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The Strategic Use of Ambiguity

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Abstract

Ambiguity can be used as a strategic device in some situations. To demonstrate this, we propose and study a framework for normal form games where players can use Knightian uncertainty strategically. In such Ellsberg games, players may use Ellsberg urns in addition to the standard objective mixed strategies. We assume that players are ambiguity–averse in the sense of Gilboa and Schmeidler. While classical Nash equilibria remain equilibria in the new game, there arise new *Ellsberg equilibria* that can be quite different from Nash equilibria. A negotiation game with three players illustrates this finding. Another class of examples shows the use of ambiguity in mediation. We also highlight some conceptually interesting properties of Ellsberg equilibria in two person games with conflicting interests.

1 Introduction

It occurs in daily life that a certain vagueness or ambiguity is in one's own favor. This might happen in very private situations where the disclosure of all information might put oneself into a bad position, but also in quite public situations when a certain ambiguity, as one intuitively feels, leads to a better outcome. Indeed, committee members at universities know only too well how

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useful it can be to play for time; in particular, it is sometimes extremely useful to *create* ambiguity where there was none before. Such strategic ambiguity might include a threat about future behavior ("I might do this or even the contrary if ..."). Past presidents of Reserve Banks became famous for being intelligently ambiguous (although the debate is not decided if this was always for society's good).

Accounts of strategic ambiguity, of a more verbal nature, can be found in different fields. This includes the use of ambiguity as a strategic instrument in US foreign policy, Benson and Niou (2000), and in organizational communication, Eisenberg (1984). Economists found the presence of strategic ambiguity to explain incompleteness of contracts, Mukerji (1998), and policies of insurance fraud detection, Lang and Wambach (2010).

This paper takes a look at such *strategic use of ambiguity* in games. The recent advances in decision theory, based on Ellsberg's famous experiments and Knight's distinction between risk and uncertainty allow to model such strategic behavior formally. Intuitively, we give players the possibility to use Ellsberg urns, that is urns with (partly) unknown proportions of colored balls, instead of merely using objective randomizing devices where all players know (ideally) the probabilities. Players can thus credibly announce: "I will base my action on the outcome of the draw from this Ellsberg urn!" Such urns are objectively ambiguous, by design; players can thus create ambiguity.

We then work under the assumption that it is common knowledge that players are ambiguity–averse. To have a concrete model, we adopt the Gilboa–Schmeidler axioms and thus get pessimistic players.

To our knowledge, the only contribution in game theory in this direction[1] is by Sophie Bade (2010) and, very early, by Robert Aumann (1974). Bade shows that in two person games, the support of "ambiguous act equilibria" is always equal to the support of some Nash equilibrium. This reminds one of Aumann's early discussion of the use of subjective randomizing devices in

[1]Of course, other applications of Knightian decision theory to normal form games with complete information are available. Dow and Werlang (1994), Lo (1996), Eichberger and Kelsey (2000) and Marinacci (2000) assume uncertainty about the players' actions and let the beliefs reflect this uncertainty. In these extensions of *equilibrium in beliefs* actual strategies in equilibrium are essentially ignored. Klibanoff (1993), Lehrer (2008) and Lo (2009) determine the actual play in equilibrium and require some consistency between actions and ambiguous beliefs. Anyhow, the actions in equilibrium are still determined by objective randomizing devices. In contrast, in our model players can *create* ambiguity by the use of subjective randomizing devices.

a Savage framework. If players evaluate Knightian uncertainty in a linear way, in other words, if they conform to the Savage axioms, no new equilibria can be generated in two person games. This might look like a quite negative, and boring, conclusion in the sense that there is no space for interesting uses of strategic ambiguity. We do not share this view.

In this short note, we exhibit three (classes of) examples that show, in our opinion, that there is a rich world to explore for economists. Our examples serve the role to highlight potential uses. In a later paper, we plan to say more on a general theory of such *Ellsberg Games*.

Our first example highlights the use of strategic ambiguity in negotiations. We take up a beautiful example of Greenberg (2000) where three parties, two small countries in conflict, and a powerful country, the mediator, are engaged in peace negotiations. There is a nice peace outcome; the game's payoffs, however, are such that in the unique Nash equilibrium of the game the bad war outcome is realized. Greenberg argues verbally that peace can be reached if the superpower "remains silent" instead of playing a mixed strategy. We show that peace can indeed be an equilibrium in the extended game where players are allowed to use Ellsberg urns. Here, the superpower leaves the other players uncertain about its actions. This induces the small countries to prefer peace over war.

As our second example we take a classic example by Aumann (1974) which has a nice interpretation. It illustrates how strategic ambiguity can be used by a mediator to achieve cooperation in situations similar to the prisoners' dilemma. In this game a mediator is able to create ambiguity about the reward in case of unilateral defection. If he creates enough ambiguity, both prisoners are afraid of punishment and prefer to cooperate. The outcome thus reached is even Pareto–improving.

A remarkable consequence of these two examples is that the strategic use of ambiguity allows to reach equilibria that are not Nash equilibria in the original game, even not in the support of the original Nash equilibrium. So here is a potentially rich world to discover.

We also take a closer look at two person 2×2 games with conflicting interests, as Matching Pennies or similar competitive situations. These games have a unique mixed Nash equilibrium. We point out two interesting features here. The first is that, surprisingly, mutual ambiguity around the Nash equilibrium distribution is not an equilibrium in competitive situations. Due to the non–linearity of the payoff functions in Ellsberg games, ambiguity around the Nash equilibrium distribution never has ambiguity as best response. Secondly, in the class of games we consider a new and interesting type of equilibria arises. In these equilibria, both players create ambiguity. They commit to strategies like, e.g.: "I play action A with probability at least 1/2, but I don't tell you anything more about my true distribution." It seems convincing that these strategies are easier to implement in reality than an objectively mixed Nash equilibrium strategy. In fact, an experiment by Goeree and Holt (2001) suggests that actual behavior is closer to Ellsberg equilibrium strategies than to Nash equilibrium strategies.

The paper is organized as follows. In Section 2 we present the negotiation example. In Section 3 we develop the framework for normal form games that allows the strategic use of ambiguity. In Section 4 we apply the concept to the negotiation example, Section 5 analyzes strategic ambiguity as a mediation tool. Finally Section 6 shows how strategic ambiguity is used in competitive situations. We conclude in Section 7.

2 Ambiguity as a Threat

Our main point here is to show that strategic ambiguity can lead to new phenomena that lie outside the scope of classical game theory. As our first example, we consider the following peace negotiation game taken from Greenberg (2000). There are two small countries who can either opt for peace, or war. If both countries opt for peace, all three players obtain a payoff of 4. If one of the countries does not opt for peace, war breaks out, but the superpower cannot decide whose action started the war. The superpower can punish one country and support the other. The game tree is in Figure 1 below.

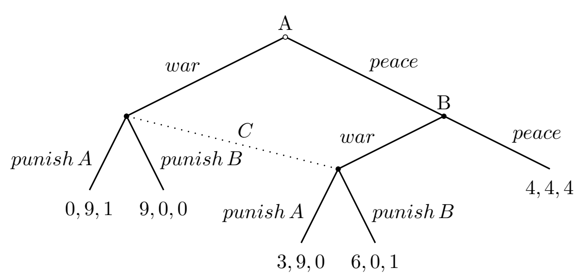

Figure 1: Peace Negotiation

As we deal here only with static equilibrium concepts, we also present the normal form, where country A chooses rows, country B columns, and the superpower chooses the matrix.

	war	*peace*
war	$0, 9, 1$	$0, 9, 1$
peace	$3, 9, 0$	$4, 4, 4$

punish A

	war	*peace*
war	$9, 0, 0$	$9, 0, 0$
peace	$6, 0, 1$	$4, 4, 4$

punish B

Figure 2: Peace Negotiation in normal form

This game possesses a unique Nash equilibrium where country A mixes with equal probabilities, and country B opts for war; the superpower has no clue who started the war given these strategies. It is thus indifferent about whom to punish and mixes with equal probabilities as well. War occurs with probability 1. The resulting equilibrium payoff vector is $(4.5, 4.5, 0.5)$.

If the superpower can create ambiguity (and if the countries A and B are ambiguity–averse), the picture changes. Suppose for simplicity, that the superpower creates maximal ambiguity by using a device that allows for any probability between 0 and 1 for its strategy *punish A*. The pessimistic players A and B are ambiguity–averse and thus maximize against the worst case. For both of them, the worst case is to be punished by the superpower, with a payoff of 0. Hence, both prefer to opt for peace given that the superpower creates ambiguity. As this leads to a very desirable outcome for the superpower, it has no incentive to deviate from this strategy. We have thus found an equilibrium where the strategic use of ambiguity leads to an equilibrium outcome outside the support of the Nash equilibrium outcome.

We present next a framework where this intuition can be formalized.

3 Ellsberg Games

Let us formalize the intuitive idea that players can create ambiguity with the help of Ellsberg urns. An Ellsberg urn is, for us, a triple $(\Omega, \mathcal{F}, \mathcal{P})$ of a nonempty set Ω of states of the world, a σ–field $\mathcal{F}$ on Ω (where one can take the power set in case of a finite Ω), and a set of probability measures $\mathcal{P}$ on the measurable space $(\Omega, \mathcal{F})$. This *set* of probability measures represents the

Knightian uncertainty of the strategy: the players do not know the probability laws that govern the state spaces Ω. A typical example would be the classical Ellsberg urn that contains 30 red balls, and 60 balls that are either black or yellow, and one ball is drawn at random. Then the state space consists of three elements $\{R, B, Y\}$, $\mathcal{F}$ is the power set, and $\mathcal{P}$ the set of probability vectors (P_1, P_2, P_3) such that $P_1 = 1/3$, $P_2 = k/60, P_3 = (60-k)/60$ for any $k = 0, \ldots, 60$.

We assume that the players of our game have access to any such Ellsberg urns; imagine that there is an independent, trustworthy laboratory that sets up such urns and reports the outcome truthfully.

We come now to the game where players can use such urns in addition to the usual mixed strategies (that correspond to roulette wheels or dice).

Let $N = \{1, ..., n\}$ be the set of players. Each player has a finite strategy set $S_i, i = 1, \ldots, N$. Let $S = \prod_{i=1}^{n} S_i$ be the set of pure strategy profiles. Players' payoffs are given by functions

$$u_i : S \to \mathbb{R} \qquad (i \in N)\,.$$

The normal form game is denoted $G = \langle N, (S_i), (u_i) \rangle_{i \in N}$.

Players can now use different devices. On the one hand, we assume that they have "roulette wheels" or "dices" at their disposal, i.e. randomizing devices with objectively known probabilities. As usual, the set of these probabilities over S_i is denoted by ΔS_i. The set of profiles of mixed strategies is denoted by $\Delta S := \prod_{i=1}^{n} \Delta S_i$. The players evaluate such devices according to expected utility, as in von Neumann–Morgenstern's axiomatization.

Moreover, and this is the new part, players can use Ellsberg urns. We imagine that all players have laboratories at their disposal that allow them to build Ellsberg urns and they can credibly commit to strategies that base actions on outcomes of draws from such urns. Technically, we model this as a triple $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ as explained above. Each player $i \in N$ has a finite set of states of the world Ω_i, together with a σ-algebra $\mathcal{F}_i$ of subsets of Ω_i, the events. An *Ellsberg strategy* (or Anscombe–Aumann act)[2] is then a measurable function $f_i : (\Omega_i, \mathcal{F}_i) \to \Delta S_i$.

We suppose that all players involved in the game G are ambiguity–averse in the sense that they prefer risky situations to ambiguous situations. To

[2]Notice that this general form of a game — without the set of probabilities $\mathcal{P}_i$ — was first introduced by Aumann (1974) in his introduction of the concept of correlated equilibrium.

evaluate utility of a profile of acts $f = (f_i, f_{-i})$ and corresponding Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$, the players use maxmin expected utility as axiomatized by Gilboa and Schmeidler (1989). The payoff of player $i \in N$ at Ellsberg strategy profile f is thus

$$U_i(f) = \min_{P_1 \in \mathcal{P}_1, \ldots, P_n \in \mathcal{P}_n} \int_{\Omega_1} \cdots \int_{\Omega_n} u_i(f(\omega)) dP_1 \ldots dP_n \,.$$

We call the described larger game an *Ellsberg game*. An *Ellsberg equilibrium* is a profile of Ellsberg urns $((\Omega^1, \mathcal{F}^1, \mathcal{P}^1), \ldots, (\Omega^n, \mathcal{F}^n, \mathcal{P}^n))$ and acts $f^* = (f_1^*, \ldots, f_n^*)$ such that no player has an incentive to deviate, i.e. for all players $i \in N$ and all Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$, and all acts f_i for player i we have

$$U_i(f^*) \geq U_i(f_i, f_{-i}^*) \,.$$

This definition depends on the particular Ellsberg urn used by each player $i \in N$. There are a large number of possible state spaces and sets of probability measures for each player. Fortunately there is a more concise way to define Ellsberg equilibrium. The procedure is similar to the reduced form of a correlated equilibrium, see Aumann (1974) or Fudenberg and Tirole (1991). Instead of working with arbitrary Ellsberg urns, we note that the players' payoff depends, in the end, on the set of distributions that the Ellsberg urns and the associated acts induce on the set of strategies. One can then work with that set of distributions directly.

Definition 1. *Let $G = \langle N, (S_i), (u_i) \rangle_{i \in N}$ be a normal form game. A* reduced form Ellsberg equilibrium *of the game G is a profile of sets of probability measures $\mathcal{P}_i^* \subseteq \Delta S_i$, such that for all players $i \in N$ and all sets of probability measures $\mathcal{Q}_i$ on S_i we have*

$$\begin{aligned} &\min_{P_1 \in \mathcal{P}_1^*, \ldots, P_n \in \mathcal{P}_n^*} \int_{\Delta S_1} \cdots \int_{\Delta S_n} u_i(s) \mathrm{d}P_1 \ldots \mathrm{d}P_n \\ \geq \quad &\min_{P_i \in \mathcal{Q}_i, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Delta S_1} \cdots \int_{\Delta S_n} u_i(s_i, s_{-i}) \mathrm{d}P_1 \ldots \mathrm{d}P_n \,. \end{aligned}$$

The two definitions of Ellsberg equilibrium are equivalent[3].

[3]This can be shown formally, the proof is available upon request. It goes essentially as follows. The maxmin expected utility representation with the associated set of probability distributions $\mathcal{P}_i \subseteq \Delta S_i$ induces for the Ellsberg equilibrium acts $f^* \in F$ a profile $\mathcal{P}^*$ of

Note that the classical game is contained in our formulation: players just choose a singleton $\mathcal{P}_i = \{\delta_{\pi_i}\}$ that puts all weight on a particular (classical) mixed strategy π.

Now let $(\pi_1, \ldots, \pi_n)$ be a Nash equilibrium of the game G. Can any player gain by creating ambiguity in such a situation? No. As players are pessimistic, they cannot gain by creating ambiguity in a situation where the other players do not react on this ambiguity because their payoffs do not depend on it. In particular, pure strategy Nash equilibria are Ellsberg equilibria.

Proposition 1. *Let $G = \langle N, (S_i), (u_i) \rangle_{i \in N}$ be a normal form game. Then a mixed strategy profile $(\pi_1, \ldots, \pi_n)$ of G is a Nash equilibrium of G if and only if the corresponding profile of singletons $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ with $\mathcal{P}_i = \{\delta_{\pi_i}\}$ is an Ellsberg equilibrium.*

In particular, Ellsberg equilibria exist. But this is not our point here. We want to show that interesting, non–Nash behavior can arise in Ellsberg games. We turn to this issue next.

4 Strategic Use of Ambiguity in Negotiations

Having defined the Ellsberg equilibrium, we return to the peace negotiation example. We show that there is the following class of Ellsberg equilibria: the superpower creates ambiguity about its decision. When this ambiguity is sufficiently large, both players fear that they will be punished by the superpower in case of war. Their best reply is thus to opt for peace.

In the case of just two strategies, we can identify an Ellsberg strategy of the superpower with an interval $[P_0, P_1]$ where $P \in [P_0, P_1]$ are the probabilities that the superpower punishes country A. Suppose the superpower plays this strategy with probability $[P_0, P_1]$ where $P_0 < \frac{4}{9}$ and $P_1 > \frac{5}{9}$. Suppose that country B opts for *peace*. If A goes for war, it uses that prior in $[P_0, P_1]$ which minimizes its expected payoff, i.e. P_1 for country A. This yields $U_1(war, war, [P_0, P_1]) = P_1 \cdot 0 + (1 - P_1) \cdot 9 < 4$. Hence, opting for peace is

sets of probability distributions on the classical mixed strategy sets ΔS_i. On the other hand, to see that every reduced form Ellsberg equilibrium $\mathcal{P}^*$ is an Ellsberg equilibrium, choose the states of the world Ω_i to be the set of mixed strategies ΔS_i, and let f_i be the identity map.

country A's best reply. The reasoning for country B is similar, but with the opposite probability P_0. If both countries A and B go for peace, the superpower gets 4 regardless of what it does; in particular, it can play strategically ambiguously as described above. We conclude that $(peace, peace, [P_0, P_1])$ is a (reduced form) Ellsberg equilibrium.

By using the strategy $[P_0, P_1]$ which is a set of probability distributions, player 3 creates ambiguity. This supports an Ellsberg equilibrium with strategies that are not in the support of the unique Nash equilibrium.

Greenberg refers to historic peace negotiations between Israel and Egypt (countries A and B in the negotiation example) mediated by the USA (superpower C) after the 1973 war. The success of the peace negotiations can be partly attributed to the fact that both Egypt and Israel were too afraid to be punished if negotiations broke down. Their fear was fed from diverging expected consequences, as pointed out by Kissinger (1982)[4]. These peace negotiations might be evidence that Ellsberg equililbria can capture real world phenomena better than Nash equilibria.

5 Strategic Ambiguity as a Mediation Tool

Next we will consider a classic example[5] from Aumann (1974), Example 2.3. He presents a three person game where one player has some mediation power to influence his opponents' choice. The original game is given by the payoff matrix in Figure 3, where we let player 1 choose rows, player 2 choose columns and player 3 choose matrices.

l	L	R
T	$0,8,0$	$3,3,3$
B	$1,1,1$	$0,0,0$

r	L	R
T	$0,0,0$	$3,3,3$
B	$1,1,1$	$8,0,0$

Figure 3: Aumann's Example.

Player 3 is indifferent between his strategies l and r, since he gets the same payoffs for both. As long as player 3 chooses l with a probability higher than

[4]See p. 802 therein, in particular.

[5]This example has also been analyzed in other literature on ambiguity in games, see Eichberger and Kelsey (2006), Lo (2009) and Bade (2010).

3/8, L is an optimal strategy for player 2 regardless what player 1 does; and player 1 would subsequently play B. By the same reasoning, as long as player 3 plays r with a probability higher than 3/8, B is optimal for player 1, and player 2 plays L then. Thus the Nash equilibria of this game are all of the form (B, L, P^*), where P^* is any objectively mixed strategy.

The example by Aumann has a nice interpretation. Suppose players 1 and 2 are prisoners, and player 3 the police officer. Let us rearrange the matrix game and put it in the form displayed in Figure 4. We swap the strategies of player 2 and rename the strategies of player 1 and 2 to $C =$ "cooperate" and $D =$ "defect" as in the classical prisoners' dilemma. We can merge the two matrices into one, because the strategy choice of the police officer is simply the choice of a probability that influences the payoffs of prisoners 1 and 2 in case of unilateral defection from (C, C). If he chooses $P = 1$ this corresponds to strategy l in the original game (i.e. prisoner 2 gets all the reward), $P = 0$ would be strategy r (i.e. prisoner 1 gets all the reward). The choice of the objectively mixed strategy $P = 1/2$ leads to the classical symmetric prisoners' dilemma with a payoff of 4 in case of unilateral defection.

In this interpretation, players 1 and 2 are facing a sort of prisoners' dilemma situation mediated by a player 3, the police officer. Given the payoffs, the police officer is most interested in cooperation between the prisoners. The police officer can influence how high the reward would be for unilateral defection by using an objective randomizing device. Nevertheless, in every Nash equilibrium of the game, the players obtain the inefficient outcome of 1.

		Prisoner 2	
		C	D
Prisoner 1	C	$3, 3, 3$	$0, 8P, 0$
	D	$8(1-P), 0, 0$	$1, 1, 1$

Figure 4: Mediated Prisoners' dilemma

Now suppose we let the players use Ellsberg strategies. The police officer could create ambiguity by announcing: "I'm not sure about who of you I will want to punish and who I will want to reward for reporting on your partner. I might also reward you both equally... I simply don't tell you what mechanism I will use to decide about this."

Let us exhibit Ellsberg strategies that support this behavior. If prisoner 2 expects the P to be lower than 3/8 and prisoner 1 expects the P to be higher than 5/8, they would prefer to cooperate[6]. This behavior corresponds to the police officer playing an Ellsberg strategy $[P_0, P_1]$ with $0 \leq P_0 < \frac{3}{8}$ and $\frac{5}{8} < P_1 \leq 1$. The ambiguity averse prisoners 1 and 2 evaluate their utility with $P = P_0$ and $P = P_1$ resp. Consequently they would prefer to play (C, C). This gives an Ellsberg equilibrium in which the prisoners cooperate.

6 Strategic Ambiguity in Competitive Situations

Let us now consider typical two person games with conflicting interests; in such a situation, one usually has no strict equilibria. We take a slightly modified version of Matching Pennies as our example; the results generalize to all such two person 2×2 games as we explain below. The payoff matrix for this game is in Figure 5.

		Player 2	
		$HEAD$	$TAIL$
Player 1	$HEAD$	$3, -1$	$-1, 1$
	$TAIL$	$-1, 1$	$1, -1$

Figure 5: Modified Matching Pennies I

We point out that the Ellsberg equilibria in such games are different from what one might expect first, on the one hand; on the other hand they allow us to emphasize an important property of Ellsberg games (or ambiguity aversion in general): the best reply functions are no longer linear in the probabilities. As a consequence, the indifference principle of classical game theory does not carry over to Ellsberg games. When a player is indifferent between two Anscombe–Aumann acts, this does not imply that she is indifferent between all mixtures over these two acts. This is due to the hedging or diversification

[6] Aumann (1974) has already commented on this behavior. He observes that (C, C) can be a "subjective equilibrium point" if players 1 and 2 have non–common beliefs about the objectively mixed strategy player 3 is going to use. In his analysis player 1 believes $P = 3/4$ and player 2 believes $P = 1/4$. Note that in Ellsberg equilibrium the players have the *common* belief $P \in [P_0, P_1]$.

effect provided by a (classical) mixed strategy when players are ambiguity–averse. We call this effect *immunization against strategic ambiguity.*

In our modified version of Matching Pennies, the unique Nash equilibrium is that player 1 mixes uniformly over his strategies, and player 2 mixes with $(1/3, 2/3)$. This yields the equilibrium payoffs $1/3$ and 0. One might guess that one can get an Ellsberg equilibrium where both players use a set of probability measures around the Nash equilibrium distribution as their strategy. This is somewhat surprisingly, at least to us, not true. The reason for this lies in the possibility to immunize oneself against ambiguity; in the modified Matching Pennies example, player 1 can use the mixed strategy $(1/3, 2/3)$ to make himself independent of any ambiguity used by the opponent. Indeed, with this strategy, his expected payoff is $1/3$ against any mixed strategy of the opponent, and a fortiori against Ellsberg strategies as well. This strategy is the unique best reply of player 2 to Ellsberg strategies with ambiguity around the Nash equilibrium; in particular, such ambiguity is not part of an Ellsberg equilibrium.

Let us explain this somewhat more formally. An Ellsberg strategy for player 2 can be identified with an interval $[Q_0, Q_1] \subseteq [0, 1]$ where $Q \in [Q_0, Q_1]$ is the probability to play HEAD. Suppose player 2 uses many probabilities around $1/3$, so $Q_0 < 1/3 < Q_1$. The (minimal) expected payoff for player 1 when he uses the mixed strategy with probability P for "HEAD" is then

$$\begin{aligned}
\min_{Q_0 \le Q \le Q_1} & 3PQ - P(1-Q) - (1-P)Q + (1-P)(1-Q) \\
&= \min\{Q_0(6P-2) + 1 - 2P, Q_1(6P-2) + 1 - 2P\} \\
&= \begin{cases} Q_1(6P-2) + 1 - 2P & \text{if } P < 1/3 \\ 1/3 & \text{if } P = 1/3 \\ Q_0(6P-2) + 1 - 2P & \text{else} \end{cases} .
\end{aligned}$$

By choosing the mixed strategy $P = 1/3$, player 1 becomes immune against any ambiguity and ensures the (Nash) equilibrium payoff of $1/3$. If there was an Ellsberg equilibrium with $P_0 < 1/2 < P_1$ and $Q_0 < 1/3 < Q_1$, then the minimal expected payoff would be below $1/3$. Hence, such Ellsberg equilibria do not exist.

Such immunization plays frequently a role in two person games, and it need not always be the Nash equilibrium strategy that plays this role. Consider, e.g., the slightly changed payoff matrix

In the unique Nash equilibrium, player 1 still plays both strategies with

		Player 2	
		$HEAD$	$TAIL$
Player 1	$HEAD$	$1, -1$	$-1, 1$
	$TAIL$	$-2, 1$	$1, -1$

Figure 6: Modified Matching Pennies II

probability 1/2 (to render player 2 indifferent); however, in order to be immune against Ellsberg strategies, he has to play HEAD with probability 3/5. Then his payoff is $-1/5$ regardless of what player II does. This strategy does not play any role in either Nash or Ellsberg equilibrium. It is only important in so far as it excludes possible Ellsberg equilibria by being the unique best reply to some Ellsberg strategies.

The question thus arises if there are any Ellsberg equilibria different from the Nash equilibrium at all. There are, and they take the following form for our first version of modified Matching Pennies (Figure 5). Player 1 plays HEAD with probability $P \in [1/2, P_1]$ for some $1/2 \leq P_1 \leq 1$ and player 2 plays HEAD with probability $Q \in [1/3, Q_1]$ for some $1/3 \leq Q_1 \leq 1/2$. This Ellsberg equilibrium yields the same payoffs 1/3 and 0 as in Nash equilibrium. We prove a more general theorem covering this case in the appendix[7].

The Ellsberg equilibrium strategy thus takes the following form. Player 1 says :"I will play HEAD with a probability of at least 50 %, but not less." And Player 2 replies: "I will play HEAD with at least 33 %, but not more than 50 %." A larger class of ambiguity thus supports equilibrium behavior in such games.

Whereas the support of the Ellsberg and Nash equilibria is obviously the same, we do think that the Ellsberg equilibria reveal a new class of behavior not encountered in game theory before. It might be very difficult for humans to play exactly a randomizing strategy with equal probabilities (indeed, some claim that this is impossible, see Dang (2009) and references therein). Our result shows that it is not necessary to randomize exactly to support a similar equilibrium outcome (with the same expected payoff). It is just enough that your opponent knows that you are randomizing with some probability, and that it could be that this probability is one half, but not less. It is thus sufficient that the player is able to control the lower bound of his random-

[7]Note that in zero–sum games there are no Ellsberg equilibria in which both players create ambiguity. We comment on this in the appendix.

izing device. This might be easier to implement than the perfectly random behavior required in classical game theory.

In addition there are experimental findings which suggest that the Ellsberg equilibrium strategy in the modified Matching Pennies game is closer to real behavior than the Nash equilibrium prediction. In an experiment run by Goeree and Holt (2001), 50 subjects play a one-shot version of the modified Matching Pennies game as in Figure 5, with the following payoffs:

		Player 2	
		$HEAD$	$TAIL$
Player 1	$HEAD$	$320, 40$	$40, 80$
	$TAIL$	$40, 80$	$80, 40$

Figure 7: Modified Matching Pennies III

The Nash equilibrium prediction of the game is $((1/2, 1/2), (1/8, 7/8))$, but in the experiment row players deviate considerably from this equilibrium. 24 of the 25 row players choose to play HEAD, and only one chooses TAIL. Furthermore, four of the column players choose HEAD, the other 21 choose to play TAIL; the aggregate observation for column players is thus closer to the Nash equilibrium strategy.

Applying the Ellsberg analysis of the modified Matching Pennies game to the payoffs of Goeree and Holt (2001)'s experiment yields an Ellsberg equilibrium

$$([1/2, P_1], [1/8, Q_1]) \text{ with } 1/2 \leq P_1 \leq 1 \text{ and } 1/8 \leq Q_1 \leq 1/2.$$

One possible equilibrium strategy of player 1 is the set of probabilities $[1/2, 1]$ to play HEAD. The aggregate observation in the experiment (that player 1 chooses HEAD with higher probability than TAIL) is thus consistent with the Ellsberg equilibrium strategy.

7 Conclusion

This article demonstrates that the strategic use of ambiguity is a relevant concept in game theory. Employing ambiguity as a strategic instrument leads to a new class of equilibria not encountered in classic game theory. We point out that players may choose to be deliberately ambiguous to gain a strategic advantage. In some games this results in equilibrium outcomes which can not be obtained as Nash equilibria.

Bade (2010) proves that in two person normal form games under some weak assumptions on the preferences Ellsberg equilibria always have the same support as some Nash equilibrium[8]. This changes as soon as one allows for three or more players. The peace negotiation example as well as the mediated prisoners' dilemma suggest that the resulting non–Nash behavior can be of economic relevance.

The point in having more than two players is that in the example superpower C is able to induce the use of different probability distributions. Although countries A and B observe the same Ellsberg strategy, due to their ambiguity aversion modeled by maxmin expected utility the countries use different probability distributions to assess their utility. This strategic possibility does not arise in two person games.

Nevertheless, 2×2 games have Ellsberg equilibria which are conceptually different from classic mixed strategy Nash equilibria. We analyze a special class of these games, where the players have conflicting interests. These games have equilibria in which both players create ambiguity. They use an Ellsberg strategy where they only need to control the lower (or upper) bound of their set of probability distributions.

We argue that this randomizing device is easier to use for a player than playing one precise probability distribution like in mixed strategy Nash equilibrium. What makes this argument attractive is on the one hand that the payoffs in these Ellsberg equilibria are the same as in the unique mixed Nash equilibrium and thus the use of ambiguous strategies in competitive games is indeed an option. On the other hand, comparing with experimental results, we point out that in the modified Matching Pennies game Ellsberg equilibrium strategies are consistent with the experimentally observed behavior.

We believe that the use of Ellsberg urns as randomizing devices is a tangible concept that does not involve more (or even less) sophistication than single probability distributions and, in connection with ambiguity–aversion,

[8]For maxmin expected utility preferences like in our model this result holds under the assumption that all beliefs in the sets $\mathcal{P}_i$ are mutually absolutely continuous. All beliefs $P_i \in \mathcal{P}_i$ are mutually absolutely continuous when $P_i(\omega) > 0$ for some $P_i \in \mathcal{P}_i$ holds if and only if $P_i'(\omega) > 0$ for all $P_i' \in \mathcal{P}_i$ and all $\omega \in \Omega$.

adds a new dimension to the analysis of strategic interaction. Ellsberg games need to be explored further to fully characterize the strategic possibilities that ambiguity offers.

A Competitive 2×2 Games

We provide here the promised proposition on 2×2 games.

Proposition 2. *Consider the competitive 2×2 game with payoff matrix*

$$\begin{array}{ccc} & & \textit{Player 2} \\ & & \begin{array}{cc} L & R \end{array} \\ \textit{Player 1} & \begin{array}{c} U \\ D \end{array} & \begin{array}{|c|c|} \hline a, d & b, e \\ \hline b, e & c, f \\ \hline \end{array} \end{array}$$

such that

$$a, c > b \text{ and } d, f < e\,.$$

Let

$$P^* = \frac{f - e}{d - 2e + f}, \quad Q^* = \frac{c - b}{a - 2b + c}$$

denote the Nash equilibrium strategies for player 1 and 2, resp. The Ellsberg equilibria of the game are of the following form:

If $Q^* < P^*$, then the Ellsberg equilibria are

$$([P^*, P_1], [Q^*, Q_1])$$

for $P^* \leq P_1 \leq 1, Q^* \leq Q_1 \leq P^*$;
and if $P^* < Q^*$, then the Ellsberg equilibria are

$$([P_0, P^*], [Q_0, Q^*])$$

for $0 \leq P_0 \leq P^*, P^* \leq Q_0 \leq Q^*$.

Proof. The Nash equilibrium strategies follow from the usual analysis. The conditions on the payoffs assure that the Nash equilibrium is completely mixed, i.e. $0 < P^* < 1$ and $0 < Q^* < 1$. Let now $[P_0, P_1]$ and $[Q_0, Q_1]$ be Ellsberg strategies of player 1 and 2, where $P \in [P_0, P_1]$ is the probability of player 1 to play U, and $Q \in [Q_0, Q_1]$ is the probability of player 2 to play L.

The maxmin expected utility of player 1 when he plays the mixed strategy P is then

$$\begin{aligned}
&\min_{Q_0 \leq Q \leq Q_1} aPQ + bP(1-Q) + b(1-P)Q + c(1-P)(1-Q) \\
&\qquad = \min_{Q_0 \leq Q \leq Q_1} Q(b-c+P(a-2b+c)) + bP + c - cP \\
&\qquad = \begin{cases} Q_1(b-c+P(a-2b+c)) + bP + c - cP & \text{if } P < \frac{c-b}{a-2b+c} \\ \frac{ac-b^2}{a-2b+c} & \text{if } P = \frac{c-b}{a-2b+c} \\ Q_0(b-c+P(a-2b+c)) + bP + c - cP & \text{else} \end{cases}.
\end{aligned}$$

Player 1's utility is constant at $\frac{ac-b^2}{a-2b+c}$ for all $P \in \left[0, \frac{c-b}{a-2b+c}\right] = [0, Q^*]$ when $Q_1 = \frac{c-b}{a-2b+c} = Q^*$, and for all $P \in \left[\frac{c-b}{a-2b+c}, 1\right] = [Q^*, 1]$ when $Q_0 = \frac{c-b}{a-2b+c} = Q^*$. This means that player 1's best response to a strategy $[Q_0, Q^*]$ where $0 \leq Q_0 \leq Q^*$ is $[P_0, P_1] \subseteq [0, Q^*]$, and player 1's best response to a strategy $[Q^*, Q_1]$ where $Q^* \leq Q_1 \leq 1$ is $[P_0, P_1] \subseteq [Q^*, 1]$.

We do the same analysis for player 2. His maxmin expected utility when he plays the mixed strategy Q is

$$\begin{aligned}
&\min_{P_0 \leq P \leq P_1} dPQ + eP(1-Q) + e(1-P)Q + f(1-P)(1-Q) \\
&\qquad = \min_{P_0 \leq P \leq P_1} P(e-f+Q(d-2e+f)) + eQ + f - fQ \\
&\qquad = \begin{cases} P_0(e-f+Q(d-2e+f)) + eQ + f - fQ & \text{if } Q < \frac{f-e}{d-2e+f} \\ \frac{df-e^2}{d-2e+f} & \text{if } Q = \frac{f-e}{d-2e+f} \\ P_1(e-f+Q(d-2e+f)) + eQ + f - fQ & \text{else} \end{cases}.
\end{aligned}$$

Player 2's utility is constant at $\frac{df-e^2}{d-2e+f}$ for all $Q \in \left[0, \frac{f-e}{d-2e+f}\right] = [0, P^*]$ when $P_0 = \frac{f-e}{d-2e+f} = P^*$, and for all $Q \in \left[\frac{f-e}{d-2e+f}, 1\right] = [P^*, 1]$ when $P_1 = \frac{f-e}{d-2e+f} = P^*$. This means player 2's best response to a strategy $[P^*, P_1]$ where $P^* \leq P_1 \leq 1$ is $[Q_0, Q_1] \subseteq [0, P^*]$, and player 2's best response to a strategy $[P_0, P^*]$ where $0 \leq P_0 \leq P^*$ is $[Q_0, Q_1] \subseteq [P^*, 1]$.

We calculate the intersections of the best response functions to find the Ellsberg equilibria. In the preceding paragraph we have found the following best response functions:

1. $B_1([Q^*, Q_1]) = [P_0, P_1] \subseteq [Q^*, 1]$

2. $B_1([Q_0, Q^*]) = [P_0, P_1] \subseteq [0, Q^*]$

3. $B_2([P^*, P_1]) = [Q_0, Q_1] \subseteq [0, P^*]$

4. $B_2([P_0, P^*]) = [Q_0, Q_1] \subseteq [P^*, 1]$

We assume that $Q^* < P^*$. Suppose player 2 plays $[Q^*, Q_1]$, then if we let player 1 choose $[P_0, P_1] = [P^*, P_1]$ with $P^* \le P_1 \le 1$ player 2 would play any strategy $[Q_0, Q_1] \subseteq [0, P^*]$ as a best response. If he picks $[Q_0, Q_1] = [Q^*, Q_1]$ with $Q^* \le Q_1 \le P^*$ this is an Ellsberg equilibrium, i.e.

$$([P^*, P_1], [Q^*, Q_1]) \text{ where } P^* \le P_1 \le 1 \text{ and } Q^* \le Q_1 \le P^*.$$

In the case $Q^* < P^*$ this is the only type of Ellsberg equilibrium in which both players play subjectively mixed strategies.

When we assume that $P^* < Q^*$, we get a similar type of Ellsberg equilibrium. Suppose player 2 plays $[Q_0, Q^*]$, then if we let player 1 pick $[P_0, P_1] \subseteq [P_0, P^*]$ with $0 \le P_0 < P^*$, player 2's best response is any subset $[Q_0, Q_1] \subseteq [P^*, 1]$. If he chooses $[Q_0, Q_1] = [Q_0, Q^*]$ with $P^* \le Q_0 \le Q*$, then this is an Ellsberg equilibrium, i.e.

$$([P_0, P^*], [Q_0, Q^*]) \text{ where } 0 \le P_0 < P^* \text{ and } P^* \le Q_0 \le Q* .$$

□

Remark 1. *1. In proposition 2 we restrict to the case with (U, D) and (L, R) giving the same payoffs (b, e) for both players. Of course the Ellsberg equilibria of competitive games with more general payoffs can easily be calculated. The nice feature of our restriction is that in Ellsberg equilibrium players use the mixed Nash equilibrium strategy of their respective opponent as upper (or lower) bound of their Ellsberg strategy.*

2. Note that for $Q^* = P^*$ there is no such equilibrium where both players create ambiguity. In this case their best response functions intersect only at the Nash equilibrium distribution. What we get are (apart from the Nash equilibrium) only the two Ellsberg equilibria where one player hedges against the ambiguity of the other. These are, if we call the symmetric Nash equilibrium strategy R^*,

$$(R^*, [Q_0, Q_1]) \text{ where } Q_0 \le R^* \le Q_1$$

and

$$([P_0, P_1], R^*) \text{ where } P_0 \leq R^* \leq P_1.$$

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