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Berno Büchel, Tim Hellmann and Michael M. Pichler

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
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The Dynamics of Continuous Cultural Traits in Social Networks*

Berno Büchel
University of Hamburg
berno.buechel@wiso.uni–hamburg.de

Tim Hellmann
Bielefeld University
tim.hellmann@uni–bielefeld.de

Michael M. Pichler
Bielefeld University and University of Valencia
mpichler@uni–bielefeld.de

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Abstract

We consider an OLG model (of a socialization process) where continuous traits are transmitted from an adult generation to the children. A weighted social network describes how children are influenced not only by their parents but also by other role models within the society. Parents can invest into the purposeful socialization of their children by strategically displaying a cultural trait (which need not coincide with their true trait). Based on Nash equilibrium behavior, we study the dynamics of cultural traits throughout generations. We provide conditions on the network structure that are sufficient for long–run convergence to a society with homogeneous subgroups. In the special case of quadratic utility, the condition is that each child is more intensely shaped by its parents than by the social environment. Our model also represents an extension of the classical DeGroot model of opinion formation for which we introduce strategic interaction in choice of expressed opinions (in our setup: traits). We show that under strategic interaction convergence is slower and for convergence we need more restrictive necessary and sufficient conditions than in the DeGroot model.

Keywords: cultural transmission, social networks, preference formation, cultural persistence, opinion dynamics

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1 Introduction

Economic behavior is fundamentally shaped by individual value systems, cultural *traits*, culturally transmitted preferences, opinions, beliefs, etc. The question of how these traits are formed is hence of central interest. We present an OLG model where children's cultural traits are formed by learning from their parents' behavior and by learning from the behavior of other individuals through local interaction. Parents are aware of their influence on the children's adopted cultural traits as the children's role model and may choose their behavior acordingly, yielding strategic interaction between parents of each generation in choice of behavior. The dynamics of the resulting traits is the central object of study in this paper.

There is already a rich body of empirical and theoretical studies on the topic of trait formation (see Bisin and Verdier, 2010 for a survey). In this paper we study a theoretical model that largely follows this literature, but deviates from it with respect to the following three aspects: (a) traits are not formalized as dichotomous variable, but as *continuous* variable. This seems to be more adequate in many applications, e.g. risk preferences, patience, political attitudes, generosity, discounting preferences, trust attitudes, etc. (b) We (do not assume that agents are equally likely to interact with any other member of the society, but) introduce a *weighted network* to capture the interaction structure. This potentially incorporates the geographic structure as well as distinctive personal relationships. (c) In our model, parents anticipate that their behavior has consequences for the formation of their children's traits and they adjust their behavior accordingly. Thereby the behavioral choices of parents can be reduced to the *strategic choice of how to display* a certain trait. In particular, in their choice of behavior, parents face a trade-off between own utility loss and improvements in their childs adopted trait (which parents evaluate with respect to their own utility function).

With these assumptions, our model has a close connection to the literature on opinion formation which is based on the model by DeGroot (1974). The DeGroot framework elegantly captures pillar (a) and (b) of our model. Pillar (c) however, goes beyond this framework and it turns out that the introduction of strategic interaction is a fruitful extension of the classic DeGroot model in both contexts dynamics of opinions and dynamics of cultural traits.

In the main results of this paper, we show that each adult deviates from its true trait into the opposite direction of the behavior in the relevant environment, in order to countervail the (subjectively negative) influence of the environment on its child (Proposition 1). That is, if an adult perceives that the environment will influence its child into some direction (e.g. that religion is less important than for the parent itself), then it adjusts its behavior into the opposite direction (e.g. by going even more frequently to church).

While existence of a Nash equilibrium in every period is guaranteed (Proposition 2), the main focus of our present work is the analysis of the dynamic evolution of the adopted traits. First, we show by a simple 2-player example (Example 1) that a sufficient condition for convergence is that children are primarily influenced by their parents (with weight larger than 1/2). Moreover, we illustrade in Example 1 that in our model the set of cultural traits does not always converge. This is true even for conditions such that opinion dynamics of the DeGroot model converge. Thus, introducing strategic interaction to the DeGroot model may lead to non-convergence.

The central question is therefore under which conditions on the interaction network and the socialization incentives the dynamics converge. We apply a number of linear algebra results to make use of results on the convergence of the left product of matrices, in particular by Lorenz (2005, 2006). With this approach we obtain a general convergence result: if the social learning matrix (which represents the interaction network) belongs to the class of so called *symmetric ultrametric matrices*, then the dynamics always converge (Proposition 4). This (very strong) assumption on the network structure can be relaxed when assuming particular functional forms of utility. Specifically, we study the case of quadratic utility functions. With quadratic utility funtions, we are able to identify some necessary and some sufficient conditions for convergence, which coincide if the interation structure (the weighted network) is symmetric: it must be positive definite, which is implied by symmetry and diagonally dominance of the underlying social learning matrix. Moreover, we show that if the socialization incentives are 'not too strong' (i.e. if the model is close to the DeGroot model), then the sequence of the transformed matrices also converges for an arbitrary aperiodic social learning matrix. Finally, we show that the speed of convergence depends on the network structure and is reduced by the parents' socialization efforts. Thus, by introducing strategic interaction speed of convergence is lower than in the DeGroot model.

Empirical Evidence In the formation of cultural traits, strong correlations have been found between the cultural traits of parents and the cultural traits of their children. This is shown in multiple studies, e.g. Dohmen et al. (2009) for risk preferences and for trust; Arrondel (2009) for risk preferences; Fernandez et al. (2004) for female labor force participation; Branas-Garza and Neuman (2007) for religious norms. This indicates the importance of parents in the process of developing a personality, particularly, in the formation of traits. But not only the parents affect the socialization of a child, there are also role models outside the family who have a significant impact on the process of trait formation. Thus, an OLG model where children are influenced by parents and neighborhood seems suitable to model the dynamics of cultural traits.

For cultural traits, empirical literature usually documents the persistence of heterogeneous traits (and opinions) in many applications. Today's heterogeneity is based on resilience of once formed traits.[1] Examples for the resilience of traits are female labor force participation (Alesina et al., 2011), the level of trust in various Italian cities (Guiso et al., 2008), anti-semitism in German cities (Voigtlaender and Voth, 2011), a preference for education (Botticini and Eckstein, 2005), and many more.

A central question is why history has not led to one homogeneous society. Our modeling approach enables us to *analyze the conditions under which homogeneous or heterogeneous societies emerge* and the speed of this process. We thereby put an emphasis on the following two variables:

(i) The interaction structure. Empirically measured heterogeneity strongly corresponds to geographic structure. E.g. Tabellini (2008) shows that trust differs substantially across countries. Geographical differences might only occasionally be made directly responsible for the observed heterogeneity. Rather the persistence is an effect of the interaction structure which is only to some extent based on geographical distances.

(ii) The socialization incentives: Persistence of heterogeneous traits has not only been found across different locations but also within one local area. In other words, the "melting–pot" hypothesis has not found empirical support, as e.g. orthodox Jewish communities in the United States show.[2] In the context of dichotomous traits, a theoretical explanation of this phenomenon is based on the idea of cultural substitution, i.e. minorities invest more into socialization to their own trait than majorities (Bisin and Verdier, 2001). In our framework of continuous traits and local interaction, we also study a similar mechanism.

Related Literature Our present work stands in close relation to two distinct strands of literature. The first one is the small existing literature on the cultural formation of continuous traits.[3] Important early treatments of the topic are Cavalli-Sforza and Feldman (1981) in a theoretical, and Otto et al. (1994) in an empirical context. More recently, Bisin and Topa (2003) proposed a representation of the formation of continuous cultural traits. Their approach is though restricted to the family's choice of its weight

[1]Or as Bisin and Verdier, 2010 put it: "the resilience of cultural traits and cultural heterogeneity are two sides of the same coin. It is not surprising then that the evidence regarding the resilience of ethnic and religious traits across generalizations is quite pervasive and it nicely complements the evidence on cultural heterogeneity."

[2]In principle, this might also be an effect of the interaction structure—if there is only limited interaction between the Jewish community and outsiders.

[3]Compared to the small literature on continuous traits, there exists a well established literature on the (probabilistic) transmission of discrete (dichotomous) traits. See Bisin and Verdier (2010) for an exhaustive overview.

in the child's socialization process.[4] The issue of the behavioral choice as socialization investment is part of the framework of Pichler (2010b) and further analyzed in Pichler (2010a).

We base our model on this framework. The main difference of our model to the existing literature on the cultural formation of continuous traits is that we explicitly introduce a social network into the model.

The second branch of literature related to our work is the literature on opinion dynamics (in social networks) introduced by DeGroot (1974) (see e.g. Jackson (2008) for a discussion). In the basic DeGroot-model individuals exchange opinions by reporting their opinions and update according to a weighted average of other individual's opinions. Convergence of opinions is then obtained under mild conditions on the interaction structure (strong connectedness and aperiodicity). A variation of this model is introduced by DeMarzo et al. (2003) where the individuals' own beliefs can vary over time. The convergence result is similar to that of DeGroot (1974) with additional assumptions on the self–trust weights. Moreover, DeMarzo et al. (2003) study the speed of convergence. In Lorenz (2005) and Lorenz (2006) the whole interaction structure is allowed to change over time. Under some conditions, i.e. type-symmetry (if i puts some weight on the opinion of j the j also puts some weight on the opinion of i), positive self-belief and non-convergence to zero of the positive entries, convergence to a consensus is obtained. Other studies on convergence of opinion dynamics include that of Krause (2000), Hegselmann and Krause (2002), Weisbuch et al. (2002), and Golub and Jackson (2010).[5] Both our evolving cultural traits and the evolving opinions in DeGroot (1974) are modeled as a variable in an interval of the real line. Moreover in both models, interaction takes place on a weighted (row stochastic) network. So far, however, the literature on opinion formation has not considered strategic interaction, i.e. it is assumed in this literature that all players report their opinion truthfully in every period. Our work presents, hence, also a generalization of the DeGroot model such that strategic interaction in expressed opinions is introduced. We show that strategic interaction leads to overstatement of opinions (Proposition 1) a similar, but less extreme behavior as in Kalai and Kalai (2001), where polarization is obtained. As a consequence, convergence cannot be as easily obtained as in the DeGroot model. For instance, we show that even in a two player setting opinions may not converge under the assumptions of DeGroot (1974) (cf. Example 1) when agents do not report their opinions truthfully. Moreover for the case of convergence, we show that the speed of which is reduced by introducing this kind of strategic interaction.

[4] The same is true for the approach of Panebianco (2010), who considers the evolution of inter–ethnic attitudes.

[5] The additional objective of the latter paper is to show conditions under which a noisy opinion profile can converge to its mean.

2 Formation of Cultural Traits

2.1 Model

Consider an overlapping generations society which is populated by the adults of a finite set of dynasties, $N = \{1, \ldots, n\}$. At the beginning of any given period $t \in \mathbb{N}$, adults reproduce asexually and have exactly one offspring, thus the population size is constant. With respect to one type of trait (e.g. risk preferences), let $\mathcal{I} \subseteq \mathbb{R}$ be a convex compact set that contains all possible intensities for that trait. Each adult is characterized by a certain value of the trait $\phi_i(t)$, which we call its *trait intensity* (henceforth: TI). The (column) vector $\Phi(t) := (\phi_1(t), \ldots, \phi_n(t))'$ represents the TIs in the population at time t.

Any adult has to make socio–economic choices each of which displays a certain trait intensity.[6] We call the choice of an adult its *displayed trait intensity* (henceforth: DTI) and denote it by $\phi_i^d(t)$. Let the (column) vector $\Phi^d(t) := \left(\phi_1^d(t), \ldots, \phi_n^d(t)\right)'$ collect the DTIs of the adults. Importantly, the DTI $\phi_i^d(t)$ is the choice variable of each adult and may be different from its true TI $\phi_i(t)$.

The society is connected by a social network represented by a $n \times n$-matrix Σ. In order to account for relative influences, we assume that Σ is a row stochastic matrix, that is $\sigma_{ij} \geq 0 \; \forall i, j \in N$, and $\sum_{j \in N} \sigma_{ij} = 1$. Σ describes a weighted, possibly directed, social network between the dynasties. A zero entry ($\sigma_{ij} = 0$) means that there is no personal interaction between two dynasties, e.g., based on geographical distance. An entry $\sigma_{ij} > 0$ represents the relative importance of adult j as a role model for child i. We can think of this as the relative cognitive impact (of the socialization interactions), which can be based on interaction time as well as on differing pre–dispositions of the children for the social learning from the adults. A diagonal element σ_{ii} (of Σ) represents the weight of the parent in the socialization process of its child. Factors that determine this *parental socialization success share* could include the social interaction time of the parent with its child, as well as the effort and devotion that the parent spends to socialize its child.[7]

Children are assumed to learn from the adults' observable behavior, i.e. the DTIs, according to the weights of relative importance. Thus, the traits of the children generation are formed by

$$\Phi(t+1) = \Sigma \Phi^d(t). \tag{1}$$

[6] Pichler (2010b) first introduced such a model. We employ here the framework of Pichler (2010b) without explicitly modeling the socio-economic choices that lead to the displayed trait intensities.

[7] See e.g. Grusec (2002) for an introductory overview of theories on determinants of parental socialization success.

Remark 1. *The basic model by DeGroot (1974) is mathematically equivalent to our model – except for one ingredient. In our model, we introduce a strategic choice variable, namely the 'articulated opinions', i.e. the DTIs. In DeGroot (1974) a simple dynamic model is studied which can be captured by $\Phi(t+1) = \Sigma\Phi(t)$. This can be interpreted in the sense that parents always tell the truth and there is no strategic interaction.*

Let us denote $\phi^d_{N_i}(t) := \sum_{j \in N \setminus \{i\}} \frac{\sigma_{ij}}{1-\sigma_{ii}} \phi^d_j(t)$, the representative DTI of child i's social environment. Then the trait formation process in (1) can be interpreted as a weighted average between the DTI of the environment ($\phi^d_{N_i}(t)$) and the DTI of the parents ($\phi^d_{N_i}(t)$):

$$\phi_i(t+1) = \sigma_{ii}\phi^d_i(t) + (1-\sigma_{ii})\phi^d_{N_i}(t).$$

We assume that all individuals carry over the TI that has been formed in their child period into their adult period.

In the adult period then, this adopted TI $\phi_i(t)$ guides socio–economic choices. Formally, an adult has to choose a DTI $\phi^d_i(t) \in \mathcal{I}$. We assume that this choice is evaluated with respect to two utility components: *own utility* and *inter–generational utility*. Let $u_i : \mathcal{I} \mapsto \mathbb{R}$ represent an adult's own utility from the DTI, $\phi^d_i(t)$, and let $v_i : \mathcal{I} \mapsto \mathbb{R}$ represent the utility of an adult derived from its child's adopted TI, $\phi_i(t+1)$, the inter–generational utility component.

The following specifies the assumptions on each adult's utility.

Assumption 1 (Parental Utility Function)**.** *The utility for an adult $i \in N$, at time $t \in \mathbb{N}$ is given by*

$$u_i\left(\phi^d_i(t)\,|\phi_i(t)\right) + v_i\left(\phi_i(t+1)\,|\phi_i(t)\,\right) \tag{2}$$

with

(a) $u_i\left(\cdot\,|\phi_i(t)\,\right)$ being single–peaked with peak $\phi_i(t)$, i.e. strictly increasing / decreasing $\forall \phi^d_i(t) \in \mathcal{I}$ such that $\phi^d_i(t) < / > \phi_i(t)$,

(b) $v_i\left(\cdot\,|\phi_i(t)\,\right)$ being single–peaked with peak $\phi_i(t)$, i.e. strictly increasing / decreasing at all $\phi_i(t+1) \in \mathcal{I}$ such that $\phi_i(t+1) < / > \phi_i(t)$,

(c) $u_i\left(\cdot\,|\phi_i(t)\,\right)$ and $v_i\left(\cdot\,|\phi_i(t)\,\right)$ being continuous and twice continuously differentiable at their peaks.

In part (a) we assume that u_i is decreasing in the difference of the DTI from the TI. Intuitively, this means that adults prefer choosing behaviors (DTIs) that are as close as possible in line with their traits (TIs). As an example, consider an adult's choice of articulated opinion. If this does not coincide with the adult's adopted opinion, then the adult is *lying*. Lying can cause dis–utilities in terms of cognitive dissonance (see Festinger, 1957) or in terms of the fear of being revealed. Intuitively, these dis–utilities are strictly increasing in the 'degree of the lie'.

In part (b) we assume that v_i is decreasing in the difference between the parent's TI and the TI that its child forms.[8] There are two basic motivations to consider this case. The first one is that parents simply have an intrinsic desire that their children develop a "personality" (TI) that is as similar as possible to their own personality. For example, empirical evidence is in line with "Protestants, Catholics, and Jews having a strong preference for children who identify with their own religious beliefs and making costly decisions to influence their children's religious beliefs." (Bisin and Verdier, 2010) The second motivation is based on a special form of parental altruism, called *imperfect empathy* (Bisin and Verdier, 1998). Parents care about the well–being of their children, but can only evaluate their child's utility under their own (not the child's) utility function—which attains its maximum at the TI of the parent.[9] Part (c) and the additive separability of the two utility components are technical assumptions which significantly reduce analytical complexity.

2.2 The Adults' Decisions

Given that we consider only the behavioral (DTI) choices of the adults, in any period $t \in \mathbb{N}$, the optimization problem of each adult $i \in N$ is

$$\begin{aligned} \max_{\phi_i^d(t) \in \mathcal{I}} u_i \left(\phi_i^d(t) \,|\phi_i(t) \right) + v_i \left(\phi_i(t+1) \,|\phi_i(t) \right) \\ \text{s.t. } \phi_i(t+1) = \sigma_{ii}\phi_i^d(t) + (1-\sigma_{ii})\phi_{N_i}^d(t). \end{aligned} \tag{3}$$

The optimization problem (3) embodies the trade-off between own utility losses (by choosing a DTI that deviates from the true TI) and eventual improvements in the location of the child's adopted TI.

In any given period, the optimization problem of an adult $i \in N$ determines the set of best reply DTI against the representative environment's DTI, subject its adopted TI (which is also the socialization target for its child). For every adult $i \in N$, we will thus denote the elements of the best reply set as $\phi_i^d(t)\left(\phi_i(t), \phi_{N_i}^d(t)\right)$, which we abbreviate as $\phi_i^d(t)(\cdot)$. Furthermore, together with the representative DTI, any such best reply DTI also determines a location of the child's TI (through the rule of trait formation (1)), $\phi_i(t+1)\left(\phi_i^d(t), \phi_{N_i}^d(t)\right)$.

Since both the own utility and inter–generational utility function are single–peaked, Assumption 1 (c) implies that both functions have zero slope

[8] See Pichler (2010b,a) for a more general representation that allows for different socialization targets.

[9] There is a form of myopia in this line of interpretation: parents do not anticipate that their children might deviate from their TI.

at their peaks. Thus, adults perceive no (own) utility losses for marginal deviations of their DTI from their TI, respectively no (inter–generational) utility losses for marginal deviations of their child's TI from the target TI.

Proposition 1 (Characterization of Best Replies)**.** *Let Assumption 1 hold. Then, $\forall t \in \mathbb{N}$, $\forall i \in N$, the sets of best reply DTIs are non–empty and satisfy the following characterization.*

(a) If $\sigma_{ii}(t) = 0$, then $\phi_i^d(t)(\cdot) = \phi_i(t)$ and $\phi_i(t+1)\left(\phi_i(t), \phi_{N_i}^d(t)\right) = \phi_{N_i}^d(t)$.

(b) If $\sigma_{ii}(t) = 1$, then $\phi_i^d(t)(\cdot) = \phi_i(t)$, thus $\phi_i(t+1)\left(\phi_i(t), \phi_{N_i}^d(t)\right) = \phi_i(t)$.

(c) Let $\sigma_{ii}(t) \in (0,1)$. Then, it holds generically that $\operatorname{sign}\left(\phi_i^d(t)(\cdot) - \phi_i(t)\right) = -\operatorname{sign}\left(\phi_{N_i}^d(t) - \phi_i(t)\right)$,[10] while it always holds that $\operatorname{sign}\left(\phi_i(t+1)\left(\phi_i^d(t)(\cdot), \phi_{N_i}^d(t)\right) - \phi_i(t)\right) = \operatorname{sign}\left(\phi_{N_i}^d(t) - \phi_i(t)\right)$.

Proof. Non–emptiness as well as parts (a) and (b) are trivial. Part (c) follows as a straightforward corollary from the proof of Proposition 1 in Pichler (2010b). □

The (generic) results of Proposition 1 (c) are illustrated in Figure 1 below. In the left interval (both intervals correspond to the set of possible DTIs $\mathcal{I}$) the context of the adult's decision problem is depicted. In the right interval a corresponding best reply choice is stylized. As can be seen both from Proposition 1 (c) directly, as well as from the graphical illustration, the results feature two dominant characteristics.

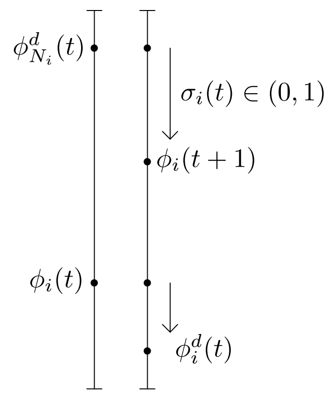

Figure 1: Characterization of Best Replies

[10] The non–generic case holds if the deviation of the best reply DTI from the adopted TI into the 'desired' direction is not possible, i.e. if the adopted TI of an adult coincides with (the relevant) one of the boundaries of $\mathcal{I}$. Then, the best reply DTI will coincide with the adopted TI (i.e. with the boundary).

The first concerns the generic location of the best reply choices. If the representative DTI does not coincide with the optimal TI, then parents countervail the respective socialization influence on their children by choosing DTI that deviates from their adopted TI.[11] This deviation is always into the opposite direction as the deviation of the representative DTI from the optimal TI (if such a choice is available). That this holds for very small deviations of the representative DTI from the optimal TI is due to the fact that marginal investments into socialization are (utility) costless (while as the resulting strictly positive decrease in the distance of the child's adopted TI from the optimal TI yields a strictly positive inter–generational utility gain).

The second dominant characteristic concerns the location of the children's adopted TIs that results out of the parental best reply choices. The adult's deviation from the utility peak is never extreme enough such as to induce that the child's adopted TI would exactly coincide with the adult's TI. Hence, there is always a strictly positive difference of the adopted TI of a child from the target TI. Thereby, the direction of this difference always accords with the direction of deviation of the representative environment's DTI from the target DTI.

To study the dynamics of cultural transmission of traits we will assume that every adult plays a best response to the behaviors (DTI choices) of others, i.e. its neighbors. Hence, we assume that a Nash equilibrium is played in every period. To ensure existence of such we assume the following.

Assumption 2 (Concavity). *For any adult $i \in N$, the functions $u_i\left(\cdot\,|a\right)$ and $v_i\left(\cdot\,|b\right)$ are concave.*[12]

Together with Assumption 1, Assumption 2 guarantees existence of Nash equilibrium which is shown using standard techniques.

Proposition 2 (Nash Equilibrium Existence). *Let Assumptions 1 and 2 hold. Then, for every $t \in \mathbb{N}$, a* Nash equilibrium *in DTI choices exists. Denote this $\Phi^{d^*}(t) := \left(\phi_1^{d^*}(t), \ldots, \phi_n^{d^*}(t)\right)'$.*

Proof. In Appendix 5.1.

3 Dynamics (of Cultural Traits)

We first describe the steady states (subsection 3.1), then study convergence to a steady state in the general case (subsection 3.2) and, finally, in a special case (subsection 3.3).

[11]Obviously, if the representative DTI exactly coincides with the optimal TI, then parents have no incentives to do so (since the adopted TI of an adult child will then anyhow coincide with the optimal TI).

[12]Note that under Assumption 1 both utility functions are already strictly quasi–concave.

3.1 Steady States

A steady state is a rest point of the dynamics of cultural traits, as defined below.

Definition 1 (Steady States)**.** *A steady state is a profile of TIs such that* $\Phi(t+1) = \Phi(t)$.

To characterize the steady states of the dynamics, we have to introduce some additional notation related to the interaction matrix Σ. We say that there exists a connection from i to j in Σ, denoted by $i \to j$, if there exists a $k \in \{0, ..., n\}$ such that $\Sigma^k_{ij} > 0$. Two dynasties communicate, denoted by $i \sim j$, if $i \to j$ and $j \to i$. A dynasty i is self-communicating if $i \to i$. Trivially, $\sim$ defines an equivalence relation on the set of self-communicating dynasties and, hence, this set can be partitioned into equivalence classes, called self-communicating classes. Denoting each non self-communicating dynasty as a single class, $\sim$ partitions the dynasty set into communication classes $\mathcal{P}(\Sigma) = \{N_1, ..., N_p\}$ such that for all $L \in \mathcal{P}(\Sigma)$, L is either a self-communicating class or a non self-communicating dynasty. A communication class $L \in \mathcal{P}(\Sigma)$ is called essential if for all $i \in L$ there does not exist a $j \notin L$ such that $i \to j$. A communication class is called inessential if it is not essential.

Proposition 3 (Steady States)**.** *Under Assumptions 1 and 2, the following holds in any steady state* $\Phi(t)$*:*

(a) $\Phi(t) = \Phi^d(t)$*, i.e. all adults behave as they are.*

(b) The TIs of the dynasties in an essential communication class $L \in \mathcal{P}(\Sigma)$ *coincide, i.e.* $\phi_i(t) = \phi_j(t)\ \forall i, j \in L$.

(c) The TIs of the dynasties in an inessential communication class $I \in \mathcal{P}(\hat{\Sigma})$ *are convex combinations of the TIs of the communication classes* $J \in \mathcal{P}(\hat{\Sigma})$ *such that* $I \to J$.

Proof. In Appendix 5.2.

To see that part (a) must hold, note that per definition, in any steady state, the children adopt the same TIs as their parents have. From Proposition 1, we know that such a constellation can only be subject to (Nash equilibrium) individual best replies if the representative DTIs of all children coincide with the parents' adopted TIs. In such a case, all parents behave as they are. Parts (b) and (c) of the Proposition are also straightforward.

3.2 Convergence (Main Result)

Given this steady state description it now remains to derive conditions under which the sequence of TIs actually converges to any such rest point. The following example shows that in case of only two connected dynasties, such a condition is easy to obtain.

Example 1 (Two Dynasties)**.** *Consider the simplest case of a non–degenerate essential communication class, i.e. that of two parent–child pairs in any given period. Assume also that for all* $i = 1, 2$, $\frac{1}{2} \leq \sigma_{ii} < 1$ *so that the parents are the 'primary socialization sources' of their children. Then, it holds that the distance between the adopted TIs* $\phi(t)$ *of the adult members of both dynasties strictly declines in every period* t *and converges to the same point. To see that this is true, let* $\phi_1(t)$ *be the dynasty with the lower TI in period* t*. Then it will be also the dynasty with the lower TI in period* $t+1$*, since* $\phi_1(t+1) = \sigma_{11}\phi_1^d(t+1) + (1-\sigma_{11})\phi_2^d(t+1)$ *and* $\phi_2(t+1) = (1-\sigma_{22})\phi_1^d(t+1) + \sigma_{22}\phi_2^d(t+1)$, $\sigma_{11} > 1 - \sigma_{22}$*, and* $\phi_1^d(t+1) < \phi_2^d(t+1)$ *by Proposition 1. Moreover, by Proposition 1,* $\phi_1(t+1) \in (\phi_1(t), \phi_2^d(t))$ *and* $\phi_2(t+1) \in (\phi_1^d(t), \phi_2(t))$*, yielding the assertion.*

Indeed, we show in Proposition 5 that given the interaction matrix Σ *is symmetric, the assumption* $\sigma_{ii} \geq 1/2$ *is also necessary for convergence in the quadratic utility case, where* $u_i\left(\phi_i^d(t)\,|\phi_i(t)\right) = -(\phi_i^d(t) - \phi_i(t))^2$ *and* $v_i\left(\phi_i(t+1)\,|\phi_i(t)\right) = -\beta_i(\phi_i(t+1) - \phi_i(t))^2$*. Consider, for instance, the latter case of quadratic utility functions and let* $\sigma_{ii} = .01$ *and* $\beta_i = 1$.[13] *Then the Nash equilibrium choices are* $\phi_i^d(t) \approx 1.11\phi_i(t) - .11\phi_j(t)$ *for distinct* $\phi_i(t)$ *and* $\phi_j(t)$ *and the adopted TIs are* $\phi_i(t+1) \approx -.1\phi_i(t) + 1.1\phi_j(t)$ *for every* $t \in \mathbb{N}$ *(we show how the Nash equilibrium DTI choices can be calculated in* (6)*). It is straightforward to see that the adopted TIs* $\phi_i(t+1)$ *and* $\phi_j(t+1)$ *lie outside the interval* $[\phi_i(t), \phi_j(t)]$ *for every period* t *(as long as the boundary conditions are not binding) and hence do not converge.*[14]

From this example, we can learn two important lessons. First, the sequence of adopted TIs does not always converge. This is in contrast with the classical DeGroot model. Second, assuming some structure on the underlying interaction network Σ can guarantee convergence. This is summarized in the following Remark.

Remark 2. *A necessary and sufficient condition for convergence (without hitting a boundary) in the two–dynasties case is that the parents of both dynasties are the 'primary socialization sources' of their children* ($\sigma_{11}, \sigma_{22} \geq 1/2$)*. In contrast, in the model by DeGroot (1974) strong connectedness and aperiodicity of* Σ *are necessary and sufficient for convergence which translates in the two–player case to* $\sigma_{11}, \sigma_{22} \in (0, 1)$*. Hence by introducing strategic interaction into the DeGroot model the conditions for convergence need to be stronger.*

[13] We show in Proposition 5 that for the symmetric case if $\sigma_{ii} < .5$ there exists a β such that the dynamics does not converge.

[14] It is easy to see that assuming boundaries does not matter for convergence in this case. If one boundary condition is binding (say the smaller boundary), then the interval of adopted TI's $[\phi_i, \phi_j]$ moves into the other direction until the boundary condition is not binding which implies that both adopted TI's lie outside the interval again. For simplicity we assume for the analysis of the quadratic utility case no boundaries.

Let us first consider the general case where the utility functions u_i and v_i only have to satisfy Assumption 1 and 2. To embed the dynamical system into a tractable (linear) form, consider the following result, which is straightforwardly implied by Propositions 1 and 2.

Corollary 1 (Nash Equilibrium Map)**.** *Let Assumptions 1–2 hold. Then, there exists a Nash equilibrium map $B^* : \mathcal{I}^n \mapsto \mathbb{R}^n_+$, such that for every $i \in N$ and for every $t \in \mathbb{N}$, $B^*(\Phi(t)) = (b_1^*(t), \ldots, b_n^*(t))'$ satisfies*

$$\phi_i^{d^*}(t) - \phi_i(t) = b_i^*(t) \cdot (\phi_i(t) - \phi_i^*(t+1))$$

where $\phi_i^*(t+1) := \sum_{j \in N} \sigma_{ij} \phi_j^{d^*}(t)$. This map has the property that if for any $i \in N$ $\sigma_{ii} = 0$, then $b_i^*(t) = 0$, $\forall t \in \mathbb{N}$.

Proof. Follows immediately from the best reply characterization of Proposition 1 and the Nash equilibrium existence of Proposition 2. □

The Nash equilibrium map simply represents the Nash equilibrium DTI choices in terms of their deviations from the adults' adopted TIs relative to the deviation of the children's adopted TIs from the socialization targets. This representation can equivalently be written as $\phi_i^{d^*}(t) + b_i^*(t)\Sigma_i \Phi^{d^*}(t) = (1 + b_i^*(t))\phi_i(t)$, for every $i \in N$. Defining $B(t) := \operatorname{diag}(b_1^*(t), \ldots, b_n^*(t))$, we thus obtain[15]

$$(I + B(t)\Sigma)\, \Phi^{d^*}(t) = (I + B(t))\Phi^*(t)$$

so that

$$\Phi^{d^*}(t) = (I + B(t)\Sigma)^{-1} (I + B(t))\Phi^*(t)$$

and hence

$$\Phi^*(t+1) = \Sigma\, (I + B(t)\Sigma)^{-1} (I + B(t))\Phi^*(t).$$

For this representation to be well defined, it is sufficient that either Σ is diagonally dominant (since then $I + B(t)\Sigma$ is strictly diagonally dominant, thus invertible) or symmetric positive semidefinite (the assumptions used in our main results will imply in particular that Σ is symmetric positive definite).[16]

Finally, denoting $M(t) := \Sigma\, (I + B(t)\Sigma)^{-1} (I + B(t))$, it follows that

$$\Phi^*(t+1) = M(t) \ldots M(0)\Phi(0) = M(t,0)\Phi(0), \ t \in \mathbb{N}\backslash\{0\} \tag{4}$$

where $M(t,0)$ denotes the *backward accumulation* $M(t,0) := M(t) \cdot M(t-1) \cdot \ldots \cdot M(0)$.

[15] $diag(y)$ denotes a diagonal matrix with diagonal entries specified by y.

[16] To see the latter, note that $I + B(t)\Sigma$ and $I + B(t)^{\frac{1}{2}}\Sigma B(t)^{\frac{1}{2}}$ have the same eigenvalues. Now since Σ is symmetric, $B(t)^{\frac{1}{2}}\Sigma B(t)^{\frac{1}{2}}$ is also symmetric and since Σ is positive semidefinite and $B(t) \geq 0$, $B(t)^{\frac{1}{2}}\Sigma B(t)^{\frac{1}{2}}$ is positive semidefinite. Thus $B(t)\Sigma$ has non-negative and real eigenvalues which implies that all eigenvalues of $I + B(t)\Sigma$ are non-zero, thus $I + B(t)\Sigma$ is invertible.

This allows us to resort to linear algebra results on the convergence of left products of matrices. Specifically, Lorenz (2005, 2006) provided convergence results for left products of row stochastic matrices—while as (for our specific context) not sufficient results are available on the left product convergence of more general matrices (that have row sum one, but with possibly negative entries). However, to guarantee that the individual $M(t)$ are row stochastic in every period $t \in \mathbb{N}$, we have to endow the social learning matrix Σ with sufficient structure which is given by the following definition.

Definition 1 (Symmetric Ultrametric Matrix)**.** *A $n \times n$-matrix Σ is* symmetric ultrametric *if*

(i) Σ *is symmetric,*

(ii) $\sigma_{ii} \geq \max\{\sigma_{ij} : j \in N_i\}$, $\forall i \in N$,

(iii) $\sigma_{ij} \geq \min\{\sigma_{ik}; \sigma_{kj}\}$, $\forall i, j, k \in N$.

To motivate the symmetry property in our context, remember the basic determinants of the relative socialization successes that different unrelated adults have with the children. These determinants consist of the relative social interaction time on the one hand, and potentially differing social learning pre–dispositions on the other hand. Thus, for any pair of children, the required symmetry can be achieved by requiring the relative social interaction time that any one of the two children has with the parent of the other child to be identical, together with the assumption that all children have identical social learning pre–dispositions. Property (ii) is the generalized 'primary parental socialization' condition. It simply means that among all adults, the parents have the largest socialization influence on their children (respectively, among all adults, they spend the largest time share with their children). In general, the third property requires a sort of consistency of the socialization patterns. It states that for any triple $i, j, k \in N$, if the socialization influence of j on child i is strictly smaller than that of k on child i, then it must not hold that k has a strictly larger socialization influence on child j than on child i (since $\sigma_{kj} = \sigma_{jk}$). This requirement can be interpreted as ruling out the existence of dynasties that have a 'too dominant' social learning influence on other dynasties.

In the case of symmetric ultrametric matrices Σ, any communication class $L \in \mathcal{P}(\Sigma)$ is essential due to the symmetry of Σ. For the following result, let $P_\Sigma(i) \subseteq N$ be such that $P_\Sigma(i) \in \mathcal{P}(\Sigma)$ and $i \in P_\Sigma(i)$ (the element of the partition $\mathcal{P}(\Sigma)$ which i belongs to). For some $n \times n$ matrix A and $J \subseteq N$, let A_J denote the matrix A restricted to the set of dynasties $J \subseteq N$. Finally, a consensus matrix is a row stochastic matrix where all rows are identical. We now get the following convergence result.

Proposition 4 (Convergence I)**.** *Let Assumptions 1 and 2 hold. If Σ is symmetric ultrametric, then $\lim_{t\to\infty} M(t,0)$ exists. Moreover, $\lim_{t\to\infty} M(t,0)_L = K(L)$ for all $L \in \mathcal{P}$, such that $K(L)$ is a consensus matrix, and $\lim_{t\to\infty} M(t,0)_{ij} = 0$ if and only if $j \notin P_\Sigma(i)$.*

Proof. In Appendix 5.3.

Endowing the social learning matrix Σ with sufficient structure, we thus arrive at a general result: In the long-run the communication classes of a society (these are the components of the social network) will end up with the same TIs. In the proof, we show first that each element $M(t)$ of the left product (4) is row stochastic. While it is straightforward to show that the rows of each $M(t)$ sum up to one, we make use of a number of linear algebra results on inverses of symmetric ultrametric matrices and inverse–positive matrices to show that $M(t)$ is positive.[17] Second, we can show that the entries of $M(t)$ corresponding to strictly positive entries of Σ can be bounded away from zero. This is due to the linearity of the determinants of the minors of $M(t)$ in all individual $b_i^*(t)$s, and the boundedness of E. In the last step, we construct a sequence of sub-accumulations of $M(t_{s+1}, t_s)_{s\in\mathbb{N}}$ such that for each element the minimal strictly positive entry can be uniformly bounded away from zero, which also implies type-symmetry and a strictly positive diagonal. For the sequence of sub-accumulations $M(t_{s+1}, t_s)_{s\in\mathbb{N}}$ we can then apply the convergence result by Lorenz (2005), which implies that the adopted TIs of each connected subset converge to the same point, respectively they reach a consensus.

Note finally that the necessity to guarantee that all $M(t)$ are row stochastic significantly reduces the convergence path types that we can analytically address. Basically, we have to restrict our glance to dynamics that are analogous to that obtained in the DeGroot–model. This follows since $M(t)$ row stochastic implies that sequence of TIs is such that all next–period TIs lie in the interval formed by the minimum and the maximum TI of the current period. However, the structure of our model is inherently more general.

3.3 The Dynamics of Cultural Traits with Quadratic Utility Functions

For the convergence result in Proposition 4, we needed the assumption of symmetric ultrametric interactions matrix Σ. This is mainly due to the generality of both utility components u_i and v_i. Suppose now that the own utility components is given by $u_i\left(\phi_i^d(t)\,|\phi_i(t)\right) = -\left(\phi_i^d(t) - \phi_i(t)\right)^2$ and the inter–generational utility component is given by $v_i\left(\phi_i(t+1)\,|\phi_i(t)\right) =$

[17] For literature on inverses of symmetric ultrametric matrices refer to Nabben and Varga (1993, 1994), Martinez et al. (1994), and for results on inverse–positive matrices see e.g. Fujimoto and Ranade (2004).

$-\beta_i\left(\phi_i(t+1)-\phi_i(t)\right)^2$. Assume further that all parents can unrestrictedly choose their displayed preference intensities ($\mathcal{I}=\mathbb{R}$), or in other words that the set of possible DTIs would be unbounded. Then, in every period $t\in\mathbb{N}$ the parents $i\in N$ face the *unrestricted* optimization problems

$$\min_{\phi_i^d(t)}\left(\phi_i^d(t)-\phi_i(t)\right)^2+\beta_i\left(\phi_i(t+1)-\phi_i(t)\right)^2. \tag{5}$$

From the first order conditions, it immediately follows that in this case $B^*(\cdot,\beta)=(\beta_1\sigma_{11},\ldots,\beta_n\sigma_{nn})'$. This has the consequence that $\forall t\in\mathbb{N}$, $B(t)=B=\operatorname{diag}(\beta_1\sigma_{11},\ldots,\beta_n\sigma_{nn})$. Thus,

$$M(t)=M=\Sigma\,(I+B\Sigma)^{-1}\,(I+B), \tag{6}$$

and finally

$$\Phi^*(t)=M^t\Phi(0).$$

Compared to our general representation, this form has a significant advantage: It transforms the problem of the convergence of the left–product of matrices into one of the convergence of the powers of one matrix.

Proposition 5 (a) and (b) give a (generically) sufficient and necessary condition on Σ to obtain convergence.

Proposition 5 (Convergence II)**.** *Let the parental optimization problems be as in (5). Then, the following holds.*

(a) If Σ is symmetric positive definite, then for every $\beta\in\mathbb{R}^n_+$ it holds that all eigenvalues of M are real and in the interval $(0,1]$ (with at least one eigenvalue equal to 1). Thus, generically $\lim_{t\to\infty}M^t\Phi(0)$ exists and is a steady state (for $\Phi(0)$ arbitrary).[18]

(b) Let Σ have a strictly positive diagonal. If for some eigenvalue λ of Σ we have $Re(\lambda)<|\lambda|^2$,[19] *then there is a $\beta\in\mathbb{R}^n_+$ such that the spectral radius of M is strictly larger than 1. Thus, the sequence $\left\{\Phi^*(t)=M^t\Phi(0)\right\}_{t\to\infty}$ does not converge (for $\Phi(0)$ arbitrary).*

Proof. In Appendix 5.4.

Proposition 5 (a) shows that symmetric positive definiteness (henceforth: PD) is generically sufficient for convergence. Even more, Σ PD guarantees that all eigenvalues of M are real and located in the interval $(0,1]$. For a symmetric matric a sufficient condition for positive definitieness is that Σ is strictly diagonally dominant, i.e. $\forall i\in N\ \Sigma_{ii}>\frac{1}{2}$. This means that each parent's influence on its child is larger than the influence of the social

[18] "Generically" applies to all cases where the geometric multiplicity of the 1–eigenvalue equals its algebraic multiplicity; see also Lemma 3 in Appendix 5.4.

[19] $Re(\lambda)$ means the real part of eigenvalue λ.

environment. Part (b) addresses matrices that are not PD and states the following necessary condition for convergence (subject to any β): $Re(\lambda) \geq |\lambda|^2$ for any eigenvalue λ of the matrix Σ, i.e. the real part of each eigenvalue is larger than the squared absolute value of this eigenvalue. (This property is violated, e.g. if the real part is negative.) Invoking symmetry of Σ, the property $Re(\lambda) \geq |\lambda|^2$ simplifies to $\lambda \geq \lambda^2$ and thus to $\lambda \in [0, 1]$ since symmetric matrices only have real eigenvalues. Moreover, Σ invertible implies that zero is not an eigenvalue of Σ such that Σ satisfies this property if and only if Σ is positive definite since symmetric matrices with strictly positive eigenvalues are PD (and eigenvalues larger than 1 are precluded by row stochasticity). Therefore, we get the following corollary for symmetric matrices Σ with a strictly positive diagonal: M^t (generically) converges if and only if Σ is positive definite.

As has been mentioned above, the present special case of our general model is basically a transformation of the DeGroot model. Given that convergence is satisfied in the latter, it is intuitive that we also obtain convergence if the transformation (as induced by the parental socialization incentives, which are embodied in β) is small enough. This is confirmed as follows.

Proposition 6 (Convergence III)**.** *Let the parental optimization problems be as in (5). Then, for every irreducible Σ with strictly positive diagonal, there exists a nonempty neighborhood $N(\mathbf{0}\,|\Sigma) \subset \mathbb{R}^n_+$,*[20] *such that $\forall \beta \in N(\mathbf{0}\,|\Sigma) \cup \mathbf{0}$, the sequence $\left\{\Phi^*(t) = M^t\Phi(0)\right\}_{t\to\infty}$ converges (for $\Phi(0)$ arbitrary).*

Proof. In Appendix 5.5.

In the proof of this Proposition, we show first that if Σ has a strictly positive diagonal, then it has a simple Perron–Frobenius eigenvalue of 1 where the absolute value of all other eigenvalues is located in the interval $(0, 1)$. Now, the eigenvalues are continuous in the underlying matrices. Thus, it must be possible to at least slightly perturb Σ such that the resulting matrix M also has a unique eigenvalue 1 with the absolute value of all other eigenvalues in the interval $(0, 1)$. Hence, M^t converges. Notably, this holds even though M might have negative entries.

Propositions 6 and 5 show that convergence either requires a special network structure (PD) or that socialization incentives are not too strong (β_i small). However, even if convergence is assured, the question remains at which rate a steady state is approached. Computing matrix powers after

[20] $N(\mathbf{0}\,|\Sigma)$ means that the size of the neighborhood around $\beta = \mathbf{0}$ depends on Σ.

diagonalizing a matrix shows that the speed of convergence is determined by the rate the power series of each eigenvalue approaches zero. The smaller the absolute eigenvalues (except eigenvalue 1), the higher the speed of convergence. To conclude this section we compare the speed of convergence of M^t with Σ^t. Since both matrices Σ and M have 1 as an eigenvalue (see Proposition 5), convergence speed is governed by the second largest eigenvalue. Let the eigenvalues of Σ and M be ordered according to size, i.e. $|\lambda_1(\Sigma)| > |\lambda_2(\Sigma)| \geq ... \geq |\lambda_K(\Sigma)|$ and $|\lambda_1(M)| > |\lambda_2(M)| \geq ... \geq |\lambda_K(M)|$, such that multiple eigenvalues may occur.[21] Then convergence of M^t is slower than convergence of Σ^t if $\lambda_2(M) > \lambda_2(\Sigma)$, which indeed holds, as is established by the following proposition.

Proposition 7 (Speed of Convergence)**.** *If Σ is symmetric positive definite and $\beta_i > 0$ for all $i \in N$, then the eigenvalues of M (which are real and positive) satisfy: $\lambda_k(M) > \lambda_k(\Sigma)$ for all $2 \leq k \leq K$.*

Proof. In Appendix 7.

We prove this result by applying a Theorem by Ostrowski (1959). In fact, it implies for any k that $\lambda_k(M) = \frac{\lambda_k(\Sigma)}{\lambda_k(\Sigma)+\theta_k(1-\lambda_k(\Sigma))}$, where θ_k is a real number in the interval $(0,1)$. More precisely, the boundaries of θ_k are $\min_{i\in N} \frac{1}{1+\sigma_{ii}\beta_i} \leq \theta_k \leq \max_{i\in N} \frac{1}{1+\sigma_{ii}\beta_i}$. The largest eigenvalue of Σ $\lambda_1(\Sigma) = 1$ leads to the largest eigenvalue of M, i.e. $\lambda_1(M) = 1$. For all other eigenvalues, we get the following.

Remark 3. *If the socialization incentives β_i approach zero for all $i \in N$, then θ_k approaches one and hence the eigenvalues of M approach the eigenvalues of Σ. Note that if $\beta_i = 0$ for all $i \in N$, we are back in the classical DeGroot model. The introduction of strategic interaction leads to overshooting and, as a consequence, to slower convergence for the strategic interaction case compared to the DeGroot model (Proposition 7). Moreover, if the socialization incentives β_i grow for all $i \in N$, then the eigenvalues of M approach 1, thus convergence speed becomes slower and slower.*

4 Conclusions

In this paper, we introduce a model of cultural transmission of traits within a finite population. Interaction ties are captured by a social network structure. In the related literature on cultural transmission of traits usually a continuous player set is assumed and interaction itself is global (see among

[21]Note that for irreducible matrices Σ and M there is always a unique largest eigenvalue. With our assumptions in Prop.7 we even show that all eigenvalues a real and positive of both Σ and M, and thus all inequalities are strict and the eigenvalues are equal to their absolute values.

others, Bisin and Verdier, 2010 and Pichler, 2010b). We show in this paper that not only the socialization incentives, but also the interaction structure matters for the question of whether a homogeneous society is observable in the long–run or not.

We identify necessary conditions on the network structure in the quadratic utility case (Proposition 5) and sufficient conditions in both the quadratic utility and the general utility case (Propositions 4 and 5). These conditions have a quite intuitive interpretation. If interaction is symmetric, then positive definiteness of the network is necessary and sufficient for convergence in the quadratic utility case. Positive definiteness is obtained, e.g. if parents have a stronger influence on their children than the social environment. Moreover, the speed of convergence is reduced by the parents' socialization efforts (Proposition 7).

This exercise yields three possible answers to the puzzle of the long–term persistence of heterogeneous cultural traits. First, if the interaction structure does not satisfy some special properties and if socialization incentives are strong enough, then the traits of a society do not converge (at all) to a steady state. Second, convergence to a steady state does not imply homogeneity of traits in the whole society but only within subgroups of it. (Those subgroups are determined by the interaction structure.) Third, even if convergence to a homogeneous society or subgroup is guaranteed, the convergence might happen at a very low pace (the speed of convergence depends on the interaction structure and it is reduced by the parents' socialization efforts) thereby matching empirical results of persistance of cultural traits.

To obtain a convergence result in the general case, we had to endow the social learning network with sufficient structure. For addressing more general convergence types of our model, we are limited by the insufficient availability of results on the convergence of the left–product of matrices that are not (in general) row–stochastic—and hope for more research on this issue in the future. Interestingly, our model is also very close to that of opinion formation dynamics of DeGroot (1974) and the succeeding literature. The opinion dynamics have been studied so far only with respect to truth telling, omitting the possibility of exaggerating as strategic choice in discussion. We show that the introduction of this kind of strategic interaction leads to cases of non-convergence while the opinion dynamics in DeGroot (1974) almost always converge. Hence, the conditions for convergence that we identify require more structure on the underlying network.

As benchmark model we studied the case of fixed interaction structure over time. Such an property is also assumed in models of opinion dynamics. While our model is robust to small vanishing perturbations on the interaction structure, it would be interesting to study also the network itself as a choice variable of parents of each generation. We leave this interesting question to further research.

5 Proofs

5.1 Proof of Proposition 2

From equation (1), it follows that $\forall i \in N$, $\phi_i(t+1)$ is concave in $\phi_i^d(t)$, thus also all $v_i\left(\phi_i(t+1)\,|\phi_i(t)\right)$ are concave in $\phi_i^d(t)$ (by Assumption 2). This implies that the target functions of the optimization problems of all parents are concave (and continuous). Since also the DTI choice set is compact and convex, a non–empty, upper hemicontinuous and convex set of DTI best replies exists for any parent (Berge's Theorem of the Maximum). Thus, a fixed point, i.e. a Nash equilibrium, exists (Kakutani's Fixed Point Theorem). □

5.2 Proof of Proposition 3

(a) That in any steady state, parents choose their adopted TI as DTI is directly implied by Proposition 1 (c).

(b) Given (a), it follows that the set of steady states given Σ coincides with the set $\{\Phi \in \mathcal{I}^n\,|\Sigma\Phi = \Phi\}$. Hence, it is immediate that if the TIs of all members of an essential communication class are identical, then $\Sigma_L\Phi_L = \Phi_L$, where Σ_L is the restriction of Σ to some essential communication class L, and Φ_L is its vector of adopted TIs. We proceed by showing that steady state TIs cannot differ within an essential communication class. To show a contradiction, suppose that for an essential communication class $L \in \mathcal{P}(\Sigma)$, $|L| \geq 2$, there exists $i, j \in L$ with $\phi_i \neq \phi_j$. Denote by $\bar{\phi}_L := max\{\phi_i | i \in L\}$ the maximal TI in communication class L. Since L is a communication class it follows that there exists an $i \in \{l \in L : \phi_l = \bar{\phi}_L\}$ and a $j \in \{l \in L | \phi_l \neq \bar{\phi}_L\}$ such that $\sigma_{ij} > 0$. Moreover, due to maximality of $\bar{\phi}_L$ and the fact that L is essential, $\sigma_{ik} = 0$ for all $k \in N$ with $\phi_k > \bar{\phi}_L$. Thus, $\Sigma_i\Phi_L \neq \phi_i$ implying that this cannot be a steady state.

(c) This is also straightforward. Suppose that for some inessential communication class $I \in \mathcal{P}(\Sigma)$ with connections to other dynasties $J := \{j \in N | i \to j,\ i \in I\}$ the set of TIs Φ_I is not included in $conv(\phi_j | j \in J)$. W.l.o.g. we have $\bar{\phi}_I := \max\{\phi_i | i \in I\} > \max\{\phi_j | j \in J\}$. Since I is a communication class and is inessential with all outside connections being to dynasties with TIs strictly less than $\bar{\phi}_I$, we get (similarly to (b)) for some player $k \in \{i \in I | \phi_i = \bar{\phi}_I\}$ that there exists $j \in N$ and $\phi_j < \bar{\phi}_I$ such that $\sigma_{kj} > 0$. Again, due to maximality of $\bar{\phi}_I$ and all other connections being to dynasties with TIs strictly less than $\bar{\phi}_I$, we get that $\Sigma_k\Phi_I \neq \phi_k$, implying that this cannot be a steady state. Hence, all TIs of the dynasties in inessential communication classes $I \in \mathcal{P}(\Sigma)$ are convex combinations of the TIs of the communication classes $J \in P(\Sigma)$ such that $I \to J$.

5.3 Proof of Proposition 4

This proof is organized in three essential steps. In the first step, we will show that if Σ is symmetric ultrametric, then $M(t)$ is row stochastic for every $t \in \mathbb{N}$. In the second step we will show that for every $i, j \in N$ with $\Sigma_{ij} > 0$ there exists a $\delta_{ij} > 0$ such that for every $t \in \mathbb{N}$, $m_{ij}(t) \geq \delta_{ij}$. We use these results to show in the third step that the backward accumulation matrices are type symmetric and have a strictly positive diagonal. This allows us to apply Theorem 2 of Lorenz (2005) to conclude that the desired convergence result holds. For the first step, we also need the following.

Lemma 1 (Unit Eigenvectors)**.** *Let Σ be positive definite. Then, $\forall x \in \mathbb{R}^n$, $\forall t \in \mathbb{N}$, $M(t)x = x$ iff $\Sigma x = x$ (i.e. x is a unit–eigenvector of $M(t)$ if and only if x is a unit–eigenvector of Σ).*

Proof. Note that $M(t) = \Sigma\left(I + B(t)\Sigma\right)^{-1}\left(I + B(t)\right) = \left(\Sigma^{-1} + B(t)\right)^{-1}(I + B(t))$. That the latter representation is well defined if Σ is positive definite follows since Σ is then invertible and also its inverse is positive definite. Thus, also $\Sigma^{-1} + B(t)$ is positive definite and invertible. Given this, both the 'if' and the 'only if' direction of the proof can be directly seen from the following sequence of transformations: $\Sigma x = x \Leftrightarrow x = \Sigma^{-1}x \Leftrightarrow (B(t) + I)x = (B(t) + \Sigma^{-1})x \Leftrightarrow M(t)x = (B(t) + \Sigma^{-1})^{-1}(B(t) + I)x = x$. $\square$

1. In the first step of the (main) proof, we show that if Σ is symmetric ultrametric, then $M(t)$ is row stochastic for every $t \in \mathbb{N}$. To do so, note first that since Σ is symmetric ultrametric, it is also positive definite (see below). Hence, by Lemma 1 (and setting $x = (1, 1, ..., 1)'$) the row entries of $M(t) = [m_{ij}(t)]$ sum up to one since the same holds for Σ. Thus, $M(t)$ is row stochastic if and only if $M(t)$ has non-negative entries (that is $M(t) \geq 0$). Now, since $I + B(t)$ is a diagonal matrix with strictly positive entries, $M(t) = \Sigma\left(I + B(t)\Sigma\right)^{-1}\left(I + B(t)\right)$ is non-negative if and only if

$$\Sigma\left(I + B(t)\Sigma\right)^{-1} = \left(\Sigma^{-1} + B(t)\right)^{-1}$$

is non-negative (that this representation is well defined if Σ is positive definite has been discussed in the proof of Lemma 1). In other words, we have to check whether $\Sigma^{-1} + B(t)$ is inverse–positive.

Now, since Σ is symmetric ultrametric, it follows that its inverse is a diagonally dominant Stieltjes matrix (see Nabben and Varga (1993, 1994), Martinez et al. (1994)), i.e. a real symmetric positive definite matrix with positive diagonal and negative off-diagonal entries. Thus, also $\Sigma^{-1} + B(t)$ is a diagonally dominant Stieltjes matrix. In particular, it is an M–matrix, the class of which is inverse–positive (on this issue, see e.g. Fujimoto and Ranade (2004)). Hence, $M(t)$ has only non-negative entries.

2. For the second step, we show first, that the map $b_i^*(t) = b_i^*(\phi_i(t), \phi_{N_i}^d(t))$ is bounded for every $i \in N$.

Lemma 2 (Boundedness of $B^*(t)$)**.** *Let Assumptions 1–2 hold. Then,* $\forall i \in N$ b_i^* *is bounded for every* $\phi_{N_i}^d(t), \phi(t) \in \mathcal{I}$*. In particular,*

$$\lim_{\phi_{N_i}^d(t)\to\phi_i(t)} b_i^*(\phi_i(t), \phi_{N_i}^d(t)) = \frac{\sigma_{ii} v_i''\big(\phi_i(t)|\phi_i(t)\big)}{u_i''(\phi_i(t)|\phi_i(t)) + \sigma_{ii}^2 v''\big(\phi_i(t)|\phi_i(t)\big)} < \infty.$$

Proof. Note that for $x := \phi_i(t)$, $y := \phi_{N_i}^d(t)$, and $f(x,y) := \phi_i^{d^*}(x,y)$ b_i^* is defined by

$$f(x,y) - x = b_i^*(x,y)\left((1-\sigma_{ii})x - (1-\sigma_{ii})y\right). \tag{7}$$

Let $x \in \mathcal{I}$ be given and without loss of generality assume that $y \geq x$. First, note that for every $y \in \mathcal{I}$ such that $x \neq y$ it holds by Proposition 1 that $0 \leq b_i^*(x,y) \leq \frac{1}{1-\sigma_{ii}} \frac{x - x_{min}}{x-y}$, where $x_{min} := \min\{z \in \mathcal{I}\}$. Further, by Proposition 1 we get for $\sigma_{ii} = 1$ that $b_i^*(x,y) = 0$ for all $y > x$.

Hence we are left to show that $\lim_{y\downarrow x} b_i^*(x,y) < \infty$ for $\sigma_{ii} < 1$. Since x is fixed, we denote $f(y) := f(x,y)$, abusing notation. We get from (7),

$$\lim_{y\downarrow x} b_i^*(x,y) = \lim_{y\downarrow x} \frac{1}{1-\sigma_{ii}} \frac{f(y)-x}{x-y} = -\frac{1}{1-\sigma_{ii}} f'(x),$$

given differentiability of f at the point x, which we show subsequently. By the first order condition, $f(y)$ solves $u_i'(f(y)|x) + \sigma_{ii} v_i'(\sigma_{ii}(f(y) + (1-\sigma_{ii})y)|x) = 0$. With the implicit function theorem,

$$f'(x) = -\frac{(1-\sigma_{ii})\sigma_{ii} v_i''\big(\sigma_{ii}(f(x) + (1-\sigma_{ii})x)|x\big)}{u_i''(f(x)|x) + \sigma_{ii}^2 v''\big(\sigma_{ii}(f(x) + (1-\sigma_{ii})x)|x\big)}. \tag{8}$$

By Proposition 1, we have $f(x) = x$, and hence by Assumption 1 the right hand side is well defined. We get

$$\begin{aligned} \lim_{y\downarrow x} b_i^*(x,y) &= -\frac{1}{1-\sigma_{ii}} f'(x) \\ &= \frac{\sigma_{ii} v_i''\big(x|x\big)}{u_i''(x|x) + \sigma_{ii}^2 v''\big(x|x\big)}, \end{aligned}$$

which is by Assumption 2 positive and bounded.

□

Now we continue to show that for every $i, j \in N$ with $\sigma_{ij} > 0$ there exists a $\delta_{ij} > 0$ such that $m_{ij}(t) \geq \delta_{ij}$ for every $t \in \mathbb{N}$. Again, since $I + B(t)$ is a diagonal matrix with strictly positive entries, we can restrict our attention to the matrix $\left(\Sigma^{-1} + B(t)\right)^{-1} =: A(t) = [a_{ij}(t)]$. Consider any $i, j \in N$

such that $\sigma_{ij} > 0$. Since $A(t)$ is non-negative by step (1), it follows that $\operatorname{sign}(a_{ij}(t)) \in \{0, \operatorname{sign}(\sigma_{ij})\}$.

Let us rule out the case $\operatorname{sign}(a_{ij}(t)) = 0$ for $\sigma_{ij}(t) > 0$. To do so, let us compare

$$a_{ij}(t) = (-1)^{i+j} \frac{\left|\Sigma^{-1} + B(t)\right|_{ij}}{|\Sigma^{-1} + B(t)|} \quad \text{vs.} \quad (-1)^{i+j} \frac{\left|\Sigma^{-1}\right|_{ij}}{|\Sigma^{-1}|} = \sigma_{ij}$$

where $|\cdot|_{ij}$ denotes the determinant of the ij adjoint matrix. Note that since Σ is positive definite, the same holds for its inverse and $\Sigma^{-1} + B(t)$. It follows that the determinants of the matrices Σ^{-1} and $\Sigma^{-1} + B(t)$ are strictly positive.

Moreover, note that for all $i, j \in N$, $\left|\Sigma^{-1} + \operatorname{diag}(b_1^*(t), \ldots, b_n^*(t))\right|_{ij}$ and $\left|\Sigma^{-1} + \operatorname{diag}(b_1^*(t), \ldots, b_n^*(t))\right|$ are linear in every individual element of $\{b_1^*(t), \ldots, b_n^*(t)\}$ (to verify this most easily, consider the Leibniz formula). Since we have $|\Sigma^{-1} + B(t)| \geq 0$ for all $b_1^*(t), \ldots, b_n^*(t) \geq 0$, it holds that

$$\frac{\partial (-1)^{i+j} |\Sigma^{-1} + B(t)|_{ij}}{\partial b_k} \geq 0 \tag{9}$$

and

$$\frac{\partial |\Sigma^{-1} + B(t)|}{\partial b_k} \geq 0, \tag{10}$$

because otherwise the determinant would switch signs due to linearity in b_k, for all $k \in \{1, ..., n\}$.

Now, since b_i^* is bounded by Lemma 2, we have $b_k(t) \in [0, \bar{b}]$ for all $t \in \mathbb{N}$. By linearity of $|\Sigma^{-1} + B(t)|_{ij}$ and $|\Sigma^{-1} + B(t)|$ in $b_k(t)$ for all $k \in \{1, ..., n\}$ and compactness of $[0, \bar{b}]$, we thus get existence of a minimum:

$$\hat{\delta}_{ij} := \min_{k \in \{1, \ldots, n\}} \min_{b_k(t) \in [0, \bar{b}]} = (-1)^{i+j} \frac{\left|\Sigma^{-1} + B(t)\right|_{ij}}{|\Sigma^{-1} + B(t)|} \leq (-1)^{i+j} \frac{\left|\Sigma^{-1}\right|_{ij}}{|\Sigma^{-1}|}$$

Note that $0 < \hat{\delta}_{ij}$ since both of nominator and denominator are bounded and strictly positive due to (9) and (10) and because of boundedness of b_k.

Thus, if $\sigma_{ij} > 0$ then $a_{ij}(t) \geq \hat{\delta}_{ij}$ for all $t \in \mathbb{N}$. Multiplication with the diagonal matrix $I + B(t)$ does not change this fact, even though the minimum might be attained at different values of $b_k \in [0, \bar{b}]$ and $k \in \{1, ..., n\}$. Thus, for all $i, j \in \mathbb{N}$ such that $\sigma_{ij} > 0$ there exists a $\delta_{ij} > 0$ such that $m_{ij}(t) \geq \delta_{ij}$ for all $t \in \mathbb{N}$.

3. In the last step, we show that given the above, the left product of the matrices $M(t)M(t-1)\ldots M(0)$ converges such that the adopted TIs of all

dynasties of a connected subset are identical (respectively, the communication classes in $\mathcal{P}(\Sigma)$ reach a consensus). Note that all communication classes of Σ are essential by symmetry of Σ. By the definition of $\mathcal{P}(\Sigma)$ we have that for all $L \in \mathcal{P}(\Sigma)$ and for all $i, j \in L$ there exists a $k \in \{0, ..., |L|\}$ such that $\Sigma^k_{ij} > 0$. Note that $\mathcal{P}(\Sigma) = \mathcal{P}(M(t))$ for all $t \in \mathbb{N}$ since $\sigma_{ij} > 0$ implies $m_{ij}(t) \geq \delta$ for all $t \in \mathbb{N}$ as shown above and, since every communication class of Σ is essential, $m_{ij}(t) = 0$ if $j \notin P_\Sigma(i)$.[22] Hence, for all $L \in \mathcal{P}(\Sigma)$ and for all $i, j \in L$ there exists a $k \in \{0, ..., |L|\}$ such that $M(t + k, t)_{ij} > 0$ for all $t \in \mathbb{N}$.[23]

Now, consider a sequence of time steps $(t_s)_{s\in\mathbb{N}}$ such that $t_0 = 0$ and $t_{s+1} = t_s + \bar{L}$, where $\bar{L} := \max\{|L| : L \in P(M)\}$, and consider the sequence of accumulations $\big(M(t_{s+1}, t_s)\big)_{s\in\mathbb{N}}$. By the rules of matrix multiplication, we get that for any two row stochastic A, B with a positive diagonal, $(AB)_{ij} > 0$ if and only if $A_{ij} > 0$ or $B_{ij} > 0$. Hence, for any $L \in \mathcal{P}(\Sigma)$ and for all $i, j \in L$, $M(t + |L|, t)_{ij} > 0$ for all $t \in \mathbb{N}$ since $M(t)$ is row stochastic with a positive diagonal. Moreover, $M(t + |L|, t)_{ij} = 0$ if $j \notin P_\Sigma(i)$ since $\mathcal{P}(\Sigma) = \mathcal{P}(M(t))$ for all $t \in \mathbb{N}$. Thus, for the accumulations $M(t_{s+1}, t_s)$ it holds that $M(t_{s+1}, t_s)_{ij} > 0$ if and only if $j \in \mathcal{P}_\Sigma(i)$. In particular, $M(t_{s+1}, t_s)$ is type-symmetric for all $s \in \mathbb{N}$.

For a non-negative matrix A let $\min^+(A)$ denote the lowest positive entry of A. We have shown above that there exists a $\delta > 0$ such that $\sigma_{ij} > 0$ implies $m_{ij}(t) \geq \delta$ for all $t \in \mathbb{N}$. Note that for any $i, j \in L \in \mathcal{P}(\Sigma)$ there exists a $k \leq |L|$ and a sequence of dynasties $(i_l)_{0\leq l\leq k}$ with $i_0 = i$ and $i_k = j$ such that $\sigma_{i_l, i_{l+1}} > 0$, implying $M(t+k, t)_{ij} \geq \prod_{l=0}^{k-1} m_{i_l, i_{l+1}}(t+l) \geq \delta^k$. Thus, for the accumulations $M(t_{s+1}, t_s)$ it holds that $M(t_{s+1}, t_s)_{ij} \geq \delta^{t_{s+1}-t_s}$ if $j \in P_\Sigma(i)$ and $M(t_{s+1}, t_s)_{ij} = 0$ else. Hence, $\min^+\big(M(t_{s+1}, t_s)\big) \geq \delta^{t_{s+1}-t_s} = \delta^{|\bar{L}|}$.

In summary, we have shown that the backward accumulation matrices $\big(M(t_{s+1}, t_s)\big)_{s\in\mathbb{N}}$ have a uniform lower bound of the positive entries $min^+\big(M(t_{s+1}, t_s)\big) \geq \delta^{|\bar{L}|}$, are type symmetric and have a strictly positive diagonal. By Lorenz (2005), Theorem 2, we get the desired result for the sequence $\big(M(t_{s+1}, t_s)\big)_{s\in\mathbb{N}}$. Since $\lim_{k\to\infty} \prod_{s=0}^{k} M(t_{s+1}, t_s) = \lim_{t\to\infty} M(t)$, we also establish the statement of the Proposition. □

5.4 Proof of Proposition 5

For both parts of the proposition, we will apply the following Lemma (see e.g. Friedberg and Insel (1992)).

Lemma 3 (Convergence)**.** *Let A be a square matrix with complex or real entries. Then, the sequence $\left\{A^t\right\}_{t\to\infty}$ converges if and only if the following*

[22]Recall, $P_\Sigma(i) \subseteq N$ is such that $P_\Sigma(i) \in \mathcal{P}(\Sigma)$ and $i \in P_\Sigma(i)$ (the element of the partition $\mathcal{P}(\Sigma)$ which i belongs to).

[23]Recall that $M(t', t)$ denotes the accumulation $M(t', t) = M(t')M(t' - 1) \dots M(t)$.

two conditions are satisfied.

(i) If λ is an eigenvalue of A, then either $\lambda = 1$ or λ lies in the open unit disc of the complex plane, i.e. $|\lambda| \in (-1, 1)$.

(ii) If 1 is an eigenvalue of A, then its algebraic multiplicity equals its geometric multiplicity.

Let us denote by $\Lambda(A)$, the set of eigenvalues of a matrix A and let $\lambda(A) \in \Lambda(A)$. Moreover, if z is a complex number, then we denote by $Re(z)$ the real part and by $Im(z)$ the imaginary part of z.

Proof of part (a) We will show that condition (i) of Lemma 3 is satisfied. To see this, note first that by definition $M = \Sigma(I + B\Sigma)^{-1}(I + B) = (B + \Sigma^{-1})^{-1}(I + B)$,[24] which implies that $M^{-1} = (I + B)^{-1}(B + \Sigma^{-1})$. Let $\tilde{B} := (I + B)^{-1}$, i.e. for every $i \in N$, $\tilde{b}_{ii} = \frac{1}{1+\sigma_{ii}\beta_i}$, hence $\tilde{B}$ is a diagonal matrix with entries in $(0, 1)$. Then, $\tilde{B}B = I - \tilde{B}$, and

$$M^{-1} = \tilde{B}(B + \Sigma^{-1}) = I - \tilde{B} + \tilde{B}\Sigma^{-1} = I + \tilde{B}(\Sigma^{-1} - I). \qquad (11)$$

First, note that since Σ is assumed to be symmetric positive definite, so is Σ^{-1} and $(\Sigma^{-1} - I)$ and the eigenvalues of all matrices are real and positive.

Second, the matrices $\tilde{B}(\Sigma^{-1} - I) = \tilde{B}^{1/2}[\tilde{B}^{1/2}(\Sigma - I)]$ and $\tilde{B}^{1/2}(\Sigma^{-1} - I)\tilde{B}^{1/2}$ have the same eigenvalues,[25] where $\tilde{B}^{1/2}$ is the diagonal matrix with entries $\left(\tilde{B}^{1/2}\right)_{ii} = \sqrt{\tilde{b}_{ii}}$. Moreover it is easily checked that $\tilde{B}^{1/2}\left(\Sigma^{-1} - I\right)\tilde{B}^{1/2}$ is positive definite and symmetric, i.e. has only positive real eigenvalues. Thus, also the eigenvalues of M^{-1} (and hence those of M) are real and positive.

Now, since Σ is row stochastic, we have $|\lambda(\Sigma)| \leq 1$, which implies that $\lambda(\Sigma^{-1}) \geq 1$. Thus, $\lambda(\Sigma^{-1} - I) \geq 0$ (subtraction of I decreases all eigenvalues by 1). By above, we have $\lambda(\tilde{B}(\Sigma^{-1} - I)) \geq 0$, which implies $\lambda(I + \tilde{B}(\Sigma^{-1} - I)) \geq 1$, i.e. $\lambda(M^{-1}) \geq 1$, and hence all eigenvalues of M are real and located in the interval $(0, 1]$. Furthermore, since M has row sum one (see Lemma 1, using $x = (1, 1, ..., 1)'$), at least one eigenvalue must be equal to 1. Thus, M^t converges, i.e. $M^\infty := \lim_{t\to\infty} M^t$ exists, and since 1 is an eigenvalue of M, $M^\infty \neq \mathbf{0}$. Denoting $\Phi(\infty) := M^\infty\Phi(0)$ it is easy to see that $\Phi(\infty)$ is a steady state since $M\Phi(\infty) = MM^\infty\Phi(0) = M^\infty\Phi(0) = \Phi(\infty)$.

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[24] That this representation is well defined if Σ is positive definite has been discussed in footnote 16.

[25] This holds since for any two $n \times n$ matrices A, B the eigenvalues of AB are the same as the eigenvalues of BA, although the eigenvectors may differ.

Proof of Part (b) Let Σ have a strictly positive diagonal and let there be an eigenvalue $\tilde{\lambda}(\Sigma)$ that satisfies $Re(\tilde{\lambda}(\Sigma)) < |\tilde{\lambda}(\Sigma)|^2$. The latter is equivalent to $Re(\tilde{\lambda}^{-1}(\Sigma)) < 1$, simply because $z^{-1} = \frac{Re(z)}{Re^2(z)+Im^2(z)} + \frac{-Im(z)}{Re^2(z)+Im^2(z)}i$ and $|z|^2 = Re^2(z) + Im^2(z)$ for any complex number $z \in \mathbb{C}$. Note that $\tilde{\lambda}^{-1}(\Sigma)$ is an eigenvalue of Σ^{-1}. Now let for each $i : \beta_i = \frac{k}{\sigma_{ii}}$, $k \in \mathbb{R}$, so that $B = kI$. We show that if k is large enough, then M has an eigenvalue with absolute value larger than 1 and hence condition (i) of Lemma 3 is violated.

To do so, we will use $M^{-1} = (I + B)^{-1}(B + \Sigma^{-1}) = (I + kI)^{-1}(kI + \Sigma^{-1}) = ((1+k)I)^{-1}(kI + \Sigma^{-1}) = \frac{1}{1+k}(kI + \Sigma^{-1})$. Now, since $Re(\tilde{\lambda}(\Sigma^{-1})) = Re(\tilde{\lambda}^{-1}(\Sigma)) < 1$, we have $Re(\tilde{\lambda}(kI + \Sigma^{-1})) < 1 + k$, because $\tilde{\lambda}(kI + \Sigma^{-1}) = k + \tilde{\lambda}(\Sigma^{-1})$. For k large enough, we must have $|\tilde{\lambda}(kI + \Sigma^{-1})| < 1 + k$.[26] To see that this must hold, denote $\epsilon := 1 - Re(\tilde{\lambda}(\Sigma^{-1}))$ and we get:

$$\begin{aligned}
|\tilde{\lambda}(kI + \Sigma^{-1})|^2 &= Re^2(\tilde{\lambda}(kI + \Sigma^{-1})) + Im^2(\tilde{\lambda}(kI + \Sigma^{-1}) \\
&= (1 - \epsilon + k)^2 + Im^2(\tilde{\lambda}(\Sigma^{-1})) \\
&= (k + 1)^2 + Im^2(\tilde{\lambda}(\Sigma^{-1})) + \epsilon^2 - 2\epsilon - 2\epsilon k,
\end{aligned}$$

which is smaller than $(1 + k)^2$ for $k > \frac{Im^2(\tilde{\lambda}(\Sigma^{-1}))+\epsilon^2-2\epsilon}{2\epsilon}$. Thus, we get for k large enough,

$$\frac{1}{1+k}|(\tilde{\lambda}(kI + \Sigma^{-1}))| = \left|\tilde{\lambda}\left(\frac{1}{1+k}(kI + \Sigma^{-1})\right)\right| = |\tilde{\lambda}(M^{-1})| < 1$$

and hence $|\tilde{\lambda}(M)| > 1$ so that condition (i) of Lemma 3 is violated. $\square$

5.5 Proof of Proposition 6

As by Lemma 3 above, for the convergence of the powers of a matrix A it is sufficient that 1 is exactly one eigenvalue of A and all other eigenvalues are in the interval $(-1, 1)$. To prove the proposition, we will in a first step apply the Perron-Frobenius Theorem (henceforth: PFT) for a regular row–stochastic matrix A: (i) The spectral radius of A is 1 (the largest eigenvalue in absolute value). (ii) For all other eigenvalues λ it holds that $|\lambda| < 1$. (iii) The eigenvalue 1 is simple.Consider any row stochastic Σ such that Σ is irreducible with strictly positive diagonal. This implies that Σ is regular, so that by the PFT for regular row stochastic matrices, Σ has simple eigenvalue 1 and all other eigenvalues are in $(-1, 1)$.

Let us now consider the transformations $M = \Sigma\,(I + B\Sigma)^{-1}\,(I+B)$. In a first step, we do have to guarantee that $I+B\Sigma$ is invertible, so that M exists. Note that strict diagonal dominance would be sufficient for non–singularity. For strict diagonal dominance, we require that $1 + \beta_i \left(\sigma_{ii} - \sum_{j\in N_i} \sigma_{ij}\right) > 0$ holds for every $i \in N$. Since Σ has a strictly positive diagonal, this is always satisfied if e.g. $\beta \leq \mathbf{1}$.

[26] If $\tilde{\lambda}^{-1}(\Sigma)$ is a real number, then this holds trivially.

Given this, it follows again by the continuity of the eigenvalues that there exists a non–empty neighborhood $N(\mathbf{0}\,|\Sigma) \subset \mathbb{R}^n_+$ such that $\forall \beta \in N(\mathbf{0}\,|\Sigma) \cup \mathbf{0}$ both $I+B\Sigma$ is strictly diagonally dominant and M has exactly one eigenvalue equal 1 and $n-1$ eigenvalues in the interval $(-1,1)$. Thus, M^t converges. $\square$

5.6 Proof of Proposition 7

For positive definite symmetric and row stochastic matrices Σ convergence of Σ^t, $t \to \infty$ is trivially implied and generic convergence of M^t, $t \to \infty$ is already established by Proposition 5. To show that convergence of M^t is slower than convergence of Σ^t for $t \to \infty$ we show that all eigenvalues of M are real and $\lambda_k(\Sigma) < \lambda_k(M)$ for all $2 \le k \le K$.

By (11) we have that $M^{-1} = I + \tilde{B}(\Sigma^{-1} - I)$ with $\tilde{B}$ being a diagonal matrix with entires $0 < \tilde{b}_{ii} < 1$ for all $\beta_i > 0$. Thus,

$$\lambda_k(M^{-1}) = 1 + \lambda_k(\tilde{B}(\Sigma^{-1} - I)) = 1 + \lambda_k(B^{1/2}(\Sigma^{-1} - I)B^{1/2}).$$

where the latter equality is again due to the fact that $\tilde{B}(\Sigma^{-1} - I)$ and $\tilde{B}^{1/2}(\Sigma^{-1} - I)\tilde{B}^{1/2}$ have the same eigenvalues. Moreover, by the proof of Proposition 5 we have that $(\Sigma^{-1} - I)$ and $\tilde{B}^{1/2}(\Sigma^{-1} - I)\tilde{B}^{1/2}$ are symmetric and positive definite.

Since $\left(\tilde{B}^{1/2}\right)^* = \tilde{B}^{1/2}$ and $\tilde{B}^{1/2}$ is non–singular,[27] and $(\Sigma^{-1} - I)$ is symmetric, we get by Theorem 1 in Ostrowski (1959) (see also Horn and Johnson (2010)) that $\lambda_k(\tilde{B}^{1/2}(\Sigma^{-1} - I)\tilde{B}^{1/2}) = \theta_k \lambda_k(\Sigma^{-1} - I)$, where θ_k are real numbers such that $\lambda_K(\tilde{B}^{1/2}\tilde{B}^{1/2}) \le \theta_k \le \lambda_1(\tilde{B}^{1/2}\tilde{B}^{1/2})$. Since $\tilde{B}^{1/2}\tilde{B}^{1/2} = \tilde{B}$ is diagonal with entries $0 < \tilde{b}_{ii} < 1$ it holds that $\lambda_k(\tilde{B}^{1/2}(\Sigma^{-1} - I)\tilde{B}^{1/2}) < \lambda_k(\Sigma^{-1} - I)$ for all k such that $\lambda_k(\Sigma^{-1} - I) > 0$. The latter is satisfied for $\lambda_k(\Sigma) < 1$, and thus for all $\lambda_k(\Sigma)$ such that $2 \le k \le K$.

Hence, for $2 \le k \le K$:

$$\begin{aligned} \lambda_k(M^{-1}) &= 1 + \lambda_k\left(\tilde{B}(\Sigma^{-1} - I)\right) \\ &< 1 + \lambda_k\left(\Sigma^{-1} - I\right) = 1 + \lambda_k(\Sigma^{-1}) - 1 = \lambda_k(\Sigma^{-1}), \end{aligned}$$

which implies that $\lambda_k(M) > \lambda_k(\Sigma)$ for all $2 \le k \le K$ proving the statement and implying that convergence of M^t is slower than Σ^t for $t \to \infty$.

[27] The asterisk denotes the complex conjugate transpose.

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