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Distorted Voronoi Languages†

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Abstract

In a recent paper, Jäger, Metzger, and Riedel (2011) study communication games of common interest when signals are simple and types complex. They characterize strict Nash equilibria as so–called Voronoi languages that consist of Voronoi tesselations of the type set and Bayesian estimators on the side of receivers. In this note, we introduce conflicts of interest in the same setting. We characterize strict Nash equilibria as *distorted Voronoi languages* that use all messages. For large conflicts, such informative equilibria need not exist. If the bias is sufficiently small, however, these equilibria do exist. This establishes the robustness of the results in Jäger, Metzger, and Riedel (2011) to biased interests. We finally give examples of strict Nash equilibria, one of them using simulations to illustrate an equilibrium with many messages and non-uniformly distributed types.

JEL subject classification: C72, D82, D83

Key words: Cheap Talk, Signaling Game, Communication Game, Voronoi tesselation, Conflict of Interest

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1 Introduction

In all kinds of situations of everyday life we rely on the decisions of others and use communication to enforce our interests. If interests are common, the sender tries to communicate her information as exact as possible. Issues like this have been widely discussed.[1] Recently, Jäger, Metzger, and Riedel (2011) study a model where the complexity of the information to be communicated is much higher than the number of messages or signals available to the sender.[2] In such a situation, one cannot have perfect separation by design. However, the authors show that strict Nash equilibria are as separating as possible, in a sense. The sender separates the type set into (usually convex) cells, a Voronoi tesselation, and the receiver interprets the signal corresponding to that cell as the best Bayesian estimator given that the type lies in that cell.

One might wonder whether such equilibria can survive if the interests of sender and receiver are not perfectly aligned. For economic applications, this is quite relevant as in many situations we are confronted with different interests, see the classic discussion in Crawford and Sobel (1982). We add some own points here. The conflict might be viewed as only a small difference between friends deciding about the restaurant to go to or as a rather huge difference between competing firms agreeing on some technical standard. To give some intuition for the behavior in such situations we consider the following example: A politician, acting as the sender, wants to win an election and therefore she needs to convince the electorate that her electoral platform is the one to vote for. Imagine that her party wants to take measures that are not popular, for example a raise of income or gasoline taxes. The electorate usually wants to get a clear prospect of what the parties want to do in case they win the election and thus there is a conflict of interest. In order to win the election, the politician might not communicate the information about the measures or certain circumstances associated with them as exact as possible, instead she might communicate a somehow different information that she thinks is advantageous for her.

Apart from that we can also interpret a small conflict of interest as a bias

[1] The seminal paper on cheap talk games as we study them here is Crawford and Sobel (1982). Robson (1990), Matsui (1991), Schlag (1993), Sobel (1993), Blume, Kim, and Sobel (1993), Wärneryd (1993), Rubinstein (2000), or Trapa and Nowak (2000) are other prominent examples.

[2] Blume and Board (2010) extend the model to allow for different degrees of language competence assuming that the knowledge of messages is private information.

due to small mistakes or slightly wrong perceptions of the players.

In this note, we allow biased interests in the model of Jäger, Metzger, and Riedel (2011). We study a cheap-talk game - or more general a signaling game - where the set of possible private information of the sender (called types) is infinite and in particular, high-dimensional, whereas the number of signals or messages - we usually refer to them as words - is only finite and, furthermore, interests are biased. This means that the sender has to aggregate many types under one word, i.e. perfect separation is not possible. After sending a word, the receiver then needs to interpret it, that is she needs to choose a prototype for the word. For example consider different weather conditions, there are literally infinite different rain intensities, but if we talk about such conditions, we usually only use very few terms like "light rain" or "shower". The interpretation then can be seen as the typical "light rain" or "shower". We call an assignment of the types to words for the sender together with interpretations of the words for the receiver a *language*.

In our setup, the sender has a certain bias $m \neq 0$ (a bias on the receiver's side can be reduced to this case as well by a shift of the type space). One might guess that conflicts of interests change the game quite a bit. Indeed, in Crawford and Sobel (1982), the bias changes the type of equilibria dramatically as we go from perfectly separating equilibria to partial information disclosure only, to give an example. In our case, the strict Nash equilibria remain qualitatively close to the Voronoi languages of Jäger, Metzger, and Riedel (2011). Indeed, we characterize strict Nash equilibria as *distorted Voronoi languages* that use all messages. In such a language, the sender uses her distortion shift m to determine the best signal to send. The receiver still uses a best Bayesian estimator.

For large conflicts, such informative equilibria need not exist as we show by example. This is plausible: large biases kill the incentives to align interests and thus preclude informative equilibria. If the bias is sufficiently small, however, these equilibria do exist. This establishes the robustness of the results in Jäger, Metzger, and Riedel (2011) to biased interests. Therefore, our result suggests that small conflicts cannot harm communication much. We finally give examples of strict Nash equilibria, one of them using simulations to illustrate an equilibrium with many messages and non-uniformly distributed types.

The paper is set up as follows. We introduce our model and notation first. In section 3 we study the behavior in equilibrium. Section 4 concludes.

2 Model and Notation

Let T, the type set, be a convex and compact subset of $\mathbb{R}^L$ with nonempty interior for some $L \geq 1$. The probability of types is described by an atomless distribution F on T with strictly positive and continuous density $f : T \to \mathbb{R}_+$. The sender chooses a word (signal) $w \in W := \{w_1, \dots, w_N\}$ from a finite vocabulary and sends it to the receiver. A sender's strategy is thus a measurable function $w : T \to W$. The receiver interprets the word w_j as some point $i_j \in \mathbb{R}^L$. Note that we allow interpretations to lie outside the type space here. With conflicting interests, we do not want to restrict the receiver's choices.

In contrast to Jäger, Metzger, and Riedel (2011) we do not assume that both players have common interests. Instead, we model the conflict of interest by a vector $m \neq 0$ in $\mathbb{R}^L$, the sender's bias. Let $l : \mathbb{R}_+ \to \mathbb{R}$ be continuous, convex and strictly increasing, and $\|\cdot\|$ be a norm on T. The expected loss or payoff of the sender (S) and the receiver (R), respectively, is then

$$L_S(w, i) := \int_T l\left(\left\|t + m - i_{w(t)}\right\|\right) F(dt)$$

and

$$L_R(w, i) := \int_T l\left(\left\|t - i_{w(t)}\right\|\right) F(dt).$$

One might ask why we model a bias on the sender's side only. Indeed, this comes without loss of generality. By a suitable translation of the type space, one can reduce the game with a receiver's bias (and also the game with biases on both sides) to the above situation, see the appendix.

3 Distorted Voronoi Languages

Recall from Jäger, Metzger, and Riedel (2011) that a language (w, i) consists of a measurable mapping $w : T \to W$ (the signaling strategy) and points $i \in (\mathbb{R}^L)^N$ (the interpretation). A language (w, i) has full vocabulary if $\operatorname{range} w = W$. A particular role in our context play Bayesian estimates of the type given that we are in a specific subset of the type set. For a subset $C \subseteq T$ with positive measure, we call $b(C) = \operatorname{argmin}_{i \in T} \int_C l\left(\|t - i\|\right) F(dt)$ the Bayesian estimator conditional on C. We know from Jäger, Metzger, and

Riedel (2011) that this estimator is unique. For quadratic loss and Euclidean norm, the Bayesian estimator is the conditional mean type.

Jäger, Metzger and Riedel show that strict Nash equilibria are given by Voronoi languages with full vocabulary: a Voronoi language (w, i) consists of a Voronoi tesselation for the sender and a Bayesian estimator interpretation for the receiver[3], i.e. we have both

$$w^*(t) = \operatorname{argmin}_{j=1,\dots,N} \|t - i_j^*\| \; F - a.s. \tag{1}$$

$$i_k = b(C_k^*) \qquad (\text{ for } C_k^* = \{t \in T : w^*(t) = w_k\})\,. \tag{2}$$

Note that the two players work nicely together in such an equilibrium. They try to coordinate their actions to produce an equilibrium that is as separating as possible. With conflict of interests, it is not clear that such a result would be attainable, at least to the second of these authors. We show now that the characterization of strict Nash equilibria generalizes indeed to our case of a biased sender.

Suppose that the receiver uses pairwise distinct interpretations $i_1, \dots, i_N$. Then for a sender of type t, it is optimal to use the signal w_j that leads to a minimal distorted distance, i.e. she chooses w_j such that

$$\|t + m - i_j\| = \min_{k=1,\dots,N} \|t + m - i_k\|\,.^4$$

Let us call the induced partition an *m–distorted Voronoi tesselation*. Note that an m–distorted Voronoi tesselation corresponds to the Voronoi tesselation generated by the translated interpretations $\iota_k = i_k - m, k = 1, \dots, N$.

In turn, if the sender uses such an m–distorted Voronoi tesselation, and as the receiver is unbiased, her best interpretation is the Bayesian estimator.

Definition 1 *An m–distorted Voronoi language (w, i) consists of an m–distorted Voronoi tessellation for the sender and a Bayesian estimator interpretation for the receiver. i.e. we have both*

$$w^*(t) = \operatorname{argmin}_{j=1,\dots,N} \|t + m - i_j^*\| \; F - a.s. \tag{3}$$

$$i_k = b(C_k^*) \qquad (\textit{ for } C_k^* = \{t \in T : w^*(t) = w_k\})\,. \tag{4}$$

[3] Note that the cells of a Voronoi tesselation are (up to a null set) convex when we use the Euclidean norm, see Okabe, Boots, and Sugihara (1992).

[4] Note that this strategy is not uniquely defined at points t that have equal distance to two or more translated interpretations. As these points form a null set for most norms, e.g. the Euclidean norm or L^p-norms, we can ignore this ambiguity. Without loss of generality, we always take the word with the smallest index in this case.

Theorem 1 *A profile of strategies (w, i) is a strict Nash equilibrium if and only if (w, i) is an m–distorted Voronoi language with full vocabulary.*

The complete proof is in the appendix.

The above result can be viewed from different angles. On the one hand, it is a robustness check for the main theorem in Jäger, Metzger, and Riedel (2011). The structure of strict Nash equilibria remains essentially the same even when we introduce small conflicts of interests. This is also important from another point of view. Real players may well have slightly wrong perceptions, or make mistakes, or interpret signals in a slightly biased way. Nevertheless, at least for small biases m, the strict Nash equilibria look similar to the case of aligned interests.

However, when the bias becomes larger, one might doubt that the players can still partially cooperate in this information transmission game. We give now an example that shows that in general, distorted Voronoi languages with full vocabulary need not exist. Our theorem then implies that the game cannot have strict Nash equilibria. As was to be expected, perfect communication breaks down when the interests are too far apart.

Example 1 *Consider the unit interval $T = [0, 1]$ with the uniform distribution $F(x) = x$, quadratic loss $l(d) := d^2$ and two words $W = \{w_1, w_2\}$. We can interpret this two words to have the usual meaning of "left" and "right". Let us choose a bias of $|m| < \frac{1}{4}$. Then the sender uses $w^*(t) = w_1$ for $t \leq \frac{1}{2} - 2m$ and $w^*(t) = w_2$ else in the m–distorted Voronoi language. The best interpretations for the receiver are $i_1^* = \frac{1}{4} - m$ and $i_2^* = \frac{3}{4} - m$. We see that $\operatorname{range} w = W$, hence the language uses full vocabulary. It is also unique up to the obvious symmetry. For a positive bias, "right" is used for a larger set of types than "left" (see Figure 1).*

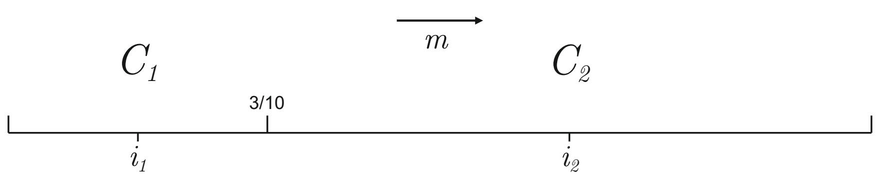

Figure 1: $\frac{1}{10}$–distorted Voronoi language on the unit interval.

Now let us show that for a bias $|m| \geq \frac{1}{4}$, no m–distorted Voronoi language with full vocabulary exists. Therefore, suppose to the contrary that (w, i) is *such a language, without loss of generality $i_1 < i_2$. A type $t \in T$ lies in C_1 if and only if*

$$|t+m-i_1| \leq |t+m-i_2| \Leftrightarrow t \leq \frac{i_1+i_2}{2} - m.$$

It follows that $i_1 = b(C_1) = \frac{1}{4} - m$ and $i_2 = b(C_2) = \frac{3}{4} - m$. Hence,

$$C_1 = \left[0, \frac{1}{2} - 2m\right] \text{ and } C_2 = \left(\frac{1}{2} - 2m, 1\right].$$

Thus, we do not have $\operatorname{range} w = W$ *since $|m| \geq \frac{1}{4}$, which is a contradiction.*

The example suggests that for small biases, we still get strict Nash equilibria. This holds true in general. As long as the bias m is sufficiently small, distorted Voronoi languages with full vocabulary exist, and hence strict Nash equilibria as well. This completes our robustness check of Jäger, Metzger, and Riedel (2011).

Proposition 1 *There exists $\epsilon > 0$ such that for all $m \in \mathbb{R}^L$ with $\|m\| < \epsilon$ m–distorted Voronoi languages with full vocabulary exist.*

The complete proof is also in the appendix.

To provide some more intuition, we want to give examples with a two-dimensional type space. First we consider uniformly distributed types and two words. The resulting distorted Voronoi language is of a similar structure compared to the one-dimensional example we studied before. We also see that the condition for existence of distorted Voronoi languages derived above is only sufficient, but not necessary. In general it can happen that even for an arbitrary large bias, measured in terms of its norm, some resulting distorted Voronoi language has full vocabulary.

Example 2 *Consider the unit square $T = [0,1]^2$ with the uniform distribution $F(x) = x$, quadratic loss $l(d) := d^2$ together with the Euclidean norm $\|\cdot\|$ and two words $W = \{w_1, w_2\}$. If we choose a bias $m \in \mathbb{R}^2$ such that $|m_1| < \frac{1}{4}$, then there is an m–distorted Voronoi language where the sender uses $w^*(t) = w_1$ for $\left\{t \in T | t_1 \leq \frac{1}{2} - 2m_1\right\}$ and $w^*(t) = w_2$ else, i.e. the square is divided vertically. The best interpretations for the receiver are*

$i_1^* = \left(\frac{1}{4} - m_1, \frac{1}{2}\right)$ *and* $i_2^* = \left(\frac{3}{4} - m_1, \frac{1}{2}\right)$ *(see Figure 2).*[5] *Furthermore, this language uses full vocabulary and is hence a strict Nash equilibrium. Since we did not impose any restrictions on the second component of the bias,* m_2*, we see that indeed the condition derived above is not necessary for the existence of a strict Nash equilibrium in this setting.*

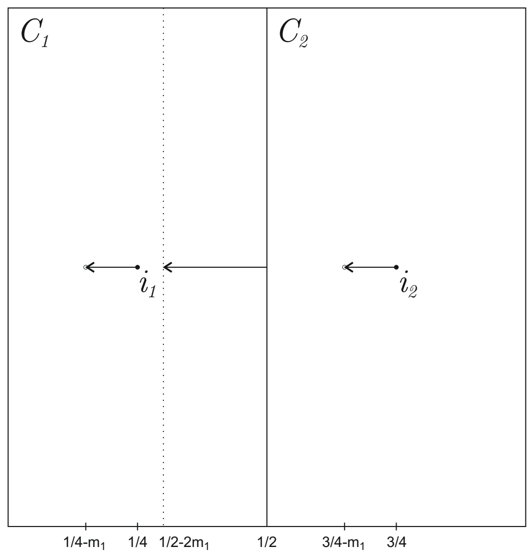

Figure 2: m–distorted Voronoi language on the unit square with $\frac{1}{4} > m_1 > 0$ and arbitrary m_2.

Finally, we want to give an example with normally distributed types and more than two words. Since the computational demand becomes already ambitious with more than two words and the uniform distribution, we rely on simulations for this example. It turns out that as in the previous examples, the interpretations are shifted into the direction which is opposite to the direction of the bias.

Example 3 *Consider the unit square* $T = [0, 1]^2$ *with the normal distribution, where we choose the variance small such that the borders receive a*

[5] Analogue for $|m_2| < \frac{1}{4}$ and the language that divides the square horizontally. Similar conditions hold for the languages that divide the square diagonally.

mass close to zero. We adapt the algorithm presented in Jäger, Metzger, and Riedel (2011) to compute a distorted Voronoi language with more than two words. But first we start with a Voronoi language or, in our terminology, a 0*–distorted Voronoi language, which is meant to serve as a measure for the impact the bias has on the tesselation. Note that the algorithm approximates those languages which are stable, i.e. it chooses one of them (cf. Jäger, Metzger, and Riedel (2011)). Then we introduce a bias m with equal and strictly positive components that are small enough such that the resulting distorted Voronoi language has full vocabulary. Thus, the algorithm approximates a stable strict Nash equilibrium (see Figure 3).*

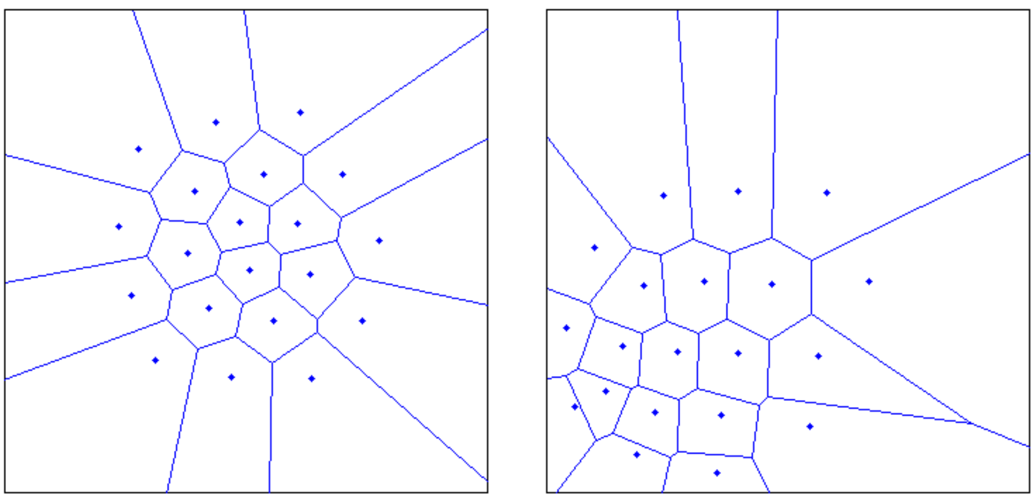

Figure 3: A Voronoi language (left) and an m–distorted Voronoi language, $m_1 = m_2 > 0$, with normally distributed types.

4 Conclusion

We study a signaling game with infinitely many types, only few signals and conflict of interest. The point of departure is the model of Jäger, Metzger, and Riedel (2011), we introduce a bias to their model and show that strict Nash equilibria are characterized by *distorted Voronoi languages* that use all messages. In these signaling systems the sender uses a *distorted Voronoi tesselation* where the prototypes that generate the tesselation are the best estimates in the Bayesian sense. This shows that the structure of strict Nash equilibria basically stays the same when we introduce a bias, the tesselation is a Voronoi tesselation generated by the prototypes translated by the bias. Thus, this result provides a robustess check for the results in Jäger, Metzger, and Riedel (2011). Furthermore, we show that strict Nash equilibria exist if the bias is small enough. For a large bias it can happen that the sender does not use all messages in equilibrium, that is perfect communication breaks down.

A Appendix

A.1 Choice of the bias

We show that using only a bias on the sender's side is without loss of generality. Therefore, consider the game with a bias $m_S \in \mathbb{R}^L$ on the sender's and $m_R \in \mathbb{R}^L$ on the receiver's side, and such that they do not coincide. Note that this also covers the case where there is only a receiver's bias by setting $m_S = 0$ and $m_R \neq 0$. The expected losses then are

$$L_S(w,i) := \int_T l\left(\left\|t + m_S - i_{w(t)}\right\|\right) F(dt)$$

and

$$L_R(w,i) := \int_T l\left(\left\|t + m_R - i_{w(t)}\right\|\right) F(dt).$$

Defining $\widetilde{T} := T + m_R = \{t + m_R | t \in T\}$, we get

$$L_R(w,i) = \int_{\widetilde{T}} l\left(\left\|t - i_{w(t)}\right\|\right) F(dt)$$

and

$$\begin{aligned} L_S(w,i) &= \int_{\widetilde{T}} l\left(\left\|t - m_R + m_S - i_{w(t)}\right\|\right) F(dt) \\ &= \int_{\widetilde{T}} l\left(\left\|t + \widetilde{m} - i_{w(t)}\right\|\right) F(dt), \end{aligned}$$

where $\widetilde{m} := m_S - m_R \in \mathbb{R}^L \backslash \{0\}$. Thus, we have reduced the game to the situation with only a sender's bias $\widetilde{m} \neq 0$.

A.2 Proof of Theorem 1

Let (w, i) be an m–distorted Voronoi language with full vocabulary. The sender chooses the word that minimizes her loss, that means this is a best reply which is also $F - a.s.$ unique since the interpretations are pairwise distinct. The receiver uses the Bayesian estimators of the partition given by the sender. This is a best reply since it minimizes her expected loss. Furthermore, it is unique because the sender uses all words and additionally the estimators are unique. Thus, (w, i) is a strict Nash equilibrium.

Contrary, suppose that (w, i) is a strict Nash equilibrium. The interpretations of the receiver are the unique best reply to the strategy of the sender, therefore the sender uses all words, because otherwise the receiver would be indifferent between interpretations for words that are not used. Since w is a best reply to the receiver's interpretations, the sender chooses the word that minimizes her loss given the interpretations of the receiver, i.e. without loss of generality

$$w^{-1}(w_k) = \left\{ t \in T \middle| \text{k is the smallest number in } \operatorname{argmin}_{j=1,\ldots,N} \|t + m - i_j\| \right\}.$$

Her reply being also $F - a.s.$ unique implies that the interpretations of the receiver are pairwise distinct, because otherwise the sender would be indifferent between the respective words if her type was close to two interpretations that are not distinct. Thus, she is using an m–distorted Voronoi tesselation. Moreover, all cells of the tesselation have positive mass with respect to F since she uses all words. For each cell of the tesselation the receiver chooses the interpretation that minimizes her expected loss given she knows the sender's type to be in this cell, i.e. she uses the Bayesian estimator interpretation for each cell, which is unique. Hence, (w, i) is an m–distorted Voronoi language with full vocabulary.

A.3 Proof of Proposition 1

First note that a 0–distorted Voronoi language is a Voronoi language in the terminology of Jäger, Metzger, and Riedel (2011). They showed that Voronoi languages with full vocabulary do always exist, and hence we have the existence of a 0–distorted Voronoi language with full vocabulary. Now we want to show that the mapping

$$f : \mathbb{R}^L \times T^N \to \mathbb{R}, (m,i) \mapsto \int_T \sum_{k=1}^N l\left(\|t - i_k\|\right) 1_{C_k^i(m)}(t) F(dt)$$

is continuous.[6] It is enough to show that the integrand is $F - a.s.$ continuous.

Let $\left(m_n, (i_j^n)_{j=1,\dots,N}\right)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^L \times T^N$ such that $(m_n, i^n) \underset{n\to\infty}{\longrightarrow} (m,i)$. Consider a type t in the interior of some $C_h^i(m)$. Since t is in the interior, for n large enough $t \in C_h^{i^n}(m_n)$, i.e. $1_{C_h^i(m)}(t) = 1_{C_h^{i^n}(m_n)}(t) = 1$. By continuity of $l(\cdot)$, we get

$$\sum_{k=1}^N l\left(\|t - i_k\|\right) 1_{C_k^i(m)}(t) = l\left(\|t - i_h\|\right) 1_{C_h^i(m)}(t) = \lim_{n\to\infty} l\left(\|t - i_h^n\|\right) 1_{C_h^i(m)}(t).$$

Therefore, $f = f(m,i)$ is continuous. Since T^N is compact we can apply the so called Maximum Theorem[7], which still holds if we replace the Maximum by the Minimum. Thus, also

$$g : \mathbb{R}^L \to \mathbb{R}, m \mapsto \min_{i\in T^N} \int_T \sum_{k=1}^N l\left(\|t - i_k\|\right) 1_{C_k^i(m)}(t) F(dt)$$

is continuous. Hence, considering a sequence $(m_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^L$ such that $m_n \to 0$ as $n \to \infty$, we get

$$\begin{aligned}&\lim_{n\to\infty} \min_{i\in T^N} \int_T \sum_{k=1}^N l\left(\|t - i_k\|\right) 1_{C_k^i(m_n)}(t) F(dt)\\ &= \min_{i\in T^N} \int_T \sum_{k=1}^N l\left(\|t - i_k\|\right) 1_{C_k^i(0)}(t) F(dt).\end{aligned}$$

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[6] The restriction of the interpretations to the type space T comes without loss of generality, the interpretations that minimize the integral are Bayesian estimators and hence lie in T.

[7] see Ok (2007)

Note that the latter is the loss of a 0–distorted Voronoi language, which always have full vocabulary. This implies[8] that there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$,

$$C_k^{i^{m_n}}(m_n) \text{ have positive mass with respect to } F \text{ for all } k = 1, \ldots, N,$$

$$\text{where } i^{m_n} \in \operatorname{argmin}_{i \in T^N} \int_T \sum_{k=1}^{N} l\left(\|t - i_k\|\right) 1_{C_k^i(m_n)}(t) F(dt).$$

Hence, if $n \geq \bar{n}$, then range $w^{m_n} = W$ and $i_k^{m_n}$ are the Bayesian estimators of the cells $C_k^{i^{m_n}}(m_n)$. This implies that there exists $\epsilon > 0$ such that for all $m \in \mathbb{R}^L$ with $\|m\| < \epsilon$ (w^m, i^m) is an m–distorted Voronoi language with full vocabulary.

[8]Suppose not, then there exists a sequence of languages $((w^{m_n}, i^{m_n}))_{n \in \mathbb{N}}$ and a subsequence $((w^{m_{n'}}, i^{m_{n'}}))_{n'}$ such that $(w^{m_{n'}}, i^{m_{n'}})$ does not use full vocabulary for all n' and by continuity

$$\lim_{n' \to \infty} L_R(w^{m_{n'}}, i^{m_{n'}}) = L_R(w^0, i^0). \left(\text{contradiction since } \left(w^0, i^0\right) \text{ has full vocabulary}\right)$$

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