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Social choice of convex risk measures through Arrovian aggregation of variational preferences*

Frederik Herzberg[†‡]

Abstract

This paper studies collective decision making with regard to convex risk measures: It addresses the question whether there exist non-dictatorial aggregation functions of convex risk measures satisfying Arrow-type rationality axioms (weak universality, systematicity, Pareto principle). Herein, convex risk measures are identified with variational preferences on account of the Maccheroni–Marinacci–Rustichini (2006) axiomatisation of variational preference relations and the Föllmer–Schied (2002, 2004) representation theorem for concave monetary utility functionals.

We prove a variational analogue of Arrow's impossibility theorem for finite electorates. For infinite electorates, the possibility of rational aggregation depends on a uniform continuity condition for the variational preference profiles; we prove variational analogues of both Campbell's impossibility theorem and Fishburn's possibility theorem. The proof methodology is based on a model-theoretic approach to aggregation theory inspired by Lauwers–Van Liedekerke (1995).

An appendix applies the Dietrich–List (2010) analysis of majority voting to the problem of variational preference aggregation.

Key words: Arrow-type preference aggregation; judgment aggregation; abstract aggregation theory; variational preferences; multiple priors preferences; convex risk measure; model theory; first-order predicate logic; ultrafilter; ultraproduct

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1 Introduction

Convex risk measures can be represented as negated maxmin expected utility functions with additive convex lower-semicontinuous penalty (Föllmer and Schied [11, 12]), which in turn are in a one-to-one correspondence with the set of so-called *variational* preference relations (Maccheroni, Marinacci and Rustichini [23]). Given such an individual decision-theoretic foundation for convex risk measures, it is only natural to study the aggregation problem for convex risk measures as an aggregation problem for variational preference relations.

Whilst classical preference aggregation theory does not provide suitable methods to study the aggregation of variational preferences, the scope of aggregation theory has developed considerably during the past decade: It now encompasses aggregation problems of very general form, including even the aggregation of logical propositions.

One of the most recent developments among these generalisations of classical (Arrovian) preference aggregation theory concerns the aggregation of relational structures (model aggregation). This approach can best be seen as a continuation of Lauwers and Van Liedekerke's far-sighted paper [20] and was elaborated systematically recently by Herzberg and Eckert [15, 16].[1]

It is a rather natural methodological choice to employ model aggregation theory in our analysis of variational preference aggregation, on account of the intrinsic emphasis which model aggregation lays on semantics (in comparison with most of the judgment aggregation literature) and also because of its historical roots in preference aggregation theory through the work of Lauwers and Van Liedekerke [20]. (Other general approaches to aggregation theory can be found in the literature on judgment aggregation, including the abstract aggregation theories of Nehring and Puppe [26] and of Dokow and Holzman [9] and, in particular, the rich body of work by List and Pettit [21], Dietrich and List [5, 6, 7], and Dietrich and Mongin [8].)

This methodology enables us to prove variational analogues of three of the most important (im)possibility theorems of social choice theory: those of Arrow, Fishburn, and Campbell. Moreover, it may well be possible to apply the same proof methodology to obtain similar results for multiple-priors preferences (which can be represented by coherent risk measures) and perhaps ultimately even for *dynamic* variational or multiple-priors preferences.

The paper is structured as follows: Section 2 reviews the axioms and the representation theorem of variational preferences and relates them to convex risk measures. Section 3 proposes a formal framework for an Arrovian aggregation theory of variational preferences, within which Section 4 formulates the main (im)possibility results of this paper. Section 5 then describes briefly the ideas behind the proof methodology (model aggregation theory), while Section 6 discusses possible extensions and future research.

In an appendix, we also apply Dietrich and List's [7] account of majority voting to the problem of variational preference aggregation. The fruit is a possibility theorem, but at the cost of considerable and — at least at first sight — rather unnatural restrictions on the domain of the variational preference aggregator.

[1] For another recent application of that approach — to the problem of representative-agent microfoundations for certain parametrised aggregator domains — see Herzberg [14].

2 Variational preferences and convex risk measures

Consider a finite set S, called the set of *states of the world*, let X be a convex subset of a vector space Y with more than one element, called the set of *consequences*, let $\mathcal{F}$ be the set of all functions from S to X. Then, $\mathcal{F}$ is a convex subset of the vector space Y^S. Let $\mathcal{F}_c$ be the set of all constant functions from S to X. Every element $x \in X$ can be identified with the constant function $s \mapsto x$ in $\mathcal{F}$ and thus with an element of $\mathcal{F}_c$.

Let us now introduce axioms for a binary relation $\succsim$ with symmetric part $\sim$ (i.e. $f \sim g$ if and only if $f \succsim g$ and $g \succsim f$) and asymmetric part $\succ$ (i.e. $f \succ g$ if and only if $f \succsim g$ but $g \not\succsim f$); our formulation of the axioms is borrowed from Maccheroni, Marinacci and Rustichini [23, p. 1453].

Definition 1. *A binary relation $\succsim$ on $\mathcal{F}$ with symmetric part $\sim$ and asymmetric part $\succ$ is a* variational preference ordering *or* convex risk-preference ordering *if and only if it satisfies all of the following axioms:*

(A1) Weak order properties. *For all $f, g \in \mathcal{F}$, either $f \succsim g$ or $g \succsim f$ (*completeness*); for all $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$, then $f \succsim h$ (*transitivity*).*

(A2) Weak certainty independence. *For all $f, g \in \mathcal{F}$, $x, y \in \mathcal{F}_c$ and $\alpha \in (0,1)$, if*

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x,$$

then

$$\alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y.$$

(A3) Continuity. *For all $f, g, h \in \mathcal{F}$, the sets*

$$\{\beta \in [0,1] \ : \ \beta f + (1-\beta)\, g \succsim h\}$$

and

$$\{\beta \in [0,1] \ : \ h \succsim \beta f + (1-\beta)\, g\}$$

are closed.

(A4) Monotonicity. *For all $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

(A5) Uncertainty aversion. *For all $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, if $f \sim g$, then $\alpha f + (1-\alpha)\, g \succsim f$.*

(A6) Non-degeneracy. *There exist $f, g \in \mathcal{F}$ such that $f \succ g$.*

Remark 2. *Let $\succsim$ be a binary relation on $\mathcal{F}$ with symmetric part $\sim$ and asymmetric part $\succ$.*

1. *If $\succsim$ satisfies completeness (A1a), then*

$$f \succsim g \Leftrightarrow f \not\prec g$$

for all $f, g \in \mathcal{F}$.

2. *If $\succsim$ satisfies completeness (A1a), then $\succsim$ satisfies continuity (A3) if and only if for all $f, g, h \in \mathcal{F}$ and all $\beta \in [0,1]$, there exist $\alpha, \gamma \in [0,1]$ such that*

   - $(\alpha, \gamma) \subseteq \{\delta \in [0,1] \ : \ \delta f + (1-\delta)\, g \succsim h\}$ *if* $\beta f + (1-\beta)\, g \succsim h$, *and*
   - $(\alpha, \gamma) \subseteq \{\delta \in [0,1] \ : \ h \succsim \delta f + (1-\delta)\, g\}$ *if* $h \succsim \beta f + (1-\beta)\, g$,

   while either

   - $0 \leq \alpha < \beta < \gamma \leq 1$ *or*
   - $0 = \alpha = \beta < \gamma \leq 1$ *or*
   - $0 \leq \alpha < \beta = \gamma = 1$.

The identification of variational preference relations with convex risk-preference orderings can be justified as follows: On the one hand, Maccheroni, Marinacci and Rustichini [23, pp. 1453, 1456] have extended previous work by Gilboa and Schmeidler [13] and established that a relation $\succsim$ satisfying axioms (A1-A6) allows for a representation in terms of a maxmin expected utility function with additive convex lower-semicontinuous penalty: A binary relation $\succsim$ on $\mathcal{F}$ is a variational preference relation if and only if there exists a nonzero linear function $u : X \to \mathbb{R}$ and a convex lower-semicontinuous function $c : \Delta \to [0, +\infty]$ (Δ being the set of all probability measures on S) whose infimum is $> -\infty$ such that for any $f, g \in \mathcal{F}$, one has

$$f \succsim g \Leftrightarrow \min_{p \in \Delta} \left(\int u \circ f \ \mathrm{d}p + c(p) \right) \geq \min_{p \in \Delta} \left(\int u \circ g \ \mathrm{d}p + c(p) \right).$$

On the other hand, Föllmer and Schied [11, 12] have demonstrated that convex risk measures can be represented as negated maxmin expected utility functions with additive convex lower-semicontinuous penalty and "real consequences" (i.e. $X \subseteq \mathbb{R}$). Therefore, variational preference relations are the ordinal equivalents of convex risk measures.

In our investigation of aggregation of variational preference orderings (i.e. convex risk-preference orderings), it will be helpful to have a more "quantitative" notion of continuity at hand, in order to distinguish degrees of continuity. For this purpose we introduce the notion of a *witness* to continuity. The following definition of being a "witness to continuity" is motivated by the role which the scalars α, γ play in the equivalent characterisation of continuity in Remark 2.

Definition 3. *Let $f, g, h \in \mathcal{F}$ and $\beta \in [0,1]$. A pair of real numbers $(\alpha, \gamma) \in [0,1]^2$ is called a* witness-pair to the continuity *of $\succsim$ along $f, g, h \in \mathcal{F}$ in β if and only if for all $\delta \in (\alpha, \gamma)$, one has*

- $\delta f + (1-\delta)\, g \prec h$ *if* $\beta f + (1-\beta)\, g \prec h$ *and*
- $h \prec \delta f + (1-\delta)\, g$ *if* $h \prec \beta f + (1-\beta)\, g$,

whilst either

- $\alpha < \beta < \gamma$ *or*
- $0 = \alpha = \beta < \gamma$ *or*
- $\alpha < \beta = \gamma = 1$.

A real number $\varepsilon \in [0,1]$ is called a witness to the continuity *of $\succsim$ along $f, g, h \in \mathcal{F}$ in β if and only if there exists some $\alpha \in [0,1]$ or $\gamma \in [0,1]$ such that either (α, ε) or (γ, ε) is a witness-pair to the continuity of $\succsim$ along $f, g, h \in \mathcal{F}$ in β.*

With this definition, we can now rephrase Remark 2:

Remark 4. *If $\succsim$ satisfies completeness (A1a), then $\succsim$ satisfies continuity (A3) if and only if for all $f, g, h \in \mathcal{F}$ and all $\beta \in [0,1]$ there exists a witness to the continuity of $\succsim$ along $f, g, h \in \mathcal{F}$ in β.*

3 Aggregation of variational preferences

Consider a set N (finite or infinite), which we shall call *population* or *electorate*. Elements of N are called *individuals*, subsets of N are called *coalitions*. Suppose that each individual $i \in N$ is endowed with a variational preference ordering $\succsim_i$ (as defined in Section 2); any such resulting N-sequence $\underset{\sim}{\succsim} = (\succsim_i)_{i \in N}$ is called a *variational preference profile*. In various circumstances — for instance, in the course of making certain policy choices — the question will arise whether one can aggregate the individual variational preference orderings and obtain a social variational preference ordering (i.e. an aggregate of the individual variational preferences $\succsim_i$ which itself happens to be variational preference relation). And if so, are there any rules, satisfying certain rationality conditions, which can be used to assign a (social) variational preference ordering to all variational preference profiles — or at least to a large class of variational preference profiles?

We shall show that any such rule whose domain encompasses a rich class of variational preference profiles must be dictatorial in the case of finite $N$and thus establish an equivalent of Arrow's [1] impossibility theorem for variational preference aggregation. For the case of infinite N, we shall first prove an impossibility result under the assumption of an even more comprehensive aggregator domain, thus obtaining an equivalent of Campbell's [3] impossibility theorem for variational preference aggregation). Secondly, we shall show a possibility result for infinite N under the assumption that the aggregator domain contains only uniformly continuous variational preference profile; this result can be seen as an variational-preference analogue of Fishburn's [10] possibility theorem.

As we shall see in an appendix, on certain restricted domains of profiles for finite electorates, the majority voting rule — which also satisfies two important rationality axioms — can be used to obtain a social variational preference ordering.

4 Main results: Variational preference aggregation for rich aggregator domains

Denote the set of all variational preference relations on $\mathcal{F}$ by $\mathcal{P}$.

In this paper, a *preference aggregator* is a map F with domain $\mathrm{dom}(F) \subseteq \mathcal{P}^N$ whose range is a set of complete binary relations on $\mathcal{F}$. A *variational preference aggregator* or *convex risk-preference aggregator* is a map F from a subset $\mathrm{dom}(F) \subseteq \mathcal{P}^N$ to $\mathcal{P}$. A preference aggregator F is said to be

- *universal* if and only if $\mathrm{dom}(F) = \mathcal{P}^N$ (so that $F : \mathcal{P}^N \to \mathcal{P}$);
- *weakly universal* if and only if $\mathrm{dom}(F)$ is a *rich aggregator domain*. Herein, a subset $\mathbb{D} \subseteq \mathcal{P}^N$ is called a *rich aggregator domain* if and only if there are $f, f', g, g' \in \mathcal{F}$ and variational preference orderings $\succsim_1, \succsim_2, \succsim_3$ such that
  - $f \succsim_1 g,\ f' \succsim_1 g', \quad f \succsim_2 g,\ f' \prec_2 g', \quad f \prec_3 g,\ f' \succsim_3 g'$, and
  - $\{\succsim_1, \succsim_2, \succsim_3\}^N \subseteq \mathbb{D}$;
- *systematic* if and only if for every $\underline{\succsim} \in \mathrm{dom}(F)$ and all $f, f', g, g' \in \mathcal{F}$ with $\{i \in N \ :\ f \succsim_i g\} = \{i \in N \ :\ f' \succsim_i g'\}$ one has

$$f\, F\left(\underline{\succsim}\right)\, g \Leftrightarrow f'\, F\left(\underline{\succsim}\right)\, g';$$

- *Paretian* if and only if for every $\underline{\succsim} \in \mathrm{dom}(F)$ and all $f, g \in \mathcal{F}$, if $f \succsim_i g$ for all $i \in N$, then $f\, F\left(\underline{\succsim}\right)\, g$;
- *dictatorial* if and only if there exists some $i \in N$ (called *dictator*) such that for every $\underline{\succsim} \in \mathrm{dom}(F)$ and all $f, g \in \mathcal{F}$,

$$f\, F\left(\underline{\succsim}\right)\, g \Leftrightarrow f \succsim_i g.$$

The modification “weakly” in “weakly universal” is justified:

Remark 5. *If S contains at least two elements, then $\mathcal{P}^N$ is a rich aggregator domain, and every universal aggregator is also weakly universal.*

(All proofs can be found in Appendix C.) Clearly, every dictatorial F can be extended to a universal, systematic and Paretian aggregator. It is remarkable that even the converse holds true:

Theorem 6. *Let N be finite and let F be a (variational) preference aggregator. F is weakly universal, systematic and Paretian if and only if it is dictatorial.*

(Theorem 6 is the variational preference analogue of Arrow’s [1] possibility theorem.)

Under an additional assumption on the richness of the domain of $\mathrm{dom}(F)$, one can even extend Theorem 6 to the case of infinite N. A profile $\underline{\succsim}$ is said to be *continuous* if and only if $\succsim_i$ is continuous for all $i \in N$. Using the terminology of Definition 3, a variational preference profile $\underline{\succsim}$ is *discontinuous in the limit* if and only if for all $f, g, h \in \mathcal{F}$ and all $\beta \in [0, 1]$, every $\alpha \in [0, 1]$ is a witness to the continuity of $\succsim_i$ along f, g, h in β for only finitely many $i \in N$. As an example one might think of a profile of variational preference relations $(\succsim_i)_{i \in \mathbb{N}}$, each with variational representation (u_i, c_i), where $u_i = i\, u_0$ and $c_i = c_0$ for all $i \in \mathbb{N}_{>0}$.

Theorem 7. *Let N be an arbitrary set (finite or infinite). Let F be a weakly universal, systematic and Paretian variational preference aggregator. Suppose that its domain $\mathrm{dom}(F)$ contains a profile $\underline{\succsim}$ that is (continuous, but) discontinuous in the limit. Then F is dictatorial.*

(Theorem 7 can be seen as the variational preference analogue of Campbell's [3] possibility theorem.)

Conversely, one can obtain a possibility result for infinite N by demanding uniform continuity rather than continuity of the variational preference profiles in the aggregator domain: A profile $(\succsim_i)_{i \in N}$ is said to be *uniformly continuous* if and only if for all $f, g, h \in \mathcal{F}$ and all $\beta \in [0, 1]$, there exist $\alpha, \gamma \in [0, 1]$ which for all $i \in N$ are a witness-pair to the continuity of $\succsim_i$ along f, g, h in β.

Theorem 8. *Let N be an infinite set, and let $\mathbb{D} \subseteq \mathcal{P}^N$ be a rich aggregator domain such that all profiles in $\mathbb{D}$ are uniformly continuous. Then there exist non-dictatorial, weakly universal, systematic and Paretian variational preference aggregators $F : \mathbb{D} \to \mathcal{P}$.*

(Theorem 8 is the variational preference analogue of Fishburn's [10] possibility theorem.)

5 Proof idea

The shortest route in proving the above theorems is to invoke recent results from model aggregation theory, due to Herzberg and Eckert [16] who generalised previous findings by Lauwers and Van Liedekerke [20]. In order to employ these results, one needs to reformulate the variational preference aggregation problem as a model aggregation problem (see Appendix B); thereafter, the proofs follow relatively easily from the model aggregation theory in Herzberg and Eckert [16] (see Appendix C). In this section, we briefly describe model aggregation theory and its application to the aggregation of variational preferences; a rigorous review can be found in Appendix A.

Model aggregation theory studies the aggregation of first-order structures (in the sense of mathematical logic). An aggregator in this setting is then just a map from a set of N-sequences of structures of a certain type to a set of structures of such type. It is not difficult to formulate analogues of Arrow's [1] rationality assumptions in this framework.

Of utmost importance is the notion of a *decisive* coalition with respect to an aggregator F: A coalition D is said to be *decisive* with respect to an aggregator F if and only if it can be written in the form $D = \{i \in N : (\mathcal{F}, \succsim_i) \models \phi\}$ for some profile $\underset{\sim}{\succsim} \in \mathrm{dom}(F)$ and some quantifier-free formula ϕ such that $\left(\mathcal{F}, F\left(\underset{\sim}{\succsim}\right)\right) \models \phi$.

Denoting the set of all decisive coalitions with respect to F by $\mathcal{D}_F$, one can next prove the following key lemma:

Lemma 9. *If F is a weakly universal, systematic and Paretian variational preference aggregator, then $\mathcal{D}_F$ is an ultrafilter on N.*[2]

[2] An *ultrafilter* on N is a nonempty set $\mathcal{D}$ of coalitions which is not equal to the powerset of N, is closed under supersets (i.e. if $D \in \mathcal{D}$ and $D' \supseteq D$, then $D' \in \mathcal{D}$), closed under intersections (i.e. if $D, D' \in \mathcal{D}$, then $D \cap D' \in \mathcal{D}$) and has the property that for any coalition D either $D \in \mathcal{D}$ or $N \setminus D \in \mathcal{D}$. A *filter* on N is a set of coalitions that has the first three properties, but may lack the last one. One can show that ultrafilters are nothing else but $\subseteq$-maximal filters.

The proof of Lemma 9 uses a slight generalisation of the main lemma in Lauwers and Van Liedekerke [20, Lemma 2]:[3] In the proof of that lemma, the ultrafilter properties (non-triviality; closure under supersets and intersections; dichotomy) are verified by constructing appropriate profiles through exploiting the richness of the aggregator domain.

Since ultrafilters on finite sets are always *principal* (i.e. systems of supersets of singletons), Lemma 9 quickly leads to a proof of the "only if" part in Theorem 6. The proof of the "if" part in Theorem 6 is straightforward.

Using the ultrafilter property of the set of decisive coalitions, Theorem 8 and Theorem 7 can now be proved through applications of Łoś's theorem: For, one can apply Lemma 9 to show that any weakly universal, systematic and Paretian preference aggregator F maps every variational preference profile to the restriction (to the original domain $\mathcal{F}$) of its ultraproduct (with respect to the ultrafilter $\mathcal{D}_F$ of decisive coalitions); and conversely, Łoś's theorem implies that every preference aggregator F which assigns to each variational preference profile in $\mathrm{dom}(F)$ the restriction of its ultraproduct with respect to a fixed ultrafilter $\mathcal{D}$ constitutes a systematic Paretian preference aggregator (which is weakly universal if $\mathrm{dom}(F)$ is a rich aggregator domain). Now, since — again by Łoś's theorem — restricted ultraproducts preserve universal formulae (also sometimes called Π_1 formulae) that hold in all factor structures, it is clear that the aggregate of a uniformly continuous variational preference profile under a weakly universal systematic Paretian preference aggregator must again be continuous and thus a variational preference profile. Hence, every weakly universal, systematic, Paretian preference aggregator whose domain only consists of uniformly continuous variational preference profiles is actually a variational preference aggregator. Now, for infinite N, there exist non-principal ultrafilters $\mathcal{U}$ on N. Choose such a $\mathcal{U}$ and let $F : \mathbb{D} \to \mathcal{P}$ be a map whose domain only contains uniformly continuous variational preference profiles and which assigns to each element of $\mathbb{D}$ the restriction of its ultraproduct with respect to $\mathcal{U}$. This F will then be a variational preference aggregator which is not dictatorial, establishing Theorem 8.

The representation of weakly systematic, Paretian preference aggregators as restricted ultraproduct constructions can also be used to show that no domain of a weakly universal, systematic, Paretian variational preference aggregator can contain a profile $\underline{\succsim}$ that is discontinuous in the limit. For, if there were such an aggregator, it would on the one hand have to preserve continuity, and on the other hand, every scalar will be a witness to the continuity of only finitely many variational preference orderings in the profile $\underline{\succsim}$ (which is discontinuous in the limit). A combination of these two facts ultimately implies that the set of decisive coalitions contains a finite set (viz. the set of all $i \in N$ such that α is a witness to the continuity of $\succsim_i$ along $f, g, h \in \mathcal{F}$ in β, wherein $\alpha, \beta \in [0,1]$ and $f, g, h \in \mathcal{F}$ have been chosen such that α is witness to the continuity of $F\left(\underline{\succsim}\right)$ along $f, g, h \in \mathcal{F}$ in β). But an ultrafilter which contains a finite set is principal, whence the corresponding aggregator is dictatorial. This proves Theorem 7.

[3] Whilst the published proof of Lauwers and Van Liedekerke's [20] lemma is incomplete, an addendum by Herzberg, Lauwers, Van Liedekerke and Fianu [17] has recently filled the gap.

6 Extensions

Using the methodology of the present paper, one can also study the aggregation of coherent risk measures for a finite set of states of the world: For, coherent risk measures can be written as negated maxmin expected utility functions, which in turn represent *multiple priors preferences*, as shown by Gilboa and Schmeidler [13]. Hence, the aggregation of coherent risk measures can be reformulated as an aggregation problem for certainty-independent, continuous, monotonic, uncertainty-averse and non-degenerate weak orders, and the theory of model aggregation can again be used to prove impossibility and possibility results.

Moreover, the approach taken in this paper might perhaps also be used to analyse the aggregation of dynamic variational preferences and thus of dynamic convex risk measures: For, the representation theorem of Föllmer and Schied [11, 12] has been extended to a dynamic setting by Detlefsen and Scandolo in a paper on dynamic convex risk measures [4] which builds upon on Riedel's seminal article on dynamic coherent risk measures [27]. Moreover, Maccheroni, Marinacci and Rustichini [24] have recently developed a dynamic generalisation of their axiomatisation of variational preferences [23]. Combining their theorem with Detlefsen and Scandolo's result, one obtains a decision-theoretic foundation of dynamic convex risk measures in terms of dynamic variational preferences.

At a more technical frontier, the systematicity condition can possibly be relaxed, since systematicity is equivalent to the weaker aggregator condition of independence if the conditional entailment relation among the set of test sentences has full transitive closure.

7 Conclusion

We have formulated Arrow-type aggregation problems for convex risk measures or variational preferences. Choosing a methodology inspired by Lauwers and Van Liedekerke [20], one can prove analogues of Arrow's impossibility theorem, Campbell's impossibility theorem, and Fishburn's possibility theorem. The proof method is sufficiently general to be applied to Arrow-type aggregation of coherent risk measures or multiple priors preferences, and perhaps even dynamic convex or dynamic coherent risk measures and their variational counterparts.

— Appendices —

A Review of model aggregation theory

The theory of model aggregation was begun by Lauwers and Van Liedekerke [20] (see also Herzberg, Lauwers Van Liedekerke and Fianu [17]) and continued recently by Herzberg and Eckert [15, 16]. In the following, we only review special cases of the most important known results from model aggregation with particular relevance for the analysis of variational preference aggregation. In fact, the presentation in this appendix is only slightly more general than the work of Lauwers and Van Liedekerke [20] — in that it allows for arbitrarily many predicate symbols rather than just one —, whence any reader who knows Lauwers and Van Liedekerke's work [20] may well skip this appendix. For proofs and a more general account of model aggregation, see Herzberg and Eckert [15, 16].

We assume in this section that the reader has some basic knowledge of model theory. The paper by Lauwers and Van Liedekerke [20] contains a short introduction to logic and model theory for social choice theorists; more comprehensive introductions can be found in textbooks such as those by Bell and Slomson [2] or Hodges [18].

Let A be a set. Let $\mathcal{L}$ be a language consisting of predicate symbols $\dot{P}_n$, $n \in \kappa$, and constant symbols $\dot{a}$ for all elements a of A. The arity of $\dot{P}_n$ will be denoted $\delta(n)$, for all $n \in \kappa$.

For the purposes of this paper, an $\mathcal{L}$-structure is a pair $\mathfrak{B} = \left\langle B, \left\langle P_n^{\mathfrak{B}} \right\rangle_{n\in\kappa} \right\rangle$ consisting of a set $B \supseteq A$ (called the *domain* of $\mathfrak{B}$) and certain sets $P_n^B \subseteq B^{\delta(n)}$ which serve to interpret the predicate symbols $\dot{P}_n$ through Tarski's definition of truth. We require that by definition any $\mathcal{L}$-structure interprets the constant symbols $\dot{a}$ canonically, i.e. by a, for any $a \in A$.

Let $\mathcal{S}$ be the set of atomic formulae in $\mathcal{L}$. Let $\mathcal{T}$ be the *Boolean closure* of $\mathcal{S}$, i.e. the closure of $\mathcal{S}$ under the logical connectives $\dot{\neg}$, $\dot{\wedge}$, $\dot{\vee}$. The elements of $\mathcal{T}$ are called *test sentences*, and the elements of $\mathcal{S}$ are called *basic test sentences.*

Let T be a consistent set of universal sentences in $\mathcal{L}$.[4], and let Ω be the collection of models of T with domain A. As is usual in model theory, the restriction of an $\mathcal{L}$-structure $\mathfrak{B}$ is the $\mathcal{L}$-structure that is obtained by restricting the interpretations of the relation symbol to the domain A; it is denoted $\mathrm{res}_A\mathfrak{B}$.

We assume that there are two sentences in $\mathcal{S}$, henceforth denoted $\mu, \nu \in \mathcal{S}$, such that each of $\mu\dot{\wedge}\nu$, $\mu\dot{\wedge}\dot{\neg}\nu$ and $\dot{\neg}\mu\dot{\wedge}\nu$ is consistent with T, in symbols,

$$T \cup \{\mu\dot{\wedge}\nu\} \not\vdash \bot, \quad T \cup \{\mu\dot{\wedge}\dot{\neg}\nu\} \not\vdash \bot, \quad T \cup \{\dot{\neg}\mu\dot{\wedge}\nu\} \not\vdash \bot \tag{1}$$

(wherein $\bot$ is shorthand for $\phi\dot{\wedge}\dot{\neg}\phi$ for some sentence ϕ).

Since $\mathcal{S}$ is the set of all atomic formulae in $\mathcal{L}$ and T is a set of universal sentences, the following propositions hold for all $\mathcal{L}$-structures $\mathfrak{A}$ and all $\mathfrak{A}_1, \mathfrak{A}_2 \in \Omega$:

$$(\forall\lambda \in \mathcal{S} \qquad (\mathfrak{A}_1 \models \lambda \Leftrightarrow \mathfrak{A}_2 \models \lambda)) \Rightarrow \mathfrak{A}_1 = \mathfrak{A}_2. \tag{2}$$

$$\mathfrak{A} \models T \Rightarrow \mathrm{res}_A\mathfrak{A} \in \Omega \tag{3}$$

$$\forall\lambda \in \mathcal{T} \qquad (\mathfrak{A} \models \lambda \Leftrightarrow \mathrm{res}_A\mathfrak{A} \models \lambda)\,. \tag{4}$$

Elements of Ω^N will be called *profiles.* An *aggregator* is a map f whose domain $\mathrm{dom}(f)$ is a subset of Ω^N and whose range is a subset of Ω.[5]

For all $\lambda \in \mathcal{T}$ and all $\omega \in \Omega^N$, we denote the *coalition supporting* λ *given profile* ω, by

$$C(\omega, \lambda) := \{i \in N \ : \ \omega_i \models \lambda\}\,.$$

Let us fix an aggregator f. Consider the following axioms:

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[4]A sentence is *universal* if it (in its prenex normal form) has the form $(\dot{\forall}\dot{v}_{k_1})\cdots(\dot{\forall}\dot{v}_{k_m})\phi$ for some formula ϕ that does not contain any quantifiers and some nonnegative integer m.

[5]We deviate from Lauwers' and Van Liedekerke's [20] notation as follows:

- Aggregators will be denoted by f (instead of AF).
- Profiles will be denoted by ω or $\langle \omega_i \ : \ i \in N\rangle$ (instead of $\langle \mathcal{A}_i \ : \ i \in N\rangle$).
- The image of a profile ω under an aggregator f will be denoted by $f(\omega)$ (instead of $\mathcal{A}(\omega)$).

(A1). $\mathrm{dom}(f) = \Omega^N$.

(A1'). There exist models $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \in \Omega$ such that

1. $\mathfrak{A}_1 \models \mu \dot{\wedge} \nu$, $\mathfrak{A}_2 \models \mu \dot{\wedge} \dot{\neg} \nu$, $\mathfrak{A}_3 \models \dot{\neg} \mu \dot{\wedge} \nu$, and
2. $\{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3\}^N \subseteq \mathrm{dom}(f)$.

(A2). For all $\omega \in \mathrm{dom}(f)$ and all $\lambda \in \mathcal{T}$, if $f(\omega) \models \lambda$, then $C(\omega, \lambda) \neq \emptyset$.

(A3). For all $\omega, \omega' \in \mathrm{dom}(f)$ and all $\lambda, \lambda' \in \mathcal{T}$ such that $C(\omega, \lambda) = C(\omega', \lambda')$, one has $f(\omega) \models \lambda$ if and only if $f(\omega') \models \lambda'$.

(A1) is the axiom of *Universality*. Axiom (A2) is a generalised *Pareto Principle*. (A3) is a generalised form of the axiom of *Systematicity*, which itself is a strong variant of the axiom of *Independence of Irrelevant Alternatives*.[6]

By our assumptions on $\mu, \nu \in \mathcal{S}$, there must be $\mathcal{L}$-structures $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ such that $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \models T$ as well as $\mathfrak{A}_1 \models \mu \dot{\wedge} \nu$, $\mathfrak{A}_2 \models \mu \dot{\wedge} \dot{\neg} \nu$, $\mathfrak{A}_3 \models \dot{\neg} \mu \dot{\wedge} \nu$. Since T is universal and so are all elements of $\mathcal{T}$, we may assume that $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ all have domain A (otherwise, take their restriction to A). Hence, Axiom (A1') is simply a weak version of (A1) because of our assumption about the sentences $\mu, \nu \in \mathcal{S}$.

Given an aggregator f, we define the set of *decisive coalitions* by

$$\mathcal{D}_f := \{C(\omega, \lambda) \ : \ \omega \in \mathrm{dom}(f), \quad \lambda \in \mathcal{T}, \quad f(\omega) \models \lambda\}.$$

It is not difficult to verify that systematic aggregators are characterised by their sets of decisive coalitions:

Remark 10. *If f satisfies (A3), then for all $\omega \in \mathrm{dom}(f)$ and $\lambda \in \mathcal{T}$,*

$$C(\omega, \lambda) \in \mathcal{D}_f \Leftrightarrow f(\omega) \models \lambda.$$

This framework is sufficiently general to cover the cases of preference aggregation, propositional judgment aggregation, and modal aggregation.[7] The general model aggregation theory in Herzberg and Eckert [15, 16] admits more general sets of test sentences $\mathcal{T}$ and relaxes the aggregator axioms (A2) and (A3).

The key result of model aggregation is the following lemma:[8]

[6] Systematicity vacuously implies Independence of Irrelevant Alternatives. The converse is true under additional hypotheses: In the preference aggregation framework, the combination of Independence of Irrelevant Alternatives and the Pareto Principle implies Systematicity if the individual preferences are complete and quasi-transitive (cf. Lauwers and Van Liedekerke [20, Section 6, p. 232]).

[7] For example, for preference aggregation, one lets $\mathcal{L}$ have a single binary predicate $\dot{P}$, modelling the preference relation. The set A will be the set of alternatives. The interpretation of $\dot{P}(\dot{a}, \dot{b})$ will be "a is preferred to b". (Thus, the interpretation of $\omega_i \models \dot{P}(\dot{a}, \dot{b})$ is "under profile ω, individual i prefers a to b", and the interpretation of $f(\omega) \models \dot{P}(\dot{a}, \dot{b})$ is "under profile ω, a is socially preferred to b".) T can be any universal theory in that language. For propositional judgment aggregation, one lets $\mathcal{L}$ have a single unary predicate $\dot{B}$, modelling a belief operator. The set A will be the agenda. The interpretation of $\dot{B}\dot{a}$ "a is accepted". (Thus, the interpretation of $\omega_i \models \dot{B}\dot{a}$ is "under profile ω, individual i accepts a", and the interpretation of $f(\omega) \models \dot{B}\dot{a}$ is "under profile ω, a is socially accepted".) T can be any universal theory in that language.

[8] Lemma 11 slightly generalises the main lemma in Lauwers and Van Liedekerke [20, Lemma 2]; a proof in a more general setting can be found in Herzberg and Eckert's first paper [15].

Lemma 11. *Let f be weakly universal, systematic, and Paretian. Then, $\mathcal{D}_f$ is an ultrafilter.*[9]

We say that f is *dictatorial* if and only if there exists some $i_f \in N$ (called the *dictator*) such that $\mathcal{D}_f = \{J \subseteq N \ : i_f \in J\}$.

Remark 12. *Let f be an aggregator, and suppose N is finite. Then, f is dictatorial if and only if $\mathcal{D}_f$ is an ultrafilter.*

As a corollary of the ultrafilter property of the set of decisive coalitions (see Lemma 11), we then get an analogue of Arrow's impossibility theorem:[10]

Corollary 13 (Impossibility theorem). *Let f be weakly universal, systematic, and Paretian. If N is finite, then f is dictatorial.*

By Łoś's theorem [22]:

Remark 14. *If $\mathcal{D}$ is an ultrafilter, then*

$$\operatorname{res}_A \prod_{i \in N} \omega_i/\mathcal{D} \models \lambda \Leftrightarrow C(\omega, \lambda) \in \mathcal{D}$$

for all $\omega \in \Omega^N$ and $\lambda \in \mathcal{T}$.

Lemma 15. *Let f be weakly universal, systematic, and Paretian, then $f(\omega) = \operatorname{res}_A \prod_{i \in N} \omega_i/\mathcal{D}_f$ for all $\omega \in \operatorname{dom}(f)$.*

Lemma 16. *Suppose $\mathcal{D}$ is an ultrafilter, and consider the aggregator $\operatorname{res}_A \prod /\mathcal{D}$, defined by*

$$\operatorname{res}_A \prod /\mathcal{D} : \Omega^N \to \Omega, \qquad \omega \mapsto \operatorname{res}_A \prod_{i \in N} \omega_i/\mathcal{D}.$$

Then $\operatorname{res}_A \prod /\mathcal{D}$ is a universal, systematic and Paretian aggregator.

Let βN denote the set of all ultrafilters on the set N, and let $\mathcal{AR}$ be the set of all universal, systematic and Paretian aggregators. Now one can prove a general version of the Kirman–Sondermann [19] correspondence:[11]

Theorem 17 (Kirman–Sondermann correspondence). *There is a bijection $\Lambda : \mathcal{AR} \to \beta N$, given by*

$$\forall f \in \mathcal{AR} \qquad \Lambda(f) = \mathcal{D}_f.$$

Its inverse is given by

$$\forall \mathcal{D} \in \beta N \qquad \Lambda^{-1}(\mathcal{D}) = \operatorname{res}_A \prod /\mathcal{D},$$

wherein, as in Lemma 16, $\operatorname{res}_A \prod /\mathcal{D} : \omega \mapsto \operatorname{res}_A \prod_{i \in N} \omega_i/\mathcal{D}$.

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[9] For the definition of an ultrafilter, see footnote 2 on page 7.

[10]

[11] This Theorem 17 is a slight generalisation of Lauwers and Van Liedekerke's main theorem; its proof — in a more general framework than that of the present paper — can be found in Herzberg and Eckert [15].

Consider an arbitrary $\mathcal{L}$-sentence which is not universal. In its prenex normal form it can be written as $\psi \equiv (\dot{\forall}\dot{x}_1)\dots(\dot{\forall}\dot{x}_m)(\dot{\exists}\dot{y})\phi(\dot{x}_1,\dots,\dot{x}_m;\dot{y})$, wherein m is a nonnegative integer and $\phi(\dot{x}_1,\dots,\dot{x}_m;\dot{y})$ is an $\mathcal{L}$-formula with $m+1$ free variables. For the rest of this section, ψ and ϕ are fixed in this manner.

We say that a profile $\omega \in \Omega^N$ has *finite witness multiplicity* with respect to ψ if and only if $\omega_i \models \psi$ for all $i \in N$, but for all $a_1,\dots,a_m,a' \in A$, the coalition $\{i \in N \ : \ \omega \models \phi(a_1,\dots,a_m;a')\}$ is finite.

An aggregator f is said to *preserve* an $\mathcal{L}$-sentence ψ if and only if for all $\omega \in \mathrm{dom}(f)$, one has $f(\omega) \models \psi$ whenever $\omega_i \models \psi$ for all $i \in N$. We then have the following theorem:[12]

Theorem 18. *Let f be weakly universal, systematic and Paretian, suppose f preserves ψ, and assume that there exists some $\omega \in \Omega^I$ with finite witness multiplicity with respect to ψ. Then, f is a dictatorship.*

B Variational preference aggregation as model aggregation

As we have remarked before, our proofs depend largely on the recent results on model aggregation by Herzberg and Eckert [15, 16] that generalise previous work by Lauwers and Van Liedekerke [20]. The key to the proofs of Theorem 6 and Theorem 7 is therefore the rephrasing of the variational preference aggregation problem in the framework of first-order model theory.

The formulation of the variational preference aggregation problem in the framework of first-order model theory can even be accomplished without appealing to multi-sorted predicate logic, as it will turn out that one can identify the closed unit interval $[0,1] \subseteq \mathbb{R}$ and the open unit interval $(0,1) \subseteq \mathbb{R}$ with subsets of $\mathcal{F}_c$ and hence of $\mathcal{F}$. The domain of the model-theoretic structures to be aggregated will thus be just $\mathcal{F}$, and individual constant and variable symbols will always be interpreted as referring to constant or variable elements of $\mathcal{F}$.

In order to embed the closed and open unit intervals of $\mathbb{R}$ into $\mathcal{F}$, choose two distinct elements $x_0, x_1 \in X$, and define for all $\alpha \in [0,1]$ a constant function $\breve{\alpha}$ by

$$\breve{\alpha} : s \mapsto \alpha x_0 + (1-\alpha)\, x_1.$$

Clearly, the map $\alpha \mapsto \breve{\alpha}$ is injective.[13] Hence, if we define

$$\bar{I} = \{\breve{\alpha} \ : \ \alpha \in [0,1]\}$$

and

$$I = \{\breve{\alpha} \ : \ \alpha \in (0,1)\} = \bar{I} \setminus \{x_0, x_1\},$$

[12] This theorem is in some sense an abstract version of a similar result by Lauwers and Van Liedekerke [20, p. 230, Property 4]; its proof can be found in Herzberg and Eckert's second paper [16].

[13] For, if

$$\alpha x_0 + (1-\alpha)\, x_1 = \beta x_0 + (1-\beta)\, x_1$$

for some $\alpha, \beta \in [0,1]$ with $\alpha \neq \beta$, then

$$(\alpha - \beta)\, x_0 = (\alpha - \beta)\, x_1$$

and thus $x_0 = x_1$, contradiction.

there is a canonical bijection between $\bar{I}$ and $[0,1] \subseteq \mathbb{R}$ as well as between I and $(0,1) \subseteq \mathbb{R}$.

This allows us to define a *mixture operator* $m : \bar{I} \times \mathcal{F}^2 \to \mathcal{F}$ as follows: For all $\alpha \in [0,1]$ and $f, g \in \mathcal{F}$, put

$$m(\check{\alpha}; f, g) = \alpha f + (1-\alpha) g \in \mathcal{F}.$$

(Recall that $\mathcal{F}$ is a convex subset of the vector space Y^S.) For every $s \in S$, let $\pi_s : \mathcal{F} \to \mathcal{F}_c$ be the projection operator which maps f to the constant function with range $\{f(s)\}$, so that $\pi_s(f) : s \mapsto f(s)$ for all $f \in \mathcal{F}$.

Finally, one can define a linear ordering $<_{\bar{I}}$ on $\bar{I}$ by

$$\check{\alpha} <_{\bar{I}} \check{\beta} \Leftrightarrow \alpha < \beta$$

for all $\alpha, \beta \in [0,1]$.

With these definitions, we may now consider the following axioms for a binary relation $\succsim$ with symmetric part $\sim$ (i.e. $f \sim g$ if and only if $f \succsim g$ and $g \succsim f$) and asymmetric part $\succ$ (i.e. $f \succ g$ if and only if $f \succsim g$ but $g \not\succsim f$):

(A1) *Weak order properties.* For all $f, g \in \mathcal{F}$, either $f \succsim g$ or $g \succsim f$; for all $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$, then $f \succsim h$.

(A2) *Weak certainty independence.* For all $f, g \in \mathcal{F}$, $x, y \in \mathcal{F}_c$ and $a \in I$, if

$$m(a; f, x) \succsim m(a; g, x),$$

then

$$m(a; f, y) \succsim m(a; g, y).$$

(A3) *Continuity.* For all $f, g, h \in \mathcal{F}$ and all $b \in I$, there exist $a, c \in \bar{I}$ such that

- if $m(b; f, g) \succsim h$, then for all $d \in I$ with $a <_{\bar{I}} d <_{\bar{I}} c$, one has $m(d; f, g) \succsim h$, and
- if $h \succsim m(b; f, g)$, then for all $d \in I$ with $a <_{\bar{I}} d <_{\bar{I}} c$, one has $h \succsim m(d; f, g)$,

while either

- $x_0 \leq_{\bar{I}} a <_{\bar{I}} b <_{\bar{I}} c \leq_{\bar{I}} x_1$ or
- $x_0 = a = b <_{\bar{I}} c \leq_{\bar{I}} x_1$ or
- $x_0 \leq_{\bar{I}} a < b = c = x_1$.

(A4) *Monotonicity.* For all $f, g \in \mathcal{F}$, if $\bigwedge_{s \in S} \pi_s(f) \succsim \pi_s(g)$, then $f \succsim g$.

(A5) *Uncertainty aversion.* For all $f, g \in \mathcal{F}$ and $a \in I$, if $f \sim g$, then $m(a; f, g) \succsim f$.

(A6) *Non-degeneracy.* There exist $f, g \in \mathcal{F}$ such that $f \succ g$.

All these axioms can be captured in a language of first-order logic with:

- two unary predicate symbols $\dot{C}, \dot{I}$ (expressing membership in the subsets $\mathcal{F}_c$ and I, respectively, of $\mathcal{F}$),

- two binary predicate symbols $\dot{\succsim}$ and $\dot{<}_{\bar{I}}$,
- card(S) operator symbols $\dot{\pi}_s$,
- and a ternary operation symbol $\dot{m}$.

Henceforth, the language with the predicate symbols $\dot{C}, \dot{I}$, the operator symbols $\dot{\pi}_s$ (for each $s \in S$), the predicate symbols $\dot{\succsim}$ and $\dot{<}_{\bar{I}}$ the operation symbol $\dot{m}$, and a constant symbol $\dot{f}$ for every element $f \in \mathcal{F}$.

Let Γ be the set of all models $\mathfrak{A} = \left\langle \mathcal{F}, \left\langle C^{\mathfrak{A}}, I^{\mathfrak{A}}, \succsim^{\mathfrak{A}}, <_{\bar{I}}{}^{\mathfrak{A}}, \langle \pi_s^{\mathfrak{A}} \rangle_{s \in S}, m^{\mathfrak{A}} \right\rangle \right\rangle$ of (A1-A6) with domain $\mathcal{F}$ such that $\mathfrak{A}$ canonically interprets

- the constant symbols $\dot{f}$ (i.e. $f^{\mathfrak{A}} = f$ for every $f \in \mathcal{F}$),
- the unary predicate symbols $\dot{C}$ and $\dot{I}$ (i.e. $C^{\mathfrak{A}} = \mathcal{F}_c$ and $I^{\mathfrak{A}} = I$),
- the binary relation symbol $\dot{<}_{\bar{I}}$ (i.e. $\dot{x}^{\mathfrak{A}} <_{\bar{I}}{}^{\mathfrak{A}} \dot{y}^{\mathfrak{A}}$ if and only if $x <_{\bar{I}} y$ for all $x, y \in X$),
- the operator symbols $\dot{\pi}_s$ (i.e. $\mathfrak{A} \models \dot{\pi}_s(\dot{f}) \dot{=} \dot{g}$ if and only if $f(s) = g$ for all $s \in S$ and $f, g \in \mathcal{F}$), and
- the ternary operation symbol m (i.e. $\mathfrak{A} \models \dot{m}\left(\dot{a}; \dot{f}, \dot{g}\right) \dot{=} \dot{h}$ if and only if $m(a; f, g) = h$ for all $f, g, h \in \mathcal{F}$ and all $a \in I$).

Then, elements of Γ are in a canonical one-to-one correspondence with variational preference orderings. Hence, variational preference aggregators are in a canonical one-to-one correspondence with maps $G : \mathrm{dom}(G) \to \Gamma$ where $\mathrm{dom}(G) \subseteq \Gamma^N$; such maps g shall also be called *model aggregators*.

One can now rephrase the variational preference aggregator axioms as model aggregator axioms. Let $\mathcal{T}$ be the Boolean closure of the set of atomic sentences. We shall call $\mathcal{T}$ the set of test sentences.

- *universal* if and only if $\mathrm{dom}(G) = \Gamma^N$ (so that $G : \Gamma^N \to \mathcal{P}$);
- *weakly universal* if and only if $\mathrm{dom}(G)$ is a *rich aggregator domain.* Herein, a set $\mathbb{D}$ is a *rich aggregator domain* if and only if there exist atomic sentences μ, ν and models $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ such that
  - $\mathfrak{A}_1 \models \mu \dot{\wedge} \nu, \quad \mathfrak{A}_2 \models \mu \dot{\wedge} \dot{\neg} \nu, \quad \mathfrak{A}_2 \models \dot{\neg} \mu \dot{\wedge} \nu$ and
  - $\{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3\}^N \subseteq \mathrm{dom}(G)$;
- *systematic* if and only if for every $\underline{\mathfrak{A}} \in \mathrm{dom}(G)$ and all test sentences λ, λ' satisfying $\{i \in N \ : \ \mathfrak{A}_i \models \lambda\} = \{i \in N \ : \ \mathfrak{A}_i \models \lambda'\}$ one has
$$G\left(\underline{\mathfrak{A}}\right) \models \lambda \Leftrightarrow G\left(\underline{\mathfrak{A}}\right) \models \lambda';$$
- *Paretian* if and only if for every $\underline{\mathfrak{A}} \in \mathrm{dom}(G)$ and all test sentences λ, if $\mathfrak{A}_I \models \lambda$ for all $i \in N$, then $G\left(\underline{\mathfrak{A}}\right) \models \lambda$;
- *dictatorial* if and only if there exists some $i \in N$ (called *dictator*) such that for every $\underline{\mathfrak{A}} \in \mathrm{dom}(G)$ and all test sentences λ,
$$G\left(\underline{\mathfrak{A}}\right) \models \lambda \Leftrightarrow \mathfrak{A}_i \models \lambda.$$

A coalition D is said to be *decisive* with respect to a model aggregator G if and only if there is some $\underline{\mathfrak{A}} \in \mathrm{dom}(G)$ and some test sentence λ such that

$$G(\underline{\mathfrak{A}}) \models \lambda, \qquad D = \{i \in N \ : \ \mathfrak{A}_i \models \lambda\}.$$

The set of all decisive coalitions with respect to G is denoted $\mathcal{D}_G$.

C Proof details

Let Δ be the set of all probability measures on S. By the Macceroni–Marinacci–Rustichini theorem [23, Theorem 3], a binary relation $\succsim$ on $\mathcal{F}$ is a variational preference relation if and only if there exists a nonzero linear function $u : X \to \mathbb{R}$ and a convex lower-semicontinuous function $c : \Delta \to [0, +\infty]$ whose infimum is a real number (rather than $-\infty$ or $+\infty$) such that for all $f, g \in \mathcal{F}$,

$$f \succsim g \Leftrightarrow \min_{p \in \Delta} \left(\int u \circ f \ \mathrm{d}p + c(p) \right) \geq \min_{p \in \Delta} \left(\int u \circ g \ \mathrm{d}p + c(p) \right).$$

In that case, we say that $\succsim$ has the *variational representation* (u, c).

Proof of Remark 5. Let s_0, s_1 be two distinct elements. Let u be a nonzero linear function. Without loss of generality, assume $u(x_0) < u(x_1)$. Let c_1, c_2, c_3 be such that $c_i(p) = +\infty$ for all $i \in \{1, 2, 3\}$ and all $p \in \Delta$ such that $p\{s\} > 0$ for some $s \in S \backslash \{s_0, s_1\}$. Then, for every $i \in \{1, 2, 3\}$, there exists some function $c'_i : [0, 1] \to \mathbb{R}$ such that

$$c'_i(q) = c_i\left(q\delta_{s_0} + (1 - q)\delta_{s_1}\right)$$

for all $q \in [0, 1]$ (wherein δ_s denotes the Dirac probability measure concentrated on the singleton $\{s\}$). Let $f, g \in \mathcal{F}$ be such that

$$f(s_0) = x_0, \quad f(s_1) = x_1, \qquad g(s_0) = x_1, \quad g(s_1) = x_0.$$

Clearly then for any $i \in \{1, 2, 3\}$, one has

$$\begin{aligned}
\min_{p \in \Delta} \left(\int u \circ f \ \mathrm{d}p + c_i(p) \right) &= \min_{q \in [0,1]} \left(qu(x_0) + (1 - q)u(x_1) + c'_i(q) \right) \\
&= u(x_1) + \min_{q \in [0,1]} \left(q\left(u(x_0) - u(x_1)\right) + c'_i(q) \right) \\
\min_{p \in \Delta} \left(\int u \circ g \ \mathrm{d}p + c_i(p) \right) &= \min_{q \in [0,1]} \left(qu(x_1) + (1 - q)u(x_0) + c'_i(q) \right) \\
&= u(x_0) + \min_{q \in [0,1]} \left(q\left(u(x_1) - u(x_0)\right) + c'_i(q) \right)
\end{aligned}$$

Let us now put $c'_1(q) = 0$ for all $q \in [0, 1]$. Then, because $u(x_1) > u(x_0)$ or equivalently $u(x_0) - u(x_1) < 0$ and $u(x_1) - u(x_0) > 0$, we have

$$\begin{aligned}
\min_{p \in \Delta} \left(\int u \circ f \ \mathrm{d}p + c_1(p) \right) &= u(x_1) + \min_{q \in [0,1]} \left(q\left(u(x_0) - u(x_1)\right) + \underbrace{c'_1(q)}_{=0} \right) \\
&= u(x_1) + u(x_0) - u(x_1) = u(x_0) \\
\min_{p \in \Delta} \left(\int u \circ g \ \mathrm{d}p + c_1(p) \right) &= u(x_0) + \min_{q \in [0,1]} \left(q\left(u(x_1) - u(x_0)\right) + \underbrace{c'_1(q)}_{=0} \right) \\
&= u(x_0)
\end{aligned}$$

hence $f \sim_1 g$ if $\succsim_1$ is chosen as the variational preference relation with variational representation (u, c_1).

Next, put $c_2' : q \mapsto q\,(u\,(x_1) - u\,(x_0))$. Then,

$$\begin{aligned}
\min_{p \in \Delta} \left(\int u \circ f \ \mathrm{d}p + c_2(p) \right) &= u\,(x_1) + \min_{q \in [0,1]} \left(q\,(u\,(x_0) - u\,(x_1)) + c_2'(q) \right) \\
&= u\,(x_1) \\
\min_{p \in \Delta} \left(\int u \circ g \ \mathrm{d}p + c_2(p) \right) &= u\,(x_0) + \min_{q \in [0,1]} \left(q\,(u\,(x_1) - u\,(x_0)) + c_2'(q) \right) \\
&= u\,(x_0) + \min_{q \in [0,1]} 2q\,(u\,(x_1) - u\,(x_0)) \\
&= u\,(x_0)\,,
\end{aligned}$$

hence $f \succ_2 g$ if $\succsim_2$ is chosen as the variational preference relation with variational representation (u, c_2).

Finally, put $c_3' : q \mapsto q\,(u\,(x_0) - u\,(x_1))$. Then,

$$\begin{aligned}
\min_{p \in \Delta} \left(\int u \circ f \ \mathrm{d}p + c_3(p) \right) &= u\,(x_1) + \min_{q \in [0,1]} \left(q\,(u\,(x_0) - u\,(x_1)) + c_3'(q) \right) \\
&= u\,(x_1) + \min_{q \in [0,1]} 2q\,(u\,(x_0) - u\,(x_1)) \\
&= u\,(x_1) + 2\,(u\,(x_0) - u\,(x_1)) \\
&= 2u\,(x_0) - u\,(x_1) < u\,(x_0) \\
\min_{p \in \Delta} \left(\int u \circ g \ \mathrm{d}p + c_3(p) \right) &= u\,(x_0) + \min_{q \in [0,1]} \left(q\,(u\,(x_1) - u\,(x_0)) + c_3'(q) \right) \\
&= u\,(x_0)
\end{aligned}$$

hence $g \succ_3 f$ if $\succsim_3$ is chosen as the variational preference relation with variational representation (u, c_3).

All in all, we have found variational preference relations $\succsim_1, \succsim_2, \succsim_3$ with

$$f \succsim_1 g \succsim_1 f, \qquad f \succ_2 g, \qquad g \succ_3 f.$$

If we put $f' = g$ and $g' = f$, then f, g, f', g' and $\succsim_1, \succsim_2, \succsim_3$ satisfy the requirements in the definition of a rich aggregator domain.

It follows that $\mathcal{P}^N$ is a rich aggregator domain. Therefore, every universal aggregator is also weakly universal. □

Proof of Theorem 6. The reformulation of variational preference aggregation as model aggregation in Appendix B permits the application of the impossibility result in Corollary 13 (a generalisation of Arrow's theorem) which in our context says that any weakly universal, systematic, Paretian model aggregator which preserves the (universal) axioms A1-A2, A4-A6 (i.e. the variational preference axioms without continuity) is a dictatorship if N is finite. Hence, a fortiori, any variational preference aggregator (which by definition even preserves all axioms A1-A6 on its domain) must be a dictatorship if N is finite. □

Proof of Theorem 7. The reformulation of variational preference aggregation in Appendix B also allows us to use the characterisation of model aggregators as restricted ultraproduct constructions (Lemma 15). In order to apply

Theorem 18, the impossibility result for aggregators on infinite populations which preserve certain non-universal formulae (for a similar result, cf. Lauwers and Van Liedekerke [20, p. 230, Property 4]), observe first that continuity of variational preferences is not a universal formula and secondly that any profile which is discontinuous in the limit has finite witness multiplicity with respect to continuity (in the terminology of Theorem 18, the abstract version of a similar). □

Proof of Theorem 8. Again, in light of Appendix B, we may use the characterisation of aggregators as restricted ultraproduct constructions (Lemma 15). Note that for fixed f, g, h and α, β, γ, the formula "$\langle \alpha, \gamma \rangle$ is a witness-pair to the continuity of $\succsim$ along f, g, h in β" is a universal formula, and all the axioms A1-A2, A4-A6 are also universal formulae. Hence, the axioms A1-A2, A4-A6 as well as the formulae "$\langle \alpha, \gamma \rangle$ is a witness-pair to the continuity of $\succsim$ along f, g, h in β" (for all fixed $f, g, h, \alpha, \beta, \gamma$) are preserved by restricted ultraproducts. Therefore, restricted ultraproduct constructions on rich domains are model aggregators which not only are weakly universal, systematic and Paretian and preserve axioms A1-A2, A4-A6, but they also aggregate uniformly continuous profiles into continuous profiles. Hence, any restricted ultraproduct construction on a rich domain with only uniformly continuous profiles constitutes weakly universal, systematic and Paretian variational preference aggregator. However, on an infinite set N there are non-principal ultrafilters and thus non-dictatorial aggregators derived from restricted ultraproduct constructions. □

Proof of Lemma 9. This is a direct consequence of Lemma 11, itself a slight generalisation of the main lemma in Lauwers and Van Liedekerke [20, Lemma 2]. □

D Variational preference aggregation with restricted domain through majority voting

We have seen that in general, universal systematic Paretian aggregation of convex risk measures is impossible. If one drops universality, then rational aggregation of risk measures is still possible, viz. through majority voting about risk measures, but at the expense of considerable restrictions on the variational preference profiles.

In analysing majority decisions about convex risk measures, one can build on the work of Dietrich and List [7] who have developed a theory of majority voting in the very general framework of judgment aggregation, including a generalisation of May's [25] theorem (which uniquely characterises majority voting by means of certain axioms such as anonymity and acceptance/rejection neutrality). In order to do so, one has to embed the aggregation problem for risk measures into the framework of judgment aggregation.

Consider the axiom system Σ consisting of the following formulae:

- The axioms (A1-A6) as reformulated in Appendix B.
- The formulae $\neg \dot{f} \dot{=} \dot{g}$ for all $f, g \in \mathcal{F}$ such that $f \neq g$.

- All formulae of the form $\dot{\pi}_s(\dot{f})\dot{=}\dot{x}$ for all $s \in S$, $f \in \mathcal{F}$ and $x \in \mathcal{F}_c$ satisfying $\pi_s(f) = x$.
- All formulae of the form $\dot{C}\dot{x}$ for all $x \in \mathcal{F}_c$, and all formulae of the form $\dot{\neg}\dot{C}\dot{f}$ for all $f \in \mathcal{F} \setminus \mathcal{F}_c$.
- All formulae of the form $\dot{I}\dot{a}$ for all $a \in I$, and all formulae of the form $\dot{\neg}\dot{I}\dot{f}$ for all $f \in \mathcal{F} \setminus I$.
- All formulae of the form $\dot{m}\left(\dot{a}; \dot{f}, \dot{g}\right) = \dot{h}$ for all $a \in I$ and $f, g, h \in \mathcal{F}$ satisfying $m(a; f, g) = h$.

For any set of $\mathcal{L}$-formulae A, completeness and consistency will be understood to mean Σ-completeness and Σ-consistency

Let

$$\Xi = \left\{ \dot{f}\dot{\succsim}\dot{g}, \dot{\neg}\dot{f}\dot{\succsim}\dot{g} \ : \ f, g \in \mathcal{F} \right\}.$$

This is an *agenda* in the terminology of judgment aggregation, i.e. a set of proposition-negation pairs. In the following, for any $p \in \Xi$, we mean by $\bar{\neg}p$ the other element of the proposition-negation pair in Ξ to which p belongs, so that $\bar{\neg}\bar{\neg}p = p$ for any $p \in \Xi$.

A *fully rational judgment set* is a complete and consistent subset of Ξ; note that by the choice of Σ, any fully rational judgment set uniquely determines a preference relation $\succsim$ on $\mathcal{F}$ that satisfies axioms (A1-A6). The set of fully rational judgment sets will be denoted by D.

Let N be finite. A *profile* is an N-tuple $\underline{A} = (A_i)_{i \in N}$ of fully rational judgment sets. For each profile $\underline{A}$ and any $p \in \Xi$, we define the *coalition supporting p under profile $\underline{A}$* by

$$\underline{A}(p) = \{ i \in N \ : \ p \in A_i \}.$$

The aggregation rule of majority voting is then defined as the map

$$F : D^N \to 2^{\Xi}, \qquad \underline{A} \mapsto \{ p \in \Xi \ : \ \mathrm{card}\,(\underline{A}\,(p)) > \mathrm{card}\,(\underline{A}\,(\bar{\neg}p)) \}.$$

At least whenever card(N) is odd, the aggregate judgment set $F(\underline{A})$ will be complete for every $\underline{A} \in D^N$. The question is whether $F(\underline{A})$ will be consistent as well; if it is, it is a fully rational judgment set and thus by our observation made above, uniquely determines a preference relation $\succsim$ on $\mathcal{F}$ that satisfies axioms (A1-A6), hence a convex risk measure.

A sufficient condition for the consistency of $F(\underline{A})$ for certain $\underline{A} \in D^N$ has been discovered by Dietrich and List [7] and is known as the value-restriction property. A profile $\underline{A} \in D^N$ *value-restricted* is for every non-singleton, minimal inconsistent subset $Y \subseteq \Xi$ there exists a two-element subset $Z \subseteq Y$ such that $Z \not\subseteq A_i$ for all $i \in N$. If there is an ordering on the agenda with respect to which every A_i (for $i \in N$) is (locally) single-plateaued or single-canyoned, then the profile is value-restricted and hence $F(\underline{A})$ is consistent (and complete anyway).

However, in the case of variational preference profiles, the conditions of single-canyonedness or single-plateauedness — let alone the value-restriction property — do not appear to be natural conditions. This gives additional weight to the impossibility results in the present paper.

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