Essays on Simulation Methods in Economic Dynamics

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Contents

1 Introduction

"[...] the problems that are solvable the way tic-tac-toe is solvable lie within two or three inches of the surface, and an ocean of problems deeper than these cannot be guaranteed of solution."
- W. B. Arthur

Whether economics is an easy or a difficult science depends to a high degree on the class of problems that researchers consider. The author of the quote understood that the array of questions that are easily tackled is only a very small subset of the questions that are of interest. Mathematics offers a large amount of tools, but even though analytical instruments are a powerful device to understand countless economic phenomena, in many areas they have sharp limitations, and sometimes their elegance and intellectual appeal lets one forget that there is a world below this surface of the ocean of problems. I want to illustrate the high potential of combining simulation with the traditional mathematical instruments of economic theory and empirical analysis to construct models, which are not only more realistic in the predictions they make, but also in the way they operate, imitating a world of high complexity that keeps constantly computing itself.

In the Guidelines published by the Association of German Engineers, simulation is defined as the replication of a system with all its dynamic properties in an experimental model in order to gain knowledge, which can be transferred to reality (VDI [2013]). I am using the engineers' definition of simulation not because it is specifically different from others, but because engineering is probably one of the disciplines where it has been applied most extensively. Simulation is often understood in conjunction with the use of computers. Strictly speaking, this does not always have to be the case. In the early 1950s, a few works were highly published in which authors solved problems by building electrical circuits to asses the dynamic behavior of certain variables (see e.g. Morehouse et al. [1950] and Enke [1951]). The rough increase in computing power, which has happened over the last decades has made simulation more powerful than it has ever been. This trend has been further amplified by the introduction of object-oriented programming languages, which started their triumph after C++ had been made publicly available in the beginning of the 90s. Featuring objects, which can independently assess and modify each others properties, object-oriented programming is particularly well suited for simulation of economic models. As one might have guessed, these methods are going to be heavily applied in this work.

In econometric models or models of economic theory with an empirical component, simulation can be used to validate ideas and conjectures about solution properties. More importantly, models can be estimated to fit empirical data sets. This is done in the first chapter of this dissertation. In particular, an interior point solver is used to solve a very large non-convex optimization problem.

Together with my coauthor, I try to find the best possible stochastic model describing the outcomes of tournaments of a set of pairwisely compared objects. The interpretation of these objects being sports teams comes very natural. We assume a transitive relation between the respective team strengths and formulate a mathematical concept for expressing stochastic transitivity in a set of constraints on the winning probabilities in our model. Maximizing the likelihood of a particular tournament (or "season") to have happened under these constraints turns out to be manageable if one fixes the ranking of teams, but highly complicated if it is endogenized. After providing a theoretical examination of the optimization problem and relating it to existing problems from the literature, we present a branch and bound algorithm which is able to find the correct ranking for sets of about 11 or 12 teams. For larger leagues a tabu search algorithm being able to find "good" solutions in short time periods is discussed. In the empirical section of the paper, we apply these algorithms and test in addition for the relative performance of some popular ranking methods. This investigation includes a specifically designed hypothesis test that assumes the correctness of one ranking scheme before simulating a large set of seasons to find out whether this assumption was correct in the first place. We find that the superiority of a ranking scheme highly depends on the type of sport as well as the form of the tournament that is used.

Lacking either the branch and bound algorithm or a modern computer it would not be possible to gauge the winning and tying probabilities of for instance the Austrian Bundesliga. Hereby, the chapter illustrates how an algorithmic reflection can be combined with computing power to find out things that seemed inaccessible.

Bringing econometric models closer to reality by applying elaborate algorithms to enhance estimation procedures, artificially generate test statistics and undertake other tasks on large data sets has long been understood as being the main application of simulation within economics. Such a view can only be held in conjunction with the opinion that standard economic models, including perfectly rational agents who's expectations are designed with the purpose of being correct, describe the reality in a satisfying manner. Such models assume that every individual is informed about everybody else and has a perfect knowledge about the future, leading to cleared markets and fulfilled expectations. This static perspective is strictly limited to the consideration of equilibria, often ignoring if or how the system would move towards them or what would happen outside. Agent based computational economics is designed to fill this gap, by setting up an economy as a collection of possibly heterogeneous individuals, who have certain limitations concerning their mental capabilities or the information they posses. Hereby models are not only enhanced in their realism, but they are generalized and can answer a broader set of questions.

In the second chapter of this work, I apply this idea to the discipline of New Economic Geography (NEG). NEG models explain empirically observed agglomeration phenomena by setting up multi country models, combining mobile and immobile factors to attain circular causalities concerning their relocation. The drawback is that most of the works in this area exclusively consider an equilibrium context of symmetric and asymmetric solutions. Fowler [2007] was the first one to build an agent based model, which is also defined out of the static equilibria. I follow his idea of augmentation, but take as a basis the model by Baldwin [1999], which seems, because of labour immobility, a lot better suited for european applications. The goal is to improve the model by substituting perfect foresight concerning future capital returns in favor of a perceived i.i.d. and later an AR1 process. This brings the necessity to make use of some ideas from the optimal consumption (also known as "buffer stock saving") literature. By means of backward induction it is possible to determine optimal consumption as a function of cash on hand. An approximation of the optimal function allows for the derivation of some analytic results concerning agglomeration. Towards the end of the paper, the model is generalized to a world of more than two regions. Apparently not only preferences and trade barriers play a role, but also the geometric structure of the global map.

When explicitly considering the additional dimension of time, economic models can become unhandy very quickly. In dynamic models in which agents adapt, learn, or predict, fully exhausting the instruments from the theory of dynamical systems is of particular significance. If there are still open questions, it is equally important to utilize numerical methods.

The third chapter takes place on a more abstract level, taking a narrower perspective to more directly address the problems I see with standard economic modeling. Usually, economic agents are assumed to deduce the future from their knowledge about the problem definition, as well as their understanding that others do the same. Individuals are designed like that because abstracting from mental models in favor of a godlike deductive ability makes life very easy for those who set up the models. However, cognitive psychology shows us that human brains mainly work by association. Our minds collect and compare patterns that we observe. Especially in situations of complication, when the human intellectual apparatus ceases to cope, we set up a competition between hypothetical predictive models to describe the environment. This very idea is in the focus of Brian Arthur's El Farol Bar Problem (see Arthur [1994]), which was later generalized as the so called Minority Game, describing the repeated competition within a group of agents for a scarce recourse. It turns out that complex dynamics are robustly generated. However, the causes of these dynamics are not yet very well understood. In this third chapter, I want to find out, whether the driving force of the rich dynamics is the predictive behavior of the agents alone. I set up a game theoretic model, in which repeatedly randomly matched players try to predict their opponents' behavior by using their personal past experience. It turns out that there is a qualitatively different answer to this question for two and three or more player games. For a single period horizon it is partly possible to arrive at these answers analytically, which is quite tedious, becoming apparent in the high proportion of proofs in this chapter. To arrive at the other part of the answers, I again use simulation to calculate time paths, pre-image plots and bifurcation diagrams. [1] The theory of dynamical systems has come a long way, but it is just yet not powerful enough to be able to describe the global behavior of the kind of nonlinear two dimensional delay system I face in this problem. However, the applied numerical methods help to aid the intuition for what is going on in the more general cases. It is certainly true that with these instruments we can only state results for sure for a grid of parameter values, and never for an infinite set. But I do not only deny the alternative of groping in the dark for these more complicated cases, but I consider it a very important step to gain knowledge of a system's behavior in this manner.

In the fourth part of this work I move closer to the original setup of the classical El Farol Bar problem, in which one observes apparently persistent fluctuations of the attendance behavior around the bar's capacity. Whether these fluctuations mirror a chaotic process, an attractor of a cycle of enormously large period, or rather a very slow convergence to a steady state solution has not been established yet. I make a first

[1] For some of these tasks I applied the simulation tool "MacroDyn" (see Böhm and Schenk-Hoppé [1998]), which I was very happy to work on as a developer together with Prof. Böhm. This tool provides a wide range of very powerful instruments for the numerical analysis of single and multidimensional dynamical systems in discrete time. I took Macrodyn from a pure linux console program to a cross platform application with a universal graphical user interface.

step to a better understanding of the very complicated dynamical system by analyzing the conditions concerning the existence of a steady state solution. The relative cardinality of the sets of predictor endowments with and without steady state solutions, representing the probability of its existence when randomly assigning prediction rules to agents, is calculated.

The motivation for this part is particularly nice, because it tries to find an answer to a question which was only possible to ask using simulation.

It is still a tough task for authors using these new methods of interest to gain wide acceptance in the community of economists. In this dissertation I want to take up the cudgels for the targeted use of technology in combination with algorithmic mathematics. Even very simple dynamic models often have properties that defy any explanation when purely using traditional analytical methods. I intent to demonstrate how simulation can aid intuition. The peculiar thing is that this intuition can potentially be fed back into mathematical and economical research to further boost the progress, the progress in answering questions of economic concern. The purpose of simulation is to not being forced to compromise on the substance of them.

2 Probabilistic Transitivity in Sports

2.1 Introduction

In many situations we are confronted with data about a certain set of objects which only include an array of comparisons about two of these objects at a time. Then all too often the task arises to find the "fairest" or "most legitimate" ranking among all of the objects in the considered set reaching from the "best" one to the "worst" one.

The probably most popular application of such paired comparisons is sports. In most sports games two opponents face each other in a duel. The result can be a win for one of the teams or, depending on the sport, also a tie.

An important attribute of a ranking is that it expresses a transitive relation between all of its objects. This means that if object or team A precedes B and B precedes C, it automatically implies that A precedes C. In contrast to this, paired comparison data can include circular relations, which seem to be inconsistent with this property. In a tournament it is possible that A beats B, B beats C, but C beats A. It is easy to imagine that as the number of teams rises, the probability of the occurrence of such inconsistencies rapidly increases. In the literature many suggestions have been made to overcome these inconsistencies and find a ranking with a good fit according to different concepts. A good overview of the classical models for obtaining rankings from data sets gives Brunk [1960]. One approach that deserves attention is the one proposed by Slater [1961]. Here the observed number of inconsistencies (in the sense mentioned above) is minimized. This nontrivial problem later became known as a particular form of the so called linear ordering problem. For a good survey on the linear ordering problem see for example Charon and Hudry [2010].

The major issue concerning the mentioned approaches is that despite all of them having some intuitive appeal, they seem to be rather arbitrary in finding the "right" ranking. The difference of our approach is that we assume that there actually *exists a correct ranking*. Of course we cannot directly observe it, but we can try to find the ranking which is most likely identical to it. To be more precise, we first of all make the assumption that the outcome of each match follows a trinomial distribution, with a fixed probability for a loss, a tie, and a win. These unobservable probabilities fulfill a certain form of transitivity. Applying the respective conditions we can then use a likelihood function to gauge the chance of the observed set of results given a particular set of probabilities. Maximizing this likelihood function while fulfilling the transitivity conditions answers the question about the most likely *correct* ranking.

In the literature there can be found plenty of works using the concepts of the so called weak and strong stochastic transitivity. These are definitions, which transfer the very intuitive concept of transitivity to the world of probabilities. Because in our model we consider ties and also home/away asymmetries, we are forced to define our own concept which goes beyond WST and SST.

At this point the optimization problem, which is the main object of the paper, is completely defined by the set of probabilities for three outcomes for each game, the likelihood function which shall be maximized, and finally the set of constraints imposed by the stochastic transitivity defined above. We are not the first authors trying to find a maximum likelihood ranking while applying probabilistic transitivity conditions. Thompson and Remage [1964] propose a similar problem of ranking pairwisely compared objects. The analysis is extended in Singh and Thompson [1968] by the incorporation of ties. However, Thompson uses only constraints of WST.[2] This contributes a lot to the simplicity of the problem and enables Decani [1969] to formulate it as a linear program and later propose in Decani [1972] a branch and bound algorithm to solve the problem even more efficiently.

Unfortunately the new set of constraints make things much more complicated. Increasing the number of teams leads to a huge number of constraints. And it is straightforward to see that the space of transitive probability sets of a particular dimension is not convex. So it is not a surprise that state of the art solvers do not succeed in finding the optimal solution to this non-linear, non-convex problem as soon as the number of teams is increased to more than 5 or 6.

This is why we split up the problem in two parts. The first one is to find the probability sets and the likelihood for a fixed ranking and the second one is to find the ranking with the greatest likelihood.

When the goal is to find probabilities for a fixed ranking, while still sticking to the transitivity definition, the constraints become much simpler.

The problem we arrive at is now very close to the so called isotonic regression problem in which a set of probabilities needs to be estimated, while one knows their order according to their magnitude (see Barlow and Brunk [1972] or Van Eeden [1996]

[2]After the incorporation of ties he naturally can't use the WST constraints, but has to alter his concept. However, it still differs substantially from ours which makes a comparison very difficult.

for an overview). A reference much closer to the subject of this paper is Brunk [1955]. Here the random variables (in our case the match results) are assumed to follow a distribution belonging to an exponential family. The single distribution parameter follows a function depending monotonically on potentially multiple variables. These variables would in this work correspond to the two teams that are playing. The very efficient method developed in this paper later became known as the pool adjacent-violators algorithm (PAVA). The major difference of Brunk's paper to our approach is that the trinomial distribution we will be using does not belong to the exponential family he is referring to. It also has not one but two distribution parameters. So we are very unfortunate to not being able to apply the PAVA. To be able to estimate not only ordered binomial but also ordered multinomial distribution parameters Jewell and Kalbfleisch [2004] developed a modification of this algorithm, the so called m-PAVA. This algorithm is technically able to solve our first problem, but turns out to be very inefficient and slow. But there is an alternative. Lim et al. [2009] find that a program of the kind we are facing can be formulated as a geometric program, which then can be transformed into a convex program. By applying state of the art interior point solvers, we are then able to find a solution very efficiently.[3]

The second part of the problem is more complicated. If we increase the number of teams, the possible number of orderings rises very quickly. For 4 teams there are 24 possibilities, for 5 teams there are 120 and for 18 teams there are more than 6×10^{15}. But even if we're not able to find the optimal ranking, we are still able to compare different rankings created by the application of empirically relevant ranking systems. And this is exactly what we do in the empirical subsection of the paper. Among the candidates are the classical "three points for a win" and "two points for a win" systems from soccer and also the Elo system applied e.g. in chess.

To be able to make a good judgment about the true quality of the systems when applied to different sports, we develop a statistical test. It assumes the trueness of the null hypothesis stating that one of two ranking systems under consideration is able to find the correct ordering. Then we estimate all the probabilities and simulate a test statistic. Combined with the empirically observed likelihoods, we are then ideally able to reject the null hypothesis which lets us state that here the considered system is not able to generate the correct ranking.

The paper proceeds as follows. In subsection 2.2 the formal model is introduced. In subsection 2.4 the problem solution for a known ranking is described, before in part 2.5 we discuss strategies for finding optimal rankings. The next two chapters then describe the sports data and provide a thorough empirical analysis. Section 2.9 concludes.

[3] In Lim et al. [2009] investigations geometric programming is more than 150 times faster.

2.2 Setup

All sports described above have in common that n teams are competing in a number of repeated one-on-one games. The results of these games should be aggregated to one final complete ranking. Let p_{ij} be the probability that team i beats team j.
Naturally, we must have $\forall\ i, j \in \{1, \ldots, n\}$

$$\begin{aligned} p_{ij} &\in [0,1] \\ p_{ij} + p_{ji} &\leq 1 \end{aligned} \tag{1}$$

It can be observed that playing at home (meaning in i's stadium) and playing away makes a difference to the winning probabilities. Therefore we introduce different probabilities for at home and away games: p_{ijh} is the probability that i beats j at home and p_{jia} that team j wins against i at i's stadium.
Therefore (1) changes to

$$p_{ijh} + p_{jia} \leq 1 \qquad \forall i, j \in \{1, \ldots, n\}$$

Since in many sports there exists the possibility of a draw, there is no strict equality. In fact, the probability of a draw is

$$q_{ijh} = q_{jia} = 1 - p_{ijh} - p_{jia}.$$

In this paper, we want to make only one assumption concerning a set of those probabilities. This assumption is based on the concept of weak and strong stochastic transitivity, which formalizes the very intuitive thought that if team i is better than team j and j is better than k then i has to be better than k, as well. In a model of symmetric paired comparison without ties this can be translated fairly easily into stochastic terms.

$$p_{ij} \geq 1/2 \quad \wedge \quad p_{jk} \geq 1/2 \implies p_{ik} \geq 1/2 \tag{WST}$$

$$p_{ij} \geq 1/2 \quad \wedge \quad p_{jk} \geq 1/2 \implies p_{ik} \geq \max\{p_{ij}, p_{jk}\} \tag{SST}$$

Where (SST) is equivalent to

$$p_{ij} \geq 1/2 \implies p_{ik} \geq p_{jk}.$$

The concept of stochastic transitivity has been widely used in the literature on paired comparisons, especially in the 60s and 70s (see e.g. Tversky [1969], Chung and Hwang [1978], Morrison [1963] or Davidson and Solomon [1973]).

The introduction of ties and in addition to that the introduction of a home/away asymmetry forbid to use this concept directly. (SST) is best interpreted by saying "if team i is better than team j, it has to have a higher chance of beating any third team k". But in a world with draws and home advantage we cannot interpret "being better" as $p_{ij} > 1/2$. Thats why one has to alter this point. This is done in the following definition.

Definition 1 (Transitivity)**.** A set of probabilities will be called transitive if the following holds for every $i, j, k, l \in \{1, \ldots, n\}, x, y \in \{a, h\}$ and $\exists i', j', k', l' \in \{1, \ldots, n\}$:

$$\begin{aligned} p_{ikx} \geq p_{jkx} &\Leftrightarrow p_{ily} \geq p_{jly} \\ p_{kix} \geq p_{kjx} &\Leftrightarrow p_{liy} \geq p_{ljy} \\ p_{i'k'x} > p_{j'k'x} &\Rightarrow p_{l'j'x} > p_{l'i'x} \end{aligned} \tag{2}$$

The set of transitive probability sets will be called $\mathcal{T}$.

The first proposition shows that our concept is in fact a generalization of SST.

Proposition 1. *Definition 1 is, when assigning* 0 *to all draw probabilities and ignoring away/home differentiation, equivalent to (SST).*

Definition 2 (Transitive Ranking)**.** A ranking will be called transitive if for all i ranked above j the following holds:

$$p_{ikh} \geq p_{jkh},\ p_{kih} \leq p_{kjh},\ p_{ika} \geq p_{jka},\ p_{kia} \leq p_{kja} \quad \forall\, k \in \{1, \ldots, n\} \backslash i, j$$

The set of probability sets according to this definition will be called $\mathcal{T}'$.

The fact that a transitive ranking has a set of transitive probabilities and every set of transitive probabilities has a transitive ranking is established in the following Proposition.

Proposition 2. *A set of probabilities P is in $\mathcal{T}$ if and only if it is in $\mathcal{T}'$.*

For the proofs of propositions 1 and 2 see appendix A.

The structure of the constraints and hereby the problem we have to solve becomes clearer, if we write down the set of p_{ijx} values in matrix form and add the constraints using one particular ranking.

$$\begin{pmatrix} * & \leq p_{12h} & \leq p_{13h} & \leq \cdots & \leq p_{1nh} \\ \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} \\ p_{21h} \leq & * & \leq p_{23h} & \leq \cdots & \leq p_{2nh} \\ \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} \\ \cdots \leq & \cdots \leq & \cdots \leq & \cdots \leq & \cdots \\ \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} \\ p_{n1h} & \leq p_{n2h} & \leq p_{n3h} & \leq \cdots \leq & * \end{pmatrix}, \begin{pmatrix} * & \leq p_{12a} & \leq p_{13a} & \leq \cdots & \leq p_{1na} \\ \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} \\ p_{21a} \leq & * & \leq p_{23a} & \leq \cdots & \leq p_{2na} \\ \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} \\ \cdots \leq & \cdots \leq & \cdots \leq & \cdots \leq & \cdots \\ \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} \\ p_{n1a} & \leq p_{n2a} & \leq p_{n3a} & \leq \cdots \leq & * \end{pmatrix}$$

Figure 1: Transitivity matrices for home and away probabilities

2.3 The Optimization Problem

By assumption, each outcome in a set of paired comparisons is trinomially distributed. The probability distribution is

$$Pr\{x_{ij} = w_{ij}\} = p_{ijh}^{w_{ijh}} p_{jia}^{w_{jia}} (1 - p_{ijh} - p_{jia})^{m_{ij} - w_{ijh} - w_{jia}} \tag{3}$$

where w_{ij} is the vector consisting of the elements w_{ijh} and w_{jia}. x_{ij} is the analogously defined vector of a realization of the corresponding random variable. (3) tells us the probability of a certain outcome of a game between two particular teams in one particular stadium. By taking the exponential of the natural logarithm of the left side, we can write the above equation as

$$\begin{aligned} Pr\{x_{ij} = w_{ij}\} = &\exp(w_{ijh} \ln(p_{ijh}) + w_{jia} \ln(p_{jia}) \\ &+ (m_{ij} - w_{ijh} - w_{jia}) \ln(1 - p_{ijh} - p_{jia})) \end{aligned}$$

Let

$$F[x_{ij}, p_{ij}] := w_{ijh} \ln(p_{ijh}) + w_{jia} \ln(p_{jia}) + (m_{ij} - w_{ijh} - w_{jia}) \ln(m_{ij} - p_{ijh} - p_{jia})$$

The likelihood of a set of particular results to occur will be

$$Pr\{(x_{ij}, \dots, x_{i'j'}) = (w_{ij}, \dots, w_{i'j'})\} = \exp(F[w_{ij}, p_{ij}] + \dots + F[w_{i'j'}, p_{i'j'}])$$

Let E be the set of all valid (i, j) combinations $E = \{(i, j)|i, j \in \{1, ..., n\}, i \neq j\}$. Then (2) implies that, in order to maximize the likelihood of a set of outcomes, we have to solve the following maximization problem

$$\max_{p_{ij}} J[p] = \sum_{(i,j)\in E} F[w_{ij}, p_{ij}] \qquad s.t. \quad \{p_{ijx} \,|\, (i, j) \in E, x \in \{h, a\}\} \in \mathcal{T}$$

This is a rather complicated optimization problem, first because the objective function (the log of the likelihood function) is not linear, and second because we have a huge number of non-linear constraints, which make the space we are dealing with highly convoluted and non-convex. We can achieve convexity by fixing a particular ranking of teams. In this case we face a total number of $2(2(n - 2)n + (n - 1))$ constraints. Note that a simple transformation of parameters cannot help us making the problem convex. Also it cannot make the problem linear after fixing a ranking. In this highly simplified case, where the untransformed constraints can be expressed in a linear form, a logarithmic transformation would make the objective function linear but take away linearity from the constraints. More details on this will follow in subsection 2.4.2.

2.4 Optimization under a known ranking

Note that the probabilities depicted in Figure 1 are only the constraints that apply for one ranking. So the optimization problem can be split into first finding the optimal (i.e., likelihood maximizing) probabilities that satisfy the monotonicity constraints from the matrix and second finding the best ranking. It should become clear that if we consider the indices as variables of the functions $p_h(i,j)$ and $p_a(i,j)$, then this function is monotone non-increasing in the first variable and monotone nondecreasing in the second one. In the considered case the two matrices are only insofar dependent on each other as the sum of an element of the upper right half of the first matrix depicted in Figure 1 and the corresponding element of the bottom left half of the second matrix has to be less than or equal to unity.

2.4.1 Transitivity without draws

Now, let us again compare the original problem to the one in the much simpler case without ties. Here, the problem of estimating the probabilities is much easier. Given the above assumptions, the number of wins when two teams play each other a particular amount of times follows an elementary binomial distribution. This instant allowed Brunk [1955] to develop an algorithmic approach, building the foundation of what later became known as the Pool Adjacent Violators Algorithm (PAVA). See also Brunk [1960] for an application to paired comparisons. It follows a short description of the estimation procedure.

A lower interval is the set of all points (i,j) for which $i \geq i'$, $j \leq j'$. So it includes a point in one of the above matrices as well as all the points in its south-west quadrant. An upper interval is analogously defined. A lower layer is a union of lower intervals and an upper layer is a union of upper intervals.

The procedure is now to find the largest upper layer within which the average number of wins is maximized. That is, we have to find an upper layer with the property that the number of wins divided by the number of games it comprises is maximal. For each p_{ij} in this layer the maximum likelihood estimate under the monotonicity constraints we defined is this average number of wins. Next step is to repeat the procedure on the remaining set of the matrix of results.

To illustrate the approach, consider the following example of a tournament of 4 teams in which each two teams played each other once. (For simplicity we only consider home games of the row teams, here.)

$$\begin{pmatrix} * & 0 & 1 & 0 \\ 1 & * & 1 & 0 \\ 0 & 0 & * & 1 \\ 1 & 1 & 0 & * \end{pmatrix} \qquad \begin{pmatrix} * & \frac{1}{2} & \frac{3}{5} & \frac{3}{5} \\ \frac{1}{2} & * & \frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} & * & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & * \end{pmatrix}$$

Figure 2: PAVA example: Result matrix and p-Matrix

On the left there is the matrix of tournament results. The solid line shows the first upper layer with an average number of wins of 3/5, giving us the p-value listed in the right matrix. The second layer includes all the numbers above and to the right of the dashed line. Here the average value is 1/2 and so on. Having the p-Matrix at hand, it is straightforward to calculate the maximum likelihood of the tournament to be 0.03888.

Please note that this algorithm, while being very efficient at finding the probabilities for a fixed ranking, does not help finding the optimal permutation of the teams. To find it, one is still forced to apply this algorithm $4! = 24$ times for this example.

Unfortunately including the chance of draws forbids to use this very simple and efficient procedure. In the next subsection we show how to arrive at a solution nonetheless.

2.4.2 Solution process for the case including draws

Again focusing on the part of the problem where the ranking is already fixed, allowing for ties makes the solution procedure much more complicated. Now, the task is not to estimate ordered binomial, but rather ordered trinomial distribution parameters. Jewell and Kalbfleisch [2004] developed an extension of the PAV algorithm discussed above. The Authors call this algorithm the modified- or m-PAV algorithm. In the process the problem is iteratively broken down into many one dimensional optimization problems. Since the number of these subproblems grows very quickly with the number of teams and also the number of adjacent violators, the required computational effort also does. This is the main reason for Lim et al. [2009] to reconsider the problem, finding that it can be formulated as a geometric program. Then it can be transformed into a convex optimization problem, for which one can find a global solution very efficiently with the help of e.g. interior-point algorithms. Lim et al. [2009] compare the computational efficiency of the two approaches and find that geometric programming is much faster than the m-PAV algorithm. These findings facilitate the choice for us in this paper.

Let us take a look at it in detail. We define w_{ijh} to be the empirically observed number of times team i beats team j at home and $t_{ijh} = t_{jia}$ as the number of times team i ties team j. Let m_{ij} be the total number of games between i and j at i's stadium. Consider the optimization problem for a fixed ranking in its raw form.[4]

$$\begin{aligned}
\min_{p} \quad & \prod_{(i,j)\in E} p_{ijh}^{-w_{ijh}} p_{jia}^{-w_{jia}} (1 - p_{ijh} - p_{jia})^{-(m_{ij}-w_{ijh}-w_{jia})} \\
s.t. \quad & \frac{p_{ijx}}{p_{ikx}} \leq 1 \quad \forall \quad (i,j) \in E, (i,k) \in E, j \succeq k, x \in \{h, a\} \\
& p_{ijh} + p_{jia} \leq 1 \\
& p_{ijx} \geq 0
\end{aligned} \tag{4}$$

This is a geometric program. The objective function as well as the left side of the first constraint are monomial and the left side of the second constraint are polynomials. The third constraint reflects the fact that the domain of our objective function is positive, as in all geometric programs. The program can easily be transformed to a convex optimization problem.

$$\begin{aligned}
\min_{p} \quad & \sum_{(i,j)\in E} -w_{ijh} \ln(p_{ijh}) - w_{jia} \ln(p_{jia}) - (m_{ij} - w_{ijh} - w_{jia}) \ln(1 - p_{ijh} - p_{jia}) \\
s.t. \quad & \ln(p_{ijx}) - \ln(p_{ikx}) \leq 0 \quad \forall \quad (i,j) \in E, (i,k) \in E, j \succeq k, x \in \{h, a\} \\
& \ln(e^{\ln(p_{ijh})} + e^{\ln(p_{jia})}) \leq 0
\end{aligned}$$

It is straightforward to show that the logarithm of a posynomial is convex in $\ln(x)$, which proves the fact that this is indeed a convex program. To solve this kind of program we make use of the software package IPOPT (see Wächter and Biegler [2006]). In addition to the program it requires the input of the Jacobian and Hessian matrices of the constraints. It then applies an interior point algorithm and solves our problem very efficiently, given a fixed ranking. This allows us to compare different ranking systems.

2.5 Ranking methods

2.5.1 The Linear Ordering Problem

At this point, before proceeding with our efforts of finding solutions to the proposed problem, it makes sense to consider a related, but as we will see, clearly different problem. As one of the classical combinatorial optimization problems the linear ordering problem (LOP) attracted many authors resulting in a huge amount of literature on it. See for example Marti and Reinelt [2011] for a good introduction to the problem as well as a review of suitable algorithms. Also feel referred to Charon and Hudry [2010] for a detailed survey.

If one is given a complete directed graph $D_n = (V_n, A_n)$ with arc weights c_{ij} for every ordered pair $(i, j) \in V_n \times V_n$, the linear ordering problem consists of finding

[4] The only change made is the conversion to a minimization instead of a maximization problem.

an acyclic tournament T (which corresponds to a permutation of the set of objects or teams), which maximizes the sum of the arcs which are in agreement with the direction of the arcs from D_n. So the sum $\sum_{(i,j)\in T} c_{ij}$ has to be maximal. Equivalently one could formulate the problem as minimizing the so called remoteness corresponding to minimizing the arc weights pointing in the opposite direction.

A more illustrative representation of the problem is the maximization of the sum of superdiagonal elements in a matrix by manipulating the row/column ordering. This is the so called Triangulation Problem.

The reader might already be able to grasp a sense of similarity here. To establish a direct connection between the LOP and the problem dealt with in this paper, consider a situation where we fix the probabilities of wins and losses at homogeneous values below and above the diagonal of the matrix independently of which teams are in question. This means we set $p_{ijh} = \overline{p}_h$ above diagonal and $p_{ijh} = \underline{p}_h$ below it and analogously for the away probabilities. Let us consider the case where $\overline{p}_h > \underline{p}_h$ and $\overline{p}_a > \underline{p}_a$. Remember that the goal is to maximize

$$\begin{aligned}
&\sum_{(ij)\in E} w_{ijh}\ln(p_{ijh}) + w_{jia}\ln(p_{jia}) + (1 - w_{ijh} - w_{jia})\ln(1 - p_{ijh} - p_{jia}) \\
=&\sum_{(ij)\in \overline{E}} w_{ijh}\ln(\overline{p}_h) + w_{ija}\ln(\overline{p}_a) + t_{ijh}\ln(1 - \overline{p}_h - \underline{p}_a) \\
&+\sum_{(ij)\in \underline{E}} w_{ijh}\ln(\underline{p}_h) + w_{ija}\ln(\underline{p}_a) + t_{ijh}\ln(1 - \underline{p}_h - \overline{p}_a)
\end{aligned}$$

where $\overline{E}$ and $\underline{E}$ represent the sets of elements above and below the diagonals, respectively.

The results of a particular team in his two games against a particular opponent makes a certain contribution to the sum. This contribution might be higher because it is multiplied by higher probabilities if the records are superdiagonal. So we are confronted with a triangulation problem just like the one described above. Many Authors suggest an application of the LOP in sports rankings (see e.g. Marti and Reinelt [2011]). And since it indeed seems well suited for our purposes, we will include it in the analysis.

2.5.2 Branch and Bound algorithm

Branch and Bound Algorithms are particularly well suited for combinatorial optimization problems. As opposed to the other methods we are proposing, this one leads with certainty to the optimal ranking. For an early survey on Branch and Bound methods feel referred to Lawler and Wood [1966].

The following steps describe the execution of the algorithm:

1. Take the next team from the list of all teams
2. Put it in the list of previously selected teams at each possible position
3. For each position calculate an upper bound $\overline{L}$ above which the likelihood cannot rise going further down the tree (i.e. after all teams were inserted)
4. Leave the team at the position with the highest upper bound
5. If all teams are inserted go to 6., otherwise go to 1.
6. Compare the likelihood to the best one found so far
7. Cut of the tree at all nodes where $\overline{L}$ is below the best likelihood
8. Go to the best of the lowest hanging nodes that could not be deleted and start with 1. from there

Before asking how the upper bound estimate $\overline{L}$ is calculated, lets first focus on the procedure itself. To understand it better, consider a simple example of three teams "a" "b" and "c".

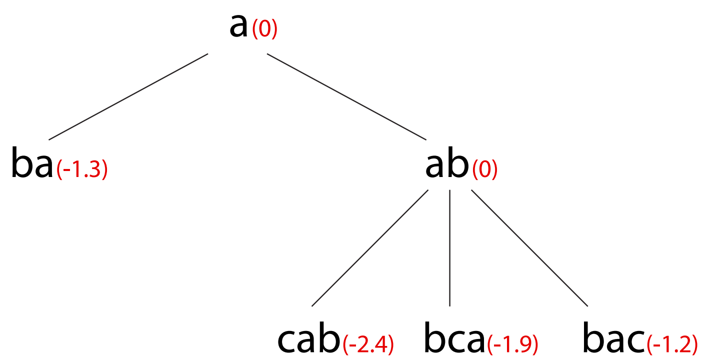


Figure 3: Branch and Bound Algorithm: An example

We start by inserting team "a". The upper bound for the log likelihood at this point is still 0, which is indicated in brackets in Figure 3. Then team "b" is added at each possible position. We see upper bounds of -1.3 and 0, respectively. So we continue by leaving "b" at the second position and then insert team c at each possible location. Since the example only includes three teams, we can now calculate the value of the real objective function instead of calculating $\overline{L}$ the way it was done previously. The highest value of the objective function is found using the ordering "bac". This value of -1.2 now enables us to cut of all hanging nodes, which have an upper bound below -1.2. So we cut of the tree at "ba", since there is no way, we could get a better likelihood going down the tree from this node. It is easy to see how the procedure can save computational effort (even in this tiny example) compared to calculating the MLE for all permutations.

The upper bound $\overline{L}$ is calculated as follows. First the optimization problem (for a fixed ranking) is applied to the teams that have been inserted so far.

Lemma 1. *Adding an additional team into an existing ranking without changing the relative order of the already existing teams can not increase* $\overline{L}$.

Proof. It is trivial to see that adding a variable (team) to the maximization problem without adding additional constraints (results) does not change the maximum likelihood (i.e., we are multiplying by 1). Now, adding additional constraints without changing the objective function or changing the other constraints can never increase the maximum likelihood and therefore the new $\overline{L}$ has to be less or equal to the $\overline{L}$ with 1 team less. □

At this stage we could already use this maximum likelihood of the considered subset of teams for $\overline{L}$. But there is a way to reduce the upper bound even further and thereby make the algorithm a lot more efficient. For each team that is still pending to be inserted we already know a subset of the constraints that will be applied to the corresponding probabilities when going further down the tree, no matter where this particular team will be inserted. Consider a situation where teams $1, ..., k$ have already been inserted. Now, for each team $l \in \{k+1, ..., n\}$ we know that $p_{ilx} \leq p_{i'lx}$ and $p_{lix} \geq p_{li'x}$ for every $i, i' \in \{1, ..., k\}$ and $x \in \{h, a\}$ such that i is ranked above i'. For $k = 3$ this is depicted in Figure 4.

$$\begin{pmatrix} * & p_{12h} & p_{13h} \\ p_{21h} & * & p_{23h} \\ p_{31h} & p_{32h} & * \end{pmatrix} \quad \begin{matrix} p_{1lh} \\ \vee\!| \\ p_{2lh} \\ \vee\!| \\ p_{3lh} \end{matrix} \qquad \begin{pmatrix} * & p_{12a} & p_{13a} \\ p_{21a} & * & p_{23a} \\ p_{31a} & p_{32a} & * \end{pmatrix} \quad \begin{matrix} p_{1la} \\ \vee\!| \\ p_{2la} \\ \vee\!| \\ p_{3la} \end{matrix}$$

$$p_{l1h} \leq p_{l2h} \leq p_{l3h} \qquad\qquad p_{l1a} \leq p_{l2a} \leq p_{l3a}$$

Figure 4: Calculation of the upper bound $\overline{L}$

For every team that has not been inserted yet, we know this subset of constraints. So we have another optimization problem for each team. The results of these optimization problems (, having the form of log likelihood values) can be added to the value $\overline{L}$.

As mentioned, the algorithm leads for sure to the optimal ordering. The drawback is that despite of the fairly sophisticated upper bound that we are suggesting, it is still not efficient enough to be applied to tournaments with more than 11-12 teams[5]. Nevertheless the branch and bound algorithm deserves to be included in the empirical subsection of this paper.

[5]Depending on the structure of results in the tournament, as well as the users patience.

2.5.3 Tabu Search

The third ranking algorithm we are suggesting is a heuristic search method. The advantage of tabu search lies in the combination of local search and a diversification mechanism. The local search systematically browses through neighborhood solutions, checking for a possible improvement of the objective value. That the algorithm doesn't get stuck in local optima is assured by a memory structure, avoiding previously visited regions of the solution space, giving a tendency for diversification. A reference with a related application is Laguna et al. [1999].

The algorithm works as follows:

1. Start from a randomly generated order of teams (call it ρ)
2. Calculate the maximum likelihood for the current ranking $L(\rho)$
3. Randomly select a team that is not on the "Tabu List" and remove it from the order
4. Insert the team at position i and calculate difference between the maximum likelihood of the new and the original ranking: $MoveValue = L(\rho') - L(\rho)$
5. Repeat 4. for $1 \leq i \leq n$ except for the original position
6. Insert the team at the position with the highest $MoveValue$
7. Put the team on the "Tabu List" so that it won't be selected for the next "TabuTenure" iterations
8. Go to 2.

Basically what the algorithm does is taking a team from the ranking and trying out every possible position for it, except for the original one. Important is that the best among the new positions is selected even if the "$MoveValue$" is negative. Different convergence criteria are possible for the procedure. Since in our analysis the computational effort in each iteration is fairly large, we use a fixed number of iterations for the algorithm, so that we can best control the amount of time it takes for the algorithm to finish.

2.5.4 Popular ranking methods

Finally, we want to take a more practical approach and compare different ranking systems, which have been used in different fields of sports. We chose the 3 point system (also known as "Three points for a win"), which awards zero points for a loss, one point for a draw and three points for a win. The sum of the points together with the goal difference as a tie breaker then decides upon the ranking. This system has been used in most soccer leagues since it was officially adopted in 1995 by FIFA.[6]

[6]England introduced the system already in 1981. The first time it was used internationally was in the 1994 World Cup finals.

Before the 3 point system was introduced, the analogously structured 2 point system had been widely used in soccer. Here the only difference is that two instead of three points are awarded for a win.

These two systems are fairly easy to apply and (unfortunately) also very similar to each other. So as a third candidate for a ranking scheme, we use the Elo rating system. The Elo rating system is a system invented by Elo [1978] originally intended as a rating system for chess. Today it is not only used as for different chess organizations, including the FIDE and the United States Chess Federation, but also the European Go Federation, many different computer games and even the National Collegiate Athletic Association, the organization which is responsible for the organization of many American college sport programs, notable college football and college basketball. The main differences to the three points for a win is that it factors in the strength of the opponent: winning against a strong opponent yields more points than winning against a weak one. This results in the major weakness for our needs: a relatively high number of games is needed to give meaningful results and the order in which the teams play matters a lot.

2.6 Comparing the explanatory power of rankings

To further enhance our comparative analysis of ranking systems, we will apply a statistical hypothesis test. In this test two ranking systems are compared, call them system a and system b. We solve problem (4) for both rankings. The p-matrix calculated with the constraints generated by one of the rankings, say system a, will yield a likelihood for the observed season at least as great as the one generated by the other one, say system b.

$$L(\hat{P}_a(w)|w) \geq L(\hat{P}_b(w)|w)$$

where $\hat{P}_a(w)$ and $\hat{P}_b(w)$ are the estimated p-matrices. So we could say, a allows one to calculate a p-matrix with a higher explanatory value, so it must be the better system. But in fact, it might have happened by chance, that this ranking system performed better than the other one. The central question concerns the degree of the odds that a performed better than b by the observed amount. Let us define the likelihood ratio as follows

$$LR_{a,b} = log(L(\hat{P}_a(w)|w))) - log(L(\hat{P}_b(w)|w)).$$

We assume a Hypothesis H_0 stating that "b is the correct ranking system". Correct means that it allows us to estimate the right p-values. Using these probabilities for each match, we simulate a complete season and get a new tournament $\hat{w}$ for which we again calculate the likelihoods given $\hat{P}_a(w)$ and $\hat{P}_b(w)$. This way a few thousand seasons are simulated and we receive a distribution over the difference of the log-likelihood. In the ideal case, the probability (suggested by the simulated distribution) of the observed difference between the likelihoods is small enough to be able to reject H_0 with this very test size α.

$$P[LR_{a,b}(\hat{w}) \leq LR_{a,b}(w)] < \alpha$$

So, roughly what we do is assuming that one of the systems is correct, and then we try to reject this hypothesis, by showing that the probability for another system to be as much better as empirically observed is very small.

The weakness of this approach is pretty obvious. We are only able to reject the hypothesis that a particular system is perfectly correct. Even though the data allows us to make a guess about it, the test does not allow us to make a statement about which of the two systems in consideration is actually better. So in fact, both of the systems might be incorrect, but we are only able to reveal the inadequacy of one of them.

2.7 Data

We obtain the data from different sources. For soccer we focus on the German Bundesliga and the British Premier League. For the former we have data from the seasons 1968/69 till 2012/13, for the latter the sample from the seasons 1997/98 till 2012/2013. Additionally, we included the season 2012/2013 from the the Austrian Bundesliga, because of its advantage of having only 10 Teams. The scores for all matches, which are translated to win/draw/loss data, are obtained from the website www.kicker.de. Notable about the soccer data is that each team plays each other team exactly once at home and once away in each season. This introduces a symmetry to the data which, even though it is not necessary, might be considered as desirable and certainly influences the results of our analysis.

Regarding tennis, we face a different situation. Since there is no league of players in which each player faces another one a fixed number of times per season we have to go a different way. We will focus only on the top 10 players according to the official ATP ranking at the end of each year (obtained from http://www.atpworldtour.com). Then we collect the data for all the ATP matches played in this season from http://www.tennis-data.co.uk. Of course these data sets will be highly asymmetric, because some players play against each other more that once, and some might not face each other at all during a season. Another special fact about the tennis data is that we don't have a real home away situation.[7] Even more importantly, in tennis there is no possibility

[7] Of course some players might feel more at home when a tournament is taking place in their country of origin. But since this is very different to the situation of a team playing in its very own stadium in its city, we will assume that every game takes place on neutral ground.

of ties. So we face only a binomial distribution for the outcome of each match which considerably facilitates the optimization procedure.

Concerning American football, we will focus exclusively on the NFL. We have data on the scores of every NFL game since 1978 from the website http://www.repole.com/sun4cast/data.html. The NFL comprises from 28 in the season 1978 to 32 teams in 2012. This is by far the largest group of teams. Almost naturally it follows that among the samples there is a huge number of teams that don't face each other during a season. Which team is playing which is determined by a complicated system, which shall not be further discussed here. In football draws are possible, but only happen very rarely. Along with the fact that American football enjoys great popularity, this makes NFL data very interesting for our analysis.

2.8 Empirical Analysis

We now want to apply the presented methods to real data from sports. Countless different types of sports are imaginable and probably the readers preferences for what he would like to see in this section are very heterogeneous. Nevertheless for reasons of space we want to focus on three types, namely soccer, tennis and American football. The main questions we seek to answer are, "Is there a tendency for one of the ranking schemes to be superior to the others according to the criterion we defined?", "If yes, which one is it?", "Does it depend on the type of sport?" and finally "Are we able to improve on the rankings found by the simple ranking methods using one of the algorithms presented in subsection 2.5?"

2.8.1 Soccer in Austria: Finding an Optimal Ranking

With the branch and bound algorithm we find our selves equipped with a very powerful instrument to find the optimal ranking. Unfortunately this algorithm can only be applied to sets of teams that have a limited size. The first object of our investigation shall be the Austrian Bundesliga. Its size of 10 teams enables us to apply the discussed bnb-method. During a season each team plays against each other team four times, two times at home and two times away. This is different from most other soccer leagues, but doesn't increase the computational complexity by much. Here, we consider the season 2012/2013. To draw a first comparison between the performances of the other ranking schemes, Table 1 shows the maximum likelihoods that have been calculated.

Method	BnB	2-Point	3-Point	LOP	Elo	Tabu-Search
MLE	-129.844	-131.742	-135.561	-140.024	-131.703	-130.465

Table 1: Log likelihood values for the Austrian Bundesliga 12/13

While the ranking corresponding to the solution to the linear ordering problem gives a relatively low likelihood, the two point system as well as the Elo-system seem to explain the results a lot better. Nevertheless, none of the systems generates the optimal ranking found by the branch and bound algorithm. The ranking produced by the Tabu Search gives a higher likelihood than all the systems, but still is not the optimal one.

Figure 5 compares the optimal ranking that we found with the actually applied order, namely the 3-Point ranking. One can see that there are indeed some differences. Perhaps most striking is that in this season SV Mattersburg was relegated, while in the optimal ranking Wacker Insbruck would have been relegated. This team was actually ranked 8th.

Figure 5: Rankings resulting from 3-point system and Branch and Bound algorithm

Unfortunately most leagues are larger than the Austrian Bundesliga. The resulting computational effort makes it virtually impossible for us to find optimal rankings. which is why in the next subsection we focus on the other methods and compare the different ranking schemes across panel data from different leagues in different sports.

2.8.2 Ranking Systems and Maximum Likelihood Estimates

To give the reader an impression of how a matrix of estimated outcome probabilities for each game looks like after the optimization, Figure 6 depicts the probabilities for home game wins for the Bundesliga season 2012/13 estimated using the "three points for a win" system.

	0.18	0.31	0.69	0.8	0.8	0.8	0.8	0.8	1	1	1	1	1	1	1	1	1	1
	0	0.31	0.69	0.69	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.92	0.92	0.92	0.92	0.92	1
	0	0.31	0.46	0.69	0.69	0.69	0.69	0.69	0.71	0.71	0.71	0.71	0.71	0.92	0.92	0.92	0.92	1
	0	0.31	0.31	0.46	0.46	0.46	0.52	0.52	0.71	0.71	0.71	0.71	0.71	0.92	0.92	0.92	0.92	0.95
	0	0.31	0.31	0.46	0.46	0.46	0.46	0.52	0.7	0.7	0.7	0.7	0.7	0.86	0.92	0.92	0.92	0.95
	0	0.31	0.31	0.46	0.46	0.46	0.46	0.49	0.49	0.49	0.49	0.61	0.7	0.85	0.85	0.85	0.85	0.85
	0	0.31	0.31	0.46	0.46	0.46	0.46	0.49	0.49	0.49	0.49	0.57	0.7	0.85	0.85	0.85	0.85	0.85
	0	0	0.31	0.31	0.31	0.46	0.46	0.48	0.49	0.49	0.49	0.53	0.58	0.58	0.85	0.85	0.85	0.85
	0	0	0.31	0.31	0.31	0.31	0.46	0.47	0.47	0.49	0.49	0.49	0.49	0.58	0.85	0.85	0.85	0.85
	0	0	0.31	0.31	0.31	0.31	0.31	0.41	0.41	0.45	0.49	0.49	0.49	0.49	0.49	0.85	0.85	0.85
	0	0	0.31	0.31	0.31	0.31	0.31	0.41	0.41	0.41	0.43	0.44	0.44	0.44	0.44	0.44	0.44	0.44
	0	0	0.19	0.31	0.31	0.31	0.31	0.41	0.41	0.41	0.41	0.42	0.42	0.43	0.44	0.44	0.44	0.44
	0	0	0.19	0.19	0.19	0.19	0.31	0.38	0.41	0.41	0.41	0.42	0.42	0.42	0.44	0.44	0.44	0.44
	0	0	0.075	0.19	0.19	0.19	0.31	0.36	0.36	0.36	0.36	0.36	0.42	0.42	0.42	0.42	0.44	0.44
	0	0	0.056	0.19	0.19	0.19	0.19	0.19	0.29	0.29	0.29	0.36	0.36	0.42	0.42	0.42	0.44	0.44
	0	0	0.037	0.19	0.19	0.19	0.19	0.19	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.36	0.44	0.44
	0	0	0	0	0	0.19	0.19	0.19	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.35	0.44
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.094	0.17	0.3

Figure 6: MLE for p_{ijh} using 3-point system

Generally, a striking feature about the structure of the estimated probably matrices is the occurrence of homogeneous values in certain areas of the matrix, reminding of the layer structure discussed in section 2.4.1. Remarkable in this particular matrix is the large number of "1"s in the upper right corner and "0"s in the lower left corner. The reader might be tempted to argue that these values are fairly unrealistic, because intuition tells us that even if the strongest team plays the weakest one, in the current case Bayern München against Greuter Fürth, the chance of the former to win against the latter will be high, but never 100%. The point is that we only hold this intuition, because probably at some point in the past we have seen top teams occasionally loosing against teams that were ranked very low. But since this kind of information is not part of our estimation procedure, it is only natural that estimates look like this.[8]

Next, we want to try to improve this ranking by using one of the algorithms presented in section 2.5. Unfortunately the sample of 18 teams is too large for an application of the branch and bound algorithm, which would technically allow us to find the optimal ranking. So we use the Tabu search method, which we run for 100 iterations. The resulting ordering as well as the corresponding maximum likelihoods are shown in Figure 7.

[8]We have to add, that in case the reader has seen Bayern München play in the season 2012/13, he most certainly would agree that estimating some probabilities in the right of the upmost row with a value of 1 most probably only involves a very small error.

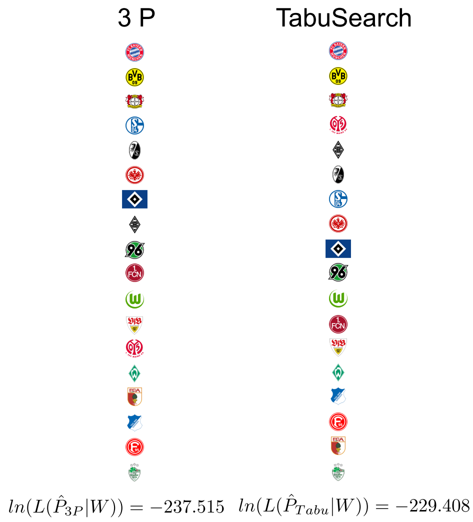


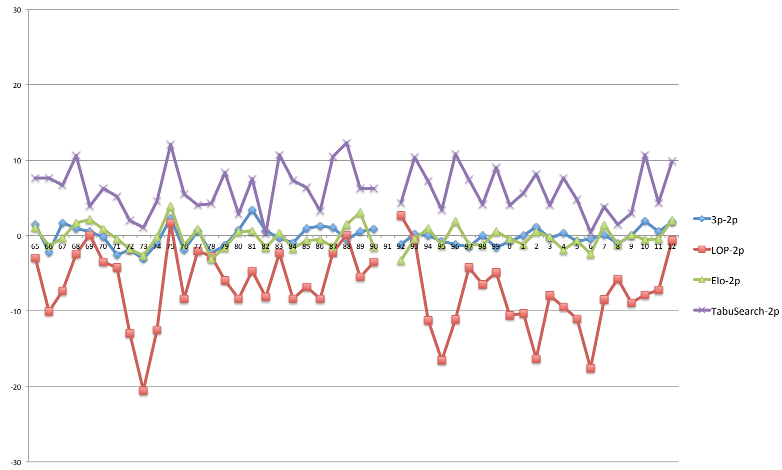
Figure 7: Rankings resulting from 3-point system and Tabu Search

The Tabu Search finds a ranking that is partly very different from the one determined using the 3-point system. The biggest difference is the position of "Mainz 05" jumping from the 13th position to the 4th. The cause of this difference can only be that "Mainz 05" has won the matches in this season that were particularly important in the sense of being in accordance with the team having fairly high winning probabilities in general. However, despite of differences in parts, a great similarity between the rankings can be observed. This similarity can be measured using Spearman's rank correlation coefficient defined as $\rho = 1 - \frac{6\sum(r_i - s_i)^2}{n(n^2-1)}$ with r_i being the original (3 point) ranking of team i and s_i the ranking with the highest maximum likelihood as calculated with the Tabu Search algorithm. The correlation between the 3-point ranking and the one found by the Tabu Search is indeed fairly high with a value of about 0.87616. The difference between the maximum likelihood values however, is in fact very large. The probabilities found using the Tabu Search ranking make the observed season 3318 times more likely compared to the probabilities found using the 3 point ranking.

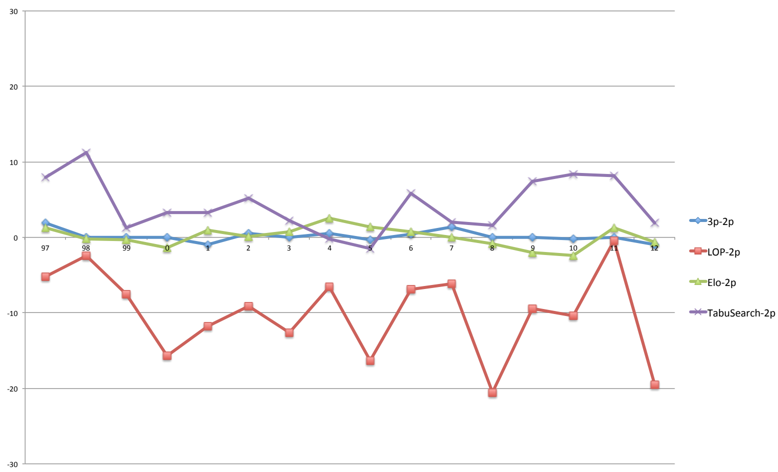
As mentioned above, we have data not only on this one Bundesliga season, but on the ones from the last 50 years.[9] For every season that we have data on, we calculated the maximum likelihood p-matrices as well as the objective function values using the "2 points for a win", the "3 points for a win", Elo system and the ranking from the solution to the linear ordering problem. Finally, we used the Tabu search method to find out, whether or not one is able to improve on one or all of the ranking schemes. Because from season to season the likelihood values fluctuate heavily, it makes sense to use the likelihood found by one of the systems as a reference value and plot the

[9]Because in the seasons 1963/64, 1964/65 and 1991/92 the number of teams in the Bundesliga was different from 18, we excluded these seasons from the sample. Sacrificing these three data points for a higher comparability seems reasonable.

differences to these values in a diagram. As opposed to just plotting the absolute likelihoods of every system in each year, this technique allows us to better compare the quality of the rankings throughout the panel data. The system of reference will be "2 points for a win".



(a) Bundesliga



(b) Premier League

Figure 8: Maximum Likelihoods for Bundesliga and Premier League panel data

Figure 8 (a) and (b) reveal that the two and three point systems are in fact very close in the maximum likelihoods they "produce". This is not least because in most cases the rankings determined by the two systems only differ in a few spots. And if the rankings do not differ much, it's only natural that the likelihood values won't be very far apart either. The two point system allows for a calculation of p-matrices that make the observed seasons on average across the Bundesliga samples by about 9.8% more likely than when using the three point system. In the Premier League the three point system has a 5.2% higher explanatory power. The Elo-system also gives us likelihoods in the same range, indicated by the green lines. Actually this is a bit of a surprise, since there were some hopes that the intuitively very reasonable mechanism of getting more points for winning against relatively strong opponents would enable us to explain the observed results better. Still it is not worse than the conventional two and three point systems. But because of its higher complexity we clearly refrain from making a recommendation for using this system. The ranking resulting from solving the linear ordering problem is by far the worst performer in the diagram. One observes it to yield likelihoods that are on average more that 1000 times smaller that the ones from the two point system. So we have to clearly reject the suggestion for a possible application of the LOP in soccer that has been made in the literature.

Another striking feature about the graphs is the position of the likelihood curves corresponding to the tabu search. The heuristic algorithm is able to improve on every single ranking from the sample, except for the Premier League seasons 04/05 and 05/06. On average it helps to explain the results about 457 times better. The graph shows us that even though the simple ranking schemes produce fairly "good" orderings in the sense of a high correlation (as seen above), they are far away from being the most likely correct ones.

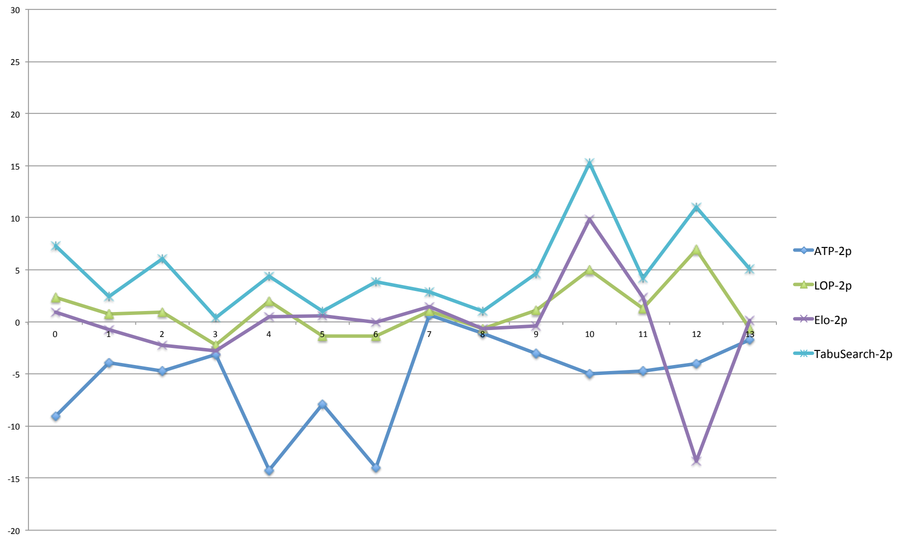


Figure 9: Maximum Likelihoods for ATP panel data

Next, Figure 9 shows the analogue results for the tennis panel data from the last 14 years. The first thing to note is that the two and three point systems produce the same likelihoods throughout the whole sample, which is why in this graph there is no curve comparing the two, since it would lie on the x-axis. The reason for this is that in tennis we do not have draws, so in both systems the players are only ranked according to their number of victories. In Figure 9, in addition to the curves from Figure 8, the likelihoods from the official ATP ranking from the end of each year are listed. This ranking is determined by awarding different amounts of points for a stage that is reached in the Grand Slam Tournaments, the ATP World Tour Finals, the Masters 1000, Olympics etc. Of course this method is very sophisticated and includes also the results of the matches of the top 10 players against others that might not be in the top 10. This data is not part of the other systems we are analyzing. According to the criterion of this work, the ATP ranking performs fairly bad in explaining the observed results. Interestingly in this tennis sample, the linear ordering ranking produces fairly high likelihoods, in fact on average higher ones than the n-point and Elo system. Again, in every year the tabu search algorithm is able to improve on all of the discussed rankings.

Finally, Figure 10 illustrates the results form the same calculations as above, now for American football results from the National Football League in the US. There is little difference between 2- and 3 point systems, because draws are very unlikely to occur. However, the 3 point system is almost at every point at least as good as the 2 point system. The LOP and Elo systems operate in the same range of likelihoods as well. With NFL data, applying the tabu search is more effortful and thus takes more time for the same number of iterations, because of the higher number of teams. However, again, the tabu search improves upon all the rankings in the sample.

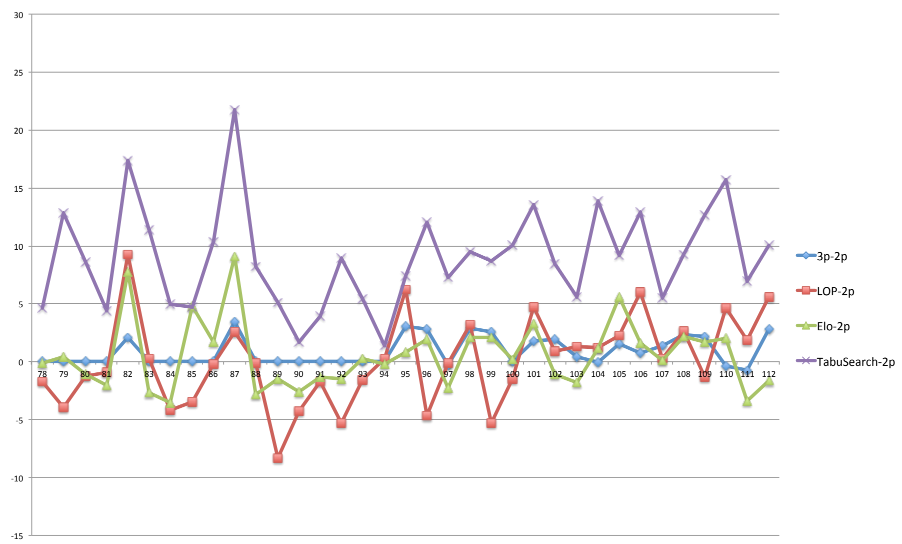


Figure 10: Maximum Likelihoods for NFL panel data

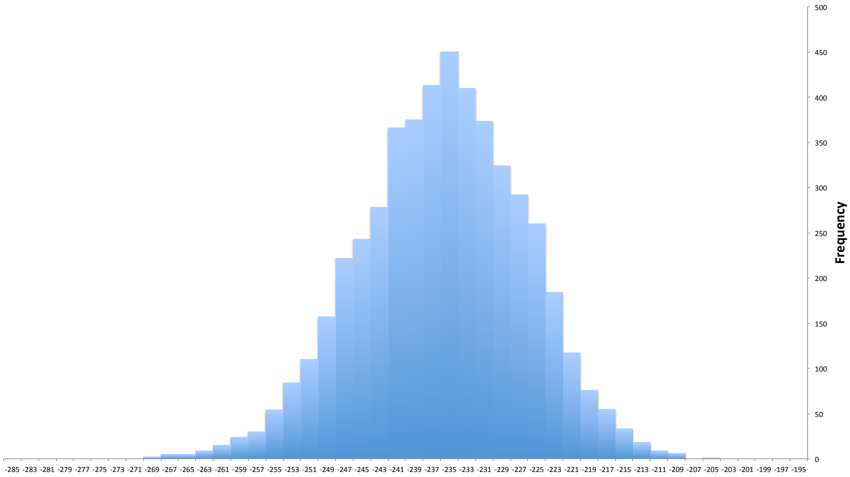
In general, the difference in the relative likelihoods when applying the Elo/LOP system and the n-point systems between soccer on the one hand and tennis and American football on the other could be due to the heterogeneity in the number of games played between the teams in tennis and football as opposed to the symmetric situation in soccer. Certainly a system like "two points for a win" doesn't seem to be particularly well suited in a situation where teams play different amounts of matches. And as explained further above, here it could be justified to give 1 and -1 points instead of 0 and 2 for a win and a loss, respectively. However, implementing this changes not much and even reduces the average likelihood a bit. Another explanation could be the sport itself. It might be due to the result generating probabilities themselves, that for one sport different ranking schemes are better suited then others. Indeed, it is easy to show that in the space of transitive probability matrices, there are areas where each of the considered systems is most likely to generate a ranking closest to the real one. This is an interesting direction for further theoretical research.

2.8.3 Hypothesis Testing

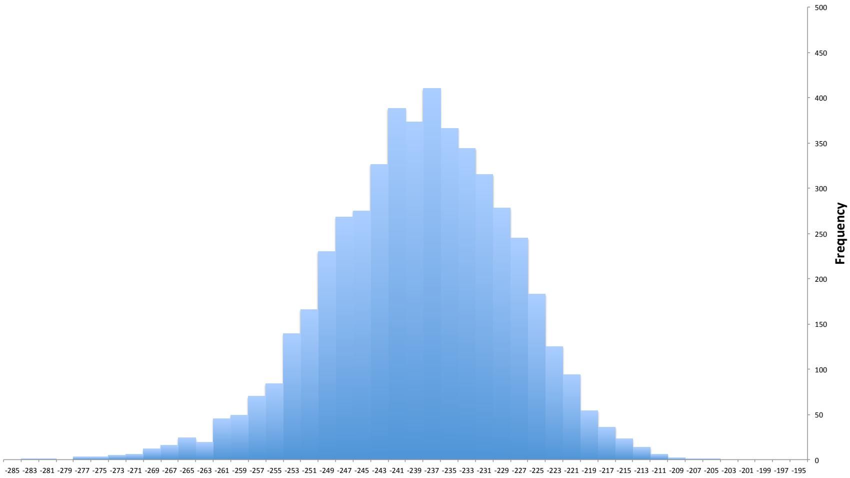
Now we are going further in the analysis of ranking systems than just observing which ordering scheme is able to generate a higher maximum likelihood value. We will consider two examples, which will help deepen the understanding of the problem, but will also clearly highlight the limitations of this hypothesis testing approach, as described in section 2.6.

Consider the Bundesliga season 2011/2012. Looking at Figure 8 reveals that for this data set the 3 point system performed better than the 2 point system. The difference between the two maximum likelihood logs is 0.564. But the central question is "did this MLE difference appear because the underlying unobservable probabilities make the 3 point system more appropriate than the 2 point system in this season or could it in fact be the other way around with the observation just happening by chance?".

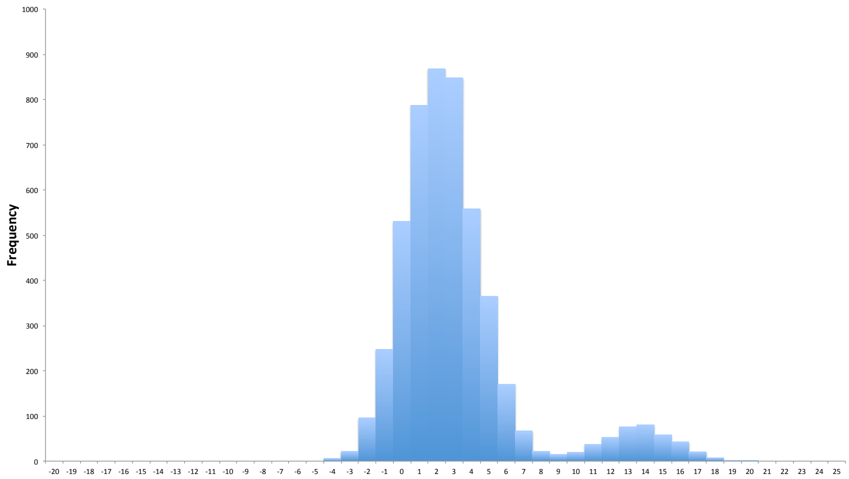
To answer this question, assume the correctness of the Hypothesis H_0: "The 2 point system puts the teams in the correct order". We will test H_0 against the alternative Hypothesis H_1: "The 3 point system puts the teams in the correct order". Now, for the two systems the probability matrices $\hat{P_{2p}}(w)$ and $\hat{P_{3p}}(w)$ are estimated. Using $\hat{P_{2p}}(w)$ 5000 seasons are simulated. Then $L(\hat{P_{2p}}(w)|\hat{w})$ and $L(\hat{P_{3p}}(w)|\hat{w})$ are calculated for each of the seasons. Their respective frequency distribution is depicted in Figure 11 (a) and (b). The distribution of their difference, which corresponds to the ratio of the likelihoods without logs is plotted in Figure 11 (c).



(a) $L(\hat{P_{2p}}(w)|\hat{w})$



(b) $L(\hat{P_{3p}}(w)|\hat{w})$



(c) $LR_{2p,3p}(\hat{w}$

Figure 11: Simulated test statistic for Bundesliga hypothesis test

Looking closely at the first two diagrams reveals that the distribution of $L(\hat{P}_{3p}(w)|\hat{w})$ is shifted a little bit to the left relative to the one of $L(\hat{P}_{2p}(w)|\hat{w})$. This is intuitively correct because it is only natural that the probability matrix that generated the seasons of the sample gives the higher likelihood values than the matrix $\hat{P}_{3p}(w)$, which has nothing to do with the season simulation. Now to find out the confidence level with which we would be able to reject H_0 one has to compare the observed likelihood ratio to the likelihood ratio distribution in Figure 11 (c). This procedure shows us that assuming the correctness of H_0, the probability of the likelihood ratio being ≤ 0.564 is only 11%. So we are able to reject the Hypothesis that the 2 point system gives the correct ranking with test size $\alpha = 0.11$, meaning that the probability of not making an error of the first kind is 0.89. One has to be careful not to misinterpret this result. It means that we are able to reject the hypothesis that the 2 point system gives the correct ranking. However, this does by no means imply that the 3 point system gives the correct ranking.

Now let us conduct a second hypothesis test, this time using tennis data. A good experiment would be to test for the correctness of the LOP system against the 3 point/2 point system in the year 2012. In this year the LOP produced a considerably higher likelihood than the 2 point system (see Figure 9), so we would like to know if this was just a random result or if we can actually conclude that the underlying probabilities favor the LOP scheme in the sense of telling us the truth about the ordering of tennis players. The hypothesis are:

H_0: "The 2 point system puts the teams in the correct order"

H_1: "The solution to the LOP puts the teams in the correct order"

Assuming the correctness of H_0, we again estimate the probability matrices and then simulate 5000 seasons. Hereby we always assume that the m_{ij} values stay constant, i.e. the amount of times players meet is the same in every simulation. We proceed as above by calculating the test statistic for the likelihood ratio and then comparing it to the empirically observed one. We have:

$$LR_{2p,LOP} = log(L(\hat{P}_{2p}(w)|w))) - log(L(\hat{P}_{LOP}(w)|w)) = -6.9103$$

The simulated test statistic tells us that in case H_0 is correct, the probability of an occurrence of such a small likelihood ratio is only 0.02%. It follows that we can reject H_0 with test size $\alpha = 0.02$ (i.e. a confidence level of 99.98%).

2.9 Conclusion

We constructed a statistical model describing the outcomes of sports matches. The model assumes a transitive relationship between the relative strengths of the teams.

The resulting constraints turn out to be very restrictive, which is illustrated by the rapidly shrinking size of the parameter space shown in appendix B. The incorporation of ties as well as home/away asymmetries makes our model much more complicated than the related isotonic regression problem. The discussed branch and bound algorithm is capable of solving the problem for up to 12 teams. For larger data sets, a tabu search heuristic has been proposed. The empirical subsection of the paper first illustrates the structure of an optimized probability matrix with an example. We have shown that in the example the maximum likelihood produced by the tabu search is more that 3000 times higher than the one resulting from an application of the 3-point system. But this does not mean that the two rankings are strongly uncorrelated as seen from the high value of Spearman's rank correlation coefficient. Panel data has been used to compare different ranking systems in three types of sports. In soccer, data from German Bundesliga and English Premier League have shown that the 2- and 3-point systems are very close to each other in the maximum likelihoods they produce, which is not a surprise when considering their structural similarity. Hopes were higher for the performance of the Elo system, because as opposed to the traditional point systems it considers the opponents strength. However, on average the generated MLEs were in the same range as the ones from the n-point systems. This result also applies for ATP tennis and NFL American football data. So the additional degree of complexity seems to be enough of a justification for not giving a recommendation towards an introduction of the Elo system. A difference worth mentioning is that the ranking, which results from the LOP performs fairly well in tennis and American football, but worse than everything else in soccer. We show that almost in every sample across all considered types of sports we are able to improve on the rankings produced by the considered systems by using tabu search. This illustrates that there might be a system that is much better at finding the most likely correct ranking, possibly without the inclusion of a great complexity. As a final remark, we want to mention that the framework presented in this paper has its natural limitations and leaves out many important aspects that should be considered when choosing or designing a ranking scheme. Things like opponents incentives during a match and the resulting effects on the observers level of thrill or the occurrence of winning decision as late as possible during a season could be interesting points for further research.

A Proofs

Proof of proposition 1: Ignoring the away/home differentiation, we can write p_{ikx} as p_{ik}. With 0 probabilities of draws, equation 1 is now

$$p_{ik} = 1 - p_{ki}$$

and therefore equation 2 is then equivalent to

$$\begin{aligned} p_{ik} \geq p_{il} &\Leftrightarrow p_{jk} \geq p_{jl} \\ p_{i'k'} > p_{j'k'} &\Rightarrow p_{l'j'} > p_{l'i'} \end{aligned} \tag{5}$$

Now we have to show that $(SST) \implies (5)$ and $(5) \implies (SST)$.

$(SST) \implies (5)$:
We are dividing this case into two cases: For $p_{ik} \geq \frac{1}{2} \geq p_{jk}$ we can see:

$$p_{ij} \geq p_{kj} = 1 - p_{jk} \geq \frac{1}{2} \overset{SST}{\Longrightarrow} p_{ix} \geq p_{jx} \quad \forall x$$

For every other case we can assume wlog that $p_{ik} \geq p_{jk} \geq \frac{1}{2}$

$$p_{ij} \geq \frac{1}{2} \overset{SST}{\Longrightarrow} p_{ix} \geq p_{jx} \quad \forall x$$

$$p_{ij} < \frac{1}{2} \implies p_{ji} > \frac{1}{2} \overset{SST}{\Longrightarrow} p_{jk} > p_{ik}$$

Which is a contradiction to the assumption, therefore $p_{ij} \geq \frac{1}{2}$.

$(5) \implies (SST)$:

$$\begin{aligned} p_{jk} > p_{ik} &\overset{(5)}{\Rightarrow} p_{li} > p_{lj} \qquad \forall l \\ \Rightarrow p_{ii} > p_{ij} &\overset{p_{ii}=1/2}{\Longrightarrow} p_{ij} < \frac{1}{2} \end{aligned}$$

□

Proof of proposition 2: Define a ranking from best to worst $\rho(i) : \{1, .., n\} \multimap \{1, ..., n\}$ such that $p_{ikx} \geq p_{jkx} \Rightarrow \rho(i) < \rho(j)$ and $p_{kix} \leq p_{kjx} \Rightarrow \rho(i) < \rho(j)$.

$$\begin{aligned} p_{ikx} \geq p_{jkx} \Leftrightarrow \rho(i) < \rho(j) &\Leftrightarrow p_{ily} \geq p_{jly} \quad \forall i, j, k, l, x, y \\ p_{kix} \geq p_{kjx} \Leftrightarrow \rho(j) < \rho(i) &\Leftrightarrow p_{ily} \geq p_{jly} \quad \forall i, j, k, l, x, y \\ p_{i'k'x} > p_{j'k'x} \Leftrightarrow \rho(j) < \rho(i) &\text{ and } \rho(i) \nleq \rho(j) \exists i', j', k', x \end{aligned}$$

□

B Parameter space

In this subsection we explore the effect of transitivity conditions on the parameter space of winning probabilities to illustrate the limitations enforced by it. To do that we compare the size of the parameter space with transitivity to the space of unrestricted winning probabilities $\overline{S}_n$, e.g. every p_{ij}, p_{ji} fulfilling $p_{ij} + p_{ji} = 1$.
The space of parameters including the transitivity conditions is a subset of this set $\overline{S}_n$. $S_n(R)$ is hereby defined as the size of this space relative to $\overline{S}_n$ only considering the restrictions for $p_{ij} \in R$. The unrestricted parameter space is in this simple case: $\overline{S}_n = [0,1]^{\frac{n(n-1)}{2}}$ which can be easily seen by the fact that every p_{ji} is completely determined by p_{ij}. The restricted space for n players and the transitivity conditions for every $(i,j) \in K_n$ with $K_n = \{(i,j)|i,j \in \{1,2,...,n\}, i < j\}$ is therefore

$$S_n(K_n) = \int_{b_{i+1,j}}^{b_{i,j+1}} S_n(K_n \backslash \{(i,j)\}) \mathrm{d}p_{ij}$$

with

$$S_n((i_0,j_0)) = \int_{b_{i_0+1,j_0}}^{b_{i_0,j_0+1}} \mathrm{d}p_{i_0 j_0}$$

and

$$b_{i,j} := \begin{cases} p_{ij}, & \text{for } (i,j) \in K_n \\ 0.5, & \text{for } i = j \\ 0, & \text{else} \end{cases}$$

As this fairly complicated recursive integral may be hard to interpret, table 2 gives the values for the relative size of the transitive parameter space for up to five teams. It can be seen that the size rapidly shrinks and it is not hard to imagine that for a league comprising e.g. 18 teams the conditions are in this sense very strict.

n	2	3	4	5	6	7
relative size	1	$\frac{1}{4}$	$\frac{1}{120}$	$\frac{1}{40320}$	$\frac{1}{203212800}$	$\frac{1}{19313344512000}$
Approximation	1	0.25	8.3×10^{-3}	2.5×10^{-5}	4.9×10^{-9}	5.2×10^{-14}

Table 2: Relative size of the transitive parameter space

C Code

The first code listing shows the problem definition of an optimization with fixed team ordering, so that the ipopt framework will understand it.

```
#include "nfl_nlp.hpp"

#include <cassert>
#include <iostream>
#include <math.h>

static int t=50;

static int w[3][50][50];
static double p[2][50][50];


using namespace Ipopt;

// constructor
nfl_NLP::nfl_NLP(int myw[][50][50], double* myp[][50][50], double*& zielwert, int myt)
{
    zielwert = &zw;
    t=myt;

    for (int h=0; h<3; h++) {
        for (int k=0; k<t; k++) {
            for (int l=0; l<t; l++) {
                w[h][k][l]=myw[h][k][l];
                myp[h][k][l]=&p[h][k][l];
            }
        }
    }

}

//destructor
nfl_NLP::~nfl_NLP()
{}

// returns the size of the problem
bool nfl_NLP::get_nlp_info(Index& n, Index& m, Index& nnz_jac_g,
                                Index& nnz_h_lag, IndexStyleEnum& index_style)
{
```

```
    // The problem described in nfl_NLP.hpp has 4 variables, x[0] through x[3]
    n = 2*pow(t,2);

    // one equality constraint and one inequality constraint
    m =  pow(t,2) + 4*t*(t-1);

    // in this example the jacobian is dense and contains 8 nonzeros
    nnz_jac_g = 2*m;

    // the hessian is also dense and has 16 total nonzeros, but we
    // only need the lower left corner (since it is symmetric)
    nnz_h_lag = 2*n-4*t;

    // use the C style indexing (0-based)
    index_style = TNLP::C_STYLE;

    return true;
}

// returns the variable bounds
bool nfl_NLP::get_bounds_info(Index n, Number* x_l, Number* x_u,
                                Index m, Number* g_l, Number* g_u)
{
    // here, the n and m we gave IPOPT in get_nlp_info are passed back to us.
    // If desired, we could assert to make sure they are what we think they are.

    // the variables have lower bounds of 0
    for (Index i=0; i<2*t*t; i++) {
        x_l[i] = 0.0;
    }

    // the variables have upper bounds of 1
    for (Index i=0; i<2*t*t; i++) {
        x_u[i] = 1.0;
    }


    Index i = 0;
    for (Index k=0; k<t; k++) {
        for (Index l=0; l<t; l++) {
            g_l[i] = -2e19;
            g_u[i] = 1.0;
            i++;
```

```
        }
    }
    for (Index h=0; h<2; h++) {
        for (Index k=0; k<t; k++) {
            for (Index l=0; l<t; l++) {
                if (l<t-1) {
                    g_l[i] = -2e19;
                    g_u[i] = 0.0;
                    i++;
                }
                if (k<t-1) {
                    g_l[i] = -2e19;
                    g_u[i] = 0.0;
                    i++;
                }
            }
        }
    }

    return true;
}

// returns the initial point for the problem
bool nfl_NLP::get_starting_point(Index n, bool init_x, Number* x,
                                  bool init_z, Number* z_L, Number* ↩
                                     z_U,
                                  Index m, bool init_lambda,
                                  Number* lambda)
{
    assert(init_x == true);
    assert(init_z == false);
    assert(init_lambda == false);

    // initialize to the given starting point
    for (Index i=0; i<2*t*t; i++) {
        x[i] = 0.4;
    }

    return true;
}

// returns the value of the objective function
bool nfl_NLP::eval_f(Index n, const Number* x, bool new_x, Number& ↩
    obj_value)
{
    assert(n == 2*t*t);
```

```
    obj_value=0;
    Index i=0;
    for (Index k=0; k<t; k++) {
        for (Index l=0; l<t; l++) {
            if (k!=l) {
                obj_value += (-w[0][k][l]*log(x[t*k+l]+0.000001) - w[1][l][k]*log(x[t*t+t*l+k]+0.000001) - (w[2][k][l]-w[0][k][l]-w[1][l][k])*log(1-x[t*k+l]-x[t*t+t*l+k]+0.000001));

                i++;
            }
        }
    }

    return true;
}

// return the gradient of the objective function grad_{x} f(x)
bool nfl_NLP::eval_grad_f(Index n, const Number* x, bool new_x, Number* grad_f)
{
    Index i=0;
    for (Index h=0; h<2; h++) {
        for (Index k=0; k<t; k++) {
            for (Index l=0; l<t; l++) {
                if (k!=l) {
                    if (h==0)
                        grad_f[i] = -w[h][k][l]/(x[t*t*h+t*k+l]+0.000001) + (w[2][k][l]-w[h][k][l]-w[1-h][l][k])/(1-x[t*t*h+t*k+l]-x[t*t*(1-h)+t*l+k]+0.000001);
                    else
                        grad_f[i] = -w[h][k][l]/(x[t*t*h+t*k+l]+0.000001) + (w[2][l][k]-w[h][k][l]-w[1-h][l][k])/(1-x[t*t*h+t*k+l]-x[t*t*(1-h)+t*l+k]+0.000001);
                }
                else{
                    grad_f[i] = 0;
                }
                i++;
            }
        }
    }
```

```

    return true;
}

// return the value of the constraints: g(x)
bool nfl_NLP::eval_g(Index n, const Number* x, bool new_x, Index m, Number* g)
{
    Index i = 0;
    for (Index k=0; k<t; k++) {
        for (Index l=0; l<t; l++) {
            g[i] = x[t*k+l]+x[t*t+t*l+k];
            i++;
        }
    }
    for (Index h=0; h<2; h++) {
        for (Index k=0; k<t; k++) {
            for (Index l=0; l<t; l++) {
                if (l<t-1) {
                    g[i] = x[t*t*h+t*k+l]-x[t*t*h+t*k+l+1];
                    i++;
                }
                if (k<t-1) {
                    g[i] = x[t*t*h+t*(k+1)+l]-x[t*t*h+t*k+l];
                    i++;
                }
            }
        }
    }
    return true;
}

// return the structure or values of the jacobian
bool nfl_NLP::eval_jac_g(Index n, const Number* x, bool new_x,
                          Index m, Index nele_jac, Index* iRow, Index *jCol,
                          Number* values)
{
    if (values == NULL) {
        // return the structure of the jacobian

        // this particular jacobian is dense


        Index z=0;
        Index r=0;
```

```
        for (Index k=0; k<t; k++) {
            for (Index l=0; l<t; l++) {
                iRow[z] = r;
                jCol[z] = t*k+l;
                z++;
                iRow[z] = r;
                jCol[z] = t*t+t*l+k;
                z++;

                r++;
            }
        }
        for (Index h=0; h<2; h++) {
            for (Index k=0; k<t; k++) {
                for (Index l=0; l<t; l++) {
                    if (l<t-1) {
                        iRow[z] = r;
                        jCol[z] = t*t*h+t*k+l;
                        z++;
                        iRow[z] = r;
                        jCol[z] = t*t*h+t*k+l+1;
                        z++;
                        r++;
                    }
                    if (k<t-1) {
                        iRow[z] = r;
                        jCol[z] = t*t*h+t*k+l;
                        z++;
                        iRow[z] = r;
                        jCol[z] = t*t*h+t*(k+1)+l;
                        z++;
                        r++;
                    }
                }
            }
        }
        assert(z==nele_jac);
    }
    else {
        Index z=0;
        Index r=0;
        for (Index k=0; k<t; k++) {
            for (Index l=0; l<t; l++) {
                values[z] = 1;
                z++;
                values[z] = 1;
```

```
                z++;

                r++;
            }
        }
        for (Index h=0; h<2; h++) {
            for (Index k=0; k<t; k++) {
                for (Index l=0; l<t; l++) {
                    if (l<t-1) {
                        values[z] = 1;
                        z++;
                        values[z] = -1;
                        z++;
                        r++;
                    }
                    if (k<t-1) {
                        values[z] = -1;
                        z++;
                        values[z] = 1;
                        z++;
                        r++;
                    }
                }
            }
        }
        assert(z==nele_jac);
    }

    return true;
}

//return the structure or values of the hessian
bool nfl_NLP::eval_h(Index n, const Number* x, bool new_x,
                     Number obj_factor, Index m, const Number* lambda,
                     bool new_lambda, Index nele_hess, Index* iRow,
                     Index* jCol, Number* values)
{
    if (values == NULL) {
        // return the structure. This is a symmetric matrix, so we fill the lower left
        // triangle only.
        Index i=0;
        for (Index h=0; h<2; h++) {
            for (Index k=0; k<t; k++) {
                for (Index l=0; l<t; l++) {
```

```
                    if (k!=l) {
                        iRow[i] = t*t*h+t*k+l;
                        jCol[i] = t*t*h+t*k+l;
                        i++;

                        iRow[i] = t*t*h+t*k+l;
                        jCol[i] = t*t*(1-h)+t*l+k;
                        i++;
                    }
                }
            }
        }
    }
    else {
        // return the values. This is a symmetric matrix, fill the lower left
        // triangle only

        Index i=0;
        for (Index h=0; h<2; h++) {
            for (Index k=0; k<t; k++) {
                for (Index l=0; l<t; l++) {
                    if (k!=l) {
                        if (h==0){
                            values[i] = obj_factor *( w[h][k][l]*pow(x[t*t*h+t*k+l]+0.000001, -2) + (w[2][k][l]-w[h][k][l]-w[1-h][l][k])*pow((1-x[t*t*h+t*k+l]-x[t*t*(1-h)+t*l+k]+0.000001),-2));
                            i++;
                            //std::cout << "values[" << i << "] = " << values[i] << std::endl;
                            values[i] = obj_factor *0.5*( (w[2][k][l]-w[h][k][l]-w[1-h][l][k])*pow((1-x[t*t*h+t*k+l]-x[t*t*(1-h)+t*l+k]+0.000001),-2));
                        }
                        else{
                            values[i] = obj_factor *( w[h][k][l]*pow(x[t*t*h+t*k+l]+0.000001, -2) + (w[2][l][k]-w[h][k][l]-w[1-h][l][k])*pow((1-x[t*t*h+t*k+l]-x[t*t*(1-h)+t*l+k]+0.000001),-2));
                            i++;
                            //std::cout << "values[" << i << "] = " << values[i] << std::endl;
                            values[i] = obj_factor *0.5*( (w[2][l][k]-w
```

```
                                                [h][k][l]-w[1-h][l][k])*pow((1-x[t*t*h+t↩
                                                *k+l]-x[t*t*(1-h)+t*l+k]+0.000001),-2));
                                    }
                                    i++;
                                }
                            }
                        }
                    }
    }

    return true;
}

void nfl_NLP::finalize_solution(SolverReturn status,
                                    Index n, const Number* x, const ↩
                                        Number* z_L, const Number* z_U,
                                    Index m, const Number* g, const ↩
                                        Number* lambda,
                                    Number obj_value,
                                    const IpoptData* ip_data,
                                    IpoptCalculatedQuantities* ip_cq)
{
    // here is where we store the solution to variables
    // so we could use the solution.

    for (Index h=0; h<2; h++) {
        for (Index k=0; k<t; k++) {
            for (Index l=0; l<t; l++) {
                //std::cout << x[t*t*h+t*k+l] << " ";
                p[h][k][l]=x[t*t*h+t*k+l];
            }
            //std::cout << " " << std::endl;
        }
        //std::cout << " " << std::endl;
    }

    zw = obj_value;
}
```

The second code snippet shows how the program reads 5 NFL seasons, puts them in different orders, and then optimizes the probabilities and prints them.

```
int main(int argv, char* argc[])
{

    // Create a new instance of the nlp
```

```
    SmartPtr<TNLP> mynlp;
    SmartPtr<TNLP> mynlp2;

    // Create a new instance of IpoptApplication
    SmartPtr<IpoptApplication> app = IpoptApplicationFactory();

    // Change some options
    app->Options()->SetIntegerValue("print_level", 0);
    app->Options()->SetNumericValue("tol", 1e-4);
    app->Options()->SetStringValue("mu_strategy", "adaptive");
    app->Options()->SetStringValue("output_file", "ipopt.out");

    // Intialize the IpoptApplication and process the options
    ApplicationReturnStatus status;
    status = app->Initialize();
    if (status != Solve_Succeeded) {
      std::cout << std::endl << std::endl << "*** Error during ↩
         initialization!" << std::endl;
      return (int) status;
    }

    srand(time(NULL));


    double* z;  //Variable for the likelihood
    int tempW50[3][50][50];//temporary result matrices

    for (Index h=0; h<3; h++)
        for (Index k=0; k<50; k++)
            for (Index l=0; l<50; l++)
                tempW50[h][k][l]=w050[h][k][l];

    double l[14][6];

    //for the years 2000 till 2005, the nfl data is read from the files
    for (int jahr=100; jahr<105; jahr++) {
        int t;
        if (jahr<95)
            t=28;
        else if (jahr<99)
            t=30;
        else if (jahr<102)
            t=31;
        else
            t=32;

```

```
        t=10;


        einlesenNfl(jahr,t);

        //order according to the 2 point system and run the optimization
        ordneNachPunkteSystem50(2,w50,t);
        mynlp2 = new nfl_NLP(w50,p50,z,t);
        app->OptimizeTNLP(mynlp2);
        l[jahr][1]=-*z;

        //order according to the 3 point system and run the optimization
        ordneNachPunkteSystem50(3,w50,t);
        mynlp2 = new nfl_NLP(w50,p50,z,t);
        app->OptimizeTNLP(mynlp2);
        l[jahr][2]=-*z;

        //order according to the LOP system and run the optimization
        ordneNachLOP50(w50,t);
        mynlp2 = new nfl_NLP(w50,p50,z,t);
        app->OptimizeTNLP(mynlp2);
        l[jahr][3]=-*z;

        //order according to the ELO system and run the optimization
        ordneNachSchach50(w50,t);
        mynlp2 = new nfl_NLP(w50,p50,z,t);
        app->OptimizeTNLP(mynlp2);
        l[jahr][4]=-*z;

        //run the tabu search for 100 iterations and then run the optimization
        l[jahr][5]=tabuSearch50(100,t);

        //print out the results
        cout<<l[jahr][0]<<" "<<l[jahr][1]<<" "<<l[jahr][2]<<" "<<l[jahr][3]<<" "<<l[jahr][4]<<" "<<l[jahr][5]<<endl;
        cout<<endl;
    }
```

3 An ACE approach to the constructed capital model

3.1 Introduction

There are countless examples to motivate the attempt to explain the logic behind economic agglomeration effects on different levels, which is the main idea of the New Economic Geography. One of the most prominent instances is the highly skewed distribution of income across the globe. In 2005, 78% of world GDP was produced in the high income counties which comprise only 16% of the world's population and 26% of the global land area (see Brakman et al. [2009]). On an intra country perspective, in the Île de France (the metropolitan area of Paris) almost a third of the French income is produced by only 18.9% of the population and on 2.2% of the area of France (see Hall and Jones [1999]). Finally, on an even smaller scale, there are clusters of industrial activities (often dominated by a certain kind of technology) in areas like Silicon Valley.

From the beginning of the 1990s on, the discipline of the New Economic Geography (NEG) evolved mainly around the core periphery model by Krugman [1990] and the vertically linked industries model by Venables [1996] and Krugman and Venables [1995]. While being very similar from a mathematical perspective, Krugman's model is better suited for application for example in the U.S., because of its assumption concerning worker mobility. The Venables model seems to fit better for the situation in Europe, where labor is not so mobile.

Both models feature an immobile agricultural sector, which fosters a dispersed economic activity. However, there are two kinds of agglomeration forces, which came to be known as demand-linked and cost-linked circular causality (which shall be explained using the logic of the core periphery model). The former describes the effect that when a firm moves from one of two initially symmetrically sized regions to the other, workers will follow, which induces expenditure shifting. Naturally, firms prefer to be in the region where expenditures are higher, which causes further production shifting and hereby closes the circle. In addition to this effect, the moving firm will lower the price index in the receiving nation, meaning that under real wage equalization nominal wages will be lower in that region. This in turn increases the competitiveness of the receiving nation fostering agglomeration even further.

Because of the high degree of complexity that these effects inevitably bring along, these original NEG models are not analytically tractable, which paved the way for a number of modifications to these approaches. Among them is the "endogenous capital" (or "constructed capital") model by Baldwin [1999]. Its roots can be found in the footloose capital model by Martin and Rogers [1995]. Baldwin assumes that the factor of labor is locally fixed in favor of an endogenization of capital. Capital can be produced in a country and it is also subject to a depreciation process. If a country produces a large amount of capital, this produces returns, which increases expenditures and potentially attracts more capital. This circular causality is very similar to the mechanisms of the original NEG models and leads to the result that there exists a critical level of trade freeness beyond which catastrophic agglomeration occurs. One of the advantages of the endogenous capital model is that this point can be explicitly determined.

The bulk of the NEG models are based on general equilibrium approaches. Their subject is the description of economic states, where markets clear, no changes in the variables involved occur, and perfect predictions concerning the future are made. Situations out of these perfect world states are not only unaddressed, but cannot even be characterized using the given equations. Fowler [2007] tackles this issue by converting the classical cp-model to an agent based framework. This enables him to analyze the dynamic behavior of the model outside of equilibrium. That the construction of an agent based model of this kind, which is very close to the original model, is not an easy task is shown by the fact that the agent based model has some severe weaknesses including the independence of worker's location and firm's labor demand. This is why Fowler [2011] improved his work by an enhanced version of the agent based framework.

Some other attempts like for instance Baldwin [2001] have been made to at least define the NEG models outside of equilibrium, most of them building on the core periphery model. Fujita and Mori [2005] have pointed out the fact that, because there has been a great progress in computational power and in the relevant software instruments, the obvious next step are numerically computable NEG models. Indeed, there has been a persistent trade of between the realism and tractability of NEG models and, while simplified, analytically solvable scenarios won't loose their importance, a numerical analysis can open the door for understanding more about distance between economic regions.

This paper builds on the constructed capital model by Baldwin [1999], makes it dynamic in nature and extends its domain to out of equilibrium states. The original model uses an optimal control approach to help agents solve the inter temporal utility optimization problem they face. This means that agents make their saving decisions with perfect knowledge about all other agents' decisions in the future, creating a self fulfilling prophecy for the level of the capital rental rate. We are more realistically assuming that agents face uncertainty concerning future rental rates. The literature on inter temporal expenditure optimization helps us to formulate an approach, by which we can determine an optimal consumption function. Including this kind of function into the original model reveals some differences as opposed to the equilibrium structure discussed by Baldwin.

Section 3.2 gives a quick introduction to the constructed capital model as in Baldwin [1999], before section 3.3 sets up and tackles the stochastic expenditure optimization problem. Section 3.4 derives some analytic results concerning the stability of the resulting equilibrium points. After that, sections 3.5 and 3.6 discuss some bifurcation diagrams for different parameter settings. Section 3.7 introduces autoregressive interest rate expectations as opposed to the i.i.d. behavior that was assumed before. Then, section 3.8 gives some examples of possible trajectories and investigates wether inter equilibrium jumps can be triggered solely by expectational shocks. Finally, section 3.9 extends the model to an n-country setting. Section 3.10 concludes.

3.2 Model

3.2.1 Basic assumptions

The basis for the investigations conducted in this work is the model by Baldwin [1999]. We are dealing with interrelated regions, each one working similar to an economy described by the Ramsey model. Important to note is that as opposed to not only Baldwin's model, but also most dynamic modifications of the standard Core-Periphery model as e.g. Krugman [1992], our model is discrete with respect to time.

In our economy there are two factors, namely labor (L) and capital (K). Both factors are immobile and non-traded. Each of the two countries under consideration has a fixed number of inhabitants, who are endowed with a fixed amount of labor in each period and an initial amount of capital which is endogenous in later periods. An agent can work in one of two sectors namely agriculture or manufacturing. In agriculture an homogeneous good is produced under constant returns to scale. To be more specific, one unit of food is produced using a_a units of L and 0 units of K. In the spirit of the Dixit-Stiglitz monopolistic competition model, in the manufacturing sector one specific kind of product from a continuum of varieties is produced requiring a fixed cost of one unit of K and a variable cost of a_m units of L. So when producing x_i units of variety i, costs of $\pi + wa_m x_i$ arise during production, where π is the nominal rental rate for a unit of capital, which shall be a central element of the model. Finally, an agent can either spent her wage w on consumption of the mentioned goods or invest the money in order to receive capital returns allowing for higher consumption levels in future periods.

3.2.1.1 Capital investment

The capital that has been invested by the agents is integrated to form the region's capital stock K. K corresponds to the number of firms in that region, because each firm producing one product requires one unit of capital. As mentioned, in every period the agents can make such an investment (or disinvestment) decision and hereby increase or decrease their own as well as the nation's capital stock. Additionally, the capital wares out and loses a proportion of δ of its value in every period. So the capital dynamics are described by

$$k_{i,t+1} = (1 + \pi_t - \delta)k_{i,t} + l_{i,t}/P_k \tag{6}$$
$$K = \sum k_i$$

where $l_{i,t} = w - e_t$ is the portion of an agent's labor income she invests in period t. P_k is the price of one unit of capital. It can be imagined as purely dictated by technology in the sense of efficiency of capital generation. The term in brackets reflects the real interest rate, which will be referred to as r_t from now on.

Manufacturing as well as the agricultural good is traded among the two regions. As usual in the new economic geography, the agricultural good is free of transportation costs, whereas the manufacturing products experience iceberg transportation costs, meaning that $\tau \geq 1$ units have to be shipped in order for one unit to arrive. No revenues are generated by the occurrence of these costs.

3.2.1.2 Agents' preferences

All consumers have the following logarithmic utility function consisting of a term $C_{A,t}$ for consumption of the agricultural good and a composite term $C_{M,t}$ comprising a CES combination of all the manufacturing goods.

$$U = \sum_{t=0}^{\infty} (1-\rho)^t ln(C_{M,t}^{\alpha} C_{At}^{1-\alpha})$$

$$C_M = \left(\int_{i=0}^{K+K^*} c_i^{1-1/\sigma} di \right)^{1/(1-1/\sigma)} \quad , \quad \sigma > 1,\, 0 < \alpha < 1$$

The elasticity of substitution parameter σ influences the well known love of variety effect. Future utility is discounted by $\rho > 0$. And since, as mentioned above, one unit of capital corresponds to one firm and one variety, integrating over varieties corresponds to integrating over the sum of home and foreign capital K and K^*.

3.2.2 Consumer and firm optimization

From the preferences given above, a set of results can be derived. Given a level of expenditure, a consumer will allocate a fraction of $1-\alpha$ of them on the agricultural good and the rest on manufacturing goods. Among the latter, the fraction spent on each one depends on the relation of its price to the overall price index.

$$c_j = \frac{s_j}{p_j}\alpha E\,, \quad s_j = \frac{p_j^{1-\sigma}}{\int_{i=0}^{K+K*} p_i^{1-\sigma} di}$$

Since these equations are rather standard, I won't go into detail concerning their derivation. The same applies for the firms' profit maximization. Before turning to the equations for optimal prices and resulting profits, consider Baldwin's [1999] very useful set of normalizations . First the wage in agriculture is normalized to 1 and units of that good are chosen such that $a_a = 1$ and because of constant returns to scale and perfect competition, we have $p_a = 1$. Now as long as in both countries the agricultural good is produced, we have wage equalization, because in this sector there are no transportation costs. The labor market is competitive so the wage in manufacturing will also be 1. The optimization of the firms' profit function implies

$$p_j = \frac{a_m \sigma w}{\sigma - 1}.$$

If one chooses units of manufacturing goods such that the unit input coefficient becomes $a_m = \frac{\sigma-1}{\sigma}$, then $p_j = 1$. Firms selling their goods in the neighbor county will forward the transport costs to consumers and charge a price of τ.

Plugging these values into the representative firms profit function, one gets

$$\pi_t = \left(\frac{\alpha}{\sigma}\right)(s_{11}E_{t-1} + s_{12}E_{t-1}^*),\ \pi_t^* = \left(\frac{\alpha}{\sigma}\right)(s_{21}E_{t-1} + s_{22}E_{t-1}^*) \tag{7}$$

$$\text{with } s_{11} = \frac{1}{K_t + \phi K_t^*},\ s_{12} = \frac{\phi^*}{\phi^* K_t + K_t^*},\ s_{21} = \frac{\phi}{K_t + \phi K_t^*},\ s_{22} = \frac{1}{\phi^* K_t + K_t^*},$$

where s_{11} is the part of home countrie's expenditures spent at home, s_{12} is the part of foreign countrie's expenditures spent on products produced in the home country and so on. [10] $\phi = \tau^{1-\sigma}$ and $\phi^* = (\tau^*)^{1-\sigma}$ represent the freeness of trade ranging from 0 for prohibitive trade barriers to 1 for free transportation.

[10] The reason for the index of E being $t-1$ instead of t is the timing in our model. First the capital is subject to the interest rate, then the wage is received and finally the agent consumes. Because capital can never be negative, consumption in t must be based on capital after being subject to the interest rate. If this very interest rate would depend on the expenditures in t, this would lead to a recursion. This reasoning will become more apparent further below.

3.3 Intertemporal expenditure optimization

Traditional economic modeling assumes perfect foresight of all agents concerning what is happening in the economy in future periods. This includes the knowledge of individuals about firms' behavior and most importantly about saving decisions of their fellow consumers. Hereby they are also assumed to know the future rental rate. This is exactly the assumption Baldwin [1999] makes. He then uses dynamic programming to solve for the Euler equation, which is well known from other models. In our discrete world, the Euler equation used by Baldwin would look like so

$$e_{t+1} = r_t(1-\rho)e_t.$$

Now, the change in consumption is set to zero, which implies $e_t = e_{t+1}$. [11] The important point is that by this step, one restricts the attention exclusively to the steady state(s) of the system. The only question answered is "what happens if everybody expects the same constant rate of return for his capital and is correct in his expectation?". Everything that happens out of this perfect world equilibrium is not addressed by the research so far. In fact we do not even know if or how the system moves towards these equilibria or if it rather ends up somewhere else in case the initial conditions are different from the equilibrium values or if an agent makes a small mistake in his behavior or expectations. While what has been done so far is certainly interesting and its justification is beyond debate, it is not a true dynamic analysis.

To change this fact, I am going to assume a more realistic consumption behavior of the agents. There is a considerable amount of literature on optimal consumption/savings behavior in case of uncertainty. Consider a setting like the one in Phelps [1962]. About the same assumptions are made in Levhari and Srinivasan [1969]. Even though the ones made here are slightly different, the derivation that follows will be very close to the latter work.

Just like it is the case in our model, the agent receives a certain and constant income in every period and faces a decision of how much to consume and how much to save. The amount saved is subject to a stochastic interest rate, which is for now assumed to be i.i.d. and the agent seeks to maximize the expected discounted sum of future utilities.

$$\max E\left[\sum_{t=0}^{\infty}(1-\rho)^t u(e_t)\right]. \tag{8}$$

[12] Before we proceed, we have to define what will be called "cash on hand". It is the amount of money that an agent can maximally spend in a given period. Note that this

[11] In the continuous framework Baldwin [1999] uses, the equivalent condition is $\dot{E} = 0$.

[12] Usually one would expect the letter c for consumption. To stay consistent with the notation from above, we continue to use e for expenditure.

amount will be different from the capital multiplied by its price, because according to our assumptions concerning the timing, capital gains are received, then the wage is received and finally the agent consumes. So cash on hand is defined as follows.

$$x_t = k_t r_t P_k + w \tag{9}$$

Plugging this definition into (6), yields the dynamics

$$x_{t+1} = (x_t - e_t) r_{t+1} + w \tag{10}$$

For the optimal consumption path the term (8) can be expressed as a so called value function. The only argument of the value function is the agent's cash on hand in period 0. This is because capital (and hereby equivalently cash on hand) is the only state variable, which is also why optimal consumption in period t can be expressed as a function of cash on hand in that same period.

$$V_t(x_t) = E\left[\sum_{\tau=t}^{T}(1-\rho)^{\tau-t}u(e_\tau)\right],$$

where

$$e_\tau = f_\tau(x_\tau)$$

is period t's optimal consumption.

Using the value function we can easily derive the Bellman equation.

$$V_t(x_t) = \max_{0\leq e_t\leq x_t}[u(e_t) + (1-\rho)EV_{t+1}[(x_t - e_t)r_{t+1} + w]] \tag{11}$$

Taking the derivative with respect to e_t leads to

$$u'(e_t) = (1-\rho)E[r_{t+1}V'_{t+1}[(x_0 - e_t)r_t + w]] \tag{12}$$

Next, we make use of the Envelope Theorem. Plugging in the optimal consumption $e_t = f_t(x_t)$ into (11) gives

$$V_t(x_t) = u(f_t(x_t)) + (1-\rho)EV_{t+1}[(x_t - f_t(x_t))r_{t+1} + w],$$

which can be differentiated on both sides with respect to x_t, yielding

$$V'_t(x_t) = f'_t(x_t)u'[f_t(x_t)] + (1-\rho)(1-f'_t(x_t))E\left[r_{t+1}V'_{t+1}[(x_t - f_t(x_t))r_{t+1} + w]\right].$$

Finally, using (12) we get

$$V'_t(x_t) = u'[f_t(x_t)]$$

and

$$u'[f_t(x_t)] = (1-\rho)E\left[r_{t+1}u'[f_{t+1}((x_t - f_t(x_t))r_{t+1} + w)]\right].$$

With the logarithmic utility assumed in the model here, this gives the expression

$$\frac{1}{f_t(x_t)} = (1-\rho)E\left[\frac{r_{t+1}}{f_{t+1}((x_t - f_t(x_t))r_{t+1} + w)}\right]. \qquad (13)$$

Attempts to find the function(s) f_t have among others been made by Zeldes [1989], Deaton [1989] and Carroll [1997]. However, all these authors have been concerned with a slightly different problem. They assume a stochastic labor income instead of a stochastic interest rate. For this problem Carroll [2004] derives conditions under which the consumption rule f_t will converge to a fixed function as the agents' lifetime gets infinitely large. Of course our stochastic interest rate problem is different. So we can not use these theoretical results. But since the structure of the problem is very similar, we can still apply the numerical methods used by the authors to find a consumption rule.

Before we apply this method, we need to make an assumption concerning the interest rate's presumed distribution. Note that this is not an assumption about the actual behavior of the interest rate. We cannot make this kind of assumption, because r is completely endogenous in the model. Rather we only determine the agent's believes about the interest rate's distribution. It is common practice to assume that returns are log-normally distributed. For our numerical purposes the lognormal distribution is approximated by a discrete grid. The method we are going to apply then works by backward induction. Since we say that there are no bequests, we know that for the agent it will be optimal to consume everything she has in the last period of her life.

$$f_T(x_T) = x_T$$

If we now choose a value for X_{T-1}, we are able to numerically determine the optimal consumption $c_{T-1} = f_{T-1}(X_{T-1})$ by using (13). We do this for a grid of X_{T-1} values and then approximate $f_{T-1}(x_{T-1})$ by a linear interpolation of the resulting points. In the next step we chose values for X_{T-2} and use the function $f_{T-1}(x_{T-1})$ in combination with (13) to determine f_{T-2} and so on.

Figure 12 shows how the consumption function, starting from a 45 degree line reflecting the recommendation to spend everything, eventually moves towards its final position.

The first thing that catches ones eye, when looking at the figure, is that the final function resembles the form of a piecewise linear function. For lower cash on hand values, the function commands the agent to spend everything she has, while for higher x values there should be some saving.

This is a very similar result to the one of Carroll. In Carroll [1996] and Carroll [2004] he finds that under certain conditions regarding the impatience of the consumer, at every instant cash on hand will be guided back to a fixed target level. This means that the optimal consumption rule can be expressed as

$$f(x_t) = e_{opt} + g(x_t - x_{opt}),$$

where x_{opt} is the optimal level of cash on hand, that the agent always seeks to attain and e_{opt} is the agent's expenditure level for the case in which it actually reaches $x_t = x_{opt}$. Of course this implies that $g(0) = 0$.

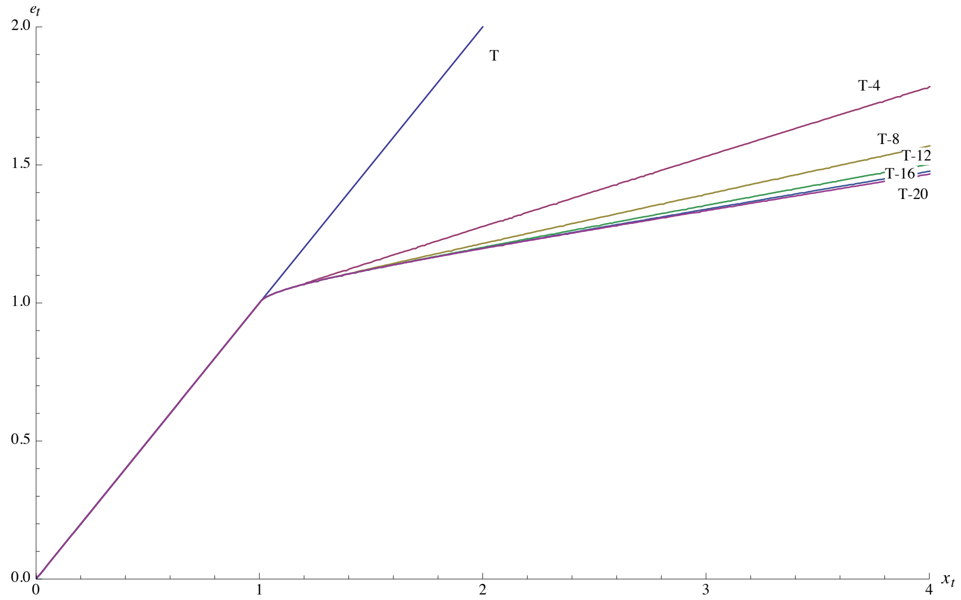


Figure 12: Convergence of the optimal consumption function for i.i.d. interest rates ($\rho = 0.1,\ \mu = 1.1,\ \sigma = 0.05,\ w = 1$)

Taking the first order taylor expansion around the point $x_t = x_{opt}$ yields

$$f(x_t) \approx e_{opt} + g'(0)(x_t - x_{opt})$$

If we change parameters, we can write this affine linear function in a form, which is better suited for the analytical challenges we will face later on.

$$f(x_t) \approx w + \beta^*(x_t - \bar{x}^*)$$

Even though the interpretation of the two new parameters of the function are not as clear as the one for e_{opt} and x_{opt} we can still say that the optimal cash on hand level is determined by $\bar{x}^*$, and the function's slope is controlled by β^*.

Since we agreed on the fact that agents cannot spend more than x_t, we need to make a definition by cases analogously to Allen and Carroll [2001].

$$f^\theta(x_t) = \begin{cases} w + \beta(x_t - \bar{x}) & \text{if} \quad w + \beta(x_t - \bar{x}) \leq x_t \\ x_t & \text{if} \quad w + \beta(x_t - \bar{x}) > x_t \end{cases} \tag{14}$$

where $\theta = \{\beta, \bar{x}\}$ are the only two parameters that describe the function. For now let us assume that every agent knows from the start about the optimal parameters θ^* leading to the correct first order taylor expansion. Parameters being "optimal" means that they are optimal corresponding to the believes the agent has about the mean and standard deviation of the random interest rate process. Perhaps this process does not really exist, but the point of this paper is to study what happens if the agents believe it does.

3.4 System behavior with piecewise linear consumption

In this subsection we make use of the piecewise linear expression in (14) to be able to derive some analytical results.

We now have a complete dynamical system defined by the equations (6), (7), (10) and (14). The system for the two country model looks as follows

$$
\begin{gathered}
K_{t+1} = \underbrace{(1 + \frac{\pi_t}{P_k} - \delta)}_{r_t} K_t + \frac{wL - E_t}{P_k}, \; K^*_{t+1} = \underbrace{(1 + \frac{\pi^*_t}{P_k} - \delta)}_{r^*_t} K^*_t + \frac{wL * - E^*_t}{P_k} \\
\text{with} \quad \pi_t = (\frac{\alpha}{\sigma})(s_{11}E_{t-1} + s_{12}E^*_{t-1}), \; \pi^*_t = (\frac{\alpha}{\sigma})(s_{21}E_{t-1} + s_{22}E^*_{t-1}) \\
s_{11} = \frac{1}{K_t + \phi K^*_t}, \; s_{12} = \frac{\phi^*}{\phi^* K_t + K^*_t}, \; s_{21} = \frac{\phi}{K_t + \phi K^*_t}, \; s_{22} = \frac{1}{\phi^* K_t + K^*_t}, \\
X_t = K_t r_t P_k + wL, \quad X^*_t = K^*_t r^*_t P_k + wL^* \\
E_t = \begin{cases} wL + \beta(X_t - \bar{X}) & \text{if} \quad wL + \beta(X_t - \bar{X}) \leq X_t \\ X_t & \text{if} \quad wL + \beta(X_t - \bar{X}) > X_t \end{cases} \\
E^*_t = \begin{cases} wL^* + \beta^*(X^*_t - \bar{X}^*) & \text{if} \quad wL^* + \beta^*(X^*_t - \bar{X}^*) \leq X^*_t \\ X^*_t & \text{if} \quad wL^* + \beta^*(X^*_t - \bar{X}^*) > X^*_t \end{cases}
\end{gathered}
\tag{15}
$$

Note that in this formulation capital letters are used for expenditures, capital and cash on hand, because agents behave homogeneously. Everybody is assumed to have the same function for expenditures and the quantities are just summed up, which leads directly to the dynamics for the aggregate values. L and L^* are the sizes of the labor forces in both countries. Although we agreed on w being normalized to 1, it is explicitly mentioned in the above equation just for clarification.

To make things simpler and to further pursue the idea that consumers are homogenous, we assume the consumption function parameters to be constant across borders as well. That is we take $\bar{X} = \bar{X}^*$ (, which is the nation's aggregated optimal cash on hand) and $\beta = \beta^*$. Furthermore we consider the case of symmetric regions for now (i.e. $L = L^*$).

Now let us consider the steady state equilibria of the dynamical system. It turns out that there are five of them. One symmetric equilibrium, two core periphery equilibria and two intermediate ones. The symmetric equilibrium has the following constant values for capital and expenditures, respectively.

$$K = K^* = \frac{L(\beta\sigma - \alpha) + \beta\bar{X}(\alpha - \sigma)}{P_k(\alpha\beta + \sigma(\delta(\beta - 1) - \beta))}, \quad E = E^* = -\frac{\sigma(L(\delta + \beta) - \delta\beta\bar{X})}{\alpha\beta + \sigma(\delta(\beta - 1) - \beta)}$$

The core periphery equilibrium with the home country as the core has the values

$$K = \frac{L(\alpha(\beta - 2) + \beta\sigma) + \beta\bar{X}(\alpha - \sigma)}{P_k(\alpha\beta + \sigma(\delta(\beta - 1) - \beta))}, \; K^* = 0,$$

$$E = \frac{\delta\beta\sigma\bar{X} - L(\alpha\beta + \sigma(\delta + \beta))}{\alpha\beta + \sigma(\delta(\beta - 1) - \beta)}, \; E^* = L.$$

Of course the other core periphery equilibrium looks exactly the same with the values between the regions interchanged, because of the symmetry between countries.

The expressions for the fourth and fifth equilibrium are very lengthy. As opposed to the other steady states of the system, the expressions for this equilibrium depend on ϕ. In the diagrammatic analysis below, we will see why this makes sense. Because of their limited range of existence, the huge mathematical terms involved, as well as their instability, these steady states shall not be further discussed here.

To draw inferences about the stability of the equilibria, we need to calculate the eigenvalues of the Jacobian of the vector valued function

$$\begin{pmatrix} K_{t+1}(K_t, K_t^*, E_t, E_t^*, E_{t-1}, E_{t-1}^*) \\ K_{t+1}^*(K_t, K_t^*, E_t, E_t^*, E_{t-1}, E_{t-1}^*) \\ E_t(K_t, K_t^*, E_{t-1}, E_{t-1}^*) \\ E_t^*(K_t, K_t^*, E_{t-1}, E_{t-1}^*) \end{pmatrix} = \begin{pmatrix} K_{t+1}(K_t, K_t^*, E_{t-1}, E_{t-1}^*) \\ K_{t+1}^*(K_t, K_t^*, E_{t-1}, E_{t-1}^*) \\ E_t(K_t, K_t^*, E_{t-1}, E_{t-1}^*) \\ E_t^*(K_t, K_t^*, E_{t-1}, E_{t-1}^*) \end{pmatrix}$$

Let J be the Jacobian of this function. Then the eigenvalues of J evaluated at the respective equilibrium point will give information about its asymptotic stability.

Of course the resulting 4×4 matrix is anything but easy to handle. But luckily J has some nice symmetry properties that enable us to calculate the eigenvalues at the two equilibrium points.

3.4.1 Stability properties of the symmetric equilibrium

In our further analysis, we must distinguish between the cases $\bar{X} < L$ and $\bar{X} \geq L$, because the system behaves fundamentally different in these parameter ranges. But how can we interpret these two parameter regions? For $\bar{X} < L$ (which means $\bar{x} < w$) the agent is impatient enough to spend his whole labor income when this is the only thing he has at hand. On the other hand, when $\bar{X} > L$, the agent is willing to save a certain part of his wage. To be more formal, at the point where $\bar{X} = L$ an agent with an empty bank account will be indifferent between consuming everything he has in each period and saving a very small amount ϵ for the future. So we have

$$log(w) + (1-\rho)log(w) = log(w-\epsilon) + (1-\rho)E_t\left[log(w + r_{t+1}\epsilon)\right].$$

If we rearrange and take the limit

$$1-\rho = \lim_{\epsilon \to 0} \frac{log(w-\epsilon) - log(w)}{log(w) - E_t\left[log(w + r_{t+1}\epsilon)\right]},$$

which would yield $1-\rho = 1/r$ for a perfectly predictable interest rate, but cannot be simplified further for the general case of a distribution of r.

In both cases that we distinguish ($\bar{X} < L$ and $\bar{X} \geq L$) the eigenvalues of the symmetric equilibrium are

$$\begin{aligned}
\lambda_{sym,1} &= \lambda_{sym,2} = 0 \\
\lambda_{sym,3} &= \frac{\alpha\beta + (1-\delta)(1-\beta)\sigma}{\sigma} \\
\lambda_{sym,4} &= \frac{1}{\sigma(\phi+1)^2(\mathrm{wL}(\beta\sigma-\alpha) + \beta\bar{X}(\alpha-\sigma))}\cdot \\
&[\mathrm{wL}(\alpha^2\beta(\phi^2-1) - \alpha\sigma(\delta(\beta-1)(\phi-1)^2 + \beta^2(\phi^2-4\phi-1) - \beta(\phi-1)^2 + (\phi+1)^2) \\
&+(\delta-1)(\beta-1)\beta\sigma^2(\phi+1)^2) + \beta\bar{X}(\alpha^2(\beta-\beta\phi^2) + \alpha\sigma(\delta(\beta-1)(\phi-1)^2 \\
&-(\phi+1)(2\beta-\phi-1)) - (\delta-1)(\beta-1)\sigma^2(\phi+1)^2)].
\end{aligned} \tag{16}$$

Proposition 3. *Under the dynamics of system (15), for* $\beta < \frac{\alpha L}{\sigma wL + \alpha\bar{X} - \sigma\bar{X}}$ *and* $wL < \bar{X}$ *there exists a level of trade freeness* ϕ_{crit} *with* $0 < \phi_{crit} < 1$ *such that for* $\phi \leq \phi_{crit}$ *the symmetric equilibrium is stable and for* $\phi > \phi_{crit}$ *it is unstable.*

Proof. For the symmetric equilibrium point to be asymptotically stable all eigenvalues must be less than 1 in absolute value. For $\lambda_{sym,3}$ this is given because $1 = \beta + (1-\beta) > \beta + (1-\delta)(1-\beta) > \frac{\alpha}{\sigma}\beta + (1-\delta)(1-\beta) = \lambda_{sym,3}$. The first inequality holds because $0 < \delta < 1$ and the second one because $0 < \alpha < 1 < \sigma$, according to our (fairly weak) assumptions. Since every term involved is nonnegative, we also have $\lambda_{sym,3} \geq 0$.

The situation concerning $\lambda_{sym,4}$ is more complex. Proposition 3 is proven if we show:

1. $\lambda_{sym,4}(\beta, \phi)$ is continuous on the interval $\beta \in [0, \frac{\alpha L}{\sigma \mathrm{wL} + \alpha\bar{X} - \sigma\bar{X}})$, $\phi \in [0, 1]$.

2. $\lambda_{sym,4}(\beta_{fix}, \phi) = 1$ has at most 2 solutions for ϕ if we fix $\beta_{fix} \in [0, \frac{\alpha L}{\sigma \mathrm{wL} + \alpha\bar{X} - \sigma\bar{X}})$.

3. $\lambda_{sym,4}(\beta, 0) < 1$ for $\beta \in [0, \frac{\alpha L}{\sigma \mathrm{wL} + \alpha\bar{X} - \sigma\bar{X}})$

4. $\lambda_{sym,4}(\beta, 1) \geq 1$ for $\beta \in [0, \frac{\alpha L}{\sigma \text{wL}+\alpha\bar{X}-\sigma\bar{X}})$

The first point is the most straightforward one, since the only discontinuity point of $\lambda_{sym,4}(\beta, \phi)$ is where its denominator crosses 0. This happens for $\beta = \frac{\alpha L}{\sigma\text{L}+\alpha\bar{X}-\sigma\bar{X}}$. For smaller values the denominator is negative and the function $\lambda_{sym,4}(\beta, \phi)$ is continuous.

Point 2 is readily observed when we set $\lambda_{sym,4} = 1$ and see that this results in a quadratic equation in ϕ. This implies that there are at most two solutions to it.

To establish point 3, let us calculate the term for $\lambda_{sym,4}(\beta, 0)$. It simplifies to $\frac{\alpha\beta+(1-\delta)(1-\beta)\sigma}{\sigma}$, which is the same term as the one for $\lambda_{sym,3}$. So we know from the initial argumentation that it is ≤ 1.

Point 4 is shown if we show that

$$\lambda_{sym,4}(\beta, 1) - 1 = \frac{\beta(\text{L} - \bar{X})(\alpha\beta + \sigma(\delta(\beta - 1) - \beta))}{\text{L}(\beta\sigma - \alpha) + \beta\bar{X}(\alpha - \sigma)}$$

is positive for $\beta \in [0, \frac{\alpha L}{\sigma\text{wL}+\alpha\bar{X}-\sigma\bar{X}})$. Setting the expression equal to 0 and solving for β gives the solutions 0 and $\frac{\delta\sigma}{\alpha+\delta\sigma-\sigma}$. It is easy to see that the latter value is never in $[0, 1]$, so it is never in the considered interval. Now if the derivative of $\lambda_{sym,4}(\beta, 1)$ at $\beta = 0$ is positive, then we are done.

$$\frac{\partial\lambda_{sym,4}}{\partial\beta}|_{\beta=0,\phi=1} = \frac{(L - \bar{X})\delta\sigma}{L\alpha}$$

This expression is clearly positive for $L > \bar{X}$.

□

Increasing β above $\frac{\alpha L}{\sigma\text{wL}+\alpha\bar{X}-\sigma\bar{X}}$ leads to an interval of $\lambda_{sym,4} < -1$ before indicating stability again. Since such high values for β are quite unrealistic (none of our simulated consumption functions came close to this value), this case won't be discussed in further detail.

For $\bar{X} > L$ the very last part of the proof does not work anymore. In fact, the considered derivative will always be negative which means that for high trade freeness the eigenvalue is less than one. But there will still be some intermediate values of ϕ for which the symmetric equilibrium gets destabilized by raising $\lambda_{sym,4}$ above 1. This is true until a critical value of $\bar{X}$, call it $\bar{X}_{crit}$ is crossed, above which the eigenvalue is always less than 1. This can be shown by demonstrating that $\lambda_{sym,4}$ is monotonically decreasing in $\bar{X}$ and in the limit for $\bar{X} \to \infty$ everywhere smaller than 1.

$\bar{X}_{crit}$ can be analytically determined by solving $\lambda_{sym,4} = 1$ for ϕ, then setting the square root component of the solution equal to zero and solving for $\bar{X}$. The analytical expression for $\bar{X}_{crit}$ can be found in the appendix.

As mentioned, beyond $\bar{X}_{crit}$ the eigenvalues are all smaller than 1. However, for some very weird parameter combinations including an unrealistically huge depreciation rate, $\lambda_{sym,4}$ can become smaller than -1. So we cannot make anymore general statements about the stability of the symmetric equilibrium here.

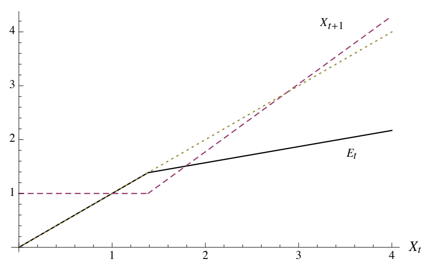
3.4.2 Stability properties of the core-periphery equilibrium

For $\bar{X} < L$, the eigenvalues of the core-periphery equilibrium are

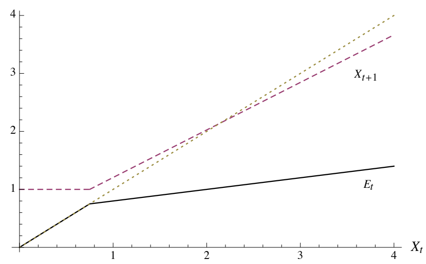
$$\begin{aligned} \lambda_{cp,1} &= \lambda_{cp,2} = \lambda_{cp,3} = 0 \\ \lambda_{cp,4} &= \frac{\alpha\beta + (1-\delta)(1-\beta)\sigma}{\sigma}. \end{aligned} \tag{17}$$

This means that for small $\bar{X}$, the cp-equilibrium is always stable, independently of the other parameter values.

The intuition for this is straightforward. Consider the consumption function in Figure 13 (a). In the diagram, the dynamical system is broken down to a single dimensional system, abstracting from the effect of X_t of r_t. In fact, because of the multiple dimensions it is not possible to depict X_{t+1} as a line in a diagram like this. Nevertheless, it can help understand what is going on. Let's assume a population size of $L = 1$. In the core-periphery case, the peripheral country will be at $X_t = 1$, which is exactly where the dashed line, indicating X_{t+1}, crosses the 45° line. After a small capital shock the country will be back in the equilibrium point by the next period, because the consumers will spend everything they have at hand, no matter if it is a little bit less or a little bit more than the steady state value. The core country will be at the point where the dashed line crosses the 45° line for the second time. At first sight this point is unstable because the slope of the X_{t+1} curve is greater than unity. But there is another important mechanism in our model, namely that the interest rate rises if the capital decreases and vice versa. Of course a change in the interest rate will also influence the X_{t+1} curve in the graph. In particular, it means that in case of a small positive capital shock for the core country, the slope of X_{t+1} will drop which will lead to the crossing point with the 45° line to shift to the right so that the capital stock will move back to its steady state value. Note that this effect is independent of trade costs, since the capital stock in the peripheral country is 0 anyway, so that the core country is the only one influencing its rate of return.



(a) $\bar{X} < L$



(b) $\bar{X} > L$

Figure 13: Stability of the asymmetric equilibrium

For completeness we also need to mention that for $\bar{X} < L$ both countries having a zero capital stock and consuming their wage income in every period is also an equilibrium. In Figure 13 this would correspond to both counties residing at $X_t = 1$. In this scenario the interest rate would rise to infinity, which makes it essentially unrealistic, because consumers would definitely be willing to save in this case. However, since their mental capabilities are kind of limited here and they continue assuming that this interest rate was just a very lucky draw out of the constant distribution, the equilibrium exists here. This is about to change later on in this paper.

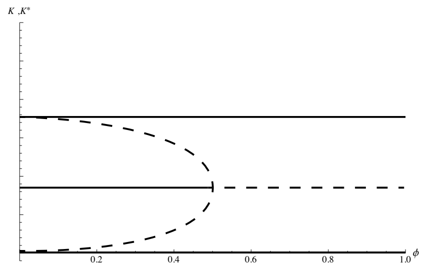
For $\bar{X} > L$ the pure cp-equilibrium does not exist anymore. Consider Figure 13 (b). Now there is only one point, where the X_{t+1} curve crosses the 45° line. This is the point that corresponds to the "core" in Figure 13 (a). This alone does not mean that there cannot exist any asymmetric equilibrium. However, now there can never be a country which is truly the "periphery" in the sense of having a zero capital stock. To see this, note that according to equation (9), we must have $x_t = w = 1$, which means $X_t = L$ in order for the capital stock to be 0. For the X_{t+1} curve to cross the 45° line at 1, we must have $r = 0$. This in turn would require an infinitely large capital stock in one of the two countries, which is impossible.

Finally, there is the boundary case for which $\bar{X} = L$. Here the kinks of the consumption curve and the future cash on hand curve coincide. For a particular rate of return the two curves can even totally coincide to the left of 1. So a core periphery situation does definitely exist, but its stability properties in this scenario are highly non-trivial.

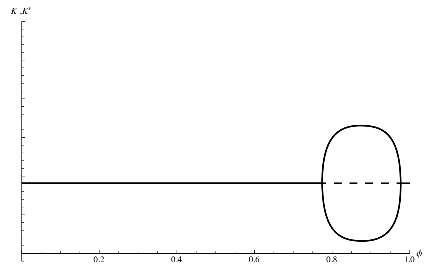
3.5 Bifurcation analysis of the piecewise linear case

The central object in the new economic geography are the transportation costs or the corresponding value of trade freeness. These transportation frictions are the channel through which the idea of distances is implemented in NEG models. For this reason the so called tomahawk diagrams have become a tradition in the field. For the case of symmetric transportation costs, they proved to be a very useful instrument for evaluating the existence and stability of equilibria.

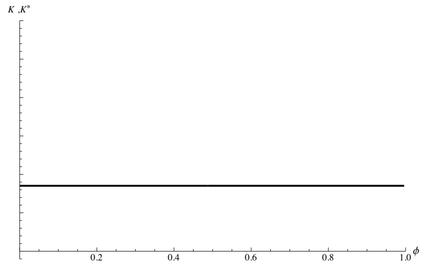
In this section, we aim to apply this instrument to our model in order to find out wether our predictions of the previous section were correct as well as to gain a better understanding of the system's behavior depending on ϕ. Figure 14 shows the equilibrium capital stocks of the two symmetrically sized nations for all levels of trade freeness. Note that (even thought the diagrams shall be of qualitative nature), as opposed to many other authors, I plot absolute values of K and K^* instead of world capital stock shares on the vertical axis. The main reason for this is better comparability with the numerical experiments further below.



(a) $\bar{X} < L$



(b) $L \leq \bar{X} \leq \bar{X}_{crit}$



(c) $\bar{X}_{crit} < \bar{X}$

Figure 14: "Tomahawk" diagram for piecewise linear consumption behavior

Figure 14 (a) shows the equilibria of our system for high consumption function intercepts $(\bar{X} < L)$. As explained, in this parameter case, the cp-equilibrium is always stable, which is indicated by the solid lines at the top and bottom of the diagram. For low trade freeness, the symmetric equilibrium is also stable before at a certain value ϕ_{crit} it becomes unstable.

When analyzing the eigenvalues we saw that for $L \leq \bar{X} \leq \bar{X}_{crit}$ the symmetric equilibrium will be stable for very high and very low trade freeness, while being unstable for some intermediate range. This is exactly what can be observed in Figure 14 (b). As stated in subsection 3.4.2, the cp-equilibrium does not exist anymore in its pure form, but vanishes in favor of an asymmetric equilibrium no longer featuring full agglomeration. Furthermore, for different levels of trade freeness, the transitions are not catastrophic anymore, but rather smooth now.

Finally, for $\bar{X}$ values beyond the critical threshold that was discussed above the only existing steady state is the symmetric one.

Now let us attempt a first comparison of these findings with the results of Baldwin [1999]. The reader shall be reminded that the central question we seek to answer is whether a more realistic consumption behavior with expectations concerning the interest rate alter the agglomeration tendencies between regions in this type of model. In this subsection we replaced the perfect foresight Euler equation resulting from an optimal control approach by an approximation to the consumption function that resembles optimal behavior when agents assume i.i.d. interest rates. For large intercepts of the non liquidity constrained part of the consumption function (corresponding to a kink further to the right) we can partly confirm the results from the original model. In our model as well as the Baldwin case, there exists a critical trade freeness above which the symmetric equilibrium gets unstable. However, a new kind of information is that in this parameter case, the cp-equilibrium is stable, regardless of the level of trade costs. Also new are the results for large $\bar{X}$, where the qualitative system behavior exhibits radically new forms.

3.6 Bifurcation analysis for simulated consumption functions

In the previous subsections, we made use of the piecewise linear approximation to the optimal consumption functions. A piecewise linear function not only resembles the form of the true consumption function very closely, but also performs well in terms of sacrificed utility, when used instead of the true function. However, taking a close look at the shape of the numerically determined solution for the optimal consumption function resulting from our inter temporal optimization problem reveals what one would expect, namely that instead of the kink there is a smooth curvature which continues for higher values of x_t. In this subsection we want to investigate whether this makes a difference with respect to the system's behavior.

When replacing the approximation with the true function, the analysis we conducted in Figure 13 will not be the same. It is hard to make any conjectures about what might happen, but even a small alternation to the form of the consumption especially around the kink might play a central role.

In Figure 15 a bifurcation diagram for our model using a numerically determined consumption function is depicted. For the consumption function the agents assume a real rate of return drawn from a lognormal distribution with mean 1.05 and standard deviation 0.05. The rest of the parameters are given in the Title of Figure 15. [13]

[13] The reason for using these apparently odd numbers for the random process is that with these parameters the distribution has a mean of 1.05 and a standard deviation of 0.05

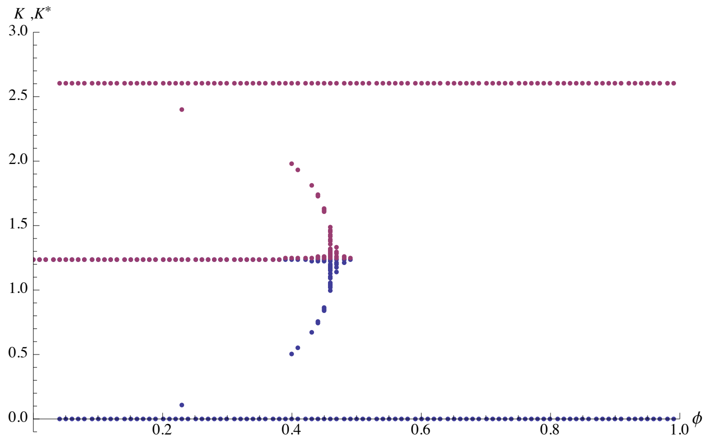


Figure 15: Equilibria depending on trade freeness for assumed i.i.d. interest rates ($\rho = 0.05,\ \alpha = 0.4,\ \sigma = 2,\ \delta = 0.08,\ \mu = 1.0488,\ \sigma = 0.04759$)

The equilibrium structure is similar to the piecewise linear case for low values of $\bar{X}$ from Figure 14. The cp-equilibrium is stable for all trade cost levels and the symmetric equilibrium gets destabilized above a certain level of ϕ. Please note that the nature of the diagram in Figure 15 is different from the one in Figure 14. While the curves in the latter result from the analytical expressions for the equilibrium points, we are now only able to numerically analyze the system's behavior for a very fine grid of initial conditions. This is the only reason why also the unstable equilibrium points partly appear in the diagram. Interestingly this also reveals the fact that the intermediate equilibrium points do not seem to be very strong repellers. Even after 10000 periods the system is still in a state very close to the unstable equilibrium, provided it started sufficiently close to it. This is in contrast to the symmetric equilibrium for high trade freeness. Even for appropriate initial conditions the system moves away very quickly. All in all, we are able to say that the structure of equilibria in this parameter case seems to resemble the well known tomahawk structure which we find in many NEG models, among them the original core periphery model by Krugman [1990].

Before we finish this subsection, let us conduct some comparative statics. Note that the sharply curved part of the consumption function just used lies somewhere around L, which means that if we were to find a piecewise linear function approximating the consumption function resulting from the parameter case just used, the value for $\bar{X}$ would also lie in this region.

If we were to increase the mean of the interest rate distribution assumed by the agents, the right part of the consumption function would shift down and the kink would move to the left. This makes perfect sense, because with a ceteris paribus higher expected interest rate it is better for the agents to consume less and entertain a higher capital stock. A recall of chapter 3.5 lets us suspect a stabilization of the symmetric equilibrium and a disappearance of the cp-solution. This is exactly what happens. First the tomahawk like area shifts to the right before vanishing, leaving the symmetric case as the only stable steady state.

A similar effect occurs, when lowering ρ reflecting the impatience parameter of the consumers. Of course a ceteris paribus more patient agent will have a lower consumption function resulting in the same effect concerning the equilibria as above.

An operation going into the opposite direction will shift the tomahawk like part of the graph to the left and then shrink the equilibrium capital stocks of both, the symmetric and the asymmetric equlibria including the cp-solution.

3.7 Autoregressive return expectations

The next step is to let the agents no longer believe that the rates of return are i.i.d., but that they follow an AR1 process. Agents now think that r_{t+1} is determined as follows.

$$log(r_{t+1}) - log(\bar{r}) = (log(r_t) - log(\bar{r}))\xi + \epsilon_t$$

where $\epsilon_t \sim N(0, \sigma)$ is a normally distributed shock and $\bar{r}$ is the interest rate that the system is expected to converge back to on average. So the larger ξ, the larger is the effect of a high or low interest rate that gets carried over to the next period. The stochastic Euler equation can be derived analogously to above, only that now the consumption function has not one but two arguments, namely the state variables x_t and r_t.

We now seek to find the function that satisfies

$$\frac{1}{f(x_t, r_t)} = (1-\rho)E\left[\frac{r_{t+1}}{f(x_{t+1}, r_{t+1})}\right].$$

Figure 16 depicts the new function $f(x_t, r_t)$ that was numerically determined using the same method from above. The difference is that now we have to iterate over a two dimensional state space, which makes the computation a lot more effortful.

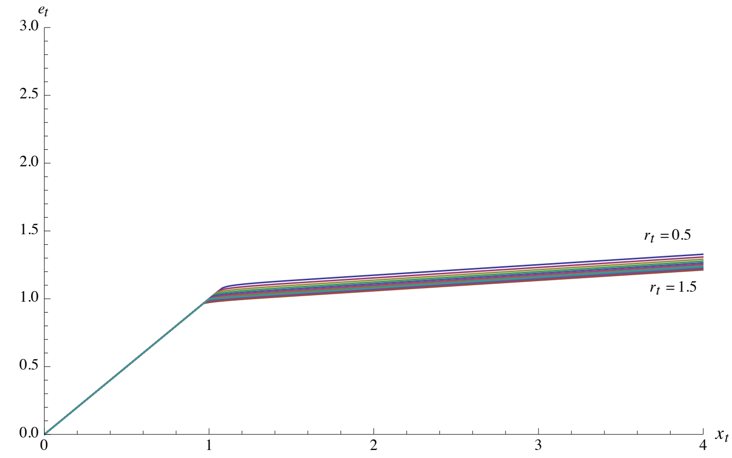
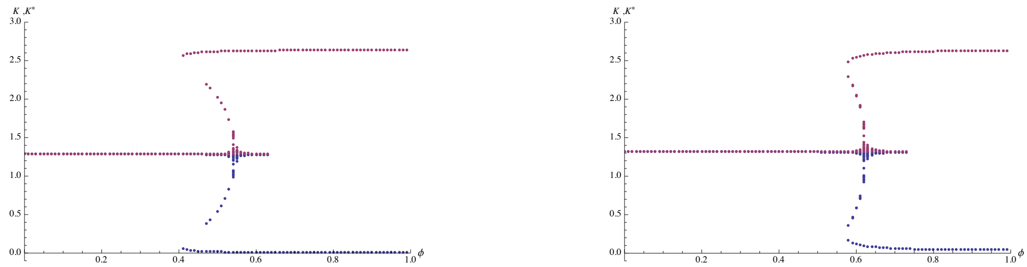


Figure 16: Optimal consumption function for AR1 interest rates ($\rho = 0.05,\ w = 1,\ \bar{r} = 1.0488,\ \sigma = 0.0476,\ \xi = 0.1$)

Let us now include this new function as the consumers behavior into our model. Figure 17 shows the bifurcation diagram using $\xi = 0.1$, $\bar{r} = 1.0488$ and $\epsilon_t \sim \mathcal{N}(0, 0.0476)$. Note that for $\xi = 0$ this process would be the same as the one considered in the previous subsection, having a mean of 1.05 and a standard deviation of 0.05. In the diagram we see that an assumed autocorrelation between the interest rates in successive periods will have a stabilizing effect on the two region system. The tomahawk like part of the graph gets shifted to the right and it gets shrinked a bit so that before the cp-equilibrium gets unstable, it looses its nature of being a true core periphery state.



(a) $\xi = 0.1$

(b) $\xi = 0.2$

Figure 17: Optimal consumption function for AR1 interest rates ($\rho = 0.05,\ w = 1,\ \bar{r} = 1.0488,\ \sigma = 0.04759$)

This trend continues for higher values of ξ until eventually only the symmetric equilibrium remains in the system. [14] This leads us to the conclusion that highly

[14]Note that the intermediate equilibria only appear in the graphs because of numerical reasons, just like above. The same applies for the small symmetric part after the intermediate equilibria vanished, where the symmetric state is just not repelling enough for the system to leave it after 10000 periods.

anticipated interest rate persistence has a strong stabilizing effect on the symmetric outcome. The intuition is that if agents believe that a high interest rate will somewhat get carried over to the next period, they will save more if the interest rate is high and save less if it is low. Since the home capital stock is the major magnitude of influence on the home interest rate, it is only natural that (at least as long as trade barriers are not extraordinarily low) it will pull the system towards the symmetric state.

3.8 Disequilibrium dynamics

In the preceding subsections we saw that under certain conditions several steady states of the system can be stable at the same time. But a very important question is wether it is possible to transfer the system from one equilibrium to another. I want to go even further and ask, whether such an equilibrium transition can be accomplished when fueled only by a temporary shock in expectations.

The numerical experiment that is conducted is fairly simple. First, the system is initialized with conditions close to or at the symmetric equilibrium. For the consumption behavior, the function $e_t = f(x_t, r_t)$ is used, which assumes an AR1 process for the returns, just like above. We use the same parameters from Figure 17 (b) ($\rho = 0.05,\ w = 1,\ \bar{r} = 1.0488,\ \sigma = 0.0476,\ \xi = 0.2$) in both countries. Then we wait for 500 periods to make sure the system has converged to the symmetric steady state, before we introduce an expectational shock in one of the countries, in this case the foreign one. For t_{shock} periods, the agents in the foreign country now believe that the fundamental rate of return $\bar{r}$ is 1.1 instead of 1.0488. After this, the original expectations are again in place.

Figure 18 shows the expenditures, the capital stocks, and the profits in both countries for 2000 periods. The solid line indicates the system's behavior for $t_{shock} = 1$, whereas for the dashed line $t_{shock} = 8$ is used. It can be seen that in both regions, especially in the foreign country, where the shock is applied, expenditures drop considerably right after period 500. As a consequence of the drastically reduced expenditures, the capital stock in the foreign country increases. On the contrary, the capital stock in the home country drops. This is because the reduced exports to the foreign region and the hereby decreased rental rate hurt the home country's capital stock more than the additional savings could compensate. Now, in the case of a single period shock (solid line), all the variables slowly converge back to their symmetric equilibrium values. As opposed to this, for a larger shock endurance (dashed line) the economic variables reside at some asymmetric values before converging to the "close to core periphery" state, discussed in the previous section.

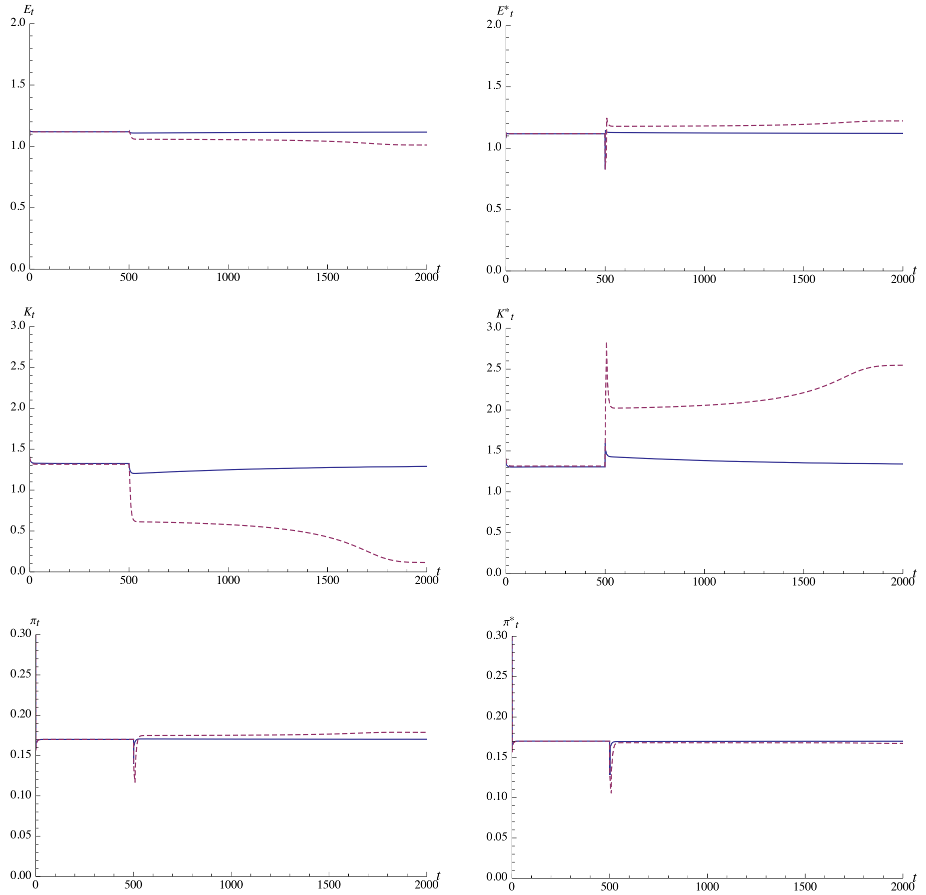


Figure 18: Dynamics starting at the symmetric steady state with a 1 (solid) and 8 (dashed) period expectational shock in the foreign country at $t = 500$ ($\phi = 0.6,\ \rho = 0.05,\ \alpha = 0.4,\ \sigma = 2,\ \delta = 0.08,\ w = 1,\ \bar{r} = 1.0488,\ \sigma = 0.0476,\ \xi = 0.2$)

In the next set of graphs in Figure 19 the opposite situation is shown. Here the system starts from a core periphery state. Then, the same expectational shock as above is applied. The question would be wether it is also possible to get from a cp-equilibrium into a symmetric steady state. As can be seen in the Figure, for a large enough shock this is indeed possible. Again, the single period shock does not suffice to leave the cp-state in the long run, whereas the longer 8 period shock pushes the system into the symmetric equilibrium, where it then resides forever. Another interesting fact, which is not depicted in any of the graphs is that if we increase the trade freeness to a level, where the symmetric equilibrium is no longer stable, the system may also be transferred from one cp-equilibirum to the other.

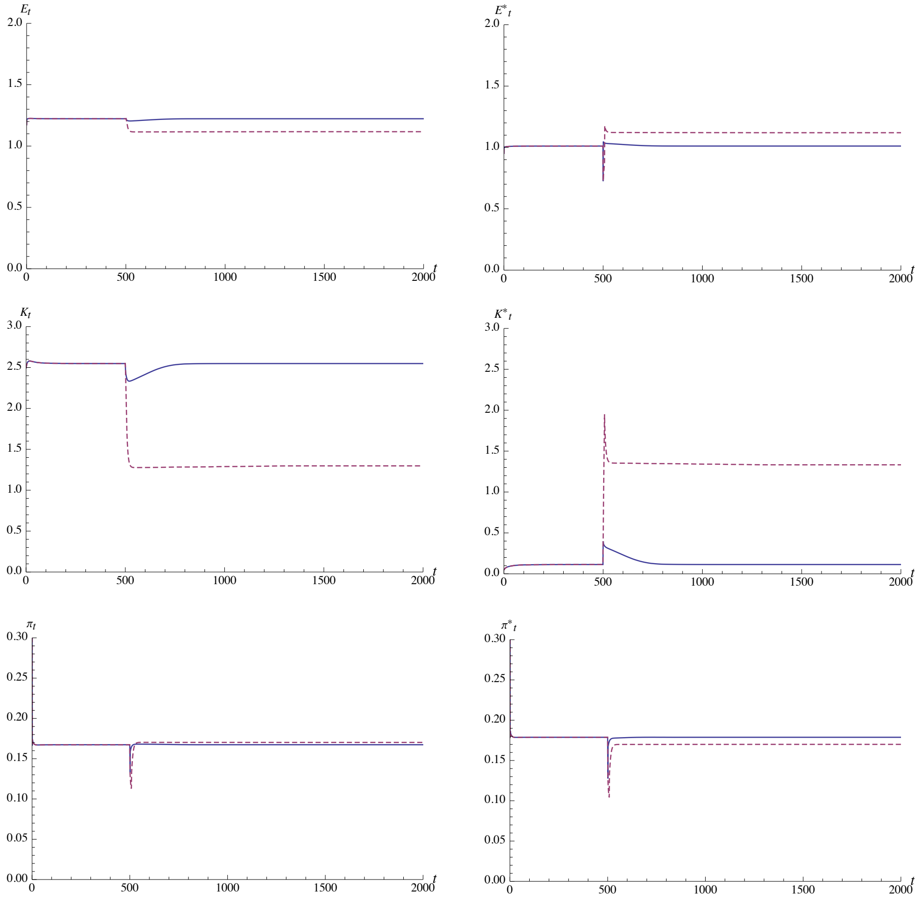


Figure 19: Dynamics starting at an asymmetric steady state with a 1 (solid) and 8 (dashed) period expectational shock in the foreign country at $t = 500$ ($\phi = 0.6,\ \rho = 0.05,\ \alpha = 0.4,\ \sigma = 2,\ \delta = 0.08,\ w = 1,\ \bar{r} = 1.0488,\ \sigma = 0.0476,\ \xi = 0.2$)

3.9 N-Region Model

So far we only considered our framework in the context of two regions or countries. A possible next step would be to ask what happens if we were to extend the model to N regions. Of course, we can no longer determine the steady states analytically when choosing a large N, but we want to consider the behavior of the model with the help of some simulations. The generalized model is

$$\begin{aligned}
K_{i,t+1} &= \underbrace{(1 + \frac{\pi_{i,t}}{P_k} - \delta)}_{r_{i,t}} K_{i,t} + \frac{w - E_{i,t}}{P_k}, \\
\text{with} \quad \pi_{i,t} &= (\frac{\alpha}{\sigma}) \sum_{j=1}^{N} s_{ij} E_{j,t-1} \\
s_{ij} &= \frac{\phi_{ij}}{\sum_{j=1}^{N} \phi_{ij} K_{j,t}}, \\
X_{i,t} &= K_{i,t} r_{i,t} P_k + wL, \\
E_{i,t} &= \begin{cases} wL + \beta(X_{i,t} - \bar{X}) & \text{if} \quad wL + \beta(X_{i,t} - \bar{X}) \leq X_{i,t} \\ X_{i,t} & \text{if} \quad wL + \beta(X_{i,t} - \bar{X}) > X_{i,t} \ . \end{cases}
\end{aligned} \tag{18}$$

As can be seen, the generalization does not complicate the formulas all too much. The law of motion for the state variable K stays the same and country i's profit now depends on the expenditure shares the country receives from *all* other countries. To calculate all shares s_{ij} we need a matrix of values reflecting the trade freeness between each two regions. Let us consider a grid of regions in which transportation costs are proportional to the distance d_{ij} between them.

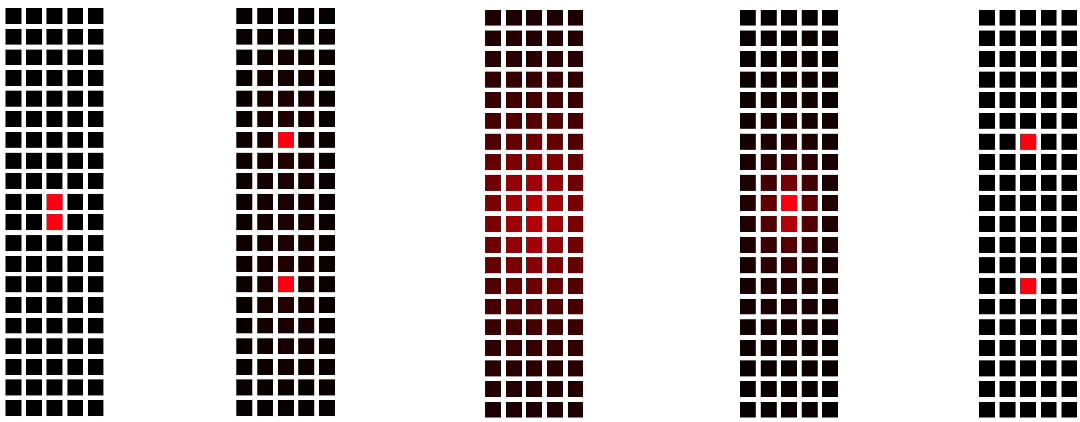
$$\begin{aligned}
\phi_{ij} &= (1 + t_{step} d_{ij})^{1-\sigma} \\
d_{ij} &= \sqrt{\Delta x^2 + \Delta y^2},
\end{aligned} \tag{19}$$

where $\Delta x = |x_i - x_j|$ and $\Delta y = |y_i - y_j|$ are the horizontal and vertical steps on the grid between two regions.

Figure 20 shows five simulations for different parameter values. A grid of 100 countries with 20 rows and 5 columns on a flat slice is considered. A flat map implies that the rightmost region faces a distance of no less than 5 steps to the leftmost region, and by the same logic, from top to bottom one has to travel 20 steps. Initially every region possesses the same capital stock with only a tiny variation (to escape a possible unstable symmetric solution). We mainly distinguish between low and high saving tendencies of the consumers reflected by $\bar{X}$ being below or 1, and high and low transportation costs controlled by the parameter t_{step}.

Under a reasonable value for t_{step} of 0.01, implying that 1% of the goods melt away per step on the grid, we observe a core in the middle of our map, which is much more dispersed if consumers tend to save more (Figure 20 (a) vs. (c)). In the case of very high transportation costs, multiple distinct cores on the map are the stable outcome.

Surprising is the model behavior under high $\bar{X}$ values when transportation costs are increased (Figure 20 (c) vs. (d)). In this scenario the core seems to get more concentrated, contradicting the intuitive result from the two country case, in which high transportation costs promoted a symmetric outcome.



(a) $\bar{X} = 0.95$, $\tau_{step} = 0.01$ (b) $\bar{X} = 0.95$, $t_{step} = 0.8$ (c) $\bar{X} = 1.05$, $t_{step} = 0.01$ (d) $\bar{X} = 1.05$, $t_{step} = 0.1$ (e) $\bar{X} = 1.05$, $t_{step} = 0.8$

Figure 20: World of N countries on a slice ($\gamma = 0.1,\ \alpha = 0.4,\ \sigma = 2,\ \delta = 0.08$)

Now, what happens if we get rid of the assumption that our world is a slice with strict boundaries on the sides? A much more realistic assumption concerning the map shape would be that if one exits on one side, one directly reenters on the other. The same applies for left and right, as well as for top and bottom. Connecting the edges like this, gives us a world in the shape of a torus. The distances are calculated just like in (19), only that now $\Delta x = Min(|x_i - x_j|, n_{colums} - |x_i - x_j|)$ and $\Delta y = Min(|y_i - y_j|, n_{rows} - |y_i - y_j|)$.

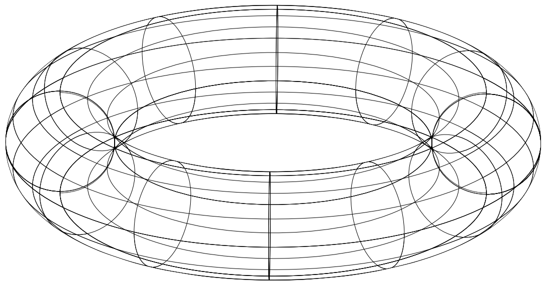
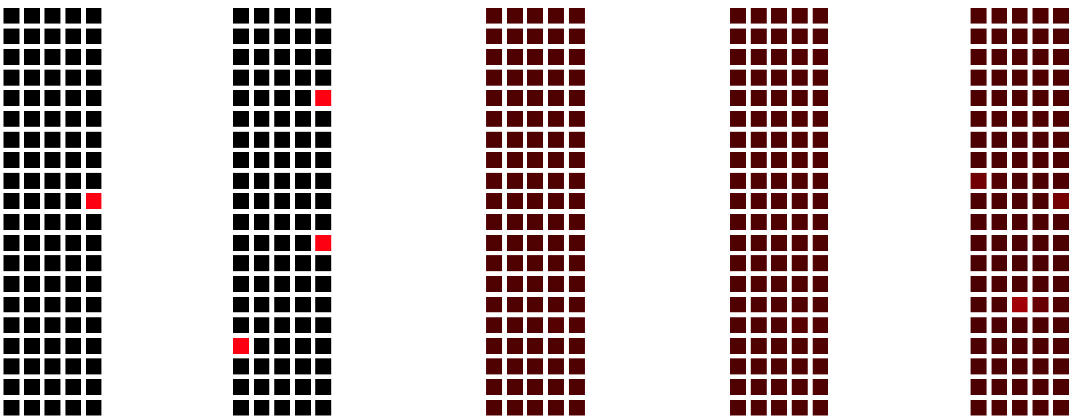


Figure 21: Torus shaped map

Using the same parameter values as in the experiment before, Figure 22 shows some differences in the equilibrium distribution of manufacturing activity across the map. In panel (a) we see again a single industrial core, just like above in Figure 20, only that now the location of the core is not in in the center. In fact, since the wold is symmetric now, depending on the initial conditions, the core can be located anywhere on the map. Increasing the transportation costs leads to the appearance of several cores of manufacturing activity, which are distributed fairly evenly. Of course this looks also different from above.

For the case of more patient consumers (corresponding to a larger $\bar{X}$), we observe a very even distribution of manufacturing activity. The torus map seems to favor the symmetry of the equilibrium outcome a lot. For higher trade barriers this seems to remain true. Only after increasing t_{step} to levels above 0.8, we begin to observe some agglomeration effects.



(a) $\bar{X} = 0.95$, $\tau_{step} = 0.01$ (b) $\bar{X} = 0.95$, $t_{step} = 0.8$ (c) $\bar{X} = 1.05$, $t_{step} = 0.01$ (d) $\bar{X} = 1.05$, $t_{step} = 0.1$ (e) $\bar{X} = 1.05$, $t_{step} = 0.8$

Figure 22: World of N countries on a torus ($\gamma = 0.1,\ \alpha = 0.4,\ \sigma = 2,\ \delta = 0.08$)

So we can state that the form of the world that is considered has an important influence on the nature and magnitude of agglomeration effects in multi country models of larger size.

3.10 Conclusion

We followed Fowler [2007] in building a dynamic new economic geography model, which is not only defined in the steady states, but also outside of it. Agents, who predict their fellows' behavior in all future periods with perfect accuracy were replaced by agents who optimize with the prospect of an uncertain future.

We observe structurally similar results for the case of impatient agents, marked by the well known tomahawk bifurcation diagram. More patient, or concerning the return distribution optimistic agents seem to foster the stability of a symmetric equilibrium and the destabilization of equilibria featuring catastrophic agglomeration. If agents assume an AR1 return process instead of i.i.d. returns, then a higher perceived autocorrelation will stabilize the symmetric equilibrium also in regions of relatively low trade costs. Also, we observed that the core periphery states gradually lose their extreme nature.

Expectational shocks are able to transfer the two region economy from cp-states to symmetric ones and the other way around.

A generalization of the two country model to more countries makes it possible to numerically analyze agglomeration effects on a 2 dimensional map of regions. Low trade costs will result in a single core, in which all manufacturing activity is located. Higher trade costs lead to multiple cores distributed across the map. Agents with a higher willingness to save make the distribution much more dispersed. However, as opposed to the two country model, there are stable solutions not featuring *perfect* symmetry. Finally, we saw that the geometry of the map plays a decisive role for the resulting agglomeration effects. A recommendation for further research into this direction will inevitably bring along a deeper integration of the disciplines of economic geography and spacial economics.

The strong need for a dynamic out of equilibrium extension of the models of the new economic geography is more than obvious. The sharp limitations of analytical instruments cannot be a justification to ignore what happens outside of symmetry, outside of perfect foresight and outside of a two country context. The last point alone showed that broadening the picture in this class of models can considerably enrich the set of results.

A Explicit solution for ϕ_{crit}

Solving $\lambda_{sym,4} = 1$ for ϕ yields the two solutions

$$\begin{aligned}\phi_{crit,1/2} = &\frac{1}{\alpha\beta+\sigma(\delta(-\beta)+\delta+\beta))(L(\alpha-\beta\sigma)+\beta\bar{X}(\sigma-\alpha))}\cdot \\ &[-\alpha\delta\beta\sigma L + \alpha\delta\sigma L - 2\alpha\beta^2\sigma L + \alpha\beta\sigma L + \alpha\delta\beta^2\sigma\bar{X} - \alpha\delta\beta\sigma\bar{X} + \alpha\beta^2\sigma\bar{X} \\ &- \delta\beta^2\sigma^2 L + \delta\beta\sigma^2 + L + \beta^2\sigma^2 L + \delta\beta^2\sigma^2\bar{X} - \delta\beta\sigma^2\bar{X} - \beta^2\sigma^2\bar{X} \pm \frac{1}{2}\sqrt{a_1}\,]\end{aligned}$$

where

$$\begin{aligned}a_1 = &4\sigma^2(L(\alpha(\delta(\beta-1)+\beta(2\beta-1))+\beta\sigma(\delta(\beta-1)-\beta))+\beta\bar{X}(\sigma(\delta(-\beta)+\delta+\beta)-\alpha(\delta(\beta-1)+\beta)))^2 \\ &+ 4(\alpha\beta + \sigma(\delta(\beta-1)-\beta))(\alpha\beta + \sigma(\delta(-\beta)+\delta+\beta))(L(\alpha-\beta\sigma)+\beta\bar{X}(\sigma-\alpha))^2\end{aligned}$$

For $\beta = 0$ we have $a_1 = 0$, which means that in this case there is only one solution. From the proof of proposition 1 we know that $\frac{\partial\lambda_{sym,4}}{\partial\beta}|_{\beta=0,\phi=1} > 0$ for $L > \bar{X}$. Together with the fact that a_1 is positive for positive β, we can say that we get the value for ϕ_{crit} that applies in our range of parameters when we subtract the square root in the above term.

B Explicit solution for $\bar{X}$ s.t. ϕ_{crit} is unique

Because of the quadratic nature of the term of $\lambda_{sym,4}$, we get two solutions for ϕ_{crit}, as seen above. When $\bar{X}$ is increased, then before $\lambda_{sym,4}$ is everywhere smaller than 1 (which is eventually the case, which can be seen taking the limit), there exists an $\bar{X}$, for which there is exactly one ϕ_{crit}. This is the case, when $a_1 = 0$. Solving $a_1 = 0$ for $\bar{X}$ yields the solution

$$\begin{aligned}\bar{X} = &\frac{1}{\lambda\left(\alpha^3\lambda^2-2\alpha^2\lambda^2\sigma+\alpha\lambda\sigma^2(4\delta(\lambda-1)+\lambda)+4\delta(\lambda-1)\sigma^3(\delta(\lambda-1)-\lambda)\right)}\cdot \\ &[\,\mathrm{wL}\left(\alpha^3\lambda^2-\alpha^2\lambda^2(\lambda+1)\sigma+\alpha\lambda\sigma^2\left(2\delta\left(\lambda^2-1\right)+\lambda(3\lambda-2)\right)+2(\lambda-1)\sigma^3\left(\delta^2\left(\lambda^2-1\right)-2\delta\lambda-\lambda^2\right)\right) \\ &\pm 2\sqrt{(\lambda-1)^2\sigma^2\mathrm{wL}^2(\sigma(\delta(\lambda-1)-\lambda)-\alpha\lambda)(\alpha\lambda+\sigma(\delta(\lambda-1)-\lambda))^3}\,].\end{aligned}$$

C Code

The following function simulates the n-country model for 150000 periods. The stored results are displayed in real time by a window object, which is not shown in the listing. The code is written in Objective-C.

```
- (void)simulation{

  T = 150000; //Total number of periods simulated
  n = 1; //number of consumers per country

  //other model parameters
  alpha = 0.4;
  sigma = 2;
  delta = 0.08;
  rho = 0.15;
  w = 1.0/(float)n;

  xi=0.0;

  //shares, profits, and capital are initialized for period 1
  for (int i=0; i<Laender; i++)
      for (int j=0; j<Laender; j++)
          s[i][j]=1;

  for (int i=0; i<Laender; i++)
      pi[i]=1;

  for (int i=0; i<Laender; i++){
      if (i<10)
          K[i]=K0;
      else if (i>89)
          K[i]=K0;
      else
          K[i]=K0;

  }

  //matrix of transportation costs is calculated using a flat map or a torus
  [self transportMatrixInit];
  [self sliderInit];

  for (int i=0; i<Laender; i++)
      E[i]=w;

  for (int i=0; i<Laender; i++)
      r[i]=1;


  //consumers in every country are initialized with a random capital endowment (also random preference have been tested)
```

```
    std::default_random_engine generator((unsigned int)time(0));
    std::normal_distribution<float> distribution1(0,0.00);
    std::normal_distribution<float> distribution2(0,0.3/(float)n);

    for (int land=0; land<Laender; land++) {
        theConsumers[land] = [[NSMutableArray alloc] init];
        float Ktemp = 0;
        for (int ci=0; ci<n; ci++) {
            Consumer* aConsumer = [[Consumer alloc] init];

            aConsumer.gamma=gamma;//+distribution1(generator);
            aConsumer.kbar=Kbar/(float)n;//+distribution2(generator);
            aConsumer.e = w;
            aConsumer.k = MAX(0, K[land]/(float)n+distribution2(↩
               generator));

            [theConsumers[land] addObject:aConsumer];

            Ktemp += aConsumer.k;
            E[land] += aConsumer.e;
        }
        K[land]=Ktemp;
    }

    //here the actual simulation begins
    for (int t=0; t<T; t++) {
        Kbar=KbarVorbereitet;
        gamma = gammaVorbereitet;
        if (matrixAktualisieren) {
            [self transportMatrixInit];
            matrixAktualisieren=false;
        }
        //view of the window object is told to update its content
        [[self.window contentView] setNeedsDisplay:YES];

        //new shares are calculated
        for (int i=0; i<Laender; i++){
            for (int j=0; j<Laender; j++){
                float temp = 0;
                for (int k=0; k<Laender; k++)
                    temp+=phi[k][j]*K[k];
                s[i][j]=phi[i][j]/temp;
            }
        }

        //new profits are calculated
```

```
        for (int i=0; i<Laender; i++) {
            for (int j=0; j<Laender; j++)
                pi[i]+=s[i][j]*E[j];
            pi[i]*=(alpha/sigma);
        }

        //iterate through all the countries
        for (int i=0; i<Laender; i++) {
            E[i]=0;
            K[i]=0;
            r[i] = 1+pi[i]/Pk-delta;

            //...and its consumers
            for (Consumer* aConsumer in theConsumers[i]) {
                aConsumer.gamma=gamma;
                aConsumer.kbar = Kbar/(float)n;

                float xt = aConsumer.k*r[i]*Pk + w; //cash on hand

                //determine the expenditures (piecewise linear function)
                aConsumer.e = MAX(MIN(xt, w+aConsumer.gamma*(xt-aConsumer.kbar)),0);

                //new capital
                aConsumer.k = r[i]*aConsumer.k + (w-aConsumer.e)/Pk;

                //expenditure and capital aggregation
                E[i] += aConsumer.e;
                K[i] += aConsumer.k;

            }
        }
    }
}


- (void) transportMatrixInit {
    for (int i=0; i<Laender; i++){
        for (int j=0; j<Laender; j++){
            //Flat world
            //float deltaX = fabsf(i%Spalten-j%Spalten);
            //float deltaY = fabsf(i/Spalten - j/Spalten);
            //Torus
            float deltaX = MIN(fabsf(i%Spalten-j%Spalten), Spalten-fabsf(i%Spalten-j%Spalten));
```

```
            float deltaY = MIN(fabs(i/Spalten - j/Spalten), (Laender/Spalten)-fabs(i/Spalten - j/Spalten));
            float tau = 1+tstep*sqrt(deltaX*deltaX+deltaY*deltaY);
            phi[i][j] = pow(tau, 1-sigma);
        }
    }

    for (int i=0; i<Laender; i++)
        phi[i][i]=1;
}
```

4 The evolution of inductive reasoning

4.1 Introduction

It is alway intriguing when a very simple model with only minor assumptions and constructual effort yields very rich and complex results. One good example for this is the El Farol Bar Problem invented by Arthur [1994]. 100 agents have to decide repeatedly wether to go to a bar which offers a special music program every thursday night. However, the bar only offers space for 60 people and if more show up, the evening is going to be unenjoyable for everyone who went there. The task that the agents face is straightforward: predict how many people will go to the bar in the next period and then attend themselves if and only if this prediction is less than or equal to the bar's capacity. The prediction is made using the attendance numbers from the recent past. Instead of deducing a certain outcome from complete information about the environment combined with an uncompromising rationality assumption, agents induce future outcomes by observing past states of the system they are part of. This way the expectations are generated by the agents' actions which are themselves a product of their expectations.

Around this issue many works appeared and a whole literature was created using a slightly generalized version of the El Farol model which later became known as the Minority Game (see Challet et al. [2013] and Coolen [2005]). The existing literature offers a very deep analysis of the occurring effects using involved mathematical instruments partly aided by statistical physics, but they fail to identify a cause in the complexity of the dynamics.

Inspired by the idea of these works, inductively reasoning agents have been introduced to artificial stock marked models, two of the most famous ones being Arthur et al. [1996] and Brock and Hommes [1997]. In these models the same feedback mechanism applies and dynamics partly resembling stylized facts from real markets are generated.

Peculiar to me is the fact that this mechanism of observing and predicting has not been widely applied in the area of game theory. There are models, in which agents remember or observe other agents' past, like e.g. Heller and Mohlin [2014]. However, agents still use this information to behave perfectly rational and the analysis is again purely equilibrium driven.

The idea of inductive reasoning very well corresponds to intellectual processes that humans apply in various decision situations. When people face a problem beyond a certain degree of complicatedness, they do no longer apply the kind of rationality that classical game theory assumes them to hold, simply because the capacity of their minds is exceeded. Also agents might not be able to collect enough information to understand the game as a whole. The result is that players can not assume others to behave rationally, hereby creating opportunities for them to exploit their mistakes. Consequently, we find ourselves describing an out of equilibrium context, a world of predicting and acting, a world that collectively reacts to its own actions.

In this paper I want to investigate the very core of this idea and strip down commonly used side components to a minimum to be able to isolatedly assess the dynamic implications of inductive reasoning. Game theory offers the kind of simple setting to meet these requirements as a framework for the analysis. I will use a model close to the classical setup of evolutionary game theory. An infinite population of individuals is repeatedly matched to play a game. As opposed to the traditional idea of agents programmed to play a fixed strategy all the time, agents are now programmed to use a fixed predictor all the time. They then apply the best response correspondence to decide which strategy to use. The dynamics of applied actions shall be one of the main objects of our investigation.

The paper is divided as follows. In section 4.2 I introduce the model and explain how agents condition on their experienced past. In section 4.2.1 I apply the model to two player games, before applying it to three and more player games in subsection 4.2.2. In part 4.3 I will discuss the implications of predictive agents on equilibrium concepts from evolutionary game theory. Section 4.4 concludes.

4.2 The Model

Imagine a world consisting of $N \to \infty$ players, who are repeatedly paired with a randomly chosen opponent to play a 2-, and later $K > 2$-player game with actions A and B. Payoffs are described by the utility function $u_i(n, s)$, where n is the number of players who play A against the agent in the current game and $s \in \{A, B\}$ are the two possible actions that one can take. This utility function can potentially be *heterogeneous* across agents. Each round every agent gets paired with a different agent. Since agents to not have an information concerning which strategy their current opponents are going to play, they use predictors to forecast their opponents' actions. A predictor maps the actions of the M previous opponents, met in the most recent encounters, to the expected number of A-players in the game that the agent is taking part in in current period, denoted by n^e.

Definition 3. A mapping $\psi : [0, K]^M \to [0, K]$,
$(n_{t-1}, ..., n_{t-M}) \to \psi(n_{t-1}, ..., n_{t-M}) = n^e$ is called a predictor.

Using the best response correspondence of the respective game, one can map the prediction of A-playing opponents to the strategy that is optimal to use. Plugging in the prediction mapping into the best response correspondence, we can simply say that agents condition their actions in the current period on their past opponents' actions. Formally, one can write the best response correspondence as

$$\xi(n^e) := \arg\max_{s \in \{A,B\}} u(n^e, s) ,$$

and the players' action as a reaction to his personal experience in the past as

$$s_t = \xi(\psi(n_{t-1}, ..., n_{t-M}))$$

The goal shall be to describe the evolution of the overall players who play A in period t, denoted by x_t. So in terms of the El Farol Bar Problem, I want to investigate the attendance behavior in a 2 (corresponding to Hawk-Dove) or more player bar game with N/K bars without learning. But for now, we are abstracting from any specific payoff structure, because we do not yet need it.

4.2.1 Two player games

Let us consider an infinitely large population. This is a model similar to the classical evolutionary game theory setup in which an infinite population repeatedly plays a game. The important difference is that in these models players are programmed to play the same fixed strategy all the time. Here I want to enable agents to think inductively and make the choice of their action dependent on their experience in the recent past.

First, consider a situation in which the players only look back $M = 1$ period. There are 4 different types of agents, namely those who

1. never play A, no matter what happened last period
2. play A, if and only if their previous opponent (the one they got paired with in $t - 1$) played B in the last period (anti-imitation)
3. play A, if and only if their previous opponent played A in the last period (imitation)

4. always play A, no matter what happened last period

The following table lists the actions taken by the different types for different observed histories.

hist. \ type	1	2	3	4
A	B	B	A	A
B	B	A	B	A

The share of players playing action A in period t, x_t determines the probability of meeting an A-player when the matching process is random. The probability of meeting a B-player will be $1 - x_t$. Let a_i be the share of type i agents in the population, which is fixed. Of course $\sum_i a_i = 1$. Now x_t will evolve as follows.

$$x_t = f(x_{t-1}) = a_1 \cdot 0 + a_2(1 - x_{t-1}) + a_3 x_{t-1} + a_4 \cdot 1 \tag{20}$$

Note that this process is deterministic. This is caused by the law of large numbers and is critically dependent on the assumption of an infinitely large population.

Now the question is how x_t will behave over time. Will it fluctuate as in the classical Bar Problem?

Theorem 1. *Suppose that $M = 1$ and that at least two types i and j exist in the population ($a_i a_j > 0\,,\ i \neq j$). Then for every set of parameters $(a_1, a_2, a_3, a_4) \in A$ independent of the initial condition x_0*

$$\lim_{t \to \infty} x_t = \bar{x} = \frac{a_2 + a_4}{1 + a_2 - a_3}$$

Proof. Let (X, d) be a metric space. According to the well known definition, a mapping $f : x \to x$ is a contraction mapping if there exists a constant c with $0 \leq c < 1$ such that

$$d(f(x), f(y)) \leq c\, d(x, y) \quad \forall\, x, y \in X \tag{21}$$

$f(.)$ is a linear equation and it is easy to see that it maps from $(0, 1)$ to $(0, 1)$. So if we can show that there exists a constant c such that (21) is fulfilled, we know that (20) is a contraction.

$$\begin{aligned} |f(x) - f(y)| &= |\,[a_2(1 - x) + a_3 x + a_4] - [a_2(1 - y) + a_3 y + a_4]\,| \\ &= |a_3 - a_2||x - y| < c|x - y| \end{aligned}$$

So the mapping $f(.)$ is a contraction. According to the Banach fixed point theorem, we can say that $f(.)$ has exactly one fixed point in $(0, 1)$. Furthermore this fixed point will be *globally* attracting. By solving $f(x) = x$ it can be determined to be $\frac{a_2 + a_4}{1 + a_2 - a_3}$. □

So unlike in the El Farol setting, in the long run we won't observe any fluctuations in the average choice of strategies across the population. However, this does not mean that, when starting outside of it, the steady state will be reached right away. There can be considerable fluctuations on the way there. Strong and fast enough learning mechanisms (corresponding to agents changing their prediction rules) known from El Farol and the Minority game might still prevent the system from reaching it. So the result does only say that in an infinite population, one period conditioning *alone* is in the long run not responsible for unsteady dynamics of any kind.

Now, let us go one step further and enable the agents not only to look back one period, but two periods into their past. Now conditioning on 4 possible personal histories, the number of agent types rapidly increases to $2^{2^2} = 16$.

hist. \ type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
AA	B	B	B	B	B	B	B	B	A	A	A	A	A	A	A	A
AB	B	B	B	B	A	A	A	A	B	B	B	B	A	A	A	A
BA	B	B	A	A	B	B	A	A	B	B	A	A	B	B	A	A
BB	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A

Analogous to above the difference equation can be constructed, now multiplying the respective shares of A and B players to calculate the required probabilities of being matched with the specific strategy users.

$$\begin{aligned}
x_t = f(x_{t-1}, x_{t-2}) = & a_2((1-x_{t-1})(1-x_{t-2}))+ \\
& a_3((1-x_{t-1})x_{t-2})+ \\
& a_4((1-x_{t-1})(1-x_{t-2}) + (1-x_{t-1})x_{t-2})+ \\
& a_5(x_{t-1}(1-x_{t-2}))+ \\
& a_6((1-x_{t-1})(1-x_{t-2}) + x_{t-1}(1-x_{t-2}))+ \\
& a_7(x_{t-1}(1-x_{t-2}) + (1-x_{t-1})x_{t-2})+ \\
& a_8(x_{t-1}(1-x_{t-2}) + (1-x_{t-1})x_{t-2} + (1-x_{t-1})(1-x_{t-2}))+ \\
& a_9(x_{t-1}x_{t-2})+ \\
& a_{10}(x_{t-1}x_{t-2} + (1-x_{t-1})(1-x_{t-2}))+ \\
& a_{11}(x_{t-1}x_{t-2} + (1-x_{t-1})x_{t-2})+ \\
& a_{12}(x_{t-1}x_{t-2} + (1-x_{t-1})x_{t-2} + (1-x_{t-1})(1-x_{t-2}))+ \\
& a_{13}(x_{t-1}x_{t-2} + x_{t-1}(1-x_{t-2}))+ \\
& a_{14}(x_{t-1}x_{t-2} + x_{t-1}(1-x_{t-2}) + (1-x_{t-1})(1-x_{t-2}))+ \\
& a_{15}(x_{t-1}x_{t-2} + x_{t-1}(1-x_{t-2}) + (1-x_{t-1})x_{t-2})+ \\
& a_{16}
\end{aligned} \tag{22}$$

This equation is more that unhandy, but it can be simplified to.

$$x_t = f(x_{t-1}, x_{t-2}) = ax_{t-1}x_{t-2} + bx_{t-1} + cx_{t-2} + d \tag{23}$$

with

$$\begin{aligned} (i) &\quad 0 \leq a+b+c+d \leq 1 \\ (ii) &\quad 0 \leq d \leq 1 \\ (iii) &\quad 0 \leq b+d \leq 1 \\ (iv) &\quad 0 \leq c+d \leq 1 \\ (v) &\quad -1 \leq a+b \leq 1 \\ (vi) &\quad -1 \leq a+c \leq 1 \end{aligned}$$

where (i) and (ii) come from the fact that if in $t-1$ and $t-2$ every (no) agent played A, we must have $0 \leq f(1,1) \leq 1$ $(0 \leq f(0,0) \leq 1)$. Similarly, (iii) and (iv) reflect the conditions $0 \leq f(1,0) \leq 1$ and $0 \leq f(0,1) \leq 1$ that have to be true in (22). Finally, (v) and (vi) can be directly calculated from 22. The set of admissible parameter combinations will be called P.

Bektešević et al. [2014] investigate a similar system and provide a thorough global analysis. However, in this reference all parameters are assumed to be non-negative, making the equation non-decreasing in both of its variables. That in turn facilitates the analysis considerably and diminishes its use for our purposes to a point of orientation.

Proposition 4. *For every parameter set with $a \neq 0$, the system (23) has at least one and at most two fixed points:*

$$\bar{x}_1 = \frac{(1-b-c) - \sqrt{(1-b-c)^2 - 4ad}}{2a}, \qquad \bar{x}_2 = \frac{(1-b-c) + \sqrt{(1-b-c)^2 - 4ad}}{2a},$$

Proof. To show that the two fixed points always exist, we have to establish that the term under the square root $\Gamma = (1-b-c)^2 - 4ad$ is always non-negative. First, consider the case $a < 0$. Because both components of the sum are now non-negative, we have $\Gamma \geq 0$. The case $a > 0$ is not quite as straight forward. To minimize Γ such that our linear restrictions are fulfilled, it is easy to see that we need to choose $a = d$, because if $a < d$ or $a > d$ we can decrease Γ by equalizing them to $a^* = d^* = \frac{a+d}{2}$, which won't violate any of the restrictions. Furthermore if $a = d$, according to (i) we must have $|b+c| \leq 1$, which means that $\frac{\partial \Gamma}{\partial a}, \frac{\partial \Gamma}{\partial b}, \frac{\partial \Gamma}{\partial c}, \frac{\partial \Gamma}{\partial d} \leq 0$. So if we make sure that at the upper bound of (i) $\Gamma \geq 0$, the existence of the fixed point(s) is proven. To see why this is true, imagine we were solving an optimization problem minimizing the objective function $\Gamma(a,b,c,d)$ s.t. (i) is true, ignoring all other constraints.

So let use $a = d = \frac{1}{2}(1-b-c)$ to get

$$\Gamma = (1-b-c)^2 - 4\left(\frac{1}{2}(1-b-c)\right)^2 = 0\,,$$

which implies that $\Gamma \geq 0$. □

To complete the picture, let me mention that for the case of $a = 0$ there is one single fixed point at $\bar{x} = \frac{d}{1-b-c}$.

Proposition 5. *$\bar{x}_1$ is always in $[0, 1]$.*

Proof. The first step is to show that $\bar{x}_1 \geq 0$. Consider the case where $a < 0$. Here, because $-4ad \geq 0$, we have $(1-b-c) \leq \sqrt{(1-b-c)^2 - 4ad}$ and since the dominator is negative, $\bar{x}_1 \geq 0$. In case $a > 0$ we know that $b + c < 1$ and so $(1-b-c) > 0$. So we see that $(1-b-c) > \sqrt{(1-b-c)^2 - 4ad}$ and thus again $\bar{x}_1 \geq 0$.

Showing that $\bar{x}_1 \leq 1$ is a bit more tedious. The fixed point only includes the sum of b and c and because we face symmetric constraints concerning the two, we can set $b = c$. Now let us consider an optimization problem only including two of our many constraints.

$$\max_{a,b,d} \frac{1 - 2b - \sqrt{(2b-1)^2 - 4ad}}{2a} \quad s.t.\ a + 2b + d \leq 1$$
$$b + d \leq 1$$

The Kuhn-Tucker approach yields the three candidate solutions $\{a, -a, a+1\}$, $\{a, b, -a-2b+1\}$ and $\{a, b, d\}$. Because the first one is a subset of the others, we can safely plug in the first solution to get the largest value the fixed point can take, while still obeying the parameter constraints.

$$\frac{1 + 2a - \sqrt{(-1-2a)^2 - 4a(1-a)}}{2a} = 1$$

□

In case of $a = 0$, our single fixed point at $\bar{x} = \frac{d}{1-b-c}$ will also be in $[0, 1]$, which is a direct consequence of (i).

Corollary. *If the proportions of agents unconditionally playing A and B are both different from zero ($a_1 a_{16} > 0$), then $\bar{x}_1$ is always in $(0, 1)$.*

Proposition 6. *$\bar{x}_2$ is never in $(0, 1)$.*

Proof. First, consider the case of $a < 0$. Because according to (ii) $0 \leq d \leq 1$, we can see that $\sqrt{(1-b-c)^2 - 4ad} \geq 1 - b - c$. Because we know that the square root is going to be positive we find a non-negative numerator and negative denominator. So $\bar{x}_2 \leq 0$.

Next, consider the case of $a > 0$. According to (i) and (ii) we must have $1-b-c > 0$. So it is easy to see that $\bar{x}_2$ is going to be positive, which means we have to show that

$\bar{x}_2 \geq 1$. In order to minimize $\bar{x}_2$, consider the partial derivatives. Directly observable, we have $\frac{\partial \bar{x}_2}{\partial a} < 0$ and (for the current case) $\frac{\partial \bar{x}_2}{\partial d} < 0$. Furthermore we have

$$\frac{\partial \bar{x}_2}{\partial b} = \frac{\frac{b+c-1}{\sqrt{(b+c-1)^2-4ad}} - 1}{2a},$$

which, using $-1 + b + c < 0$, is also negative. The symmetry makes sure that also $\frac{\partial \bar{x}_2}{\partial c} < 0$. The negative partial derivatives indicate that the minimal value of our fixed point must be at the upper bound of (i), meaning that we can search at $a+b+c+d = 1$.

Plugging in $1 - a - d$ for $b + c$ leads to

$$\frac{a + d + \sqrt{(a-d)^2}}{2a} = \begin{cases} 1 & \text{if} \quad a \geq d \\ \frac{d}{a} & \text{if} \quad a < d. \end{cases}$$

Since $\min_{a<d} \frac{d}{a} > 1$, we have established the result that $\bar{x}_2 \notin (0, 1)$. □

So apparently our system has at most one fixed point (henceforth called $\bar{x} = \bar{x}_1$) inside of the domain we are interested in. So, except for boundary cases, if players use second order predictors, there is only one single proportion of the population playing a certain strategy that reproduces itself, just like it was the case for first order predictors. The question remains as to wether this state is stable.

In order to tackle this question, let us write the second order difference equation in vector form, so that it becomes a system of first order.

$$X_t = \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} f(x_{t-1}, x_{t-2}) \\ x_{t-1} \end{pmatrix} = F(X_{t-1})$$

Lemma 2. *If at least one of the proportions of agents unconditionally playing A and B is different from zero ($a_1 + a_{16} > 0$) and $\Gamma \neq 0$, then for each parameter set $(a, b, c, d) \in P$ $\exists$ (c_1, c_2, c_3) such that $V(X) = c_1(x_1 - \bar{x})^2 + c_2(x_2 - \bar{x})^2 + c_3(x_1 - \bar{x})(x_2 - \bar{x})$ is a Liapunov function of (22) on an ϵ-neighborhood of the fixed point $\mathcal{S}_\epsilon = \{X \mid ||X - \bar{X}|| < \epsilon\}$*

Proof. Let $\delta_1 = x_{t-1} - \bar{x}$ and $\delta_2 = x_{t-2} - \bar{x}$. Then we can write (23) as

$$\begin{aligned} f(x_{t-1}, x_{t-2}) &= ax_{t-1}x_{t-2} + bx_{t-1} + cx_{t-2} + d \\ &= a(\bar{x} + \delta_1)(\bar{x} + \delta_2) + b(\bar{x} + \delta_1) + c(\bar{x} + \delta_2) + d \\ &= \bar{x} + (a\bar{x} + b)\delta_1 + (a\bar{x} + c)\delta_2 + a\delta_1\delta_2 \end{aligned}$$

Because we only consider an arbitrarily small neighborhood of the fixed point, implying an arbitrarily small δ_i, we are save to drop the last term of the sum, since $|a\delta_1\delta_2| \ll |\bar{x} + (a\bar{x} + b)\delta_1 + (a\bar{x} + c)\delta_2|$.

Now we can calculate

$$V(X) = V((x_1, x_2)) = c_1\delta_1^2 + c_2\delta_2^2 + c_3\delta_1\delta_2$$

$$\begin{aligned} V(F(X)) &= V(f(x_1, x_2), x_1) \\ &= c_1\left[\bar{x} + (a\bar{x} + b)\delta_1 + (a\bar{x} + c)\delta_2\right]^2 + c_2\delta_1^2 + c_3\left[\bar{x} + (a\bar{x} + b)\delta_1 + (a\bar{x} + c)\delta_2\right]\delta_1 \end{aligned}$$

We have to show that $V(F(X)) \leq V(X)\ \forall X \in \mathcal{S}_\epsilon$ and $V(F(X)) < V(X)\ \forall X \in \mathcal{S}_\epsilon \setminus \{\bar{X}\}$. Clearly $V(F(\bar{X})) = V(\bar{X})$. So to accomplish the condition it will be enough to establish the strict concavity of $W(X) = V(F(X)) - V(X)$ at $\bar{X}$. For that I will set up the Hessian Matrix of $W(X)$ and show that it is negative definite. The Hessian is

$$\mathbf{H_W} = \begin{pmatrix} 2\left(c_2 + c_3(b + a\bar{x}) + c_1\left(b^2 + 2a\bar{x}b + a^2\bar{x}^2 - 1\right)\right) & c_3(c + a\bar{x} - 1) + 2bc_1(c + a\bar{x}) + 2ac_1\bar{x}(c + a\bar{x}) \\ c_3(c + a\bar{x} - 1) + 2bc_1(c + a\bar{x}) + 2ac_1\bar{x}(c + a\bar{x}) & 2\left(c_1c^2 + 2ac_1\bar{x}c + a^2c_1\bar{x}^2 - c_2\right) \end{pmatrix}$$

We need $z^T\mathbf{H_W}z = h_{11}z_1^2 + h_{12}z_1z_2 + h_{21}z_1z_2 + h_{22}z_2^2 = h_{11}z_1^2 + 2h_{12}z_1z_2 + h_{22}z_2^2 \leq 0$ for every non zero column vector z.

I (arbitrarily) choose the sufficient conditions $h_{11} = h_{22} = -1$ and $h_{12} = 0$. Soving this system of equations for the Lyapunov Function parameters (c_1, c_2, c_3) yields

$$\begin{aligned} c_1 &= -\frac{a\bar{x} + c - 1}{\Gamma(b - c + 1)(a\bar{x} + c + 1)} \\ c_2 &= -\frac{(a\bar{x} + c - 1)(a\bar{x} + c)^2}{\Gamma(b - c + 1)(a\bar{x} + c + 1)} + \frac{1}{2} \\ c_3 &= \frac{2(a\bar{x} + b)(a\bar{x} + c)}{\Gamma(b - c + 1)(a\bar{x} + c + 1)} \end{aligned}$$

To see that the common denominator is different from 0, note that $b-c = (b+d)-(c+d)$ and since $a_1 + a_{16} > 0$ we must have either $b + d > 0$ or $c + d < 1$ and so $b - c + 1 > 0$. Concerning $(a\bar{x} + c + 1)$ note that $\frac{\partial(a\bar{x}+c+1)}{\partial c} \geq \frac{\partial(a\bar{x}+c+1)}{\partial a,b,d}$, which means that minimizing this term we can set $c = -1$. This simplifies the term to be minimized to ad. Similar to above, since $a_1 + a_{16} > 0$ we must have either $c + d > 0$ or $b + d < 1$. In the former case we must have $d > 1$, which is impossible, so the latter case leads us to $d = 1$ and $b < 0$. For (i) to be fulfilled a must now be strictly positive. So $ad > 0$ and also $(a\bar{x} + c + 1) > 0$.

We have established that a Lyapunov Function exists under the required conditions. □

Lemma 3. *If $a_1 + a_{16} > 0$ and $\Gamma = 0$, then the fixed point $\bar{x}$ is globally attracting.*

Proof. If $\Gamma = 0$ then $\bar{x}_1 = \bar{x}_2$, which implies that the single fixed point will be at 0 or 1.

If $\bar{x} = 0$, then $d = 0$. But since $\Gamma = 0$, $b+c = 1$ and so $a < 0$ because of $a_1+a_{16} > 0$.

Just as in the above proof, let $\delta_1 = x_{t-1} - \bar{x}$, $\delta_2 = x_{t-2} - \bar{x}$ and

$$\begin{aligned} f(x_{t-1}, x_{t-2}) &= \bar{x} + (a\bar{x} + b)\delta_1 + (a\bar{x} + c)\delta_2 + a\delta_1\delta_2 \\ &= b\delta_1 + c\delta_2 + a\delta_1\delta_2 \\ &< Max[\delta_1, \delta_2] \ , \end{aligned}$$

where the last step follows directly from the above conditions and the fact that $\delta_{1,2} \geq 0$.

In case $\bar{x} = 1$, we must have $a + b + c + d = 1$. So there is no agent unconditionally playing B, implying there must be agents unconditionally playing A, such that $d > 0$. For $\Gamma = 0$, a must be strictly positive. In fact, for Γ to be minimal, it must be equal to d. So we can write $2a + b + c = 1$.

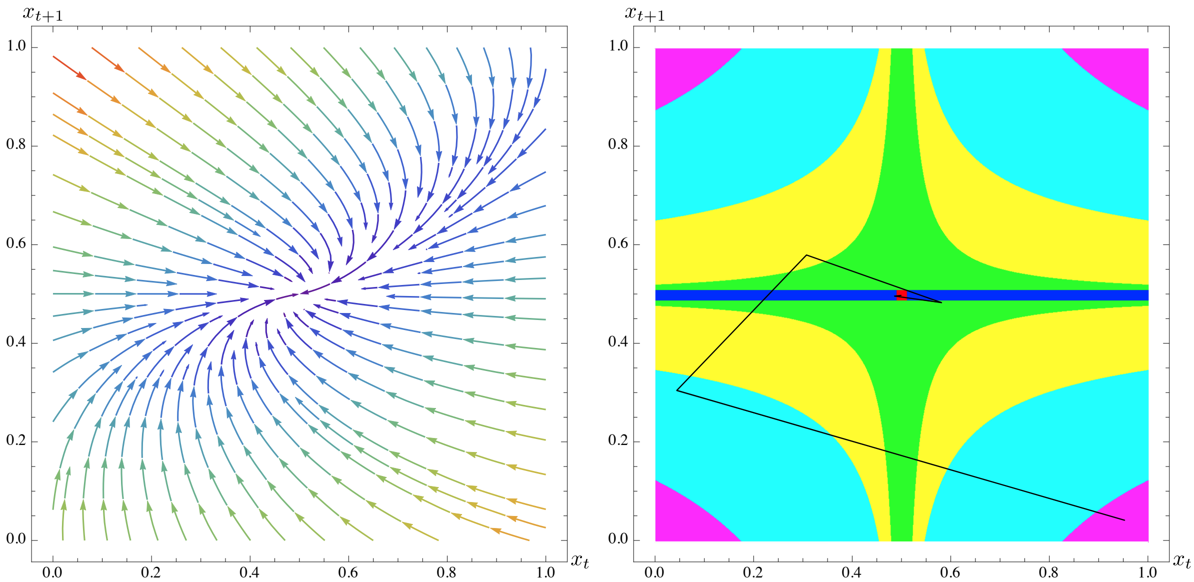
$$\begin{aligned} f(x_{t-1}, x_{t-2}) - 1 &= (a + b)\delta_1 + (a + c)\delta_2 + a\delta_1\delta_2 \\ &> Min[\delta_1, \delta_2] \ , \end{aligned}$$

where the last step follows directly from the above conditions and the fact that $\delta_{1,2} \leq 0$. □

The next result directly follows from the lemmas concerning the existence of a Lyapunov function on a neighboring set of the fixed point and the global stability in the special case of $\Gamma = 0$.

Theorem 2. *If $M = 2$ and $a_1 + a_{16} > 0$, then $\bar{x}$ is asymptotically stable.*

An extensive numerical analysis indicates that the fixed point $\bar{x}$ is not only locally, but globally stable on $(0, 1)$. This seems to be true even though the map for a two-period horizon is not a contraction anymore. Experiments over a very fine grid of the parameter space of the considered difference equation have been carried out, unfailingly exhibiting $\lim_{t\to\infty} = \bar{x}$. Figure 23 illustrates the typical system behavior for a more or less random parameter set.



(a) Vector plot, displaying $F(X) - X$

(b) Pre-images of an ϵ-square around $\bar{x}$

Figure 23: Global attraction for $a_{10} = \frac{1}{2}$ and $a_i = \frac{1}{30}$ for $i \in \{1, ...16\} \setminus 10$

Here every predictor is equally popular within the population, except for predictor number 10, which is favored a bit. The vectors in picture (a) clearly show a converging behavior to the fixed point which lies in this case at 1/2. Graph (b) shows a very small neighborhood of $\bar{x}$ in a red color. The set of points that is directly mapped into this square $(F^{-1}(X)\ X \in \mathcal{S}_\epsilon)$ is colored in blue. The next preimages are then drawn in green, yellow, cyan and purple, respectively. In addition, a sample trajectory starting at the bottom right corner is delineated.

In Bekteŝević et al. [2014] the global attractivity of system (23) is shown analytically for positive parameters. Because the system loses its monotonicity property when allowing for negative coefficients, one cannot use the known theorems for such systems anymore making it a very difficult task to prove the global attractivity. However, the results of Bekteŝević et al. [2014] are not completely useless since they enable us to state (with certainty) that for great parts of the space of predictor distributions (the ones corresponding to $a, b, c, d \geq 0$) the system possesses a fixed point with basin of attraction extending over the whole interval $(0, 1)^2$.

In the numerical analysis I went one step further and analyzed the dynamics for predictors with a maximal horizon of 3 periods. This increases the number of possible predictors to $2^{2^3} = 256$. The experiments indicate that the proportion of strategy A playing agents again converges to a constant level. However, since the parameter space is very large, we cannot guarantee a robust behavior. This leaves me to classify this last point as a conjecture.

4.2.2 Three and more player games

Up to now players in the infinite population were randomly drawn together in pairs to play a two player game. In this subsection I want to extend this idea to games of three and more players in order to find out wether the dynamics of the proportion of people playing a specific strategy qualitatively changes.

Let us consider an infinite population of agents who are randomly arranged into groups of three to play a game with payoffs who shall, just like above, be arbitrary for now. To keep things simple, let the considered game be symmetric. [15] The game still will have two strategies. A typical player in this scenario faces now the task to not only predict whether or not his single opponent is going to be choosing strategy A, but whether zero, one, or all of his opponents will do so. Let us consider the case of a single period memory. Here, the number of possible predictors is $2^{3^1} = 8$. Let $k_{A,t}$ be the number of A playing opponents of an agent. The predictors(, or better: "their behavioral consequences") can be seen in the table below.

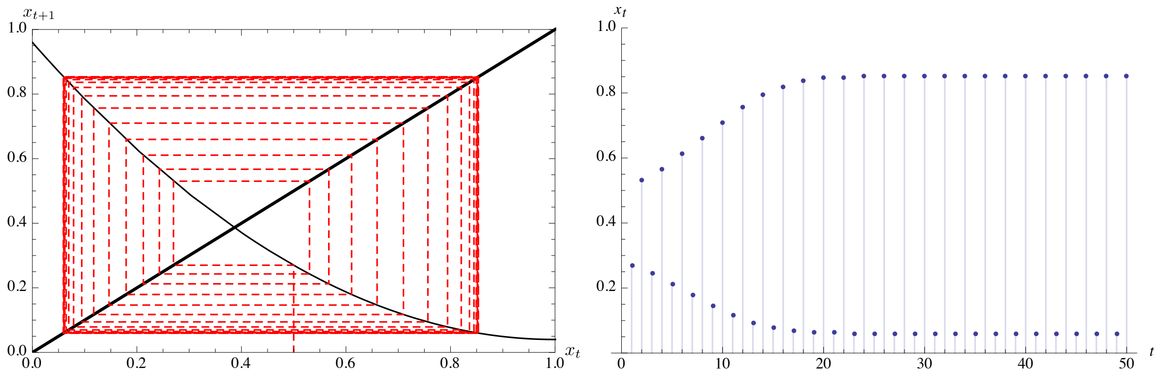
hist. \ type	1	2	3	4	5	6	7	8
$k_{A,t} = 2$	B	B	B	B	A	A	A	A
$k_{A,t} = 1$	B	B	A	A	B	B	A	A
$k_{A,t} = 0$	B	A	B	A	B	A	B	A

The difference equation becomes

$$x_t = f(x_{t-1}) = a_2(1 - x_{t-1})^2 + a_3 2(1 - x_{t-1})x_{t-1} + a_4((1 - x_{t-1})^2 + 2(1 - x_{t-1})x_{t-1}) + a_5 x_{t-1}^2 + a_6(x_{t-1}^2 + (1 - x_{t-1})^2) + a_7(x_{t-1}^2 + 2(1 - x_{t-1})x_{t-1}) + a_8 \, .$$

I want to choose a parameter set in order to demonstrate that there can be indeed a qualitative change in the dynamics. In the considered scenario most of the agents in this population will use predictor type 2, which means they will choose A if and only if none of their opponents played A in the previous round. The rest of the predictors are evenly distributed across the population.

[15] I do this for simplicity, but also to stay close to the El Farol game, which is a symmetric n-player two-strategy game. However, note that for asymmetric games the results will be very similar.



(a) cobweb plot

(b) x_t over time

Figure 24: Two cycle for $a_2 = 0.93$ and $a_i = 0.01$ for $i \in \{1, ...8\} \setminus 2$, $x_0 = 0.5$

Figure 24 shows the existence of a two cycle. Solving $f^2(x) = x$, this two-cycle can be determined to go through the points $x_{1/2} = \frac{3}{46}\left(7 \pm \sqrt{37}\right)$. As can be seen in the graph, the fixed point at about 0.4 exists as well. However, it is unstable for the considered parameter set.

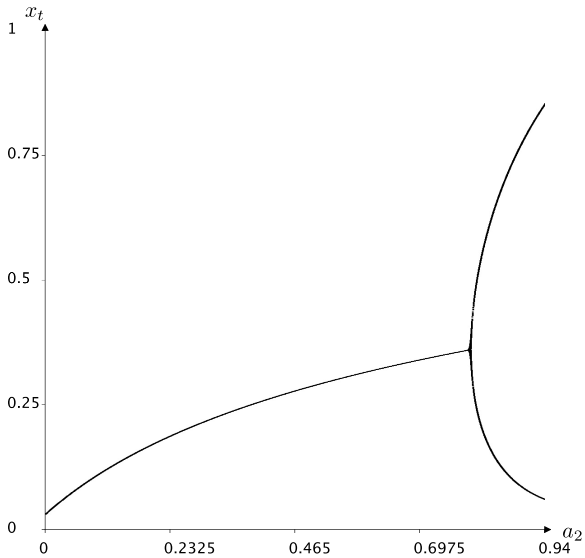


Figure 25: Bifurcation diagram for a variation of $a_1 = 0.94 - a_2$ with $a_i = 0.01$ for $i \in \{1, ...8\} \setminus \{1, 2\}$

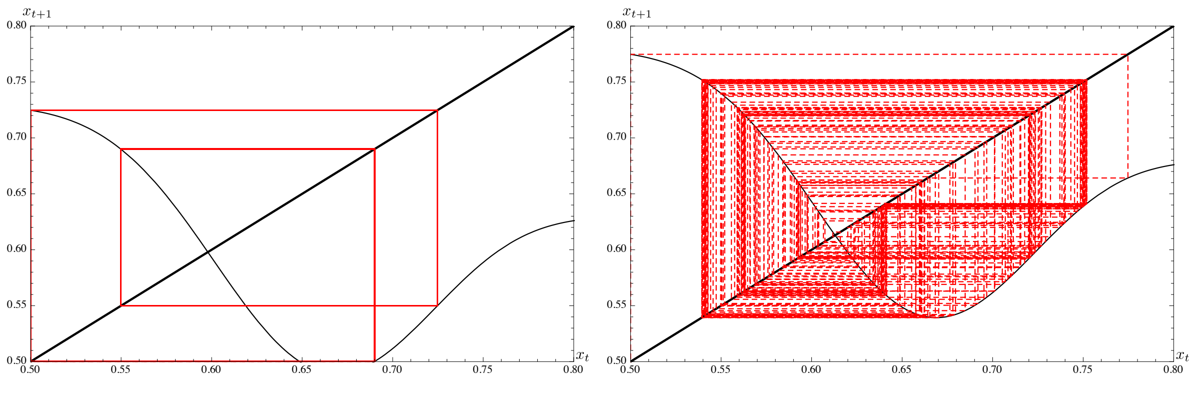
Figure 25 shows what happens when varying the shares of predictors 1 and 2. In the bifurcation diagram we start at the same parameter set as above and then relocate the share of people using predictor 2 from 0 to 0.94, resulting in type 1 shares ranging from 0.94 to 0.

So apparently there is a fundamental difference in the infinite population strategy dynamics which appear in 2 player games vs. those appearing in 3 player games. While in the former we saw a convergence to a stable steady state, in the latter we observe two cycles for some parameter sets. But is that all that can happen? The natural question arises whether it is possible to create other, more sophisticated dynamics as well.

In the El Farol Bar Problem and in the Minority Game the observed dynamics are complex. As this paper is somehow driven by the fascination concerning the causes of these complex dynamics, I want to stay close to the original scenario. [16] Now we change our 3-player game to a 100-player game, just like Arthur's El Farol Bar Problem is played by 100 people. Of course with 100 players, even if the horizon angents are conditioning upon is only one period, the number of possible predictors will be huge, which is why I only want to consider 4 of them. Type 1 will imply always playing strategy A (or, in the El Farol terminology, "go to the bar"), type 2 will imply never playing A, type 3 will imply choosing A if and only if 60 or less of the opponents in the last round played A and type 4 will imply choosing A if and only if more than 70 of the last rounds opponents did so. [17] The resulting difference equation is

$$x_t = a_1 + a_3 \sum_{k=0}^{60} \binom{100}{k} x_{t-1}^k (1 - x_{t-1})^{100-k} + a_4 \sum_{k=71}^{100} \binom{100}{k} x_{t-1}^k (1 - x_{t-1})^{100-k}$$

The following figures show what can happen if we choose different values for the population proportions who use these four predictors.



(a) $a_1 = 0.43$, $a_2 = 0.07$

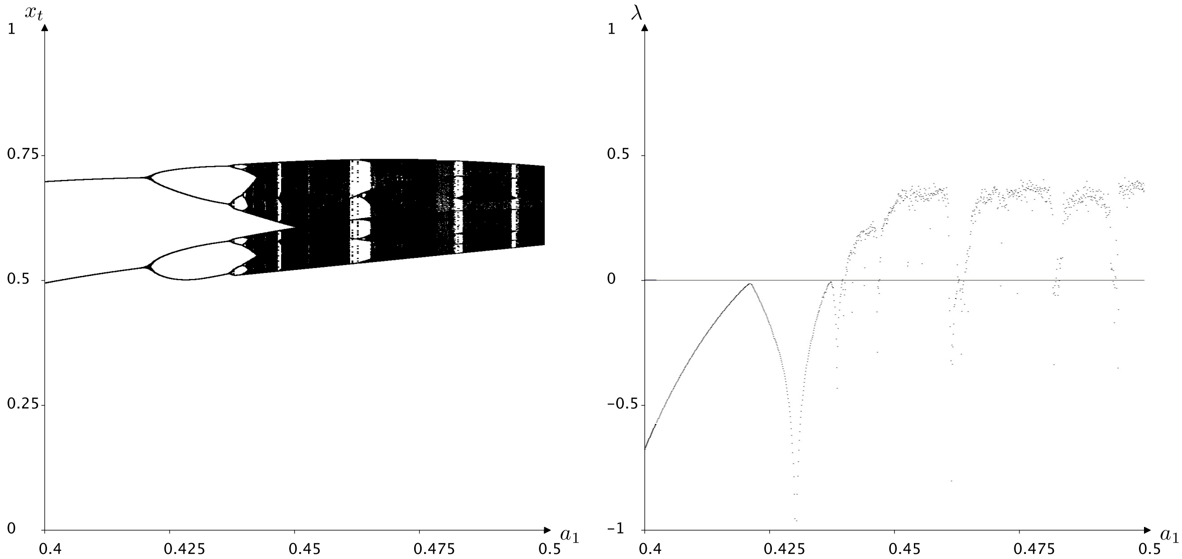
(b) $a_1 = 0.48$, $a_2 = 0.02$

Figure 26: Cobweb plot for $a_3 = 0.3$, $a_4 = 0.2$, $x_0 = 0.5$

In Figure 26 (a) we see a cycle of period 4. When shifting a few people from type 2 to type 1 predictors, we observe a much richer dynamic behavior.

[16] Of course the infinite population assumption I make, makes still a considerable difference.

[17] Choosing these prediction models may seem a bit arbitrary and in fact it is. But it is possibly one of the simplest choices to illustrate the point I want to make.



(a) bifurcation diagram (b) Lyapunov Exponent

Figure 27: $a_1 = 0.5 - a_2$, $a_3 = 0.3$, $a_4 = 0.2$, $x_0 = 0.5$

Figure 27 (a) shows the observed phenomenon from the cobweb plot in a bifurcation diagram. Several period doubling bifurcations are taking place, before the complex orbits occur. The Lyapunov Exponents λ which are plotted in Figure 27 (b) exhibit positive values for these parameter ranges, which is a strong indicator for topological chaos.

To wrap up this section we can state that in contrast to the attracting constant orbits occurring in 2-player games played by an infinite population using prediction models, much richer dynamics are evoked if more that 2 players are included in the game.

4.3 Evolutionary Stability

So far we looked at the dynamics that occur when agents in an infinite population use predictors to decide which strategy they use in a repeated setting. Apart from the number of possible strategies, no actual game was considered. This is about to change in this subsection. I want to apply the described model to the classical notion of evolutionary stable equilibria.

The concept of an evolutionary stable strategy (ESS) was developed to formalize the idea of evolutionary forces. In a large population individuals are programmed to play the same pure or mixed strategy over and over again to be repeatedly matched with each other. If now some small proportion of mutants playing a different strategy is created, evolutionary forces will select against them, if their expected payoff is smaller than the one of the incumbents. Any strategy that by this mechanism is able to prevent any mutant strategy from invading it, is considered evolutionary stable. One can sort of project the interpretation of a large population of randomly matched individuals, who each play a fixed strategy to two individuals each playing mixed strategies. The same can be done in a world of agents who condition on their past. So let us call the set of mixed strategies of agents using predictors of order M by Δ_M. So the traditionally used set of mixed strategies will be denoted by Δ_0 and its elements will be the probabilities of playing strategy A, denoted by $x \in \Delta_0$. For $M \geq 1$ we need to name the prediction type and an initial condition at which the agents start, because at the moment of their creation there will be no past to condition on. An element of Δ_M will thus be a tuple consisting of the prediction type and an initial condition. So for example $(a_1, a_2, a_3, x_0) \in \Delta_1$, where a_i is the proportion or probability of the agent's prediction type (naturally $a_4 = 1-a_1-a_3-a_3$) and x_0 is the "virtual past", with which he is created. The sequence x_t of mixed strategies that the agent then actually applies over time is not defined by the agent alone, but is rather a result of the interplay with the rest of the population[18]. Note that even though caused by the vector notation mathematically it is not quite correct, we effectively have $\Delta_M \subset \Delta_{M+1}$. For example a traditional mixing behavior without consideration of the past can also be achieved by mixing between agent types 1 and 4, which again is included in Δ_1.

The concept of evolutionary stability was defined by Smith and Price [1973] and Smith [1974].

Definition 4. x is an evolutionary stable strategy (ESS) if for every strategy $y \in \Delta_0 \backslash \{x_{ESS}\}$ there exists some $\bar{\epsilon}_y \in (0,1)$ such that $u[x,\, \epsilon y+(1-\epsilon)x] > u[y,\, \epsilon y+(1-\epsilon)x]$ holds for all $\epsilon \in (0, \bar{\epsilon}_y)$.

As mentioned above, in this classical evolutionary game theory setting agents are programmed to play a fixed mixed or pure strategy all the time. But what if we allow agents to be programmed in a more sophisticated way. Just like described in the preceding subsections, agents could be able to take into account their past experience in order to decide which strategy to apply. An important difference is that in the traditional setting expected payoffs do not change over time. In the new situation, payoffs can vary over time because even apart from evolutionary forces, an equilibrium might not be reached right away. In subsection 4.2.1 we saw that a steady state will (or for $M \geq 2$ "most probably will") be reached eventually. But on the way to this steady state there might be fluctuations or other unsteady dynamics which will affect the agents' utility. To capture these effects of time variant expected utility, it makes sense to use a time discounted sum of all future utilities.

$$U[(x_t)_{t=0}^{\infty}, (y_t)_{t=0}^{\infty}] \;=\; \sum_{t=0}^{\infty} (1-\rho)^t \, u[x_t, y_t]$$

[18] Or the single other opponent, depending on the interpretation.

Let us consider a symmetric 2by2 game with the following payoffs.

	A	B
A	p_{AA}/p_{AA}	p_{AB}/p_{BA}
B	p_{BA}/p_{AB}	p_{BB}/p_{BB}

Then the utility function will be

$$U[(x_t)_{t=0}^{\infty},(y_t)_{t=0}^{\infty}] = \sum_{t=0}^{\infty}(1-\rho)^t\left[x_t y_t p_{AA}+x_t(1-y_t)p_{AB}+(1-x_t)y_t p_{BA}+(1-x_t)(1-y_t)p_{BB}\right].$$

For the first result of this chapter, we do not yet really need this new utility formulation and can still stick to the static variant. It addresses the question wether an evolutionary stable strategy in the classical sense still fulfills the criterion if we add predictive agents to the picture.

Proposition 7. *In any symmetric 2by2 game, a pure or mixed strategy evolutionary stable equilibrium can be invaded by agents who imitate their last opponent. Formally:* $\exists\, y \in \Delta_1$ *s.t.* $U[(y_t)_{t=0}^{\infty}, (\epsilon y_t + (1-\epsilon)x_{ESS})_{t=0}^{\infty}] \geq U[(x_{ESS})_{t=0}^{\infty}, (\epsilon y_t + (1-\epsilon)x_{ESS})_{t=0}^{\infty}]$.

Proof. Let x_{ESS} be the evolutionary stable strategy. Now assume a small fraction of agents is created who imitate the behavior of their last opponent. Then the dynamics for the share of A-playing agents will be

$$x_t = \epsilon x_{t-1} + (1-\epsilon)x_{ESS}\,. \tag{24}$$

It is easily observable that the only steady state of this system is $\bar{x} = x_{ESS}$. Because we have $u[x_{ESS}, \epsilon y + (1-\epsilon)x_{ESS}] > u[y, \epsilon y + (1-\epsilon)x_{ESS}]$ for all $y \in \Delta_0 \backslash \{x_{ESS}\}$ and small enough ϵ, we would require the steady state to be exactly at $\bar{x} = x_{ESS}$ anyway. So if we picked an initialization of $x_0 = x_{ESS}$, the steady state would be reached right away and will never be left. This implies that there won't be any fluctuations created in the dynamics and the steady state value of agents who effectively play A is not changed even though the composition of the population changed.

So we found that for $(0,0,1,x_{ESS}) \in \Delta_1$ such that $u[y_t, \epsilon y_t + (1-\epsilon)x_{ESS}] = u[x_{ESS}, \epsilon y_t + (1-\epsilon)x_{ESS}]\ \forall t$ and so we also have $U[(y_t)_{t=0}^{\infty}, (\epsilon y_t + (1-\epsilon)x_{ESS})_{t=0}^{\infty}] = U[(x_{ESS})_{t=0}^{\infty}, (\epsilon y_t + (1-\epsilon)x_{ESS})_{t=0}^{\infty}]$. □

I have to add that we were only able to take away the property of evolutionary stability from the pure strategies when adding conditioning agents. However, since no one is able to beat a Nash equilibrium, it is clear that the ESSs remain neutrally stable. [19] Think of some minor punishment for an intellectual process, because e.g.

[19] Replacing ">" by "≥" in the definition of ESS yields the definition of NSS.

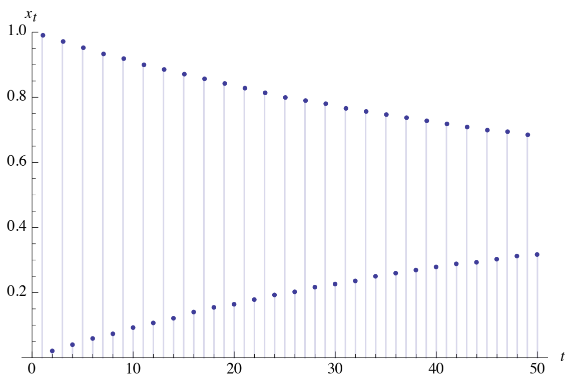
conditioning on period $t-1$ might be more effortful than just playing a fixed strategy all the time. Binmore and Samuelson [1992] analyze finite automata in repeated games and work with a lexicographic order, marginally handicapping more complex automata. If we were to introduce a similar rule, then the new agents were not anymore able to invade.

Now I want to ask the question in the other direction. Can there be compositions of predictive agents that make it impossible for non-predictive agents to invade?

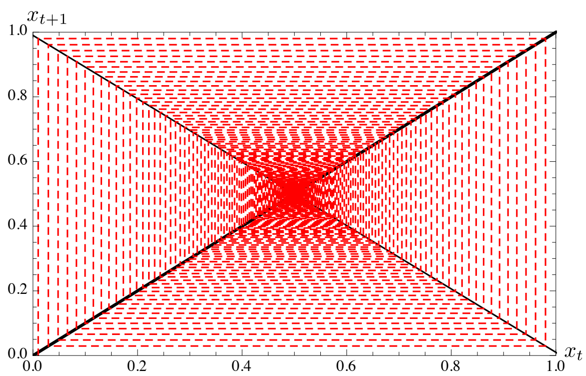
Imagine a symmetric 2by2 coordination game. That means in our payoff matrix we have $p_{AA} > p_{BA}$, $p_{BB} > p_{AB}$. Now, suppose there is a population of agents who act opposite to their last opponents' behavior (type 2 agents). The corresponding difference equation for the strategy dynamics is

$$x_t = (1 - x_t)$$

It is straightforward to see that for each starting value (except for 1/2) we will have a cycle of minimal period 2. Starting for instance at $x_0 = 0$ will make sure that agents alternately coordinate on the two pure strategy Nash equilibria of the game. Now, if a group of mutants of any type composition is created, the difference equation changes. Take for example an invasion of a few constantly B-playing and a few constantly A-playing individuals. This leads to a convergence to the steady state, as was shown in Theorem 1 in subsection 4.2.1.



(a)



(b)

Figure 28: $a_1 = \frac{1}{100}$, $a_2 = \frac{98}{100}$, $a_3 = 0$, $a_4 = \frac{1}{100}$, $x_0 = \frac{1}{100}$

Initially, when the fluctuations are still strong, it is impossible for other agent types to coordinate with the incumbents on playing the two Nash equilibria, one after the other. As the graphs of Figure 28 indicate, convergence to the steady state can potentially be quite slow. And since we discount the state utility of future periods, intuition suggests that the mutants might die out before enough convergence took place.

The next theorem formalizes this intuition.

Theorem 3. *In any symmetric 2by2 coordination game, acting opposite to one's last opponent's strategy is evolutionary stable.*

Formally: If $z = (0, 1, 0, z_0)$ *with* $z_0 \neq \frac{1}{2}$ *is the strategy of purely opposing the last opponents behavior* ($z_t = (1 - x_{t-1})$)*, then for each* $y \in \Delta_1 \backslash \{z\}$ *and* $\delta \in (0, 1)$ $\exists \bar{\epsilon}_y \in (0, 1)$ *s.t.*

$$U[(z_t)_{t=0}^{\infty}, (\epsilon y_t + (1-\epsilon)z_t)_{t=0}^{\infty}] > U[(y_t)_{t=0}^{\infty}, (\epsilon y_t + (1-\epsilon)z_t)_{t=0}^{\infty}]$$

holds for all $\epsilon \in (0, \bar{\epsilon}_y)$*.*

Proof. When a proportion of size ϵ of a mutant group of general composition invades the incumbent population consisting exclusively of type 2 agents, then the dynamics become

$$\begin{aligned} & x_t = (1-\epsilon)z_t + \epsilon y_t \\ \Leftrightarrow \quad & x_t = (1-\epsilon)(1 - x_{t-1}) + \epsilon(a_2(1 - x_{t-1}) + a_3 x_{t-1} + a_4) \\ \Leftrightarrow \quad & x_t + ((1-\epsilon) + \epsilon(a_2 - a_3))x_{t-1} = (1-\epsilon) + \epsilon(a_2 + a_4) \end{aligned}$$

Let us set the initial condition to be $x_0 = 0$. Then the solution to this inhomogeneos first order difference equation can by the usual methods be determined to be

$$x_t = \frac{\epsilon(a_2 + a_4) + (1-\epsilon)}{\epsilon(a_2 - a_3) + (2-\epsilon)} - \frac{(\epsilon(a_2 + a_4) + (1-\epsilon))(\epsilon(a_3 - a_2) + (\epsilon - 1))^t}{\epsilon(a_2 - a_3) + (2-\epsilon)}$$

This solution is well defined for $\epsilon = 0$, as well as for small positive values. Also we find that the static utility function $u[x_t, y_t] = x_t y_t p_{AA} + x_t(1 - y_t)p_{AB} + (1 - x_t)y_t p_{BA} + (1 - x_t)(1 - y_t)p_{BB}$ is continuous over the whole range of its arguments. These facts considerably simplify the analysis.

For the incumbents' utility, we have

$$\lim_{\epsilon \to 0} U[(z_t)_{t=0}^{\infty}, (\epsilon y_t + (1-\epsilon)z_t)_{t=0}^{\infty}] = \frac{(1-\delta)p_{AA} + p_{BB}}{2\delta - \delta^2} \tag{25}$$

And for the mutants utility, we have

$$\lim_{\epsilon \to 0} U[(y_t)_{t=0}^{\infty}, (\epsilon y_t + (1-\epsilon)z_t)_{t=0}^{\infty}]$$

$$= \frac{((a_2 + a_4)(\delta - 1)p_{AA} - (a_3 + a_4)p_{AB}) + ((a_2 + a_4)(1-\delta) - 1)p_{BA} + (-1 + a_3 + a_4)p_{BB}}{(\delta - 2)\delta} \tag{26}$$

Subtracting (25) and (26) gives us

$$\frac{(1-\delta)(1 - a_2 - a_4)(p_{AA} - p_{BA}) + (a_3 + a_4)(p_{BB} - p_{AB})}{(2-\delta)\delta}$$

Since we are dealing with a coordination game, the payoffs have to fulfill $p_{AA} > p_{BA}$ and $p_{BB} > p_{AB}$. The only way that this utility difference is not strictly positive is when $a_2 = 1$. But this case is excluded by $y \in \Delta_1 \backslash \{z\}$. So we have established that the incumbents utility is always strictly greater than the mutants utility. □

So there can be both, static behavioral forms loosing their property of being evolutionary stable as well as dynamic forms being able to prohibit any form of invasion, be it by inductive or by traditionally mixing agents.

4.4 Conclusion

Within an infinitely large population, even without the presence of evolutionary forces, inductive behavior can lead to nontrivial dynamics in the shares of applied strategies. However, we saw that in two player games in the case of agents who look back only a single period, the dynamics always converge if at least two different agent types exist in the population. When extending the depth of the agents' memory to 2 periods we are still able to analytically show local stability for the unique interior fixed point. Numerically we also observed global attractivity of this fixed point. Experiments also suggest that the same result applies for $M = 3$. If more than two players are paired in a game, the results qualitatively change and cycles of higher periods emerge. Also a complex, possibly chaotic behavior can be observed. In the final subsection we saw that when including inductive agents into the framework of evolutionary game theory, evolutionary stable strategies can loose their defining property.

The theory of difference equations is very close to enabling us concluding a global stability for higher order cases. In this respect the suggestion for further research is very concrete and clear. But there are other, more creative extensions, if you will, like embedding inductive agents into a form of replicator dynamics. This is only one of many imaginable aspects expanding on this topic.

The traditional theory of repeated games might in parts be an elegant construct, but it is incredibly far away from giving a plausible description of how people behave in *complex* situations. It is not a necessity that individuals start playing repeatedly all at the same time, assume that everybody else behaves rationally and then behave rationally themselves. Removing one brick from this wall takes the theory of repeated games from a static concept to a dynamic, needless to say, more realistic one. This is why I personally consider it inevitable that future research goes deeper into this very promising direction of inductive reasoning that I was only able to touch at its surface.

A Code for pre pre-image map

These functions, written in Objective-C, generate a random parameter set for the system with a horizon of two periods, and then check for a fine grid of initial conditions how many iterations are necessary to arrive at an epsilon neighborhood of the fixed point.

```
- (void) urbild{
    double xbar = ((1-b-c)+sqrt((1-b-c)*(1-b-c)-4*a*d))/(2*a); //fixed point
    [self defineNewMap]; //randomly a new map is generated

    double epsilon = 0.01; //size of the square, where the points are checked to land in

    //initialize the array to store the number of iterations unitl the trajectory is close enough
    for(int x1=0;x1<1000;x1++){
        for(int x2=0;x2<1000;x2++){
            schritteBisInEpsilonQadrat[x1][x2]=0;
        }
    }

    //iterate over the whole state space to classify the preimages
    for(double x1=0;x1<1.0;x1+=0.001){
        for(double x2=0;x2<1.0;x2+=0.001){
            x[0]=x2;
            x[1]=x1;
            int ti=1;
            //apply the map until we are epsilon close
            while (MAX(fabsf(x[ti-1]-xbar),fabsf(x[ti-2]-xbar)) > epsilon) {
                ti++;
                schritteBisInEpsilonQadrat[(int)(x1*1000)][(int)(x2*1000)] = ti-2;
                x[ti] = a*x[ti-1]*x[ti-2]+b*x[ti-1]+c*x[ti-2]+d;
            }
        }
    }

    //tell the window to display the changes
    dasFenster->urbildNeu=true;
    [[self.window contentView] setNeedsDisplay:YES];
}
```

```
//new model parameters are generated in this function
- (void) defineNewMap{
        a = -2.0+3.0*(double)rand()/RAND_MAX;
        b = -1.0+2.0*(double)rand()/RAND_MAX;
        c = -1.0+2.0*(double)rand()/RAND_MAX;
        d = (double)rand()/RAND_MAX;
        //check for the conditions on the parameters
        while ( !(0<a+b+c+d && a+b+c+d<1 && 0<b+d && b+d<1 && 0<c+d && ↩
            c+d<1 && -1<a+c && a+c<1 && -1<a+b && a+b<1) ) {
            a = -2.0+3.0*(double)rand()/RAND_MAX;
            b = -1.0+2.0*(double)rand()/RAND_MAX;
            c = -1.0+2.0*(double)rand()/RAND_MAX;
            d = (double)rand()/RAND_MAX;
        }
        x[0]=(double)rand()/RAND_MAX;
        x[1]=(double)rand()/RAND_MAX;
}
```

5 A steady state attendance at El Farol

5.1 Introduction

Experiments with the El Farol Bar problem as originally proposed by Arthur [1994] show the seemingly very robust existence of a fluctuating behavior concerning the bar attendance. These fluctuations appear so random, that many subsequent authors (e.g. Casti [1996]) conjectured a chaotic process. However, Shalizi and Albers [2002] argue that what is observed is not topological chaos. It is well known, that in high-dimensional dynamical systems the approach to periodic attractors can be so slow that they are never reached in simulations (Dyson [1979]). Wether or not the orbits that occur are chaotic or rather reflect a very slow convergence to periodic attractors hasn't been fully established as of this writing. Could it be that eventually the system approaches a very regular, perhaps even a steady behavior? Böhm [2015] establishes the existence of an ϵ-perfect predictor and argues that the failure of agents finding rational prediction rules which stabilize is not due to a non-existence of perfect rules, but rather to the failure of agents to identify the correct class of predictors from which the perfect ones can be chosen.

I seek to analyze the impact of certain limitations of the agents' mental capabilities on the existence of a stable attendance behavior. To be more specific, one of the goals shall be to answer the question as to wether it is a coincidence that in the typical El Farol experiment we do not observe steady state orbits.

5.2 The formal model

In the usual setting, all agents from the set $I = \{1, ..., N\}$ are equipped with a number of randomly chosen predictors. Here, a predictor ψ is a map from the last $T \geq 1$ periods' attendances, denoted by $(n_{t-1}, ..., n_{t-T})$ to the expected number of people being at the bar in the upcoming period n_t^e. But if agents only have a subset from the (depending on the history depth T) very large space of possible predictors at hand, then it is clear that it might not be possible to generate certain orbits, among which possibly being several or all orbits featuring a constant attendance. I will identify a necessary and sufficient condition for the set of predictor sets that agents are equipped with, such that a steady state solution exists under very weak assumptions about the learning algorithm that agents apply to select among their forecasting models.

A player's utility depends on his own action $x \in \{1, 0\}$, where 1 is attending the bar and 0 is staying at home, and the number of people at the bar n. I assume the standard utility function, featuring a fixed high utility for an evening at a non overcrowded bar with capacity c, a fixed low utility for an evening at an overcrowded bar and an intermediate utility for an evening at home which is normalized to 0.

$$u(x,\ n) = \begin{cases} 0 & \text{if} \quad x = 0 \\ u_f & \text{if} \quad x = 1 \text{ and } n > c \\ u_s & \text{if} \quad x = 1 \text{ and } n \leq c, \end{cases} \tag{27}$$

with $u_s > 0 > u_f$. The resulting best response correspondence is

$$\xi(n^e) = \begin{cases} 0 & \text{if} \quad n^e > c \\ 1 & \text{if} \quad n^e \leq c. \end{cases} \tag{28}$$

Using the defined notation, facing a specific attendance history $(n_{t-1}, ..., n_{t-T})$ agent i's action using predictor $\psi_{i,j}$ will be

$$x_{i,t} = \xi(\psi_{i,j}(n_{t-1}, ..., n_{t-T}))\,, \quad i \in I\,,\ \psi_{i,j} \in \omega_i$$

where ω_i is the set of prediction models that agent i can choose from. It is a subset of the set of predictors, $\omega_i \subset \Psi$. Concerning the question which predictors are selected I make the assumption of a rule under which the predictor with the highest valuation is deterministically chosen. Valuations of the models with the lowest absolute prediction errors are monotonically increased, meaning that in the long run, the predictor with the lowest prediction error is favored. In case of two prediction errors of equal size, the continually chosen predictor is determined by the initial conditions. Note that this naturally includes the learning mechanism Arthur [1994] uses, where the valuation of a predictor is calculated by forming a convex combination between last period's valuation and the inverse prediction error of the current period.

The intuition for the condition for the existence of a steady state is the following. If a steady state with an attendance of $\bar{n}$ exists, we must be able to find at least $\bar{n}$ agents who's most accurate predictor (being fed with the constant history of $\bar{n}$) predicts an attendance lower or equal to the bar's capacity. Conversely there must be at least $N - \bar{n}$ agents who's best predictor predicts an attendance strictly greater than the bar's capacity. This is formalized in the following proposition.

Proposition 8. *A steady state solution with constant attendance of* $\bar{n} = \sum_{i\in I} x_{i,t}$ *exists if and only if:*

$$|I_{0,\bar{n}}| \geq N - \bar{n}$$
$$|I_{1,\bar{n}}| \geq \bar{n} \ ,$$

where

$$I_{0,\bar{n}} = \{i \in I \ : \ \exists\, \psi_{i,j} \in \omega_i \wedge \psi_{ij}(\bar{n},...,\bar{n}) > c \wedge |\psi_{ij}(\bar{n},...,\bar{n}) - \bar{n}| \leq |\psi_{ij'}(\bar{n},...,\bar{n}) - \bar{n}| \ \forall\, \psi_{ij'} \in \omega_i \backslash \psi_{i,j}\}$$

$$I_{1,\bar{n}} = \{i \in I \ : \ \exists\, \psi_{i,j} \in \omega_i \wedge \psi_{ij}(\bar{n},...,\bar{n}) \leq c \wedge |\psi_{ij}(\bar{n},...,\bar{n}) - \bar{n}| \leq |\psi_{ij'}(\bar{n},...,\bar{n}) - \bar{n}| \ \forall\, \psi_{ij'} \in \omega_i \backslash \psi_{i,j}\}$$

Proof. The direction "$\Leftarrow$" is straightforward. According to the assumption concerning the learning rule, in the long run, agents must use their most accurate predictors. The two sets $I_{0,\bar{n}}$ and $I_{1,\bar{n}}$ identify all agents who's best predictors recommend staying at home and going to the bar, respectively, when confronted with a constant history of $\bar{n}$ attendances. So in the set of agents there are $\bar{n}$ who have an optimal predictor telling them to go, so in the long run they will go. Conversely, there are $N - \bar{n}$ agents who have an optimal predictor telling them to stay at home, which in the long run they will do. Choosing the appropriate initial conditions will then implement the attendance of exactly $\bar{n}$.

To establish "$\Rightarrow$" first consider a case where $|I_{0,\bar{n}}| < N - \bar{n}$. This implies that the set of agents who's best predictor commanding them to go is *strictly* better than their best predictor commanding them not to go has cardinality $> \bar{n}$. Formally

$$|\{i \in I \ : \ \exists\, \psi_{i,j} \in \omega_i \wedge \psi_{ij}(\bar{n},...,\bar{n}) \leq c \wedge |\psi_{ij}(\bar{n},...,\bar{n}) - \bar{n}| < |\psi_{ij'}(\bar{n},...,\bar{n}) - \bar{n}| \ \forall\, \psi_{ij'} \in \omega_i \backslash \psi_{i,j}\}| > \bar{n}$$

Since agents must use their most accurate predictors. This means that the attendance will eventually rise above $\bar{n}$.

For the case of $|I_{1,\bar{n}}| < \bar{n}$, an analogous reasoning applies.

□

Let us denote the set of all subsets of the predictor space Ψ by Ω. An element $\omega_i = \{\psi_1, ..., \psi_m\} \in \Omega$ represents a mental model endowment of one agent. The subset of Ω in which all predictor sets have the property that when facing a constant attendance history of $\bar{n}$ one of the most accurate predictors predicts an above (below or equal) capacity attendance will be called $\Omega_{\bar{n}c0}$ ($\Omega_{\bar{n}c1}$). Note that $\Omega_{\bar{n}c0} \cup \Omega_{\bar{n}c1} = \Omega$, but $\Omega_{\bar{n}c0} \cap \Omega_{\bar{n}c1} \neq \emptyset$.

As mentioned above, in the El Farol and Minority Game literature it is common to equip all agents with a number of uniformly chosen random predictors. I am interested in the probability that such a model setup can generate a constant attendance orbit.

For that we need the set of predictor endowments of all N agents, which will be called Φ_N. These are all N dimensional vectors who's components are taken from Ω. Formally

$$\Phi_N = \{(\omega_1, ..., \omega_N) \,:\, \omega_i \in \Omega \,\forall\, i \in I\}.$$

The predictor endowments with the property that they can generate a constant orbit for a bar with capacity c will be called $\Phi_{N\,c\,steady}$. All elements of $\Phi_{N\,c\,steady}$ have at least $\bar{n}$ entries which are also elements of $\Omega_{\bar{n}c1}$ and at least $N - \bar{n}$ entries which are also elements of $\Omega_{\bar{n}c0}$ for some $\bar{n}$.

$$\begin{aligned}\Phi_{N\,c\,steady} = & \{\phi \in \Phi_N \,:\, \exists\, \bar{n}\, s.t.\, |\{i \in I \,|\, \omega_i \in \{\phi_k : k \in \{1, ..., N\}\} \cap \Omega_{\bar{n}c0}\}| \geq N - \bar{n} \\ & \wedge\, |\{i \in I \,|\, \omega_i \in \{\phi_k : k \in \{1, ..., N\}\} \cap \Omega_{\bar{n}c1}\}| \geq \bar{n}\}\end{aligned}$$

I want to determine $\frac{|\Phi_{N\,c\,steady}|}{|\Phi_N|}$.

5.3 Calculation

5.3.1 Analytic approach

Consider an agent, who gets assigned two random predictors. If the steady state is $\bar{n}$, then this corresponds to simply choosing two integers on the interval $[0, N]$. The predictor, which is chosen in the long run, is the one with the lowest absolute prediction error, corresponding to the distance between its prediction and $\bar{n}$. Here it becomes very obvious that the desired magnitude $\frac{|\Phi_{N\,c\,steady}|}{|\Phi_N|}$ is independent of the horizon agents look back in the past.

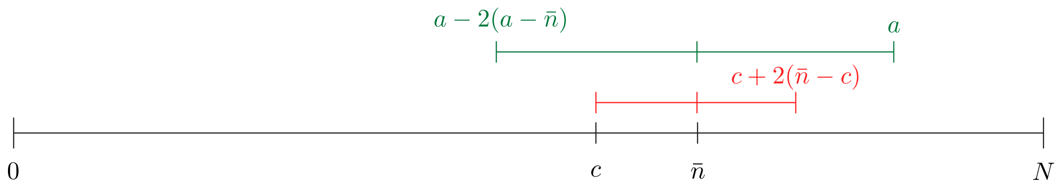


Figure 29: Two predictors with steady state $\bar{n}$

Let p_0 and p_1 be the probabilities that the best of two randomly chosen predictors (uniform distribution), facing a constant history of $\bar{n}$ will command an agent to go to the bar or to stay at home, respectively. p_{01} is the overlap between the two, corresponding to the probability that one of the best predictors recommends going and the other recommends staying at home. This would mean that, choosing the right initial conditions, both actions are possible.

$$p_0 = \left(\frac{N-c}{N+1}\right)^2 + \binom{2}{1}\frac{c+1}{N+1}\frac{N-c}{N+1}\frac{1}{N-c}\left(\min\left(2(\bar{n}-c)-1, 1\right) + \sum_{a=\min(c+2(\bar{n}-c),N)}^{N} \max\left(\frac{a-2(a-\bar{n})+1}{c+1}, 0\right)\right)$$

$$p_1 = \left(\frac{c+1}{N+1}\right)^2 + \binom{2}{1}\frac{c+1}{N+1}\frac{N-c}{N+1}\frac{1}{N-c}\left(\sum_{a=\min(c+2(\bar{n}-c),N)}^{N} \min\left(\frac{c-(a-2(a-\bar{n}))+1}{c+1},1\right)\right)$$

$$p_{01} = p_0 + p_1 - 1$$

Figure 29 illustrates the idea behind the above formulas. For the best of two predictors recommending to stay at home we can either have that both predictors predict a number greater than c. This event has a probability of $\left(\frac{N-c}{N+1}\right)^2$. The other way it could happen is if one of the two predictors lies between 0 and c and the other between $c+1$ and N. If now the greater prediction lies in the interval drawn in red, then it is for sure the best. If it lies outside the red interval, we have to make sure the lower prediction is small enough to be further away from $\bar{n}$. So the green line shall indicate the variable a, handled in the sum in the last term. For the p_1 formula an analogous logic applies. Please note that the above formulas are only correct if $\bar{n} \geq c$. For the complementary case, the rightmost terms have to be altered.

To find out the probability that the steady state with attendance $\bar{n}$ exists, we need to aggregate these probabilities over the whole population of N individuals. Each agent is assigned two predictors and the aggregate outcome, where each agent uses his best predictor, has to be an attendance of exactly $\bar{n}$. Otherwise this particular steady state does not exist.

According to the proposition in section 5.2, the size of group 0 (agents who's best model indicates staying at home is optimal) has to be at least $N - \bar{n}$ and of group 1 (agents who's best model indicates going to the bar is optimal) has to be at least $\bar{n}$. Since the two groups overlap, one has to use a multinomial distribution.

Let $F(n_1, n_2, n_3)$ be the cumulative distribution function of the multinomial distribution with N trials and probabilities $p_0 - p_{01}$, $p_1 - p_{01}$ and p_{01} [20]. The function gives the probably of at most n_1 individuals belonging *not* to group 1, at most n_2 individuals belonging *not* to group 0, and at most n_3 individuals belonging to both groups.

Then

$$F(N - \bar{n}, \bar{n}, N)$$

gives the probability that the steady state at $\bar{n}$ exists.

Let us consider an example. As in Arthur's original setting, let $N = 100$ and $c = 60$. Then the probabilities that an agent will or won't attend at a steady state of $\bar{n} = 65$ are $p_1 = \frac{4713}{10201}$ and $p_0 = \frac{5550}{10201}$, respectively. Plugging this into the multinomial distribution yields a value of 0.0000866548. So the relative size of the space of predictor endowments across 100 agents, in which this particular steady state of an attendance of 65 exists, is very small.

[20] Note that these three numbers sum to 1.

To find out the chance of *any* steady state to exist, one has to repeat this procedure and then sum over all possible $\bar{n}$. Since this involves a lot of effort, including a case differentiation concerning the relative location of $\bar{n}$ and because we can only cover the case of agents being equipped with exactly 2 predictors, I suggest switching to a simulation approach at this point.

5.3.2 Numerical approach

Numerically determining $\frac{|\Phi_{N\,c\,steady}|}{|\Phi_N|}$ under the restriction of a particular amount of prediction models that agents are endowed with will be accomplished by a Monte Carlo Simulation approach. The program is written in C and its task is to save $\frac{|\Phi_{N\,c\,steady}|}{|\Phi_N|}$ for all predictor quantities between 2 and 200. The code is shown in the listing below.

```
void probabilityForSteadyStateExistance() {
    int N = 100;                //Number of agents
    int psi[100][201][101];     //Predictors
    int c = 60;                 //Bar's capacity
    double x[2] ={0,0};         //Number of Agents who stay at home/go to ↩
        the bar
    bool ssPossible = false;
    double pSs = 0;             //Steady state probability
    int numberOfRuns = 10000;

    FILE *file;
    file = fopen("../pSs.csv","w");

    for (int numberOfPredictors =2; numberOfPredictors<=200; ↩
        numberOfPredictors+=1) {
        pSs=0;
        for (int a=0; a<numberOfRuns; a++) {
            ssPossible = false;

            //Predictors are randomly assigned
            for (int i=0; i<N; i++)    //Agents
                for (int j=0; j<numberOfPredictors; j++)    //Predictors
                    for (int n=0; n<N+1; n++)       //Histories
                        psi[i][j][n] = floor((double)rand()/RAND_MAX*(N↩
                            +1));


            for (int n=0; n<101; n++) {
                x[0]=0;
                x[1]=0;
                for (int i=0; i<N; i++) {
```

```
                    //Best predictor is found
                    int minDistance=999999;
                    for (int j=0; j<numberOfPredictors; j++)
                        if (abs(psi[i][j][n]-n)<minDistance)
                            minDistance = abs(psi[i][j][n]-n);

                    for (int j=0; j<numberOfPredictors; j++) {
                        if (abs(psi[i][j][n]-n)==minDistance) {
                            if (psi[i][j][n]<=c)
                                x[1]++;
                            else
                                x[0]++;
                        }
                    }
                }
                //Check whether attendance number fits
                if(x[0]>=(N-n) && x[1]>=n)
                    ssPossible = true;
            }

            if (ssPossible)
                pSs++;
        }

        pSs/=(double)numberOfRuns;
        fprintf(file,"%d, %lf \n", numberOfPredictors, pSs);
    }

    fclose(file);
}
```

Agents will simply be equipped with random predictors before the program will run through all possible steady state solutions. For each one the best predictor(s) of each agent will be determined. Then the program checks what would happen if the agents actually applied these predictors and acted according to the best response correspondence. The number of agents who attend the bar or stay and home will be stored in an array called "x". These numbers are then compared to the assumed steady state. If the attendance fits to it, then the variable "pSs" is incremented. This is repeated "numberOfRuns" times and in the end the value is divided by the number of runs before being saved to the file.

The output of the program, using the parameters from Brian Arthur's original setup, can be seen in the diagram shown in Figure 30. Note that the qualitative properties of the solution are robust for other capacities and population sizes.

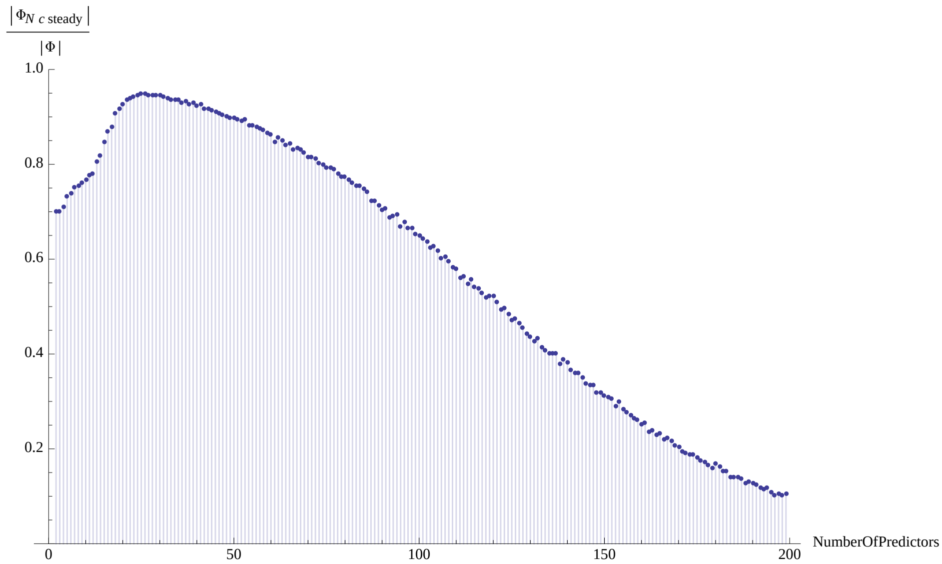


Figure 30: Relative cardinality of predictor endowments with steady state for different amounts of predictors per agent ($N = 100$, $c = 60$)

One could have guessed that the probability of a steady state existence goes to zero when increasing the number of predictors per agent to really high values. When every agent is able to make any prediction, then everybody will be able to predict exactly $\bar{n}$ and as long as $\bar{n} \leq c$, everybody will go to the bar, falsifying the initial assumption of an attendance of $\bar{n}$ and hereby excluding the existence of that steady state. Similarly, if $\bar{n} > c$ everybody will predict an overcrowded bar and nobody will go so again the steady state won't exist. [21]

All this is not true for lower values of the number of predictors. Interestingly for two predictors per agent $\frac{|\Phi_{N\ c\ steady}|}{|\Phi_N|}$ starts at 70% and then it increases when increasing the agents mental capabilities, so to say, by providing them with more predictors. At about 27 predictors per agent there seems to be a maximal value of around 95% for a steady state to exist, before the numbers drop monotonically towards zero.

So apparently, the existence of a steady state critically depends on the agents' mental capabilities. Agents being able to choose from intermediate amounts of randomly generated prediction models make sure that the existence of a steady state attendance orbit is almost certain. More interestingly, increasing the agents' prediction skills, lowers the relative size of the predictor endowment space with the steady state existence property. From a game theoretic perspective, smarter agents thus make the implementation of any of the pure Nash equilibrium solutions essentially impossible.

[21] Note that in my program, I do not exclude the perfect resemblance of two prediction models of one agent. As long as the agents consider at least one period of the past attendance, the space of prediction models is large enough to justify this simplification. For the case of every agent actually possessing *every* possible predictor, the relative cardinality is 0.

5.4 Conclusion

I built a formal model to find out the conditions that are necessary and sufficient for a steady state solution in the classical setup of the El Farol Bar Problem. The analysis was directed towards the relative size of the space of predictor endowments across the agents with and without the existence of a steady state. An analytic approach is surely possible and was able to guide our intuition concerning what the conditions structurally imply. A numerical simulation revealed that (using Arthur's parameter set) the probability of a steady state existence when uniformly randomly assigning prediction rules to the agents starts at intermediate values for 2 predictors per agent, then rises close to 1 before dropping to eventually approach 0 for large predictor endowments.

It would be intriguing to know whether these steady states, given their existence, are attracting at least on a local neighborhood of attendance histories and predictor valuations. Also, an object of further research could be the existence of low period cycles.

Not much is known about the dynamic behavior of classifier systems, especially in the context of a from an agent's perspective ill-defined competition for a scarce resource, like the El Farol Bar Problem and the Minority Game. However, I doubt that these systems have to remain black boxes.

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