L_t^{p_2} L_x^3} \\ &\quad \times \prod_{j=4}^{k+1} \|P_{N_j} e^{it\Delta_\theta} \phi_j\|_{L_t^{p_3} L_x^{q_2}}, \end{aligned} \tag{2.32}$$

where $L_t^r L_x^s := L^r(\tau_1, L^s(\mathbb{T}^2))$ and $L_{t,x}^2 := L_t^2 L_x^2$. Let $f_j := |P_{N_j} e^{it\Delta_\theta} \phi_j|$, $j = 2, 3$. Then we treat the bilinear term as follows:

$$\|f_2 f_3\|_{L_t^{p_2} L_x^3}^2 = \|f_2^2 f_3^2\|_{L_t^{\frac{p_2}{2}} L_x^{\frac{3}{2}}} \leq \|f_2 f_3\|_{L_t^r L_x^s} \|f_2\|_{L_t^s L_x^6},$$

where $s > 6$ and $\frac{2}{p_2} = \frac{1}{r} + \frac{1}{s}$. Note that $p_2 \leq \frac{24}{5}$ ensures that $r \leq 4$. By Lemma 2.15 and Corollary 2.16, we have for all $\eta > 0$,

$$\|P_{N_2} e^{it\Delta_\theta} \phi_2 P_{N_3} e^{it\Delta_\theta} \phi_3\|_{L_t^{p_2} L_x^3} \leq N_2^{\frac{1}{6} + \eta} N_3^{\frac{7}{6} - \frac{2}{p_2} - \eta}. \tag{2.33}$$

Corollary 2.16, (2.33), and Bernstein's inequality imply

$$\begin{aligned} (2.32) &\lesssim \left(\frac{M}{N_2} \right)^{\frac{1}{2} - \frac{3}{q_1}} N_2^{\frac{7}{6} - \frac{2}{p_1} - \frac{2}{q_1} + \eta} N_3^{\frac{7}{6} - \frac{2}{p_2} - \eta} \prod_{j=4}^{k+1} N_j^{1 - \frac{2}{k-2} (\frac{2}{3} - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{q_1})} \\ &\quad \times \|P_{\mathcal{R}} P_{N_1} \phi_1\|_{L^2(\mathbb{T}^2)} \prod_{j=2}^{k+1} \|P_{N_j} \phi_j\|_{L^2(\mathbb{T}^2)}. \end{aligned}$$

For all $0 < \nu_1, \nu_2 \ll 1$, there exist $\delta > 0$ and $p_1, q_1 > 6$ sufficiently close to 6 as well as $p_2 > 3$ sufficiently close to 3 such that

$$\begin{aligned} \text{(i)} \quad \left(\frac{M}{N_2} \right)^{\frac{1}{2} - \frac{3}{q_1}} &= \left(\frac{M}{N_2} \right)^\delta, & \text{(ii)} \quad N_2^{\frac{7}{6} - \frac{2}{p_1} - \frac{2}{q_1} + \eta} &= N_2^{\frac{1}{2} + \nu_1 + \eta}, \\ \text{(iii)} \quad N_3^{\frac{7}{6} - \frac{2}{p_2} - \eta} &= N_3^{\frac{1}{2} + \nu_2 - \eta}, & \text{(iv)} \quad N_j^{1 - \frac{2}{k-2} (\frac{2}{3} - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{q_1})} &= N_j^{1 - \frac{\nu_1 + \nu_2}{k-2}}, \end{aligned}$$

where $j \in \{4, \dots, k+1\}$. Since $\frac{1}{2} < s_c < 1$ for $k \geq 3$, we may choose $0 < \nu_1, \nu_2, \eta \ll 1$ small enough to get

$$\left\| P_{\mathcal{R}} P_{N_1} e^{it\Delta_\theta} \phi_1 \prod_{j=2}^{k+1} P_{N_j} e^{it\Delta_\theta} \phi_j \right\|_{L_{t,x}^2} \lesssim \left(\frac{N_{k+1}}{N_1} + \frac{1}{N_2} \right)^\delta \|P_{\mathcal{R}} P_{N_1} \phi_1\|_{L_x^2} \prod_{j=2}^{k+1} N_j^{s_c} \|P_{N_j} \phi_j\|_{L_x^2},$$

where $L_x^2 := L^2(\mathbb{T}^2)$. \square

2.5 Product of spheres

[Assumption 2.1](#) is verified for M being a product of \mathbb{S} with a two-dimensional sphere with arbitrary radius. In this thesis, we give the first proof of this result.

2.5.1 Selected results

There are only few known results about well-posedness on products of spheres. Let $\rho > 0$ and \mathbb{S}_ρ^2 be the embedded sphere of radius ρ in \mathbb{R}^3 , then Burq–Gérard–Tzvetkov [[BGT05b](#), Theorem 1] proved that

$$i\partial_t u + \Delta_g u = |u|^{\alpha-1}u$$

with initial data $\phi \in H^1(\mathbb{S} \times \mathbb{S}_\rho^2)$ is globally well-posed whenever $\alpha < 5$. To accomplish that they proved a weak bilinear estimate [[BGT05b](#), Proposition 5.3], which implies well-posedness for $1 < \alpha \leq 4$, and a stronger trilinear estimate [[BGT05b](#), Proposition 5.1], which allows to get the well-posedness in the case $\alpha = 5$ but only for data in $H^s(\mathbb{S} \times \mathbb{S}_\rho^2)$ with $s > 1$. A suitable interpolation between those two approaches yields the claimed well-posedness. Moreover, they rely on certain multilinear spectral cluster estimates [[BGT05b](#), Theorem 2], which have been proved by themselves as well.

In a joint work of Herr and the author [[HS15](#), Theorem 1], local well-posedness and small data global well-posedness on $\mathbb{S} \times \mathbb{S}^2$ was established. Using almost orthogonality and replacing the number of lattice points estimate in [[BGT05b](#), Proposition 5.1] by a new exponential sum estimate, it was possible to verify [Assumption 2.1](#) [[HS15](#), Proposition 2.6]. However, the exponential sum estimate in [[HS15](#), Lemma 2.3] can not be extended to handle the case of products of spheres with different radii.

In this section, we are going to replace the exponential sum estimate in [[HS15](#), Lemma 2.3] by [Corollary 2.20](#) and use a more refined almost orthogonality argument to overcome the problems described in [[HS15](#), Remark 1]⁴.

2.5.2 Set-up

We take the notation for the spectrum and the spectral projectors that has been used in [[BGT05b](#), Section 5]: Set $M := \mathbb{S} \times \mathbb{S}_\rho^2$ for brevity. The eigenvalues of $-\Delta := -\Delta_g$ are given by $\{\lambda_{m,n}\}_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0}$, where

$$\lambda_{m,n} := m^2 + \kappa(n^2 + n), \quad (m, n) \in \mathbb{Z} \times \mathbb{N}_0$$

and $\kappa := \rho^{-2}$. This follows simply from the fact that the spectrum of a product manifold equals the sum of the spectra of the individual manifolds, cf. [[Cha84](#), Section 2.1], and the behavior of the eigenvalues under scaling of the underlying manifold, see e.g. [[Han12](#), Section 2.2]. The spectral projector onto spherical harmonics of degree n on \mathbb{S}_ρ^2 shall be denoted by $\Pi_n: L^2(\mathbb{S}_\rho^2) \rightarrow L^2(\mathbb{S}_\rho^2)$. For a function $f: \mathbb{S} \times \mathbb{S}_\rho^2 \rightarrow \mathbb{C}$ we write $\mathbb{S} \times \mathbb{S}_\rho^2 \ni (\theta, \omega) \mapsto f(\theta, \omega)$. For fixed $\omega \in \mathbb{S}_\rho^2$ the m th Fourier coefficient of $f(\cdot, \omega)$ shall be defined by

$$\Theta_m f(\omega) := \frac{1}{2\pi} \int_0^{2\pi} f(\theta, \omega) e^{-im\theta} d\theta, \quad m \in \mathbb{Z}.$$

⁴[[HS15](#), Remark 1] is repeated at the beginning of [Section 2.5.5](#).

For $f \in L^2(M)$ we have the following representation

$$f(\theta, \omega) = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0} e^{im\theta} \Pi_n \Theta_m(f)(\omega)$$

in the L^2 -sense. For dyadic $N \geq 1$ the projectors are given by

$$P_N f(\theta, \omega) = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0} \eta_N(\sqrt{\lambda_{m,n}}) e^{im\theta} \Pi_n \Theta_m(f)(\omega), \quad (\theta, \omega) \in M,$$

where η_N is defined in (1.11). Given a second function $g \in L^2(M)$ and a point-set $\mathcal{S} \subseteq \mathbb{Z}^4$, we define the bilinear projector

$$Q_{\mathcal{S}}(f, g)(\theta, \omega) := \sum_{(m_1, n_1, m_2, n_2) \in \mathcal{S} \cap (\mathbb{Z} \times \mathbb{N}_0)^2} e^{i(m_1+m_2)\theta} \Pi_{n_1} \Theta_{m_1}(f)(\omega) \Pi_{n_2} \Theta_{m_2}(g)(\omega)$$

for $(\theta, \omega) \in M$.

Recall that the Sobolev norm, which was defined in Definition 1.52, is given by

$$\|f\|_{H^s(M)}^2 = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0} \langle \sqrt{\lambda_{m,n}} \rangle^{2s} \|\Pi_n \Theta_m f\|_{L^2(M)}^2 \approx \sum_{N \geq 1} N^{2s} \|P_N f\|_{L^2(M)}^2.$$

In view of (1.20), the linear Schrödinger evolution is given by

$$e^{it\Delta} f(\theta, \omega) = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0} e^{-i\lambda_{m,n}t} e^{im\theta} \Pi_n \Theta_m(f)(\omega).$$

2.5.3 A trilinear estimate for spherical harmonics

The succeeding trilinear estimate for eigenfunctions of the Laplace–Beltrami operator on the 2-sphere is stated in [BGT05b, Theorem 2]. It is deduced as a consequence of the more general trilinear spectral cluster estimate in [BGT05b, Theorem 3] that holds on any two-dimensional, compact, smooth, boundaryless Riemannian manifold. Bilinear and higher-dimensional versions are provided as well.

Proposition 2.18. *There exists $C_\rho > 0$ such that for all integers $n_1 \geq n_2 \geq n_3 \geq 0$ and $f_1, f_2, f_3 \in L^2(\mathbb{S}_\rho^2)$ the following trilinear estimate holds true*

$$\|\Pi_{n_1} f_1 \Pi_{n_2} f_2 \Pi_{n_3} f_3\|_{L^2(\mathbb{S}_\rho^2)} \leq C_\rho (\langle n_2 \rangle \langle n_3 \rangle)^{\frac{1}{4}} \prod_{j=1}^3 \|\Pi_{n_j} f_j\|_{L^2(\mathbb{S}_\rho^2)}.$$

Remark. We want to highlight Remark 2.1 in [BGT05b]. If one is interested in estimating products of single eigenfunctions, the spectral cluster estimates seem only to be relevant for “sphere like manifolds”. On the one hand, they are far from being optimal in the case of the torus. Indeed, the spectral cluster estimate in [BGT05b, formula (2.5)] states that there is a constant $C > 0$ such that for every two eigenfunctions f and g of the Laplace–Beltrami operator on \mathbb{T}^2 with eigenvalues n respectively m ,

$$\|fg\|_{L^2(\mathbb{T}^2)} \leq C \min\{n, m\}^{\frac{1}{2}} \|f\|_{L^2(\mathbb{T}^2)} \|g\|_{L^2(\mathbb{T}^2)},$$

whereas the dual statement of the classical result of Zygmund [Zyg74, Theorem 1] shows that

$$\|fg\|_{L^2(\mathbb{T}^2)} \leq C \|f\|_{L^2(\mathbb{T}^2)} \|g\|_{L^2(\mathbb{T}^2)}.$$

On the other hand, Burq–Gérard–Tzvetkov discussed the optimality of this estimate on \mathbb{S}^2 [BGT05b, Section 2.1]: Choosing the spherical harmonics $R_n := (x_1 + ix_2)^n$, one easily calculates

$$\|R_n\|_{L^2(\mathbb{S}^2)} \approx n^{-\frac{1}{4}}, \quad n \gg 1$$

and $R_{n_1}R_{n_2}R_{n_3} = R_{n_1+n_2+n_3}$ for $n_1, n_2, n_3 \geq 0$. Hence, for $n_1 \geq n_2 \geq n_3 \gg 1$,

$$\|R_{n_1}R_{n_2}R_{n_3}\|_{L^2(\mathbb{S}^2)} \gtrsim (n_1 + n_2 + n_3)^{-\frac{1}{4}} \gtrsim (n_2n_3)^{-\frac{1}{4}} \prod_{j=1}^3 \|R_{n_j}\|_{L^2(\mathbb{S}^2)}.$$

The spectral cluster estimates given in [BGT05b, Theorem 3] have successfully been applied to gain energy-critical well-posedness of the NLS posed on three-dimensional Zoll manifolds, see [Her13, Proposition 3.6]. \diamond

2.5.4 Two exponential sum estimates

The next exponential sum estimate is used for handling a term that arises from the time and \mathbb{S} component of the two high-frequency functions. We are only interested in p close to $\frac{8}{3}$ since it serves as the lower endpoint of the interpolation with an estimate in $L^\infty(\tau_0 \times \mathbb{S})$, which takes the precise size of \mathcal{S} into account. The strategy is similar to the proof of the linear Strichartz estimates by Bourgain for free solutions on \mathbb{T}_θ^3 [Bou07, Proposition 1.1]. This lemma replaces [HS15, Lemma 2.3] and allows to treat the case $\rho \neq 1$.

Lemma 2.19. *Let $\frac{8}{3} < p \leq 4$ and $\tau_0 \subset \mathbb{R}$ be a bounded time interval. Then, there exists a constant $C > 0$ such that for any $a \in \ell^2(\mathbb{Z}^4)$, $N \geq 1$, and all $\mathcal{S} \in \mathcal{C}_N^4$ the estimate*

$$\left\| \sum_{(m_1, n_1, m_2, n_2) \in \mathcal{S}} e^{-i(\lambda_{m_1, n_1} + \lambda_{m_2, n_2})t} e^{i(m_1 + m_2)\theta} a_{m_1, n_1, m_2, n_2} \right\|_{L_t^p(\tau_0, L_\theta^2(\mathbb{S}))} \leq CN^{\frac{3}{2} - \frac{2}{p}} \|a\|_{\ell^2}$$

holds true.

Proof. For $p \geq 2$ Plancherel's identity with respect to θ as well as Minkowski's inequality allow to estimate the left-hand side by

$$\begin{aligned} & \left\| \left[\sum_{\xi \in \mathbb{Z}} \left| \sum_{(m_1, n_1, \xi - m_1, n_2) \in \mathcal{S}} e^{-i(\lambda_{m_1, n_1} + \lambda_{\xi - m_1, n_2})t} a_{m_1, n_1, \xi - m_1, n_2} \right|^2 \right]^{\frac{1}{2}} \right\|_{L_t^p} \\ & \leq \left[\sum_{\xi \in \mathbb{Z}} \left\| \sum_{(m_1, n_1, \xi - m_1, n_2) \in \mathcal{S}} e^{-i(\lambda_{m_1, n_1} + \lambda_{\xi - m_1, n_2})t} a_{m_1, n_1, \xi - m_1, n_2} \right\|_{L_t^p}^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (2.34)$$

Fix $\xi \in \mathbb{Z}$. An application of the Hausdorff–Young inequality, see Proposition 1.36 (ii), yields

$$\|\cdots\|_{L_t^p} \lesssim \left[\sum_{\tau \in \mathbb{N}_0} \left\| \sum_{\substack{(m_1, n_1, \xi - m_1, n_2) \in \mathcal{S}: \\ |\lambda_{m_1, n_1} + \lambda_{\xi - m_1, n_2} - \tau| \leq \frac{1}{2}}} |a_{m_1, n_1, \xi - m_1, n_2}| \right\|_{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}. \quad (2.35)$$

By rewriting $|\lambda_{m_1, n_1} + \lambda_{\xi - m_1, n_2} - \tau| \leq \frac{1}{2}$ as

$$|2(2m_1 - \xi)^2 + \kappa(2n_1 + 1)^2 + \kappa(2n_2 + 1)^2 - (4\tau - 2\xi^2 + 2\kappa)| \leq 2,$$

we observe that there exists some rectangular set $\mathcal{C} = a + [0, 10N]^3$ such that for

$$\mathfrak{S}_{\tau,\xi} := \{(m, n, \tilde{n}) \in \mathcal{C} : |2m^2 + \kappa n^2 + \kappa \tilde{n}^2 - (4\tau - 2\xi^2 + 2\kappa)| \leq 2\}$$

we have

$$|\{(m_1, n_1, \xi - m_1, n_2) \in \mathcal{S} : |\lambda_{m_1, n_1} + \lambda_{\xi - m_1, n_2} - \tau| \leq \frac{1}{2}\}| \leq |\mathfrak{S}_{\tau,\xi}|.$$

Thus, applying Hölder's inequality twice, we get

$$\begin{aligned} (2.35) &\lesssim \left[\sum_{\tau \in \mathbb{N}_0} |\mathfrak{S}_{\tau,\xi}|^{\frac{p}{2(p-1)}} \left(\sum_{\substack{(m_1, n_1, \xi - m_1, n_2) \in \mathcal{S}: \\ |\lambda_{m_1, n_1} + \lambda_{\xi - m_1, n_2} - \tau| \leq \frac{1}{2}}} |a_{m_1, n_1, \xi - m_1, n_2}|^2 \right)^{\frac{p}{2(p-1)}} \right]^{\frac{p-1}{p}} \\ &\lesssim \left(\sum_{\tau \in \mathbb{N}_0} |\mathfrak{S}_{\tau,\xi}|^{\frac{p}{p-2}} \right)^{\frac{p-2}{2p}} \left[\sum_{(m_1, n_1, \xi - m_1, n_2) \in \mathcal{S}} |a_{m_1, n_1, \xi - m_1, n_2}|^2 \right]^{\frac{1}{2}} \end{aligned}$$

since the inner sum is essentially disjoint for different values of τ . Plugging this into (2.34) provides the bound

$$(2.34) \lesssim \sup_{\xi \in \mathbb{Z}} \left(\sum_{\tau \in \mathbb{N}_0} |\mathfrak{S}_{\tau,\xi}|^{\frac{p}{p-2}} \right)^{\frac{p-2}{2p}} \|a\|_{\ell^2}.$$

Hence, we are left to estimate the first term on the right-hand side by $N^{\frac{3}{2} - \frac{2}{p}}$. Since $p \leq 4$, we have $\frac{p}{p-2} \geq 2$, and we may apply [Corollary 1.37](#) to estimate

$$\left(\sum_{\tau \in \mathbb{N}_0} |\mathfrak{S}_{\tau,\xi}|^{\frac{p}{p-2}} \right)^{\frac{p-2}{p}} \lesssim \left\| \sum_{(m, n, \tilde{n}) \in \mathcal{C}} e^{i(2m^2 + \kappa n^2 + \kappa \tilde{n}^2)t} \right\|_{L_t^{\frac{p}{2}}(I)}$$

for some compact interval $I \subset \mathbb{R}$. Due to the rectangular structure of \mathcal{C} , the sum can be factorized and Hölder's estimate leads to

$$\begin{aligned} \left(\sum_{\tau \in \mathbb{N}_0} |\mathfrak{S}_{\tau,\xi}|^{\frac{p}{p-2}} \right)^{\frac{p-2}{p}} &\lesssim \left\| \sum_{m \in a_1 + [0, 10N]} e^{2im^2 t} \right\|_{L_t^{\frac{3p}{2}}(I)} \left\| \sum_{n \in a_2 + [0, 10N]} e^{\kappa i n^2 t} \right\|_{L_t^{\frac{3p}{2}}(I)} \\ &\quad \times \left\| \sum_{\tilde{n} \in a_3 + [0, 10N]} e^{\kappa i \tilde{n}^2 t} \right\|_{L_t^{\frac{3p}{2}}(I)} \end{aligned}$$

uniformly in ξ . Since $p > \frac{8}{3}$, we have $\frac{3p}{2} > 4$ and [Corollary 1.39](#) yields

$$\sup_{\xi \in \mathbb{Z}} \left(\sum_{\tau \in \mathbb{N}_0} |\mathfrak{S}_{\tau,\xi}|^{\frac{p}{p-2}} \right)^{\frac{p-2}{2p}} \lesssim N^{\frac{3}{2} - \frac{2}{p}}$$

as asserted. \square

As mentioned before, interpolating with $L^\infty(\tau_0 \times \mathbb{S})$ leads to the next estimate we shall rely on later. The factor of $|\mathcal{S}|$ plays a crucial role in the upcoming arguments.

Corollary 2.20. *Let $p > \frac{8}{3}$, $2 \leq q < \frac{3p}{4}$ and $\tau_0 \subset \mathbb{R}$ be a bounded time interval. Then, there exists $C > 0$ such that for any $a \in \ell^2(\mathbb{Z}^4)$, $N \geq 1$, and all sets $\mathcal{S} \in \mathcal{C}_N^4$ the estimate*

$$\left\| \sum_{(m_1, n_1, m_2, n_2) \in \mathcal{S}} e^{-i(\lambda_{m_1, n_1} + \lambda_{m_2, n_2})t} e^{i(m_1 + m_2)\theta} a_{m_1, n_1, m_2, n_2} \right\|_{L_t^p(\tau_0, L_\theta^q(\mathbb{S}))} \leq CN^{\frac{3}{q} - \frac{2}{p}} |\mathcal{S}|^{\frac{1}{2} - \frac{1}{q}} \|a\|_{\ell^2}$$

holds.

Proof. For brevity set

$$f(t, \theta) := \left| \sum_{(m_1, n_1, m_2, n_2) \in \mathcal{S}} e^{-i(\lambda_{m_1, n_1} + \lambda_{m_2, n_2})t} e^{i(m_1 + m_2)\theta} a_{m_1, n_1, m_2, n_2} \right|.$$

Let $\varepsilon := \frac{2p}{q} - \frac{8}{3} > 0$ and $\vartheta := \frac{2}{q} \leq 1$. By Hölder's inequality and Lemma 2.19 we have

$$\|f\|_{L_t^p L_\theta^q} = \|f^\vartheta f^{1-\vartheta}\|_{L_t^p L_\theta^q} \leq \|f\|_{L_t^{\frac{8}{3}+\varepsilon} L_\theta^2}^\vartheta \|f\|_{L_{t,\theta}^\infty}^{1-\vartheta} \lesssim N^{\frac{3}{q}-\frac{2}{p}} |\mathcal{S}|^{\frac{1}{2}-\frac{1}{q}} \|a\|_{\ell^2}.$$

Here, we used

$$\|f\|_{L_{t,\theta}^\infty} \leq |\mathcal{S}|^{\frac{1}{2}} \|a\|_{\ell^2},$$

which follows immediately from the Cauchy–Schwarz inequality. \square

2.5.5 Almost orthogonality

The subsequent lemmas exploit almost orthogonality in space and time. In contrast to what has been done before, we gain some factor of the lowest frequency (see (2.37)) from the two high-frequency terms in Lemma 2.21. Since we need to get a factor $N_1^{-\delta}$, this seems to work only in the case $N_1 \leq N_3^2$. The idea of using this kind of almost orthogonality to achieve a spectral localization of the two high-frequency terms in terms of the lowest frequency seems to be new in this context. Without such an argument it is not obvious how one could obtain a sufficiently high power of N_3 in Proposition 2.24 using the estimate in Lemma 2.19. To show this, we repeat the author's argument in [HS15, Remark 1]: We start with a trilinear $L^2(\tau_0 \times M)$ estimate and proceed as in the proof of Proposition 2.24 until (2.43). Then, using Hölder's inequality to put the two functions with the highest frequencies to $L_t^{16/3+} L_\theta^4$, and thus, the function with the lowest frequency, say N_3 , to $L_t^{8-} L_\theta^\infty$. We treat the latter term as follows: Applying Bernstein's inequality to bound it by the $L_t^{8-} L_\theta^4$ -norm gives a factor $N_3^{1/4}$. The exponential sum estimate in Lemma 2.19 gives $N_3^{1/2-}$ and from the trilinear estimate for spherical harmonics we get another $N_3^{1/4}$ as in (2.43). All in all, we obtain N_3^{-} , and hence, the power on the lowest frequency is too low to conclude well-posedness from Section 2.2.1.

The remaining case $N_1 > N_3^2$ is treated in Lemma 2.22. By exploiting almost orthogonality in space and time, we restrict the spectrum only of the high-frequency term. In the proof of Proposition 2.24 below, it turns out that this case is in fact sub-critical.

Given dyadic numbers $N_1, N_2, N_3 \geq 1$, we define the point-sets

$$\mathcal{N}_j := \{(m, n) \in \mathbb{Z} \times \mathbb{N}_0 : \eta_{N_j}(\sqrt{\lambda_{m,n}}) > 0\}, \quad j = 1, 2, 3. \quad (2.36)$$

Lemma 2.21. *Let $\nu > 0$ and $\tau_0 \subset \mathbb{R}$ be a bounded interval. Furthermore, let $\tau_1 \supset \overline{\tau_0}$ be an open interval. Then, for all $\phi_1, \phi_2, \phi_3 \in L^2(M)$ and dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ with $N_1 \leq N_3^2$ there are finitely many sets $\mathcal{S}_\ell \subseteq \mathcal{N}_1 \times \mathcal{N}_2$ of size*

$$|\mathcal{S}_\ell| \lesssim \min_{0 \leq \delta \leq 1} N_1^{-\delta} N_2^{1+2\delta} N_3^{3-\delta}, \quad (2.37)$$

with the property $\mathcal{N}_1 \times \mathcal{N}_2 = \dot{\bigcup}_{\ell \in \mathbb{Z}} \mathcal{S}_\ell$ such that

$$\begin{aligned} \left\| \prod_{j=1}^3 P_{N_j} e^{it\Delta} \phi_j \right\|_{L^2(\tau_0 \times M)}^2 &\lesssim \sum_{\ell \in \mathbb{Z}} \|Q_{\mathcal{S}_\ell}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_1 \times M)}^2 \\ &\quad + N_2^{\frac{3}{2}} N_3^{-\nu} \prod_{j=1}^3 \|P_{N_j} \phi_j\|_{L^2(M)}^2. \end{aligned}$$

Proof. We prove this result in four steps. In the first three steps, we exploit almost orthogonality in the \mathbb{S} and the \mathbb{S}_ρ^2 component, respectively. We then use almost orthogonality in time to conclude the claim. Note that we may assume $N_1 \gg N_2$.

In this proof, we agree on the notation

$$\sum_{\mathcal{A}} := \sum_{(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{A}} \quad \text{and} \quad \sum_{\substack{\mathcal{A}, \\ \mathcal{B}}} := \sum_{\substack{(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{A}, \\ (\tilde{m}_1, \tilde{n}_1, \tilde{m}_2, \tilde{n}_2, \tilde{m}_3, \tilde{n}_3) \in \mathcal{B}}} \quad (2.38)$$

for given sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{Z}^6$. First, we recall that for $t \in \tau_0$ and $(\theta, \omega) \in \mathbb{S} \times \mathbb{S}_\rho^2$,

$$\prod_{j=1}^3 P_{N_j} e^{it\Delta} \phi_j(\theta, \omega) = \sum_{\mathcal{N}_1 \times \mathcal{N}_2 \times \mathcal{N}_3} \prod_{j=1}^3 \eta_{N_j}(\sqrt{\lambda_{m_j, n_j}}) e^{-i\lambda_{m_j, n_j} t} e^{im_j \theta} \Pi_{n_j} \Theta_{m_j} \phi_j(\omega).$$

Step 1. Due to spatial almost orthogonality induced by the \mathbb{S} component, it suffices to prove the desired estimate in the case

$$P_{\mathcal{R}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3,$$

where $\mathcal{R} \subseteq \mathcal{N}_1 \cap ([b, b + N_2] \times \mathbb{N}_0)$ for some $|b| \leq 2N_1$. To prove that, we consider the partition

$$\mathbb{Z} = \bigcup_{k \in \mathbb{Z}} I_k, \quad \text{where} \quad I_k := [kN_2, (k+1)N_2] \cap \mathbb{Z}.$$

Indeed, for fixed $\omega \in \mathbb{S}_\rho^2$ and $t \in \tau_0$ we can show

$$\left\| \prod_{j=1}^3 P_{N_j} e^{it\Delta} \phi_j(\omega) \right\|_{L^2(\mathbb{S})}^2 \approx \sum_{k \in \mathbb{Z}} \|P_{\mathcal{R}_k} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\omega)\|_{L^2(\mathbb{S})}^2,$$

where $\mathcal{R}_k := \mathcal{N}_1 \cap (I_k \times \mathbb{N}_0)$. For $k, \tilde{k} \in \mathbb{Z}$ we have

$$\begin{aligned} & \langle P_{\mathcal{R}_k} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\omega), P_{\mathcal{R}_{\tilde{k}}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\omega) \rangle_{L^2(\mathbb{S})} \\ &= \sum_{\substack{\mathcal{R}_k \times \mathcal{N}_2 \times \mathcal{N}_3, \\ \mathcal{R}_{\tilde{k}} \times \mathcal{N}_2 \times \mathcal{N}_3}} I_{\mathbf{m}, \mathbf{n}} \prod_{j=1}^3 e^{-i(\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) t} \Pi_{n_j} \Theta_{m_j} \phi_j(\omega) \overline{\Pi_{\tilde{n}_j} \Theta_{\tilde{m}_j} \phi_j(\omega)}, \end{aligned}$$

where $\mathbf{m} := (m_1, m_2, m_3, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$, $\mathbf{n} := (n_1, n_2, n_3, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$, and

$$I_{\mathbf{m}, \mathbf{n}} := \prod_{j=1}^3 \eta_{N_j}(\sqrt{\lambda_{m_j, n_j}}) \eta_{N_j}(\sqrt{\lambda_{\tilde{m}_j, \tilde{n}_j}}) \int_{\mathbb{S}} e^{i(m_1 + m_2 + m_3 - \tilde{m}_1 - \tilde{m}_2 - \tilde{m}_3)\theta} d\theta.$$

Since m_j, \tilde{m}_j , $j = 1, 2, 3$, are integers, we may conclude $I_{\mathbf{m}, \mathbf{n}} = 0$ provided $|k - \tilde{k}| > 8$.

Step 2. Now, we use almost orthogonality that comes from the \mathbb{S}_ρ^2 component. It is well-known that the product of a spherical harmonic of degree m with another of degree ℓ can be expanded in terms of spherical harmonics of degree less or equal to $m + \ell$. Furthermore, two spherical harmonics of different degree are orthogonal in $L^2(\mathbb{S}_\rho^n)$, $n \in \mathbb{N}$. We finally remark that complex conjugation does not change the degree of a spherical harmonic. Details may be found in [SW71, Section VI.2]. These facts applied to \mathbb{S} would lead to the same result

that we obtained in Step 1. In Step 1, however, we wanted to point out that no theory about spherical harmonics is required to conclude almost orthogonality in \mathbb{S} .

We now prove that it suffices to consider the case where n_1 is located in an interval of the size of the second highest frequency N_2 . To this end, we define similarly as above a partition of \mathbb{N}_0 :

$$\mathbb{N}_0 = \bigcup_{k \in \mathbb{N}_0} I_k, \quad \text{where } I_k := [kN_2, (k+1)N_2) \cap \mathbb{N}_0.$$

Fix $\theta \in \mathbb{S}$ and $t \in \tau_0$, then it holds that

$$\begin{aligned} & \|P_{\mathcal{R}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\theta)\|_{L^2(\mathbb{S}_\rho^2)}^2 \\ & \approx \sum_{k \in \mathbb{N}_0} \|P_{\mathcal{C}_k} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\theta)\|_{L^2(\mathbb{S}_\rho^2)}^2, \end{aligned}$$

where $\mathcal{C}_k := \mathcal{R} \cap (\mathbb{Z} \times I_k)$. To see this, let $k, \tilde{k} \in \mathbb{N}_0$ and write

$$\begin{aligned} & \langle P_{\mathcal{C}_k} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\theta), P_{\mathcal{C}_{\tilde{k}}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\theta) \rangle_{L^2(\mathbb{S}_\rho^2)} \\ & = \sum_{\substack{\mathcal{C}_k \times \mathcal{N}_2 \times \mathcal{N}_3, \\ \mathcal{C}_{\tilde{k}} \times \mathcal{N}_2 \times \mathcal{N}_3}} I_{\mathbf{m}, \mathbf{n}} \prod_{j=1}^3 e^{-i(\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j})t} e^{i(m_j - \tilde{m}_j)\theta}, \end{aligned}$$

where $\mathbf{m} := (m_1, m_2, m_3, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$, $\mathbf{n} := (n_1, n_2, n_3, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$, and $I_{\mathbf{m}, \mathbf{n}}$ is defined by

$$I_{\mathbf{m}, \mathbf{n}} := \prod_{j=1}^3 \eta_{N_j}(\sqrt{\lambda_{m_j, n_j}}) \eta_{N_j}(\sqrt{\lambda_{\tilde{m}_j, \tilde{n}_j}}) \left\langle \prod_{j=1}^3 \Pi_{n_j} \Theta_{m_j} \phi_j, \prod_{j=1}^3 \Pi_{\tilde{n}_j} \Theta_{\tilde{m}_j} \phi_j \right\rangle_{L^2(\mathbb{S}_\rho^2)}.$$

Without loss of generality, we may assume $n_1 > \tilde{n}_1$. Then,

$$Y_{\mathbf{m}, \mathbf{n}} := \Pi_{\tilde{n}_1} \Theta_{\tilde{m}_1} \phi_1 \prod_{j=2}^3 \overline{\Pi_{n_j} \Theta_{m_j} \phi_j} \Pi_{\tilde{n}_j} \Theta_{\tilde{m}_j} \phi_j \in L^2(\mathbb{S}_\rho^2)$$

can be expanded in terms of spherical harmonics of degree less or equal to $\tilde{n}_1 + 8N_2$. Hence, if $|k - \tilde{k}| > 8$, then

$$I_{\mathbf{m}, \mathbf{n}} = \prod_{j=1}^3 \eta_{N_j}(\sqrt{\lambda_{m_j, n_j}}) \eta_{N_j}(\sqrt{\lambda_{\tilde{m}_j, \tilde{n}_j}}) \langle \Pi_{n_1} \Theta_{m_1} \phi_1, Y_{\mathbf{m}, \mathbf{n}} \rangle_{L^2(\mathbb{S}_\rho^2)} = 0.$$

As a consequence, we are left to show

$$\begin{aligned} & \left\| P_{\mathcal{C}} P_{N_1} e^{it\Delta} \phi_1 \prod_{j=1}^3 P_{N_j} e^{it\Delta} \phi_j \right\|_{L^2(\tau_0 \times M)}^2 \lesssim \sum_{\ell \in \mathbb{Z}} \|Q_{S_\ell}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_1 \times M)}^2 \\ & \quad + N_2^{\frac{3}{2}} N_3^{-\nu} \|P_{\mathcal{C}_k} P_{N_1} \phi_1\|_{L^2(M)}^2 \prod_{j=2}^3 \|P_{N_j} \phi_j\|_{L^2(M)}^2 \end{aligned}$$

for any fixed $\mathcal{C} := \mathcal{C}_k$, $k \in \mathbb{N}_0$.

Step 3. Analogously as in the first step, we see that

$$\begin{aligned} & \|P_{\mathcal{C}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3(\theta)\|_{L^2(\mathbb{S}_\rho^2)}^2 \\ & \approx \sum_{k \in \mathbb{Z}} \|Q_{\mathcal{R}_k^{(2)}}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\mathbb{S}_\rho^2)}^2, \end{aligned}$$

where $\mathcal{R}_k^{(2)} := \{(m_1, n_1, m_2, n_2) \in \mathcal{C} \times \mathcal{N}_2 : m_1 + m_2 \in [kN_3, (k+1)N_3]\}$. Again, this is a consequence of almost orthogonality in \mathbb{S} . We omit details since the argument is pretty close to Step 1.

Step 4. Let $k \in \mathbb{Z}$ be fixed and $M := N_3^2/N_1 \geq 1$. We define the partition

$$\mathbb{N}_0 = \dot{\bigcup}_{\ell \in \mathbb{N}_0} J_\ell, \quad \text{where } J_\ell := [\ell M, (\ell+1)M) \cap \mathbb{N}_0.$$

Inspired by the proof of [HTT11, Proposition 3.5], we consider the following partition: Let ξ_0 be the center of $\mathcal{C}_2 := \mathcal{C} \times \mathcal{N}_2$, and define disjoint strips of width M that are orthogonal to ξ_0 :

$$\mathcal{S}_{k,\ell} := \{(m_1, n_1, m_2, n_2) \in \mathcal{R}_k^{(2)} : (m_1, n_1, m_2, n_2) \cdot_\kappa \xi_0 \in [|\xi_0|\ell M, |\xi_0|(\ell+1)M)\},$$

where $\xi \cdot_\kappa \zeta := \xi_1\zeta_1 + \kappa\xi_2\zeta_2 + \xi_3\zeta_3 + \kappa\xi_4\zeta_4$. We observe from the construction that the angle⁵ $\angle((m_1, \kappa n_1, m_2, \kappa n_2), \xi_0) \lesssim_\kappa \frac{N_2}{N_1}$. Since $N_1 \gg N_2$, we have $\angle((m_1, \kappa n_1, m_2, \kappa n_2), \xi_0) \leq \frac{1}{2}$. From this, we get

$$(m_1, n_1, m_2, n_2) \cdot_\kappa \xi_0 = |(m_1, \kappa n_1, m_2, \kappa n_2)| |\xi_0| \cos \angle((m_1, \kappa n_1, m_2, \kappa n_2), \xi_0) \approx N_1^2.$$

Thus, $\ell \geq 0$ and $\ell \approx \frac{N_1}{M}$ since $|\xi_0| \approx N_1$. Since $\mathcal{R}_k^{(2)} = \dot{\bigcup}_{\ell \approx N_1/M} \mathcal{S}_{k,\ell}$, we see that

$$Q_{\mathcal{R}_k^{(2)}}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) = \sum_{\ell \approx N_1/M} Q_{\mathcal{S}_{k,\ell}}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2).$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be a non-negative cut-off function satisfying $\chi(t) = 1$ for all $t \in \tau_0$ and $\chi(t) = 0$ for all $t \in \mathbb{R} \setminus \tau_1$. Obviously,

$$\begin{aligned} & \|Q_{\mathcal{R}_k^{(2)}}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_0 \times M)}^2 \\ & \leq \|\sqrt{\chi}(t) Q_{\mathcal{R}_k^{(2)}}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_1 \times M)}^2 \lesssim I_{k,1} + I_{k,2}, \end{aligned}$$

where

$$I_{k,1} := \sum_{\ell \approx N_1/M} \|Q_{\mathcal{S}_{k,\ell}}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_1 \times M)}^2$$

and

$$\begin{aligned} I_{k,2} := & \sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \langle \chi(t) Q_{\mathcal{S}_{k,\ell}}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3, \\ & Q_{\mathcal{S}_{k,\tilde{\ell}}}(P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3 \rangle_{L^2(\mathbb{R} \times M)}. \end{aligned}$$

It then suffices to show

$$\sum_{k \in \mathbb{Z}} |I_{k,2}| \lesssim N_2^{\frac{3}{2}} N_3^{-\nu} \|P_C P_{N_1} \phi_1\|_{L^2(M)}^2 \prod_{j=2}^3 \|P_{N_j} \phi_j\|_{L^2(M)}^2. \quad (2.39)$$

⁵ $\angle(\xi, \zeta)$ denotes the angle between ξ and ζ with respect to the standard inner product $\xi \cdot \zeta$.

The benefit of extending the integration with respect to t to \mathbb{R} is that we may interpret this integration as Fourier transform on \mathbb{R} . Doing so and taking the absolute value, we obtain

$$|I_{k,2}| \lesssim \sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \sum_{\substack{\mathcal{S}_{k,\ell} \times \mathcal{N}_3, \\ \mathcal{S}_{k,\tilde{\ell}} \times \mathcal{N}_3}} |\hat{\chi}| \left(\sum_{j=1}^3 (\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) \right) \int_M \prod_{j=1}^3 |\Pi_{n_j} \Theta_{m_j} \phi_j \Pi_{\tilde{n}_j} \Theta_{\tilde{m}_j} \phi_j(\omega)| d(\theta, \omega). \quad (2.40)$$

The term $|\hat{\chi}|$ provides us with arbitrarily fast decay in N_3 . To prove this, we define the quadratic form

$$Q(\xi) := \xi_1^2 + \kappa \xi_2^2 + \xi_3^2 + \kappa \xi_4^2, \quad \xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{Z}^4$$

and observe that for $\xi := (m_1, n_1 + \frac{1}{2}, m_2, n_2 + \frac{1}{2})$ we have

$$\lambda_{m_1, n_1} + \lambda_{m_2, n_2} = Q(\xi) - \frac{\kappa}{2}.$$

Motivated by the proof of [HTT11, Proposition 3.5], we write

$$Q(\xi) = \frac{1}{Q(\xi_0)} |\xi \cdot \kappa \xi_0|^2 + Q(\xi - \xi_0) - \frac{1}{Q(\xi_0)} |(\xi - \xi_0) \cdot \kappa \xi_0|^2.$$

We note from the restriction to $\mathcal{S}_{k,\ell}$ and $N_3^2 \lesssim M^2 \ell$ that

$$Q(\xi) = M^2 \ell^2 + \mathcal{O}(M^2 \ell).$$

The same result holds true for the elements in $\mathcal{S}_{k,\tilde{\ell}}$:

$$Q((\tilde{m}_1, \tilde{n}_1 + \frac{1}{2}, \tilde{m}_2, \tilde{n}_2 + \frac{1}{2})) = M^2 \tilde{\ell}^2 + \mathcal{O}(M^2 \tilde{\ell}).$$

Assuming $|\ell - \tilde{\ell}| \gg 1$, we see

$$\left| \sum_{j=1}^3 (\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) \right| = M^2 (\ell + \tilde{\ell}) |\ell - \tilde{\ell}| + \mathcal{O}(M^2 \ell) + \mathcal{O}(M^2 \tilde{\ell}) \gtrsim N_3^2 |\ell - \tilde{\ell}|$$

since $\ell, \tilde{\ell} \approx \frac{N_1}{M}$. Thus, for any $\mu > 0$,

$$|\hat{\chi}| \left(\sum_{j=1}^3 (\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) \right) \lesssim_\mu N_3^{-2\mu} \langle \ell - \tilde{\ell} \rangle^{-\mu}.$$

Now, we proceed to estimate (2.40). Cauchy–Schwarz with respect to (θ, ω) , $\mathcal{S}_{k,\ell} \times \mathcal{N}_3$, and $\mathcal{S}_{k,\tilde{\ell}} \times \mathcal{N}_3$ as well as the trilinear estimate for spherical harmonics in Proposition 2.18 yield

$$\begin{aligned} |I_{k,2}| &\lesssim (N_2 N_3)^{\frac{1}{2}} \sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \left(\sum_{\substack{\mathcal{S}_{k,\ell} \times \mathcal{N}_3, \\ \mathcal{S}_{k,\tilde{\ell}} \times \mathcal{N}_3}} |\hat{\chi}| \left(\sum_{j=1}^3 (\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) \right) \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{\mathcal{S}_{k,\ell} \times \mathcal{N}_3} \prod_{j=1}^3 \|\Pi_{n_j} \Theta_{m_j} \phi_j\|_{L^2(M)}^2 \right)^{\frac{1}{2}} \left(\sum_{\mathcal{S}_{k,\tilde{\ell}} \times \mathcal{N}_3} \prod_{j=1}^3 \|\Pi_{\tilde{n}_j} \Theta_{\tilde{m}_j} \phi_j\|_{L^2(M)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

Assume for the moment that (2.37) holds. Then, choosing $\delta = 0$, we see that the square root of the sum over $\mathcal{S}_{k,\ell} \times \mathcal{N}_3$ and $\mathcal{S}_{k,\tilde{\ell}} \times \mathcal{N}_3$ is bounded by $N_2 N_3^{3-2\mu} \langle \ell - \tilde{\ell} \rangle^{-\mu} \lesssim N_2 N_3^{-\nu} \langle \ell - \tilde{\ell} \rangle^{-\mu}$ provided $2\mu - 3 \geq \nu$. Finally, Schur's lemma and Cauchy-Schwarz with respect to k imply

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \langle \ell - \tilde{\ell} \rangle^{-\mu} \left(\sum_{\mathcal{S}_{k,\ell} \times \mathcal{N}_3} \prod_{j=1}^3 \|\Pi_{n_j} \Theta_{m_j} \phi_j\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{\mathcal{S}_{k,\tilde{\ell}} \times \mathcal{N}_3} \prod_{j=1}^3 \|\Pi_{\tilde{n}_j} \Theta_{\tilde{m}_j} \phi_j\|_{L^2}^2 \right)^{\frac{1}{2}} \\ \lesssim \|P_{\mathcal{C}} P_{N_1} \phi_1\|_{L^2(M)}^2 \|P_{N_2} \phi_2\|_{L^2(M)}^2 \|P_{N_3} \phi_3\|_{L^2(M)}^2 \end{aligned}$$

provided $\mu > 1$. This proves (2.39).

It remains to prove (2.37). Although there are two directions of size N_3 and M introduced in the third and fourth step, respectively, we can not expect $|\mathcal{S}_{k,\ell}| \lesssim MN_2^2 N_3$ to be true since these directions might be not orthogonal. If we just take the restriction of $\mathcal{S}_{k,\ell}$ into account, it is obvious that $|\mathcal{S}_{k,\ell}| \lesssim N_2^3 M$. We then obtain the asserted estimate by interpolating with a second estimate for $|\mathcal{S}_{k,\ell}|$. From the restriction to $\mathcal{R}_k^{(2)}$, we see that there are $CN_2^2 N_3$ combinations of (m_1, m_2, n_2) . The definition of $\mathcal{S}_{k,\ell}$ implies

$$A \leq \kappa n_1 \xi_{0,2} / |\xi_0| \leq A + M, \quad (2.41)$$

where $\xi_0 = (\xi_{0,1}, \xi_{0,2}, \xi_{0,3}, \xi_{0,4})$ and $A := \ell M - (m_1 \xi_{0,1} + m_2 \xi_{0,3} + \kappa n_2 \xi_{0,4}) / |\xi_0|$. Recall that ξ_0 was chosen to be the center of $\mathcal{C} \times \mathcal{N}_2$, where $\mathcal{C} \subseteq \mathcal{N}_1$ is a cube of size N_2 and the second component is a subset of $\{0, \dots, 2N_1\}$. We deduce $\xi_{0,2} \gtrsim N_2$ and consequently, $\xi_{0,2} / |\xi_0| \gtrsim N_2 / N_1$. Hence, (2.41) implies that there are CN_2^3 / N_2 possible values for n_1 (depending on κ). All in all, we proved $|\mathcal{S}_{k,\ell}| \lesssim N_2 N_3^3$. Now, (2.37) follows from interpolating the two bounds on $|\mathcal{S}_{k,\ell}|$. \square

The following lemma treats the remaining case, where the highest frequency is larger than the square of the lowest frequency.

Lemma 2.22. *Let $\nu > 0$ and $\tau_0 \subset \mathbb{R}$ be a bounded interval. Furthermore, let $\tau_1 \supset \overline{\tau_0}$ be an open interval. Then, for all $\phi_1, \phi_2, \phi_3 \in L^2(M)$ and dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ with $N_1 > N_3^2$ there are finitely many sets $\mathcal{T}_\ell \subseteq \mathcal{N}_1$ with the properties that $\mathcal{T}_\ell \in \mathcal{R}_{N_2, M}^2$, where $M := \max\{N_2^2 / N_1, 1\}$, and $\mathcal{N}_1 = \dot{\bigcup}_{\ell \in \mathbb{Z}} \mathcal{T}_\ell$ such that*

$$\begin{aligned} \left\| \prod_{j=1}^3 P_{N_j} e^{it\Delta} \phi_j \right\|_{L^2(\tau_0 \times M)}^2 &\lesssim \sum_{\ell \in \mathbb{Z}} \left\| P_{\mathcal{T}_\ell} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3 \right\|_{L^2(\tau_1 \times M)}^2 \\ &\quad + N_2^{-\nu} \prod_{j=1}^3 \|P_{N_j} \phi_j\|_{L^2(M)}^2. \end{aligned}$$

Proof. We use the notation introduced in (2.38). From Step 1 and Step 2 of the proof of the previous lemma, we see that we may consider $P_{\mathcal{C}} P_{N_1} e^{it\Delta} \phi_1$ instead of $P_{N_1} e^{it\Delta} \phi_1$, where $\mathcal{C} \subseteq \mathcal{N}_1$ and $\mathcal{C} \in \mathcal{C}_{N_2}^2$.

In what follows, we omit details since we argue along the lines of Step 4. Define the partition

$$\mathbb{N}_0 = \dot{\bigcup}_{\ell \in \mathbb{N}_0} J_\ell, \quad \text{where } J_\ell := [\ell M, (\ell + 1)M) \cap \mathbb{N}_0 \text{ and } M := \max \left\{ \frac{N_2^2}{N_1}, 1 \right\}.$$

Let ξ_0 be the center of \mathcal{C} , and define disjoint strips of width M that are orthogonal to ξ_0 :

$$\mathcal{T}_\ell := \{(m_1, n_1) \in \mathcal{C} : (m_1, \kappa n_1) \cdot \xi_0 \in [|\xi_0|\ell M, |\xi_0|(\ell + 1)M)\},$$

where $\ell \geq 0$ and $\ell \approx \frac{N_1}{M}$. We clearly have

$$P_{\mathcal{C}} P_{N_1} e^{it\Delta} \phi_1 = \sum_{\ell \approx N_1/M} P_{\mathcal{T}_\ell} P_{N_1} e^{it\Delta} \phi_1.$$

We denote by χ the same non-negative cut-off function as in Step 4. We compute

$$\begin{aligned} & \|P_{\mathcal{C}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_0 \times M)}^2 \\ & \leq \|\sqrt{\chi}(t) P_{\mathcal{T}_\ell} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_1 \times M)}^2 \lesssim I_1 + I_2, \end{aligned}$$

where

$$I_1 := \sum_{\ell \approx N_1/M} \|P_{\mathcal{T}_\ell} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_1 \times M)}^2$$

and I_2 is defined as

$$\sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \langle \chi(t) P_{\mathcal{T}_\ell} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3, P_{\mathcal{T}_{\tilde{\ell}}} P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3 \rangle_{L^2(\mathbb{R} \times M)}.$$

We are left to estimate

$$|I_2| \lesssim N_2^{-\nu} \|P_{\mathcal{C}} P_{N_1} \phi_1\|_{L^2(M)}^2 \|P_{N_2} \phi_2\|_{L^2(M)}^2 \|P_{N_3} \phi_3\|_{L^2(M)}^2.$$

By the same argument that we used to obtain (2.40), we deduce

$$|I_2| \lesssim \sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \sum_{\substack{\mathcal{T}_\ell \times \mathcal{N}_2 \times \mathcal{N}_3, \\ \mathcal{T}_{\tilde{\ell}} \times \mathcal{N}_2 \times \mathcal{N}_3}} |\hat{\chi}| \left(\sum_{j=1}^3 (\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) \right) \int_M \prod_{j=1}^3 |\Pi_{n_j} \Theta_{m_j} \phi_j \Pi_{\tilde{n}_j} \Theta_{\tilde{m}_j} \phi_j(\omega)| d(\theta, \omega).$$

For $|\ell - \tilde{\ell}| \gg 1$ and $\ell, \tilde{\ell} \approx \frac{N_1}{M}$, we get

$$\left| \sum_{j=1}^3 (\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) \right| = M^2(\ell + \tilde{\ell})|\ell - \tilde{\ell}| + \mathcal{O}(M^2\ell) + \mathcal{O}(M^2\tilde{\ell}) \gtrsim N_2^2|\ell - \tilde{\ell}|.$$

Thus, for any $\mu > 0$,

$$|\hat{\chi}| \left(\sum_{j=1}^3 (\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) \right) \lesssim_\mu N_2^{-2\mu} (\ell - \tilde{\ell})^{-\mu}.$$

Cauchy–Schwarz with respect to (θ, ω) , $\mathcal{T}_\ell \times \mathcal{N}_2 \times \mathcal{N}_3$, and $\mathcal{T}_{\tilde{\ell}} \times \mathcal{N}_2 \times \mathcal{N}_3$ as well as the trilinear estimate for spherical harmonics in Proposition 2.18 yield

$$\begin{aligned} |I_2| & \lesssim (N_2 N_3)^{\frac{1}{2}} \sum_{\substack{\ell, \tilde{\ell} \approx N_1/M: \\ |\ell - \tilde{\ell}| \gg 1}} \left(\sum_{\substack{\mathcal{T}_\ell \times \mathcal{N}_2 \times \mathcal{N}_3, \\ \mathcal{T}_{\tilde{\ell}} \times \mathcal{N}_2 \times \mathcal{N}_3}} |\hat{\chi}| \left(\sum_{j=1}^3 (\lambda_{m_j, n_j} - \lambda_{\tilde{m}_j, \tilde{n}_j}) \right) \right)^{\frac{1}{2}} \\ & \times \left(\sum_{\mathcal{T}_\ell \times \mathcal{N}_2 \times \mathcal{N}_3} \prod_{j=1}^3 \|\Pi_{n_j} \Theta_{m_j} \phi_j\|_{L^2(M)}^2 \right)^{\frac{1}{2}} \left(\sum_{\mathcal{T}_{\tilde{\ell}} \times \mathcal{N}_2 \times \mathcal{N}_3} \prod_{j=1}^3 \|\Pi_{\tilde{n}_j} \Theta_{\tilde{m}_j} \phi_j\|_{L^2(M)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since $|\mathcal{T}_\ell \times \mathcal{N}_2 \times \mathcal{N}_3| \lesssim N_2^6$ for any $\ell \in \mathbb{N}_0$, we conclude

$$|I_2| \lesssim N_2^{7-2\mu} \|P_{\mathcal{C}} P_{N_1} \phi_1\|_{L^2(M)}^2 \|P_{N_2} \phi_2\|_{L^2(M)}^2 \|P_{N_3} \phi_3\|_{L^2(M)}^2$$

using Schur's lemma as done in Step 4. Choosing μ large enough implies the asserted result. \square

Remark. In the third step of the proof of [HS15, Proposition 2.6], an annular smallness condition was derived. A similar restriction could have been determined in the previous two lemmas, which was avoided here due to the more complicated number of lattice points estimate for annular sets. \diamond

2.5.6 The trilinear Strichartz estimate

Before we turn to the proof of [Assumption 2.1](#), we state the following estimate of the number of lattice points solving a Diophantine equation. The proof is similar to the proof of (1.8) and can be found in [BGT05a, Lemma 3.2], for instance.

Lemma 2.23. *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for every $\tau \in \mathbb{N}_0$ and $N \in \mathbb{N}$,*

$$|\{(n_1, n_2) \in [0, N] \times \mathbb{N}_0 : n_1^2 + n_2^2 = \tau\}| \leq C_\varepsilon N^\varepsilon.$$

Now, we have everything we need to conclude the trilinear Strichartz estimate, which in turn implies the local well-posedness result in [Theorem 2.3](#).

Proposition 2.24. *There exists $\delta > 0$ such that for all $\phi_1, \phi_2, \phi_3 \in L^2(M)$ and dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ the following estimate holds:*

$$\|P_{N_1} e^{it\Delta} \phi_1 P_{N_2} e^{it\Delta} \phi_2 P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_0 \times M)} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2} \right)^\delta N_2 N_3 \prod_{j=1}^3 \|P_{N_j} \phi_j\|_{L^2(M)}.$$

Proof. According to our almost orthogonality results, we have to treat the cases $N_1 \leq N_3^2$ and $N_1 > N_3^2$ separately. The latter case can be considered as sub-critical since a gain of a small power of N_1^{-1} allows compensate a loss of a small power of N_3 . This is exploited at the end of this proof.

Case $N_1 \leq N_3^2$. Let $\tau_1 \supset \overline{\tau_0}$ be an open interval. Thanks to [Lemma 2.21](#), we may replace the left-hand side by

$$\left(\sum_{\ell \in \mathbb{Z}} \|Q_{\mathcal{S}_\ell} (P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3\|_{L^2(\tau_1 \times M)}^2 \right)^{\frac{1}{2}}.$$

To be definite we choose $\delta = \frac{1}{12}$ which gives that $\mathcal{S}_\ell \subseteq \mathcal{N}_1 \times \mathcal{N}_2$ are sets of size $M^{1/12} N_2^{7/6} N_3^{11/4}$. The \mathcal{N}_j are defined as in (2.36). Recall that for $t \in \tau_0$ and $(\theta, \omega) \in \mathbb{S} \times \mathbb{S}_\rho^2$,

$$\begin{aligned} & Q_{\mathcal{S}_\ell} (P_{N_1} e^{it\Delta} \phi_1, P_{N_2} e^{it\Delta} \phi_2) P_{N_3} e^{it\Delta} \phi_3(\theta, \omega) \\ &= \sum_{(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{M}_\ell} \prod_{j=1}^3 \eta_{N_j}(\sqrt{\lambda_{m_j, n_j}}) e^{-i\lambda_{m_j, n_j} t} e^{im_j \theta} \Theta_{m_j} \phi_j(\omega), \end{aligned}$$

where $\mathcal{M}_\ell := \mathcal{S}_\ell \times \mathcal{N}_3$. In the next step we treat the $L^2(\mathbb{S}_\rho^2)$ -norm separately without losing oscillations in the \mathbb{S} component and in time. This was already used by Burq–Gérard–Tzvetkov in the proof of [BGT05b, Proposition 5.1]. Plancherel’s identity with respect to t (see Proposition 1.36 (ii)) and θ , and the triangle inequality for the $L^2(\mathbb{S}_\rho^2)$ -norm yield

$$\begin{aligned} & \|Q_{\mathcal{S}_\ell}(P_{N_1}e^{it\Delta}\phi_1, P_{N_2}e^{it\Delta}\phi_2)P_{N_3}e^{it\Delta}\phi_3\|_{L^2(\tau_1 \times M)}^2 \\ & \lesssim \sum_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \left\| \sum_{\substack{(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{M}_\ell: \\ |\lambda_{m_1, n_1} + \lambda_{m_2, n_2} + \lambda_{m_3, n_3} - \tau| \leq \frac{1}{2}, \\ \xi = m_1 + m_2 + m_3}} \prod_{j=1}^3 \|\Pi_{n_j} \Theta_{m_j} \phi_j\|_{L^2(\mathbb{S}_\rho^2)} \right\|_{L^2(\mathbb{S}_\rho^2)}^2 \\ & \lesssim \sum_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \left[\sum_{\substack{(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{M}_\ell: \\ |\lambda_{m_1, n_1} + \lambda_{m_2, n_2} + \lambda_{m_3, n_3} - \tau| \leq \frac{1}{2}, \\ \xi = m_1 + m_2 + m_3}} \left\| \prod_{j=1}^3 \Pi_{n_j} \Theta_{m_j} \phi_j \right\|_{L^2(\mathbb{S}_\rho^2)} \right]^2. \end{aligned} \quad (2.42)$$

In contrast to [BGT05b, Proposition 5.1], we do not estimate the number of terms of the inner sum, but we go back to the physical space: We set $a_{m_j, n_j}^{(j)} := \|\Pi_{n_j} \Theta_{m_j} \phi_j\|_{L^2(\mathbb{S}_\rho^2)}$ for $j = 1, 2, 3$ and apply Proposition 2.18 as well as Plancherel’s identity with respect to θ and Proposition 1.36 (i) with respect to t to obtain

$$\begin{aligned} & \|Q_{\mathcal{S}_\ell}(P_{N_1}e^{it\Delta}\phi_1, P_{N_2}e^{it\Delta}\phi_2)P_{N_3}e^{it\Delta}\phi_3\|_{L^2(\tau_1 \times M)}^2 \\ & \lesssim (N_2 N_3)^{\frac{1}{2}} \sum_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \left(\sum_{\substack{(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{M}_\ell: \\ |\lambda_{m_1, n_1} + \lambda_{m_2, n_2} + \lambda_{m_3, n_3} - \tau| \leq \frac{1}{2}, \\ \xi = m_1 + m_2 + m_3}} \prod_{j=1}^3 a_{m_j, n_j}^{(j)} \right)^2 \\ & \lesssim (N_2 N_3)^{\frac{1}{2}} \left\| \sum_{(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{M}_\ell} \prod_{j=1}^3 e^{-i\lambda_{m_j, n_j} t} e^{im_j \theta} a_{m_j, n_j}^{(j)} \right\|_{L_{t, \theta}^2(\tau_0 \times \mathbb{S})}^2. \end{aligned} \quad (2.43)$$

Hölder’s estimate yields

$$\begin{aligned} (2.43) & \lesssim (N_2 N_3)^{\frac{1}{4}} \left\| \sum_{(m_1, n_1, m_2, n_2) \in \mathcal{S}_\ell} \prod_{j=1}^2 e^{-i\lambda_{m_j, n_j} t} e^{im_j \theta} a_{m_j, n_j}^{(j)} \right\|_{L_t^3(\tau_1, L_\theta^{9/4}(\mathbb{S}))} \\ & \times \left\| \sum_{(m_3, n_3) \in \mathcal{N}_3} e^{-i\lambda_{m_3, n_3} t} e^{im_3 \theta} a_{m_3, n_3}^{(3)} \right\|_{L_t^6(\tau_1, L_\theta^{18}(\mathbb{S}))}. \end{aligned} \quad (2.44)$$

Applying Bernstein’s inequality to the last term in (2.44) and then Corollary 2.20 to both terms. This leads to

$$\left(\sum_{\ell \in \mathbb{Z}} \|Q_{\mathcal{S}_\ell}(P_{N_1}e^{it\Delta}\phi_1, P_{N_2}e^{it\Delta}\phi_2)P_{N_3}e^{it\Delta}\phi_3\|_{L^2(\tau_1 \times M)}^2 \right)^{\frac{1}{2}} \lesssim N_1^{-\frac{1}{216}} N_2^{\frac{53}{54}} N_3^{\frac{221}{216}} \prod_{j=1}^3 \|\phi_j\|_{L^2(M)},$$

which immediately implies the desired result in the first case.

Case $N_1 > N_3^2$. We follow the strategy of Burq–Gérard–Tzvetkov in the proof of [BGT05b, Proposition 5.1]. The only difference is the estimate (ii) below and how we exploit it.

In light of Lemma 2.22, it suffices to show

$$\begin{aligned} & \|P_{\mathcal{T}}P_{N_1}e^{it\Delta}\phi_1P_{N_2}e^{it\Delta}\phi_2P_{N_3}e^{it\Delta}\phi_3\|_{L^2(\tau_1\times M)} \\ & \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2}\right)^\delta N_2N_3\|P_{\mathcal{T}}P_{N_1}\phi_1\|_{L^2(M)} \prod_{j=2}^3 \|P_{N_j}\phi_j\|_{L^2(M)} \end{aligned}$$

for some open interval $\tau_1 \supset \overline{\tau_0}$ and $\mathcal{T} \subseteq \mathcal{N}_1$ with $\mathcal{T} \in \mathcal{R}_{N_2, M}^2$, where $M := \max\{N_2^2/N_1, 1\}$. For $\mathcal{M} := \mathcal{T} \times \mathcal{N}_2 \times \mathcal{N}_3$ we estimate

$$\begin{aligned} & \|P_{\mathcal{T}}P_{N_1}e^{it\Delta}\phi_1P_{N_2}e^{it\Delta}\phi_2P_{N_3}e^{it\Delta}\phi_3\|_{L^2(\tau_1\times M)}^2 \\ & \lesssim \sum_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \left\| \sum_{\substack{(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{M}: \\ |\lambda_{m_1, n_1} + \lambda_{m_2, n_2} + \lambda_{m_3, n_3} - \tau| \leq \frac{1}{2}, \\ \xi = m_1 + m_2 + m_3}} \prod_{j=1}^3 |\Pi_{n_j} \Theta_{m_j} \phi_j| \right\|_{L^2(\mathbb{S}_\rho^2)}^2 \end{aligned}$$

as in (2.42) above. The triangle inequality for the $L^2(\mathbb{S}^2)$ -norm, Cauchy–Schwarz in the summation over $(m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{M}$, and Proposition 2.18 yield

$$\|P_{\mathcal{T}}P_{N_1}e^{it\Delta}\phi_1P_{N_2}e^{it\Delta}\phi_2P_{N_3}e^{it\Delta}\phi_3\|_{L^2(\tau_1\times M)}^2 \lesssim (N_2N_3)^{\frac{1}{2}} \sup_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \Lambda(\xi, \tau) \prod_{j=1}^3 \|\phi_j\|_{L^2(M)}^2,$$

where $\Lambda(\xi, \tau)$ is defined as

$$\left| \left\{ (m_1, n_1, m_2, n_2, m_3, n_3) \in \mathcal{M} : \xi = m_1 + m_2 + m_3, |\lambda_{m_1, n_1} + \lambda_{m_2, n_2} + \lambda_{m_3, n_3} - \tau| \leq \frac{1}{2} \right\} \right|.$$

We are left to bound $\Lambda(\xi, \tau)$ uniformly in ξ and τ by

$$C \left(\frac{N_3}{N_1} + \frac{1}{N_2} \right)^{2\delta} N_2^{\frac{3}{2}} N_3^{\frac{3}{2}}.$$

In fact, we shall prove that there exists $\eta > 0$ such that

$$\sup_{\tau \in \mathbb{N}_0, \xi \in \mathbb{Z}} \Lambda(\xi, \tau) \leq CN_2^{\frac{3}{2}-\eta} N_3^{\frac{3}{2}}. \quad (2.45)$$

In contrast to [BGT05b], we will use the smallness properties of \mathcal{T} introduced by almost orthogonality in space and time to gain a small power of M . For any $\varepsilon > 0$ we get the following two estimates:

$$(i) \quad \Lambda(\xi, \tau) \leq C_\varepsilon N_2^{1+\varepsilon} N_3^2, \quad (ii) \quad \Lambda(\xi, \tau) \leq C_\varepsilon M N_2 N_3^{1+\varepsilon}.$$

The estimates can be proved as follows:

- (i) Here, we neglect the restriction to \mathcal{T} . The number of possible triples (m_2, m_3, n_3) is bounded by $CN_2N_3^2$. Now, we fix a possible triple (m_2, m_3, n_3) and eliminate m_1 by $m_1 = \xi - m_2 - m_3$. Then (n_1, n_2) has to satisfy

$$\left| (2n_1 + 1)^2 + (2n_2 + 1)^2 - r \right| \leq \frac{2}{\kappa}, \quad (2.46)$$

with $r := 2 + \frac{4}{\kappa}(\tau - (\xi - m_2 - m_3)^2 - m_2^2 - \lambda_{m_3, n_3})$. Hence, Lemma 2.23 implies that the number of integer solutions $(n_1, n_2) \in [0, 2N_1] \times [0, 2N_2]$ of (2.46) is bounded by $C_\varepsilon N_2^\varepsilon$. From this we deduce (i).

- (ii) From the definition of \mathcal{T} , we see that the number of possible triples (m_1, n_1, m_3) can be estimated by CMN_2N_3 . We fix a possible triple (m_1, n_1, m_3) and eliminate m_2 by $m_2 = \xi - m_1 - m_3$. In order to evaluate $\Lambda(\xi, \tau)$, we observe that (n_2, n_3) satisfies

$$|(2n_2 + 1)^2 + (2n_3 + 1)^2 - r| \leq \frac{2}{\kappa},$$

with $r := 2 + \frac{4}{\kappa}(\tau - \lambda_{m_1, n_1} - (\xi - m_1 - m_3)^2 - m_3^2)$. By Lemma 2.23, we can estimate the number of integer solutions by CN_3^ε .

Note that the estimates (i) and (ii) above have an additional loss of power ε and thus, should not be useful in our energy-critical study. However, since $N_1 > N_3^2$, the factor $M = \max\{N_2^2/N_1, 1\}$ in (ii) allows to compensate this loss in either case.

On the one hand, if $M = 1$, then (ii) clearly yields

$$\Lambda(\xi, \tau) \leq CN_2N_3^{\frac{3}{2}},$$

which immediately implies (2.45).

On the other hand, if $M = N_2^2/N_1$, then we bound

$$\Lambda(\xi, \tau) \leq C_\varepsilon(N_2^{1+\varepsilon}N_3^2)^{\frac{9}{10}}(N_1^{-1}N_2^3N_3^{1+\varepsilon})^{\frac{1}{10}} \leq C_\varepsilon N_1^{-\frac{1}{10}}N_2^{\frac{6}{5}+\varepsilon}N_3^{\frac{19}{10}}$$

for some $\varepsilon > 0$. Observe that $N_1 > N_3^2$ implies $N_1^{-1/10} \leq N_3^{-1/5}$. Therefore,

$$\Lambda(\xi, \tau) \leq C_\varepsilon N_2^{\frac{6}{5}+\varepsilon}N_3^{\frac{17}{10}}.$$

Choosing $\varepsilon < \frac{1}{20}$ yields (2.45) since $\frac{17}{10} > \frac{3}{2}$ and $\frac{6}{5} + \frac{1}{20} + \frac{17}{10} < 3$. \square

2.6 Further results on other manifolds and remarks

Apart from the energy-critical local and small data global well-posedness results proved above, there is only very little knowledge about energy-critical well-posedness on compact manifolds.

As mentioned before, well-posedness on the class of Zoll manifolds, which are manifolds for which all geodesics are simple and closed with a common minimal period, has been studied. The most important example of a Zoll manifold is \mathbb{S}^n . To the authors knowledge, Burq–Gérard–Tzvetkov were the first who obtained energy-sub-critical well-posedness results for the NLS on two- and three-dimensional Zoll manifolds as well as $\mathbb{S} \times M$ where M is a two-dimensional Zoll manifold, see [BGT05a, Theorem 1] and [BGT05b, Theorem 1.1]⁶. Herr [Her13, Theorem 1.1] finally established energy-critical local and small data global well-posedness for three-dimensional Zoll manifolds. The proof relies on the stronger (compared to Corollary 1.39) exponential sum estimate

$$\left\| \sum_{n \in J} c_n e^{-itn^2} \right\|_{L_t^p(I)} \leq CN^{\frac{1}{2} - \frac{2}{p}} \left(\sum_{n \in J} |c_n|^2 \right)^{\frac{1}{2}},$$

where $4 < p \leq \infty$ and J is an interval in \mathbb{Z} of size $N \geq 1$ [Her13, Lemma 3.1]. Using an almost orthogonality argument, the trilinear estimate in Assumption 2.1 is obtained implicitly in the

⁶ Even though [BGT05b, Theorem 1.1] is only stated for \mathbb{S}^3 and $\mathbb{S}_\rho^2 \times \mathbb{S}$, it is mentioned in the introduction of Sections 4.2 and 5.2 that it applies to three-dimensional Zoll manifolds and $\mathbb{S} \times M$ as well.

proof of [Her13, Proposition 3.6]. Besides this trilinear estimate, another very important novelty in this article is the treatment of the minor contribution term Σ_2 in the proof of [Her13, Proposition 4.2], which corresponds Σ_2 in the proof of Lemma 2.9. This contribution was zero in the previously considered case \mathbb{T}^3 .⁷ It is worth to mention that the well-posedness study on Zoll manifolds does not rely on the geometrical property but on the fact that the spectrum is clustered around a sequence of squares.

One might ask whether the proof of Proposition 2.24 can be extended to the class of $\mathbb{S} \times M$, where M is a two-dimensional Zoll manifold. First, one should mention that the spectrum of two-dimensional Zoll manifolds is—like in the three-dimensional case—clustered around square numbers, see [BGT05a, Proposition 3.3] and [Gui77, Theorem 6]. As a consequence, the spectrum does not change much compared to the previous section if one considers a Laplace–Beltrami operator $\tilde{\Delta}$ similar as in [Her13, Lemma 2.2] instead. Hence, it seems likely that an argument as in the proof of Lemma 2.19 allows to get a similar result. A fundamental change has to be done in the proof of Lemma 2.21. One does not have almost orthogonality of eigenfunctions of Zoll manifolds, though, in light of Lemma 1.54, the additional contribution should be negligible. Hence, we strongly expect Proposition 2.24 to hold even for the product of \mathbb{S} with any two-dimensional Zoll manifold.

In higher dimensions even less is known. So far, there is only one energy-critical well-posedness result on a four-dimensional compact manifold, namely on \mathbb{T}^4 . This result is due to Herr–Tataru–Tzvetkov [HTT14, Theorem 1.1] and relies heavily on the Strichartz estimates given in [Bou13, formula (0.11)]. A natural domain to be considered next is \mathbb{S}^4 . In this case, new ideas seem to be needed due to the failure of the scale invariant $L_{t,x}^4$ -Strichartz estimate [BGT04, Theorem 4]. However, Gérard–Pierfelice [GP10] proved that the quadratic NLS is locally well-posed in $H_{\text{zonal}}^s(\mathbb{S}^4)$ for every $s > \frac{1}{2}$, where $H_{\text{zonal}}^s(\mathbb{S}^4)$ is the space of all zonal functions in $H^s(\mathbb{S}^4)$.

⁷In the special case of \mathbb{S}^3 —due to orthogonality reasons—the term Σ_2 is zero, too.

3 Global well-posedness for large data

Having discussed the local and small data global theory in the previous chapter, we shall now address the energy-critical large data global well-posedness theory on rectangular tori. For any initial data in H^1 we prove that the *defocusing* nonlinear Schrödinger equation with quintic nonlinearity is globally well-posed. This result, which has already been published in [Str15] by the author of the present thesis, extends results of Ionescu–Pausader [IP12b].

3.1 Set-up and main result

Analogously to the local theory in Section 2.3, we study the following defocusing nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_{\theta} u = u|u|^4 \\ u(0, \cdot) = \phi \in H^1(\mathbb{T}^3) \end{cases} \quad (3.1)$$

with base space \mathbb{T}^3 and modified Laplace–Beltrami operator Δ_{θ} instead of the equivalent equation on \mathbb{T}_{θ}^3 ,

$$\begin{cases} i\partial_t v + \Delta_g v = v|v|^4 \\ v(0, \cdot) = \tilde{\phi} \in H^1(\mathbb{T}_{\theta}^3). \end{cases}$$

Recall the definition of the modified Laplace operator Δ_{θ} given in (2.24), the notion of the evolution operator $e^{it\Delta_{\theta}}$ in (2.25), and the conservation of mass and energy, see (2.26).

For notational convenience we write $\nabla = \nabla_g$, and this time, we use the equivalent H^1 -norm which is given by

$$\|f\|_{H^s(\mathbb{T}^3)} := \left(\sum_{N \geq 1} N^{2s} \|P_N f\|_{L^2(\mathbb{T}^3)}^2 \right)^{\frac{1}{2}}.$$

This affects only constants and in some cases changes them to one as in Proposition 3.3 (ii).

We also specify the frequency localization operators P_N : We fix a smooth, non-negative, even function $\eta^1: \mathbb{R} \rightarrow [0, 1]$ with $\eta^1(y) = 1$ for $|y| \leq 1$ and $\text{supp } \eta^1 \subseteq (-2, 2)$. Then, let $\eta^3: \mathbb{R}^3 \rightarrow [0, 1]$ be defined via $\eta^3(x) := \eta^1(x_1)\eta^1(x_2)\eta^1(x_3)$. For a dyadic number $N > 1$ we set

$$\eta_N^3(x) := \eta^3\left(\frac{|x|}{N}\right) - \eta^3\left(\frac{2|x|}{N}\right) \quad \text{and} \quad \eta_1^3(x) := \eta^3(|x|).$$

Then, we define the frequency localization operators $P_N: L^2(\mathbb{T}^3) \rightarrow L^2(\mathbb{T}^3)$ as the Fourier multiplier with symbol η_N^3 . Furthermore, we set $P_{\leq N} := \sum_{1 \leq M \leq N} P_M$. More generally, given a set $\mathcal{S} \subseteq \mathbb{Z}^3$, we define $P_{\mathcal{S}}$ to be the Fourier multiplier with symbol $\mathbf{1}_{\mathcal{S}}$, where $\mathbf{1}_{\mathcal{S}}$ denotes the characteristic function of \mathcal{S} .

Using the space $X_r^1(I)$, which is defined in Definition 3.2 below, we may formulate the main result of this chapter.

Theorem 3.1 (Global well-posedness). *If $\phi \in H^1(\mathbb{T}^3)$, then there exists a unique global solution $u \in C(\mathbb{R}, H^1(\mathbb{T}^3)) \cap X_r^1(\mathbb{R})$ of the initial value problem (3.1). Moreover, the mapping $\phi \mapsto u$ extends to a continuous mapping from $H^1(\mathbb{T}^3)$ to $C([-T, T], H^1(\mathbb{T}^3)) \cap X_r^1([-T, T])$ for any $T \in [0, \infty)$, and the quantities $M(u)$ and $E(u)$ defined in (2.26) are conserved along the flow.*

Important results regarding well-posedness on tori have been summarized in Section 2.3.1. Now, we want to put the results presented here better into context. In a series of papers, Ionescu–Pausader [IPS12, IP12a, IP12b] ([IPS12] is a joint work with Staffilani) developed a method to obtain energy-critical large data global well-posedness on \mathbb{T}^3 . This was the first critical result of this kind on a compact manifold. So far, the corresponding result has been only obtained on \mathbb{S}^3 [PTW14] by Pausader–Tzvetkov–Wang and on rectangular tori [Str15] by the author. A variant of the proof of the latter result is given in this chapter. Our proof is closely tied to the strategy developed by Ionescu–Pausader [IP12b], which itself relies on ideas that have been applied on \mathbb{R}^3 [Bou99, CKS⁺08, KM06]. Since some proofs are omitted in [IP12b] as they follow analogously as on the previously considered domains in [IPS12, IP12a], we take the opportunity to review the whole argument.

Our first step is to refine the large data local well-posedness theory presented in Section 2.3. For that purpose, we introduce a variant of the resolution spaces X^s and Y^s , which give a local-in-time control, and a weaker critical space-time norm Z . On the one hand, it is proved that the nonlinear solution stays regular as long as the Z -norm is finite. On the other hand, we show that concentration of a large amount of the Z -norm in finite time is self-defeating. The reason is that a concentration of the Z -norm in finite time is equivalent to the fact of undergoing a self-similar Euclidean concentration, which is prevented by the Euclidean theory. This is a consequence of the following: Concentration of a large amount the Z -norm in finite time can only happen around a point in space-time, which itself must occur in a way that is comparable to Euclidean solutions. Finally, it is known that Euclidean-like solutions can only concentrate a bounded, finite amount of space-time norm [CKS⁺08]. To implement this, we perform a profile decomposition of the initial data with profiles that concentrate in a point. Such profiles are studied in detail.

We finally highlight the novelties. The main new ingredients for extending the result in [IP12b] are the extinction lemma (Lemma 3.21) and Lemma 3.32. Unlike in the case of \mathbb{T}^3 , we can not apply the Weyl inequality in Lemma 1.41 to $|K_M(t, x)|$, which is defined in (3.41). However, it turns out that throwing away the oscillations in two components and using the Weyl inequality in one dimension, is still strong enough to obtain a similar extinction lemma as in [IP12b, Lemma 4.3]. The main novelty in Lemma 3.32, which estimates the interaction of a high-frequency linear solution with a low-frequency profile, is the way we estimate (3.98). This, however, was already done in the author’s work [Str15].

In [Str15], the author already mentioned that the range of Strichartz estimates in Lemma 2.10 suffice to not only conclude small data global well-posedness but even global well-posedness for arbitrary large initial data in H^1 . This is remarkable since the proof of Lemma 2.10 requires no sophisticated arguments. Indeed, the essential tools are the exponential sum estimates proved in Section 1.3, see [Bou07, Proposition 1.1]. This is accomplished by modifying the Z -norm, which mainly effects the local theory that is developed here and the extinction lemma. Motivated by the fact that the conditional result in Section 2.2 used dyadic scale resolution spaces, we are going to define related resolution spaces X_r^s and Y_r^s with dyadic scales as well. This differs from [IP12b, Str15], where resolution spaces with unit scales have been used.

The argument given in [IP12b] relies heavily on earlier works and therefore, we take the opportunity to review the whole proof here.

3.2 Basic definitions and statements

This section is devoted to introduce functions spaces with some of their properties that we shall rely on. Furthermore, strong solutions are defined and dispersive estimates are recalled.

Recall the definition of the resolution spaces X^s and Y^s in Definition 2.2. Based on this spaces, we define the norms X_r^s and Y_r^s by restricting to time intervals of length at most one.

Definition 3.2 (Resolution spaces X_r^s and Y_r^s). Let $s \in \mathbb{R}$. For a time interval $I \subseteq \mathbb{R}$ we define $X_r^s(I)$ and $Y_r^s(I)$ to be the restriction spaces defined as

$$X_r^s(I) := \left\{ u: I \rightarrow H^1(\mathbb{T}^3) : \|u\|_{X_r^s(I)} := \sup_{\substack{J \subseteq I, \\ |J| \leq 1}} \inf_{\substack{v \in X^s: \\ v \cdot \mathbf{1}_J = u \cdot \mathbf{1}_J}} \|v\|_{X^s} < \infty \right\},$$

$$Y_r^s(I) := \left\{ u: I \rightarrow H^1(\mathbb{T}^3) : \|u\|_{Y_r^s(I)} := \sup_{\substack{J \subseteq I, \\ |J| \leq 1}} \inf_{\substack{v \in Y^s: \\ v \cdot \mathbf{1}_J = u \cdot \mathbf{1}_J}} \|v\|_{Y^s} < \infty \right\}.$$

Remark. In [IP12b, Str15], the spaces X_r^s and Y_r^s were defined to consist of functions that are continuous in time. We omitted this to be consistent with our small data theory in the previous chapter. Therefore, we add this property to the definition of a strong solution, see Definition 3.6. Besides of the aforementioned scale of resolution, this is another small difference to [IP12b, Str15]. \diamond

Similarly as in Proposition 2.4, we have the following basic properties of our resolution spaces.

Proposition 3.3 (Properties of X_r^s and Y_r^s). Let $I \subseteq \mathbb{R}$ be a bounded time interval and $s \in \mathbb{R}$.

(i) We have

$$X_r^s(I) \hookrightarrow Y_r^s(I) \hookrightarrow L^\infty(I, H^s(\mathbb{T}^3)).$$

(ii) Let $0 \in I$, $s \geq 0$, and $\phi \in H^s(\mathbb{T}^3)$, then $e^{it\Delta_\theta} \phi \in X_r^s(I)$ and

$$\|e^{it\Delta_\theta} \phi\|_{X_r^s(I)} \leq \|\phi\|_{H^s(\mathbb{T}^3)}.$$

(iii) Suppose $|I| \leq 1$ and $u \in Y_r^s(I)$ for some $s \in \mathbb{R}$. Then,

$$\left(\sum_{N \geq 1} N^{2s} \|P_N u\|_{Y_r^0(I)}^2 \right)^{\frac{1}{2}} \lesssim \|u\|_{Y_r^s(I)}.$$

Proof. The first two statements follow from the same argument as in Proposition 2.4.

To prove (iii), we first observe that since $|I| \leq 1$,

$$\|u\|_{Y_r^s(I)} = \inf_{v \cdot \mathbf{1}_I(t) = u \cdot \mathbf{1}_I(t)} \|v\|_{Y^s}.$$

Indeed, on the one hand,

$$\sup_{J \subseteq I} \inf_{v \cdot \mathbf{1}_J(t) = u \cdot \mathbf{1}_J(t)} \|v\|_{Y^s} \leq \sup_{J \subseteq I} \inf_{v \cdot \mathbf{1}_I(t) = u \cdot \mathbf{1}_I(t)} \|v\|_{Y^s} = \inf_{v \cdot \mathbf{1}_I(t) = u \cdot \mathbf{1}_I(t)} \|v\|_{Y^s}$$

and on the other hand,

$$\inf_{v \cdot \mathbf{1}_I(t) = u \cdot \mathbf{1}_I(t)} \|v\|_{Y^s} \leq \sup_{J \subseteq I} \inf_{v \cdot \mathbf{1}_J(t) = u \cdot \mathbf{1}_J(t)} \|v\|_{Y^s}$$

since the left-hand side is the special case where $J = I$. Hence,

$$\begin{aligned} \sum_{N \geq 1} N^{2s} \|P_N u\|_{Y_r^0(I)}^2 &\leq \sum_{N \geq 1} N^{2s} \inf_{v \cdot \mathbf{1}_J(t) = u \cdot \mathbf{1}_J(t)} \sum_{M \geq 1} \|e^{-it\Delta_\theta} P_M P_N v\|_{V^2}^2 \\ &\lesssim \inf_{v \cdot \mathbf{1}_J(t) = u \cdot \mathbf{1}_J(t)} \sum_{N \geq 1} N^{2s} \|e^{-it\Delta_\theta} P_N v\|_{V^2}^2. \end{aligned}$$

The last term equals $\|u\|_{Y_r^s(I)}$ as shown above. \square

We introduce a critical norm Z which is weaker than X_r^1 . It is also related to the Z -norm appearing in [IP12b, Str15], which was defined as

$$\|u\|_{Z(I)} = \sum_{p \in \{4+1/10, 100\}} \sup_{J \subseteq I, |J| \leq 1} \left(\sum_{N \geq 1} N^{5-p/2} \|P_N u\|_{L^p(J \times \mathbb{T}^3)}^p \right)^{\frac{1}{p}}.$$

This modification is due to the attempt to use only the Strichartz estimates provided by Lemma 2.10.

Definition 3.4 (Z -norm). Let

$$p_0 := \frac{16}{3} + \frac{1}{6} = \frac{11}{2}, \quad q_0 := 4 \quad \text{and} \quad p_1 := 100, \quad q_1 := 100, \quad (3.2)$$

then we define $\mathcal{P} := \{(p_0, q_0), (p_1, q_1)\}$ and the norm

$$\|u\|_{Z(I)} := \sum_{(p,q) \in \mathcal{P}} \sup_{J \subseteq I, |J| \leq 1} \left(\sum_{N \geq 1} N^{(\frac{2}{p} + \frac{3}{q} - \frac{1}{2})p} \|P_N u\|_{L^p(J, L^q(\mathbb{T}^3))}^p \right)^{\frac{1}{p}}.$$

The following properties follow immediately:

Corollary 3.5 (Properties of the Z -norm). *Let $I \subseteq \mathbb{R}$ be a bounded interval.*

(i) *For all $\phi \in H^1(\mathbb{T}^3)$ we have*

$$\|e^{it\Delta_\theta} \phi\|_{Z(I)} \lesssim \|\phi\|_{H^1(\mathbb{T}^3)}.$$

(ii) *Let $|I| \leq 1$. For all $p \in [p_0, p_1]$ and $q \geq q_p := \frac{pq_0 q_1 (p_1 - p_0)}{p_0 q_1 (p_1 - p) + p_1 q_0 (p - p_0)}$ the following holds true:*

$$\|P_N u\|_{L^p(I, L^q(\mathbb{T}^3))} \lesssim N^{\frac{1}{2} - \frac{2}{p} - \frac{3}{q}} \|P_N u\|_{Z(I)}.$$

(iii) *For all $u \in X_r^1(I)$ we have*

$$\|u\|_{Z(I)} \lesssim \|u\|_{X_r^1(I)}$$

and thus, $X_r^1(I) \hookrightarrow Z(I)$.

Proof. The first statement follows from Strichartz estimates, see Lemma 2.10, and the fact that $\ell^2 \subseteq \ell^p$ for $p \geq 2$:

$$\|e^{it\Delta_\theta}\phi\|_{Z(I)} \lesssim \sum_{(p,q) \in \mathcal{P}} \left(\sum_{N \geq 1} N^p \|P_N \phi\|_{L^2(\mathbb{T}^3)}^p \right)^{\frac{1}{p}} \lesssim \|\phi\|_{H^1(\mathbb{T}^3)}.$$

Claim (ii) follows essentially from interpolation: Since $q_p \geq 2$, we may apply Bernstein's inequality, cf. Lemma 1.53 (iii), to obtain

$$\|P_N u\|_{L^p(I, L^q(\mathbb{T}^3))} \lesssim N^{3(\frac{1}{q_p} - \frac{1}{q})} \|P_N u\|_{L^p(I, L^{q_p}(\mathbb{T}^3))}.$$

Let $\vartheta := \frac{p_0(p-p_1)}{p(p_0-p_1)}$, then we use Hölder's inequality and Young's inequality for products to deduce

$$\begin{aligned} \|P_N u\|_{L_t^p L_x^{q_p}} &\leq \|P_N u\|_{L_t^{p_0} L_x^{q_0}}^\vartheta \|P_N u\|_{L_t^{p_1} L_x^{q_1}}^{1-\vartheta} \\ &= N^{\frac{1}{2} - \frac{2}{p} - \frac{3}{q_p}} \left(N^{\frac{2}{p_0} + \frac{3}{q_0} - \frac{1}{2}} \|P_N u\|_{L_t^{p_0} L_x^{q_0}} \right)^\theta \left(N^{\frac{2}{p_1} + \frac{3}{q_1} - \frac{1}{2}} \|P_N u\|_{L_t^{p_1} L_x^{q_1}} \right)^{1-\theta} \\ &\lesssim N^{\frac{1}{2} - \frac{2}{p} - \frac{3}{q_p}} \|P_N u\|_{Z(I)}. \end{aligned}$$

In order to prove (iii), we first observe that

$$\|u\|_{Z(I)} = \sum_{(p,q) \in \mathcal{P}} \sup_{J \subseteq I, |J| \leq 1} \inf_{v \cdot \mathbf{1}_J = u \cdot \mathbf{1}_J} \left(\sum_{N \geq 1} N^{(\frac{2}{p} + \frac{3}{q} - \frac{1}{2})p} \|P_N v\|_{L^p(J, L^q(\mathbb{T}^3))}^p \right)^{\frac{1}{p}}.$$

Hence, Corollary 3.7 below implies

$$\|u\|_{Z(I)} \lesssim \sum_{(p,q) \in \mathcal{P}} \sup_{J \subseteq I, |J| \leq 1} \inf_{v \cdot \mathbf{1}_J = u \cdot \mathbf{1}_J} \left(\sum_{N \geq 1} N^p \|e^{-it\Delta_\theta} P_N v\|_{U^p}^p \right)^{\frac{1}{p}}.$$

This immediately implies (iii), since $U^2 \hookrightarrow U^p$ and $\ell^2 \subset \ell^p$ for any $p \geq 2$. \square

We now state the notion of a strong solution.

Definition 3.6 (Strong solution).

- (i) Let $I \subseteq \mathbb{R}$ be an interval, $t_0 \in I$, and $f \in L^1(I, L^2(\mathbb{T}^3))$, then we define the *Duhamel term* as

$$\mathcal{I}_{t_0}(f)(t) := \int_{t_0}^t e^{i(t-s)\Delta_\theta} f(s) ds$$

for $t \in I \cup \{\inf I\}$, $\mathcal{I}_{t_0}(f)(t) := 0$ for $t < \inf I$, and $\mathcal{I}_{t_0}(f)(t) := \lim_{s \rightarrow \sup I} \mathcal{I}_{t_0}(f)(s)$ for $t \geq \sup I$.

- (ii) We call $u \in C(I, H^1(\mathbb{T}^3))$ a *strong solution* to

$$i\partial_t u + \Delta_\theta u = F(u)$$

if $u \in X_r^1(I)$ and u satisfies

$$u(t) = e^{i(t-t_0)\Delta_\theta} u(t_0) - i\mathcal{I}_{t_0}(F(u))(t)$$

for all $t, t_0 \in I$.

The Strichartz estimates in Lemma 2.10 immediately imply the following result.

Corollary 3.7. *Let $I \subseteq \mathbb{R}$ be any interval with $|I| \leq 1$ and $p > \frac{16}{3}$, then for any cube $\mathcal{C} \subset \mathbb{Z}^3$ of size $N \geq 1$ and any $e^{-it\Delta_\theta} P_{\mathcal{C}}u \in U^p$ we have*

$$\|P_{\mathcal{C}}u\|_{L^p(I, L^4(\mathbb{T}^3))} \lesssim N^{\frac{3}{4} - \frac{2}{p}} \|P_{\mathcal{C}}e^{-it\Delta_\theta}u\|_{U^p}$$

and

$$\|P_{\mathcal{C}}u\|_{L^p(I \times \mathbb{T}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|P_{\mathcal{C}}e^{-it\Delta_\theta}u\|_{U^p}. \quad (3.3)$$

In particular, if $P_{\mathcal{C}}u \in Y_r^0(I)$, then

$$\begin{aligned} \|P_{\mathcal{C}}u\|_{L^p(I, L^4(\mathbb{T}^3))} &\lesssim N^{\frac{3}{4} - \frac{2}{p}} \|P_{\mathcal{C}}u\|_{Y_r^0(I)}, \\ \|P_{\mathcal{C}}u\|_{L^p(I \times \mathbb{T}^3)} &\lesssim N^{\frac{3}{2} - \frac{5}{p}} \|P_{\mathcal{C}}u\|_{Y_r^0(I)}. \end{aligned}$$

Proof. We only prove the estimate (3.3) since the bound of the $L_t^p L_x^4$ -norm follows from a similar argument.

For a function $P_{\mathcal{C}}v \in U^p$ which is defined on \mathbb{R} , we have

$$\|P_{\mathcal{C}}e^{it\Delta_\theta}v\|_{L^p(I \times \mathbb{T}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|P_{\mathcal{C}}v\|_{U^p}. \quad (3.4)$$

It suffices to prove (3.4) for a U^p -atom

$$P_{\mathcal{C}}v(t, x) = \sum_{k=1}^K \mathbf{1}_{[t_{k-1}, t_k)}(t) P_{\mathcal{C}}e^{it\Delta_\theta} \phi_k, \quad \sum_{k=1}^K \|P_{\mathcal{C}}\phi_k\|_{L^2(\mathbb{T}^3)}^p = 1.$$

Bernstein's inequality and the Strichartz estimate in Lemma 2.10 yield

$$\|P_{\mathcal{C}}v\|_{L^p(I \times \mathbb{T}^3)} \leq \left(\sum_{k=1}^K \|P_{\mathcal{C}}e^{it\Delta_\theta} \phi_k\|_{L^p(I \times \mathbb{T}^3)}^p \right)^{\frac{1}{p}} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \left(\sum_{k=1}^K \|P_{\mathcal{C}}\phi_k\|_{L^2(\mathbb{T}^3)}^p \right)^{\frac{1}{p}} \lesssim N^{\frac{3}{2} - \frac{5}{p}}.$$

This proves (3.4).

One may obtain the bound in $Y_r^0(I)$ from the bound in U^p as follows: Since $P_{\mathcal{C}}u \in Y_r^0(I)$, we see that for any $\varepsilon > 0$ there is $J_0 \subseteq I$ and an extension $v \in Y^0$ of $P_{\mathcal{C}}u|_{J_0}$ with

$$\|v\|_{Y^0} \leq \|P_{\mathcal{C}}u\|_{Y_r^0(I)} + \varepsilon.$$

Now, inequality (3.4) and the embedding $V^2 \hookrightarrow U^p$ give

$$\|P_{\mathcal{C}}u\|_{L^p(I \times \mathbb{T}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|v\|_{Y^0} \lesssim N^{\frac{3}{2} - \frac{5}{p}} (\|P_{\mathcal{C}}u\|_{Y_r^0(I)} + \varepsilon).$$

For $\varepsilon > 0$ tending to zero this implies

$$\|P_{\mathcal{C}}u\|_{L^p(I \times \mathbb{T}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|P_{\mathcal{C}}u\|_{Y_r^0(I)}. \quad \square$$

The following statement is an analogue of Lemma 2.5 and may be proved similarly.

Lemma 3.8. *Let $I \subseteq \mathbb{R}$ be a bounded interval. Furthermore, let $P_N f \in L^1(I, L^2(\mathbb{T}^3))$ for all $N \geq 1$ and $t_0 \in I$. Then, $\sum_{N \geq 1} \mathcal{I}_{t_0}(P_N f) =: \mathcal{I}_{t_0}(f) \in X_r^1(I)$ and*

$$\|\mathcal{I}_{t_0}(f)\|_{X_r^1(I)} \lesssim \sup_{\substack{v \in Y_r^{-1}(I): \\ \|v\|_{Y_r^{-1}(I)} \leq 1}} \sum_{N \geq 1} \left| \int_I \int_{\mathbb{T}^3} P_N f(t, x) \overline{v(t, x)} dx dt \right|$$

provided that the right-hand side is finite. In particular, if $f \in L^1(I, H^1(\mathbb{T}^3))$, then

$$\|\mathcal{I}_{t_0}(f)\|_{X_r^1(I)} \lesssim \|f\|_{L^1(I, H^1(\mathbb{T}^3))}. \quad (3.5)$$

Consequently, for any $g \in C^1(I, C^2(\mathbb{T}^3))$ we have

$$\|g\|_{X_r^1(I)} \lesssim \|g(t_0)\|_{H^1(\mathbb{T}^3)} + \left(\sum_{N \geq 1} \|P_N(i\partial_t + \Delta_{\theta})g\|_{L^1(I, H^1(\mathbb{T}^3))}^2 \right)^{\frac{1}{2}}. \quad (3.6)$$

Proof/Reference. As in the proof of Lemma 2.5, it suffices to show for any $L \geq 1$,

$$\|\mathcal{I}_{t_0}(P_{\leq L} f)\|_{X^s} \lesssim \sup_{\substack{v \in Y^{-s}: \\ \|v\|_{Y^{-s}} = 1}} \sum_{N \geq 1} \left| \int_{\tau_0} \int_M P_N f(t, x) \overline{v(t, x)} dx dt \right|.$$

This follows along the lines of the proof of (2.7), in which we observe that (2.8) holds if we replace a by any $t_0 \in I$.

Inequality (3.6) is an immediate consequence of Proposition 3.3 (ii), (3.5), and the identity

$$g(t) = e^{i(t-t_0)\Delta_{\theta}} g(t_0) - i\mathcal{I}_{t_0}((i\partial_t + \Delta_{\theta})g)(t)$$

for $t_0 \in I$. □

3.3 Local well-posedness and stability theory

Large data local well-posedness and stability results are addressed in this section. Similar results have been obtained in [IP12b, Section 3] for \mathbb{T}^3 , in [Str15, Section 3] for rectangular tori, and in [PTW14, Section 3] for the 3-sphere. Note that the local results proved here are slightly more precise compared to Chapter 2, see Corollary 3.13 below.

We introduce another norm that interpolates between X_r^1 and Z . We use this norm to obtain estimates that are linear in a norm controlling $L^\infty(I, H^1(\mathbb{T}^3))$.

Definition 3.9. Let $I \subseteq \mathbb{R}$ be an interval. For $u \in X_r^1(I)$ we define the Z' -norm

$$\|u\|_{Z'(I)} := \|u\|_{Z(I)}^{\frac{1}{2}} \|u\|_{X_r^1(I)}^{\frac{1}{2}}.$$

3.3.1 Estimates on the Duhamel term

Lemma 3.10. *There exists $\delta > 0$ such that for every interval I with $|I| \leq 1$, all dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$, and $P_{N_1}u_1, P_{N_2}u_2, P_{N_3}u_3 \in X_r^1(I)$ the following trilinear estimate holds*

$$\|P_{N_1}u_1 P_{N_2}u_2 P_{N_3}u_3\|_{L^2(I \times \mathbb{T}^3)} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2}\right)^\delta \|P_{N_1}u_1\|_{Y_r^0(I)} \|P_{N_2}u_2\|_{Z'(I)} \|P_{N_3}u_3\|_{Z'(I)}. \quad (3.7)$$

Moreover, for $p_0 = \frac{11}{2}$ and $q_0 = 4$ defined as in (3.2) we have

$$\|P_{N_1}u_1 P_{N_2}u_2 P_{N_3}u_3\|_{L^2(I \times \mathbb{T}^3)} \lesssim (N_1 N_2)^{\frac{1}{2} - \frac{2}{p_0} - \frac{3}{q_0}} N_3^{\frac{4}{p_0} - \frac{1}{2}} \|P_{N_1}u_1\|_{Z(I)} \|P_{N_2}u_2\|_{Z(I)} \|P_{N_3}u_3\|_{Z(I)}. \quad (3.8)$$

Proof. For notational convenience we write $L_{t,x}^p$ and $L_t^p L_x^q$ for $L^p(I \times \mathbb{T}^3)$ and $L^p(I, L^q(\mathbb{T}^3))$, respectively.

First, we prove inequality (3.7): This follows from interpolation between

$$\|P_{N_1}u_1 P_{N_2}u_2 P_{N_3}u_3\|_{L_{t,x}^2} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2}\right)^{2\delta} \|P_{N_1}u_1\|_{Y_r^0(I)} \|P_{N_2}u_2\|_{X_r^1(I)} \|P_{N_3}u_3\|_{X_r^1(I)} \quad (3.9)$$

and

$$\|P_{N_1}u_1 P_{N_2}u_2 P_{N_3}u_3\|_{L_{t,x}^2} \lesssim \|P_{N_1}u_1\|_{Y_r^0(I)} \|P_{N_2}u_2\|_{Z(I)} \|P_{N_3}u_3\|_{Z(I)}. \quad (3.10)$$

Hence, it remains to prove (3.9) and (3.10). Inequality (3.9) follows in a well-known fashion: From the definition of the spaces and since $X_r^1(I) \hookrightarrow Y_r^1(I)$, we see that

$$N_j \|P_{N_j}u_j\|_{Y_r^0(I)} \lesssim \|P_{N_j}u_j\|_{Y_r^1(I)} \lesssim \|P_{N_j}u_j\|_{X_r^1(I)}$$

for $j = 1, 2, 3$. Hence, to prove (3.9), it suffices to show

$$\|P_{N_1}u_1 P_{N_2}u_2 P_{N_3}u_3\|_{L_{t,x}^2} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2}\right)^{2\delta} N_2 N_3 \|P_{N_1}u_1\|_{Y_r^0(I)} \|P_{N_2}u_2\|_{Y_r^0(I)} \|P_{N_3}u_3\|_{Y_r^0(I)},$$

which follows from Proposition 2.13 and Lemma 2.8.

Next we prove (3.10). Thanks to spatial orthogonality (see Step 1 in the proof of Lemma 2.12), we may replace $P_{N_1}u_1$ in (3.10) by $P_{\mathcal{C}}P_{N_1}u_1$, where $\mathcal{C} \subset \mathbb{Z}^3$ is a cube of side length N_2 . Using Hölder's inequality, we obtain

$$\|P_{\mathcal{C}}P_{N_1}u_1 P_{N_2}u_2 P_{N_3}u_3\|_{L_{t,x}^2} \leq \|P_{\mathcal{C}}P_{N_1}u_1\|_{L_t^{p_0} L_x^4} \|P_{N_2}u_2\|_{L_t^{p_0} L_x^4} \|P_{N_3}u_3\|_{L_t^p L_x^\infty},$$

where $p := \frac{2p_0}{p_0-4}$. Now, Corollary 3.7 implies

$$\|P_{\mathcal{C}}P_{N_1}u_1\|_{L_t^{p_0} L_x^4} \lesssim N_2^{\frac{3}{4} - \frac{2}{p_0}} \|P_{\mathcal{C}}P_{N_1}u_1\|_{Y_r^0(I)},$$

and from the definition of the Z -norm, we infer

$$\|P_{N_2}u_2\|_{L_t^{p_0} L_x^4} \lesssim N_2^{-\frac{1}{4} - \frac{2}{p_0}} \|P_{N_2}u_2\|_{Z(I)}.$$

We apply Corollary 3.5 (ii) to treat the remaining term:

$$\|P_{N_3}u_3\|_{L_t^p L_x^\infty} \lesssim N_3^{\frac{4}{p_0} - \frac{1}{2}} \|P_{N_3}u_3\|_{Z(I)}.$$

All in all, we deduce

$$\|P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L_{t,x}^2} \lesssim \left(\frac{N_3}{N_2}\right)^{\frac{4}{p_0}-\frac{1}{2}} \|P_{N_1} u_1\|_{Y_T^0(I)} \|P_{N_2} u_2\|_{Z(I)} \|P_{N_3} u_3\|_{Z(I)},$$

which implies (3.10) because of $\frac{4}{p_0} - \frac{1}{2} > 0$. This proves (3.7).

The bound (3.8) follows from

$$\|P_C P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L_{t,x}^2} \leq \|P_C P_{N_1} u_1\|_{L_t^{p_0} L_x^{q_0}} \|P_{N_2} u_2\|_{L_t^{p_0} L_x^{q_0}} \|P_{N_3} u_3\|_{L_t^p L_x^\infty},$$

the definition of the Z -norm, and Corollary 3.5 (ii). \square

The previous lemma allows us to prove an important nonlinear estimate for the Duhamel term, which is stronger than Lemma 2.9.

Lemma 3.11. *Let $I \subset \mathbb{R}$ be an interval with $|I| \leq 1$. Then, for any $t_0 \in I$ and $u_j \in X_T^1(I)$, $j = 1, \dots, 5$, the estimate*

$$\left\| \mathcal{I}_{t_0} \left(\prod_{j=1}^5 \tilde{u}_j \right) \right\|_{X_T^1(I)} \lesssim \sum_{k=1}^5 \|u_k\|_{X_T^1(I)} \prod_{\substack{j=1 \\ j \neq k}}^5 \|u_j\|_{Z'(I)}$$

holds true, where \tilde{u}_j denotes either u_j or its complex conjugate.

Proof. To prove the lemma, we closely follow the arguments in the proofs of Lemma 2.9 and [IP12a, Lemma 3.2].

We decompose $\prod_{j=1}^5 \tilde{u}_j$ as

$$\sum_{N_1 \geq 1} P_{N_1} \tilde{u}_1 \prod_{j=2}^5 P_{\leq N_1} \tilde{u}_j + \sum_{k=2}^5 \sum_{N_k \geq 2} P_{N_k} \tilde{u}_k \prod_{j=1}^{k-1} P_{< N_k} \tilde{u}_j \prod_{j=k+1}^5 P_{\leq N_k} \tilde{u}_j. \quad (3.11)$$

This can be easily seen as follows: For a quintuple $(N_1, N_2, N_3, N_4, N_5)$ we denote $N_{\max} := \max_{j=1, \dots, 5} N_j$, then

$$(2^{\mathbb{N}_0})^5 = \bigcup_{k=1, \dots, 5} \{(N_1, \dots, N_5) \in (2^{\mathbb{N}_0})^5 : N_j < N_{\max}, j < k, \text{ and } N_k = N_{\max}\}$$

is a disjoint partition. Each of this sets corresponds to one of the sums in (3.11). Hence, by symmetry, it suffices to prove the more precise estimate

$$\left\| \mathcal{I}_{t_0} \left(\sum_{N_1 \geq 1} P_{N_1} \tilde{u}_1 \prod_{j=2}^5 P_{\leq BN_1} \tilde{u}_j \right) \right\|_{X_T^1(I)} \lesssim_B \|u_1\|_{X_T^1(I)} \prod_{j=2}^5 \|u_j\|_{Z'(I)} \quad (3.12)$$

for any $B \geq 1$.

By Lemma 3.8, it suffices to show that for any $u_0 \in Y_T^{-1}(I)$ we have

$$\sum_{N_0 \geq 1} \left| \int_I \int_{\mathbb{T}^3} P_{N_0} u_0 \sum_{N_1 \geq 1} P_{N_1} \tilde{u}_1 \prod_{j=2}^5 P_{\leq BN_1} \tilde{u}_j dx dt \right| \lesssim \|u_0\|_{Y_T^{-1}(I)} \|u_1\|_{X_T^1(I)} \prod_{j=2}^5 \|u_j\|_{Z'(I)} \quad (3.13)$$

in order to verify (3.12). To prove this, we decompose u_k dyadically in space

$$u_k = \sum_{N_k \geq 1} P_{N_k} u_k, \quad k = 1, \dots, 5.$$

Note that the L^2 -norm does not change under complex conjugation and the integral is non-trivial only if the two highest frequencies are comparable. Hence, by the Cauchy–Schwarz inequality and symmetry, it suffices to replace the left-hand side of (3.13) by

$$\Sigma := \sum_{\mathcal{N}} \|P_{N_1} u_1 P_{N_3} u_3 P_{N_5} u_5\|_{L^2(I \times \mathbb{T}^3)} \|P_{N_0} u_0 P_{N_2} u_2 P_{N_4} u_4\|_{L^2(I \times \mathbb{T}^3)},$$

where \mathcal{N} is the set of all sextuples (N_0, N_1, \dots, N_5) such that

$$N_1 \approx_B \max\{N_0, N_2\} \geq N_2 \geq N_3 \geq N_4 \geq N_5.$$

We subdivide the sum into two parts $\Sigma = \Sigma_1 + \Sigma_2$, where Σ_1 and Σ_2 are defined via the constraints $N_2 \leq N_0 \approx N_1$ and $N_0 < N_2 \approx N_1$, respectively. The trilinear estimate (3.7) implies

$$\Sigma_1 \lesssim \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ N_2 \leq N_0 \approx N_1}} \left(\frac{N_5}{N_1} + \frac{1}{N_3}\right)^\delta \left(\frac{N_4}{N_0} + \frac{1}{N_2}\right)^\delta \|P_{N_0} u_0\|_{Y_r^0(I)} \|P_{N_1} u_1\|_{Y_r^0(I)} \prod_{j=2}^5 \|P_{N_j} u_j\|_{Z'(I)}$$

for some $\delta > 0$. Summing up with respect to N_2, N_3, N_4, N_5 , and finally with respect to $N_0 \approx N_1$ (using Cauchy–Schwarz) yields

$$\Sigma_1 \lesssim_B \|u_0\|_{Y_r^{-1}(I)} \|u_1\|_{X_r^1(I)} \prod_{j=2}^5 \|u_j\|_{Z'(I)}.$$

The remaining case $N_0 < N_2 \approx N_1$ can be treated as follows: From (3.7) and Hölder’s estimate, we get

$$\begin{aligned} \Sigma_2 &\lesssim_B \sum_{\substack{(N_0, \dots, N_5) \in \mathcal{N}: \\ N_0 < N_2 \approx N_1}} \left(\frac{N_5}{N_1} + \frac{1}{N_3}\right)^\delta \|P_{N_1} u_1\|_{Y_r^0(I)} \|P_{N_3} u_3\|_{Z'(I)} \|P_{N_5} u_5\|_{Z'(I)} \\ &\quad \times \|P_{N_0} u_0\|_{L^{33/5}(I, L^5(\mathbb{T}^3))} \|P_{N_2} u_2\|_{L^{p_0}(I, L^{q_0}(\mathbb{T}^3))} \|P_{N_4} u_4\|_{L^6(I, L^{20}(\mathbb{T}^3))}. \end{aligned}$$

We observe that $\frac{33}{5} > \frac{16}{3}$, and hence, from Sobolev’s inequality and Corollary 3.7 we may estimate

$$\|P_{N_0} u_0\|_{L^{33/5}(I, L^5(\mathbb{T}^3))} \lesssim N_0^{\frac{3}{20}} \|P_{N_0} u_0\|_{L^{33/5}(I, L^4(\mathbb{T}^3))} \lesssim N_0^{\frac{197}{330}} \|P_{N_0} u_0\|_{Y_r^0(I)}.$$

Noting that q_6 , given in Corollary 3.5 (ii), is less than 20, we may deduce from Corollary 3.5 (ii) that

$$\|P_{N_2} u_2\|_{L^{p_0}(I, L^{q_0}(\mathbb{T}^3))} \lesssim N_2^{-\frac{27}{44}} \|P_{N_2} u_2\|_{Z(I)} \lesssim N_2^{-\frac{27}{44}} \|P_{N_2} u_2\|_{Z'(I)}$$

and

$$\|P_{N_4} u_4\|_{L^6(I, L^{20}(\mathbb{T}^3))} \lesssim N_4^{\frac{1}{60}} \|P_{N_4} u_4\|_{Z(I)} \lesssim N_4^{\frac{1}{60}} \|P_{N_4} u_4\|_{Z'(I)}.$$

Summing with respect to N_2, N_3, N_4 , and N_5 yields

$$\begin{aligned} \Sigma_2 &\lesssim_B \sum_{\substack{N_0, N_1 \geq 1: \\ N_0 \leq N_1}} \left(\frac{N_0}{N_1}\right)^{\frac{197}{330}} \|P_{N_0} u_0\|_{Y_r^0(I)} \|P_{N_1} u_1\|_{Y_r^0(I)} \prod_{j=2}^5 \|u_j\|_{Z'(I)} \\ &\lesssim_B \|u_0\|_{Y_r^{-1}(I)} \|u_1\|_{X_r^1(I)} \prod_{j=2}^5 \|u_j\|_{Z'(I)}, \end{aligned}$$

which proves (3.13). \square

3.3.2 Local well-posedness

The foregoing estimates allow us to obtain a local existence result and a criterion for global existence. Statement (iii) states that the solution stays regular as long as the Z -norm stays finite.

Proposition 3.12 (Local well-posedness I). *Let $E \geq 1$ and $\rho \in [-1, 1]$ be given.*

(i) *There exists $\delta_0 = \delta_0(E) < 1$ such that if $\|\phi\|_{H^1(\mathbb{T}^3)} \leq E$ and*

$$\|e^{i(t-t_0)\Delta_\theta}\phi\|_{Z(I)} + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(I)} \leq \delta_0$$

on some interval $I \ni t_0$ with $|I| \leq 1$, then there exists a unique strong solution $u \in C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I)$ to the approximate nonlinear Schrödinger equation

$$i\partial_t u + \Delta_\theta u = \rho u|u|^4 + e \quad (3.14)$$

with initial data $u(t_0) = \phi$. Besides,

$$\|u(t) - e^{i(t-t_0)\Delta_\theta}\phi\|_{X_r^1(I)} \lesssim_E \|e^{i(t-t_0)\Delta_\theta}\phi\|_{Z(I)}^{\frac{3}{2}} + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(I)}. \quad (3.15)$$

If $e = 0$ and $\rho = \pm 1$, then the quantities $E(u)$ and $M(u)$, defined in (2.26), are conserved on I .

(ii) *Suppose that $I \subset \mathbb{R}$ is an open bounded interval and $u \in C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I)$ is a strong solution to the approximate nonlinear Schrödinger equation (3.14) on I with*

$$\|u\|_{L^\infty(I, H^1(\mathbb{T}^3))} \leq E.$$

There exists $\varepsilon_0 = \varepsilon_0(E) > 0$ with the property that if

$$\|u\|_{Z(I)} \leq \varepsilon_0 \quad \text{and} \quad \sup_{t_0 \in I} \|\mathcal{I}_{t_0}(e)\|_{X_r^1(I)} \leq \varepsilon_0,$$

then the following holds true for all $t_0 \in I$:

$$\|e^{i(t-t_0)\Delta_\theta}u(t_0)\|_{Z(I)} \lesssim \varepsilon_0.$$

(iii) *If $u \in C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I)$ is a strong solution to (3.1) on some bounded open interval $I \subset \mathbb{R}$ and*

$$\|u\|_{Z(I)} < +\infty,$$

then u can be extended as a nonlinear solution to a neighborhood of \bar{I} and

$$\|u\|_{X_r^1(I)} \leq C(E(u), \|u\|_{Z(I)})$$

for some function C depending on $E(u)$ and $\|u\|_{Z(I)}$.

Proof. Let $E \geq 1$ and $\rho \in [-1, 1]$ be given.

Ad (i). We prove the first claim by a standard fixed-point argument. Let $\phi \in H^1(\mathbb{T}^3)$ with $\|\phi\|_{H^1(\mathbb{T}^3)} \leq E$. We define the complete space (since it is closed in $X_r^1(I)$)

$$\mathcal{S}_I := \{u \in C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I) : \|u\|_{X_r^1(I)} \leq 2E, \|u\|_{Z'(I)} \leq a\},$$

where $0 < a = a(E) < 1$ will be chosen later. Define the mapping

$$\Phi(v)(t) := e^{i(t-t_0)\Delta_\theta} \phi - i\mathcal{I}_{t_0}(\rho v|v|^4 + e)(t). \quad (3.16)$$

First, we verify that Φ is a contraction on \mathcal{S}_I provided a is small enough. Let $u, v \in \mathcal{S}_I$, then it follows that

$$\|\Phi(u) - \Phi(v)\|_{X_r^1(I)} \leq \|\mathcal{I}_{t_0}(u|u|^4 - v|v|^4)\|_{X_r^1(I)}.$$

Consequently, thanks to Lemma 3.11, we have

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{X_r^1(I)} &\lesssim (\|u\|_{X_r^1(I)} + \|v\|_{X_r^1(I)}) (\|u\|_{Z'(I)} + \|v\|_{Z'(I)})^3 \|u - v\|_{X_r^1(I)} \\ &\lesssim Ea^3 \|u - v\|_{X_r^1(I)}. \end{aligned}$$

We choose $0 < a < 1$ such that $\|\Phi(u) - \Phi(v)\|_{X_r^1(I)} < \frac{1}{2}\|u - v\|_{X_r^1(I)}$. Using the triangle inequality, Proposition 3.3 (ii), and Lemma 3.11, we obtain

$$\|\Phi(u)\|_{X_r^1(I)} \leq \|\Phi(0)\|_{X_r^1(I)} + \|\Phi(u) - \Phi(0)\|_{X_r^1(I)} \leq \|\phi\|_{H^1(\mathbb{T}^3)} + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(I)} + CEa^4$$

for some $C \geq 1$. If necessary, we decrease a further such that $Ca^4 \leq \frac{1}{2}$, and we choose $\delta_0 < \frac{1}{2}$. This implies $\|\Phi(u)\|_{X_r^1(I)} \leq 2E$. To show $\|\Phi(u)\|_{Z'(I)} \leq a$, we estimate

$$\|\Phi(u)\|_{Z'(I)} \lesssim (\|\Phi(0)\|_{Z(I)} + \|\Phi(u) - \Phi(0)\|_{X_r^1(I)})^{\frac{1}{2}} \|\Phi(u)\|_{X_r^1(I)}^{\frac{1}{2}}.$$

Then, we use the sub-linearity of $x \mapsto x^{\frac{1}{2}}$ as well as the bounds

$$\|\Phi(0)\|_{Z(I)} \lesssim \delta_0 \quad \text{and} \quad \|\Phi(u) - \Phi(0)\|_{X_r^1(I)} \lesssim Ea^4$$

to get

$$\|\Phi(u)\|_{Z'(I)} \leq C(E^{\frac{1}{2}}\delta_0^{\frac{1}{2}} + Ea^2).$$

By possibly decreasing a again, we may obtain that $2CEa^2 < a$. Now, we choose $\delta_0 = \delta_0(E)$ to be small enough such that $2CE^{\frac{1}{2}}\delta_0^{\frac{1}{2}} < a$. Therefore,

$$\|\Phi(u)\|_{Z'(I)} \leq a < 1.$$

Consequently, Φ is a contraction on \mathcal{S}_I , and hence, there exists a unique fixed-point $\bar{u} \in \mathcal{S}_I$. This argument only gives uniqueness in \mathcal{S}_I . Nevertheless, we justify uniqueness in $C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I)$. For that purpose, assume that two solutions $u, v \in C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I)$ satisfy $u(t_0) = v(t_0)$. From the continuity in time, it is clear that the set $\{t \in I : u(t) = v(t)\}$ is closed in I . We prove that this set is also open in I , what finishes the proof of (i). Let $t_1 \in \{t \in I : u(t) = v(t)\}$. One may choose an open interval $J \subseteq I$ with $t_1 \in J$ such that $u|_J, v|_J \in \mathcal{S}_J$. Indeed, with $E := \max\{\|u\|_{X_r^1(I)}, \|v\|_{X_r^1(I)}\}$ we choose $a = a(E)$ as above and take J small enough such that $\max\{\|u\|_{Z(J)}, \|v\|_{Z(J)}\} < E^{-\frac{1}{2}}a$. From the uniqueness in \mathcal{S}_J , we obtain $u|_J = v|_J$. Thus, $\{t \in I : u(t) = v(t)\}$ is open in I and hence, is equal to I .

As noted above, for a strong solution to (3.14) and sufficiently small $\delta_0(E) < 1$ we have

$$\|\bar{u}(t) - e^{i(t-t_0)\Delta_\theta} \phi\|_{X_r^1(I)} = \|\Phi(\bar{u}) - \Phi(0)\|_{X_r^1(I)} + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(I)} \lesssim Ea^4 + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(I)}.$$

Inequality (3.15) then follows from choosing a such that

$$0 < a \leq (\|e^{i(t-t_0)\Delta_\theta} \phi\|_{Z(I)} + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(I)})^{\frac{3}{8}}$$

provided the right-hand side is larger than zero. Otherwise, the left-hand side of (3.15) is zero in which case we have nothing to show.

Ad (ii). Let $\varepsilon_0 > 0$, which shall be chosen later. Furthermore, let $u \in C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I)$ be a strong solution of (3.14) on some bounded interval I , and assume that

$$\|u\|_{Z(I)} \leq \varepsilon_0, \quad \sup_{t_0 \in I} \|\mathcal{I}_{t_0}(e)\|_{X_r^1(I)} \leq \varepsilon_0, \quad \text{as well as} \quad \|u\|_{L^\infty(I, H^1(\mathbb{T}^3))} \leq E.$$

It suffices to consider intervals of length at most one. If that is not the case, then we subdivide I into finitely many intervals of length less or equal to one, and run the following argument on each interval separately. In the sequel, we write $I = (T_-, T^+)$. Now, we show that the assumptions imply

$$\|e^{i(t-t_0)\Delta_\theta} u(t_0)\|_{Z(I)} \lesssim \varepsilon_0$$

for all $t_0 \in I$ provided $\varepsilon_0 > 0$ is sufficiently small depending on E . Let $t_0 \in I$ be arbitrary, and define

$$h: [0, T^+ - T_-] \rightarrow \mathbb{R}, \quad h(s) := \|e^{i(t-t_0)\Delta_\theta} u(t_0)\|_{Z(T_-, T_-+s)}.$$

The function h is continuous in s and satisfies $h(0) = 0$. We choose $2\varepsilon_0 \leq \delta_0 = \delta_0(E)$. Then we use (i) as long as $h(s) \leq \frac{1}{2}\delta_0(E)$ and we get that

$$\|u(t) - e^{i(t-t_0)\Delta_\theta} u(t_0)\|_{X_r^1(T_-, T_-+s)} \lesssim_E h(s)^{\frac{3}{2}} + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(T_-, T_-+s)}.$$

For the same range of s we deduce

$$\begin{aligned} h(s) &\leq \|u\|_{Z(T_-, T_-+s)} + \tilde{C} \|u(t) - e^{i(t-t_0)\Delta_\theta} u(t_0)\|_{X_r^1(T_-, T_-+s)} \leq \varepsilon_0 + C(h(s)^{\frac{3}{2}} + \varepsilon_0) \\ &\leq C_0\varepsilon_0 + Ch(s)^{\frac{3}{2}}. \end{aligned} \quad (3.17)$$

We use (3.17) to conclude $h(s) \leq \frac{\delta_0}{2}$ for all $s \in [0, T^+ - T_-]$ provided ε_0 is small enough. To this end, we consider $f: [0, \infty) \rightarrow \mathbb{R}$, $f(x) = x - Cx^{\frac{3}{2}}$, which increases from 0 to its maximum value $\frac{4}{27C}$. Moreover, one easily sees that $f(x) \geq \frac{x}{2}$ on the interval $[0, (4C^2)^{-1}]$. Hence, we proved in (3.17) that $\frac{1}{2}h(s) \leq f(h(s)) \leq C_0\varepsilon_0$ provided $h(s) \leq \tilde{\delta} := \min\{\delta_0/2, (4C^2)^{-1}\}$. We choose $\varepsilon_0 = \varepsilon_0(E)$ to be small enough such that $C_0\varepsilon_0 < \frac{\tilde{\delta}}{4}$. Suppose there is $0 < s_0 < T^+ - T_-$ such that $h(s_0) \leq \tilde{\delta}$ and $h(s) > \tilde{\delta}$ for all $s_0 < s < T^+ - T_-$. Then, the argument above shows that $h(s_0) \leq \frac{\tilde{\delta}}{2}$, which contradicts the assumption since h is continuous. Thus, $h(s) \leq \tilde{\delta}$ for any $s \in [0, T^+ - T_-]$ and from $\frac{1}{2}h(s) \leq f(h(s)) \leq C_0\varepsilon_0$, we obtain the desired result

$$\|e^{i(t-t_0)\Delta_\theta} u(t_0)\|_{Z(I)} \leq 2C_0\varepsilon_0.$$

Ad (iii). We apply the argument that was used to prove (ii). Since the Z -norm is bounded, for any $\varepsilon_0 > 0$ there exists $T_1 \in (T^+ - 1, T^+)$ such that $(T_1, T^+) \subseteq I$ and

$$\|u\|_{Z(T_1, T^+)} \leq \varepsilon_0.$$

Hence, for some $t_0 \in (T_1, T^+)$ and δ_0 as in (i) there exists $\varepsilon_0 > 0$ small enough such that the argument above is applicable on (T_1, T^+) , and we obtain

$$\|e^{i(t-t_0)\Delta_\theta} u(t_0)\|_{Z(T_1, T^+)} \leq \frac{1}{2}\delta_0.$$

The continuity of h implies the existence of a larger time $T_2 > T^+$ such that $T_2 - T_1 < 1$ and

$$\|e^{i(t-t_0)\Delta_\theta} u(t_0)\|_{Z(T_1, T_2)} \leq \frac{3}{4}\delta_0.$$

Hence, we may apply (i). From the uniqueness, we obtain the existence of a nonlinear solution $\tilde{u} \in C((T_-, T_2), H^1(\mathbb{T}^3)) \cap X_r^1(T_-, T_2)$. A similar argument allows to extend the solution to the left-hand side.

Finally, we prove the estimate stated in (iii). Since $\|u(t)\|_{H^1(\mathbb{T}^3)} \lesssim E(u) + E(u)^{\frac{1}{3}}$ for any $t \in I$, we observe that

$$\|u\|_{L^\infty(I, H^1(\mathbb{T}^3))} \lesssim E(u) + E(u)^{\frac{1}{3}} < +\infty.$$

Let $\varepsilon_0 = \varepsilon_0(E(u)) > 0$ be the ε_0 given by (ii). We subdivide the interval I into $N = \mathcal{O}(\|u\|_{Z(I)}/\varepsilon_0)$ many subintervals I_k such that for every $k = 1, \dots, N$ we have

$$\|u\|_{Z(I_k)} \leq \varepsilon_0.$$

Let $t_k \in I_k$. Applying the triangle inequality yields

$$\|u\|_{X_r^1(I_k)} \leq \|e^{i(t-t_k)\Delta_\theta} u(t_k)\|_{X_r^1(I_k)} + \|u(t) - e^{i(t-t_k)\Delta_\theta} u(t_k)\|_{X_r^1(I_k)}.$$

Now, (ii) implies the smallness of the free solution, i.e.

$$\|e^{i(t-t_k)\Delta_\theta} u(t_k)\|_{Z(I_k)} \lesssim \varepsilon_0.$$

Choosing ε_0 possibly smaller (still depending on $E(u)$), we may apply (i) to obtain

$$\|u(t) - e^{i(t-t_k)\Delta_\theta} u(t_k)\|_{X_r^1(I_k)} \lesssim \|e^{i(t-t_k)\Delta_\theta} u(t_k)\|_{Z(I_k)}^{\frac{3}{2}} \leq 1.$$

We may conclude that $\|u\|_{X_r^1(I_k)}$ is bounded uniformly in k . Summing over k gives the desired estimate, where the right hand side only depends on the number of intervals N . Here N depends on $\|u\|_{Z(I)}$ and $E(u)$ as pointed out above. \square

The previous well-posedness result also implies the local well-posedness in [Theorem 2.3](#). We state it for later references.

Corollary 3.13 (Local well-posedness II). *Let $\rho \in [-1, 1]$. For every $\phi_* \in H^1(\mathbb{T}^3)$ there exists $\varepsilon > 0$ and $T = T(\phi_*) > 0$ such that for all initial data $\phi \in H^1(\mathbb{T}^3)$ with $\|\phi - \phi_*\|_{H^1(\mathbb{T}^3)} < \varepsilon$ the Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta_\theta u = \rho u|u|^4 \\ u(0, \cdot) = \phi \in H^1(\mathbb{T}^3) \end{cases}$$

has a unique solution $u \in C((-T, T), H^1(\mathbb{T}^3)) \cap X_r^1(-T, T)$.

Proof. Let $E > 1$ be such that $\|\phi_*\|_{H^1(\mathbb{T}^3)} < E - 1$. Then, for all $0 < \varepsilon < 1$ and all $\phi \in H^1(\mathbb{T}^3)$ with $\|\phi - \phi_*\|_{H^1(\mathbb{T}^3)} < \varepsilon$ it holds that

$$\|\phi\|_{H^1(\mathbb{T}^3)} \leq \|\phi_*\|_{H^1(\mathbb{T}^3)} + \varepsilon \leq E.$$

Now, [Corollary 3.5](#) (i) implies

$$\|e^{it\Delta_\theta} \phi_*\|_{Z(I)} \lesssim \|\phi_*\|_{H^1(\mathbb{T}^3)} < +\infty$$

for any interval $I \ni 0$. Hence, for any $\delta > 0$ there is I with $0 \in I$ and $|I| \leq 1$ such that

$$\|e^{it\Delta_\theta} \phi_*\|_{Z(I)} \leq \delta.$$

We easily get the smallness of the free solution $e^{it\Delta_\theta} \phi$ in the Z -norm from

$$\|e^{it\Delta_\theta} \phi\|_{Z(I)} - \|e^{it\Delta_\theta} \phi_*\|_{Z(I)} \leq C\|\phi - \phi_*\|_{H^1(\mathbb{T}^3)} < C\varepsilon.$$

Indeed, this immediately leads to $\|e^{it\Delta_\theta} \phi\|_{Z(I)} < \delta + C\varepsilon$. We now may choose ε and δ small enough such that $\delta + C\varepsilon < \delta_0$, where $\delta_0 = \delta_0(E)$ is given by [Proposition 3.12](#) (i). Finally, we may apply [Proposition 3.12](#) (i) to obtain the desired result. \square

3.3.3 Small data global well-posedness

In the proof of [Theorem 3.1](#) below, we shall rely on the following small data global well-posedness result.

Lemma 3.14 (Global well-posedness for small initial data). *There exists $\delta_0 > 0$ such that for all initial data $\phi \in H^1(\mathbb{T}^3)$ with $\|\phi\|_{H^1(\mathbb{T}^3)} =: \delta \leq \delta_0$ and every $T > 0$ the Cauchy problem [\(3.1\)](#) has a unique solution*

$$u \in C((-T, T), H^1(\mathbb{T}^3)) \cap X_r^1(-T, T).$$

Moreover, the solution satisfies

$$\|u\|_{X_r^1(-T, T)} \leq 2\delta \quad \text{and} \quad \|u(t) - e^{it\Delta_\theta} \phi\|_{X_r^1(-T, T)} \lesssim \delta^2.$$

Furthermore, the quantities $E(u)$ and $M(u)$, which are defined in [\(2.26\)](#), are conserved on $(-T, T)$.

Proof. The global existence follows from the same a priori bound on solutions in H^1 given in [\(2.19\)](#). For small enough initial data, this implies that there is a uniform in time bound on the H^1 -norm of the solution. Thus, the local well-posedness result may be iterated indefinitely many times.

The bounds in the X_r^1 -norm may be similarly obtained as in the proof of [Proposition 3.12](#). Indeed, let Φ be defined as in [\(3.16\)](#) (with $t_0 = 0$) and u the solution to [\(3.1\)](#) with initial data ϕ , i.e. $u = \Phi(u)$. Recall that $u \in \mathcal{S}_{(-T, T)}$, where \mathcal{S} is defined in the beginning of the proof of [Proposition 3.12](#). Thus, $\|u\|_{X_r^1(-T, T)} \leq 2\delta$ and $\|u\|_{Z'(-T, T)} \leq a$ for a sufficiently small. Then, we have

$$\|u\|_{X_r^1(-T, T)} \leq \|\Phi(0)\|_{X_r^1(-T, T)} + \|\Phi(u) - \Phi(0)\|_{X_r^1(-T, T)} \leq \delta + C\delta a^4 \leq 2\delta$$

provided $a^4 \leq C^{-1}$. The second bound may be obtained similarly as inequality [\(3.15\)](#). \square

3.3.4 Stability

We close our study of the local well-posedness theory with a stability result.

Proposition 3.15 (Stability). *Assume that I is an open bounded interval, $\rho \in [-1, 1]$, and $\tilde{u} \in C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I)$ satisfies the approximate Schrödinger equation*

$$i\partial_t \tilde{u} + \Delta_\theta \tilde{u} = \rho \tilde{u} |\tilde{u}|^4 + e \quad \text{on } I \times \mathbb{T}^3. \quad (3.18)$$

Suppose in addition that

$$\|\tilde{u}\|_{Z(I)} + \|\tilde{u}\|_{L^\infty(I, H^1(\mathbb{T}^3))} \leq M \quad (3.19)$$

for some $M \in [1, \infty)$. Assume that $t_0 \in I$ and $\phi \in H^1(\mathbb{T}^3)$ are such that the smallness condition

$$\|\phi - \tilde{u}(t_0)\|_{H^1(\mathbb{T}^3)} + \sup_{t_1 \in I} \|\mathcal{I}_{t_1}(e)\|_{X_r^1(I)} \leq \varepsilon \quad (3.20)$$

holds for some $0 < \varepsilon < \varepsilon_1$, where $\varepsilon_1 \leq 1$ is a small constant depending on M .

Then, there exists a strong solution $u \in C(I, H^1(\mathbb{T}^3)) \cap X_r^1(I)$ of the Schrödinger equation

$$i\partial_t u + \Delta_\theta u = \rho u |u|^4 \quad \text{on } I \times \mathbb{T}^3 \quad (3.21)$$

such that $u(t_0) = \phi$ and

$$\|u\|_{X_r^1(I)} + \|\tilde{u}\|_{X_r^1(I)} \leq C(M), \quad (3.22)$$

$$\|u - \tilde{u}\|_{X_r^1(I)} \leq C(M)\varepsilon. \quad (3.23)$$

Remark. Note that the bound $\|\tilde{u}\|_{X_r^1(I)} \leq M$ implies

$$\|\tilde{u}\|_{Z(I)} + \|\tilde{u}\|_{L^\infty(I, H^1(\mathbb{T}^3))} \lesssim M. \quad \diamond$$

Proof. We argue close to the proof of [IP12a, Proposition 3.5] and proceed in four steps:

Step 1. From Proposition 3.12 (i) it follows that there is $\delta_1 = \delta_1(M)$ such that if for some interval $J \subseteq I$ and $t_0 \in J$

$$\|e^{i(t-t_0)\Delta_\theta} \tilde{u}(t_0)\|_{Z(J)} + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(J)} \leq \delta_1,$$

then \tilde{u} is the only solution of (3.18) in $C(J, H^1(\mathbb{T}^3)) \cap X_r^1(J)$ and

$$\|\tilde{u}(t) - e^{i(t-t_0)\Delta_\theta} \tilde{u}(t_0)\|_{X_r^1(J)} \lesssim \|e^{i(t-t_0)\Delta_\theta} \tilde{u}(t_0)\|_{Z(J)}^{\frac{3}{2}} + \|\mathcal{I}_{t_0}(e)\|_{X_r^1(J)}.$$

Step 2. Proposition 3.12 (ii) implies the existence of $\varepsilon_1 = \varepsilon_1(M)$ such that if the inequalities

$$\|\tilde{u}\|_{Z(J)} \leq \varepsilon_1 \quad \text{and} \quad \sup_{t_0 \in J} \|\mathcal{I}_{t_0}(e)\|_{X_r^1(J)} \leq \varepsilon_1 \quad (3.24)$$

hold on an interval $J := [T_-, T^+] \subseteq I$, then

$$\|e^{i(t-T_-)\Delta_\theta} \tilde{u}(T_-)\|_{Z(J)} \lesssim \varepsilon_1. \quad (3.25)$$

Step 3. Let \tilde{u} be as stated in the proposition. We still consider the interval $J = [T_-, T^+]$ and assume

$$\begin{aligned} \|e^{i(t-T_-)\Delta_\theta} \tilde{u}(T_-)\|_{Z(J)} &\leq \varepsilon_1, \\ \|\tilde{u}\|_{Z(J)} &\leq \varepsilon_1, \\ \sup_{t_0 \in J} \|\mathcal{I}_{t_0}(e)\|_{X_r^1(J)} &\leq \varepsilon_1 \end{aligned} \quad (3.26)$$

for some sufficiently small constant $\varepsilon_1 = \varepsilon_1(M)$ such that the first two steps are applicable. Using Step 1, the X_r^1 -norm of \tilde{u} on J can be estimated by

$$\|\tilde{u}\|_{X_r^1(J)} \leq \|e^{i(t-T_-)\Delta_\theta} \tilde{u}(T_-)\|_{X_r^1(J)} + \|\tilde{u}(t) - e^{i(t-T_-)\Delta_\theta} \tilde{u}(T_-)\|_{X_r^1(J)} \leq M + 1. \quad (3.27)$$

The local well-posedness (Corollary 3.13) implies that there is an interval $K_u \ni T_-$, and a strong solution $u \in C(K_u, H^1(\mathbb{T}^3)) \cap X_r^1(K_u)$ to (3.21) such that

$$\|u(T_-) - \tilde{u}(T_-)\|_{H^1(\mathbb{T}^3)} \leq \varepsilon_1. \quad (3.28)$$

We set $\omega(t) := u(t) - \tilde{u}(t)$ for $t \in J \cap K_u$. Let $K := [T_-, T_- + \bar{s}] \cap J \cap K_u$, where

$$\bar{s} := \max\{s \in \mathbb{R} : \|\omega\|_{Z'([T_-, T_- + s] \cap J \cap K_u)} \leq 5C_0\varepsilon_1\}, \quad (3.29)$$

and $C_0 \geq 1$ is the constant of the embedding $X_r^1 \hookrightarrow Z'$. The maximum, and hence \bar{s} , exists since $s \mapsto \|\omega\|_{Z'([T_-, T_- + s] \cap J \cap K_u)}$ vanishes for $s = 0$ and is finite and continuous for all $s \geq 0$. One easily verifies that ω is a strong solution to

$$i\partial_t \omega + \Delta_\theta \omega = \rho((\tilde{u} + \omega)|\tilde{u} + \omega|^4 - \tilde{u}|\tilde{u}|^4) - e.$$

Duhamel's formula yields

$$\begin{aligned} \|\omega\|_{X_T^1(K)} &\leq \|e^{i(t-T_-)\Delta_\theta} (u(T_-) - \tilde{u}(T_-))\|_{X_T^1(K)} \\ &\quad + \|\mathcal{I}_{T_-}((\tilde{u} + \omega)|\tilde{u} + \omega|^4 - \tilde{u}|\tilde{u}|^4)\|_{X_T^1(K)} + \|\mathcal{I}_{T_-}(e)\|_{X_T^1(K)}. \end{aligned}$$

Lemma 3.11 then implies

$$\begin{aligned} \|\omega\|_{X_T^1(K)} &\leq \|u(T_-) - \tilde{u}(T_-)\|_{H^1(\mathbb{T}^3)} + \|\mathcal{I}_{T_-}(e)\|_{X_T^1(K)} \\ &\quad + C\|\omega\|_{X_T^1(K)} \left(\|\omega\|_{Z'(K)}^4 + \|\tilde{u}\|_{X_T^1(K)} \sum_{j=0}^3 \|\omega\|_{Z'(K)}^j \|\tilde{u}\|_{Z'(K)}^{3-j} \right). \end{aligned}$$

If ε_1 fulfills $5C_0\varepsilon_1 \leq (M+1)^{-2}$, we get from (3.26)–(3.29) that

$$\|\omega\|_{X_T^1(K)} \leq 2\varepsilon_1 + \tilde{C}\varepsilon_1^{\frac{1}{4}}\|\omega\|_{X_T^1(K)}.$$

Hence, we conclude for $\varepsilon_1 < (2\tilde{C})^{-4}$ that

$$\|\omega\|_{Z'(K)} \leq C_0\|\omega\|_{X_T^1(K)} \leq 4C_0\varepsilon_1. \quad (3.30)$$

It then follows that $K = J \cap K_u$ and (3.30) holds on $J \cap K_u$. Thus,

$$\|u\|_{Z(J \cap K_u)} \leq C\|u\|_{X_T^1(J \cap K_u)} \leq C_1,$$

and we get from Proposition 3.12 (iii) that u can be extended to the entire interval J . Also the bounds (3.29) and (3.30) remain true with $K = J$.

Step 4. Now, we conclude the statement of the proposition. Take $\varepsilon_2(M) < \varepsilon_1(M)$ sufficiently small and suppose that

$$\sup_{t_0 \in I_k} \|\mathcal{I}_{t_0}(e)\|_{X_T^1(I_k)} \leq \varepsilon_2.$$

Subdivide the interval I into finitely many intervals $I_k = [T_k, T_{k+1})$ such that

$$\|\tilde{u}\|_{Z(I_k)} \leq \varepsilon_2.$$

Note that the number of intervals is of size $\mathcal{O}(\|\tilde{u}\|_{Z(I)}/\varepsilon_2)$ and, in particular, independent of $|I|$. On each of those intervals, we have (3.24) and hence (3.25). The latter implies (3.26) and consequently the bounds (3.27) and (3.30) hold true on each interval. (3.30) immediately implies (3.23). Estimate (3.22) follows from the reverse triangle inequality, (3.23), and (3.27). \square

3.4 Euclidean profiles

This section is devoted to prove estimates, which compare Euclidean and periodic solutions of both linear and nonlinear Schrödinger equations. This kind of comparison is meaningful only in the case of rescaled data that concentrate in a point, and then only for short time. This short time interval is called *Euclidean window*. Beyond the Euclidean window the nonlinear solution can be compared to linear Euclidean solutions with initial data that are related to the Euclidean scattering data. For the study beyond the Euclidean window, the extinction lemma plays a fundamental role. In the present section, we argue closely to [IPS12, Section 3].

Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a fixed spherically symmetric function with $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$.

Definition 3.16. For $\phi \in \dot{H}^1(\mathbb{R}^3)$ and $N \geq 1$ we define

$$\begin{aligned} Q_N\phi &\in H^1(\mathbb{R}^3), & Q_N\phi(x) &:= \eta(N^{-\frac{1}{2}}x)\phi(x), \\ \phi_N &\in H^1(\mathbb{R}^3), & \phi_N(x) &:= N^{\frac{1}{2}}Q_N\phi(Nx), \\ T_N\phi &\in H^1(\mathbb{T}^3), & T_N\phi(y) &:= \phi_N(\Psi^{-1}(y)), \end{aligned}$$

where Ψ is the projection on the torus defined by

$$\Psi: (-\pi, \pi]^3 \rightarrow \mathbb{T}^3, \quad (\Psi(x))_j := x_j = \begin{cases} x_j & 0 \leq x_j \leq \pi, \\ 2\pi - x_j & -\pi < x_j < 0, \end{cases} \quad j = 1, 2, 3.$$

$Q_N\phi$ equals ϕ in the ball of radius $N^{\frac{1}{2}}$ and is supported in the ball of radius $2N^{\frac{1}{2}}$. ϕ_N is an \dot{H}^1 -invariant rescaling of $Q_N\phi$ with support in the ball of radius $2N^{-\frac{1}{2}}$. The function $T_N\phi$ is obtained by transferring ϕ_N to a neighborhood of zero in \mathbb{T}^3 . We make the following observations about T_N :

Corollary 3.17. *The operator $T_N: \dot{H}^1(\mathbb{R}^3) \rightarrow H^1(\mathbb{T}^3)$ is linear and satisfies the estimate*

$$\|T_N\phi\|_{H^1(\mathbb{T}^3)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^3)}.$$

Furthermore, there exists sufficiently large $N_0 = N_0(\phi) \geq 1$ such that for any $N \geq N_0$,

$$\|\phi\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|T_N\phi\|_{H^1(\mathbb{T}^3)}.$$

Proof. The linearity is obvious. From a version of Poincaré's inequality, see e.g. [Eva10, Section 5.6, Theorem 3], we have for every $N \geq 1$ that

$$\|T_N\phi\|_{H^1(\mathbb{T}^3)} = \|\phi_N\|_{H^1(\mathbb{R}^3)} \lesssim \|\phi_N\|_{\dot{H}^1(\mathbb{R}^3)}$$

since $\text{supp } \phi_N \subseteq [-2, 2]^3$ for any $N \geq 1$. Now, the claim follows from the fact that ϕ_N is an \dot{H}^1 -invariant rescaling of $Q_N\phi$ and $\|Q_N\phi\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^3)}$. The latter may be proved as follows:

$$\begin{aligned} \|\eta(N^{-\frac{1}{2}} \cdot)\phi\|_{\dot{H}^1(\mathbb{R}^3)} &\lesssim N^{-\frac{1}{2}}\|(\nabla_{\mathbb{R}^3}\eta)(N^{-\frac{1}{2}} \cdot)\phi\|_{L^2(\mathbb{R}^3)} + \|\phi\|_{\dot{H}^1(\mathbb{R}^3)} \\ &\lesssim N^{\frac{1}{4}}\|(\nabla_{\mathbb{R}^3}\eta)\phi(N^{\frac{1}{2}} \cdot)\|_{L^2(\mathbb{R}^3)} + \|\phi\|_{\dot{H}^1(\mathbb{R}^3)} \\ &\lesssim \|\nabla_{\mathbb{R}^3}\eta\|_{L^3(B_3(0))}\|\phi \cdot \mathbf{1}_{\text{supp } \eta}\|_{L^6(B_3(0))} + \|\phi\|_{\dot{H}^1(\mathbb{R}^3)}. \end{aligned}$$

Now, Sobolev's embedding and Poincaré's inequality imply $\|\phi \cdot \mathbf{1}_{\text{supp } \eta}\|_{L^6(B_3(0))} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^3)}$.

The second bound follows immediately from the observation that there exists $N_0 = N_0(\phi)$ such that for any $N \geq N_0$

$$\|\phi\|_{\dot{H}^1(\mathbb{R}^3)} \leq 2\|Q_N\phi\|_{\dot{H}^1(\mathbb{R}^3)}. \quad \square$$

3.4.1 Global well-posedness on the Euclidean space

In this subsection, we recall the global well-posedness result that is known for the Euclidean space \mathbb{R}^3 . Furthermore, we show that this result holds true even if we replace the standard Laplace operator on \mathbb{R}^3 , which shall be denoted by $\Delta^{\mathbb{R}^3}$, with a Laplace operator corresponding to Δ_{θ} .

Definition 3.18.

(i) We define the modified Laplace operator on \mathbb{R}^3 corresponding to Δ_{θ} as

$$\Delta_{\theta}^{\mathbb{R}^3} := \sum_{j=1}^3 \theta_j \frac{\partial^2}{\partial x_j^2}.$$

(ii) Given $\phi \in \dot{H}^1(\mathbb{R}^3)$, we define the *Euclidean energy* with respect to $\Delta_{\theta}^{\mathbb{R}^3}$ as

$$E_{\mathbb{R}^3}(\phi) := \frac{1}{2} \int_{\mathbb{R}^3} \sum_{j=1}^3 \theta_j^{\frac{1}{2}} \left| \frac{\partial \phi}{\partial x_j}(x) \right|^2 dx + \frac{1}{6} \int_{\mathbb{R}^3} |\phi(x)|^6 dx.$$

The proof of [Theorem 3.1](#) relies heavily on the following results that were essentially proved by Colliander–Keel–Staffilani–Takaoka–Tao [[CKS⁺08](#)]. We summarize some of their results in the following theorem.

Theorem 3.19 (Global well-posedness on \mathbb{R}^3). *For any $\phi \in \dot{H}^1(\mathbb{R}^3)$, there is a unique global solution $v \in C(\mathbb{R}, \dot{H}^1(\mathbb{R}^3))$ of the initial value problem*

$$i\partial_t v + \Delta_{\theta}^{\mathbb{R}^3} v = v|v|^4, \quad v(0) = \phi, \quad (3.31)$$

and the solution satisfies the estimate

$$\|\nabla_{\mathbb{R}^3} v\|_{(L_t^{\infty} L_x^2 \cap L_t^2 L_x^6)(\mathbb{R} \times \mathbb{R}^3)} \leq \tilde{C}(E_{\mathbb{R}^3}(\phi)). \quad (3.32)$$

Moreover, this solution scatters in the sense that there exists $\phi^{\pm\infty} \in \dot{H}^1(\mathbb{R}^3)$ such that

$$\|v(t) - e^{it\Delta_{\theta}^{\mathbb{R}^3}} \phi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \rightarrow 0 \quad (3.33)$$

as $t \rightarrow \pm\infty$. Furthermore, if $\phi \in H^s(\mathbb{R}^3)$ for some $s > 1$, then $v \in C(\mathbb{R}, H^s(\mathbb{R}^3))$ and

$$\sup_{t \in \mathbb{R}} \|v(t)\|_{H^s(\mathbb{R}^3)} \lesssim \|\phi\|_{H^s(\mathbb{R}^3)}^1. \quad (3.34)$$

Proof. The proof in case of the standard Laplacian may be found in [[CKS⁺08](#), Theorem 1.1 and Corollary 1.2]. We reduce the statement for the modified Laplace operator to this result.

Let $\Theta := \text{diag}(\theta_1, \theta_2, \theta_3)^{\frac{1}{2}}$. There exists a unique global solution $v \in C(\mathbb{R}, \dot{H}^1(\mathbb{R}^3))$ of the initial value problem

$$i\partial_t v + \Delta^{\mathbb{R}^3} v = v|v|^4, \quad v(0) = \psi,$$

where $\psi := \phi \circ \Theta$. The rescaled function $u(t, x) := v(t, \Theta^{-1}x)$ solves

$$i\partial_t u + \Delta_{\theta}^{\mathbb{R}^3} u = u|u|^4, \quad u(0) = \phi.$$

By a change of variables, it is easy to see that the estimates (3.32) and (3.34) hold true.

Let $\psi^{\pm\infty} \in \dot{H}^1(\mathbb{R}^3)$ be the scattering data corresponding to v . We claim that $\phi^{\pm\infty} := \psi^{\pm\infty} \circ \Theta^{-1}$ are the scattering data corresponding to u . Indeed, $e^{it\Delta_{\theta}^{\mathbb{R}^3}} \phi^{\pm\infty}(x) = e^{it\Delta^{\mathbb{R}^3}} \psi^{\pm\infty}(\Theta^{-1}x)$ since

$$\begin{aligned} \int_{\mathbb{R}^3} e^{2\pi i x \cdot \xi} e^{it|\Theta\xi|^2} \widehat{\phi^{\pm\infty}}(\xi) d\xi &= |\det \Theta| \int_{\mathbb{R}^3} e^{2\pi i x \cdot \xi} e^{it|\Theta\xi|^2} \widehat{\psi^{\pm\infty}}(\Theta\xi) d\xi \\ &= \int_{\mathbb{R}^3} e^{2\pi i \Theta^{-1}x \cdot \xi} e^{it|\xi|^2} \widehat{\psi^{\pm\infty}}(\xi) d\xi. \end{aligned}$$

Hence,

$$\|u(t) - e^{it\Delta_{\theta}^{\mathbb{R}^3}} \phi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} = \|v(t) - e^{it\Delta^{\mathbb{R}^3}} \psi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \rightarrow 0$$

as $t \rightarrow \pm\infty$. □

3.4.2 Connection between solutions on tori and Euclidean solutions

We now turn to one of the fundamental observations. We discuss the connection between Euclidean solutions and solutions on tori of both linear and nonlinear Schrödinger equations. For $\phi \in \dot{H}^1(\mathbb{R}^3)$ we consider solutions on tori with initial data $T_N\phi$. There exists a large $T > 0$ such that for all large $N \geq 1$ we distinguish the behavior of solutions on tori in the Euclidean window, that is $(-TN^{-2}, TN^{-2})$, and beyond the Euclidean window, namely in $(-T^{-1}, T^{-1}) \setminus (-TN^{-2}, TN^{-2})$. We find that within the Euclidean window solutions on tori stay close to Euclidean-like solutions, see [Lemma 3.20](#). Outside of the Euclidean window, the crucial extinction lemma, stability, and the Euclidean scattering property show that nonlinear solutions on tori can be compared to the linear evolution with initial data $T_N\phi^{\pm\infty}$, where $\phi^{\pm\infty}$ are the scattering data of ϕ given by [Theorem 3.19](#).

Comparison to Euclidean solutions within the Euclidean window

Similarly as in [[IP12b](#), Lemma 4.2], we obtain the following lemma comparing the linear and nonlinear evolution on tori with the Euclidean evolution within the Euclidean window.

Lemma 3.20. *Let $\phi \in \dot{H}^1(\mathbb{R}^3)$, $T_0 > 0$, and $\rho \in \{0, 1\}$ be given. Then the following conclusions hold:*

- (i) *There is $N_0 = N_0(\phi, T_0)$ such that for any $N \geq N_0$ there is a unique strong solution $U_N \in C((-T_0N^{-2}, T_0N^{-2}), H^1(\mathbb{T}^3)) \cap X_{\Gamma}^1(-T_0N^{-2}, T_0N^{-2})$ of the initial value problem*

$$i\partial_t U_N + \Delta_{\theta} U_N = \rho U_N |U_N|^4, \quad U_N(0) = T_N \phi. \quad (3.35)$$

Moreover, for any $N \geq N_0$,

$$\|U_N\|_{X_{\Gamma}^1(-T_0N^{-2}, T_0N^{-2})} \lesssim_{E_{\mathbb{R}^3}(\phi)} 1.$$

- (ii) *Given $\phi' \in H^s(\mathbb{R}^3)$ for some $s \geq 5$, let $v' \in C(\mathbb{R}, H^s(\mathbb{R}^3))$ denote the solution of the initial value problem*

$$i\partial_t v' + \Delta_{\theta}^{\mathbb{R}^3} v' = \rho v' |v'|^4, \quad v'(0) = \phi'.$$

Furthermore, we define for $N \geq R \geq 1$,

$$\begin{aligned} v'_R(t, x) &= \eta\left(\frac{x}{R}\right) v'(t, x), & (t, x) &\in (-T_0, T_0) \times \mathbb{R}^3, \\ v'_{R,N}(t, x) &= N^{\frac{1}{2}} v'_R(N^2 t, Nx), & (t, x) &\in (-T_0 N^{-2}, T_0 N^{-2}) \times \mathbb{R}^3, \\ V_{R,N}(t, y) &= v'_{R,N}(t, \Psi^{-1}(y)), & (t, y) &\in (-T_0 N^{-2}, T_0 N^{-2}) \times \mathbb{T}^3. \end{aligned} \quad (3.36)$$

Then there exists $\varepsilon_2 = \varepsilon_2(E_{\mathbb{R}^3}(\phi)) > 0$ such that for all $0 < \varepsilon < \varepsilon_2$ and $\phi' \in H^s(\mathbb{R}^3)$ with $\|\phi - \phi'\|_{\dot{H}^1(\mathbb{R}^3)} \leq \varepsilon$ there exists $R_0 = R_0(T_0, \phi') \geq 1$ such that for any $R \geq R_0$,

$$\lim_{N \rightarrow \infty} \|U_N - V_{R,N}\|_{X_{\Gamma}^1(-T_0N^{-2}, T_0N^{-2})} \lesssim_{E_{\mathbb{R}^3}(\phi)} \varepsilon.$$

Proof. The proof follows the arguments in [[IPS12](#), Lemma 4.2] and [[IP12a](#), Lemma 4.2].

We prove (i) by showing that $V_{R,N}$ is an almost-solution to (3.35), which implies the asserted statement by applying our stability result. Throughout this proof, $\lesssim_{E_{\mathbb{R}^3}(\phi)}$ denotes that the implicit constant may depend on the large constant $\tilde{C}(E_{\mathbb{R}^3}(\phi))$ in (3.32). We also denote $I_N := (-T_0N^{-2}, T_0N^{-2})$ for brevity.

Let $\phi \in \dot{H}^1(\mathbb{R}^3)$, $T_0 > 0$, and $\rho \in \{0, 1\}$ be given as in the assumptions. For any $\varepsilon > 0$, we may choose some $\phi' \in H^s(\mathbb{R}^3)$ which satisfies $\|\phi - \phi'\|_{\dot{H}^1(\mathbb{R}^3)} \leq \varepsilon$. Let $v' \in C(\mathbb{R}, H^s(\mathbb{R}^3))$ be as given in the lemma. The existence of the global solution is implied by [Theorem 3.19](#) and so are the estimates

$$\|\nabla_{\mathbb{R}^3} v'\|_{(L_t^\infty L_x^2 \cap L_t^2 L_x^6)(\mathbb{R} \times \mathbb{R}^3)} \lesssim_{E_{\mathbb{R}^3}(\phi')} 1 \quad \text{and} \quad \sup_{t \in \mathbb{R}} \|v'(t)\|_{H^s(\mathbb{R}^3)} \lesssim_{\|\phi'\|_{H^s(\mathbb{R}^3)}} 1. \quad (3.37)$$

Furthermore, we remark that we even have $v' \in C(\mathbb{R}, C^3(\mathbb{R}^3))$ from Sobolev's embedding.

Step 1. In the following, we prove that there exists $R_0 = R_0(T_0, \phi') \geq 1$ such that $V_{R,N}$ is an almost-solution to [\(3.35\)](#) for any $N \geq R \geq R_0$. For $R \geq 1$ we set

$$\begin{aligned} e_R(t, x) &:= ((i\partial_t + \Delta_{\theta}^{\mathbb{R}^3})v'_R - \rho v'_R |v'_R|^4)(t, x) \\ &= \rho \left(\eta\left(\frac{x}{R}\right) - \eta\left(\frac{x}{R}\right)^5 \right) v'(t, x) |v'(t, x)|^4 + R^{-2} v'(t, x) (\Delta_{\theta}^{\mathbb{R}^3} \eta)\left(\frac{x}{R}\right) \\ &\quad + 2R^{-1} \sum_{j=1}^3 \partial_j v'(t, x) \partial_j \eta\left(\frac{x}{R}\right). \end{aligned}$$

It follows from [\(3.37\)](#) and Sobolev's embedding that $|v'(t, x)| \lesssim_{\|\phi'\|_{H^s(\mathbb{R}^3)}} 1$. Hence, for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$ we have

$$|e_R(t, x)| + \sum_{k=1}^3 |\partial_k e_R(t, x)| \lesssim \mathbf{1}_{[R, 2R]}(|x|) \left(|v'(t, x)| + \sum_{k=1}^3 |\partial_k v'(t, x)| + \sum_{k,j=1}^3 |\partial_k \partial_j v'(t, x)| \right),$$

where the implicit constant depends on $\|\phi'\|_{H^s(\mathbb{R}^3)}$. In view of this estimate and from the fact that $v' \in C(\mathbb{R}, H^s(\mathbb{R}^3))$, we see that there exists $R_0 = R_0(T_0, \phi', \varepsilon)$ such that

$$\| |e_R| + |\nabla_{\mathbb{R}^3} e_R| \|_{L^2((-T_0, T_0) \times \mathbb{R}^3)} \leq T_0^{-\frac{1}{2}} \varepsilon \quad (3.38)$$

for any $R \geq R_0$. If $N \geq R \geq 1$, then we may define

$$e_{R,N}(t, x) := ((i\partial_t + \Delta_{\theta}^{\mathbb{R}^3})v'_{R,N} - \rho v'_{R,N} |v'_{R,N}|^4)(t, x) = N^{\frac{5}{2}} e_R(N^2 t, Nx).$$

For $N \geq 1$ and $R \geq R_0$ Hölder's inequality with respect to t and [\(3.38\)](#) yield

$$\| |e_{R,N}| + |\nabla_{\mathbb{R}^3} e_{R,N}| \|_{L^1(I_N, L^2(\mathbb{R}^3))} \lesssim T_0^{\frac{1}{2}} \| |e_R| + |\nabla_{\mathbb{R}^3} e_R| \|_{L^2((-T_0, T_0) \times \mathbb{R}^3)} \lesssim \varepsilon. \quad (3.39)$$

Note that $v'_{R,N}$ is supported in a ball of radius $2\frac{R}{N}$. Now, we define

$$E_{R,N}(t, y) := ((i\partial_t + \Delta_{\theta})V_{R,N} - \rho V_{R,N} |V_{R,N}|^4)(t, y) = e_{R,N}(t, \Psi^{-1}(y))$$

for $N \geq R$. From the bound [\(3.39\)](#), we deduce that there exists $R_0 = R_0(T_0, \phi', \varepsilon)$ such that $V_{R,N}$ is an almost-solution to [\(3.35\)](#) for $N \geq R \geq R_0$, i.e.

$$\sup_{t_0 \in I_N} \|\mathcal{I}_{t_0}(E_{R,N})\|_{X^1(I_N)} \lesssim \|E_{R,N}\|_{L^1(I_N, H^1(\mathbb{T}^3))} \lesssim \varepsilon. \quad (3.40)$$

Step 2. Here we verify the assumptions of the stability result in [Proposition 3.15](#). Assumption [\(3.19\)](#) follows from the definition of $V_{R,N}$ and [\(3.37\)](#). Indeed, for every $R \geq 1$ and $N \geq R$ we have

$$\|V_{R,N}\|_{L^\infty(I_N, H^1(\mathbb{T}^3))} \lesssim \|v'_{R,N}\|_{L^\infty(I_N, H^1(\mathbb{R}^3))} \lesssim \|v'_{R,N}\|_{L^\infty(I_N, \dot{H}^1(\mathbb{R}^3))} \lesssim_{E_{\mathbb{R}^3}(\phi')} 1.$$

Moreover, the bound on the Z -norm is obtained by using Littlewood-Paley theory (e.g. [ST87, Section 3.5.4]) and (3.37): For $N \geq R \geq 1$ and N large enough (depending on T_0) such that $|I_N| \leq 1$, we apply Bernstein's inequality and compute

$$\begin{aligned} \|V_{R,N}\|_{Z(I_N)} &\leq \sum_{(p,q) \in \mathcal{P}} \left\| \left(\sum_{M \geq 1} M^{(\frac{2}{p} + \frac{3}{q} - \frac{1}{2})p} \|P_M V_{R,N}\|_{L^q(\mathbb{T}^3)}^p \right)^{\frac{1}{p}} \right\|_{L^p(I_N)} \\ &\lesssim \sum_{(p,q) \in \mathcal{P}} \left\| \left(\sum_{M \geq 1} M^{(\frac{2}{p} + 1)p} \|P_M V_{R,N}\|_{L^2(\mathbb{T}^3)}^p \right)^{\frac{1}{p}} \right\|_{L^p(I_N)} \\ &\lesssim \sum_{(p,q) \in \mathcal{P}} \|V_{R,N}\|_{L^p(I_N, H^{2/p+1}(\mathbb{T}^3))}. \end{aligned}$$

Thus, we showed that

$$\|V_{R,N}\|_{Z(I_N)} \lesssim \sum_{(p,q) \in \mathcal{P}} \|v'_{R,N}\|_{L^p(I_N, H^{2/p+1}(\mathbb{R}^3))}.$$

Note that $\text{supp } v'_{R,N} \subseteq B_2(0)$ for all $N \geq R$. Hence, by interpolating the first bound in (3.37), we obtain

$$\|v'_{R,N}\|_{L^p(I_N, H^{2/p+1}(\mathbb{R}^3))} \lesssim \|\nabla_{\mathbb{R}^3} v'_{R,N}\|_{L^p(I_N, L^{r_p}(\mathbb{R}^3))} \lesssim_{E_{\mathbb{R}^3}(\phi')} 1,$$

where $r_p := \frac{6p}{3p-4}$.

All in all, we have

$$\|V_{R,N}\|_{L^\infty(I_N, H^1(\mathbb{T}^3))} + \|V_{R,N}\|_{Z(I_N)} \leq C(E_{\mathbb{R}^3}(\phi)).$$

We remark that if necessary, we can decrease ε to satisfy $\varepsilon < \varepsilon_1(C(E_{\mathbb{R}^3}(\phi)))$, where ε_1 is given in Proposition 3.15.

We still have to verify assumption (3.20). Consider the first term of (3.20). From Poincaré's inequality we deduce

$$\|T_N \phi - V_{R,N}(0)\|_{H^1(\mathbb{T}^3)} \lesssim \|\phi_N - v'_{R,N}(0)\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|Q_N \phi - v'_R(0)\|_{\dot{H}^1(\mathbb{R}^3)}$$

provided $N \geq R$. Clearly, we can find $R_0 = R_0(\phi', \varepsilon)$ and $N_0 = N_0(\phi, \varepsilon)$ such that for all $R \geq R_0$ and $N \geq N_0$ with $N \geq R$ it holds that

$$\|Q_N \phi - v'_R(0)\|_{\dot{H}^1(\mathbb{R}^3)} \leq \|Q_N \phi - \phi\|_{\dot{H}^1(\mathbb{R}^3)} + \|\phi - \phi'\|_{\dot{H}^1(\mathbb{R}^3)} + \|\phi' - v'_R(0)\|_{\dot{H}^1(\mathbb{R}^3)} \leq \varepsilon.$$

The bound on the second term of (3.20) was already proved in (3.40). Possibly, we decrease $\varepsilon > 0$ further such that

$$\|T_N \phi - V_{R,N}(0)\|_{H^1(\mathbb{T}^3)} + \sup_{t_0 \in I_N} \|\mathcal{I}_{t_0}(E_{R,N})\|_{X^1_r(I_N)} < \varepsilon_1,$$

where ε_1 is as defined above. This proves that the assumptions in Proposition 3.15 are fulfilled.

Step 3. Finally, we apply our stability result and obtain the existence of a strong solution $U_N \in C(I_N, H^1(\mathbb{T}^3)) \cap X^1_r(I_N)$ to (3.35) for every $N \geq N_0(\phi, T_0)$ satisfying

$$\|U_N\|_{X^1_r(I_N)} \lesssim_{E_{\mathbb{R}^3}(\phi)} 1.$$

Furthermore, if $R \geq R_0$, then

$$\lim_{N \rightarrow \infty} \|U_N - V_{R,N}\|_{X^1_r(I_N)} \lesssim_{E_{\mathbb{R}^3}(\phi)} \varepsilon. \quad \square$$

Comparison to Euclidean solutions beyond the Euclidean window

To understand the behavior of solutions on tori beyond the Euclidean window, we have to work a bit harder. The next lemma is fundamental for our analysis since it helps to understand the linear and consequently (cf. [Proposition 3.12](#) (i)) the nonlinear solution beyond the Euclidean window. In contrast to [[IP12b](#), Lemma 4.3], we have to deal with two additional difficulties. The Z -norm used here makes the arguments a bit more delicate compared to [[IP12b](#)] and due to the modified Laplace operator, we use the weaker estimate ([3.42](#)). Nevertheless, we show that both difficulties can be dealt with. We want to point out that the following argument can easily be modified to treat a general three-dimensional manifold $\mathbb{T} \times M$.

Lemma 3.21 (Extinction lemma).

(i) Let $\phi \in \dot{H}^1(\mathbb{R}^3)$. For any $\varepsilon > 0$ there exists $T = T(\phi, \varepsilon)$ and $N_0 = N_0(\phi, \varepsilon)$ such that for all $N \geq N_0$ it holds that

$$\|e^{it\Delta_\theta}(T_N\phi)\|_{Z(TN^{-2}, T^{-1})} \lesssim \varepsilon.$$

(ii) Let $\phi \in C_0^\infty(\mathbb{R}^3)$, $p \in [4, \infty]$, and $1 \leq T \leq N$, then

$$\sup_{|t| \in [TN^{-2}, T^{-1}]} \|e^{it\Delta_\theta}(T_N\phi)\|_{L^p(\mathbb{T}^3)} \lesssim_\phi T^{-\frac{1}{10}} N^{\frac{1}{2} - \frac{3}{p}}.$$

Proof. First, we prove (i) by modifying the argument in [[IP12b](#), Lemma 4.3]. For $M \geq 1$ we have that

$$(P_{\leq M} e^{it\Delta_\theta}(T_N\phi))(t, x) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} K_M(t, x - y) T_N\phi(y) dy,$$

where K_M is given by

$$K_M(t, x) := \sum_{\xi \in \mathbb{Z}^3} e^{i(x \cdot \xi - tQ(\xi))} \eta^3 \left(\frac{\xi}{M} \right). \quad (3.41)$$

The Weyl type estimate given in [Lemma 1.41](#) yields

$$|K_M(t, x)| \lesssim M^2 \left| \sum_{\xi_1 \in \mathbb{Z}} e^{i(x_1 \xi_1 - t\xi_1^2)} \eta^1 \left(\frac{\xi_1}{M} \right) \right|^2 \lesssim \frac{M^3}{\sqrt{q} (1 + M |\frac{t}{2\pi} - \frac{a}{q}|^{1/2})} \quad (3.42)$$

provided

$$\frac{t}{2\pi} = \frac{a}{q} + \beta, \quad \text{where } q \in \{1, \dots, M\}, a \in \mathbb{Z}, (a, q) = 1, |\beta| \leq (Mq)^{-1}.$$

Dirichlet's lemma, see [Lemma 1.42](#), and (3.42) imply for $1 \leq S \leq M$,

$$\|K_M\|_{L^\infty([SM^{-2}, S^{-1}] \times \mathbb{T}^3)} \lesssim S^{-\frac{1}{2}} M^3. \quad (3.43)$$

Indeed, assume that $|t| \leq \frac{1}{S}$, and write $\frac{t}{2\pi} = \frac{a}{q} + \beta$. Since $|\beta| \leq \frac{1}{M} \leq \frac{1}{S}$, it follows that $|\frac{a}{q}| \leq \frac{2}{S}$. Therefore, either $|a| \geq 1$, which implies $q \geq \frac{S}{4}$, or $a = 0$, and hence, $q = 1$ because $(a, q) = 1$. In the first case, (3.43) follows from (3.42):

$$|K_M(t, x)| \lesssim q^{-\frac{1}{2}} M^3 \lesssim S^{-\frac{1}{2}} M^3.$$

In the second case, we have $|\frac{t}{2\pi} - \frac{a}{q}|^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} |t|^{\frac{1}{2}}$, and we obtain from (3.42) that

$$|K_M(t, x)| \lesssim |t|^{-\frac{1}{2}} M^2 \lesssim S^{-\frac{1}{2}} M^3$$

for $t \in [SM^{-2}, S^{-1}]$.

Since the Z -norm is based on L^p -spaces with $1 \leq p < \infty$, we may assume that $\phi \in C_0^\infty(\mathbb{R}^3)$. From the definition of T_N (Definition 3.16), we get

$$\|T_N \phi\|_{L^p(\mathbb{T}^3)} \lesssim_\phi N^{\frac{1}{2} - \frac{3}{p}} \quad (3.44)$$

and

$$\|P_L(T_N \phi)\|_{L^2(\mathbb{T}^3)} \lesssim_\phi \left(1 + \frac{L}{N}\right)^{-10} N^{-1}.$$

The latter estimate in combination with the Strichartz estimates in Lemma 2.10 leads to

$$\begin{aligned} \|e^{it\Delta_\theta} P_L(T_N \phi)\|_{L^p([-1,1], L^q(\mathbb{T}^3))} &\lesssim_{p,q} L^{\frac{3}{2} - \frac{2}{p} - \frac{3}{q}} \|P_L(T_N \phi)\|_{L^2(\mathbb{T}^3)} \\ &\lesssim_{\phi,p,q} L^{\frac{3}{2} - \frac{2}{p} - \frac{3}{q}} \left(1 + \frac{L}{N}\right)^{-10} N^{-1} \end{aligned} \quad (3.45)$$

for $p > \frac{16}{3}$ and $q \geq 4$. If $1 \leq T \leq N$ and $(p, q) \in \mathcal{P}$, then this allows us to bound

$$\begin{aligned} \sum_{L \notin [NT^{-1/1000}, NT^{1/1000}]} L^{(\frac{2}{p} + \frac{3}{q} - \frac{1}{2})p} \|e^{it\Delta_\theta} P_L(T_N \phi)\|_{L^p([-1,1], L^q(\mathbb{T}^3))}^p \\ \lesssim_\phi \sum_{1 \leq L < NT^{-1/1000}} L^p N^{-p} + \sum_{L > NT^{1/1000}} L^{-9p} N^{9p} \lesssim_\phi T^{-\frac{p}{1000}}. \end{aligned}$$

Here, we sum over dyadic numbers.

Now, we use the inequalities at the beginning of the proof to estimate the remaining sum over $L \in [NT^{-1/1000}, NT^{1/1000}]$. Young's inequality for convolutions, (3.44), and (3.43) give for all $L \geq 1$,

$$\begin{aligned} \|e^{it\Delta_\theta} P_L(T_N \phi)\|_{L^\infty([TN^{-2}, T^{-1}] \times \mathbb{T}^3)} &\leq \|K_L - K_{L/2}\|_{L^\infty([T(\max\{L, N\})^{-2}, T^{-1}] \times \mathbb{T}^3)} \|T_N \phi\|_{L^1(\mathbb{T}^3)} \\ &\lesssim_\phi T^{-\frac{1}{2}} (L + N)^3 N^{-\frac{5}{2}}. \end{aligned}$$

Interpolating this with the estimate given in (3.45) (with $p = \frac{16}{3} +$ and $q = 4$), we obtain for $L \in [NT^{-1/1000}, NT^{1/1000}]$ and $(p, q) \in \mathcal{P}$,

$$\|e^{it\Delta_\theta} P_L(T_N \phi)\|_{L^p([TN^{-2}, T^{-1}], L^q(\mathbb{T}^3))} \lesssim_\phi T^{-\frac{1}{1000}} N^{\frac{1}{2} - \frac{2}{p} - \frac{3}{q}}.$$

Then, (i) follows from

$$\begin{aligned} \sum_{(p,q) \in \mathcal{P}} \sum_{L \in [NT^{-1/1000}, NT^{1/1000}]} L^{(\frac{2}{p} + \frac{3}{q} - \frac{1}{2})p} \|e^{it\Delta_\theta} P_L(T_N \phi)\|_{L^p([TN^{-2}, T^{-1}], L^q(\mathbb{T}^3))}^p \\ \lesssim_\phi T^{(\frac{2}{p_0} + \frac{3}{q_0} - \frac{3}{2})p_0/1000} + T^{-(\frac{2}{p_1} + \frac{3}{q_1} + \frac{1}{2})p_1/1000}. \end{aligned}$$

The result follows for $T = T(\varepsilon, \phi)$ sufficiently large since both exponents are negative.

Now, we turn to the proof of (ii). From (3.43) and (3.44), we get

$$\begin{aligned} \sup_{t \in [TN^{-2}, T^{-1}]} \|e^{it\Delta_\theta} P_{\leq T^{1/10}N}(T_N \phi)\|_{L^\infty(\mathbb{T}^3)} \\ \leq \|K_{NT^{1/10}}\|_{L^\infty([T(NT^{1/10})^{-2}, T^{-1}] \times \mathbb{T}^3)} \|T_N \phi\|_{L^1([-1,1] \times \mathbb{T}^3)} \\ \lesssim_\phi T^{-\frac{1}{5}} N^{\frac{1}{2}} \end{aligned}$$

as well as

$$\sup_{t \in \mathbb{R}} \|e^{it\Delta_\theta} P_{\leq T^{1/10}N}(T_N\phi)\|_{L^2(\mathbb{T}^3)} \lesssim_\phi N^{-1}.$$

Interpolating these estimates, we obtain for $2 \leq p \leq \infty$,

$$\sup_{t \in [TN^{-2}, T^{-1}]} \|e^{it\Delta_\theta} P_{\leq T^{1/10}N}(T_N\phi)\|_{L^p(\mathbb{T}^3)} \lesssim_\phi T^{\frac{2}{5p}-\frac{1}{5}} N^{\frac{1}{2}-\frac{3}{p}}. \quad (3.46)$$

Note that for $p \geq 4$, we have $T^{\frac{2}{5p}-\frac{1}{5}} \leq T^{-\frac{1}{10}}$. From the estimates

$$\|P_L T_N\phi\|_{L^1(\mathbb{T}^3)} \lesssim_\phi \left(\frac{N}{L}\right)^{10} N^{-\frac{5}{2}} \quad \text{and} \quad \|P_L T_N\phi\|_{L^2(\mathbb{T}^3)} \lesssim_\phi \left(\frac{N}{L}\right)^{10} N^{-1}$$

for $L \geq N$ and Sobolev's embedding, we infer

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|e^{it\Delta_\theta} P_L(T_N\phi)\|_{L^2(\mathbb{T}^3)} &\lesssim_\phi N^{-1} \left(\frac{N}{L}\right)^{10}, \\ \sup_{t \in \mathbb{R}} \|e^{it\Delta_\theta} P_L(T_N\phi)\|_{L^\infty(\mathbb{T}^3)} &\leq L^3 \|P_L(T_N\phi)\|_{L^1(\mathbb{T}^3)} \lesssim_\phi N^{\frac{1}{2}} \left(\frac{N}{L}\right)^7. \end{aligned}$$

Consequently, for $2 \leq p \leq \infty$,

$$\sup_{t \in \mathbb{R}} \|e^{it\Delta_\theta} P_L(T_N\phi)\|_{L^p(\mathbb{T}^3)} \lesssim_\phi N^{\frac{1}{2}-\frac{3}{p}} \left(\frac{N}{L}\right)^{7+\frac{6}{p}}.$$

Hence, we may estimate

$$\sup_{t \in \mathbb{R}} \sum_{L > NT^{1/10}} \|e^{it\Delta_\theta} P_L(T_N\phi)\|_{L^p(\mathbb{T}^3)} \lesssim_\phi N^{\frac{1}{2}-\frac{3}{p}} \sum_{L > NT^{1/10}} \left(\frac{N}{L}\right)^{7+\frac{6}{p}} \lesssim_\phi N^{\frac{1}{2}-\frac{3}{p}} T^{-\frac{1}{10}}. \quad (3.47)$$

We are now able to conclude the lemma using (3.46) and (3.47): For all $4 \leq p \leq \infty$ and $t \in [TN^{-2}, T^{-1}]$, we have

$$\begin{aligned} \|e^{it\Delta_\theta}(T_N\phi)\|_{L^p(\mathbb{T}^3)} &\leq \|e^{it\Delta_\theta} P_{\leq T^{1/10}N}(T_N\phi)\|_{L^p(\mathbb{T}^3)} + \sum_{L > NT^{1/10}} \|e^{it\Delta_\theta} P_L(T_N\phi)\|_{L^p(\mathbb{T}^3)} \\ &\lesssim_\phi T^{-\frac{1}{10}} N^{\frac{1}{2}-\frac{3}{p}}. \end{aligned} \quad \square$$

Now, we shall bring everything together to compare Euclidean solutions with initial data $\phi \in \dot{H}^1(\mathbb{R}^3)$ and solutions on tori with initial data $T_N\phi$ in a certain time frame. We begin with some notation and the definition of renormalized Euclidean frames.

Given $f \in L^2(\mathbb{T}^3)$, $t_0 \in \mathbb{R}$, and $x_0 \in \mathbb{T}^3$, we define

$$\begin{aligned} (\pi_{x_0} f)(x) &:= f(x - x_0), \\ (\Pi_{t_0, x_0} f)(x) &:= (e^{-it_0\Delta_\theta} f)(x - x_0) = (\pi_{x_0} e^{-it_0\Delta_\theta} f)(x). \end{aligned}$$

Definition 3.22 (Renormalized Euclidean frames). We define the set of *renormalized Euclidean frames* as

$$\begin{aligned} \tilde{\mathcal{F}}_E &:= \left\{ (N_k, t_k, x_k)_{k \geq 1} : N_k \geq 1, N_k \rightarrow +\infty, t_k \rightarrow 0, x_k \in \mathbb{T}^3, \right. \\ &\quad \left. \text{and either } t_k = 0 \text{ for all } k \geq 1 \text{ or } \lim_{k \rightarrow \infty} N_k^2 |t_k| = +\infty \right\}. \end{aligned}$$

Remark. In [Definition 3.24](#) below, we introduce a slightly more general class of frames, called *Euclidean frames*. As we show in the beginning of the proof of [Proposition 3.30](#), it is enough to prove the following proposition under the stronger assumption of a renormalized Euclidean frame. \diamond

Proposition 3.23. *Let $\mathcal{O} = (N_k, t_k, x_k)_k \in \tilde{\mathcal{F}}_E$ and $\phi \in \dot{H}^1(\mathbb{R}^3)$.*

(i) *There exist $\tau = \tau(\phi)$ and $k_0 = k_0(\phi, \mathcal{O})$ such that for all $k \geq k_0$ there is a strong solution $U_k \in C((-\tau, \tau), H^1(\mathbb{T}^3)) \cap X_r^1(-\tau, \tau)$ of the nonlinear equation (3.1) with initial data $U_k(0) = \Pi_{t_k, x_k}(T_{N_k} \phi)$. Moreover, the solution satisfies the bound*

$$\|U_k\|_{X_r^1(-\tau, \tau)} \lesssim_{E_{\mathbb{R}^3}(\phi)} 1.$$

(ii) *For any $s \geq 1$ there exists a Euclidean solution $u \in C(\mathbb{R}, \dot{H}^s(\mathbb{R}^3))$ of*

$$i\partial_t u + \Delta_{\theta}^{\mathbb{R}^3} u = u|u|^4 \quad (3.48)$$

with scattering data $\phi^{\pm\infty} \in \dot{H}^1(\mathbb{R}^3)$ defined as in [Theorem 3.19](#) such that the following holds up to a subsequence: For any $\varepsilon > 0$ there exists $T_0 = T_0(\phi, \varepsilon)$ such that for all $T \geq T_0$ there is $R_0 = R_0(\phi, \varepsilon, T)$ such that for all $R \geq R_0$ there is $k_0 = k_0(\phi, \varepsilon, T, R)$ with the property that for any $k \geq k_0$, it holds that

$$\|U_k - \tilde{u}_k\|_{X_r^1(\{|t-t_k| \leq TN_k^{-2}\} \cap \{|t| \leq T^{-1}\})} \leq \varepsilon, \quad (3.49)$$

where

$$(\pi_{-x_k} \tilde{u}_k)(t, x) = N_k^{\frac{1}{2}} \eta \left(\frac{N_k \Psi^{-1}(x)}{R} \right) u(N_k^2(t - t_k), N_k \Psi^{-1}(x)). \quad (3.50)$$

In addition, up to a subsequence, we have

$$\|U_k(t) - \Pi_{t_k - t, x_k} T_{N_k} \phi^{\pm\infty}\|_{X_r^1(\{\pm(t-t_k) \geq TN_k^{-2}\} \cap \{|t| \leq T^{-1}\})} \leq \varepsilon \quad (3.51)$$

for $k \geq k_0$.

Proof. The comparison within the Euclidean window was essentially done in [Lemma 3.20](#). For the comparison beyond the Euclidean window we make use of the previous extinction lemma and our stability result. In this interval, the general idea is as follows:

$$\begin{aligned} U_k(t) &\approx e^{it\Delta_{\theta}} U_k(TN_k^{-2}) && \text{(extinction lemma and [Proposition 3.12](#) (i))} \\ &\approx e^{it\Delta_{\theta}} \tilde{u}_k(TN_k^{-2}) && \text{(stability and (3.49))} \\ &\approx e^{it\Delta_{\theta}} T_{N_k} \phi^{\pm\infty}. && \text{(Euclidean scattering property)} \end{aligned}$$

Let $\mathcal{O} = (N_k, t_k, x_k)_k \in \tilde{\mathcal{F}}_E$, $\phi \in \dot{H}^1(\mathbb{R}^3)$, and $\varepsilon > 0$ be fixed. Without loss of generality, we may assume $x_k = 0$.

Case 1. Assume $t_k = 0$ for all $k \geq 1$. Let $s' := \max\{5, s\}$. Given any $0 < \varepsilon' \ll \varepsilon$ we may choose $\phi' \in H^{s'}(\mathbb{R}^3)$ to satisfy $\|\phi - \phi'\|_{\dot{H}^1(\mathbb{R}^3)} < \varepsilon'$. Let $u \in C(\mathbb{R}, H^{s'}(\mathbb{R}^3))$ be the solution to the nonlinear Euclidean Schrödinger equation (3.31) with initial data $u(0) = \phi' \in H^{s'}(\mathbb{R}^3)$ and scattering data $\phi^{\pm} \in \dot{H}^1(\mathbb{R}^3)$. The existence of such a solution is guaranteed by [Theorem 3.19](#).

Let $T > 0$ be arbitrary. If $\varepsilon' = \varepsilon'(E_{\mathbb{R}^3}(\phi), \varepsilon)$ is small enough such that [Lemma 3.20](#) (ii) can be applied, then there is $R_0 = R_0(\phi, \varepsilon, T) \geq 1$ such that for any $R \geq R_0$ there exists $k_0 = k_0(\phi, \varepsilon, T, R)$ with the property that for any $k \geq k_0$ there is a unique strong solution

$$U_k \in C((-2TN_k^{-2}, 2TN_k^{-2}), H^1(\mathbb{T}^3)) \cap X_r^1(-2TN_k^{-2}, 2TN_k^{-2}) \quad (3.52)$$

such that the estimate

$$\|U_k - \tilde{u}_k\|_{X^1_{\mathbb{R}^3}(-2TN_k^{-2}, 2TN_k^{-2})} \lesssim_{E_{\mathbb{R}^3}(\phi)} \varepsilon' < \varepsilon \quad (3.53)$$

holds true. This implies (3.49).

For notational convenience we prove existence of U_k beyond the Euclidean window and (3.51) only in the case $t > 0$. By Lemma 3.21, there exists $T_0 = T_0(\phi, \varepsilon')$ and $k_0 = k_0(\phi, \varepsilon')$ such that for all $T \geq T_0$ and $k \geq k_0$,

$$\|e^{it\Delta_{\theta}}(TN_k\phi^{+\infty})\|_{Z([TN_k^{-2}, T^{-1}])} \leq \varepsilon'. \quad (3.54)$$

In view of (3.52), we may conclude the existence of a unique solution U_k on $(-T_0^{-1}, T_0^{-1})$ from Proposition 3.12 (i) by showing

$$\|e^{i(t-T_0N_k^{-2})\Delta_{\theta}}U_k(T_0N_k^{-2})\|_{Z([T_0N_k^{-2}, T_0^{-1}])} \leq \delta_0, \quad (3.55)$$

where $\delta_0 = \delta_0(\|U_k(T_0N_k^{-2})\|_{H^1(\mathbb{T}^3)})$ is given by Proposition 3.12 (i).

Let $T \geq T_0$, $R \geq R_0$, as well as $k \geq k_0$, and define the interval $I_k := [TN_k^{-2}, T^{-1}]$. For $J_k := [0, T^{-1} - TN_k^{-2}]$ we deduce,

$$\begin{aligned} \|e^{it\Delta_{\theta}}U_k(TN_k^{-2})\|_{Z(J_k)} &\leq \|e^{it\Delta_{\theta}}(U_k(TN_k^{-2}) - \tilde{u}_k(TN_k^{-2}))\|_{Z(J_k)} \\ &\quad + \|e^{it\Delta_{\theta}}(\tilde{u}_k(TN_k^{-2}) - e^{iTN_k^{-2}\Delta_{\theta}}(TN_k\phi^{+\infty}))\|_{Z(J_k)} \\ &\quad + \|e^{it\Delta_{\theta}}(TN_k\phi^{+\infty})\|_{Z(I_k)}. \end{aligned}$$

The first term is small since Corollary 3.5 (i) and (3.53) imply

$$\|U_k(TN_k^{-2}) - \tilde{u}_k(TN_k^{-2})\|_{H^1(\mathbb{T}^3)} \lesssim_{E_{\mathbb{R}^3}(\phi)} \varepsilon'. \quad (3.56)$$

The smallness of the last term is given by (3.54). It remains to estimate second term. We see from Corollary 3.5 (i) that

$$\|e^{it\Delta_{\theta}}(\tilde{u}_k(TN_k^{-2}) - e^{iTN_k^{-2}\Delta_{\theta}}(TN_k\phi^{+\infty}))\|_{Z(J_k)} \lesssim \|\tilde{u}_k(TN_k^{-2}) - e^{iTN_k^{-2}\Delta_{\theta}}(TN_k\phi^{+\infty})\|_{H^1(\mathbb{T}^3)}. \quad (3.57)$$

For $v \in C(\mathbb{R}, \dot{H}^1(\mathbb{R}^3))$ we denote by $V_{R,N}(v)$ the function constructed in (3.36). Let $\phi'' \in H^5(\mathbb{R}^3)$ be such that $\|\phi'' - \phi^{+\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \leq \varepsilon'$. The triangle inequality and Poincaré's inequality allow to bound

$$\begin{aligned} (3.57) &\lesssim \|\tilde{u}_k(TN_k^{-2}) - V_{R,N_k}(e^{it\Delta_{\theta}^{\mathbb{R}^3}}\phi^{+\infty})(TN_k^{-2})\|_{\dot{H}^1(\mathbb{T}^3)} \\ &\quad + \|V_{R,N_k}(e^{it\Delta_{\theta}^{\mathbb{R}^3}}\phi^{+\infty})(TN_k^{-2}) - V_{R,N_k}(e^{it\Delta_{\theta}^{\mathbb{R}^3}}\phi'')(TN_k^{-2})\|_{\dot{H}^1(\mathbb{T}^3)} \\ &\quad + \|V_{R,N_k}(e^{it\Delta_{\theta}^{\mathbb{R}^3}}\phi'')(TN_k^{-2}) - e^{iTN_k^{-2}\Delta_{\theta}}(TN_k\phi^{+\infty})\|_{H^1(\mathbb{T}^3)}. \end{aligned}$$

All terms may be bounded by $C\varepsilon'$ provided T_0 is large enough. Indeed, from the scattering property (3.33), it follows that there exists a possibly larger $T_0 = T_0(\phi, \varepsilon')$ such that for all $T \geq T_0$,

$$\|u(T) - e^{iT\Delta_{\theta}^{\mathbb{R}^3}}\phi^{+\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \leq \varepsilon'.$$

A computation shows that this implies the boundedness of the first term by $C\varepsilon'$. The second term is small because ϕ'' approximates $\phi^{+\infty}$ in $\dot{H}^1(\mathbb{R}^3)$. Finally, the smallness of the last term follows from Lemma 3.20 (ii) with $\rho = 0$. Hence, we have proved

$$\|e^{i(t-TN_k^{-2})\Delta_{\theta}}U_k(TN_k^{-2})\|_{Z(I_k)} \leq C\varepsilon' \quad (3.58)$$

for any $T \geq T_0$. This implies (3.55) for small enough ε' and therefore, we have shown the existence of a unique solution $U_k \in C((-T_0^{-1}, T_0^{-1}), H^1(\mathbb{T}^3)) \cap X_r^1(-T_0^{-1}, T_0^{-1})$.

Next, we prove (3.51) for $T \geq T_0$, $R \geq R_0$, and $k \geq k_0$. Applying the triangle inequality twice gives

$$\begin{aligned} \|U_k(t) - e^{it\Delta_\theta} T_{N_k} \phi^{+\infty}\|_{X_r^1(I_k)} &\leq \|U_k(t) - e^{i(t-TN_k^{-2})\Delta_\theta} U_k(TN_k^{-2})\|_{X_r^1(I_k)} \\ &\quad + \|e^{i(t-TN_k^{-2})\Delta_\theta} (U_k(TN_k^{-2}) - \tilde{u}_k(TN_k^{-2}))\|_{X_r^1(I_k)} \\ &\quad + \|e^{it\Delta_\theta} (e^{-iTN_k^{-2}\Delta_\theta} \tilde{u}_k(TN_k^{-2}) - T_{N_k} \phi^{+\infty})\|_{X_r^1(I_k)} \\ &=: S_1 + S_2 + S_3. \end{aligned}$$

We are left to prove $S_1 + S_2 + S_3 < \varepsilon$. In the following steps, we might decrease $\varepsilon' > 0$ further, which may increase T_0 , R_0 , and k_0 . First, we consider S_1 . We apply Proposition 3.12 (i) and use (3.58) to obtain

$$\|U_k(t) - e^{i(t-TN_k^{-2})\Delta_\theta} U_k(TN_k^{-2})\|_{X_r^1(I_k)} \lesssim \varepsilon'^{\frac{3}{2}} < \frac{\varepsilon}{3},$$

which proves the desired smallness of S_1 . The smallness of S_2 is a consequence of (3.56) and Proposition 3.3 (ii):

$$S_2 \leq \|U_k(TN_k^{-2}) - \tilde{u}_k(TN_k^{-2})\|_{H^1(\mathbb{T}^3)} < \frac{\varepsilon}{3}.$$

Finally, we consider S_3 . We have

$$S_3 \leq \|\tilde{u}_k(TN_k^{-2}) - e^{iTN_k^{-2}\Delta_\theta} (T_{N_k} \phi^{+\infty})\|_{H^1(\mathbb{T}^3)}.$$

However, this term has already appeared in (3.57) and was shown to be smaller than $\varepsilon/3$ provided ε' is small enough. That gives the desired estimate (3.51) provided $t_k = 0$ for all $k \geq 1$.

Case 2. Assume $\lim_{k \rightarrow +\infty} N_k^2 |t_k| = +\infty$. We may even assume $\lim_{k \rightarrow +\infty} N_k^2 t_k = +\infty$ by symmetry. From the existence of the wave operator and Theorem 3.19, we see that there is a solution u to (3.48) such that

$$\|u(t) - e^{it\Delta_\theta^{\mathbb{R}^3}} \phi\|_{\dot{H}^1(\mathbb{R}^3)} \rightarrow 0$$

as $t \rightarrow -\infty$. In other words, $\phi^{-\infty} = \phi$. We set $\tilde{\phi} := u(0)$ and apply the result of the proposition to the frame $\mathcal{O}' := (N_k, 0, 0)_{k \geq 1}$. Note that this frame fulfills the assumptions of the first case. Hence, there exists a solution to (3.1) on $(-T_0^{-1}, T_0^{-1})$, say V_k , with initial data $V_k(0) = T_{N_k} \tilde{\phi}$. From $\lim_{k \rightarrow +\infty} N_k^2 t_k = +\infty$, we have for sufficiently large k that $t_k \geq T_0 N_k^{-2}$. Hence, (3.51) implies

$$\|V_k(-t_k) - \Pi_{t_k, 0} T_{N_k} \phi\|_{H^1(\mathbb{T}^3)} \lesssim \|V_k(t) - \Pi_{-t, 0} T_{N_k} \phi\|_{X_r^1(\{-t \geq T_0 N_k^{-2}\} \cap \{|t| \leq T_0^{-1}\})} \rightarrow 0$$

as $k \rightarrow +\infty$. Recall that, by definition, $U_k(0) = \Pi_{t_k, 0} T_{N_k} \phi$. This allows us to apply our stability result (Proposition 3.15), and we observe

$$\|V_k(\cdot - t_k) - U_k\|_{X_r^1(-T_0^{-1}, T_0^{-1})} \rightarrow 0.$$

Note that U_k inherits the estimates (3.49) and (3.51) from V_k . \square

3.5 Profile decomposition

We show that for every given bounded sequence of functions in $H^1(\mathbb{T}^3)$, we can construct suitable Euclidean profiles and up to a subsequence, express the sequence as an almost orthogonal sum of these profiles, the sequence's weak limit, and a remainder term. The study of Euclidean profiles in the previous section makes this decomposition meaningful. We adapt the strategies in [IPS12, Lemma 5.7] and [IP12a, Section 5] in which analogue statements were proved for the nonlinear Schrödinger equation on the hyperbolic space \mathbb{H} and $\mathbb{R} \times \mathbb{T}^3$, respectively. The profile decomposition discussed here is an analogue of Keraani's theorem [Ker01] on rectangular tori.

3.5.1 Definition and properties

The previously introduced class of renormalized Euclidean frames $\widetilde{\mathcal{F}}_E$ is extended now to the class of *Euclidean frames*. Here, we drop the assumption that either $t_k = 0$ for all $k \geq 1$ or $\lim_{k \rightarrow \infty} N_k^2 |t_k| = +\infty$.

Definition 3.24 (Euclidean frames).

- (i) The set of *Euclidean frames* is defined as

$$\mathcal{F}_E := \left\{ (N_k, t_k, x_k)_{k \geq 1} : N_k \geq 1, N_k \rightarrow +\infty, t_k \rightarrow 0, x_k \in \mathbb{T}^3 \right\}.$$

We say that two frames, $(N_k, t_k, x_k)_k$ and $(N'_k, t'_k, x'_k)_k$, are *orthogonal* if

$$\lim_{k \rightarrow +\infty} \left(\left| \ln \frac{N_k}{N'_k} \right| + N_k^2 |t_k - t'_k| + N_k |x_k - x'_k| \right) = +\infty.$$

Two frames that are not orthogonal are called *equivalent*.

- (ii) If $\mathcal{O} = (N_k, t_k, x_k)_k$ is a Euclidean frame and if $\psi \in \dot{H}^1(\mathbb{R}^3)$, we define the *Euclidean profile associated to (ψ, \mathcal{O})* as the sequence $(\widetilde{\psi}_{\mathcal{O}_k})_k$ in $H^1(\mathbb{T}^3)$ with

$$\widetilde{\psi}_{\mathcal{O}_k} := \Pi_{t_k, x_k}(T_{N_k} \psi). \quad (3.59)$$

In the following lemma, we summarize the basic properties of profiles associated to equivalent and orthogonal frames. The proof follows the strategy in [IPS12, Lemma 5.7].

Lemma 3.25 (Properties of frames).

- (i) If \mathcal{O} and \mathcal{O}' are equivalent Euclidean frames, then there is an isometry $S: \dot{H}^1(\mathbb{R}^3) \rightarrow \dot{H}^1(\mathbb{R}^3)$ such that for any profile $(\widetilde{\psi}_{\mathcal{O}'_k})_k$, up to a subsequence, it holds that

$$\lim_{k \rightarrow +\infty} \|\widetilde{S}\widetilde{\psi}_{\mathcal{O}_k} - \widetilde{\psi}_{\mathcal{O}'_k}\|_{H^1(\mathbb{T}^3)} = 0. \quad (3.60)$$

- (ii) If \mathcal{O} and \mathcal{O}' are orthogonal frames and $(\widetilde{\psi}_{\mathcal{O}_k})_k, (\widetilde{\phi}_{\mathcal{O}'_k})_k$ are corresponding profiles, then, up to a subsequence:

$$\lim_{k \rightarrow +\infty} \langle \widetilde{\psi}_{\mathcal{O}_k}, \widetilde{\phi}_{\mathcal{O}'_k} \rangle_{H^1(\mathbb{T}^3)} = 0, \quad (3.61)$$

$$\lim_{k \rightarrow +\infty} \|\widetilde{\psi}_{\mathcal{O}_k} \widetilde{\phi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{T}^3)} = 0. \quad (3.62)$$

(iii) If \mathcal{O} is a Euclidean frame and $(\tilde{\psi}_{\mathcal{O}_k})_k, (\tilde{\phi}_{\mathcal{O}_k})_k$ are two profiles corresponding to \mathcal{O} , then

$$\begin{aligned} \lim_{k \rightarrow +\infty} (\|\tilde{\psi}_{\mathcal{O}_k}\|_{L^2(\mathbb{T}^3)} + \|\tilde{\phi}_{\mathcal{O}_k}\|_{L^2(\mathbb{T}^3)}) &= 0, \\ \lim_{k \rightarrow +\infty} \langle \tilde{\psi}_{\mathcal{O}_k}, \tilde{\phi}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)} &= \langle \psi, \phi \rangle_{H^1(\mathbb{R}^3)}. \end{aligned} \quad (3.63)$$

(iv) If \mathcal{O} is a renormalized Euclidean frame and $(\tilde{\psi}_{\mathcal{O}_k})_k$ a profile corresponding to \mathcal{O} , then for every $g \in H^1(\mathbb{T}^3)$,

$$\limsup_{k \rightarrow +\infty} \|\tilde{\psi}_{\mathcal{O}_k} g\|_{L^3(\mathbb{T}^3)} = 0.$$

Proof. We prove every claim individually.

Ad (i). Let $\mathcal{O} = (N_k, t_k, x_k)_k$ and $\mathcal{O}' = (N'_k, t'_k, x'_k)_k$ be equivalent Euclidean frames. After passing to a subsequence, we may assume

$$\lim_{k \rightarrow \infty} \frac{N'_k}{N_k} = \bar{N}, \quad \lim_{k \rightarrow \infty} N'_k{}^2(t'_k - t_k) = \bar{t}, \quad \text{and} \quad \lim_{k \rightarrow \infty} N'_k \Psi^{-1}(x'_k - x_k) = \bar{x}$$

for some $\bar{N}, \bar{t} \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^3$. Note that there exists $T_0 > 0$ such that $|t_k - t'_k| < T_0 N'_k{}^{-2}$ for all k . Given $\psi \in \dot{H}^1(\mathbb{R}^3)$ we define $S: \dot{H}^1(\mathbb{R}^3) \rightarrow \dot{H}^1(\mathbb{R}^3)$ via

$$(S\psi)(x) := \bar{N}^{\frac{1}{2}} \Pi_{\bar{t}, \bar{x}} \psi(\bar{N}x) = \bar{N}^{\frac{1}{2}} (e^{-i\bar{t}\Delta_{\theta}} \psi)(\bar{N}x - \bar{x})$$

and remark that S is an isometry on $\dot{H}^1(\mathbb{R}^3)$. Furthermore, we define $\widetilde{S\psi}_{\mathcal{O}_k}$ as in (3.59). By definition, (3.60) follows from

$$\lim_{k \rightarrow \infty} \|\Pi_{t_k, x_k}(T_{N_k}(S\psi)) - \Pi_{t'_k, x'_k}(T_{N'_k}\psi)\|_{H^1(\mathbb{T}^3)} = 0,$$

which is equivalent to

$$\lim_{k \rightarrow \infty} \|\pi_{x_k - x'_k}(T_{N_k}(S\psi)) - e^{i(t_k - t'_k)\Delta_{\theta}}(T_{N'_k}\psi)\|_{H^1(\mathbb{T}^3)} = 0. \quad (3.64)$$

In order to prove (3.64), we may assume $S\psi \in C_0^\infty(\mathbb{R}^3)$ and $\psi \in H^5(\mathbb{R}^3)$ because of density and the $\dot{H}^1(\mathbb{R}^3) \rightarrow H^1(\mathbb{T}^3)$ boundedness of the operator T_N (Corollary 3.17). Set $v(t, x) := e^{it\Delta_{\theta}^{\mathbb{R}^3}} \psi(x)$, and define v_R, v_{R, N'_k} , and V_{R, N'_k} as in (3.36). Now, we apply Lemma 3.20 (ii) with $\rho = 0$ and T_0 as defined above. We deduce that for any $\varepsilon > 0$ small enough there exists $R_0 = R_0(T_0, \psi, \varepsilon)$ such that for all $R \geq R_0$,

$$\lim_{k \rightarrow \infty} \|e^{i(t_k - t'_k)\Delta_{\theta}}(T_{N'_k}\psi) - V_{R, N'_k}(t_k - t'_k)\|_{H^1(\mathbb{T}^3)} \lesssim \varepsilon.$$

This, indeed, is true for any $k \geq 1$ since, from the choice of T_0 , the evolution stays inside the Euclidean window. By the triangle inequality, the last estimate implies that (3.64) follows if we prove

$$\lim_{k \rightarrow \infty} \|\pi_{x_k - x'_k}(T_{N_k}(S\psi)) - V_{R, N'_k}(t_k - t'_k)\|_{H^1(\mathbb{T}^3)} \lesssim \varepsilon$$

for sufficiently large R . From the definitions and since $S\psi \in C_0^\infty(\mathbb{R}^3)$, this inequality is equivalent to

$$\lim_{k \rightarrow \infty} \|N_k^{\frac{1}{2}}(S\psi)(N_k \Psi^{-1}(y - (x_k - x'_k))) - N'_k{}^{\frac{1}{2}} v_R(N'_k{}^2(t_k - t'_k), N'_k \Psi^{-1}(y))\|_{H_y^1(\mathbb{T}^3)} \lesssim \varepsilon.$$

Note that η can be dropped in the first term because for k sufficiently large, we have that $\text{supp}((S\psi)(N_k \cdot)) \subset \text{supp}(\eta(N_k^{1/2} \cdot))$. We substitute $y := \Psi(x)$, then the inequality above is equivalent to

$$\lim_{k \rightarrow \infty} \left\| N_k^{\frac{1}{2}} (S\psi)(N_k x - N_k \Psi^{-1}(x_k - x'_k)) - N_k^{\frac{1}{2}} v_R(N_k'^2(t_k - t'_k), N_k' x) \right\|_{\dot{H}_x^1(\mathbb{R}^3)} \lesssim \varepsilon. \quad (3.65)$$

One easily calculates that the left-hand side is equal to

$$\lim_{k \rightarrow \infty} \left\| (S\psi)(x) - \left(\frac{N_k'}{N_k} \right)^{\frac{1}{2}} v_R \left(N_k'^2(t_k - t'_k), \frac{N_k'}{N_k} x - N_k' \Psi^{-1}(x'_k - x_k) \right) \right\|_{\dot{H}_x^1(\mathbb{R}^3)}.$$

By definition, $(S\psi)(x) = \overline{N}^{\frac{1}{2}} v(-\bar{t}, \overline{N}x - \bar{x})$, and since Sobolev's embedding implies $v_R \in C(\mathbb{R}, C_0^1(\mathbb{R}^3))$, we deduce from dominated convergence that (3.65) is equivalent to

$$\left\| \overline{N}^{\frac{1}{2}} v(-\bar{t}, \overline{N}x - \bar{x}) - \overline{N}^{\frac{1}{2}} v_R(-\bar{t}, \overline{N}x - \bar{x}) \right\|_{\dot{H}_x^1(\mathbb{R}^3)} \lesssim \varepsilon,$$

which is obviously true for R sufficiently large.

Ad (ii). Let $\mathcal{O} = (N_k, t_k, x_k)_k$ and $\mathcal{O}' = (N_k', t'_k, x'_k)_k$ be orthogonal Euclidean frames. Without loss of generality, we may assume $\psi, \phi \in C_0^\infty(\mathbb{R}^3)$. Since $N_k, N_k' \rightarrow +\infty$ as $k \rightarrow +\infty$, we obtain from (3.44) that

$$\lim_{k \rightarrow \infty} \left| \int_{\mathbb{T}^3} \tilde{\psi}_{\mathcal{O}_k}(x) \overline{\tilde{\phi}_{\mathcal{O}'_k}(x)} dx \right| \leq \lim_{k \rightarrow \infty} \|T_{N_k} \psi\|_{L^2(\mathbb{T}^3)} \|T_{N_k'} \phi\|_{L^2(\mathbb{T}^3)} = 0.$$

As a consequence, we reduced (3.61) to

$$\lim_{k \rightarrow +\infty} \left| \int_{\mathbb{T}^3} \nabla \tilde{\psi}_{\mathcal{O}_k}(x) \cdot \overline{\nabla \tilde{\phi}_{\mathcal{O}'_k}(x)} dx \right| = 0. \quad (3.66)$$

To prove the remaining estimates, we select a subsequence such that either

$$\lim_{k \rightarrow \infty} \frac{N_k'}{N_k} = 0, \quad (3.67)$$

or

$$\lim_{k \rightarrow \infty} \frac{N_k'}{N_k} = \overline{N}, \quad \lim_{k \rightarrow \infty} N_k'^2 |t'_k - t_k| = \infty \quad (3.68)$$

for some $\overline{N} \in (0, \infty)$, or

$$\lim_{k \rightarrow \infty} \frac{N_k'}{N_k} = \overline{N}, \quad \lim_{k \rightarrow \infty} N_k'^2 (t'_k - t_k) = \bar{t}, \quad \lim_{k \rightarrow \infty} N_k' |x'_k - x_k| = \infty \quad (3.69)$$

for some $\overline{N} \in (0, \infty)$ and $\bar{t} \in \mathbb{R}$.

First, we assume the case (3.67). We deduce from Green's formula (cf. [Jos11, formula (3.1.7)]), the definition of a Euclidean profile, and Hölder's inequality that

$$\left| \int_{\mathbb{T}^3} \nabla \tilde{\psi}_{\mathcal{O}_k}(x) \cdot \overline{\nabla \tilde{\phi}_{\mathcal{O}'_k}(x)} dx \right| = \left| \int_{\mathbb{T}^3} \tilde{\psi}_{\mathcal{O}_k}(x) \overline{\Delta_g \tilde{\phi}_{\mathcal{O}'_k}(x)} dx \right| \lesssim \|T_{N_k} \psi\|_{L^2(\mathbb{T}^3)} \|\Delta_g(T_{N_k'} \phi)\|_{L^2(\mathbb{T}^3)}.$$

One easily computes that $\|\Delta_g(T_{N_k'} \phi)\|_{L^2(\mathbb{T}^3)} \lesssim \phi N_k'$, and together with (3.44), we obtain

$$\left| \int_{\mathbb{T}^3} \nabla \tilde{\psi}_{\mathcal{O}_k}(x) \cdot \overline{\nabla \tilde{\phi}_{\mathcal{O}'_k}(x)} dx \right| \lesssim_{\psi, \phi} \frac{N_k'}{N_k}. \quad (3.70)$$

Furthermore, using Sobolev embeddings,

$$\begin{aligned} \|\tilde{\psi}_{\mathcal{O}_k} \tilde{\phi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{T}^3)} &\leq \|\Pi_{t_k, x_k}(T_{N_k} \psi)\|_{L^{\frac{9}{2}}(\mathbb{T}^3)} \|\Pi_{t'_k, x'_k}(T_{N'_k} \phi)\|_{L^9(\mathbb{T}^3)} \\ &\lesssim \|(-\Delta_g)^{\frac{5}{6}} T_{N_k} \psi\|_{L^2(\mathbb{T}^3)} \|(-\Delta_g)^{\frac{7}{6}} T_{N'_k} \phi\|_{L^2(\mathbb{T}^3)} \\ &\lesssim_{\psi, \phi} \left(\frac{N'_k}{N_k}\right)^{\frac{1}{6}}. \end{aligned} \quad (3.71)$$

Now, (3.66) and (3.62) follow from (3.70) and (3.71) as $k \rightarrow \infty$ provided (3.67).

We consider the case (3.68) now. We first prove the following statement: For any $f \in \dot{H}^1(\mathbb{R}^3)$ and all sequences $M_k \geq 1$ and $s_k \rightarrow 0$ with $M_k^2 |s_k| \rightarrow +\infty$ as $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} \|e^{is_k \Delta_\theta}(T_{M_k} f)\|_{L^6(\mathbb{T}^3)} = 0 \quad (3.72)$$

for a subsequence.

This is accomplished by applying Lemma 3.21 (ii) in either of the following two sub-cases: We may choose a subsequence such that either $0 < M_k |s_k| \leq 1$ or $M_k |s_k| > 1$ for any $k \geq 1$.

Lets first assume $M_k |s_k| \leq 1$ for all $k \geq 1$, and define $T_k := M_k^2 |s_k|$. Note that $1 \leq T_k \leq M_k$ for large k . Since $|s_k| \in [T_k M_k^{-2}, T_k^{-1}]$, we may apply Lemma 3.21 (ii) from which we deduce

$$\|e^{is_k \Delta_\theta}(T_{M_k} f)\|_{L^6(\mathbb{T}^3)} \lesssim_f (1 + M_k^2 |s_k|)^{-\frac{1}{10}}$$

provided k is sufficiently large.

On the other hand, if $M_k |s_k| > 1$ for any $k \geq 1$, we define $T_k := |s_k|^{-1}$. Obviously, for k large enough, $1 \leq T_k < M_k$ and $|s_k| \in [T_k M_k^{-2}, T_k^{-1}]$. Thus, Lemma 3.21 (ii) implies

$$\|e^{is_k \Delta_\theta}(T_{M_k} f)\|_{L^6(\mathbb{T}^3)} \lesssim_f |s_k|^{\frac{1}{10}},$$

and the claim is proved.

We conclude that for k large enough,

$$\begin{aligned} \left| \int_{\mathbb{T}^3} \nabla \tilde{\psi}_{\mathcal{O}_k}(x) \cdot \overline{\nabla \tilde{\phi}_{\mathcal{O}'_k}(x)} dx \right| &= \left| \int_{\mathbb{T}^3} \Delta_g(T_{N_k} \psi)(x) \overline{\Pi_{t'_k - t_k, x'_k - x_k}(T_{N'_k} \phi)(x)} dx \right| \\ &\lesssim \|\Delta_g(T_{N_k} \psi)\|_{L^{\frac{6}{5}}(\mathbb{T}^3)} \|\Pi_{t'_k - t_k, x'_k - x_k}(T_{N'_k} \phi)\|_{L^6(\mathbb{T}^3)}. \end{aligned}$$

Using $\|\Delta_g(T_{N_k} \psi)\|_{L^{\frac{6}{5}}(\mathbb{T}^3)} \lesssim_\phi 1$, see (3.44), and (3.72), we obtain (3.61). The claim (3.62) is implied by

$$\|\tilde{\psi}_{\mathcal{O}_k} \tilde{\phi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{T}^3)} \leq \|T_{N_k} \psi\|_{L^6(\mathbb{T}^3)} \|\Pi_{t'_k - t_k, x'_k - x_k}(T_{N'_k} \phi)\|_{L^6(\mathbb{T}^3)},$$

(3.44), and (3.72).

We now assume (3.69). First, we claim that for all sequences $y_k \in \mathbb{T}^3$, $M_k \geq 1$ with the properties $\lim_{k \rightarrow \infty} M_k = \infty$, $\lim_{k \rightarrow \infty} M_k |y_k| = \infty$, and all $f, g \in \dot{H}^1(\mathbb{R}^3)$, it holds that

$$\lim_{k \rightarrow \infty} \left(\left| \int_{\mathbb{T}^3} (\pi_{y_k} \nabla(T_{M_k} f))(x) \cdot \overline{\nabla(T_{M_k} g)(x)} dx \right| + \|\pi_{y_k}(T_{M_k} f)(T_{M_k} g)\|_{L^3(\mathbb{T}^3)} \right) = 0. \quad (3.73)$$

Assuming this, we may prove (3.61) and (3.62) in the case (3.69). Indeed, thanks to (3.64), we have for $f \in \dot{H}^1(\mathbb{R}^3)$ and a sequence $(s_k)_k$ with the property $\lim_{k \rightarrow \infty} N_k^2 s_k = \bar{s} \in \mathbb{R}$ that

$$\lim_{k \rightarrow \infty} \|T_{N_k}(Sf) - e^{-is_k \Delta_\theta}(T_{N'_k} f)\|_{H^1(\mathbb{T}^3)} = 0, \quad (3.74)$$

where $(Sf)(x) := \overline{N}^{\frac{1}{2}}(e^{-i\overline{s}\Delta_{\theta}^{\mathbb{R}^3}} f)(\overline{N}x)$. We estimate

$$\begin{aligned} \left| \int_{\mathbb{T}^3} \nabla \tilde{\psi}_{\mathcal{O}_k}(x) \cdot \overline{\nabla \tilde{\phi}_{\mathcal{O}'_k}(x)} dx \right| &= \left| \int_{\mathbb{T}^3} \nabla(T_{N_k}\psi)(x) \cdot \overline{\nabla(\Pi_{t'_k-t_k, x'_k-x_k} T_{N'_k}\phi)(x)} dx \right| \\ &\lesssim \left| \int_{\mathbb{T}^3} \nabla(T_{N_k}\psi)(x) \cdot \overline{\pi_{x'_k-x_k} \nabla(T_{N_k}(S\phi))(x)} dx \right| \\ &\quad + \|\psi\|_{\dot{H}^1(\mathbb{R}^3)} \|T_{N_k}(S\phi) - e^{-i(t'_k-t_k)\Delta_{\theta}}(T_{N'_k}\phi)\|_{H^1(\mathbb{T}^3)}. \end{aligned}$$

From (3.73), we see that the first term tends to 0 as $k \rightarrow \infty$, and from (3.74), we obtain the same for the second term. If either $N_k^2|t_k| \rightarrow \infty$ or $N'_k{}^2|t'_k| \rightarrow \infty$ as $k \rightarrow \infty$, we get from (3.72) that

$$\|\tilde{\psi}_{\mathcal{O}_k} \tilde{\phi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{T}^3)} \leq \|\Pi_{t_k, x_k}(T_{N_k}\psi)\|_{L^6(\mathbb{T}^3)} \|\Pi_{t'_k, x'_k}(T_{N'_k}\phi)\|_{L^6(\mathbb{T}^3)} \rightarrow 0$$

as $k \rightarrow \infty$. Otherwise, if $\lim_{k \rightarrow \infty} N_k^2|t_k| = \overline{T} \in \mathbb{R}$ and $\lim_{k \rightarrow \infty} N'_k{}^2|t'_k| = \overline{T}' \in \mathbb{R}$, we estimate

$$\begin{aligned} \|\tilde{\psi}_{\mathcal{O}_k} \tilde{\phi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{T}^3)} &= \left\| (\pi_{x_k-x'_k} e^{-it_k\Delta_{\theta}}(T_{N_k}\psi)) e^{-it'_k\Delta_{\theta}}(T_{N'_k}\phi) \right\|_{L^3(\mathbb{T}^3)} \\ &\lesssim \left\| \pi_{x_k-x'_k} (e^{-it_k\Delta_{\theta}}(T_{N_k}\psi) - T_{N'_k}(S\psi)) \right\|_{H^1(\mathbb{T}^3)} \|\phi\|_{\dot{H}^1(\mathbb{R}^3)} \\ &\quad + \|\psi\|_{\dot{H}^1(\mathbb{R}^3)} \|e^{-it'_k\Delta_{\theta}}(T_{N'_k}\phi) - T_{N'_k}(\tilde{S}\phi)\|_{H^1(\mathbb{T}^3)} \\ &\quad + \|\pi_{x_k-x'_k}(T_{N'_k}(S\psi))T_{N'_k}(\tilde{S}\phi)\|_{L^3(\mathbb{T}^3)}, \end{aligned}$$

where $\tilde{S}: \dot{H}^1(\mathbb{R}^3) \rightarrow \dot{H}^1(\mathbb{R}^3)$, $(\tilde{S}\phi)(x) := (e^{-i\overline{T}'\Delta_{\theta}^{\mathbb{R}^3}} \phi)(x)$. Each term tends to zero because of (3.74) and (3.73).

We turn to the proof of (3.73). Because of density and the $\dot{H}^1(\mathbb{R}^3) \rightarrow H^1(\mathbb{T}^3)$ boundedness of T_N (Corollary 3.17), we may assume that $f, g \in C_0^\infty(\mathbb{R}^3)$ and replace $T_{M_k}f$ and $T_{M_k}g$ by $\tilde{f}(x) := M_k^{\frac{1}{2}}f(M_k\Psi^{-1}(x))$ and $\tilde{g}(x) := M_k^{\frac{1}{2}}g(M_k\Psi^{-1}(x))$, respectively. We have

$$\left| \int_{\mathbb{T}^3} (\pi_{y_k} \nabla \tilde{f})(x) \cdot \overline{\nabla \tilde{g}(x)} dx \right| = M_k^3 \left| \int_{\mathbb{R}^3} \nabla_{\mathbb{R}^3} f(M_k(x-y_k)) \cdot \overline{\nabla_{\mathbb{R}^3} g(M_k x)} dx \right|$$

as well as

$$\|(\pi_{y_k} \tilde{f}) \tilde{g}\|_{L^3(\mathbb{T}^3)} = M_k \|f(M_k(\cdot - y_k))g(M_k \cdot)\|_{L^3(\mathbb{R}^3)}.$$

That either term tends to zero as $k \rightarrow \infty$ follows from the fact that the support of these functions become disjoint for large k , which is due to the assumption $\lim_{k \rightarrow \infty} M_k|y_k| = \infty$.

Ad (iii). Let $\mathcal{O} = (N_k, t_k, x_k)_k$ be a Euclidean frame and $(\tilde{\psi}_{\mathcal{O}_k})_k, (\tilde{\phi}_{\mathcal{O}_k})_k$ be two profiles corresponding to \mathcal{O} . Again, the $\dot{H}^1(\mathbb{R}^3) \rightarrow H^1(\mathbb{T}^3)$ boundedness of T_N allows to assume $\psi, \phi \in C_0^\infty(\mathbb{R}^3)$. Since Π_{t_k, x_k} is an isometry on $L^2(\mathbb{T}^3)$, we easily get from (3.44) that

$$\|\tilde{\psi}_{\mathcal{O}_k}\|_{L^2(\mathbb{T}^3)} = \|T_{N_k}\psi\|_{L^2(\mathbb{T}^3)} \lesssim_{\psi} N_k^{-1},$$

which in turn implies (3.63).

By the unitarity of Π_{t_k, x_k} , it suffices to prove

$$\lim_{k \rightarrow \infty} \langle \nabla(T_{N_k}\psi), \nabla(T_{N_k}\phi) \rangle_{L^2(\mathbb{T}^3)} = \langle \nabla_{\mathbb{R}^3}\psi, \nabla_{\mathbb{R}^3}\phi \rangle_{L^2(\mathbb{R}^3)}.$$

For $f \in C_0^\infty(\mathbb{R}^3)$, we have

$$\|\nabla(T_{N_k}f - N_k^{\frac{1}{2}}f(N_k\Psi^{-1}))\|_{L^2(\mathbb{T}^3)} \rightarrow 0$$

as $k \rightarrow \infty$, and consequently, we may replace the functions $T_{N_k}\psi$ and $T_{N_k}\phi$ by $N_k^{\frac{1}{2}}\psi(N_k\Psi^{-1})$ and $N_k^{\frac{1}{2}}\phi(N_k\Psi^{-1})$, respectively. Thus, the desired estimate is implied, if we show

$$N_k \langle \nabla(\psi(N_k\Psi^{-1}\cdot)), \nabla(\phi(N_k\Psi^{-1}\cdot)) \rangle_{L^2(\mathbb{T}^3)} = \langle \nabla_{\mathbb{R}^3}\psi, \nabla_{\mathbb{R}^3}\phi \rangle_{L^2(\mathbb{R}^3)}$$

for sufficiently large k . However, this follows from a change of variables.

Ad (iv). Without loss of generality, we may assume $g \in C^\infty(\mathbb{T}^3)$ and $\psi \in C_0^\infty(\mathbb{R}^3)$. Let $\mathcal{O} = (N_k, t_k, x_k)_k$. We use (3.44) to estimate

$$\|\tilde{\psi}_{\mathcal{O}_k} g\|_{L^3(\mathbb{T}^3)} \leq \|\tilde{\psi}_{\mathcal{O}_k}\|_{L^{\frac{9}{2}}(\mathbb{T}^3)} \|g\|_{L^9(\mathbb{T}^3)} \lesssim_{\psi, g} N_k^{-\frac{1}{6}}.$$

Letting $k \rightarrow \infty$, this implies the claim. \square

Definition 3.26 (Absence from a frame). We say that a sequence of functions $(f_k)_k \subseteq H^1(\mathbb{T}^3)$ is *absent from a frame* \mathcal{O} , if for every profile $(\tilde{\psi}_{\mathcal{O}_k})_k$ associated to \mathcal{O} ,

$$\langle f_k, \tilde{\psi}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)} \rightarrow 0$$

as $k \rightarrow +\infty$.

Remark. Note that (3.61) implies that a profile associated to a frame \mathcal{O} is absent from any frame orthogonal to \mathcal{O} . \diamond

3.5.2 Extracting profiles from a sequence

The profile decomposition in the next proposition is the main statement of this subsection.

Proposition 3.27. *Let $(f_k)_k$ be a sequence of functions in $H^1(\mathbb{T}^3)$ satisfying*

$$\limsup_{k \rightarrow +\infty} \|f_k\|_{H^1(\mathbb{T}^3)} \lesssim E$$

and up to a subsequence, $f_k \rightharpoonup g \in H^1(\mathbb{T}^3)$. Furthermore, let $I_k = (-T_k, T_k)$ be a sequence of intervals around the origin such that $|I_k| \rightarrow 0$ as $k \rightarrow +\infty$. Then, there exist a sequence of pairwise orthogonal Euclidean frames $(\mathcal{O}^\alpha)_\alpha$ and a subsequence of profiles $(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha)_k$ associated to \mathcal{O}^α such that, after extracting a subsequence, for every $J \geq 0$,

$$f_k = g + \sum_{\alpha=1}^J \tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha + R_k^J,$$

where R_k^J is absent from the frames \mathcal{O}^α , $1 \leq \alpha \leq J$, and is small in the sense that

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{N \geq 1, t \in I_k, x \in \mathbb{T}^3} N^{-\frac{1}{2}} |(e^{it\Delta_\theta} P_N R_k^J)(x)| = 0. \quad (3.75)$$

Besides, we also have the following orthogonality relations:

$$\begin{aligned} \|f_k\|_{L^2(\mathbb{T}^3)}^2 &= \|g\|_{L^2(\mathbb{T}^3)}^2 + \|R_k^J\|_{L^2(\mathbb{T}^3)}^2 + o_k(1), \\ \|\nabla f_k\|_{L^2(\mathbb{T}^3)}^2 &= \|\nabla g\|_{L^2(\mathbb{T}^3)}^2 + \sum_{\alpha=1}^J \|\nabla_{\mathbb{R}^3}\psi^\alpha\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla R_k^J\|_{L^2(\mathbb{T}^3)}^2 + o_k(1), \end{aligned} \quad (3.76)$$

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \left| \|f_k\|_{L^6(\mathbb{T}^3)}^6 - \|g\|_{L^6(\mathbb{T}^3)}^6 - \sum_{\alpha=1}^J \|\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha\|_{L^6(\mathbb{T}^3)}^6 \right| = 0,$$

where $o_k(1) \rightarrow 0$ as $k \rightarrow +\infty$, possibly depending on J .

Before we turn to its proof, we prove two auxiliary results, which are similar to [IP12a, Lemma 2.3] and [IP12a, Lemma 5.4].

Lemma 3.28. *For every $f \in H^1(\mathbb{T}^3)$,*

$$\|f\|_{L^6(\mathbb{T}^3)}^6 \lesssim \|f\|_{H^1(\mathbb{T}^3)}^2 \left(\sup_{N \geq 1} N^{-\frac{1}{2}} \|P_N f\|_{L^\infty(\mathbb{T}^3)} \right)^4$$

holds true.

Proof. We dyadically decompose f in its frequencies, $f = \sum_{N \geq 1} P_N f$, and obtain

$$\|f\|_{L^6(\mathbb{T}^3)}^6 \leq \sum_{N_1, \dots, N_6 \geq 1} \left| \int_{\mathbb{T}^3} P_{N_1} f \overline{P_{N_2} f} P_{N_3} f \overline{P_{N_4} f} P_{N_5} f \overline{P_{N_6} f} dx \right|.$$

The integral is zero, unless there are elements in the support of the Fourier transforms which add up to zero. Hence, we may assume the two highest frequencies to be comparable. We order the frequencies to get

$$\|f\|_{L^6(\mathbb{T}^3)}^6 \lesssim \sum_{N_1 \approx N_2 \geq \dots \geq N_6} \int_{\mathbb{T}^3} |P_{N_1} f P_{N_2} f P_{N_3} f P_{N_4} f P_{N_5} f P_{N_6} f| dx.$$

Estimating the two high-frequency terms in $L^2(\mathbb{T}^3)$ and the rest in $L^\infty(\mathbb{T}^3)$, we obtain

$$\begin{aligned} \|f\|_{L^6(\mathbb{T}^3)}^6 &\lesssim \left(\sup_{N \geq 1} N^{-\frac{1}{2}} \|P_N f\|_{L^\infty(\mathbb{T}^3)} \right)^4 \\ &\quad \times \sum_{N_1 \approx N_2 \geq N_3 \geq \dots \geq N_6} (N_3 N_4 N_5 N_6)^{\frac{1}{2}} \|P_{N_1} f\|_{L^2(\mathbb{T}^3)} \|P_{N_2} f\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

Summing over N_6, N_5, N_4 , and N_3 yields

$$\|f\|_{L^6(\mathbb{T}^3)}^6 \lesssim \left(\sup_{N \geq 1} N^{-\frac{1}{2}} \|P_N f\|_{L^\infty(\mathbb{T}^3)} \right)^4 \sum_{N_1 \approx N_2} N_1 N_2 \|P_{N_1} f\|_{L^2(\mathbb{T}^3)} \|P_{N_2} f\|_{L^2(\mathbb{T}^3)},$$

which, after applying Cauchy–Schwarz, implies the claim. \square

Lemma 3.29. *Let $\delta > 0$ be fixed, and let $(f_k)_k$ be a sequence of functions in $H^1(\mathbb{T}^3)$ satisfying*

$$\limsup_{k \rightarrow +\infty} \|f_k\|_{H^1(\mathbb{T}^3)} \lesssim E \tag{3.77}$$

and up to passing to a subsequence, $f_k \rightharpoonup g \in H^1(\mathbb{T}^3)$. Furthermore, let $I_k = (-T_k, T^k)$ be a sequence of intervals around the origin such that $|I_k| \rightarrow 0$ as $k \rightarrow +\infty$. Then, there exist $J \lesssim \delta^{-2}$ pairwise orthogonal frames \mathcal{O}^α , $1 \leq \alpha \leq J$, and profiles $(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha)_k$ associated to \mathcal{O}^α such that, after extracting a subsequence,

$$f_k = g + \sum_{\alpha=1}^J \tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha + R_k,$$

where R_k is absent from all frames \mathcal{O}^α and is small in the sense that

$$\sup_{N \geq 1, t \in I_k, x \in \mathbb{T}^3} N^{-\frac{1}{2}} |(e^{it\Delta_\theta} P_N R_k)(x)| \leq \delta.$$

Besides, the following orthogonality relations hold true:

$$\begin{aligned} \|f_k\|_{L^2(\mathbb{T}^3)}^2 &= \|g\|_{L^2(\mathbb{T}^3)}^2 + \|R_k\|_{L^2(\mathbb{T}^3)}^2 + o_k(1) \\ \|\nabla f_k\|_{L^2(\mathbb{T}^3)}^2 &= \|\nabla g\|_{L^2(\mathbb{T}^3)}^2 + \sum_{\alpha=1}^J \|\nabla_{\mathbb{R}^3} \psi^\alpha\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla R_k\|_{L^2(\mathbb{T}^3)}^2 + o_k(1), \end{aligned}$$

where $o_k(1) \rightarrow 0$ as $k \rightarrow +\infty$.

Proof. We subdivide the proof into three steps.

Step 1. In this step, we extract a frame under the additional assumption that $f_k \rightharpoonup 0$ in $H^1(\mathbb{T}^3)$. So, let $(f_k)_k$ be a sequence satisfying the assumptions of the lemma, and assume that $(f_k)_k$ converges weakly in $H^1(\mathbb{T}^3)$ to zero. We define the functional Λ via

$$\Lambda((f_k)_k) := \limsup_{k \rightarrow +\infty} \sup_{N \geq 1, t \in I_k, x \in \mathbb{T}^3} N^{-\frac{1}{2}} |(e^{it\Delta_\theta} P_N f_k)(x)|.$$

Claim. If $\Lambda((f_k)_k) \geq \delta$, then there exist a frame \mathcal{O} and an associated profile $(\tilde{\psi}_{\mathcal{O}_k})_k$ satisfying

$$\limsup_{k \rightarrow +\infty} \|\tilde{\psi}_{\mathcal{O}_k}\|_{H^1(\mathbb{T}^3)} \lesssim \|\psi\|_{\dot{H}^1(\mathbb{R}^3)} \quad (3.78)$$

and

$$\limsup_{k \rightarrow +\infty} |\langle f_k, \tilde{\psi}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)}| \geq \frac{\delta}{2}. \quad (3.79)$$

Furthermore, if $(f_k)_k$ was absent from a family of frames $(\mathcal{O}^\alpha)_\alpha$, then \mathcal{O} is orthogonal to all the frames \mathcal{O}^α .

We now prove the claim: The bound (3.78) follows for every Euclidean frame \mathcal{O} immediately from the definition of a Euclidean profile and the properties of T_N (Corollary 3.17). It remains to select a frame as well as an associated profile, and to show (3.79). Since $\Lambda((f_k)_k) \geq \delta$, there exists a subsequence, which we still denote by $(f_k)_k$, such that there exists a sequence $(N_k, t_k, x_k)_k$ with $(N_k, t_k, x_k) \in [1, \infty) \times I_k \times \mathbb{T}^3$ for all k and such that for all k ,

$$\frac{2}{3}\delta \leq N_k^{-\frac{1}{2}} |(e^{it_k \Delta_\theta} P_{N_k} f_k)(x_k)|. \quad (3.80)$$

From the definition of Λ , we have, after passing to a subsequence, $t_k \rightarrow 0$, $x_k \rightarrow x_\infty$, and either $N_k \rightarrow N_\infty \in [1, \infty)$ or $N_k \rightarrow +\infty$.

We claim that the first case, namely $N_k \rightarrow N_\infty \in [1, \infty)$, does not occur. Indeed, it holds for $g_{x,t,N} \in C^\infty(\mathbb{T}^3)$,

$$g_{x,t,N}(y) := \sum_{\xi \in \mathbb{Z}^3} e^{i((x-y)\cdot\xi - tQ(\xi))} \left[\eta^3 \left(\frac{\xi}{N} \right) - \eta^3 \left(\frac{2\xi}{N} \right) \right]$$

that

$$|(e^{it_k \Delta_\theta} P_{N_k} f_k)(x_k)| = (2\pi)^{-3} \left| \int_{\mathbb{T}^3} f_k(y) g_{x_k, t_k, N_k}(y) dy \right| \lesssim |\langle f_k, \overline{g_{x_k, t_k, N_k}} \rangle_{H^1 \times H^{-1}(\mathbb{T}^3)}|.$$

We also observe that g_{x_k, t_k, N_k} converges point-wise to

$$g(y) := g_{x_\infty, 0, N_\infty}(y) = \sum_{\xi \in \mathbb{Z}^3} e^{i(x_\infty - y)\cdot\xi} \left[\eta^3 \left(\frac{\xi}{N_\infty} \right) - \eta^3 \left(\frac{2\xi}{N_\infty} \right) \right] \in C^\infty(\mathbb{T}^3)$$

as $k \rightarrow \infty$, and thus, strongly in $H^{-1}(\mathbb{T}^3)$. Finally, we see that

$$|\langle f_k, \overline{g_{x_k, t_k, N_k}} \rangle_{H^1 \times H^{-1}(\mathbb{T}^3)}| \leq |\langle f_k, \overline{g_{x_k, t_k, N_k} - \bar{g}} \rangle_{H^1 \times H^{-1}(\mathbb{T}^3)}| + |\langle f_k, \bar{g} \rangle_{H^1 \times H^{-1}(\mathbb{T}^3)}| \rightarrow 0$$

as $k \rightarrow \infty$, which contradicts (3.80).

In the remaining case, $N_k \rightarrow +\infty$, we define the Euclidean frame $\mathcal{O} = (N_k, t_k, x_k)_k$ and the function

$$\psi := \mathcal{F}_{\mathbb{R}^3}^{-1}(|\cdot|^{-2}[\eta^3 - \eta^3(2\cdot)]) \in H^1(\mathbb{R}^3).$$

We prove (3.79) now. By definition,

$$|\langle f_k, \tilde{\psi}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)}| = N_k^{\frac{1}{2}} |\langle f_k, \Pi_{t_k, x_k}(\eta(N_k^{\frac{1}{2}} \Psi^{-1})\psi(N_k \Psi^{-1})) \rangle_{H^1(\mathbb{T}^3)}|,$$

and it is easy to verify that this is equal to

$$N_k^{\frac{1}{2}} \left| \sum_{\xi \in \mathbb{Z}^3} |\xi|^2 e^{i(x_k \cdot \xi - t_k Q(\xi))} \mathcal{F}(f_k)(\xi) \overline{\mathcal{F}_{\mathbb{R}^3}(\eta(N_k^{\frac{1}{2}} \cdot)\psi(N_k \cdot))(\xi)} \right| + o_k(1).$$

Here, it is important to notice that from the compact support of η , we have

$$\mathcal{F}(\eta(N_k^{\frac{1}{2}} \Psi^{-1})\psi(N_k \Psi^{-1}))(\xi) = \mathcal{F}_{\mathbb{R}^3}(\eta(N_k^{\frac{1}{2}} \cdot)\psi(N_k \cdot))(\xi)$$

for all $\xi \in \mathbb{Z}^3$ and for sufficiently large k . Using the scaling properties of the Fourier transform, we deduce that

$$N_k^{\frac{1}{2}} \left| \sum_{\xi \in \mathbb{Z}^3} |\xi|^2 e^{i(x_k \cdot \xi - t_k Q(\xi))} \mathcal{F}(f_k)(\xi) \overline{\mathcal{F}_{\mathbb{R}^3}(\psi(N_k \cdot))(\xi)} \right| = N_k^{-\frac{1}{2}} |(e^{it_k \Delta_\theta} P_{N_k} f_k)(x_k)| \geq \frac{2}{3} \delta.$$

Hence, (3.79) follows if we show that

$$S_k := N_k^{\frac{1}{2}} \left| \sum_{\xi \in \mathbb{Z}^3} |\xi|^2 e^{i(x_k \cdot \xi - t_k Q(\xi))} \mathcal{F}(f_k)(\xi) \overline{\mathcal{F}_{\mathbb{R}^3}((\eta(N_k^{\frac{1}{2}} \cdot) - 1)\psi(N_k \cdot))(\xi)} \right| \rightarrow 0$$

as $k \rightarrow \infty$. From the Cauchy–Schwarz inequality and the scaling properties of the Fourier transform, we get that

$$S_k \lesssim N_k^{-1} \|f_k\|_{L^2(\mathbb{T}^3)} \left(\sum_{\xi \in \mathbb{Z}^3} |\xi|^4 [\mathcal{F}_{\mathbb{R}^3}((\eta - 1)\psi(N_k^{\frac{1}{2}} \cdot))(N_k^{-\frac{1}{2}} \xi)]^2 \right)^{\frac{1}{2}}.$$

Observing that $\psi \in \mathcal{S}(\mathbb{R}^3)$, an oscillatory phase type argument yields for any $N \geq 1$ and any $\mu \geq 1$,

$$|\mathcal{F}_{\mathbb{R}^3}((\eta - 1)\psi(N_k^{\frac{1}{2}} \cdot))(\xi)| \lesssim_{N, \mu} \frac{N_k^{\frac{N}{2} - \mu}}{(1 + |\xi|)^N}, \quad \xi \in \mathbb{R}^3.$$

Choosing, for instance, $N = \mu = 4$, we obtain $S_k \rightarrow 0$ as $k \rightarrow \infty$. This finally proves claim (3.79).

To prove the last part of the claim, assume $(f_k)_k$ is absent from a family of frames $(\mathcal{O}^\alpha)_\alpha$, i.e. for all α and every profile $(\tilde{\psi}_{\mathcal{O}_k^\alpha})_k$ associated to \mathcal{O}^α ,

$$\langle f_k, \tilde{\psi}_{\mathcal{O}_k^\alpha} \rangle_{H^1(\mathbb{T}^3)} \rightarrow 0$$

as $k \rightarrow +\infty$. We argue by contradiction: Suppose there is $\mathcal{O}^\beta \in (\mathcal{O}^\alpha)_\alpha$ such that \mathcal{O} and \mathcal{O}^β are equivalent. From Lemma 3.25 (i), we see that

$$\limsup_{k \rightarrow +\infty} \|\tilde{\psi}_{\mathcal{O}_k} - \widetilde{S}\psi_{\mathcal{O}_k^\beta}\|_{H^1(\mathbb{T}^3)} = 0,$$

where S is the isometry given in Lemma 3.25 (i). In view of (3.79), we obtain

$$\begin{aligned} \frac{\delta}{2} &\leq \limsup_{k \rightarrow +\infty} |\langle f_k, \tilde{\psi}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)}| \\ &\leq \limsup_{k \rightarrow +\infty} (\|f_k\|_{H^1(\mathbb{T}^3)} \|\tilde{\psi}_{\mathcal{O}_k} - \widetilde{S}\psi_{\mathcal{O}_k^\beta}\|_{H^1(\mathbb{T}^3)} + |\langle f_k, \widetilde{S}\psi_{\mathcal{O}_k^\beta} \rangle_{H^1(\mathbb{T}^3)}|). \end{aligned}$$

Since $(f_k)_k$ is absent from all the frames in $(\mathcal{O}^\alpha)_\alpha$, there exists a subsequence such that the right-hand side tends to zero as $k \rightarrow \infty$, which in turn leads to a contradiction. Hence, \mathcal{O} and $(\mathcal{O}^\alpha)_\alpha$ are pairwise orthogonal.

Step 2. Let $(f_k)_k$ be as in the first step. Now that step one provides us with a Euclidean frame \mathcal{O} , we may select the localization of $(f_k)_k$ in \mathcal{O} as a linear profile. For $R \geq 1$ there exists k_0 such that for any $k \geq k_0$ we may define

$$\psi_k^R: \mathbb{R}^3 \rightarrow \mathbb{C}, \quad \psi_k^R(y) := N_k^{-\frac{1}{2}} \eta^3 \left(\frac{y}{R} \right) (\Pi_{-t_k, -x_k} f_k) \left(\Psi \left(\frac{y}{N_k} \right) \right).$$

One easily checks that

$$\|\psi_k^R\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|f_k\|_{H^1(\mathbb{T}^3)}$$

uniformly in R . Assumption (3.77) allows us to extract a subsequence that converges weakly to a function $\psi^R \in \dot{H}^1(\mathbb{R}^3)$ with the property

$$\|\psi^R\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim 1.$$

Because of this, we may assume that, after taking a subsequence, $\psi^R \rightharpoonup \psi \in \dot{H}^1(\mathbb{R}^3)$, and by the uniqueness of the weak limit, we see that for every $R \geq 1$,

$$\psi^R(x) = \eta^3 \left(\frac{x}{R} \right) \psi(x).$$

For some $\gamma \in C_0^\infty(\mathbb{R}^3)$ we choose $R \geq 1$ to be large enough such that $\text{supp } \gamma \subset B_{R/2}(0)$. Then, we calculate for k sufficiently large,

$$\begin{aligned} \langle f_k, \tilde{\gamma}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)} &= \langle \Pi_{-t_k, -x_k} f_k, T_{N_k} \gamma \rangle_{H^1(\mathbb{T}^3)} = \langle \psi^R, \gamma \rangle_{\dot{H}^1(\mathbb{R}^3)} + o_k(1) \\ &= \langle \psi, \gamma \rangle_{\dot{H}^1(\mathbb{R}^3)} + o_k(1). \end{aligned} \tag{3.81}$$

This in combination with (3.78) and (3.79) implies for k sufficiently large,

$$\langle \psi, \gamma \rangle_{\dot{H}^1(\mathbb{R}^3)} = \langle f_k, \tilde{\gamma}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)} + o_k(1) \geq \frac{\delta}{4},$$

and therefore, using a density argument,

$$\|\psi\|_{\dot{H}^1(\mathbb{R}^3)} \gtrsim \delta. \tag{3.82}$$

Moreover, $f_k - \tilde{\psi}_{\mathcal{O}_k}$ is absent from the Euclidean frame \mathcal{O} : For every $\varphi \in C_0^\infty(\mathbb{R}^3)$ there exists $R \geq 1$ such that for any k sufficiently large we get

$$\langle f_k - \tilde{\psi}_{\mathcal{O}_k}, \tilde{\varphi}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)} = \langle f_k, \tilde{\varphi}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)} - \langle \tilde{\psi}_{\mathcal{O}_k}, \tilde{\varphi}_{\mathcal{O}_k} \rangle_{H^1(\mathbb{T}^3)} = o_k(1), \tag{3.83}$$

where we used Lemma 3.25 (iii) and (3.81). By density, the statement holds true also for $\varphi \in \dot{H}^1(\mathbb{R}^3)$. This implies on the one hand,

$$\begin{aligned} \|f_k - \tilde{\psi}_{\mathcal{O}_k}\|_{L^2(\mathbb{T}^3)}^2 &= \|f_k\|_{L^2(\mathbb{T}^3)}^2 - \langle f_k, \tilde{\psi}_{\mathcal{O}_k} \rangle_{L^2(\mathbb{T}^3)} - \langle \tilde{\psi}_{\mathcal{O}_k}, f_k - \tilde{\psi}_{\mathcal{O}_k} \rangle_{L^2(\mathbb{T}^3)} \\ &= \|f_k\|_{L^2(\mathbb{T}^3)}^2 + o_k(1), \end{aligned} \quad (3.84)$$

which we deduce from Lemma 3.25 (iii) and (3.83). On the other hand,

$$\begin{aligned} \|\nabla(f_k - \tilde{\psi}_{\mathcal{O}_k})\|_{L^2(\mathbb{T}^3)}^2 &= \|\nabla f_k\|_{L^2(\mathbb{T}^3)}^2 - 2\langle \nabla f_k, \nabla \tilde{\psi}_{\mathcal{O}_k} \rangle_{L^2(\mathbb{T}^3)} + \|\nabla \tilde{\psi}_{\mathcal{O}_k}\|_{L^2(\mathbb{T}^3)}^2 \\ &= \|\nabla f_k\|_{L^2(\mathbb{T}^3)}^2 - \|\nabla_{\mathbb{R}^3} \psi\|_{L^2(\mathbb{R}^3)}^2 + o_k(1). \end{aligned} \quad (3.85)$$

Step 3. Now, we can conclude the statement of the lemma. Let $(f_k)_k$ be as stated in the lemma. We pass to a subsequence such that $f_k \rightharpoonup g$ in $H^1(\mathbb{T}^3)$ and define $f_k^1 := f_k - g$. For $\alpha \geq 1$ and as long as $\Lambda((f_k^\alpha)_k) > \delta$, we do the following: We apply the first two steps to get a Euclidean frame \mathcal{O}^α and an associated profile $(\tilde{\psi}_{\mathcal{O}^\alpha}^\alpha)_k$. Then, we define

$$f_k^{\alpha+1} := f_k^\alpha - \tilde{\psi}_{\mathcal{O}^\alpha}^\alpha, \quad k \geq 1.$$

Note that in Step 1 we proved that \mathcal{O}^α is orthogonal to all previous Euclidean frames \mathcal{O}^β , $\beta < \alpha$, and by induction, all frames \mathcal{O}^β , $\beta \leq \alpha$, are pairwise orthogonal. Furthermore, Step 2 implies that $f_k^{\alpha+1}$ is absent from \mathcal{O}^α . It is an easy task to show that $f_k^{\alpha+1}$ is absent from \mathcal{O}^β for every $\beta \leq \alpha$: Let $\varphi \in \dot{H}^1(\mathbb{R}^3)$ be arbitrary and $\beta < \alpha$, then

$$\langle f_k^{\alpha+1}, \tilde{\varphi}_{\mathcal{O}_k^\beta}^\beta \rangle_{H^1(\mathbb{T}^3)} = \langle f_k^{\beta+1}, \tilde{\varphi}_{\mathcal{O}_k^\beta}^\beta \rangle_{H^1(\mathbb{T}^3)} - \sum_{\nu=\beta+1}^{\alpha} \langle \tilde{\psi}_{\mathcal{O}_k^\nu}^\nu, \tilde{\varphi}_{\mathcal{O}_k^\beta}^\beta \rangle_{H^1(\mathbb{T}^3)}.$$

This expression tends to zero by the induction hypothesis and Lemma 3.25 (ii). Note also that, since $f_k = f_k^1 + g$ and $f_k^1 \rightharpoonup 0$ in $H^1(\mathbb{T}^3)$, we have

$$\|f_k\|_{L^2(\mathbb{T}^3)}^2 = \|f_k^1\|_{L^2(\mathbb{T}^3)}^2 + 2\langle f_k^1, g \rangle_{L^2(\mathbb{T}^3)} + \|g\|_{L^2(\mathbb{T}^3)}^2 = \|f_k^1\|_{L^2(\mathbb{T}^3)}^2 + \|g\|_{L^2(\mathbb{T}^3)}^2 + o_k(1).$$

By the same argument, we also obtain

$$\|\nabla f_k\|_{L^2(\mathbb{T}^3)}^2 = \|\nabla f_k^1\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla g\|_{L^2(\mathbb{T}^3)}^2 + o_k(1).$$

Hence, applying (3.84) and (3.85) inductively, we conclude

$$\|f_k\|_{L^2(\mathbb{T}^3)}^2 = \|g\|_{L^2(\mathbb{T}^3)}^2 + \|f_k^{\alpha+1}\|_{L^2(\mathbb{T}^3)}^2 + o_k(1)$$

and

$$\|\nabla f_k\|_{L^2(\mathbb{T}^3)}^2 = \|\nabla g\|_{L^2(\mathbb{T}^3)}^2 + \sum_{\beta=1}^{\alpha} \|\nabla_{\mathbb{R}^3} \psi^\beta\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla f_k^{\alpha+1}\|_{L^2(\mathbb{T}^3)}^2 + o_k(1).$$

We still have to prove that this method stops after $O(\delta^{-2})$ applications. From Strichartz inequalities, we obtain

$$\sup_{N \geq 1, t \in I_k, x \in \mathbb{T}^3} N^{-\frac{1}{2}} |(e^{it\Delta_\theta} P_N f_k^{\alpha+1})(x)| \lesssim \sup_{N \geq 1} N \|P_N f_k^{\alpha+1}\|_{L^2(\mathbb{T}^3)} \lesssim \|\nabla f_k^{\alpha+1}\|_{L^2(\mathbb{T}^3)}.$$

The orthogonality relations, (3.77), and (3.82) imply that there exists some large $M > 0$ such that for k large enough,

$$\|\nabla f_k^{\alpha+1}\|_{L^2(\mathbb{T}^3)}^2 = \left| \|\nabla(f_k - g)\|_{L^2(\mathbb{T}^3)}^2 - \sum_{\beta=1}^{\alpha} \|\nabla_{\mathbb{R}^3} \psi^\beta\|_{L^2(\mathbb{R}^3)}^2 \right| + o_k(1) \lesssim |M - \alpha \delta^2|$$

We deduce that it takes $\mathcal{O}(\delta^{-2})$ steps until we have $\Lambda((f_k^{\alpha+1})) \leq \delta$. In this case, we set $\alpha_{\text{end}} := \alpha$ and $R_k := f_k^{\alpha_{\text{end}}+1}$, what finishes the proof. \square

Using the two foregoing lemmas, we are finally able to conclude the main statement of this section.

Proof of Proposition 3.27. We apply Lemma 3.29 iteratively with $\delta_\ell = 2^{-\ell}$, which provides us with a sequence of Euclidean frames $(\mathcal{O}^\alpha)_\alpha$ and profiles $(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha)_k$. The first two orthogonality relations in (3.76) are given by Lemma 3.29, too. It only remains to prove the last equality of (3.76).

By Lemma 3.25 (ii), we have that for $\alpha_j \geq 1$, $j = 1, \dots, 6$, such that at least two of them are different, say $\alpha_1 \neq \alpha_2$,

$$\int_{\mathbb{T}^3} \prod_{j=1}^6 |\tilde{\psi}_{\mathcal{O}_k^{\alpha_j}}^{\alpha_j}(x)| dx \leq \|\tilde{\psi}_{\mathcal{O}_k^{\alpha_1}}^{\alpha_1} \tilde{\psi}_{\mathcal{O}_k^{\alpha_2}}^{\alpha_2}\|_{L^3(\mathbb{T}^3)} \prod_{j=3}^6 \|\tilde{\psi}_{\mathcal{O}_k^{\alpha_j}}^{\alpha_j}\|_{L^6(\mathbb{T}^3)}^6 \leq o_k(1).$$

Similarly, for $\alpha \geq 1$ we deduce

$$\int_{\mathbb{T}^3} |g(x)| |\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha(x)|^5 dx \leq \|\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha g\|_{L^3(\mathbb{T}^3)} \|\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha\|_{L^6(\mathbb{T}^3)}^4 \leq o_k(1)$$

and

$$\int_{\mathbb{T}^3} |g(x)|^5 |\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha(x)| dx \leq \|\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha g\|_{L^3(\mathbb{T}^3)} \|g\|_{L^6(\mathbb{T}^3)}^4 \leq o_k(1)$$

from Lemma 3.25 (iv). Moreover, we use Lemma 3.28 to see that

$$\|R_k^J\|_{L^6(\mathbb{T}^3)}^6 \lesssim \|R_k^J\|_{H^1(\mathbb{T}^3)}^2 \left(\sup_{N \geq 1} N^{-\frac{1}{2}} \|P_N R_k^J\|_{L^\infty(\mathbb{T}^3)} \right)^4,$$

and we conclude from (3.75) that

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \left(\|f_k\|_{L^6(\mathbb{T}^3)}^6 - \|f_k - R_k^J\|_{L^6(\mathbb{T}^3)}^6 + \|R_k^J\|_{L^6(\mathbb{T}^3)}^6 \right) = 0. \quad (3.86)$$

To see this, note that

$$|f_k|^6 - |f_k - R_k^J|^6 = |f_k|^6 - (|f_k|^2 - f_k \overline{R_k^J} - \overline{f_k} R_k^J + |R_k^J|^2)^3$$

point-wise and thus, each term contains a factor of R_k^J that can be put in the $L^6(\mathbb{T}^3)$ -norm.

From the point-wise estimate,

$$\begin{aligned} \left| |f_k - R_k^J|^6 - |g|^6 - \sum_{\alpha=1}^J |\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha|^6 \right| &\lesssim_J \sum_{\alpha=1}^J \left(|g| |\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha|^5 + |g|^5 |\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha| \right) \\ &\quad + \sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^J \left(|\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha| |\tilde{\psi}_{\mathcal{O}_k^\beta}^\beta|^5 + |\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha|^5 |\tilde{\psi}_{\mathcal{O}_k^\beta}^\beta| \right), \end{aligned}$$

and the estimates above, we get by integration and (3.86)

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \left| \|f_k\|_{L^6(\mathbb{T}^3)}^6 - \|g\|_{L^6(\mathbb{T}^3)}^6 - \sum_{\alpha=1}^J \|\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha\|_{L^6(\mathbb{T}^3)}^6 \right| = 0. \quad \square$$

3.6 Proof of the main theorem

In order to prove [Theorem 3.1](#), we proceed quite similar as in [[IP12b](#), Section 6]: We introduce a functional Λ_* , which controls the global existence of solutions and is suitable for the local and small data global theory. This functional decomposes the set of initial data into sub-level sets of the energy and we are looking at the supremum of the functional on these sub-level sets. If the functional increases too quickly, then the maximizers form a sequence that is bounded in Z , which leads to a contradiction. The main obstruction to the boundedness of the sequence comes from solutions that concentrate in a point in space-time. These solutions have been studied in [Section 3.4](#). The principal idea is induction on energy. Assume that nonlinear solutions with energy less than E_{\max} are global. That $E_{\max} > 0$ follows from the small data global theory. We decompose the initial data of the maximizers according to the profile decomposition in [Section 3.5](#). If one of the terms has energy E_{\max} , then it is easy to show that the sequence of maximizers stays bounded. Otherwise, nonlinear solutions to the weak limit g and for every profile exist globally in time. It is then shown that the sum of these nonlinear global solutions plus the linear evolution of the remainder R_k^J is an approximate solution. We conclude from stability that the sequence of maximizers is bounded in Z .

3.6.1 The main argument

We see from [Proposition 3.12 \(iii\)](#) that it suffices to show that solutions remain bounded in Z on intervals of length at most one. To prove this, we induct on the energy $E(u)$.

We define the quantity

$$\Lambda(L, \tau) := \sup\{\|u\|_{Z(I)}^2 : E(u) \leq L, |I| \leq \tau\}, \quad L, \tau > 0,$$

where the supremum is taken over all strong solutions u of [\(3.1\)](#) with $E(u) \leq L$ and all intervals I of length at most τ . If L or τ increases, the supremum is taken over a larger set, and hence, the function Λ is increasing in both its arguments. Obviously,

$$\begin{aligned} \Lambda(L, \tau_1 + \tau_2) &\lesssim \sup\{\|u\|_{Z(I_1)}^2 + \|u\|_{Z(I_2)}^2 : E(u) \leq L, |I_j| \leq \tau_j, j = 1, 2\} \\ &\lesssim \Lambda(L, \tau_1) + \Lambda(L, \tau_2). \end{aligned}$$

The last two properties imply that if we define

$$\Lambda_*(L) := \lim_{\tau \rightarrow 0} \Lambda(L, \tau),$$

then we have for all $\tau > 0$,

$$\Lambda(L, \tau) < +\infty \quad \Leftrightarrow \quad \Lambda_*(L) < +\infty. \tag{3.87}$$

Finally, we define the maximal energy such that $\Lambda_*(L)$ is finite:

$$E_{\max} := \sup\{L \in \mathbb{R}_+ : \Lambda_*(L) < +\infty\}.$$

Note that our small data global well-posedness result ([Lemma 3.14](#)) ensures that $E_{\max} > 0$. All in all, we have that [Theorem 3.1](#) is equivalent to the following statement.

Proposition 3.30. *We have that $E_{\max} = +\infty$. In particular, every solution of [\(3.1\)](#) is global in the sense given in [Theorem 3.1](#).*

Proof. We argue by contradiction and assume $E_{\max} < +\infty$. By the definition of E_{\max} , there exists a sequence $(u_k)_k$ of strong solutions to (3.1) such that

$$E(u_k) \rightarrow E_{\max} \quad \text{and} \quad \|u_k\|_{Z(I_k)} \rightarrow +\infty \quad (3.88)$$

for intervals $I_k \ni 0$ with $|I_k| \rightarrow 0$ as $k \rightarrow +\infty$. Since $(u_k(0))_k$ is bounded in $H^1(\mathbb{T}^3)$, there is a subsequence that converges weakly to, say, $g \in H^1(\mathbb{T}^3)$. We decompose the sequence of initial data $(u_k(0))_k$ in profiles using Proposition 3.27. This provides us with a sequence of pairwise orthogonal frames $(\mathcal{O}^\alpha)_{\alpha \in \mathbb{N}}$ and with a sequence of corresponding profiles $(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha)_k$ such that, after extracting a subsequence, for any $J \geq 1$,

$$u_k(0) = g + \sum_{\alpha=1}^J \tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha + R_k^J.$$

To be able to apply Proposition 3.23 later, we have to switch to *renormalized* Euclidean profiles. We show that every $\mathcal{O}^\alpha \in \mathcal{F}_E \setminus \tilde{\mathcal{F}}_E$ may be replaced by some $\tilde{\mathcal{O}}^\alpha \in \tilde{\mathcal{F}}_E$. To accomplish this, consider $\mathcal{O}^\alpha = (N_k, t_k, x_k)_k \in \mathcal{F}_E \setminus \tilde{\mathcal{F}}_E$. Then, after passing to a subsequence, $N_k^2 |t_k| \rightarrow C$ for some $0 \leq C < \infty$. We define $\tilde{\mathcal{O}}^\alpha := (N_k, 0, x_k)_k \in \tilde{\mathcal{F}}_E$ and observe that this frame is equivalent to \mathcal{O}^α . Furthermore, Lemma 3.25 (i) yields the existence of a profile $(\widetilde{S}_\alpha \tilde{\psi}_{\tilde{\mathcal{O}}_k^\alpha}^\alpha)_k$, $k \in \mathbb{N}$, such that, up to a subsequence,

$$\lim_{k \rightarrow +\infty} \|\widetilde{S}_\alpha \tilde{\psi}_{\tilde{\mathcal{O}}_k^\alpha}^\alpha - \tilde{\psi}_{\tilde{\mathcal{O}}_k^\alpha}^\alpha\|_{H^1(\mathbb{T}^3)} = 0,$$

and hence,

$$\lim_{k \rightarrow +\infty} \|u_k(0) - \tilde{u}_k(0)\|_{H^1(\mathbb{T}^3)} = 0, \quad \text{where } \tilde{u}_k(0) := g + \sum_{\alpha=1}^J \widetilde{S}_\alpha \tilde{\psi}_{\tilde{\mathcal{O}}_k^\alpha}^\alpha + R_k^J.$$

Let \tilde{u}_k be the solution to (3.1) on I_k with initial data $\tilde{u}_k(0)$. The existence follows from our stability result in Proposition 3.15 provided k is sufficiently large. Suppose now that $\|\tilde{u}_k\|_{Z(I_k)}$ is uniformly bounded, then $\|\tilde{u}_k\|_{X_r^1(I_k)}$ is uniformly bounded (see Proposition 3.12 (iii)). As a consequence, there exists $M > 0$ such that for all k large enough,

$$\|\tilde{u}_k\|_{Z(I_k)} + \|\tilde{u}_k\|_{L^\infty(I_k, H^1(\mathbb{T}^3))} \leq M.$$

We now conclude from stability (see (3.23)) and

$$\|u_k\|_{Z(I_k)} \lesssim \|u_k - \tilde{u}_k\|_{X_r^1(I_k)} + \|\tilde{u}_k\|_{Z(I_k)}$$

that $\|u_k\|_{Z(I_k)}$ is uniformly bounded. Hence, from now on, we may assume each frame \mathcal{O}^α to be renormalized.

By the same argument, we may also assume that for every $\alpha \neq \beta$ either $|\ln(N_k^\alpha/N_k^\beta)| \rightarrow +\infty$ as $k \rightarrow \infty$ or $N_k^\alpha = N_k^\beta$ for all k . In the latter case, we may further assume that either $t_k^\alpha = t_k^\beta$ for all k or $(N_k^\alpha)^2 |t_k^\alpha - t_k^\beta| \rightarrow +\infty$ as $k \rightarrow \infty$.

The conservation of energy implies $E(u_k) = E(u_k(0))$ in I_k , and the orthogonality relations (3.76), (3.86) and Lemma 3.25 (iii) yield that, after passing to a subsequence,

$$\lim_{J \rightarrow +\infty} \left(\sum_{\alpha=1}^J E(\alpha) + \lim_{k \rightarrow +\infty} E(R_k^J) \right) \leq E_{\max} - E(g), \quad (3.89)$$

where

$$E(\alpha) := \lim_{k \rightarrow +\infty} E(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha) \in (0, E_{\max}].$$

We use the Bernstein's inequality and the Strichartz estimates given in Lemma 2.10 to compute for the remainder R_k^J , $p \in \{p_0, p_1\}$ and $q = \frac{p_0+16/3}{2}$,

$$\begin{aligned} & \sum_{N \geq 1} N^{5-\frac{p}{2}} \|P_N e^{it\Delta_\theta} R_k^J\|_{L_{t,x}^p}^p \\ & \leq \left(\sup_{N \geq 1} N^{-\frac{1}{2}} \|P_N e^{it\Delta_\theta} R_k^J\|_{L_{t,x}^\infty} \right)^{p-q} \sum_{N \geq 1} \left(N^{\frac{5}{q}-\frac{1}{2}} \|P_N e^{it\Delta_\theta} R_k^J\|_{L_{t,x}^q} \right)^q \\ & \lesssim \left(\sup_{N \geq 1} N^{-\frac{1}{2}} \|P_N e^{it\Delta_\theta} R_k^J\|_{L_{t,x}^\infty} \right)^{p-q} \sum_{N \geq 1} N^q \|P_N R_k^J\|_{L^2(\mathbb{T}^3)}^q \\ & \lesssim \left(\sup_{N \geq 1} N^{-\frac{1}{2}} \|P_N e^{it\Delta_\theta} R_k^J\|_{L_{t,x}^\infty} \right)^{p-q} \|R_k^J\|_{H^1(\mathbb{T}^3)}^q, \end{aligned}$$

where $L_{t,x}^r := L_{t,x}^r(I_k \times \mathbb{T}^3)$. In view of (3.75) and (3.76), it follows

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \|e^{it\Delta_\theta} R_k^J\|_{Z(I_k)} = 0. \quad (3.90)$$

We consider three cases. The first two cases deal with the situations, where there is a term in the profile decomposition with energy E_{\max} . Once we dealt with them, we may apply the induction hypothesis in the remaining third case.

Case 1. Assume $(u_k(0))_k$ converges strongly in $H^1(\mathbb{T}^3)$ to its limit $g \in H^1(\mathbb{T}^3)$, which satisfies $E(g) = E_{\max}$. We have that

$$\|e^{it\Delta_\theta} u_k(0)\|_{Z(I_k)} \leq \|e^{it\Delta_\theta} (u_k(0) - g)\|_{Z(I_k)} + \|e^{it\Delta_\theta} g\|_{Z(I_k)},$$

and we deduce from Corollary 3.5 (i) that

$$\|e^{it\Delta_\theta} (u_k(0) - g)\|_{Z(I_k)} \lesssim \|u_k(0) - g\|_{H^1(\mathbb{T}^3)}.$$

Therefore there exists some small $\eta > 0$ such that for k large enough,

$$\|e^{it\Delta_\theta} u_k(0)\|_{Z(I_k)} \leq \|e^{it\Delta_\theta} g\|_{Z(-\eta, \eta)} + o_k(1) \leq \delta_0,$$

where δ_0 is the δ_0 given by the local well-posedness result in Proposition 3.12 (i). This proposition yields for k sufficiently large,

$$\|u_k\|_{Z(I_k)} \lesssim \|u_k(t) - e^{it\Delta_\theta} u_k(0)\|_{X_T^1(I_k)} + \|e^{it\Delta_\theta} u_k(0)\|_{Z(I_k)} \lesssim_{E_{\max}} \delta_0.$$

Consequently, $\|u_k\|_{Z(I_k)}$ is bounded, which contradicts (3.88).

Case 2a. Assume $g = 0$ and there are no profiles. Then, by (3.90), we may choose J sufficiently large such that we get for k large enough,

$$\|e^{it\Delta_\theta} u_k(0)\|_{Z(I_k)} = \|e^{it\Delta_\theta} R_k^J\|_{Z(I_k)} \leq \delta_0,$$

where δ_0 is as in the first case. Applying Proposition 3.12 (i), this contradicts (3.88) as discussed in Case 1.

Case 2b. Assume $g = 0$ and there is only one Euclidean profile $(\tilde{\psi}_{\mathcal{O}_k})_k$ such that

$$\limsup_{k \rightarrow +\infty} \|u_k(0) - \tilde{\psi}_{\mathcal{O}_k}\|_{H^1(\mathbb{T}^3)} = 0,$$

where \mathcal{O} is a renormalized Euclidean frame. Let U_k be the solution of (3.1) with initial data $U_k(0) = \tilde{\psi}_{\mathcal{O}_k}$. By Proposition 3.23 (i), we see that there is $\tau > 0$ such that for k large enough,

$$\|U_k\|_{X^1(I_k)} \leq \|U_k\|_{X^1(-\tau, \tau)} \lesssim_{E_{\mathbb{R}^3}(\psi)} 1.$$

Hence, by the embeddings Proposition 3.3 (i) and Corollary 3.5 (iii),

$$\|U_k\|_{Z(-\tau, \tau)} + \|U_k\|_{L^\infty((-\tau, \tau), H^1(\mathbb{T}^3))} \lesssim_{E_{\mathbb{R}^3}(\psi)} 1,$$

and the assumption implies for a subsequence,

$$\lim_{k \rightarrow +\infty} \|u_k(0) - U_k(0)\|_{H^1(\mathbb{T}^3)} \rightarrow 0.$$

From stability, see Proposition 3.15, we get for large k that

$$\|u_k\|_{Z(I_k)} \lesssim \|u_k\|_{X^1(I_k)} \lesssim_{E_{\mathbb{R}^3}(\psi)} 1.$$

This is a contradiction to (3.88).

Case 3. In the remaining case, we assume, up to passing to subsequences,

$$\lim_{k \rightarrow \infty} \|u_k(0) - g\|_{H^1(\mathbb{T}^3)} > 0,$$

and furthermore, if $g = 0$, then we assume that there exists a profile $(\tilde{\psi}_{\mathcal{O}_k^\beta}^\beta)_k$ with the property that $\lim_{k \rightarrow \infty} \|u_k(0) - \tilde{\psi}_{\mathcal{O}_k^\beta}^\beta\|_{H^1(\mathbb{T}^3)} > 0$. We claim that in each case $E(g) < E_{\max}$ and for any $\alpha \in \mathbb{N}$, $E(\alpha) < E_{\max}$.

Indeed, if $g \neq 0$, then $E(g) > 0$, which already implies $E(\alpha) < E_{\max}$ by (3.89). It remains to show that $E(g) < E_{\max}$, which, in view of (3.89), follows from

$$\lim_{J \rightarrow +\infty} \left(\sum_{\alpha=1}^J E(\alpha) + \lim_{k \rightarrow +\infty} E(R_k^J) \right) > 0. \quad (3.91)$$

This in turn is a consequence of the fact that $(u_k(0))_k$ does not converge strongly in $H^1(\mathbb{T}^3)$ to g : There is $\delta > 0$ such that we have

$$\delta < \lim_{k \rightarrow \infty} \|u_k(0) - g\|_{H^1(\mathbb{T}^3)} \leq \lim_{k \rightarrow \infty} \left(\sum_{\alpha=1}^J \|\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha\|_{H^1(\mathbb{T}^3)} + \|R_k^J\|_{H^1(\mathbb{T}^3)} \right)$$

uniformly in J , and consequently, there exists either a profile with positive energy or

$$\lim_{J \rightarrow +\infty} \lim_{k \rightarrow +\infty} E(R_k^J) > 0. \quad (3.92)$$

Hence, (3.91) is shown provided $g \neq 0$. If on the contrary $g = 0$, then we see from $\lim_{k \rightarrow \infty} \|u_k(0) - \tilde{\psi}_{\mathcal{O}_k^\beta}^\beta\|_{H^1(\mathbb{T}^3)} > 0$ by the same argument that there is either another non-trivial profile with positive energy or (3.92) holds true. Hence, (3.89) yields $E(\alpha) < E_{\max}$.

By relabeling the profiles, we can assume that for all $\alpha \in \mathbb{N}$, $E(\alpha) \leq E(1) < E_{\max} - \eta$ and $E(g) < E_{\max} - \eta$ for some $\eta > 0$. For any $\alpha \in \mathbb{N}$ let U_k^α be the maximal strong solution of (3.1)

with initial data $U_k^\alpha(0) = \tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha$. U_k^α can be understood as a nonlinear profile corresponding to the linear profile $\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha$. Analogously, let W be the maximal strong solution to (3.1) with initial data g .

We apply the induction hypothesis: From the definition of E_{\max} and (3.87), we see that all nonlinear profiles and W are global and up to a subsequence, satisfy

$$\|W\|_{Z(I)} + \lim_{k \rightarrow +\infty} \|U_k^\alpha\|_{Z(I)} \lesssim \Lambda \left(E_{\max} - \frac{\eta}{2}, 1 \right)^{\frac{1}{2}} \lesssim 1, \quad I := \left(-\frac{1}{2}, \frac{1}{2} \right).$$

From now on, all implicit constants may depend on $\Lambda(E_{\max} - \frac{\eta}{2}, 1)$. Since W is a (global) strong solution in X_r^1 , we know that

$$\|W\|_{L^\infty(I, H^1(\mathbb{T}^3))} \lesssim 1.$$

Using Lemma 3.25 (iii) and $\lim_{k \rightarrow +\infty} E(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha) = E(\alpha) < E_{\max}$, we also have that

$$\|U_k^\alpha\|_{L^\infty(I, H^1(\mathbb{T}^3))} \lesssim E(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha) + E(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha)^{\frac{1}{3}} \lesssim_{E_{\max}} 1$$

for every $\alpha \in \mathbb{N}$ and $k > k_0(\alpha)$ large enough. Hence, stability implies that for every $\alpha \in \mathbb{N}$ and $k > k_0(\alpha)$ large enough,

$$\|W\|_{X_r^1(I)} + \|U_k^\alpha\|_{X_r^1(I)} \lesssim 1. \quad (3.93)$$

For $J, k \in \mathbb{N}$ we define

$$U_{\text{prof}, k}^J := W + \sum_{\alpha=1}^J U_k^\alpha.$$

First, we prove that for all $k \geq k_0(J)$ sufficiently large,

$$\|U_{\text{prof}, k}^J\|_{X_r^1(I)} \lesssim_{E_{\max}} 1 \quad (3.94)$$

uniformly in J . Thanks to (3.89), we know that for every $0 < \delta < 1$ there are finitely many profiles $(\tilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha)_k$ such that $E(\alpha) > \delta$. After relabeling, we may assume that for all $\alpha \geq A$ it holds $E(\alpha) \leq \delta$. We also have $\|U_k^\alpha(0)\|_{H^1(\mathbb{T}^3)} \lesssim E(\alpha)^{\frac{1}{2}} \lesssim \delta^{\frac{1}{2}}$ for any $\alpha \geq A$ and k large enough, as we may observe from

$$\|U_k^\alpha(0)\|_{H^1(\mathbb{T}^3)}^2 \lesssim \|U_k^\alpha(0)\|_{L^6(\mathbb{T}^3)}^2 + \|\nabla U_k^\alpha(0)\|_{L^2(\mathbb{T}^3)}^2 \lesssim \|\nabla U_k^\alpha(0)\|_{L^2(\mathbb{T}^3)}^2 \lesssim E(U_k^\alpha(0)).$$

Now, we choose δ small enough such that the small data global well-posedness result in Lemma 3.14 can be applied. Using (3.93) and Lemma 3.14,

$$\begin{aligned} \|U_{\text{prof}, k}^J\|_{X_r^1(I)} &\leq \|W\|_{X_r^1(I)} + \sum_{\alpha=1}^{A-1} \|U_k^\alpha\|_{X_r^1(I)} + \sum_{\alpha=A}^J \|U_k^\alpha(t) - e^{it\Delta_\theta} U_k^\alpha(0)\|_{X_r^1(I)} \\ &\quad + \left\| e^{it\Delta_\theta} \sum_{\alpha=A}^J U_k^\alpha(0) \right\|_{X_r^1(I)} \\ &\lesssim 1 + A + \sum_{\alpha=A}^J E(\alpha) + \left\| \sum_{\alpha=A}^J U_k^\alpha(0) \right\|_{H^1(\mathbb{T}^3)}. \end{aligned}$$

From (3.89), we know that $\sum_{\alpha=A}^J E(\alpha) \leq E_{\max}$ uniformly in J . The boundedness of the last term is implied by Lemma 3.25 (ii) and (3.89):

$$\left\| \sum_{\alpha=A}^J U_k^\alpha(0) \right\|_{H^1(\mathbb{T}^3)}^2 = \sum_{\alpha=A}^J \|U_k^\alpha(0)\|_{H^1(\mathbb{T}^3)}^2 + o_k(1) \lesssim \sum_{\alpha=A}^J E(\alpha) + o_k(1) \lesssim E_{\max}$$

for k large enough. Hence, we proved $\|U_{\text{prof},k}^J\|_{X_T^1(I)} \lesssim 1$ for large k .

Define for $J, k \in \mathbb{N}$,

$$U_{\text{app},k}^J(t) := U_{\text{prof},k}^J(t) + e^{it\Delta_\theta} R_k^J = W(t) + \sum_{\alpha=1}^J U_k^\alpha(t) + e^{it\Delta_\theta} R_k^J.$$

We claim for any $J \geq J_0$ and any $k \geq k_0(J)$ sufficiently large that $U_{\text{app},k}^J$ is an approximate solution of (3.1) on I_k . We note from (3.89) that for sufficiently large k and any J the $H^1(\mathbb{T}^3)$ -norm of R_k^J is bounded by $C(E_{\max})$ uniformly in k and J . From this and (3.94), it follows that there exists $C_0 > 0$ such that

$$\|U_{\text{app},k}^J\|_{Z(I)} + \|U_{\text{app},k}^J\|_{L^\infty(I, H^1(\mathbb{T}^3))} \leq C \|U_{\text{app},k}^J\|_{X_T^1(I)} \leq C_0.$$

Now, we choose $\varepsilon_1 = \varepsilon_1(C_0) \leq 1$ to be the constant of our stability result in Proposition 3.15. Writing $F(z) := z|z|^4$, we set

$$e_k^J := (i\partial_t + \Delta_\theta)U_{\text{app},k}^J - F(U_{\text{app},k}^J) = F(W) + \sum_{\alpha=1}^J F(U_k^\alpha) - F(U_{\text{app},k}^J)$$

and compute

$$e_k^J(t) = \left(F(U_{\text{prof},k}^J(t)) - F(U_{\text{prof},k}^J(t) + e^{it\Delta_\theta} R_k^J) \right) + \left(F(W(t)) + \sum_{\alpha=1}^J F(U_k^\alpha(t)) - F(U_{\text{prof},k}^J(t)) \right).$$

Applying Lemma 3.31, we get

$$\limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \|\mathcal{I}_{t_0}(e_k^J)\|_{X_T^1(I_k)} \leq \frac{\varepsilon_1}{2}$$

for $J \geq J_0(\varepsilon_1)$. Hence, by stability, we obtain that

$$\|u_k\|_{X_T^1(I_k)} \lesssim 1.$$

Note that this contradicts (3.88), which finishes the proof. \square

Thus, Proposition 3.30 and Theorem 3.1 are proved once we prove the following lemma.

Lemma 3.31. *With the notation in Case 3 of the proof of Proposition 3.30, we have that*

$$\limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} \left(F(U_{\text{prof},k}^J) - F(W) - \sum_{\alpha=1}^J F(U_k^\alpha) \right) \right\|_{X_T^1(I_k)} = 0, \quad (3.95)$$

for fixed $J \in \mathbb{N}$, and

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} \left(F(U_{\text{prof},k}^J(t) + e^{it\Delta_\theta} R_k^J) - F(U_{\text{prof},k}^J(t)) \right) \right\|_{X_T^1(I_k)} = 0. \quad (3.96)$$

3.6.2 Proof of Lemma 3.31

Before we turn to the proof of Lemma 3.31, we provide two more lemmas concerning the interaction of a high-frequency linear solution with a low-frequency profile on the one hand and the interaction of two profiles corresponding to two orthogonal frames on the other hand. The general strategy is the same as for the standard tours in [IP12b, Section 7]. Due to the modified Laplace–Beltrami operator Δ_θ , the arguments in Lemma 3.32 are adapted, though.

We fix the following notation: For a given vector $p \in \mathbb{N}^n$ we denote by $\mathfrak{D}_{p_1, \dots, p_n}(a_1, \dots, a_n)$ a $|p|$ -linear expression which is a product of p_1 terms that are either equal to a_1 or its complex conjugate $\overline{a_1}$ and similarly for $p_j, a_j, 2 \leq j \leq n$.

Interaction of a high-frequency linear solution with a low-frequency profile

The following lemma shows that a high-frequency linear solution does not interact significantly with a low-frequency profile.

Lemma 3.32. *Assume that $B, N \geq 2$ are dyadic numbers and that $\omega: (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{T}^3 \rightarrow \mathbb{C}$ is a function satisfying*

$$|\omega| \leq N^{\frac{1}{2}} \mathbf{1}_{\{|x| \leq N^{-1}, |t| \leq N^{-2}\}} \quad \text{and} \quad |\nabla \omega| \leq N^{\frac{3}{2}} \mathbf{1}_{\{|x| \leq N^{-1}, |t| \leq N^{-2}\}}.$$

Then, for any $f \in H^1(\mathbb{T}^3)$,

$$\left\| \mathfrak{D}_{4,1}(\omega(t), e^{it\Delta_\theta} P_{>BN} f) \right\|_{L^1((-\frac{1}{2}, \frac{1}{2}), H^1(\mathbb{T}^3))} \lesssim (B^{-\frac{1}{200}} + N^{-\frac{1}{200}}) \|f\|_{H^1(\mathbb{T}^3)}.$$

Proof. For brevity, we assume that $f = P_{>BN} f$. By scaling, and we may also assume $\|f\|_{H^1(\mathbb{T}^3)} = 1$. Using the product rule and Hölder's inequality, we see that

$$\begin{aligned} & \left\| \mathfrak{D}_{4,1}(\omega(t), e^{it\Delta_\theta} f) \right\|_{L^1((-\frac{1}{2}, \frac{1}{2}), H^1(\mathbb{T}^3))} \\ & \lesssim \left\| \mathfrak{D}_{4,1}(\omega(t), \nabla e^{it\Delta_\theta} f) \right\|_{L_t^1 L_x^2} + \left\| |\nabla \omega| + |\omega| \right\|_{L_t^4 L_x^\infty} \|\omega\|_{L_t^4 L_x^\infty}^3 \|e^{it\Delta_\theta} f\|_{L_t^\infty L_x^2}. \end{aligned}$$

Obviously, from $f = P_{>BN} f$ and $\|f\|_{H^1(\mathbb{T}^3)} = 1$, we obtain

$$\|e^{it\Delta_\theta} f\|_{L_t^\infty L_x^2} = \|f\|_{L^2(\mathbb{T}^3)} \lesssim (BN)^{-1} \|\nabla f\|_{L^2(\mathbb{T}^3)} \lesssim (BN)^{-1}.$$

Furthermore,

$$\begin{aligned} \|\omega\|_{L_t^4 L_x^\infty} & \leq N^{\frac{1}{2}} \|\mathbf{1}_{\{|x| \leq N^{-1}, |t| \leq N^{-2}\}}\|_{L_t^4 L_x^\infty} \lesssim 1, \\ \|\nabla \omega\|_{L_t^4 L_x^\infty} & \leq N^{\frac{3}{2}} \|\mathbf{1}_{\{|x| \leq N^{-1}, |t| \leq N^{-2}\}}\|_{L_t^4 L_x^\infty} \lesssim N. \end{aligned}$$

All in all, we get

$$\left\| \mathfrak{D}_{4,1}(\omega(t), e^{it\Delta_\theta} f) \right\|_{L^1((-\frac{1}{2}, \frac{1}{2}), H^1(\mathbb{T}^3))} \lesssim \left\| \mathfrak{D}_{4,1}(\omega(t), \nabla e^{it\Delta_\theta} f) \right\|_{L^1((-\frac{1}{2}, \frac{1}{2}), L^2(\mathbb{T}^3))} + B^{-1}.$$

Set

$$W: \mathbb{R} \times \mathbb{T}^3 \rightarrow \mathbb{R}, \quad W(t, x) := N^4 \eta^1(N^2 t) \eta^3(N \Psi^{-1}(x)),$$

and note that $|\omega|^4 \leq W^{\frac{1}{2}}$. Hence, we estimate

$$\begin{aligned} \left\| \mathfrak{D}_{4,1}(\omega(t), \nabla e^{it\Delta_\theta} f) \right\|_{L^1((-\frac{1}{2}, \frac{1}{2}), L^2(\mathbb{T}^3))} & \leq \|W(t)^{\frac{1}{2}} \nabla e^{it\Delta_\theta} f\|_{L^1((-\frac{1}{2}, \frac{1}{2}), L^2(\mathbb{T}^3))} \\ & \lesssim N^{-1} \|W(t)^{\frac{1}{2}} \nabla e^{it\Delta_\theta} f\|_{L^2((-\frac{1}{2}, \frac{1}{2}) \times \mathbb{T}^3)} \end{aligned}$$

using Hölder's inequality. The latter expression can be rewritten as

$$\begin{aligned} \|W(t)^{\frac{1}{2}} \nabla e^{it\Delta_{\theta}} f\|_{L^2((-\frac{1}{2}, \frac{1}{2}) \times \mathbb{T}^3)}^2 &= \sum_{j=1}^3 \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle e^{it\Delta_{\theta}} \partial_j f, W(t) e^{it\Delta_{\theta}} \partial_j f \rangle_{L^2(\mathbb{T}^3)} dt \\ &= \sum_{j=1}^3 \left\langle \partial_j f, \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-it\Delta_{\theta}} W(t) e^{it\Delta_{\theta}} dt \right] \partial_j f \right\rangle_{L^2(\mathbb{T}^3)}. \end{aligned}$$

The theorem is proved if we show

$$\|K\|_{L^2(\mathbb{T}^3) \rightarrow L^2(\mathbb{T}^3)} \lesssim N^2 (B^{-\frac{1}{100}} + N^{-\frac{1}{100}}), \quad (3.97)$$

where $K: L^2(\mathbb{T}^3) \rightarrow L^2(\mathbb{T}^3)$,

$$K(f)(x) := P_{>BN} \int_{\mathbb{R}} e^{-it\Delta_{\theta}} W(t, x) P_{>BN} e^{it\Delta_{\theta}} f(x) dt.$$

To that purpose, we calculate the Fourier coefficients of K : Let $p, q \in \mathbb{Z}^3$, then

$$\begin{aligned} c_{p,q} &:= \langle e^{ip \cdot x}, K(e^{iq \cdot x})(x) \rangle_{L_x^2(\mathbb{T}^3)} \\ &= \int_{\mathbb{R}} \langle P_{>BN} e^{it\Delta_{\theta}} e^{ip \cdot x}, W(t, x) P_{>BN} e^{it\Delta_{\theta}} e^{iq \cdot x} \rangle_{L_x^2(\mathbb{T}^3)} dt. \end{aligned}$$

One immediately sees

$$\mathcal{F}(P_{>BN} e^{it\Delta_{\theta}} e^{ip \cdot x})(p) = (1 - \eta^3) \left(\frac{p}{BN} \right) e^{-itQ(p)}$$

and $\mathcal{F}(P_{>BN} e^{it\Delta_{\theta}} e^{ip \cdot x})(\xi) = 0$ for any $\xi \in \mathbb{Z}^3 \setminus \{p\}$. Hence, we compute

$$\begin{aligned} c_{p,q} &= (1 - \eta^3) \left(\frac{p}{BN} \right) (1 - \eta^3) \left(\frac{q}{BN} \right) \int_{\mathbb{R}} e^{it(Q(q) - Q(p))} \mathcal{F}(W(t))(p - q) dt \\ &= C (1 - \eta^3) \left(\frac{p}{BN} \right) (1 - \eta^3) \left(\frac{q}{BN} \right) (\mathcal{F}_{t,x} W)(Q(p) - Q(q), p - q). \end{aligned}$$

From the definition of W and scaling in t and x , we get the estimate

$$|c_{p,q}| \lesssim N^{-1} \left(1 + \frac{|Q(p) - Q(q)|}{N^2} \right)^{-10} \left(1 + \frac{|p - q|}{N} \right)^{-10} \mathbf{1}_{[BN, \infty)}(|p|) \mathbf{1}_{[BN, \infty)}(|q|). \quad (3.98)$$

Using Schur's lemma and Young's inequality for products, we see that

$$\|K\|_{L^2(\mathbb{T}^3) \rightarrow L^2(\mathbb{T}^3)} \lesssim \sup_{p \in \mathbb{Z}^3} \sum_{q \in \mathbb{Z}^3} |c_{p,q}| + \sup_{q \in \mathbb{Z}^3} \sum_{p \in \mathbb{Z}^3} |c_{p,q}|.$$

In view of (3.98), it suffices to prove

$$\sup_{|p| \geq BN} \sum_{v \in \mathbb{Z}^3} \left(1 + \frac{|Q(p) - Q(p+v)|}{N^2} \right)^{-10} \left(1 + \frac{|v|}{N} \right)^{-10} \lesssim N^3 (B^{-\frac{1}{100}} + N^{-\frac{1}{100}}) \quad (3.99)$$

to obtain (3.97).

Define $\theta_{\max} := \max\{\theta_1, \theta_2, \theta_3\}$ and $\Theta := \text{diag}(\theta_1, \theta_2, \theta_3)$, then we split the sum over $v \in \mathbb{Z}^3$ into three parts:

$$S_1 + S_2 + S_3 := \sum_{|v| \geq N \min\{N, B\}^{1/100}} + \sum_{\substack{|v| < N \min\{N, B\}^{1/100}, \\ |p \cdot \Theta v| \geq \theta_{\max} N^2 \min\{N, B\}^{1/10}}} + \sum_{\substack{|v| < N \min\{N, B\}^{1/100}, \\ |p \cdot \Theta v| < \theta_{\max} N^2 \min\{N, B\}^{1/10}}}.$$

Thus, it suffices to show (3.99), where we replace the sum by any of the sums above. One easily verifies that $S_1 \lesssim N^3 \min\{N, B\}^{-1/100}$ because

$$\begin{aligned} S_1 &\leq \sum_{|v| \geq N \min\{N, B\}^{\frac{1}{100}}} \left(1 + \frac{|v|}{N}\right)^{-10} \leq N^{10} \sum_{|v| \geq N \min\{N, B\}^{\frac{1}{100}}} |v|^{-10} \\ &\lesssim N^{10} (N \min\{N, B\}^{\frac{1}{100}})^{-7}. \end{aligned}$$

In order to treat S_2 , we observe that

$$Q(v) \leq \theta_{\max} |v|^2 < \theta_{\max} N^2 \min\{N, B\}^{\frac{1}{50}} < \theta_{\max} N^2 \min\{N, B\}^{\frac{1}{10}},$$

and thus,

$$\left(1 + \frac{|Q(p) - Q(p+v)|}{N^2}\right)^{-1} \leq \frac{N^2}{2|p \cdot \Theta v| - Q(v)} \leq \frac{N^2}{|p \cdot \Theta v|}.$$

We may bound S_2 by

$$\begin{aligned} N^{20} \sum_{\substack{|v| < N \min\{N, B\}^{1/100} \\ |p \cdot \Theta v| \geq \theta_{\max} N^2 \min\{N, B\}^{1/100}}} |p \cdot \Theta v|^{-10} &\leq \min\{N, B\}^{-1} \sum_{|v| < N \min\{N, B\}^{1/100}} 1 \\ &\lesssim N^3 \min\{N, B\}^{-\frac{1}{100}}. \end{aligned}$$

Finally, it remains to bound S_3 . For that purpose, we set $\hat{p} := \frac{p}{|p|}$. Since $|p| \geq BN$, it suffices to prove that

$$|\{v \in \mathbb{Z}^3 : |v| < N \min\{N, B\}^{\frac{1}{100}}, |\hat{p} \cdot \Theta v| < \theta_{\max} N \min\{N, B\}^{-\frac{9}{10}}\}| \lesssim N^3 \min\{N, B\}^{-\frac{1}{100}}.$$

This point-set is covered by a rectangle in \mathbb{R}^3 with two sides of length $N \min\{N, B\}^{\frac{1}{100}}$ and one side of length $\lesssim_{\Theta} N \min\{N, B\}^{-\frac{9}{10}}$. Therefore, the point-set is bounded by

$$(N \min\{N, B\}^{\frac{1}{100}})^2 N \min\{N, B\}^{-\frac{9}{10}} \lesssim N^3 \min\{N, B\}^{-\frac{1}{100}},$$

which proves (3.99). \square

Interaction of two profiles corresponding to two orthogonal frames

In the proof of Lemma 3.31, we also rely on the following result, which shows that two profiles corresponding to two orthogonal frames do interact very little with each other.

Lemma 3.33. *Assume that $\mathcal{O}^\alpha = (N_k^\alpha, t_k^\alpha, x_k^\alpha)_k \in \mathcal{F}_E$, $\alpha = 1, 2$, are two orthogonal frames, $I \subseteq (-\frac{1}{2}, \frac{1}{2})$ is a fixed open interval with $0 \in I$, and $T_1, T_2, R \in [1, \infty)$ are fixed numbers satisfying $R \geq T_1 + T_2$. For $\alpha = 1, 2$ and k large enough let*

$$\mathfrak{S}_k^\alpha := \{(t, x) \in I \times \mathbb{T}^3 : |t - t_k^\alpha| < T_\alpha (N_k^\alpha)^{-2}, |x - x_k^\alpha| \leq R (N_k^\alpha)^{-1}\}.$$

Assume that $(\omega_k^1, \omega_k^2, f_k, g_k, h_k)_k$ is a sequence of quintuples of functions in $X_r^1(I)$ with the properties that $\omega_k^1, \omega_k^2 \in C^1(I, C^4(\mathbb{T}^3))$ and

$$\begin{aligned} |\partial_x^\nu \omega_k^\alpha| + (N_k^\alpha)^{-2} \mathbf{1}_{\mathfrak{S}_k^\alpha} |\partial_t \partial_x^\nu \omega_k^\alpha| &\leq R (N_k^\alpha)^{\frac{1}{2} + |\nu|} \mathbf{1}_{\mathfrak{S}_k^\alpha}, \quad |\nu| \leq 4, \quad \alpha = 1, 2, \\ \|f_k\|_{X_r^1(I)} &\leq 1, \quad \|g_k\|_{X_r^1(I)} \leq 1, \quad \|h_k\|_{X_r^1(I)} \leq 1 \end{aligned} \tag{3.100}$$

for any k sufficiently large. Then,

$$\lim_{k \rightarrow +\infty} \sup_{t_0 \in I} \|\mathcal{I}_{t_0}(\omega_k^1 \omega_k^2 f_k g_k h_k)\|_{X_r^1(I)} = 0.$$

Proof. We fix some small $0 < \varepsilon < 1$. If

$$\frac{N_k^1}{N_k^2} + \frac{N_k^2}{N_k^1} \leq 4\varepsilon^{-2}$$

for any k sufficiently large, then the orthogonality of the frames implies $\mathfrak{S}_k^1 \cap \mathfrak{S}_k^2 = \emptyset$ provided k is large enough. Indeed, since \mathcal{O}^1 and \mathcal{O}^2 are orthogonal, we know that either $(N_k^1)^2 |t_k^1 - t_k^2| \rightarrow +\infty$ or $N_k^1 |x_k^1 - x_k^2| \rightarrow +\infty$. Suppose that $(N_k^1)^2 |t_k^1 - t_k^2| \rightarrow +\infty$. Then, we may conclude for $t \in \mathfrak{S}_k^1 \cap \mathfrak{S}_k^2$ that

$$|t_k^1 - t_k^2| \leq |t - t_k^1| + |t - t_k^2| \leq T_1(N_k^1)^{-2} + T_2(N_k^2)^{-2}.$$

This implies

$$(N_k^1)^2 |t_k^1 - t_k^2| \leq T_1 + T_2 \left(\frac{N_k^1}{N_k^2} \right)^2 < \infty,$$

which contradicts our assumption. The same argument leads to a contradiction if instead $N_k^1 |x_k^1 - x_k^2| \rightarrow +\infty$. From (3.100), we see that for k sufficiently large,

$$\omega_k^1 \omega_k^2 f_k g_k h_k \equiv 0.$$

By symmetry, it suffices to consider the case

$$\frac{N_k^1}{N_k^2} > 2\varepsilon^{-2} \tag{3.101}$$

for any sufficiently large k . We define $\tilde{\omega}_k^2(t) := \omega_k^2(t) \mathbf{1}_{(t_k^1 - T_1(N_k^1)^{-2}, t_k^1 + T_1(N_k^1)^{-2})}(t)$, and we note that $\omega_k^1 \omega_k^2 = \omega_k^1 \tilde{\omega}_k^2$. Furthermore, we claim that for k sufficiently large

$$\|\tilde{\omega}_k^2\|_{X_{\Gamma}^1(I)} \lesssim_R 1, \quad \|\tilde{\omega}_k^2\|_{Z(I)} \lesssim_R \varepsilon^{\frac{1}{50}}, \quad \text{and} \quad \|P_{>\varepsilon^{-1}N_k^2} \tilde{\omega}_k^2\|_{X_{\Gamma}^1(I)} \lesssim_R \varepsilon. \tag{3.102}$$

The first bound may be computed using the estimate (3.6): If we define

$$\mathfrak{S}_{k,t}^{\alpha} := \{x \in \mathbb{T}^3 : (t, x) \in \mathfrak{S}_k^{\alpha}\}, \quad t \in I,$$

then we deduce from (3.6) and (3.100) that

$$\begin{aligned} \|\tilde{\omega}_k^2\|_{X_{\Gamma}^1(I)} &\lesssim \|\tilde{\omega}_k^2(0)\|_{H^1(\mathfrak{S}_{k,0}^2)} + \left(\sum_{N \geq 1} \|P_N(i\partial_t + \Delta_{\theta}) \tilde{\omega}_k^2\|_{L_t^1(I, H^1(\mathbb{T}^3))}^2 \right)^{\frac{1}{2}} \\ &\lesssim_R 1 + (N_k^2)^{-2} \sup_{t \in I} (\|\partial_t \tilde{\omega}_k^2(t)\|_{H^1(\mathfrak{S}_{k,t}^{\alpha})} + \|\Delta_{\theta} \tilde{\omega}_k^2(t)\|_{H^1(\mathfrak{S}_{k,t}^{\alpha})}) \\ &\lesssim_R 1. \end{aligned}$$

The same argument combined with the Bernstein inequality,

$$\begin{aligned} \|P_{>\varepsilon^{-1}N_k^2} f\|_{H^s(\mathbb{T}^3)}^2 &= \sum_{N \geq 1} N^{2s} \|P_N P_{>\varepsilon^{-1}N_k^2} f\|_{L^2}^2 \lesssim \varepsilon^2 (N_k^2)^{-2} \sum_{N \geq 1} N^{2(s+1)} \|P_N P_{>\varepsilon^{-1}N_k^2} f\|_{L^2}^2 \\ &\lesssim \varepsilon^2 (N_k^2)^{-2} \|P_{>\varepsilon^{-1}N_k^2} f\|_{H^{s+1}(\mathbb{T}^3)}^2 \end{aligned} \tag{3.103}$$

for $f \in H^{s+1}(\mathbb{T}^3)$, yields the third inequality of (3.102). To gain the smallness of $\tilde{\omega}_k^2$ in the $Z(I)$ -norm, we first observe from

$$\|\tilde{\omega}_k^2\|_{Z(I)} \lesssim \|P_{\leq \varepsilon^{-1}N_k^2} \tilde{\omega}_k^2\|_{Z(I)} + \|P_{>\varepsilon^{-1}N_k^2} \tilde{\omega}_k^2\|_{X_{\Gamma}^1(I)}$$

that we only have to consider $P_{\leq \varepsilon^{-1}N_k^2} \tilde{\omega}_k^2$. Let $(p, q) \in \mathcal{P}$. Applying Bernstein's inequality with respect to x yields

$$\sum_{N \leq 2\varepsilon^{-1}N_k^2} N^{(\frac{2}{p} + \frac{3}{q} - \frac{1}{2})p} \|P_N \tilde{\omega}_k^2\|_{L^p(I, L^q(\mathbb{T}^3))}^p \lesssim_R (N_k^1)^{-2} \sup_{t \in I} \sum_{N \leq 2\varepsilon^{-1}N_k^2} N^{(1 + \frac{2}{p})p} \|P_N \tilde{\omega}_k^2(t)\|_{L^2(\mathbb{T}^3)}^p.$$

Estimating $N^{(1 + \frac{2}{p})p} \leq (2\varepsilon^{-1}N_k^2)^2 N^p$ and using $\ell^2 \subset \ell^p$, we may bound

$$\sum_{N \leq 2\varepsilon^{-1}N_k^2} N^{(\frac{2}{p} + \frac{3}{q} - \frac{1}{2})p} \|P_N \tilde{\omega}_k^2\|_{L^p(I, L^q(\mathbb{T}^3))}^p \lesssim_R \left(\frac{N_k^2}{N_k^1}\right)^2 \varepsilon^{-2} \sup_{t \in I} \|\tilde{\omega}_k^2(t)\|_{H^1(\mathfrak{S}_{k,t}^\alpha)}^p \lesssim_R \varepsilon^2.$$

This immediately implies

$$\|\tilde{\omega}_k^2\|_{Z(I)} \lesssim_R \varepsilon^{\frac{1}{50}}.$$

We also decompose ω_k^1 in low-frequency and high-frequency terms and get for sufficiently large k :

$$\begin{aligned} \omega_k^1 &= P_{\leq \varepsilon N_k^1} \omega_k^1 + P_{> \varepsilon N_k^1} \omega_k^1, \\ \|\omega_k^1\|_{X_r^1(I)} &\lesssim_R \quad \text{and} \quad \|P_{\leq \varepsilon N_k^1} \omega_k^1\|_{X_r^1(I)} \lesssim_R \varepsilon. \end{aligned} \tag{3.104}$$

The first bound follows as above, and the second estimate follows from

$$\|P_{\leq \varepsilon N_k^1} f\|_{H^1(\mathbb{T}^3)} \lesssim \varepsilon N_k^1 \|P_{\leq \varepsilon N_k^1} f\|_{L^2(\mathbb{T}^3)}, \quad f \in H^1(\mathbb{T}^3).$$

Indeed, for sufficiently large k (depending on ε),

$$\|P_{\leq \varepsilon N_k^1} \omega_k^1(0)\|_{H^1(\mathbb{T}^3)} \lesssim \varepsilon N_k^1 \|P_{\leq \varepsilon N_k^1} \omega_k^1(0)\|_{L^2(\mathfrak{S}_{k,0}^1)} \lesssim_R \varepsilon.$$

The two remaining terms, $\|\partial_t P_{\leq \varepsilon N_k^1} \omega_k^1(0)\|_{H^1(\mathfrak{S}_{k,0}^1)}$ and $\|\Delta_\theta P_{\leq \varepsilon N_k^1} \omega_k^1(0)\|_{H^1(\mathfrak{S}_{k,0}^1)}$, can be estimated along the same lines.

Now, an application of the triangle inequality yields

$$\begin{aligned} \|\mathcal{I}_{t_0}(\omega_k^1 \omega_k^2 f_k g_k h_k)\|_{X_r^1(I)} &\lesssim \|\mathcal{I}_{t_0}((P_{\leq \varepsilon N_k^1} \omega_k^1) \tilde{\omega}_k^2 f_k g_k h_k)\|_{X_r^1(I)} \\ &\quad + \|\mathcal{I}_{t_0}((P_{> \varepsilon N_k^1} \omega_k^1) (P_{> \varepsilon^{-1} N_k^2} \tilde{\omega}_k^2) f_k g_k h_k)\|_{X_r^1(I)} \\ &\quad + \|\mathcal{I}_{t_0}((P_{> \varepsilon N_k^1} \omega_k^1) (P_{\leq \varepsilon^{-1} N_k^2} \tilde{\omega}_k^2) f_k g_k h_k)\|_{X_r^1(I)} \\ &=: I_1 + I_2 + I_3 \end{aligned}$$

for every $t_0 \in I$. Applying [Lemma 3.11](#), [\(3.102\)](#), and [\(3.104\)](#), we may bound the first term for k sufficiently large as follows

$$I_1 \lesssim \|P_{\leq \varepsilon N_k^1} \omega_k^1\|_{X_r^1(I)} \|\tilde{\omega}_k^2\|_{X_r^1(I)} \|f_k\|_{X_r^1(I)} \|g_k\|_{X_r^1(I)} \|h_k\|_{X_r^1(I)} \lesssim_R \varepsilon.$$

I_2 can be bounded similarly for large k :

$$I_2 \lesssim \|\omega_k^1\|_{X_r^1(I)} \|P_{> \varepsilon^{-1} N_k^2} \tilde{\omega}_k^2\|_{X_r^1(I)} \|f_k\|_{X_r^1(I)} \|g_k\|_{X_r^1(I)} \|h_k\|_{X_r^1(I)} \lesssim_R \varepsilon.$$

To estimate I_3 , we have to use the more precise estimate [\(3.12\)](#) instead of [Lemma 3.11](#). From the relation of N_k^1 and N_k^2 (see [\(3.101\)](#)), we get that $\varepsilon N_k^1 > 2\varepsilon^{-1} N_k^2$. Thus, we have $P_{\leq \varepsilon^{-1} N_k^2} = P_{\leq \varepsilon N_k^1} P_{\leq \varepsilon^{-1} N_k^2}$. We decompose the product as in [\(3.11\)](#), and remark that

$$\sum_{N_2 \geq 2} P_{N_2} (P_{\leq \varepsilon^{-1} N_k^2} \tilde{\omega}_k^2) P_{< N_2} (P_{> \varepsilon N_k^1} \omega_k^1) P_{\leq N_2} f_k P_{\leq N_2} g_k P_{\leq N_2} h_k = 0.$$

This expression corresponds to the first summand in the second term of (3.11) if we identify $\tilde{u}_1 = P_{>\varepsilon N_k^1} \omega_k^1$, $\tilde{u}_2 = P_{\leq \varepsilon^{-1} N_k^2} \tilde{\omega}_k^2$, $\tilde{u}_3 = f_k$, $\tilde{u}_4 = g_k$, and $\tilde{u}_5 = h_k$. We conclude from (3.12) that the factor $P_{\leq \varepsilon^{-1} N_k^2} \tilde{\omega}_k^2$ can be estimated in $Z'(I)$. Hence, $I_3 \lesssim_R \varepsilon^{\frac{1}{100}}$. All in all, we proved that

$$\sup_{t_0 \in I} \|\mathcal{I}_{t_0}(\omega_k^1 \omega_k^2 f_k g_k h_k)\|_{X_r^1(I)} \lesssim_R \varepsilon^{\frac{1}{100}}$$

for all $\varepsilon > 0$ and k large enough, which implies the desired result. \square

Conclusion

We finally turn to the proof of Lemma 3.31.

Proof of Lemma 3.31. In this proof, we use the successive decomposition of a nonlinear profile U_k^γ several times.

Claim. For all $\theta > 0$ there is some $T_{\theta,\gamma}^0 = T(\psi^\gamma, \theta)$ sufficiently large such that for every $T_{\theta,\gamma} \geq T_{\theta,\gamma}^0$ there is $R_{\theta,\gamma}$ sufficiently large such that for any k large enough (depending on $R_{\theta,\gamma}$) we may decompose, up to a subsequence,

$$\mathbf{1}_{I_{\theta,\gamma}} U_k^\gamma = \omega_k^{\theta,\gamma,-\infty} + \omega_k^{\theta,\gamma} + \omega_k^{\theta,\gamma,+\infty} + \rho_k^{\theta,\gamma} + \rho_k^{\theta,\gamma,-\infty} + \rho_k^{\theta,\gamma,+\infty}, \quad (3.105)$$

where $I_{\theta,\gamma} = (-T_{\theta,\gamma}^{-1}, T_{\theta,\gamma}^{-1})$ and every function is in $X_r^1(I_{\theta,\gamma})$. Furthermore, the following estimates hold

$$\begin{aligned} \|\omega_k^{\theta,\gamma,\pm\infty}\|_{Z'(I_{\theta,\gamma})} + \|\rho_k^{\theta,\gamma}\|_{X_r^1(I_{\theta,\gamma})} + \|\rho_k^{\theta,\gamma,\pm\infty}\|_{X_r^1(I_{\theta,\gamma})} &\leq \theta, \\ \|\omega_k^{\theta,\gamma,\pm\infty}\|_{X_r^1(I_{\theta,\gamma})} + \|\omega_k^{\theta,\gamma}\|_{X_r^1(I_{\theta,\gamma})} &\lesssim 1, \\ |\partial_x^\nu \omega_k^{\theta,\gamma}| + (N_k^\gamma)^{-2} \mathbf{1}_{\mathfrak{S}_k^{\theta,\gamma}} |\partial_t \partial_x^\nu \omega_k^{\theta,\gamma}| &\leq R_{\theta,\gamma} (N_k^\gamma)^{\frac{1}{2}+|\nu|} \mathbf{1}_{\mathfrak{S}_k^{\theta,\gamma}}, \end{aligned} \quad (3.106)$$

for $|\nu| \leq 6$ and

$$\mathfrak{S}_k^{\theta,\gamma} := \{(t, x) \in I_{\theta,\gamma} \times \mathbb{T}^3 : -T_{\theta,\gamma} (N_k^\gamma)^{-2} \leq t - t_k^\gamma < T_{\theta,\gamma} (N_k^\gamma)^{-2}, |x - x_k^\gamma| \leq R_{\theta,\gamma} (N_k^\gamma)^{-1}\}.$$

Moreover, we have that

$$\omega_k^{\theta,\gamma,\pm\infty}(t) = \mathbf{1}_{\{\pm(t-t_k^\gamma) \geq T_{\theta,\gamma} (N_k^\gamma)^{-2}\} \cap I_{\theta,\gamma}}(t) \cdot e^{i(t-t_k^\gamma)\Delta_\theta} \pi_{x_k^\gamma}(T_{N_k^\gamma} \phi^{\theta,\gamma,\pm\infty}),$$

where $\phi^{\theta,\gamma,\pm\infty} = P_{\leq R_{\theta,\gamma}} \phi^{\theta,\gamma,\pm\infty} \in \mathcal{S}(\mathbb{R}^3)$ and

$$\|\phi^{\theta,\gamma,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim 1, \quad \|\phi^{\theta,\gamma,\pm\infty}\|_{L^2(\mathbb{R}^3)} \lesssim R_{\theta,\gamma}. \quad (3.107)$$

Here, $\omega_k^{\theta,\gamma}$ describes the solution in the Euclidean window, which, by Proposition 3.23, can be expressed in terms of a solution to the nonlinear Schrödinger equation on \mathbb{R}^3 . The terms $\omega_k^{\theta,\gamma,\pm\infty}$ characterize the behavior of the solution beyond the Euclidean window, which can be written in terms of the scattering data of a solution to the nonlinear Schrödinger equation on \mathbb{R}^3 as proved in Proposition 3.23. Terms that have small X_r^1 -norm are collected in the error terms $\rho_k^{\theta,\gamma}$ and $\rho_k^{\theta,\gamma,\pm\infty}$.

Now, we turn to the proof of the claim. Proposition 3.23 (ii) states that for all $\theta > 0$ there is a $T_{\theta,\gamma}^0 = T(\psi^\gamma, \theta)$ sufficiently large such that for every $T_{\theta,\gamma} \geq T_{\theta,\gamma}^0$ there is $R_{\theta,\gamma}$ sufficiently

large such that for any k large enough (depending on $R_{\theta,\gamma}$) we may decompose, after passing to a subsequence,

$$U_k^\gamma(t) - \omega_k^{\theta,\gamma}(t) = \rho_k^{\theta,\gamma}(t), \quad t \in J_k^{\theta,\gamma} := \{t \in I_{\theta,\gamma} : -T_{\theta,\gamma}(N_k^\gamma)^{-2} \leq t - t_k^\gamma < T_{\theta,\gamma}(N_k^\gamma)^{-2}\}$$

where $\omega_k^{\theta,\gamma}, \rho_k^{\theta,\gamma} \in C(J_k^{\theta,\gamma}, H^1(\mathbb{T}^3)) \cap X_{\Gamma}^1(J_k^{\theta,\gamma})$ and on the remaining time interval

$$U_k^\gamma(t) - \omega_k^{\theta,\gamma,\pm\infty}(t) = \rho_k^{\theta,\gamma,\pm\infty}(t), \quad t \in J_k^{\theta,\gamma,\pm\infty} := \{t \in I_{\theta,\gamma} \setminus J_k^{\theta,\gamma} : \pm t > 0\}$$

where $\omega_k^{\theta,\gamma,\pm\infty}, \rho_k^{\theta,\gamma,\pm\infty} \in C(J_k^{\theta,\gamma,\pm\infty}, H^1(\mathbb{T}^3)) \cap X_{\Gamma}^1(J_k^{\theta,\gamma,\pm\infty})$. Moreover, [Proposition 3.23](#) implies $\|\rho_k^{\theta,\gamma}\|_{X_{\Gamma}^1(J_k^{\theta,\gamma})} \leq \theta$ and $\|\rho_k^{\theta,\gamma,\pm\infty}\|_{X_{\Gamma}^1(J_k^{\theta,\gamma,\pm\infty})} \leq \theta$. In the decomposition above, $\omega_k^{\theta,\gamma}$ plays the role of \tilde{u}_k in [\(3.49\)](#), and from [\(3.51\)](#), we have

$$\omega_k^{\theta,\gamma,\pm\infty}(t) = \pi_{x_k^\gamma} e^{i(t-t_k^\gamma)\Delta_\theta} (T_{N_k^\gamma} \phi^{\theta,\gamma,\pm\infty}), \quad t \in J_k^{\theta,\gamma,\pm\infty}.$$

From the uniform bound on U_k^γ in $X_{\Gamma}^1(-\frac{1}{2}, \frac{1}{2})$, see [\(3.93\)](#), we deduce

$$\|\omega_k^{\theta,\gamma,\pm\infty}\|_{X_{\Gamma}^1(J_k^{\theta,\gamma,\pm\infty})} + \|\omega_k^{\theta,\gamma}\|_{X_{\Gamma}^1(J_k^{\theta,\gamma})} \lesssim 1$$

uniformly in γ and k .

The last bound in [\(3.106\)](#) is immediate from [\(3.50\)](#) and the sufficient smoothness (even in time) of $\omega_k^{\theta,\gamma}$ is implied by [Proposition 3.23 \(ii\)](#) if $s \geq 1$ is chosen large enough.

We show that it suffices to assume $\phi^{\theta,\gamma,\pm\infty} \in \mathcal{S}(\mathbb{R}^3)$. Indeed, for any given $\varepsilon > 0$ we may choose $\tilde{\phi}^{\theta,\gamma,\pm\infty} \in \mathcal{S}(\mathbb{R}^3)$ such that $\|\phi^{\theta,\gamma,\pm\infty} - \tilde{\phi}^{\theta,\gamma,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \leq \varepsilon$. Define

$$\tilde{\omega}_k^{\theta,\gamma,\pm\infty}(t) := \pi_{x_k^\gamma} e^{i(t-t_k^\gamma)\Delta_\theta} (T_{N_k^\gamma} \tilde{\phi}^{\theta,\gamma,\pm\infty}), \quad t \in J_k^{\theta,\gamma,\pm\infty},$$

then we compute

$$\begin{aligned} \|\omega_k^{\theta,\gamma,\pm\infty} - \tilde{\omega}_k^{\theta,\gamma,\pm\infty}\|_{X_{\Gamma}^1(J_k^{\theta,\gamma,\pm\infty})} &\lesssim \|T_{N_k^\gamma} \phi^{\theta,\gamma,\pm\infty} - T_{N_k^\gamma} \tilde{\phi}^{\theta,\gamma,\pm\infty}\|_{H^1(\mathbb{T}^3)} \\ &\lesssim \|\phi^{\theta,\gamma,\pm\infty} - \tilde{\phi}^{\theta,\gamma,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \varepsilon. \end{aligned}$$

Hence, by putting $\omega_k^{\theta,\gamma,\pm\infty} - \tilde{\omega}_k^{\theta,\gamma,\pm\infty}$ in the error term $\rho_k^{\theta,\gamma,\pm\infty}$, we see that we may assume $\phi^{\theta,\gamma,\pm\infty} \in \mathcal{S}(\mathbb{R}^3)$.

Using [Corollary 3.17](#), we obtain the uniform bound on $\phi^{\theta,\gamma,\pm\infty}$ in $\dot{H}^1(\mathbb{R}^3)$ for sufficiently large k :

$$\|\phi^{\theta,\gamma,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|T_{N_k^\gamma} \phi^{\theta,\gamma,\pm\infty}\|_{H^1(\mathbb{T}^3)} \lesssim \|\omega_k^{\theta,\gamma,\pm\infty}\|_{X_{\Gamma}^1(J_k^{\theta,\gamma,\pm\infty})} \lesssim 1.$$

The smallness of the Z' -norm follows from $\|\omega_k^{\theta,\gamma,\pm\infty}\|_{Z(J_k^{\theta,\gamma,\pm\infty})} \lesssim \theta^2$, which is a direct consequence of the extinction lemma, cf. [Lemma 3.21 \(i\)](#), after possibly increasing $T_{\theta,\gamma}^0$. For possibly larger $R_{\theta,\gamma}$, we have

$$\|\pi_{x_k^\gamma} e^{i(t-t_k^\gamma)\Delta_\theta} (T_{N_k^\gamma} P_{>R_{\theta,\gamma}} \phi^{\theta,\gamma,\pm\infty})\|_{X_{\Gamma}^1(J_k^{\theta,\gamma,\pm\infty})} \lesssim \|P_{>R_{\theta,\gamma}} \phi^{\theta,\gamma,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} + o_k(1) \lesssim \theta$$

for sufficiently large k . We add this to the error term $\rho_k^{\theta,\gamma,\pm\infty}$ and assume $\phi^{\theta,\gamma,\pm\infty} = P_{\leq R_{\theta,\gamma}} \phi^{\theta,\gamma,\pm\infty}$. As a consequence, we can conclude the bound on the $L^2(\mathbb{R}^3)$ -norm from Hölder's inequality now:

$$\|P_{\leq R_{\theta,\gamma}} \phi^{\theta,\gamma,\pm\infty}\|_{L^2(\mathbb{R}^3)} \lesssim R_{\theta,\gamma} \|\mathcal{F}_{\mathbb{R}^3}(\phi^{\theta,\gamma,\pm\infty})\|_{L^6(\mathbb{R}^3)} \lesssim R_{\theta,\gamma} \|\phi^{\theta,\gamma,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim R_{\theta,\gamma}.$$

We extend $\omega_k^{\theta,\gamma}$, $\omega_k^{\theta,\gamma,\pm\infty}$, $\rho_k^{\theta,\gamma}$, and $\rho_k^{\theta,\gamma,\pm\infty}$ (without changing the notation) similarly as \mathcal{I}_{t_0} in Definition 3.6 (i) to functions defined on $I_{\theta,\gamma}$. Note that the extensions are in $X_r^1(I_{\theta,\gamma})$ and the $X_r^1(I_{\theta,\gamma})$ -norm equals the X_r^1 -norm on the respective support of those functions. This finishes the proof of the claim.

Furthermore, we remark that, since $\|W\|_{X_r^1(-\frac{1}{2},\frac{1}{2})} \lesssim 1$ (see (3.93)), we may choose for any $\theta > 0$ some $T_{\theta,g} > 0$ such that

$$\|W\|_{Z'(-T_{\theta,g}^{-1},T_{\theta,g}^{-1})} \leq \theta \quad \text{and} \quad \|W\|_{X_r^1(-T_{\theta,g}^{-1},T_{\theta,g}^{-1})} \lesssim 1.$$

Proof of (3.95). Since $F(z) = z|z|^4 = z^3\bar{z}^2$, for a fixed $J \geq 1$,

$$F(U_{\text{prof},k}^J) - F(W) - \sum_{\alpha=1}^J F(U_k^\alpha)$$

may be written as a finite linear combination of products of the form

$$V_k^1 V_k^2 V_k^3 V_k^4 V_k^5, \tag{3.108}$$

where $V_k^j \in \{W, \bar{W}, U_k^\alpha, \bar{U}_k^\alpha, 1 \leq \alpha \leq J\}$, $j = 1, \dots, 5$, and at least two terms differ by more than just complex conjugation.

We now assume $\theta > 0$ to be fixed, and we decompose every profile U_k^α , $1 \leq \alpha \leq J$, as in (3.105). We may assume that $T_{\theta,\alpha} = T_{\theta,\beta} = T_{\theta,g}$ for $1 \leq \alpha, \beta \leq J$. Set $T_\theta := T_{\theta,1}$, and note that $I_k \subset (-T_\theta^{-1}, T_\theta^{-1})$ for large k . Whenever a product as in (3.108) contains an error term $\rho_k^{\theta,\gamma}$ or $\rho_k^{\theta,\gamma,\pm\infty}$, then we have

$$\sup_{t_0 \in I_k} \|\mathcal{I}_{t_0}(V_k^1 V_k^2 V_k^3 V_k^4 V_k^5)\|_{X_r^1(I_k)} \lesssim \theta,$$

which follows from Lemma 3.11 and (3.106). Analogously, we obtain the same bound if the expression contains at least one of the following:

- two scattering terms $\omega_k^{\theta,\alpha,\pm\infty}$,
- W and one scattering term $\omega_k^{\theta,\alpha,\pm\infty}$, or
- two terms W .

Lemma 3.33 shows that the $X_r^1(I_k)$ -norm of $\mathcal{I}_{t_0}(V_k^1 V_k^2 V_k^3 V_k^4 V_k^5)$ converges to zero for any $t_0 \in I_k$, whenever the product contains two different $\omega_k^{\theta,\alpha}$ and $\omega_k^{\theta,\beta}$, $\alpha \neq \beta$. Hence, in order to finish the proof, it suffices to show

$$\limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \|\mathcal{I}_{t_0}(\mathfrak{D}_{4,1}(\omega_k^{\theta,\beta}, \omega_k^{\theta,\alpha,\pm\infty}))\|_{X_r^1(I_k)} \lesssim \theta, \tag{3.109}$$

for any $\alpha = 0, 1, \dots, J$, $\beta = 1, 2, \dots, J$, with $\alpha \neq \beta$ and $\omega_k^{\theta,0,\pm\infty}(t) := W(t)$.

We set $N_k^0 := 1$. Assuming that

$$\lim_{k \rightarrow +\infty} \frac{N_k^\alpha}{N_k^\beta} = \infty,$$

we may deduce (3.109) essentially from Lemma 3.32. The lemma ensures the existence of $B > 0$ such that if we decompose

$$\omega_k^{\theta,\alpha,\pm\infty} = P_{\leq BN_k^\beta} \omega_k^{\theta,\alpha,\pm\infty} + P_{> BN_k^\beta} \omega_k^{\theta,\alpha,\pm\infty},$$

then

$$\sup_{t_0 \in I_k} \|\mathcal{I}_{t_0}(\mathfrak{D}_{4,1}(\omega_k^{\theta,\beta}, P_{>BN_k^\beta} \omega_k^{\theta,\alpha,\pm\infty}))\|_{X_r^1(I_k)} \lesssim \theta.$$

Since

$$\begin{aligned} \|P_{\leq BN_k^\beta} \omega_k^{\theta,\alpha,\pm\infty}\|_{X_r^1(I_k)} &\lesssim \|P_{\leq BN_k^\beta} (T_{N_k^\alpha} \phi^{\theta,\alpha,\pm\infty})\|_{H^1(\mathbb{T}^3)} \lesssim (1 + BN_k^\beta) \|T_{N_k^\alpha} \phi^{\theta,\alpha,\pm\infty}\|_{L^2(\mathbb{T}^3)} \\ &\lesssim (1 + BN_k^\beta) (N_k^\alpha)^{-1} \|\phi^{\theta,\alpha,\pm\infty}\|_{L^2(\mathbb{R}^3)} \lesssim o_k(1), \end{aligned}$$

we may conclude (3.109) from Lemma 3.11.

If in contrast¹

$$\lim_{k \rightarrow +\infty} \frac{N_k^\beta}{N_k^\alpha} = \infty,$$

then we proceed as follows: First, we derive for any $\alpha \in \{1, \dots, J\}$ and any $B \geq 1$ that

$$\|P_{>BN_k^\alpha} \omega_k^{\theta,\alpha,\pm\infty}\|_{X_r^1(I_k)} \lesssim \|P_{>BN_k^\alpha} (T_{N_k^\alpha} \phi^{\theta,\alpha,\pm\infty})\|_{H^1(\mathbb{T}^3)} \lesssim_{\theta,\alpha} B^{-1} + o_k(1). \quad (3.110)$$

A simple argument as in (3.103) allows to compute

$$\|P_{>BN_k^\alpha} (T_{N_k^\alpha} \phi^{\theta,\alpha,\pm\infty})\|_{H^1(\mathbb{T}^3)} \lesssim (BN_k^\alpha)^{-1} \|P_{>BN_k^\alpha} (T_{N_k^\alpha} \phi^{\theta,\alpha,\pm\infty})\|_{H^2(\mathbb{T}^3)}.$$

We only consider the highest order term that is if both derivatives fall on $\phi^{\theta,\alpha,\pm\infty}$. This term may be estimated by

$$(BN_k^\alpha)^{-1} (N_k^\alpha)^{\frac{1}{2}} \|\phi^{\theta,\alpha,\pm\infty}(N_k^\alpha \cdot)\|_{\dot{H}^2(\mathbb{R}^3)} \lesssim B^{-1} \|\phi^{\theta,\alpha,\pm\infty}\|_{\dot{H}^2(\mathbb{R}^3)} \lesssim_{\theta,\alpha} B^{-1},$$

where we used $\phi^{\theta,\alpha,\pm\infty} = P_{\leq R_{\theta,\alpha}}(\phi^{\theta,\alpha,\pm\infty})$, Bernstein's inequalities and (3.107).

If $\alpha = 0$, i.e. $\omega_k^{\theta,\alpha,\pm\infty}(t) = W(t)$, then we note from the definition of the X_r^1 -norm that there is $B_0(\theta) > 0$ such that for any $B \geq B_0$,

$$\|P_{>B} W\|_{X_r^1(I_k)} \leq \|P_{>B} W\|_{X_r^1(-T_\theta^{-1}, T_\theta^{-1})} < \theta.$$

Hence, we deduce (3.110) in this case, too.

From (3.110), we may deduce for any $\alpha \in \{0, \dots, J\}$ and any $t_0 \in I_k$,

$$\|\mathcal{I}_{t_0}(\mathfrak{D}_{4,1}(\omega_k^{\theta,\beta}, \omega_k^{\theta,\alpha,\pm\infty}))\|_{X_r^1(I_k)} \leq \|\mathcal{I}_{t_0}(\mathfrak{D}_{4,1}(\omega_k^{\theta,\beta}, P_{\leq BN_k^\alpha} \omega_k^{\theta,\alpha,\pm\infty}))\|_{X_r^1(I_k)} + \theta + o_k(1)$$

provided $B = B(\theta, \alpha)$ is sufficiently large. We deal with the first term as in the end of the proof of Lemma 3.33: Given $\delta > 0$, we decompose one factor of $\omega_k^{\theta,\beta}$ similarly as in (3.104),

$$\omega_k^{\theta,\beta} = P_{\leq \delta N_k^\beta} \omega_k^{\theta,\beta} + P_{> \delta N_k^\beta} \omega_k^{\theta,\beta},$$

and again, we get the bound

$$\|P_{\leq \delta N_k^\beta} \omega_k^{\theta,\beta}\|_{X_r^1(I_k)} \lesssim_{\theta,\beta} \delta$$

for k sufficiently large. Hence, $\|\mathcal{I}_{t_0}(\mathfrak{D}_{4,1}(\omega_k^{\theta,\beta}, P_{\leq BN_k^\alpha} \omega_k^{\theta,\alpha,\pm\infty}))\|_{X_r^1(I_k)}$ is less or equal to

$$\begin{aligned} \|\mathcal{I}_{t_0}(\mathfrak{D}_{1,3,1}(P_{\leq \delta N_k^\beta} \omega_k^{\theta,\beta}, \omega_k^{\theta,\beta}, P_{\leq BN_k^\alpha} \omega_k^{\theta,\alpha,\pm\infty}))\|_{X_r^1(I_k)} \\ + \|\mathcal{I}_{t_0}(\mathfrak{D}_{1,3,1}(P_{> \delta N_k^\beta} \omega_k^{\theta,\beta}, \omega_k^{\theta,\beta}, P_{\leq BN_k^\alpha} \omega_k^{\theta,\alpha,\pm\infty}))\|_{X_r^1(I_k)}. \end{aligned}$$

¹Note that the case $\alpha = 0$ is included here.

The smallness of the first term follows immediately from [Lemma 3.11](#). The second term can be treated as in the term I_3 in the end of the proof of [Lemma 3.33](#).

If

$$N_k^\alpha = N_k^\beta \quad \text{and} \quad t_k^\alpha = t_k^\beta \quad \text{for any } k$$

(see the reduction at the beginning of the proof of [Proposition 3.30](#)), then $\omega_k^{\theta,\beta} \omega_k^{\theta,\alpha,\pm\infty} = 0$ since the supports in time of those functions become disjoint for large enough k .

The remaining case is

$$N_k^\alpha = N_k^\beta \quad \text{and} \quad \lim_{k \rightarrow +\infty} (N_k^\alpha)^2 |t_k^\alpha - t_k^\beta| = \infty.$$

For any $\varepsilon > 0$ we may choose $\tilde{\phi}^{\theta,\alpha,\pm\infty} \in C_0^\infty(\mathbb{R}^3)$ such that $\|\phi^{\theta,\alpha,\pm\infty} - \tilde{\phi}^{\theta,\alpha,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \leq \varepsilon$. Define

$$\tilde{\omega}_k^{\theta,\alpha,\pm\infty}(t) := \pi_{x_k^\alpha} e^{i(t-t_k^\alpha)\Delta_\theta} (T_{N_k^\alpha} \tilde{\phi}^{\theta,\alpha,\pm\infty}), \quad t \in J_k^{\theta,\alpha,\pm\infty},$$

then we have

$$\begin{aligned} \|\omega_k^{\theta,\alpha,\pm\infty} - \tilde{\omega}_k^{\theta,\alpha,\pm\infty}\|_{X_{\mathbb{R}^1}^1(J_k^{\theta,\alpha,\pm\infty})} &\lesssim \|T_{N_k^\alpha} \phi^{\theta,\alpha,\pm\infty} - T_{N_k^\alpha} \tilde{\phi}^{\theta,\alpha,\pm\infty}\|_{H^1(\mathbb{T}^3)} \\ &\lesssim \|\phi^{\theta,\alpha,\pm\infty} - \tilde{\phi}^{\theta,\alpha,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \varepsilon. \end{aligned}$$

Because of [Lemma 3.11](#) and (3.5), it suffices to prove

$$\limsup_{k \rightarrow +\infty} \|\mathfrak{D}_{4,1}(\omega_k^{\theta,\beta}, \tilde{\omega}_k^{\theta,\alpha,\pm\infty})\|_{L^1(I_k, H^1(\mathbb{T}^3))} = 0.$$

By Hölder's inequality, the $L^1(I_k, H^1(\mathbb{T}^3))$ -norm is bounded by

$$\|\nabla \omega_k^{\theta,\beta}\|_{L_t^4 L_x^8} \|\omega_k^{\theta,\beta}\|_{L_t^4 L_x^8}^3 \|\tilde{\omega}_k^{\theta,\alpha,\pm\infty}\|_{L_{t,x}^\infty} + \|\omega_k^{\theta,\beta}\|_{L_t^4 L_x^8}^4 (\|\nabla \tilde{\omega}_k^{\theta,\alpha,\pm\infty}\|_{L_t^\infty L_x^2} + \|\tilde{\omega}_k^{\theta,\alpha,\pm\infty}\|_{L_t^\infty L_x^2}). \quad (3.111)$$

We apply [Lemma 3.21](#) (ii) with $T = N_k^\alpha |t_k^\alpha - t_k^\beta|^{\frac{1}{2}}$, and we use the third inequality of (3.106), then

$$(3.111) \lesssim_{\theta,\beta} (N_k^\beta)^{-\frac{1}{2}} (N_k^\alpha)^{\frac{1}{2}} (N_k^\alpha |t_k^\alpha - t_k^\beta|^{\frac{1}{2}})^{-\frac{1}{10}} + o_k(1) \lesssim_{\theta,\beta} (N_k^\alpha |t_k^\alpha - t_k^\beta|^{\frac{1}{2}})^{-\frac{1}{10}} + o_k(1).$$

This finishes the proof of (3.95).

Proof of (3.96). It is easy to see that for fixed $J \geq 1$ and $t_0 \in I_k$,

$$\begin{aligned} \left\| \mathcal{I}_{t_0} \left(F(U_{\text{prof},k}^J(t) + e^{it\Delta_\theta} R_k^J) - F(U_{\text{prof},k}^J(t)) \right) \right\|_{X_{\mathbb{R}^1}^1(I_k)} \\ \lesssim \sum_{p=0}^4 \left\| \mathcal{I}_{t_0} (\mathfrak{D}_{p,5-p}(U_{\text{prof},k}^J(t), e^{it\Delta_\theta} R_k^J)) \right\|_{X_{\mathbb{R}^1}^1(I_k)} \end{aligned}$$

holds true. If $p \leq 3$, then we can control the terms easily: Indeed, from [Lemma 3.11](#) and (3.94), we see

$$\begin{aligned} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} (\mathfrak{D}_{p,5-p}(U_{\text{prof},k}^J(t), e^{it\Delta_\theta} R_k^J)) \right\|_{X_{\mathbb{R}^1}^1(I_k)} &\lesssim \|R_k^J\|_{H^1(\mathbb{T}^3)}^{4-p} \|e^{it\Delta_\theta} R_k^J\|_{Z'(I_k)} \|U_{\text{prof},k}^J\|_{X_{\mathbb{R}^1}^1(I_k)}^p \\ &\lesssim \|e^{it\Delta_\theta} R_k^J\|_{Z'(I_k)}. \end{aligned}$$

Now, (3.90) implies that

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \sum_{p=0}^3 \left\| \mathcal{I}_{t_0} (\mathfrak{D}_{p,5-p}(U_{\text{prof},k}^J(t), e^{it\Delta_\theta} R_k^J)) \right\|_{X_{\mathbb{R}^1}^1(I_k)} = 0.$$

Hence, we are left to prove

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} (\mathfrak{D}_{4,1}(U_{\text{prof},k}^J(t), e^{it\Delta_\theta} R_k^J)) \right\|_{X_{\Gamma}^1(I_k)} = 0.$$

Let $\varepsilon > 0$ be fixed and $A' \leq J$. We define $U_{\text{prof},k}^{A'}$ via $U_{\text{prof},k}^J - U_{\text{prof},k}^{A'} = \sum_{\alpha=A'+1}^J U_k^\alpha$ and hence,

$$\begin{aligned} \|U_{\text{prof},k}^J - U_{\text{prof},k}^{A'}\|_{X_{\Gamma}^1(-\frac{1}{2}, \frac{1}{2})} &\leq \sum_{\alpha=A'+1}^J \|U_k^\alpha(t) - e^{it\Delta_\theta} U_k^\alpha(0)\|_{X_{\Gamma}^1(-\frac{1}{2}, \frac{1}{2})} \\ &\quad + \left\| e^{it\Delta_\theta} \sum_{\alpha=A'+1}^J U_k^\alpha(0) \right\|_{X_{\Gamma}^1(-\frac{1}{2}, \frac{1}{2})}. \end{aligned}$$

As seen in the proof of (3.94), this can be further estimated by

$$\sum_{\alpha=A'+1}^J E(\alpha) + \left\| \sum_{\alpha=A'+1}^J U_k^\alpha(0) \right\|_{H^1(\mathbb{T}^3)},$$

which is bounded uniformly in J . From the uniform bound of this expression, we see that there exists $A' = A'(\varepsilon)$ such that for any $J \geq A'$ and all $k \geq k_0(J)$,

$$\|U_{\text{prof},k}^J - U_{\text{prof},k}^{A'}\|_{X_{\Gamma}^1(-\frac{1}{2}, \frac{1}{2})} \leq \varepsilon$$

Thus, by Lemma 3.11, it remains to show

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} (\mathfrak{D}_{4,1}(U_{\text{prof},k}^{A'}(t), e^{it\Delta_\theta} R_k^J)) \right\|_{X_{\Gamma}^1(I_k)} \lesssim \varepsilon.$$

By the definition of $U_{\text{prof},k}^{A'}$, it suffices to prove that for any $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{0, 1, \dots, A'\}$,

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} (\mathfrak{D}_{1,1,1,1,1}(U_k^{\alpha_1}(t), U_k^{\alpha_2}(t), U_k^{\alpha_3}(t), U_k^{\alpha_4}(t), e^{it\Delta_\theta} R_k^J)) \right\|_{X_{\Gamma}^1(I_k)} \lesssim \varepsilon' \quad (3.112)$$

holds true, where we set $U_k^0 := W$ and $\varepsilon' := \varepsilon A'^{-4}$.

Decompose all nonlinear profiles U_k^α , $\alpha = 1, \dots, A'$, as in (3.105). As done before, we may assume

$$T_{\theta,\alpha} = T_{\theta,g} =: T_\theta \quad \text{and} \quad R_{\theta,\alpha} =: R_\theta \quad \text{for all } \alpha = 1, \dots, A',$$

and that the bounds (3.106) and (3.107) hold. We apply Lemma 3.11 to the left-hand side of (3.112) and from (3.90), we see that whenever there is an error term $\rho_k^{\theta,\alpha}$ or $\rho_k^{\theta,\alpha,\pm\infty}$, a scattering term $\omega^{\theta,\alpha,\pm\infty}$, or W , then (3.112) holds true. Hence, it suffices to prove

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} (\mathfrak{D}_{1,1,1,1,1}(\omega_k^{\theta,\alpha_1}(t), \omega_k^{\theta,\alpha_2}(t), \omega_k^{\theta,\alpha_3}(t), \omega_k^{\theta,\alpha_4}(t), e^{it\Delta_\theta} R_k^J)) \right\|_{X_{\Gamma}^1(I_k)} \lesssim \varepsilon'$$

for any $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{1, \dots, A'\}$. Thanks to Lemma 3.33, we may assume $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, which means that (3.112) reduces to

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} (\mathfrak{D}_{4,1}(\omega_k^{\theta,\alpha}(t), e^{it\Delta_\theta} R_k^J)) \right\|_{X_{\Gamma}^1(I_k)} \lesssim \varepsilon' \quad (3.113)$$

for any $\alpha \in \{1, \dots, A'\}$. Let $B > 0$ be fixed, we decompose

$$e^{it\Delta_\theta} R_k^J = P_{>BN_k^\alpha} e^{it\Delta_\theta} R_k^J + P_{\leq BN_k^\alpha} e^{it\Delta_\theta} R_k^J.$$

With B sufficiently large (depending on R_θ), we apply [Lemma 3.32](#) and get

$$\limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} \left(\mathfrak{D}_{4,1}(\omega_k^{\theta,\alpha}(t), P_{>BN_k^\alpha} e^{it\Delta_\theta} R_k^J) \right) \right\|_{X_\Gamma^1(I_k)} \lesssim \varepsilon'$$

for every $J \geq A'$. By possibly increasing B further, we may assume

$$\|P_{\leq B^{-1}N_k^\alpha} \omega_k^{\theta,\alpha}\|_{X_\Gamma^1(-\frac{1}{2}, \frac{1}{2})} \leq \varepsilon'$$

as shown in [\(3.104\)](#). Hence, [Lemma 3.11](#) yields

$$\limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} \left(\mathfrak{D}_{4,1}(P_{\leq B^{-1}N_k^\alpha} \omega_k^{\theta,\alpha}(t), P_{\leq BN_k^\alpha} e^{it\Delta_\theta} R_k^J) \right) \right\|_{X_\Gamma^1(I_k)} \lesssim \varepsilon'.$$

Thus, [\(3.113\)](#) is proved, provided we show

$$\limsup_{J \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \sup_{t_0 \in I_k} \left\| \mathcal{I}_{t_0} \left(\mathfrak{D}_{4,1}(P_{>B^{-1}N_k^\alpha} \omega_k^{\theta,\alpha}(t), P_{\leq BN_k^\alpha} e^{it\Delta_\theta} R_k^J) \right) \right\|_{X_\Gamma^1(I_k)} = 0,$$

which follows from [\(3.12\)](#) and [\(3.90\)](#) in the well-known fashion. \square

3.7 Further remarks

Since one has a rather good knowledge of the local and small data global well-posedness theory on $\mathbb{S} \times \mathbb{S}_\rho^2$ and on Zoll manifolds, it is natural to ask for the global theory for large data in these cases.

On $\mathbb{S} \times \mathbb{S}_\rho^2$, the main obstruction to study global well-posedness is the lack of linear Strichartz estimates for a wide range of L^p -spaces. [Proposition 2.24](#) only implies the L^6 -estimate

$$\|P_N e^{it\Delta_g} \phi\|_{L^6(I \times \mathbb{S} \times \mathbb{S}_\rho^2)} \lesssim N^{\frac{2}{3}} \|P_N \phi\|_{L^2(\mathbb{S} \times \mathbb{S}_\rho^2)}.$$

However, taking a closer look at the implicit linear version of [Lemma 2.19](#) and at the proof of [Proposition 2.24](#), one may show for $p > \frac{16}{3}$,

$$\|P_N e^{it\Delta_g} \phi\|_{L^6(\mathbb{S}_\rho^2, L^p(I, L^4(\mathbb{S})))} \lesssim N^{\frac{7}{12} - \frac{2}{p}} \|P_N \phi\|_{L^2(\mathbb{S} \times \mathbb{S}_\rho^2)}. \quad (3.114)$$

Using our approach to treat the \mathbb{S} and \mathbb{S}^2 component separately, as it was done in the proof of [Proposition 2.24](#), it seems unlikely that one can get anything better than L^6 in the \mathbb{S}^2 component. The reason is that for $f \in L^2(\mathbb{S}^2)$ the scaling invariant estimate

$$\|\Pi_n f\|_{L^p(\mathbb{S}^2)} \lesssim \langle n \rangle^{\frac{1}{2} - \frac{2}{p}} \|f\|_{L^2(\mathbb{S}^2)}$$

is known to fail for $p < 6$, cf. [[Sog86](#), page 55].

The linear Strichartz estimate [\(3.114\)](#), however, seems to be insufficient for estimating the contribution Σ_2 in the proof of [Lemma 3.11](#).

Moreover, the extinction lemma, more precisely [Lemma 3.21 \(i\)](#), has to be adapted. In the given proof, the extinction argument essentially relies on a decay in time introduced by a one-dimensional torus component, which is also present in $\mathbb{S} \times \mathbb{S}_\rho^2$. As a consequence, the proof can be modified to cover $\mathbb{S} \times \mathbb{S}_\rho^2$.

The last thing one has to take care of is [Lemma 3.32](#). A combination of the arguments given in the proofs of [Lemma 3.32](#) and [[PTW14](#), Lemma 5.3] might allow to get the desired result.

Even though the small data global well-posedness theory has been developed on Zoll manifolds, large data global well-posedness was only obtained in the special case \mathbb{S}^3 [PTW14]. The linear Strichartz estimates obtained in [Her13, Lemma 3.5] allow to gain the necessary local well-posedness and stability results in Section 3.3. The difficulties again arise in proving the extinction lemma and an analogue of Lemma 3.32. In [PTW14], the proofs rely on explicit formulas of the eigenprojectors and the particular localization of the spectrum.

Summary

Large parts of the introductory [Chapter 1](#) are a review of well-known material. Though, there have been some new aspects. Aside from short introductions to function spaces, the Fourier transform, Riemannian manifolds, and dispersive partial differential equations, we have given a new detailed proof of a variant of the Hausdorff–Young inequality for non-periodic exponential sums and have related it to a lattice point counting problem. In addition, we have applied a Weyl type inequality due to Bourgain [[Bou93a](#)], to give a proof of the exponential sum estimate in [Corollary 1.39](#), which we have heavily relied on. Although the statement is not new, as it may be seen as a special case of the stronger exponential sum estimate in [[Her13](#), Lemma 3.1], we show that the proof of this particular estimate does not require sophisticated arguments. We want to emphasize that this exponential sum estimate have been used in all our subsequent results.

In [Chapter 2](#), local and small data global well-posedness of nonlinear Schrödinger equations posed on compact, smooth Riemannian manifolds (M, g) without boundary have been discussed. We have started this chapter with the conditional energy-critical well-posedness result in [Theorem 2.3](#). It states that given the trilinear Strichartz estimate in [Assumption 2.1](#) for any given 3-manifold (M, g) , we have that the quintic nonlinear Schrödinger equation is locally well-posedness in H^1 and even globally well-posedness provided the initial data are small in H^1 . The proof of this result, that is essentially due to Herr [[HS15](#)], has been reviewed. This is valuable since the proof given in [[HS15](#)] is strongly tied to earlier works. Furthermore, we have verified [Assumption 2.1](#) for rectangular tori, which extends previous results in [[HTT11](#), [GOW14](#)]. The present author published this result in [[Str14](#)]. Also, the first proof of [Assumption 2.1](#) on products of spheres has been provided, which expands the result given in [[HS15](#)] to a general radius. Moreover, we have shown a multilinear Strichartz estimate for free solutions on two-dimensional rectangular tori that implies—by standard arguments—local well-posedness of some scaling-critical nonlinear Schrödinger equations with power type nonlinearities.

[Chapter 3](#) has been devoted to prove large data global well-posedness of the energy-critical nonlinear Schrödinger equation on rectangular 3-tori. This extends the earlier result in [[IP12b](#)] for the standard torus. The author of the present thesis published this result in [[Str15](#)], we relied on the L^4 -Strichartz estimate given in [[KV14](#)]. However, we have presented a modified proof here, which shows that Strichartz estimates for a smaller range of L^p -norms, which can be obtained essentially using the exponential sum estimates in [Chapter 1](#), suffice to conclude global well-posedness in H^1 of the quintic nonlinear Schrödinger equation on rectangular 3-tori.

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