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American options with multiple priors in continuous time

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Abstract

We investigate American options in a multiple prior setting of continuous time and determine optimal exercise strategies form the perspective of an ambiguity averse buyer. The multiple prior setting relaxes the presumption of a known distribution of the stock price process and captures the idea of incomplete information of the market data leading to model uncertainty. Using the theory of (reflected) backward stochastic differential equations we are able to solve the optimal stopping problem under multiple priors and identify the particular worst-case scenario in terms of the worst-case prior. By means of the analysis of exotic American options we highlight the main difference to classical single prior models. This is characterized by a resulting endogenous dynamic structure of the worst-case scenario generated by model adjustments of the agent due to particular occurring events that change the agent's beliefs.

Key words and phrases: optimal stopping for exotic American options, uncertainty aversion, multiple priors, robustness, (reflected) BSDEs

JEL subject classification: G13, D81, C61

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1 Introduction

This paper builds on a previous analysis of optimal stopping problems for American exotic options under ambiguity, Chudjakow and Vorbrink [5]. The motivations and the economic relevance of this study are similar to before, although we move from discrete to continuous time.

In finance it is more appropriate to use continuous time models. Closed-form solutions have the advantage of being easier to interpret, and as such, tend to predominate. They allow for comparative statics that would be otherwise difficult to interpret. In our analysis continuous time also provides a direct relationship to the famous Black-Scholes model, Black and Scholes [1].

We analyze American options from the perspective of an ambiguity averse buyer in the sense of Ellsberg's paradox. The task of the buyer holding the option is to exercise it optimally realizing the highest possible utility. The valuation reflects the agent's personal utility as it depends on investment horizon, objective, and on risk, as well as ambiguity attitude. Generally this valuation is not related to the market value directly.

Given a classical stochastic model in continuous time such as the Black-Scholes model, one can solve the optimal stopping problem of the buyer using classical theory on optimal stopping, or the relation to free-boundary problems. Despite the abundance of literature on the issue, e.g. Peskir and Shiryaev [20] or El Karoui [9], these settings impose the assumption of a unique probability measure that drives stock price processes. This assumption might be too strong in many cases since it requires perfect understanding of the market and complete agreement on one particular model. To incorporate uncertainty we drop this assumption. We consider a Black-Scholes-like market whose stock price $X = (X_t)$ evolves according to

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{1}$$

where $W = (W_t)$ represents standard Brownian motion under some reference measure P.[1] The various beliefs of the agent are reflected by a set of multiple priors (probability measures) $\mathcal{P}$. Thus she considers the dynamics in (1) under each prior Q of the set $\mathcal{P}$ which provides a family of models that come into question to evaluate the claims.

[1] Later we change this point of view slightly, cf. page 11.

As an example we have in mind a bank which holds an American claim in its trading book. The trading strategy of the bank depends on the underlying model used by the bank. If the model specification is error-prone the bank faces model uncertainty. Being unable to completely specify the model, traders rather use multiple priors model instead of choosing one particular model. If the uncertainty cannot be resolved and the accurate model specification is impossible, traders prefer more robust strategies as they perform well even if the model is specified slightly incorrect.

Also, a risk controlling unit assigning the portfolio value and riskiness uses rather a multiple priors model in order to test for model robustness and to measure model risk. Taking several models into account, while performing portfolio distress tests, allows to check the sensitivity of the portfolio to model misspecification. Again in a situation of model uncertainty more robust riskiness assignment is desirable as it minimizes model risks.

Similar reasoning can be applied to accounting issues. An investment funds manager making his annual valuation is interested in the value of options in the book that are not settled yet. In case the company applies coherent risk measures as standard risk evaluation tool for future cash flows on the short side, it is plausible to use a multiple priors model evaluating long positions. Finally, a private investor holding American claims in his depot might exhibit ambiguity aversion in the sense of Ellsberg paradox or Knightian uncertainty. Such behavior may arise from lack of expertise or bad quality of information that is available to the decision maker.

Although for different reasons, all the market participants described above face problems that should not be analyzed in a single prior model and need to be formulated as multiple priors problems.

As to the ambiguity model, we use κ-ignorance, see Chen and Epstein [2]. It models uncertainty in the drift rate of the stock price. Under each prior, the stock price in (1) obtains an additional drift rate term varying within the interval $[-\kappa, \kappa]$, where κ measures the degree of ambiguity/uncertainty. As noted in Cheng and Riedel [4], it is essential that the additional terms be allowed to be stochastic and time-varying as this guarantees dynamic consistency.[2]

Dynamic consistency allows the agent to adapt the model according to changing beliefs induced by occurring events. In this setting, the agent hold-

[2]See Cheng and Riedel [4] and Delbaen [6] for a discussion of the concept of dynamic consistency in dynamic models.

ing an American option who is uncertain about the correct drift of the underlying stock price faces the optimization problem

$$V_t := \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right). \tag{2}$$

To clarify, at the current time t, the agent aims to optimize her expected discounted payoff $H_\tau \gamma_{\tau-t}^{-1}$ in a worst-case scenario by exercising the claim prior to maturity.

In our analysis the optimization problem is solved by using the relationship to reflected backward stochastic differential equations (RBSDEs).[3] To obtain this relation, the generator of the (reflected) BSDE should be chosen as $f(t, y, z) = -ry - \kappa|z|$ where $-\kappa|z|$ describes the ambiguity aversion and $-ry$ the discounting. This was first established by Chen and Epstein [2] who used the generator $f(z) = -\kappa|z|$ for a BSDE to derive a generalized stochastic differential utility. A similar BSDE framework to is used in El Karoui and Quenez [12] in the context of pricing and hedging under constraints.

BSDEs provide a powerful method for analyzing problems in mathematical finance, (El Karoui, Peng, and Quenez [11] and Duffie and Epstein [8]), or in stochastic control and differential games (Hamadene and Lepeltier [13] & Pham [21]). BSDEs, in conjunction with g-expectations, play an important role in the theory of dynamic risk measures, (Peng [18]) and dynamic convex risk measures, respectively, (Delbaen, Peng, and Gianin [7]). By means of "reflection", the solution is maintained above a given stochastic process, in our case, the payoff process of the respective American claim.

We analyze the problem in (2) for several American options exemplifying the effect of ambiguity. As described in Chudjakow and Vorbrink [5] the effect of ambiguity depends highly on the payoff structure of the claim. If the payoff satisfies certain monotonicity behavior as is the case for the American call and put option, the situation resembles the classical one without the emergence of ambiguity. The agent's worst-case scenario is specified by the least favorable drift rate of the stock price process that affects the performance of the agent's option. This scenario is identified by the worst-case prior. In the above described monotone case, the worst-case prior leads to the lowest possible drift rate for the stock price process in case of a call, and the highest possible drift rate in the case of a put option.

[3] Another approach is the characterization of the value function (V_t) by Cheng and Riedel [4] as the smallest right-continuous g-supermartingale that dominates the payoff from exercising the claim.

For options with more complex payoffs, the worst-case prior generates a stochastic drift rate in (1) which is path-dependent and produces endogenous dynamics in the model. These are induced by the ambiguity averse agent and her reaction to the latest information by adjusting the model from time to time as necessary depending on her changing beliefs, or fears, respectively. As such, in the multiple prior setting, changing fears due to transpired events are taken into account when American claims are evaluated and early exercise strategies are determined.

This central difference to classical models is exemplified with the help of barrier options and shout options. In the latter case, the agent will change her beliefs directly after taking action, when she fixes the strike price. In the case of barrier options, here exemplified by means of an up-and-in put option, she adapts the model as a consequence of the trigger event when the underlying stock price reaches the barrier specified in the claim's contractual terms.

From decision theoretical point of view, our examples expose that optimal stopping under ambiguity aversion is behaviorally distinguishable from optimal stopping under subjective expected utility. For example, the holder of an American up-and-in put behaves as two readily distinguishable expected utility maximizers.

The paper is structured as follows. The following section introduces the ambiguity setup in continuous time and relates the resulting multiple prior framework to the financial market. Section 3 presents the decision problem of an ambiguity averse agent who holds an American option. It contains a short detour to reflected BSDEs and explains their relationship to the decision problem of the ambiguity averse agent. This section also provides the solution to the optimal stopping problem for American options featuring some monotone payoff structure (see Section 3.2). This section builds the base for the subsequent analysis in Section 4 concerning American claims with more complex payoffs such as up-and-in put options or shout options. Extensive proofs are given in the appendix, Section 5 concludes.

2 The setting

We introduce the ambiguity framework in continuous time. We focus on κ-*ignorance*, a particular ambiguity setting, as described by Chen and Epstein [2] who introduced various ambiguity models. Throughout this paper we consider an ambiguity framework for a fixed finite time horizon $T > 0$.

First, we depict the ambiguity model κ-ignorance as in Chen and Epstein [2]. Second we introduce the financial market within this ambiguity framework.

Remark 2.1 *Given an infinite time horizon, one faces additional technical difficulties according to the underlying filtration arising from Girsanov's theorem and a Brownian motion environment.*[4] *This leads to weaker assumptions on filtration. In particular, the usual conditions on filtration should be relaxed.*[5] *This sometimes causes technical problems since the theory of stochastic calculus and backward stochastic differential equations is usually developed under these conditions.*[6]

2.1 The ambiguity model κ-ignorance

Let $W = (W_t)$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is the completed Borel σ-algebra on Ω. We denote by $(\mathcal{F}_t)_{0 \le t \le T}$ the filtration generated by the process W and augmented with respect to P. We have $\mathcal{F}_T = \mathcal{F}$ and the filtration satisfies the usual conditions. P serves as a reference measure in the ambiguity model. As we shall see, under κ-ignorance all occurring probability measures $Q \in \mathcal{P}$ are equivalent. So, P has the role of fixing the events of measure zero. Hence, there will be no uncertainty about the events of measure zero.

Remark 2.2 *Throughout the analysis, unless stated otherwise, all equalities and inequalities will hold almost surely. The "almost-sure-statements" are to be understood with respect to the reference measure P. Due to the equivalence of all priors $Q \in \mathcal{P}$ the statements will also hold almost surely with respect to any prior $Q \in \mathcal{P}$. If we write $\mathbb{E}$ without any measure we will mean the expectation with respect to the reference measure P.*

Let us depict the construction of the ambiguity model κ-ignorance, Chen and Epstein [2], Delbaen [6]. It relies heavily on Girsanov's theorem. We

[4] See Remark 2.4 as an illustration.

[5] Usually the filtration is assumed to satisfy the usual conditions. This means that the filtration is right-continuous and augmented, cf. Karatzas and Shreve [14].

[6] The interested reader is referred to von Weizsäcker and Winkler [23] who develop stochastic calculus in particular Itô calculus without assuming the usual conditions.

only focus on the one-dimensional case. The d-dimensional case works in a straightforward way.

First consider $\mathbb{R}$-valued measurable, $(\mathcal{F}_t)$-adapted, and square-integrable processes $\theta = (\theta_t)$ such that the process $z^\theta = (z_t^\theta)$ defined by

$$dz_t^\theta = -\theta_t z_t^\theta dW_t, \quad z_0^\theta = 1,$$

that is,

$$z_t^\theta = \exp\Big\{ -\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dW_s \Big\} \quad \forall t \in [0, T] \tag{3}$$

is a P-martingale. Given $\kappa > 0$ we define the set of density generators Θ by

$$\Theta = \{\theta | \, \theta \text{ progressively measurable and } |\theta_t| \leq \kappa, \ t \in [0, T]\}.^7 \tag{4}$$

κ is called the degree of ambiguity (uncertainty). Obviously, for each $\theta \in \Theta$ the Novikov condition $\mathbb{E}\big(\exp\{\frac{1}{2}\int_0^T \theta_s^2 ds\}\big) < \infty$ is satisfied. Therefore, $\mathbb{E}(z_T^\theta) = z_0^\theta = 1$ and z_T^θ is a P-density on $\mathcal{F}$, Karatzas and Shreve [14]. Consequently, each $\theta \in \Theta$ induces a probability measure Q^θ on $(\Omega, \mathcal{F})$ that is equivalent to P where Q^θ is defined by

$$Q^\theta(A) := \mathbb{E}(\mathbb{1}_A z_T^\theta) \quad \forall \text{A} \in \mathcal{F}. \tag{5}$$

In other words,

$$\frac{dQ^\theta}{dP}\bigg|_{\mathcal{F}_t} = z_t^\theta \quad \forall t \in [0, T].$$

According to Girsanov's theorem (cf. Karatzas and Shreve [14]) we define the set of probability measures $\mathcal{P} := \mathcal{P}^\Theta$ on $(\Omega, \mathcal{F})$ generated by Θ by

$$\mathcal{P}^\Theta := \{Q^\theta \mid \theta \in \Theta \text{ and } Q^\theta \text{ is defined by (5)}\}. \tag{6}$$

[7] Since we work in a Brownian motion environment we do not need to require predictability in (4) as in Delbaen [6], cf. Theorem 6.3.1 in von Weizsäcker and Winkler [23].

Note that we allow for stochastic and time-varying Girsanov kernels θ. This is important to ensure the dynamic consistency. We otherwise lose this important property.[8]

Additionally, by Girsanov's theorem, the process $W^\theta = (W_t^\theta)$ defined by

$$W_t^\theta := W_t + \int_0^t \theta_s ds \quad \forall t \in [0, T] \tag{7}$$

is a standard Brownian motion on $(\Omega, \mathcal{F})$ with respect to the measure Q^θ.

Remark 2.3 *κ-ignorance as an ambiguity model has important properties. It allows for explicit results when evaluating financial claims since the range of values of the density processes θ does not change over time as is the case for other models like IID-ambiguity in Chen and Epstein [2]. Consequently we shall see that the worst-case densities become very simple in some examples, meaning without any formal difficulties. Furthermore, under κ-ignorance, the set of priors $\mathcal{P}$ possesses important properties like m-stability or time-consistency, Delbaen [6], and the existence of worst-case priors, Chen and Epstein [2].*[9]

Regarding Remark 2.1 the following remark illustrates the importance of relaxing the usual conditions for filtration when κ-ignorance is constructed on an infinite time horizon.

Remark 2.4 *(cf. Karatzas and Shreve [14]) Let P be Wiener measure on $(\Omega, \mathcal{F}) := (\mathcal{C}([0,\infty), \mathbb{R}), \mathcal{B}(\mathcal{C}([0,\infty), \mathbb{R})))$ such that the canonical process $W = (W_t), W_t(\omega) := \omega(t), 0 \leq t < \infty, \omega \in \Omega$ is a standard Brownian motion. Denote by $(\mathcal{F}_t^W)$ the (not augmented) filtration generated by W such that $\mathcal{F}_\infty^W = \mathcal{F}$. Let $\theta = (\theta_t)$ be a progressively measurable process with corresponding filtration $(\mathcal{F}_t^W)$, and square-integrable for each $T \in [0, \infty)$. Assume that the process $z^\theta = (z_t^\theta)$ defined as in (3) is a P-martingale. Then Girsanov's theorem for an infinite time horizon*[10] *states that there exists a probability measure Q^θ satisfying*

$$Q^\theta(A) = \mathbb{E}(z_T^\theta 1_A), \;\; A \in \mathcal{F}_T^W, \;\; T \in [0, \infty) \tag{8}$$

[8] See Chen and Epstein [2] for details. Also the examples in Section 4 illustrate this fact.

[9] See also Chudjakow and Vorbrink [5].

[10] See Corollary 5.2 in Karatzas and Shreve [14].

and the process $W^\theta = (W_t^\theta)$ defined as in Equation (7) with corresponding filtration $(\mathcal{F}_t^W)$ is a Brownian motion on $(\Omega, \mathcal{F}, Q^\theta)$.

It is essential that $(\mathcal{F}_t^W)$ be raw, unaugmented filtration. Therefore, κ-ignorance can only be constructed with respect to a filtration that does not fulfill the usual conditions.

The difference to the finite time horizon is that now P and Q^θ are only mutually locally absolutely continuous, i.e., equivalent on each $\mathcal{F}_T^W, T \in [0,\infty)$. Viewed as probability measures on $\mathcal{F}$, P and Q^θ are equivalent if and only if z^θ is uniformly integrable. To understand why (8) is only required to hold for $A \in \mathcal{F}_T^W, T \in [0,\infty)$, consider the following example.

Example 2.5 *Let $\mu > 0$ and fix a process θ with $\theta_t := -\mu \ \forall t \in [0,\infty)$. For this θ consider the P-martingale z^θ defined by*

$$z_t^\theta = \exp\{-\frac{1}{2}\mu^2 t + \mu W_t\} \quad \forall t \in [0,\infty).$$

z^θ is not uniformly integrable. By Girsanov's theorem and the law of large numbers for Brownian motion, Karatzas and Shreve [14] we obtain for $A := \{\lim_{t\to\infty} \frac{W_t}{t} = \mu\} \in \mathcal{F}$

$$Q^\theta(A) = 1 \quad \text{and} \quad P(A) = 0.$$

Clearly, the P-null event A is in the augmented σ-field $\mathcal{F}_T$ for every $T \in [0,\infty)$. This is the reason why (8) is only required to hold for all $A \in \mathcal{F}_T^W, T < \infty$. Otherwise P and Q^θ were mutually singular on $\mathcal{F}_T$ for every $T \geq 0$.

Therefore, κ-ignorance in a Brownian motion environment with infinite time horizon must be set up on a filtration that is not augmented by the P-null sets of $\mathcal{F}$.

2.2 The financial market under κ-ignorance

Throughout this paper we consider a Black-Scholes-like market consisting of two assets, a riskless bond γ and a risky stock X. Their prices evolve according to

$$\begin{aligned} d\gamma_t &= r\gamma_t dt, \quad \gamma_0 = 1, \\ dX_t &= \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x > 0 \end{aligned} \tag{9}$$

where r is a constant interest rate, μ a constant drift rate, and $\sigma > 0$ a constant volatility rate for the stock price.[11] The dynamics in (9) are obviously free of ambiguity. To incorporate ambiguity, the decision maker considers Equation (9) under multiple priors. She uses the set of priors $\mathcal{P}$ as defined in (6). As we shall see, by utilizing the set $\mathcal{P}$ she tries to capture her uncertainty about the true drift rate of the stock.

Let $Q \in \mathcal{P}$, if Q is equal to Q^θ for $\theta \in \Theta$ then the stock price dynamics under Q become

$$dX_t = \mu X_t dt - \sigma X_t \theta_t dt + \sigma X_t dW_t^\theta.$$

This illustrates that κ-ignorance just models uncertainty about the true drift rate of the stock price.

At this point it is worthwhile mentioning that by changing the prior under consideration, the stock price's volatility rate remains completely unchanged. Based on the equivalence of all priors and Girsanov's theorem, κ-ignorance cannot be used to model volatility uncertainty. This requires a set of mutually singular priors. For a detailed study of this issue see Peng [19] or Vorbrink [24].

In the next section, we consider American contingent claims from the perspective of an ambiguity averse decision maker who holds a long position in the claims. The decision maker, a private investor or financial institution, for example, may seek to evaluate or liquidate their position. Both may happen with respect to their subjective probability distribution. They may use their subjective probability distribution to evaluate the claim and to figure out an optimal exercise strategy due to the claim's American feature. In addition, in real option investment decisions, the subjective probability law appears naturally when coming to a decision.[12]

All decision problems are considered under Knightian uncertainty. We focus on a decision maker who is uncertain about market data. As a consequence she does not believe completely in the dynamics proposed in (9). For instance she is uncertain about the stock's drift rate which in turn affects the market price of risk.

Contingent claims in finance are typically evaluated with respect to risk-neutral probability measures. Therefore, we assume that the agent will con-

[11] As it is often possible we may also consider a price process with non-constant and stochastic coefficients. To avoid later distinctions of cases and missing the point we assume constant coefficients.

[12] See McDonald and Siegel [17], for example.

sider the stock's dynamics in (9) under the risk-neutral probability measure. Since she does not completely trust in the market, nor all the data, she allows for various market prices of risk.[13] She takes into account prices surfacing around $\frac{\mu-r}{\sigma}$ currently observed at the market. Expanding on this idea, if $Q = Q^\theta$ for some θ defined by $\theta_t = \frac{\mu-r}{\sigma} + \psi_t, \forall t \in [0,T]$, with $\psi = (\psi_t) \in \Theta$ then the dynamics in (9) become

$$dX_t = \mu X_t dt - \sigma X_t \theta_t dt + \sigma X_t dW_t^\theta = rX_t dt - \sigma X_t \psi_t dt + \sigma X_t dW_t^\theta.$$

To stay in the framework of κ-ignorance, as introduced above, we need to change the reference measure. To avoid this step, we prefer to model the stock price dynamics directly under the risk-neutral probability measure, i.e., the agent starts with the reference dynamics

$$dX_t = rX_t dt + \sigma X_t dW_t. \tag{10}$$

Now, if she considers (10) under $Q = Q^\theta$ for some $\theta \in \Theta$ the dynamics become

$$dX_t = rX_t dt - \sigma X_t \theta_t dt + \sigma X_t dW_t^\theta. \tag{11}$$

Throughout the paper, Equation (11) for varying $\theta \in \Theta$ represents the dynamics our decision maker will take into account when studying optimal stopping problems under the ambiguity aversion modeled by κ-ignorance.

3 American options under ambiguity aversion

We focus on American contingent claims under ambiguity aversion.[14] For this issue, we analyze optimal stopping problems under multiple priors. Formally, the optimal stopping problem under ambiguity aversion is defined as

$$V_t := \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right), \quad t \in [0,T] \tag{12}$$

[13]As mentioned above the subjective evaluation appears natural. By the variety of considered models subjective beliefs are nevertheless contained. If one prefers the subjective in place of the risk-neutral probability measure as a reference one may also use the model in (9) with drift rate μ as a reference.

[14]A detailed economic motivation is given in Chudjakow and Vorbrink [5].

where $\gamma_{\tau-t}^{-1}$ is the discounting from current time t up to stopping time τ when the claim is exercised. $H = (H_t)$ represents the payoff process.

We only consider claims with maturity T. The "ess inf" accords with ambiguity aversion which leads to worst-case pricing. The "ess sup" imposes the goal of the agent to optimize the claim's payoff by finding an optimal exercise strategy in the worst-case scenario. All stopping times τ that will come into question in (12) are naturally bounded by the time horizon and claim's maturity T. Without ambiguity, V_t represents the unique price for the claim at time t, see Peskir and Shiryaev [20] for example.

We analyze American options written on X. In general, the claim's payoff from exercising depends on the whole history of the price process. To ensure that the value $V_t, t \in [0, T]$ is well-defined, we impose the following assumption on the claim's payoff process.

Assumption 3.1 *Given an American contingent claim H, the payoff from exercising $H = (H_t)$ is an adapted, measurable, nonnegative process with continuous sample paths*[15] *satisfying* $\mathbb{E}\left(\sup_{0\leq t\leq T} H_t^2\right) < \infty$.

To solve the optimal stopping problem under multiple priors in (12) we utilize the methodology of reflected backward stochastic differential equations (RBSDEs).

3.1 A detour: reflected backward stochastic differential equations

At this point we briefly introduce the notion of RBSDEs and point out its relationship to the optimal stopping problem under ambiguity aversion. The proof can be found in Appendix A. The Markovian framework contains a very useful connection to partial differential equations (PDEs), a generalization of the Feynman-Kac formula. As a reference for the particular case of backward stochastic differential equations (BSDEs) see El Karoui, Peng, and Quenez [11]. In Section 3.2 we employ the results of Chen, Kulperger, and Wei [3] which strongly exploit the relationship to PDEs.

In this detour we use the same stochastic foundation introduced above. The introduction is taken from El Karoui, Kapoudjian, Pardoux, Peng, and

[15] It is possible to relax the assumption, see Cheng and Riedel [4].

Quenez [10].[16] We also introduce the following notation, cf. Pham [21]:

$$\mathbb{L}^2 := \{\xi \mid \xi \text{ is an } \mathcal{F}\text{-measurable random variable with } \mathbb{E}(|\xi|^2) < \infty\},$$
$$\mathbb{H}^2 := \left\{(\varphi_t)|(\varphi_t) \text{ is a progressively mb. process s.t. } \mathbb{E}\int_0^T |\varphi_t|^2 dt < \infty\right\},$$
$$\mathcal{S}^2 := \left\{(\varphi_t)|(\varphi_t) \text{ is a progressively mb. process s.t. } \mathbb{E}\left(\sup_{0\le t\le T} |\varphi_t|^2\right) < \infty\right\}.$$

Given a progressively measurable process $S = (S_t)$, interpreted as an obstacle, the aim is to control a process $Y = (Y_t)$ such that it remains above the obstacle and satisfies equality at terminal time, i.e., $Y_T = S_T$. This is achieved by a RBSDE. We briefly state the definition.

Let $S = (S_t)$ be a real-valued process in $\mathcal{S}^2$, and a generator $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, y, z) \in \mathbb{H}^2\ \forall (y, z) \in \mathbb{R} \times \mathbb{R}$, and

$$|f(t, y, z) - f(t, y', z')| \le C(|y - y'| + |z - z'|) \quad \forall t \in [0, T]$$

for some constant $C > 0$ and all $y, y' \in \mathbb{R}, z, z' \in \mathbb{R}$.

Definition 3.2 *The solution of the RBSDE with parameters (f, S) is a triple $(Y, Z, K) = (Y_t, Z_t, K_t)$ of $(\mathcal{F}_t)$-progressively measurable processes taking values in $\mathbb{R}, \mathbb{R},$ and $\mathbb{R}_+$, respectively, and satisfying:*

(i) $Y_t = S_T + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s, \quad t \in [0, T]$

(ii) $Y_t \ge S_t, \quad t \in [0, T]$

(iii) $K = (K_t)$ *is continuous, increasing,* $K_0 = 0$, *and* $\int_0^T (Y_t - S_t)dK_t = 0$

(iv) $Z = (Z_t) \in \mathbb{H}^2, Y = (Y_t) \in \mathcal{S}^2$, *and* $K_T \in \mathbb{L}^2$

The dynamics in (i) are often expressed in differential form. That is

$$-dY_t = f(t, Y_t, Z_t)dt + dK_t - Z_t dW_t, \quad Y_T = S_T. \tag{13}$$

Intuitively, the process K "pushes Y upwards" such that the constraint (ii) is satisfied, but minimally in the sense of condition (iii). From (i) and (iii)

[16] The framework is based on predictable processes. But the arguments rely only on progressive measurability, cf. Pham [21]. Therefore we require the measurability conditions as in Pham [21].

it follows that (Y_t) is continuous. El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10] proved the existence and uniqueness of a solution to the RBSDE as defined here.

Let us consider equation (12) for a fixed probability measure Q omitting the operator "ess inf". If $Q = Q^\theta \in \mathcal{P}$ then the process Y^θ defined as the unique solution of the reflected BSDE with obstacle $S = H$[17]

$$Y_t^\theta = H_T + \int_t^T (-rY_s^\theta - \theta_s Z_s^\theta) dt + K_T^\theta - K_t^\theta - \int_t^T Z_s^\theta dW_s, \quad t \in [0,T]$$

also solves Equation (12) without ambiguity under the single prior $Q = Q^\theta$. Hence $Y_t^\theta = V_t^Q$ with

$$V_t^Q := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^Q \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right), \quad t \in [0,T].$$

This follows by Proposition 7.1 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10] together with Girsanov's theorem. It illustrates that for each $\theta \in \Theta$ the decision maker faces a RBSDE induced by the parameters (f^θ, H) with $f^\theta(t,y,z) = -ry - \theta_t z \ \forall t \in [0,T]$.

The following theorem establishes the link to the optimal stopping problem defined in (12). It presents the key to solving the optimal stopping problem under ambiguity aversion.

Theorem 3.3 (Duality) *Given a payoff process H, define $f^\theta(t,y,z) := -ry - \theta_t z$ for each $t \in [0,T]$ and consider the unique solution $(Y_t^\theta, Z_t^\theta, K_t^\theta)$ to the RBSDE associated with (f^θ, H) for each $\theta \in \Theta$.*
Let (Y_t, Z_t, K_t) denote the solution of the RBSDE with parameters (f, H) where $f(t,y,z) := \operatorname{ess\,inf}_{\theta\in\Theta} f^\theta(t,y,z) \ \forall t \in [0,T], \forall \ y,z \in \mathbb{R}$. Then there exists $\theta^\star \in \Theta$ such that

$$\begin{aligned} f(t, Y_t, Z_t) :=& \operatorname*{ess\,inf}_{\theta\in\Theta} f^\theta(t, Y_t, Z_t) = f^{\theta^\star}(t, Y_t, Z_t) \\ =& -rY_t - \max_{\theta\in\Theta} \theta_t Z_t = -rY_t - \kappa|Z_t| \quad dt \otimes P \ a.e. \end{aligned}$$

Hence,

$$\begin{aligned} (Y_t, Z_t, K_t) = (Y_t^{\theta^\star}, Z_t^{\theta^\star}, K_t^{\theta^\star}) \quad \forall t \in [0,T] \ a.s. \ and \\ Y_t = \operatorname*{ess\,inf}_{\theta\in\Theta} Y_t^\theta = \operatorname*{ess\,inf}_{Q\in\mathcal{P}} V_t^Q \quad \forall t \in [0,T] \ a.s. \end{aligned}$$

[17] Since we assumed $H = (H_t)$ to be adapted, measurable, and continuous it is progressively measurable, cf. Proposition 1.13 in Karatzas and Shreve [14].

Furthermore,

$$Y_t = \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}^Q(H_\tau \gamma_{\tau-t}^{-1}|\mathcal{F}_t) = \operatorname*{ess\,sup}_{\tau\geq t} \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q(H_\tau \gamma_{\tau-t}^{-1}|\mathcal{F}_t) = V_t \quad a.s.$$

Hence, Y also solves the optimal stopping problem of the ambiguity averse decision maker in (12)*. In particular we have*

$$\max_{\tau\geq 0} \min_{Q\in\mathcal{P}} \mathbb{E}^Q(H_\tau \gamma_\tau^{-1}) = \min_{Q\in\mathcal{P}} \max_{\tau\geq 0} \mathbb{E}^Q(H_\tau \gamma_\tau^{-1}).$$

An optimal stopping rule is given by

$$\tau_t^\star := \inf\{s \geq t | V_s = H_s\} \quad \forall t \in [0,T].$$

The subscript t indicates that $\tau_t^\star$ is an optimal stopping time when we begin at time t.

PROOF: The proof is mostly given in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10], Theorem 7.2. Since it is not directly related to multiple priors under κ-ignorance, we present the main ideas in Appendix A. □

Remark 3.4 *The infimum above is an infimum of random variables. Therefore it must be seen as an essential infimum. For time zero there is no ambiguity in the definitions since the σ-algebra $\mathcal{F}_0$ is trivial.*

By interpreting the theorem, the ambiguity averse agent solves the optimal stopping problem under a worst-case prior $Q^\star := Q^{\theta^\star} \in \mathcal{P}$. That is, she first determines the worst-case scenario and then solves a classical optimal stopping problem with respect to this scenario.

The theorem states the relevance of RBSDEs for solving the optimal stopping problem under ambiguity aversion. As indicated in Theorem 3.3, from this point on, the payoff process of the claim H will represent the obstacle for the associated RBSDEs. We are interested in the solution of the RBSDE associated with the parameters (f, H). In particular, we target understanding the process $\theta^\star$ that induces the worst-case measure.

3.2 Options with monotone payoffs

We focus on American claims whose current payoff can be expressed by a function only depending on the current stock price of the claim's underlying.

We assume $H_t = \Phi_t(X_t)$ for each $t \in [0,T]$.[18] In this case the RBSDE with parameters (f, H) becomes a reflected forward backward stochastic differential equation (RFBSDE), cf. El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10]. The solution for (12) is given by the process Y determined as the solution for

$$\begin{cases} dX_t = rX_t dt + \sigma X_t dW_t, \quad X_0 = x \\ -dY_t = \underbrace{\min_{\theta \in \Theta}(-rY_t - \theta_t Z_t)}_{=-rY_t - \kappa|Z_t| = f(t,Y_t,Z_t)} dt + dK_t - Z_t dW_t, \; Y_T = \Phi_T(X_T) \end{cases} \tag{14}$$

with obstacle $H_t = \Phi_t(X_t)\ \forall t \in [0,T]$.

From this point forward, the mapping $(t,x) \mapsto \Phi_t(x)$ is assumed to be jointly continuous for all $(t,x) \in [0,T] \times \mathbb{R}_+$, and $\Phi_t(X_t) \in L^2(\Omega, \mathcal{F}_t, P)\ \forall t \in [0,T]$. The latter is for instance true if each Φ_t is of polynomial growth (see for example Malliavin [15], p. 6).

Remark 3.5 *If the payoff is zero for each $t \in [0,T)$, i.e., the obstacle only consists of the terminal condition $Y_T = \Phi(X_T)$ the process K is set equal to zero and* (14) *just becomes a forward BSDE without reflection. In this case, the solution Y of* (14) *solves the "optimal stopping problem" under ambiguity aversion for a European contingent claim.*

In order to solve the optimal stopping problem in (12) we focus on the RFBSDE in (14). The characteristic of this setting is that the generator and the obstacle are deterministic. The only randomness of the parameters (f, H) comes from the state of the forward SDE X, a Markov process. We will make use of this observation in the next results. First we derive a result which characterizes the process Z of the solution to (14).

Lemma 3.6 *Consider the RFBSDE in* (14) *with obstacle $H_t = \Phi_t(X_t)\ \forall t \in [0,T]$. Let (Y_t, Z_t, K_t) be the unique solution.*

(i) If Φ_t is increasing for all $t \in [0,T]$, we have

$$Z \geq 0 \quad dt \otimes P \ a.e.$$

[18] Since it is assumed that $H = (H_t)$ has continuous sample paths the mapping $(t,x) \mapsto \Phi_t(x)$ has to be jointly continuous for all $(t,x) \in [0,T] \times \mathbb{R}_+$.

(ii) If Φ_t is decreasing for all $t \in [0,T]$, we have

$$Z \leq 0 \quad dt \otimes P \ a.e.$$

PROOF: We only prove (i); (ii) follows analogously.

Without the obstacle requirement in (14), and just the terminal condition $Y_T = \Phi_T(X_T)$, it follows from a result in Chen, Kulperger, and Wei [3][19] that $Z \geq 0 \ dt \otimes P \ a.e.$ To achieve the passage to reflected BSDEs we employ a penalization method.[20]

Let $n \in \mathbb{N}$, and $(Y_t^{(n)}, Z_t^{(n)})$ be the unique solution of the penalized BSDE with dynamics

$$Y_t^{(n)} = \Phi_T(X_T) + \int_t^T [\underbrace{f(s, Y_s^{(n)}, Z_s^{(n)}) + n(Y_s^{(n)} - \Phi_s(X_s))^-}_{=:\tilde{f}(s,X_s,Y_s,Z_s)}]ds - \int_t^T Z_s^{(n)} dW_s,$$

$t \in [0,T], (x)^- := \max\{-x, 0\}$, and $f(t,y,z) = -ry - \kappa|z|$ as above.

$\tilde{f}$ satisfies the assumptions of a generator for a BSDE as stated in the detour for (reflected) BSDEs.[21] In Chen, Kulperger, and Wei [3] the generator of the BSDE considered does not depend on $X_\cdot$. Fortunately, the map $x \mapsto \tilde{f}(t,x,y,z)$ is increasing for all $t \in [0,T], \ y,z \in \mathbb{R}$ if and only if $x \mapsto \Phi_t(x)$ is increasing for all $t \in [0,T]$. Thus, a larger x leads to larger generator $\tilde{f}$ and larger terminal payoff. This monotonicity behavior is compatible with the application of the comparison theorem for BSDEs which is necessary to derive the result in Chen, Kulperger, and Wei [3]. Thus, the result in Chen, Kulperger, and Wei [3] can also be derived for this penalized BSDE. Hence,

$$Z^{(n)} \geq 0 \quad dt \otimes P \ a.e.$$

Now we let n go to infinity. Then $Z^{(n)}$ converges to Z in $L^2(dt \otimes P)$, cf. Section 6 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10]. By

[19] See Theorem 2 in Chen, Kulperger, and Wei [3]. It is proved by a generalization of the Feynman-Kac formula for BSDEs in connection with the comparison theorem for BSDEs, cf. Peng [18].

[20] Approximation via penalization is a standard method to transfer results on BSDEs to RBSDEs, see El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10].

[21] The additional dependence on $X_\cdot$ in terms of the function $\Phi_\cdot$ does not exhibit any further difficulty here, cf. El Karoui, Peng, and Quenez [11].

standard subsequence argument we also obtain $Z \geq 0$ $dt \otimes P$ *a.e.* □

Using the lemma we can prove the following theorem.

Theorem 3.7 (Claims with monotone payoffs) *Consider an American claim H with payoff at current time t given by $H_t = \Phi_t(X_t)$ $\forall t \in [0,T]$. The value of the optimal stopping problem under ambiguity aversion in* (12) *is given by*

$$V_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^{Q^\star} \left(\Phi_\tau(X_\tau)\, \gamma_{\tau - t}^{-1} | \mathcal{F}_t \right), \quad t \in [0,T].$$

The worst-case prior $Q^\star$ can be specified by its Girsanov density $z_T^{\theta^\star}$.

(i) If Φ_t is increasing for all $t \in [0,T]$, we have $Q^\star = Q^\kappa$, $z_T^{\theta^\star} = z_T^\kappa$ with

$$z_T^\kappa = \exp\{-\frac{1}{2}\kappa^2 T - \kappa W_T\}.$$

(ii) If Φ_t is decreasing for all $t \in [0,T]$, we have $Q^\star = Q^{-\kappa}$, $z_T^{\theta^\star} = z_T^{-\kappa}$ with

$$z_T^{-\kappa} = \exp\{-\frac{1}{2}\kappa^2 T + \kappa W_T\}.$$

In both cases, an optimal stopping time is given by

$$\tau_t^\star := \inf\{s \in [t,T] | V_s = \Phi_s(X_s)\}.$$

PROOF: Let (Y_t, Z_t, K_t) be the unique solution of (14). For $t \in [0,T]$ we have $V_t = Y_t = Y_t^{\theta^\star} = \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E}^{Q^\star} \left(\Phi_\tau(X_\tau)\, \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right)$ by duality, see Theorem 3.3. This also verifies the statement about an optimal stopping time.

In case (i), by Lemma 3.6 we know that $Z \geq 0$ $dt \otimes P$ a.e. Hence,

$$f(t, Y_t, Z_t) = -rY_t - \kappa Z_t \quad dt \otimes P \text{ a.e.}$$

which implies

$$f(t, Y_t, Z_t) = f^{\theta^\star}(t, Y_t, Z_t) \quad dt \otimes P \text{ a.e.}$$

for $\theta^\star = (\kappa) \in \Theta$. So, the worst-case prior is given by $Q^\star = Q^\kappa$ where Q^κ is identified by its Girsanov density

$$z_T^\kappa = \exp\{-\frac{1}{2}\kappa^2 T - \kappa W_T\}.$$

In case (ii), $f(t, Y_t, Z_t) = -rY_t + \kappa Z_t \; dt \otimes P$ a.e. Therefore we identify $Q^\star = Q^{-\kappa}$ as the worst-case prior. □

The preceding theorem's proof relies heavily on the close relationship between optimal stopping problems and RBSDEs, the comparison theorem for (reflected) BSDEs, and the Markovian framework which is essential for Lemma 3.6. In discrete time, the corresponding theorem has been proven by a generalized backward induction and first-order stochastic dominance, Riedel [22]. As a direct application, we quickly collect the conclusions for the American call and put option.

Corollary 3.8 (American call) *Given $L > 0$, let the payoff from exercising the claim be $H_t := (X_t - L)^+$ for all $t \in [0, T]$. Then Q^κ is the worst-case measure. Thus, a risk-neutral buyer of an American call option determines an optimal stopping rule under the prior Q^κ.*

Corollary 3.9 (American put) *Given $L > 0$, let $H_t := (L - X_t)^+$ for all $t \in [0, T]$. Then $Q^{-\kappa}$ is the worst-case measure and a risk-neutral buyer of an American put option utilizes an optimal stopping rule for the prior $Q^{-\kappa}$.*

The interpretation of these results is as follows. Exactly as in the corresponding discrete time setting, the ambiguity averse buyer uses for her valuation of a call option for example the prior under which the underlying stock price possesses the lowest possible drift rate among all priors of the set. That is, under the worst-case prior Q^κ, the stock evolves according to the dynamics of

$$dX_t = (r - \sigma\kappa)X_t dt + \sigma X_t dW_t^\kappa.$$

In the case of an American put option she assumes the highest possible drift rate corresponding to the following stochastic evolution of the stock with respect to $Q^{-\kappa}$

$$dX_t = (r + \sigma\kappa)X_t dt + \sigma X_t dW_t^{-\kappa}.$$

Since X is a Markov process, we write $X_s^{t,x}, s \geq t$ to indicate the price of the stock at time s under the presumption that it is equal to x at time t, i.e., $X_t^{t,x} = x$. As discussed above, by the Markovian structure of (14) and X as the only randomness, we also write $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ for the solution of (14) to indicate the Markovian framework. That is, the solution $Y_\cdot$ can be written as a function of time and state $X_\cdot$, (see Section 4 in El Karoui, Peng, and Quenez [11] or Section 8 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10]).

Using the Markovian structure, the value function $V_t, t \in [0,T]$ in Theorem 3.7 simplifies to a function depending solely on the present time and present stock price. That is, under the assumption of $X_t = x$ at time t the value of the optimal stopping problem under ambiguity aversion in (12) reduces to

$$\begin{aligned} V_t = Y_t^{t,x} &= \operatorname*{ess\,sup}_{\tau\geq t} \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(\Phi_\tau(X_\tau)\, \gamma_{\tau-t}^{-1} | X_t = x\right) \\ &= \sup_{\tau\geq t} \mathbb{E}^{Q^\star} \left(\Phi_\tau(X_\tau^{t,x})\, \gamma_{\tau-t}^{-1}\right) =: u(t,x). \end{aligned}$$

Remark 3.10 *The value in* (12) *is strictly a function in the above setting, i.e. u of the present time t and the present stock price X_t. Note that we did not assume this to determine the worst-case prior. In particular we did not assume that the value function $u(t,x)$ is differential with respect to x and increasing in x, decreasing, respectively, an assumption often made. The proofs of Lemma 3.6 and Theorem 3.7 do not require these assumptions, see also Chen, Kulperger, and Wei [3].*

Besides, the monotonicity of $x \mapsto u(t,x)$ follows directly by comparison theorem. In case (i) of Theorem 3.7 for instance, the mapping $x \mapsto \Phi_s(X_s^{t,x})$ increases because $x \mapsto X_s^{t,x}$ increases[22] *for each $s \in [t,T]$. Then, by comparison theorem for RBSDEs, we obtain that $u(t,x)$ is monotone increasing in x.*

The usual characterization of Markovian processes yields the following result concerning the remaining maturity of an American put option. The option's American style as well as the fact that the payoff from exercising is just a function depending on the current stock price is essential for this result.

[22] See the comparison result for forward SDEs in for example Karatzas and Shreve [14].

Lemma 3.11 *Consider an American put option with strike price L. Given $t \in [0,T]$, the solution of the optimal stopping problem under ambiguity aversion at time t V_t decreases in t.*

PROOF: Let $(t,x) \in [0,T] \times \mathbb{R}_+$ and $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ be the unique solution of the RFBSDE in (14) with obstacle $H_s = (L - X_s^{t,x})^+ \ \forall s \in [t,T]$. The Markov property of X and Y, Corollary 3.9, and Theorem 3.3 yield

$$Y_t^{t,x} = \sup_{0 \le \tau \le T-t} \mathbb{E}^{Q^{-\kappa}} \left((L - X_\tau^{0,x})^+ \gamma_\tau^{-1} \right).$$

Now let $\varepsilon > 0$ with $t + \varepsilon \le T$. Again,

$$Y_{t+\varepsilon}^{t+\varepsilon,x} = \sup_{0 \le \tau \le T-t-\varepsilon} \mathbb{E}^{Q^{-\kappa}} \left((L - X_\tau^{0,x})^+ \gamma_\tau^{-1} \right).$$

Hence, $Y_{t+\varepsilon}^{t+\varepsilon,x} \le Y_t^{t,x}$ and the claim follows by duality, cf. Theorem 3.3. □

For later use let us denote for $t \in [0,T]$ the value in (12) for an American put option with strike price L under the assumption of $X_t = x$ by

$$\underline{Y}_t^{t,x} = \sup_{\tau \ge t} \mathbb{E}^{Q^{-\kappa}} \left((L - X_\tau^{t,x})^+ \gamma_{\tau - t}^{-1} \right). \tag{15}$$

4 Exotic options

In this section we leave the world of Markovian claims with monotone payoffs in the current stock price. We move on to consider the problem in (12) for exotic American claims. With the help of two particular examples, we analyze the effect of ambiguity aversion on the optimal stopping behavior in this more involved situation. Examples are a shout option and an American barrier option in terms of an up-and-in put.

Similar to the discrete time setting in Chudjakow and Vorbrink [5], the analysis of these examples demonstrates one of the main differences to the classical situation without ambiguity. Even though multiple priors lead to a more complex evaluation, the approach is more appropriate in the sense of investment evaluation for accounting and risk measurement.

We will see that dynamical model adjustments occur. With these adjustments the agent takes into account changing beliefs based on realized events within the evaluation period. As such, the multiple priors setting induces particular endogenous dynamics. The agent evaluates her stopping behavior under the worst-case scenario, the worst-case prior. This prior will depend crucially on the payoff process as well as on events occurring during the lifetime of the claim under consideration.

4.1 American up-and-in put option

An American up-and-in put presents its owner the right to sell a specified underlying stock at a predetermined strike price under the condition that the underlying stock first has to rise above a given barrier level.

Formally, the payoff from exercising the option at time $t \in [0, T]$ is defined as

$$H_t := (L - X_t)^+ \mathbb{1}_{\{\tau_H \leq t\}} \tag{16}$$

where $\tau_H := \inf\{0 \leq s \leq T | X_s \geq H\} \wedge T$ denotes the knock-in time at which the option becomes valuable. This is the first time that the underlying reaches the barrier. L defines the strike price and H the barrier. We assume $H > L$ to focus on the most interesting case. We hope not to confuse the reader by the ambiguous use of the letter H denoting the barrier and the claim's payoff process at the same time.

Using previous results and first-order stochastic dominance, we obtain the following evaluation scheme for the American up-and-in put option.

Theorem 4.1 (Up-and-in put) *Consider an American up-and-in put with payoff as defined in* (16)*. The function*

$$V_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^{Q^\star} \left(H_\tau \gamma_{\tau - t}^{-1} | \mathcal{F}_t \right)$$

solves the optimal stopping problem under ambiguity aversion in (12) *whereas the worst-case prior* $Q^\star = Q^{\theta^\star}$ *is specified by the Girsanov density*

$$z_T^{\theta^\star} := \exp\left\{ -\frac{1}{2} \int_0^T (\theta_s^\star)^2 ds - \int_0^T \theta_s^\star dW_s \right\}$$

with $\theta^\star$ *defined as*

$$\theta_t^\star := \begin{cases} \kappa, & \text{if } t < \tau_H \\ -\kappa, & \text{if } \tau_H \leq t \leq T \end{cases}.$$

An optimal stopping time is given by

$$\tau_t^\star := \inf\left\{t \vee \tau_H \leq s \leq T | V_s = (L - X_s)^+\right\}.$$

The theorem states that the agent considers the stopping problem under the measure $Q^{\theta^\star}$. It is the pasting of the measures Q^κ and $Q^{-\kappa}$ at the time of knock-in. Thus, she assumes the stock to evolve according to the least favorable drift rate $r - \sigma\kappa$ at the beginning of the contract. During the contract's lifetime, she changes her beliefs and assumes the highest possible drift rate $r + \sigma\kappa$ for the underlying. That is, she adapts her beliefs based on transpired events corresponding to her pessimistic point of view. So at τ_H, the point in time when the option knocks in the agent's beliefs or fears change abruptly. From a decision theoretical point of view, this result illustrates that optimal stopping under ambiguity aversion is behaviorally distinguishable from optimal stopping under expected utility. The buyer of an American up-and-in put for example behaves as two readily distinguishable expected utility maximizers. This is so because the worst-case measure $\hat{P}$ depends on the payoff process.

Proof: In this section we provide an overview of the main ideas. More details can be found in Appendix B.

Given the event $\{\tau_H \leq t\}$ the claim equals the usual American put option. Hence, $V_t = \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E}^{Q^{-\kappa}}\left((L - X_\tau)^+ \gamma_{\tau-t}^{-1} | \mathcal{F}_t\right)$.

On $\{\tau_H > t\}$ we have $V_t = \operatorname{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q\left(V_{\tau_H} \gamma_{\tau_H - t}^{-1} | \mathcal{F}_t\right)$, (see the appendix for more details). V_{τ_H} represents the value of the optimal stopping problem under ambiguity aversion at the specific time of knock-in.

Let us write $g(s) := \underline{Y}_s^{s,H}$ where $\underline{Y}_s^{s,H}$ is the value of the American put option under ambiguity aversion, see (15). By Lemma 3.11 the function $s \mapsto g(s)$ decreases, as is $s \mapsto \gamma_{s-t}^{-1}$. In the appendix we show that τ_H is stochastically largest under Q^κ in the set of all priors $\mathcal{P}$. That is, for all t, s with $t < s \leq T$, we have on $\{\tau_H > t\}$ and for all $\theta \in \Theta$

$$Q^\kappa\left(\tau_H \leq s | \mathcal{F}_t\right) \leq Q^\theta\left(\tau_H \leq s | \mathcal{F}_t\right).$$

Then the usual characterization of first-order stochastic dominance, Mas-Colell, Whinston, and Green [16], yields on $\{\tau_H > t\}$

$$\mathbb{E}^{Q^\kappa}\left(g(\tau_H) \gamma_{\tau_H - t}^{-1} | \mathcal{F}_t\right) \leq \mathbb{E}^{Q^\theta}\left(g(\tau_H) \gamma_{\tau_H - t}^{-1} | \mathcal{F}_t\right).$$

Thus the worst-case prior $Q^\star$ is equal to Q^κ on $\{\tau_H > t\}$. Setting both together, $Q^\star$ is given by $Q^{\theta^\star}$ with $\theta^\star$ as defined in the theorem. Since $\theta^\star$ is right-continuous, it is progressively measurable, per Proposition 1.13 in Karatzas and Shreve [14]. Hence $\theta^\star \in \Theta$, which finishes the proof. □

An analogous result holds for the American down-and-in call option. In that case, the agent solves the stopping problem under the worst-case scenario $Q^\star = Q^{\theta^\star}$ where $\theta^\star$ is now defined as

$$\theta^\star_t := \begin{cases} -\kappa, & \text{if } t < \tau_H \\ \kappa, & \text{if } \tau_H \le t \le T \end{cases}.$$

Here, τ_H denotes the initial time when the underlying stock price breaks from above through the barrier H.

4.2 Shout option

A shout option gives its owner the right to determine the strike price of a corresponding call or put option. We focus on the European put option version. That is, we consider a shout option that gives its buyer the right to freeze the asset price at any time τ^S before maturity to insure herself against later losses. At maturity the buyer obtains the payoff

$$H_T = \begin{cases} X_{\tau^S} - X_T, & \text{if } X_T < X_{\tau^S} \\ 0, & \text{else} \end{cases}. \tag{17}$$

The value of the optimal stopping problem under ambiguity aversion for a shout option at time $t \le \tau^S \le T$ is defined as

$$V_t = \operatorname*{ess\,sup}_{\tau^S \ge t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left((X_{\tau^S} - X_T)^+ \gamma^{-1}_{T-t} | \mathcal{F}_t \right). \tag{18}$$

We only consider the problem for times $t \le \tau^S$. This is the most interesting case since the owner has not fixed the strike price yet. She still faces the optimal stopping decision which is the decision of shouting.

To evaluate this contract under ambiguity aversion, we first mention the following observation already made in the discrete time setting, Chudjakow and Vorbrink [5]. This option is equivalent to the following: upon shouting the owner receives a European put option (at the money) with strike X_{τ^S}

and remaining time to maturity $T-\tau^S$. We obtain the following evaluation scheme.

Theorem 4.2 (Shout option) *Consider a shout option at its starting time zero with a payoff as defined in* (17). *The solution of* (18) *at time zero simplifies to*

$$V_0 = \sup_{\tau^S \geq 0} \mathbb{E}^{Q^\star}\left((X_{\tau^S} - X_T)^+ \gamma_T^{-1}\right)$$

where the worst-case prior $Q^\star = Q^{\theta^\star}$ is specified by the Girsanov density $z^{\theta^\star}$ with $\theta^\star$ defined by

$$\theta_t^\star := \begin{cases} \kappa, & \text{if } t < \tau^S \\ -\kappa, & \text{if } \tau^S \leq t \leq T \end{cases}.$$

An optimal shouting time is given by

$$\tau^S := \inf\left\{0 \leq t \leq T | V_t = \mathbb{E}^{Q^{-\kappa}}\left((X_t - X_T)^+ \gamma_{T-t}^{-1} | \mathcal{F}_t\right)\right\}.$$

So in this case the ambiguity averse agent changes her beliefs after taking action. Before shouting she assumes the lowest drift rate $(r - \sigma\kappa)$, and the highest rate $(r + \sigma\kappa)$ afterwards. Both rates correspond to the respective least favorable rate, see also Chudjakow and Vorbrink [5]. Similarly to the up-and-in put, her pessimistic perspective leads to fearing the lowest possible returns of the risky asset before shouting and the highest possible returns hence.

Proof: As noted above, at the time of shouting, the value of the contract in (18) is

$$\operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q\left((X_{\tau^S} - X_T)^+ \gamma_{T-\tau^S}^{-1} | \mathcal{F}_{\tau^S}\right).$$

This is a European type of monotone problem. The payoff at maturity T is $\Phi_T(x) := (X_{\tau^S} - x)^+$ which is monotone decreasing in x. As a special case of Theorem 3.7 we derive the value at the time of action as

$$\operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q\left((X_{\tau^S} - X_T)^+ \gamma_{T-\tau^S}^{-1} | \mathcal{F}_{\tau^S}\right) = g(\tau^S, X_{\tau^S})$$

where $g(t,x) := \mathbb{E}^{Q^{-\kappa}}\left((x - X_T^{t,x})^+ \gamma_{T-t}^{-1}|\mathcal{F}_t\right)$.

To determine the value before shouting, consider the following reflected FBSDE with obstacle $(g(t,X_t))_{t\in[0,T]}$

$$\begin{cases} dX_t = rX_t dt + \sigma X_t dW_t, \quad X_0 = x \\ -dY_t = -rY_t - \kappa|Z_t|dt + dK_t - Z_t dW_t,\ Y_T = g(T, X_T) \end{cases} . \tag{19}$$

At this point it is important to note that the function $g(t,X_t)$ satisfies the assumptions for presenting an obstacle for a reflected BSDE. The joint continuity in (t,x) follows by the properties of solutions to (reflected) BSDEs.[23]

Since g can be rewritten in the following form

$$g(t,x) = x\mathbb{E}^{Q^{-\kappa}}\left(\left(1 - \exp\{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)\}\right)^+ \gamma_{T-t}^{-1}\big|\mathcal{F}_t\right)$$

we deduce that the function $x \mapsto g(t,x)$ is increasing for all $t \in [0,T]$. Using Theorem 3.7 we conclude

$$V_0 = Y_0 = \sup_{\tau\geq 0} \mathbb{E}^{Q^{\kappa}}\left(g(\tau, X_\tau)\gamma_\tau^{-1}\right) = \sup_{\tau\geq 0} \mathbb{E}^{Q^{\star}}\left((X_\tau - X_T)^+ \gamma_T^{-1}\right).$$

The last equality follows from the law of iterated expectation. Additionally we obtain an optimal shouting time τ^S. It is determined as the first time that value $V_.$ is equal to $g(\cdot, X_.)$, the value of the European put under ambiguity aversion. This proves the theorem. $\theta^\star \in \Theta$ since it is right-continuous, again implying progressive measurability. □

5 Conclusion

The paper studies the optimal stopping problem of the buyer of various American options in a framework of model uncertainty in continuous time. Model uncertainty induced by imprecise information is mirrored in a set of multiple probability measures.

Each measure corresponds to a specific drift rate for the stock price process in the respective market model. The agent then is allowed to adapt

[23] The value for the European put option is obtained as the solution of a BSDE. Due to the European version of the put option g even belongs to $C^{1,2}([0,T]\times\mathbb{R}_+)$, cf. El Karoui, Peng, and Quenez [11].

the model she uses to assign a value to the claim according to the worst possible model due to her ambiguity averse attitude. We characterize the worst possible model by determining a worst-case measure that drives the processes within this model. We established a link to the calculus of reflected BSDEs to solve the optimal stopping problem from arising given the options' American style under multiple priors.

While the solution for plain vanilla options is straightforward, the situation differs if the payoff of the option is more complex. The buyer of such option adapts her beliefs to the state of the world, and to the overall effect of Knightian uncertainty. This leads to dynamical structure of the worst-case measure highlighting the structural differences between standard models in finance and the multiple priors models.

The characteristics are exemplified by solving the problem explicitly for an American barrier option and a shout option. Particularly with regard to risk management objectives, these models are more appropriate since the valuation becomes less sensitive in terms of varying model data and provides more robust exercise strategies.

A Proof of Theorem 3.3.3

This proof is a slight modification of the proof of Theorem 7.2 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10]. Since their formulation of the theorem is not directly related to multiple priors, we present the main ideas here.

Let (H_t) define the obstacle and H_T the terminal payoff of all regarded RBSDEs.

Consider the unique solution (Y_t^0, Z_t^0, K_t^0) of the RBSDE with dynamics

$$-dY_t^0 = \underbrace{-rY_t^0}_{=f^0(t,Y_t^0,Z_t^0)} dt + dK_t^0 - Z_t^0 dW_t.$$

Then for each $t \in [0,T]$

$$Y_t^0 = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left(\int_t^\tau -rY_s^0 ds + H_\tau | \mathcal{F}_t\right),$$

see Proposition 2.3 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10]. Analogously for any $\theta \in \Theta$, the solution $(Y_t^\theta, Z_t^\theta, K_t^\theta)$ of the RBSDE with dynamics

$$-dY_t^\theta = \underbrace{(-rY_t^\theta - \theta_t Z_t^\theta)}_{=f^\theta(t,Y_t^\theta,Z_t^\theta)}dt + dK_t^\theta - Z_t^\theta dW_t$$

satisfies for $t \in [0,T]$

$$Y_t^\theta = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left(\int_t^\tau (-rY_s^\theta - \theta_s Z_s^\theta)ds + H_\tau | \mathcal{F}_t\right). \tag{20}$$

Now consider for $t \in [0,T]$ and any probability measure Q the equation

$$Y_t^Q = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^Q\left(\int_t^\tau -rY_s^Q ds + H_\tau | \mathcal{F}_t\right). \tag{21}$$

If $Q = Q^\theta$ for some $\theta \in \Theta$ then the solution (Y_t^Q, Z_t^Q, K_t^Q) of the RBSDE with dynamics

$$-dY_t^Q = -rY_t^Q dt + dK_t^Q - Z_t^Q dW_t^\theta$$

satisfies Equation (21). Using Girsanov's theorem, $W^\theta = W + \int_0^\cdot \theta_s ds$, we can rewrite the dynamics as

$$-dY_t^Q = (-rY_t^Q - \theta_t Z_t^Q)dt + dK_t^Q - Z_t^Q dW_t.$$

Thus, by uniqueness, we obtain $Y^Q = Y^\theta$, and as a consequence

$$\operatorname*{ess\,inf}_{Q \in \mathcal{P}} Y_t^Q = \operatorname*{ess\,inf}_{\theta \in \Theta} Y_t^\theta.$$

Since $f(t,y,z) \leq f^\theta(t,y,z)\ \forall y,z \in \mathbb{R}, \forall \theta \in \Theta$, we obtain by comparison for RBSDEs, Theorem 4.1 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10] that

$$Y_t \leq Y_t^\theta \quad \forall \theta \in \Theta.$$

Since Θ is weakly compact in $L^1([0,T] \times \Omega)$,[24] for any real-valued measurable process Z there exists $\theta^\star \in \Theta$ such that $\theta_t^\star Z_t = \max_{\theta \in \Theta} \theta_t Z_t =$

[24]See Chen and Epstein [2]. This again induces the weak compactness of $\mathcal{P}$ which is that induced by the set of bounded measurable functions.

$\kappa|Z_t|\ \forall t \in [0,T]$, by Lemma B.1 in Chen and Epstein [2]. Hence $f(t,y,z) = f^{\theta^\star}(t,y,z)\ \ dt \otimes P\ a.e.\ \forall y,z \in \mathbb{R}$ and

$$Y_t = Y_t^{\theta^\star} \geq \operatorname*{ess\,inf}_{\theta\in\Theta} Y_t^{\theta},\ \ t \in [0,T].$$

In brief,

$$\begin{aligned} Y_t =& \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}^Q \left(\int_t^\tau -rY_s ds + H_\tau | \mathcal{F}_t \right) \\ =& \operatorname*{ess\,inf}_{\theta\in\Theta} \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E} \left(\int_t^\tau -rY_s - \theta_s Z_s ds + H_\tau | \mathcal{F}_t \right) \\ =& \operatorname*{ess\,inf}_{\theta\in\Theta} Y_t^\theta. \end{aligned}$$

Using Proposition 7.1 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10] and Bayes' rule (Lemma 5.3 in Karatzas and Shreve [14]) we obtain for each $\theta \in \Theta$

$$\begin{aligned} Y_t^\theta =& \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E} \left(H_\tau \gamma_{\tau-t}^{-1} \exp\{-\int_t^\tau \theta_s dW_s - \frac{1}{2}\int_t^\tau \theta_s^2 ds\} | \mathcal{F}_t \right) \\ =& \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E} \left(H_\tau \gamma_{\tau-t}^{-1} \frac{z_\tau^\theta}{z_t^\theta} \left| \mathcal{F}_t \right.\right) = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}^{Q^\theta} \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right). \end{aligned}$$

Hence,

$$Y_t = \operatorname*{ess\,inf}_{\theta\in\Theta} \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E} \left(H_\tau \gamma_{\tau-t}^{-1} \frac{z_\tau^\theta}{z_t^\theta} \left| \mathcal{F}_t \right.\right) = \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}^{Q} \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right).$$

We clearly have

$$Y_t \geq \operatorname*{ess\,sup}_{\tau\geq t} \operatorname*{ess\,inf}_{\theta\in\Theta} \mathbb{E} \left(H_\tau \gamma_{\tau-t}^{-1} \frac{z_\tau^\theta}{z_t^\theta} \left| \mathcal{F}_t \right.\right).$$

To obtain the other inequality, we use the stopping time $D_t^\theta := \inf\{s \in [t,T] | Y_s^\theta = H_s\}$ which is optimal in Equation (20) for each fixed $\theta \in \Theta$, see

El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [10], Theorem 7.2. Then

$$\begin{aligned}
Y_t &= \mathbb{E}^{Q^{\theta^\star}}\left(H_{D_t^{\theta^\star}}\gamma^{-1}_{D_t^{\theta^\star}-t}|\mathcal{F}_t\right) \\
&= \operatorname*{ess\,inf}_{\theta\in\Theta}\mathbb{E}\left(H_{D_t^{\theta}}\gamma^{-1}_{D_t^{\theta}-t}\exp\{-\int_t^{D_t^\theta}\theta_s dB_s - \frac{1}{2}\int_t^{D_t^\theta}\theta_s^2 ds\}|\mathcal{F}_t\right) \\
&\leq \operatorname*{ess\,sup}_{\tau\geq t}\operatorname*{ess\,inf}_{\theta\in\Theta}\mathbb{E}\left(H_\tau\gamma^{-1}_{\tau-t}\frac{z_\tau^\theta}{z_t^\theta}|\mathcal{F}_t\right) \\
&= \operatorname*{ess\,sup}_{\tau\geq t}\operatorname*{ess\,inf}_{Q\in\mathcal{P}}\mathbb{E}^Q\left(H_\tau\gamma^{-1}_{\tau-t}|\mathcal{F}_t\right).
\end{aligned}$$

This proves for $t\in[0,T]$

$$Y_t = \operatorname*{ess\,sup}_{\tau\geq t}\operatorname*{ess\,inf}_{Q\in\mathcal{P}}\mathbb{E}^Q\left(H_\tau\gamma^{-1}_{\tau-t}|\mathcal{F}_t\right) = V_t.$$

By a continuity argument $Y_t = V_t\ \forall t\in[0,T]\ a.s.$,[25] and $\tau_t^\star$ is optimal for V_t. Since the minimum for f is attained we conclude the claim for $t=0$.

B Proof of Theorem 3.4.1

We start with a lemma yielding that τ_H is stochastically largest under Q^κ in the set of priors $\mathcal{P}$ in the following sense.

Lemma B.1 *On $\{\tau_H > t\}$ we have for all t,s with $t<s\leq T$ and all $\theta\in\Theta$*

$$Q^\kappa(\tau_H\leq s|\mathcal{F}_t)\leq Q^\theta(\tau_H\leq s|\mathcal{F}_t).$$

PROOF: Throughout this proof, all results are conditioned on the event $\{\tau_H > t\}$. Consider for any $u\in(t,s]$ the set $\{X_u\geq H\}$ and define $M_u := \frac{1}{\sigma}[\ln\frac{H}{X_t} - (r-\frac{\sigma^2}{2})(u-t)]$. Let $\theta\in\Theta$. By definition and construction of Q^θ and W^θ by means of Girsanov's theorem we have

$$\begin{aligned}
X_u &= X_t\exp\{(r-\frac{\sigma^2}{2})(u-t)+\sigma(W_u-W_t)\} \\
&= X_t\exp\{(r-\frac{\sigma^2}{2})(u-t)+\sigma(W_u^\theta-W_t^\theta)-\sigma\int_t^u\theta_s ds\}
\end{aligned}$$

[25] Cheng and Riedel [4] showed that there exists a version of (V_t) that is right-continuous. Using this version we can deduce the claim.

for any $\theta \in \Theta$. Furthermore,

$$\begin{aligned} Q^\theta\big(\{X_u \geq H\}|\mathcal{F}_t\big) &= Q^\theta\big(\{W_u - W_t \geq M_u\}|\mathcal{F}_t\big) \\ &= Q^\theta\Big(\{W_u^\theta - W_t^\theta - \int_t^u \theta_s ds \geq M_u\}|\mathcal{F}_t\Big) \\ &\geq Q^\theta\big(\{W_u^\theta - W_t^\theta - \kappa(u-t) \geq M_u\}|\mathcal{F}_t\big) \\ &= Q^\kappa\big(\{W_u^\kappa - W_t^\kappa - \kappa(u-t) \geq M_u\}|\mathcal{F}_t\big) \\ &= Q^\kappa\big(\{W_u - W_t \geq M_u\}|\mathcal{F}_t\big) = Q^\kappa\big(\{X_u \geq H\}|\mathcal{F}_t\big). \end{aligned}$$

The inequality holds since for any $\theta \in \Theta$

$$\{W_u^\theta - W_t^\theta - \int_t^u \theta_s ds \geq M_u\} \supseteq \{W_u^\theta - W_t^\theta - \kappa(u-t) \geq M_u\}, \tag{22}$$

the subsequent equality holds since both W^θ under Q^θ and W^κ under Q^κ are standard Brownian motions and M_u is deterministic on $\mathcal{F}_t$. Due to

$$\bigcup_{u \in (t,s]} \{X_u \geq H\} = \{\tau_H \leq s\} \in \mathcal{F}_s$$

and since the inclusion in (22) also holds for the union we conclude the result. □

Cheng and Riedel [4] verified that the optimal stopped value process is a $\mathcal{P}$-multiple priors martingale in the sense that it, say (M_t) satisfies $M_t = \operatorname{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q\left(M_s|\mathcal{F}_t\right)$ $\forall s, t \in [0,T]$ with $s \geq t$.[26]

To avoid any confusion, let us denote their value process by $(\bar{V}_{t \wedge \tau^\star})_{t \in [0,T]}$, where $\tau^\star$ is an optimal stopping time. In their setting, $\bar{V}_t$ denotes the value of the optimal stopping problem after time t at time zero.[27] In our setting, V_t denotes the value of the optimal stopping problem after time t at time t.

That is, $(V_{t\wedge\tau^\star}\gamma_{t\wedge\tau^\star}^{-1}) = (\bar{V}_{t\wedge\tau^\star})$ is a $\mathcal{P}$-multiple priors martingale. By optional sampling for $\mathcal{P}$-multiple priors martingales, Cheng and Riedel [4] or Peng [18] for any stopping time σ with $\sigma \geq t$ a.s.

$$\bar{V}_{t\wedge\tau^\star} = \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q\left(\bar{V}_{\sigma\wedge\tau^\star}|\mathcal{F}_{t\wedge\tau^\star}\right)$$

[26] Cheng and Riedel [4] called this a g-martingale. See also Peng [18].

[27] To fit into our setting the payoff for $(\bar{V}_t)$ has to be the discounted payoff which is $H_t\gamma_t^{-1}$ for each $t \in [0,T]$.

which yields

$$V_{t\wedge\tau^\star} = \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\sigma\wedge\tau^\star} \gamma^{-1}_{\sigma\wedge\tau^\star - t\wedge\tau^\star} | \mathcal{F}_{t\wedge\tau^\star} \right). \tag{23}$$

Using (23) we can rewrite the optimal stopped value process as follows.

Lemma B.2 *Given $t \in [0,T]$. We have*

$$V_{t\wedge\tau^\star} = V_{\tau^\star} \mathbb{1}_{\{\tau^\star \le t\}} + V_t \mathbb{1}_{\{\tau_H \le t\}} \mathbb{1}_{\{\tau^\star > t\}} + \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\tau_H} \gamma^{-1}_{\tau_H - t} | \mathcal{F}_t \right) \mathbb{1}_{\{\tau_H > t\}}.$$

Proof: First note that exercising the option before knock-in yields payoff zero and therefore cannot be optimal. Hence $\tau^\star \ge \tau_H$ a.s. While keeping this in mind, consider the equality in (23) for the stopping time $\sigma := \tau_H \vee t$ yielding

$$\begin{aligned}
V_{t\wedge\tau^\star} &= \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\tau^\star} \mathbb{1}_{\{\tau^\star \le t\}} + V_{\tau_H \vee t}\, \gamma^{-1}_{\tau_H \vee t - t} \mathbb{1}_{\{\tau^\star > t\}} | \mathcal{F}_{t\wedge\tau^\star} \right) \\
&= V_{\tau^\star} \mathbb{1}_{\{\tau^\star \le t\}} + \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\tau_H \vee t}\, \gamma^{-1}_{\tau_H \vee t - t} | \mathcal{F}_t \right) \mathbb{1}_{\{\tau^\star > t\}} \\
&= V_{\tau^\star} \mathbb{1}_{\{\tau^\star \le t\}} + \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_t \mathbb{1}_{\{\tau_H \le t\}} + V_{\tau_H} \gamma^{-1}_{\tau_H - t} \mathbb{1}_{\{\tau_H > t\}} | \mathcal{F}_t \right) \mathbb{1}_{\{\tau^\star > t\}} \\
&= V_{\tau^\star} \mathbb{1}_{\{\tau^\star \le t\}} + V_t \mathbb{1}_{\{\tau_H \le t\}} \mathbb{1}_{\{\tau^\star > t\}} + \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\tau_H} \gamma^{-1}_{\tau_H - t} | \mathcal{F}_t \right) \mathbb{1}_{\{\tau_H > t\}}
\end{aligned}$$

which proves the claim. Besides optional sampling, which heavily requires time-consistency of $\mathcal{P}$, we used that τ_H and $\tau^\star$ are stopping times, and $\operatorname{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q (S + \eta | \mathcal{F}_t) = \eta + \operatorname{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q (S | \mathcal{F}_t)$ for any $\mathcal{F}_t$-measurable random variable η and square-integrable $\mathcal{F}$-measurable S. □

The expectation occurring in Lemma B.2 remains to be evaluated. V_{τ_H} corresponds to the value of the American put option under ambiguity aversion. At knock-in when $s = \tau_H$, we know the value is given by

$$g(s) := V_s = \underline{Y}^{s,H}_s = \operatorname*{ess\,sup}_{\tau \ge s} \mathbb{E}^{Q^{-\kappa}} \left((L - X^{s,H}_\tau)^+ \gamma^{-1}_{\tau - s} \right).$$

γ^{-1}_{s-t} and $g(s)$ are decreasing in s, per Lemma 3.11. Therefore, by Lemma B.1 and the usual characterization of first-order stochastic dominance, Mas-Colell, Whinston, and Green [16] we deduce on $\{\tau_H > t\}$ for any $\theta \in \Theta$

$$V_t = \mathbb{E}^{Q^\kappa} \left(g(\tau_H) \gamma^{-1}_{\tau_H - t} | \mathcal{F}_t \right) \le \mathbb{E}^{Q^\theta} \left(g(\tau_H) \gamma^{-1}_{\tau_H - t} | \mathcal{F}_t \right).$$

On the complementary event $\{\tau_H \leq t\}$ the claim equals the usual American put option. Hence, it is evaluated with respect to $Q^{-\kappa}$. Setting both together, $\theta^\star$ is as claimed in the theorem. By right-continuity it is progressively measurable. Therefore, $\theta^\star \in \Theta$ and $Q^{\theta^\star}$ is the worst-case prior for the American up-and-in put problem. This finishes the proof.

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