Center for
Mathematical Economics
Working Papers

485

February 2015

Continuous-Time Public Good Contribution under Uncertainty

Giorgio Ferrari, Frank Riedel and Jan-Henrik Steg

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

Continuous-Time Public Good Contribution under Uncertainty*

Giorgio Ferrari † Frank Riedel ‡ Jan-Henrik Steg §

February 5, 2015

Abstract. We study a continuous-time problem of public good contribution under uncertainty for an economy with a finite number of agents. Each agent aims to maximize his expected utility allocating his initial wealth over a given time period between private consumption and repeated but irreversible contributions to increase the stock of some public good. We study the corresponding social planner problem and the case of strategic interaction between the agents. These problems are set up as stochastic control problems with both monotone and classical controls representing the cumulative contribution into the public good and the consumption of the private good, respectively. We characterize the optimal investment policies by a set of necessary and sufficient stochastic Kuhn-Tucker conditions, which in turn allow to identify a universal signal process that triggers the public good investments. Further we show that our model exhibits a dynamic *free rider* effect. We explicitly evaluate it in a symmetric Black-Scholes setting with Cobb-Douglas utilities and we show that uncertainty and irreversibility of public good provisions need not affect the degree of free-riding.

MSC2010 subject classification: 93E20, 91B70, 91A15, 91A25, 60G51

JEL subject classification: C02, C61, C62, C73

Keywords: irreversible investment, public good contribution, free-riding, singular stochastic control, first order conditions for optimality, stochastic games, Nash equilibrium, Lévy processes

1 Introduction

We study a general stochastic continuous-time problem of public good contribution under portfolio constraints for an economy with a fixed number of agents. Each agent maximizes his expected utility choosing how to allocate his initial wealth over a fixed time period $[0, T]$, $T \in (0, \infty]$,

*Financial support by the German Research Foundation (DFG) via grant Ri 1128-4-1, *Singular Control Games: Strategic Issues in Real Options and Dynamic Oligopoly under Knightian Uncertainty*, is gratefully acknowledged.

†Corresponding author. Center for Mathematical Economics (IMW), Bielefeld University, Universitätsstrasse 25, D-33615 Bielefeld, Germany; Tel.: +49 521 106 5642; giorgio.ferrari@uni-bielefeld.de

‡Center for Mathematical Economics (IMW), Bielefeld University, Universitätsstrasse 25, D-33615 Bielefeld, Germany; frank.riedel@uni-bielefeld.de

§Center for Mathematical Economics (IMW), Bielefeld University, Universitätsstrasse 25, D-33615 Bielefeld, Germany; jsteg@uni-bielefeld.de

between private consumption and repeated but irreversible contributions to increase the stock of some public good. Examples of public goods include clean environment, national security, academic research and accessible public capital. In order to determine the (unique) efficient allocation we first consider the corresponding social planner problem. As in other settings, our analysis reveals that its solution cannot be obtained by strategic interaction between the agents because of a classical *free rider effect*: agents enjoy the contributions of others but do not take into account other's benefits when making their own contributions (see, e.g., Cornes and Sandler (1996) or Laffont (1988)).

In the economic literature there is a long tradition of research on public good contribution and free rider problems started by the static analyses of Olson (1965) and Samuelson (1954), and further developed by Bergstrom et al. (1986), Groves and Ledyard (1977) (in the context of a general equilibrium model), Palfrey and Rosenthal (1984a), Palfrey and Rosenthal (1984b), Varian (1994), among others[1]. Free rider problems in continuous time are studied for instance in the early papers Fershtman and Nitzan (1991) and Levhari and Mirman (1980); see also the more recent Wirl (1996) and Wirl (2007). Irreversibility constraints on the public good contribution are introduced in the literature on 'monotone games' by assuming that players' individual actions can only increase over time. We refer to Lockwood and Thomas (2002), Matthews (2011) and, more recently, to Battaglini et al. (2014), among others. Several papers also considered public good provision problems under uncertainty. We refer to Austen-Smith (1980), Gradstein et al. (1993), Sandler et al. (1987) and Eichberger and Kelsey (1999). Recently, the originally deterministic setting of Fershtman and Nitzan (1991) has been extended by Ewald and Wang (2010) including a diffusion term. Subgame consistent cooperative solutions for public good provisions in a stochastic differential game framework are finally considered in Yeung and Petrosyan (2013).

Our model differs from the existing related literature on public goods for two main aspects. First of all, we consider a *stochastic, dynamic* model of *intertemporal* investment choice with general concave utilities, not necessarily separable, which thus allow us to account for cross effects between the public and the private good. To the best of our knowledge, this is a novelty with respect to the classical models in which usually the investor has a quasilinear utility and can choose how to divide a budget given in each period between instantaneous private consumption and public good investment, such that only the stock of the public good allows intertemporal transfers.

Secondly, we take into account *irreversibility* of the investments which together with uncertainty typically induces reluctance to invest. We are able to analyze the interplay of this dynamic effect with free riding. Interestingly it is not necessarily the case that uncertainty and irreversibility of public good contributions aggravate the degree of free-riding. We indeed explicitly evaluate the free rider effect in a symmetric Black-Scholes setting with Cobb-Douglas utilities and we show that uncertainty and irreversibility of public good provisions do not affect the degree of free-riding.

From the mathematical point of view our problem falls into the class of continuous-time, optimal stochastic control problems with both monotone and classical control processes. The monotone controls represent the cumulative contributions into the public good, whereas the

[1] Public good contribution with imperfect information about individual actions is considered in Marx and Matthews (2000); a direct extension to a Bayesian setting of the model in Varian (1994) is addressed in Bag and Roy (2011)

instantaneous consumption of the private good is modeled through a classical nonnegative and adapted control process. We analyze such an optimal control problem by a first order condition approach that may be thought of as a stochastic, infinite-dimensional generalization of the classical Kuhn-Tucker conditions. Our method does not require any Markovian or diffusive hypothesis, and in this sense it represents a substitute in non-Markovian frameworks for the Hamilton-Jacobi-Bellman equation. This approach is very powerful in solving general singular control problems as it has been shown in a quite recent literature. We refer to Bank and Riedel (2001), Bank and Riedel (2003) for an intertemporal utility maximization problem with Hindy, Huang and Kreps preferences; to Bank (2005), Chiarolla and Ferrari (2014), Ferrari (2015) and Riedel and Su (2011) for the irreversible investment problem of a monopolistic firm with both limited and unlimited resources; to Chiarolla et al. (2013) for the social planner problem in a market with N firms and limited resources; to Steg (2012) for a capital accumulation game.

We start analyzing the public good contribution problem by taking the point of view of a fictitious social planner who aims to maximize the expected total welfare of the economy under a social budget constraint. Assuming prices of the public good and of the private consumption given by discounted exponential martingales, we prove existence and uniqueness of the social planner's optimal policy via a suitable application of Komlós' classical theorem (cf. Komlós (1967)) and a further generalization of it due to Kabanov (cf. Lemma 3.5 in Kabanov (1999); see also the functional analytic version of Balder (1990)). The optimal investment strategy is completely characterized by a set of necessary and sufficient stochastic Kuhn-Tucker conditions which in turn lead to the identification of a universal signal process that triggers the public good investments. Such a signal process is the unique solution of a backward stochastic equation in the spirit of Bank and El Karoui (2004). We then consider strategic interaction between the agents in our economy and we show that any Nash equilibrium is again the solution of a set of first order conditions for optimality. We restrict our attention to equilibria with *open-loop* strategies (see also Back and Paulsen (2009) and Steg (2012)) in which agents do react to the evolving exogenous uncertainty, take the contribution processes of others as given and do not react to deviations from announced (equilibrium) play. Indeed, as pointed out in Back and Paulsen (2009) there are serious conceptual problems defining a stochastic continuous-time game of singular controls as ours with more explicit feedback (closed loop) strategies. Our approach allows also to provide explicit results in a symmetric homogeneous setting where prices are driven by Lévy uncertainty. We find the explicit forms of the social planner's optimal policy and of the Nash equilibrium which enable a detailed analysis of the degree of free riding in our model.

The paper is organized as follows. In Section 2 we set up the model. In Section 3 we consider the social planner problem, proving existence and uniqueness of its solution and introducing the stochastic Kuhn-Tucker conditions for optimality. The public good contribution game is addressed in Section 4, whereas explicit results are obtained in Section 5. Finally we refer to Appendix A for some technical proofs.

2 The Model

We consider a continuous-time stochastic economy with a finite number n of agents over a fixed time horizon $0 < T \leq \infty$. Each agent, indexed by $i = 1, \ldots, n$, chooses how to allocate his initial wealth $w^i > 0$ between private consumption x^i and arbitrary but nondecreasing cumulative contributions C^i to increase the stock of some public good. We assume a continuous revelation of information about an exogenous source of uncertainty and we allow the agents to condition their decisions on the accumulated information. Formally, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness. For the moment we do not make any Markovian assumption.

One may think that the agents are financed entirely by their labour or by holding a portfolio of financial instruments. Hence they are part of a more complex financial market that, however, we do not model explicitly. At initial time each agent can buy one unit of the private good for contingent delivery at time $t \in [0,T]$ and state $\omega \in \Omega$ at forward price $\psi_x(\omega, t)$. Analogously, the contingent state-price for the contribution to the public good is $\psi_c(\omega, t)$. Both ψ_x and ψ_c are strictly positive (more technical conditions are listed in Assumption 2 below). The forward price of an investment plan (x^i, C^i) is therefore

$$E\bigg[\int_0^T \psi_x(t)x^i(t)dt + \int_0^T \psi_c(t)dC^i(t)\bigg],$$

where (x^i, C^i) can be chosen in the nonempty, convex budget-feasible set

$$\begin{aligned}\mathcal{B}_{w^i} := \bigg\{&(x^i, C^i) : \Omega \times [0,T] \mapsto \mathbb{R}^2_+ \text{ adapted, s.t. } C^i \text{ is right-continuous, nondecreasing,}\\ &C^i(0-) = 0 \ P\text{-a.s., and } E\bigg[\int_0^T \psi_x(t)x^i(t)dt + \int_0^T \psi_c(t)dC^i(t)\bigg] \le w^i\bigg\}.\end{aligned} \tag{2.1}$$

Here $E[\int_0^T \psi_x(t)x^i(t)dt + \int_0^T \psi_c(t)dC^i(t)] \le w^i$ defines therefore the budget constraint of agent i. Notice that we can have jumps as well as singular increases in C^i. This allows for contributions into the public good in bulks, as well as in a singular way. Clearly, the contributions in rates are allowed for as well. Recalling that each $t \mapsto C^i(\omega, t)$ is a.s. nondecreasing, from now on we will denote by $\int_0^T \psi_c(t)dC^i(t)$ the Lebesgue-Stiltjes integral $\int_{[0,T]} \psi_c(t)dC^i(t)$, to include a possible initial contribution's bulk.

The agents are assumed to derive some expected, time-separable utility from the private good and the aggregate public good process $C := \sum_{i\in\{1,\dots,n\}} C^i$. Given a combination of strategies from $\prod_{i=1}^n \mathcal{B}_{w^i}$, agent i's utility is

$$U^i(x^i, C^i; C^{-i}) := E\left[\int_0^T e^{-\int_0^t r(s)\,ds} u^i(x^i(t), C(t))\,dt\right], \tag{2.2}$$

where $C^{-i} := \sum_{j\in\{1,\dots,n\}\setminus i} C^j$, r is an exogenous stochastic discount factor and the random fields $u^i : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ describe instantaneous utilities. In the economic literature on public good contribution it is customary to assume quasilinear utilities (see, e.g., the early paper by Varian (1994) and the very recent by Battaglini et al. (2014)). Here, instead, we work with general concave utilities, allowing to account for cross effects between the public and the private good.

Assumption 1.

i. For any $\omega \in \Omega$, the mapping $(x, c) \mapsto u^i(\omega, x, c)$ is increasing and strictly concave on $\mathbb{R}^2_+$, as well as twice continuously differentiable on the open cone $\mathbb{R}^2_{++}$. Moreover, it satisfies the Inada conditions

$$\lim_{x \downarrow 0} u^i_x(\omega, x, c) = +\infty \quad \text{and} \quad \lim_{x \uparrow \infty} u^i_x(\omega, x, c) = 0$$

for any $\omega \in \Omega$ and $c > 0$.

ii. For any given $(x, C) \in \prod_{i=1}^n \mathcal{B}_{w^i}$, the process $(\omega, t) \mapsto u^i(\omega, x(\omega, t), C(\omega, t))$ is progressively measurable.

iii. The family $\left(e^{-\int_0^t r(\omega,s)\,ds} u^i(\omega, x(\omega, t), C(\omega, t)),\ (x, C) \in \prod_{i=1}^n \mathcal{B}_{w^i}\right)$ is $P \otimes dt$-uniformly integrable.

From now on, to simplify exposition, we will suppress the dependence on ω in the random utility functions of the agents and in all the stochastic processes. Moreover, we make the next standing assumption.

Assumption 2.

i. The optional process $\psi_c := \{\psi_c(t), t \in [0, T]\}$ is of class (D), lower semicontinuous in expectation[2] and such that $\psi_c(t) := e^{-\int_0^t r_c(s)ds} \mathcal{E}_c(t)$, for some continuous, uniformly bounded and (strictly) positive process $r_c := \{r_c(t), t \in [0, T]\}$, and for some exponential martingale $\mathcal{E}_c := \{\mathcal{E}_c(t), t \in [0, T]\}$. Moreover, if $T = \infty$, one has $\psi_c(T) = 0$ a.s.

ii. The optional process $\psi_x := \{\psi_x(t), t \in [0, T]\}$ is such that $\psi_x(t) := e^{-\int_0^t r_x(s)ds} \mathcal{E}_x(t)$ for some continuous, uniformly bounded and (strictly) positive process $r_x := \{r_x(t), t \in [0, T]\}$, and for some exponential martingale $\mathcal{E}_x := \{\mathcal{E}_x(t), t \in [0, T]\}$. Moreover, if $T = \infty$, one has $\psi_x(T) = 0$ a.s.

iii. The optional (strictly) positive continuous process $r := \{r(t), t \in [0, T]\}$ is uniformly bounded.

Under Assumptions 1 and 2 the payoff in (2.2) is well defined and finite for any $i = 1, \ldots, n$.

Remark 2.1.

i. The Inada conditions of Assumption 1.i. guarantee that there will be an interior solution for optimal private consumption.

[2] A stochastic process X is:

(a) *optional* if it is measurable with respect to the optional sigma-field $\mathcal{O}$ on $\Omega \times [0, T]$ generated, e.g., by the right-continuous adapted processes;

(b) of *class (D)* if $\{X(\tau),\ \tau \text{ stopping time}\}$ defines a uniformly integrable family of random variables on $(\Omega, \mathscr{F}, P)$.

(c) *lower-semicontinuous in expectation* if for any stopping time τ one has $\liminf_{n\uparrow\infty} E[X(\tau_n)] \geq E[X(\tau)]$, whenever $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of monotone stopping times converging to τ.

We refer the reader to Dellacherie and Meyer (1978), among others, for further details.

ii. Note that since u^i is concave in c, by Assumption 1.iii. $e^{-\int_0^t r(s)\,ds} u^i_c(x(t), C(t))$ is $P \otimes dt$-integrable for any $(x, C) \in \prod_{i=1}^n \mathcal{B}_{w^i}$ such that $C(0) > 0$ a.s.

iii. Thanks to the budget constraint (cf. (2.1)), when $T < \infty$ one can easily adapt the arguments in the proof of Lemma 2.1 in Bank and Riedel (2001) to show that Assumption 1.iii. is satisfied if

(a) for some $\alpha, \beta \in (0, 1)$ one has

$$u^i(\omega, x, c) \leq \text{const.}(1 + x^\alpha + c^\beta), \quad \text{for all } x, c \geq 0 \text{ uniformly in } \omega \in \Omega;$$

(b)

$$\mathcal{E}_x^{-1} \in L^{\hat{p}}(P), \quad \text{and} \quad \mathcal{E}_c^{-1} \in L^{\hat{q}}(P),$$

for some $\hat{p} > \frac{\alpha}{1-\alpha}$ and $\hat{q} > \frac{\beta}{1-\beta}$, and with $\mathcal{E}_x$ and $\mathcal{E}_c$ as in Assumption 2.

iv. It is easy to see that Assumption 2.i. and 2.ii. are satisfied, for example, by the classical benchmark cases of geometric Brownian motions (for discount factors r_c and r_x suitably chosen to have $\psi_c(T) = 0 = \psi_x(T)$ a.s. when $T = +\infty$).

3 The Social Planner Problem

We start our analysis by studying a social planner problem for the economy described in Section 2. Throughout this section, denote by $(\underline{x}, \underline{C})$ a vector of investment processes valued in $\mathbb{R}^{2n}_+$ with components $(x^1, \dots, x^n, C^1, \dots, C^n)$ and introduce the nonempty, convex, social budget-feasible set

$$\mathcal{B}_w := \Big\{ (\underline{x}, \underline{C}) : \Omega \times [0, T] \mapsto \mathbb{R}^{2n}_+ \text{ adapted s.t. } C^i \text{ is right-continuous, nondecreasing,}$$
$$C^i(0-) = 0,\ i = 1, \dots, n,\ P\text{-a.s. and } \sum_{i=1}^n E\Big[\int_0^T \psi_x(t) x^i(t) dt + \int_0^T \psi_c(t) dC^i(t) \Big] \leq w \Big\}$$

with $w := \sum_{i=1}^n w^i$. We say that $(\underline{x}, \underline{C})$ is admissible if $(\underline{x}, \underline{C}) \in \mathcal{B}_w$. Suppose that there exists a fictitious social planner aiming to maximize the aggregate expected utility by allocating efficiently the available initial wealth. This amounts to solving the optimization problem with value function

$$V_{SP} := \sup_{(\underline{x}, \underline{C}) \in \mathcal{B}_w} U_{SP}(\underline{x}, \underline{C}) = \sup_{(\underline{x}, \underline{C}) \in \mathcal{B}_w} \sum_{i=1}^n \gamma^i U^i(x^i, C^i; C^{-i}) \tag{3.1}$$

with $U^i(x^i, C^i; C^{-i})$ as in (2.2) and for positive weights γ^i, $i = 1, \dots, n$, such that $\sum_{i=1}^n \gamma^i = 1$.

Theorem 3.1. *Under Assumptions 1 and 2 there exists $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$ which solves the social planner problem* (3.1) *and which is unique up to indistinguishability.*

Proof. The proof is organized in three steps.

Step 1. In this first step we let $T < \infty$. Recall that $\psi_x(t) = e^{-\int_0^t r_x(s)ds}\mathcal{E}_x(t)$ and $\psi_c(t) = e^{-\int_0^t r_c(s)ds}\mathcal{E}_c(t)$, for some continuous and uniformly bounded processes r_x and r_c, and for some exponential martingales $\mathcal{E}_x$ and $\mathcal{E}_c$ (cf. Assumption 2.i.). Let $\tilde{E}_c[\cdot]$ and $\tilde{E}_x[\cdot]$ be the expectations under the measures $\tilde{P}_c$ and $\tilde{P}_x$ with Radon-Nikodym derivative $\mathcal{E}_c(T)$ and $\mathcal{E}_x(T)$, respectively, with respect to P. Since $\mathcal{E}_x(T) > 0$ and $\mathcal{E}_c(T) > 0$ a.s., the measure P is equivalent both to $\tilde{P}_c$ and $\tilde{P}_x$. Denote by $\mathcal{V}_T$ the space of all optional random measures on $[0, T]$ endowed with the weak-topology in the probabilistic sense[3], by $\mathbb{L}^1(d\mu_x)$ the space of all functions integrable with respect to the measure $d\mu_x := d\tilde{P}_x \otimes dt$ and set $x := \sum_{i=1}^n x^i$. Then $\mathcal{B}_w \subset \mathbb{L}^1(d\mu_x)^n \otimes \mathcal{V}_T^n$. Indeed, for any $i = 1, \dots, n$, and for some constant $K_1 > 0$

$$\begin{aligned} w \geq E\bigg[\int_0^T \psi_x(t)x^i(t)dt\bigg] &= E\bigg[\int_0^T e^{-\int_0^t r_x(s)ds} E[\mathcal{E}_x(T)|\mathscr{F}_t]\, x^i(t)dt\bigg] \\ &= E\bigg[\mathcal{E}_x(T)\int_0^T e^{-\int_0^t r_x(s)ds} x^i(t)dt\bigg] \geq K_1\tilde{E}_x\bigg[\int_0^T x^i(t)dt\bigg], \end{aligned} \tag{3.2}$$

where Girsanov's Theorem and the uniform boundedness of r_x imply the last step. Also, each component of $\underline{C}$ is the cumulative distribution of an optional random measure, being an adapted, right-continuous, nonnegative process.

Let now $\{(\underline{x}_m, \underline{C}_m)\}_{m\in\mathbb{N}} \subset \mathcal{B}_w$ be a maximizing sequence; that is, a sequence of investment plans such that

$$\lim_{m\to\infty}\sum_{i=1}^n \gamma^i U^i(x_m^i, C_m^i; C_m^{-i}) = V_{SP}.$$

The two sequences $\{\tilde{E}_x[\int_0^T x_m^i(t)dt]\}_{m\in\mathbb{N}}$ and $\{\tilde{E}_c[C_m^i(T)]\}_{m\in\mathbb{N}}$ are uniformly bounded in m for every $i = 1, \dots, n$, because of (3.2) and since, analogously,

$$\begin{aligned} w \geq E\bigg[\int_0^T \psi_c(t)dC^i(t)\bigg] &= E\bigg[\int_0^T e^{-\int_0^t r_c(s)ds} E[\mathcal{E}_c(T)|\mathscr{F}_t]\, dC^i(t)\bigg] \\ &= E\bigg[\mathcal{E}_c(T)\int_0^T e^{-\int_0^t r_c(s)ds} dC^i(t)\bigg] = \tilde{E}_c\bigg[\int_0^T e^{-\int_0^t r_c(s)ds} dC^i(t)\bigg] \geq K_2\tilde{E}_c[C^i(T)], \end{aligned}$$

where the second equality follows from Theorem 1.33 in Jacod (1979), and with $K_2 > 0$ a suitable constant by the uniform boundedness of r_c. Hence by Komlós' theorem (see Komlós (1967) for its standard formulation and Kabanov (1999), Lemma 3.5, for a version of Komlós' theorem for optional random measures), for every $i = 1, \dots, n$ there exist two subsequences $\{\tilde{x}_m^i\}_{m\in\mathbb{N}} \subset \{x_m^i\}_{m\in\mathbb{N}}$ and $\{\tilde{C}_m^i\}_{m\in\mathbb{N}} \subset \{C_m^i\}_{m\in\mathbb{N}}$ and a $d\mu_x$-integrable and adapted process x_*^i and some optional random measure dC_*^i, $i = 1, \dots, n$ such that as $k \uparrow \infty$

$$X_k^i(t) := \frac{1}{k+1}\sum_{m=0}^k \tilde{x}_m^i \to x_*^i(t), \;\; d\mu_x\text{-a.e.} \tag{3.3}$$

[3] An optional random measure on $[0, T]$ is simply a random variable ν valued in the space of positive finite measures on $[0, T]$ (endowed with the topology of weak*-convergence) such that the process $\nu(\omega, t) := \nu(\omega, [0, t])$ is adapted.

and

$$I_k^i(t) := \frac{1}{k+1}\sum_{m=0}^{k} \tilde{C}_m^i \to C_*^i(t), \ \ \tilde{P}_c\text{-a.s., for every point of continuity of } C_*^i(\cdot) \text{ and } t = T. \tag{3.4}$$

From now on, with a slight abuse of notation, we will denote by C_*^i as well the right-continuous modification of C_*^i. Notice that having $\lim_{k\to\infty} I_k^i(t) = C_*^i(t)$ $\tilde{P}_c$-a.s. for every point of continuity of $C_*^i(\cdot)$ and for $t = T$ (cf. (3.4)) means that the sequence of measures on $[0,T]$ $dI_k^i(\omega,\cdot)$ converges weakly to $dC_*^i(\omega,\cdot)$ $\tilde{P}_c$-a.e. ω; that is (see, e.g., Billingsley (1986))

$$\lim_{k\to\infty}\int_0^T f(t)dI_k^i(t) = \int_0^T f(t)dC_*^i(t), \ \ \tilde{P}_c - a.s., \tag{3.5}$$

for every continuous and bounded function f. We now claim that the Komlós' limit $(\underline{x}_*, \underline{C}_*) := (x_*^1, \dots, x_*^n, C_*^1, \dots, C_*^n)$ belongs to $\mathcal{B}_w$ and that it is optimal for the social planner's problem (3.1). Indeed, $(\underline{X}_k, \underline{I}_k) := (X_k^1, \dots, X_k^n, I_k^1, \dots, I_k^n) \in \mathcal{B}_w$ by convexity of $\mathcal{B}_w$, and (3.3), (3.5) and Fatou's Lemma imply

$$\begin{aligned}
w &\geq \liminf_{k\to\infty}\sum_{i=1}^n E\Big[\int_0^T \psi_x(t)X_k^i(t)dt + \int_0^T \psi_c(t)dI_k^i(t)\Big] \\
&= \liminf_{k\to\infty}\sum_{i=1}^n \Big(\tilde{E}_x\Big[\int_0^T e^{-\int_0^t r_x(s)ds}X_k^i(t)dt\Big] + \tilde{E}_c\Big[\int_0^T e^{-\int_0^t r_c(s)ds}dI_k^i(t)\Big]\Big) \\
&\geq \sum_{i=1}^n \Big(\tilde{E}_x\Big[\int_0^T e^{-\int_0^t r_x(s)ds}x_*^i(t)dt\Big] + \tilde{E}_c\Big[\int_0^T e^{-\int_0^t r_c(s)ds}dC_*^i(t)\Big]\Big) \\
&= \sum_{i=1}^n E\Big[\int_0^T \psi_x(t)x_*^i(t)dt + \int_0^T \psi_c(t)dC_*^i(t)\Big];
\end{aligned} \tag{3.6}$$

that is, $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$. Recall now that $P_x \sim P$ and $P_c \sim P$. Then (3.4) and (3.3) also hold P-a.s. and $dP \otimes dt$-a.e., respectively, and therefore we may write

$$\sum_{i=1}^n \gamma^i U^i(x_*^i, C_*^i; C_*^{-i}) \geq \lim_{k\to\infty}\sum_{i=1}^n \gamma^i U^i(X_k^i, I_k^i; I_k^{-i}) = V_{SP}$$

by the uniform integrability assumed in Assumption 1.iii. and because $(\underline{X}_k, \underline{I}_k)$ is also a maximizing sequence by concavity of each U^i. Hence $(\underline{x}_*, \underline{C}_*)$ is optimal.

Step 2. The above arguments also extend to the infinite horizon case $T = +\infty$. We only sketch the proof, since some of the arguments are similar to those employed in *Step 1*.

Let $T = +\infty$ and $\{(\underline{x}_m, \underline{C}_m)\}_{m\in\mathbb{N}}$ be an admissible maximizing sequence for V_{SP}. For each $K \in \mathbb{N}$ we can apply the construction of *Step 1* to the interval $[0,K]$ to obtain measures $\tilde{P}_x^K$ and $\tilde{P}_c^K$ equivalent to P on $\mathscr{F}_K$ and limit processes $x_*^{K,i}$ and optional random measures $dC_*^{K,i}$ such that the convergence (3.3) resp. (3.4) holds on $[0,K]$. As the convergence occurs also $dP \otimes dt$ a.e., resp. P-a.s., we can aggregate the limits $\{x_*^{K,i}\}_{K\in\mathbb{N}}$ and $\{dC_*^{K,i}\}_{K\in\mathbb{N}}$ consistently to adapted processes x_*^i and optional measures dC_*^i while maintaining convergence. Now we can apply the budget estimate (3.6) on each $[0,K]$, let $K \to \infty$ and conclude by monotone convergence that $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$. Optimality obtains exactly as in *Step 1.*

Step 3. Finally, uniqueness of $(\underline{x}_*, \underline{C}_*)$ up to indistinguishability follows as usual (both in the finite and in the infinite time-horizon case) from strict concavity of the utility functions u^i, $i = 1, \dots, n$ and from convexity of $\mathcal{B}_w$. □

We now aim to characterize the social planner's optimal policy by means of a set of first order conditions for optimality. These conditions may be thought of as a stochastic, infinite dimensional generalization of the classical Kuhn-Tucker method and they have been used in various instances to solve singular stochastic control problems of the monotone follower type modeling irreversible investment problems or consumption problems with Hindy-Huang-Kreps preferences (see Bank (2005), Bank and Riedel (2001), Chiarolla et al. (2013), Chiarolla and Ferrari (2014), Ferrari (2015), Riedel and Su (2011) and Steg (2012), among others).

For any $(\underline{x}, \underline{C}) \in \mathcal{B}_w$ and for some Lagrange multiplier $\lambda > 0$, define the *Lagrangian functional* of problem (3.1) as

$$\begin{aligned}\mathcal{L}^w(\underline{x}, \underline{C}; \lambda) := &\sum_{i=1}^n \gamma^i E\left[\int_0^T e^{-\int_0^t r(s)\,ds} u^i(x^i(t), C(t))\,dt\right] \\ &+ \lambda\Big\{w - E\Big[\int_0^T \psi_x(t)x(t)dt + \int_0^T \psi_c(t)dC(t)\Big]\Big\},\end{aligned}$$

where again $x := \sum_{i=1}^n x^i$ and $C := \sum_{i=1}^n C^i$. Moreover, let $\mathcal{T}$ be the set of all $(\mathscr{F}_t)$-stopping times with values in $[0,T]$ a.s. and denote by $\nabla_c \mathcal{L}^w$ the Lagrangian functional's supergradient with respect to the aggregated public good's contribution; that is, the unique optional process such that

$$\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)(\tau) := E\left[\int_\tau^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i\, u^i_c(x^i(t), C(t))\,dt \,\middle|\, \mathscr{F}_\tau\right] - \lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}}$$

for any $\tau \in \mathcal{T}$. $\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)(\tau)$ may be interpreted as the future marginal net expected utility the social planner would have making an infinitesimal investment into the public good at time $\tau \in \mathcal{T}$ when the investment plan is $(\underline{x}, \underline{C}) \in \mathcal{B}_w$ and the Lagrange multiplier is λ.

On the other hand, an additional consumption of the private good x^i affects marginal utility only at those times at which consumption actually occurs, thus leading to

$$\nabla_x \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)(\tau) := \gamma^i e^{-\int_0^\tau r(s)ds} u^i_x(x^i(\tau), C(\tau)) - \lambda\psi_x(\tau), \quad \tau \in \mathcal{T}.$$

Remark 3.2. Following Remark 3.1 in Bank and Riedel (2001), $\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)$ is the Riesz representation of the Lagrangian gradient at C. More precisely, for any arbitrary but fixed $\lambda > 0$, define $\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)$ as the optional projection of the product-measurable process

$$\Phi(\omega, t) := \int_t^T e^{-\int_0^s r(\omega,u)\,du} \sum_{i=1}^n \gamma^i\, u^i_c(\omega, x^i(\omega, s), C(\omega, s))\,ds - \lambda\psi_c(\omega, t)\mathbb{1}_{\{t<T\}}$$

for $\omega \in \Omega$ and $t \in [0,T]$. Hence $\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)$ is uniquely determined up to P-indistinguishability and it holds

$$\mathbb{E}\Big\{\int_0^T \nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)dC(t)\Big\} = \mathbb{E}\Big\{\int_0^T \Phi(t)dC(t)\Big\}$$

for all admissible C (cf. Jacod (1979), Theorem 1.33).

Proposition 3.3. *Let Assumptions 1 and 2 hold. An admissible policy $(\underline{x}_*, \underline{C}_*)$ is optimal for the social planner's problem (3.1) if and only if there exists a Lagrange multiplier $\lambda > 0$ such that the following first order conditions hold true for any stopping time $\in \mathcal{T}$*

$$\begin{cases} E\Big[\int_0^T \psi_x(t)x_*(t)dt + \int_0^T \psi_c(t)dC_*(t)\Big] = w, \\ E\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_c^i(x_*^i(t), C_*(t))\,dt\Big|\mathscr{F}_\tau\Big] \le \lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}}, \;\; P-a.s., \\ E\Big[\int_0^T \Big(E\Big[\int_t^T e^{-\int_0^s r(u)\,du}\sum_{i=1}^n \gamma^i u_c^i(x_*^i(s), C_*(s))\,ds\Big|\mathscr{F}_t\Big] - \lambda\psi_c(t)\Big)dC_*(t)\Big] = 0, \\ \gamma^i e^{-\int_0^\tau r(s)ds}u_x^i(x_*^i(\tau), C_*(\tau)) \le \lambda\psi_x(\tau), \;\; P-a.s. \text{ with equality whenever } x_*^i(\tau) > 0. \end{cases} \tag{3.7}$$

The proof of Proposition 3.3 is given in Appendix A, Section A.1. It generalizes that of Theorem 3.2 in Bank and Riedel (2001), to the present setting of a multidimensional optimal consumption problem with both classical and monotone controls and it is obtained suitably adapting in the stochastic, infinite-dimensional setting the proof of the classical Kuhn-Tucker conditions. Indeed, concavity of the utility functions u^i, $i = 1, \dots, n$ yields sufficiency, whereas the proof of the necessity part is more delicate. One has indeed to linearize the original problem (3.1) around its optimal solution $(\underline{x}_*, \underline{C}_*)$ and then to show that $(\underline{x}_*, \underline{C}_*)$ solves the linearized problem as well. Finally, one must prove that any solution to the linearized problem (and therefore $(\underline{x}_*, \underline{C}_*)$ as well) satisfies some flat-off conditions as the third and the fourth ones of (3.7).

Notice that because of the Inada conditions (cf. Assumption 1.i.) the fourth one of (3.7) is binding at any $\tau \in \mathcal{T}$, i.e.

$$\gamma^i e^{-\int_0^\tau r(s)ds}u_x^i(x_*^i(\tau), C_*(\tau)) = \lambda\psi_x(\tau), \quad P-a.s.$$

Recalling that $(x, c) \mapsto u^i(x, c)$ is strictly concave (see again Assumption 1), and denoting by $g^i(\cdot, c)$ the inverse of $u_x^i(\cdot, c)$, we may write for any $\tau \in \mathcal{T}$

$$x_*^i(\tau) = g^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^\tau r(s)ds}\psi_x(\tau), C_*(\tau)\Big), \quad P-a.s. \tag{3.8}$$

Then, by plugging (3.8) into (3.7) we obtain the equivalent formulation of the first order conditions for optimality

$$\begin{cases} E\Big[\int_0^T \psi_x(t)x_*(t)dt + \int_0^T \psi_c(t)dC_*(t)\Big] = w, \\ E\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i h^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t), C_*(t)\Big)\,dt\Big|\mathscr{F}_\tau\Big] \leq \lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}}, \;\; P-a.s., \\ E\Big[\int_0^T \Big(E\Big[\int_t^T e^{-\int_0^s r(u)\,du}\sum_{i=1}^n \gamma^i h^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^s r(u)du}\psi_x(s), C_*(s)\Big)\,ds\Big|\mathscr{F}_t\Big] - \lambda\psi_c(t)\Big)dC_*(t)\Big] = 0, \\ e^{-\int_0^\tau r(s)ds}\gamma^i u_x^i(x_*^i(\tau), C_*(\tau)) = \lambda\psi_x(\tau), \;\; P-a.s., \end{cases} \tag{3.9}$$

for any $\tau \in \mathcal{T}$ and with $h^i(\psi, c) := u_c^i(g^i(\psi, c), c)$.

Although the first order conditions of Proposition 3.3 (or those in (3.9)) completely characterize the optimal policy, they are not binding at all times and so they cannot be directly used to determine C_* and consequently $\underline{x}_*$ by (3.8). As usual in the literature on stochastic control problems with monotone controls (see, e.g., Chiarolla and Haussmann (1994), El Karoui and Karatzas (1991), Karatzas (1981), Karatzas and Shreve (1984) as classical references), the optimal policy consists of keeping the controlled process close to some barrier (which is the free boundary of an associated optimal stopping problem in a Markovian setting) in a 'minimal way' (i.e. according to a Skorohod's reflection principle). Here we derive the social planner's optimal investment into the public good C_* in terms of the running supremum of an index process representing the desirable value of investment or consumption the agents would like to have. Mathematically, such an index process is the optional solution of a stochastic backward equation in the spirit of Bank-El Karoui (cf. Bank and El Karoui (2004), Theorem 1 and Theorem 3) and it may be represented in terms of the value functions of a family of standard optimal stopping problems (see also Bank and Föllmer (2003) for further details and applications).

Theorem 3.4. *Let Assumptions 1 and 2 hold and define for every $i = 1, \dots, n$ $g^i(\cdot, c)$ as the inverse of $u_x^i(\cdot, c)$, as well as $h^i(\psi, c) := u_c^i(g(\psi, c), c)$ for any $\psi, c > 0$. Then the unique solution of the social planner's problem (3.1) is*

$$\begin{cases} C_*(t) = \sum_{i=1}^n C_*^i(t) = (\sup_{0\leq u\leq t} l^*(u)) \vee 0 \\ x_*^i(t) = g^i\Big(\frac{\lambda}{\gamma^i}\psi_x(t), C_*(t)\Big), \quad i = 1, \dots, n \end{cases} \tag{3.10}$$

for a suitable Lagrange multiplier $\lambda > 0$ and where the optional, upper right-continuous process l^* uniquely solves the stochastic backward equation

$$E\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i h^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t), \sup_{\tau\leq u\leq t} l^*(u)\Big)\,dt\Big|\mathscr{F}_\tau\Big] = \lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}} \tag{3.11}$$

for any stopping time $\tau \in [0, T]$, P-a.s.

Proof. Existence of a unique optional, upper right-continuous solution l^* to (3.11) is shown in Appendix A, Proposition A.2. To show optimality of $(\underline{x}_*(t), C_*(t))$ as in (3.10) it then suffices to verify that it is admissible and it satisfies the sufficient and necessary first order conditions (3.9). By Theorem 33 in Chapter IV of Dellacherie and Meyer (1978), C_* as in (3.10) is adapted since l^* is optional; also, it has right-continuous sample paths since l^* is upper right-continuous. On the other hand, x^i_*, $i = 1, 2, \dots, n$, is adapted and positive, since g^i is continuous and positive. Moreover, for any $\tau \in \mathcal{T}$ we have

$$\begin{aligned} &E\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i h^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t), (\sup_{0\le u\le t} l^*(u))\vee 0\Big)\,dt\Big|\mathscr{F}_\tau\Big] \\ &\le E\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i h^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t), \sup_{\tau\le u\le t} l^*(u)\Big)\,dt\Big|\mathscr{F}_\tau\Big] = \lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}}, \qquad (3.12)\end{aligned}$$

where the first inequality follows from the fact that $c \mapsto h^i(\psi, c)$ is strictly decreasing, whereas (3.11) implies the last equality. On the other hand, if $\tau \in \mathcal{T}$ is a time of investment, i.e. such that $C_*(\tau+\varepsilon) - C_*(\tau) > 0$[4] for any $\varepsilon > 0$, we have $(\sup_{0\le u\le t} l^*(u)) \vee 0 = \sup_{\tau\le u\le t} l^*(u)$ and equality holds in (3.12). Therefore the second line of (3.9) is satisfied as well. The optimal private good consumption x^i_* of (3.10) is then determined by means of (3.8). □

Remark 3.5. The process l^* may be found numerically by backward induction on a discretized version of problem (3.11) (see Bank and Föllmer (2003), Section 4). In some cases, when $T = +\infty$, (3.11) has a closed form solution as in the case of a Cobb-Douglas utility function (see Section 5 below).

4 The Public Good Contribution Game

In Section 3 we have taken the point of view of a fictitious social planner aiming to efficiently maximize the social welfare. Here we aim to study strategic interaction between the agents of our economy.

Determining agent i's optimal choice of a strategy against a given process C^{-i} specifying aggregate contributions by the opponents amounts to solving the stochastic control problem with value function

$$V^i(C^{-i}) := \sup_{(x^i,C^i)\in\mathcal{B}_{w^i}} U^i(x^i, C^i; C^{-i}), \qquad i = 1, 2, \dots, n, \qquad (4.1)$$

where $\mathcal{B}_{w^i}$ and U^i are as in (2.1) and (2.2), respectively. The description of the game is completed by the introduction of a standard Nash equilibrium concept.

Definition 4.1. $(\hat{x}^1, \dots, \hat{x}^n, \hat{C}^1, \dots, \hat{C}^n)$ *is a* Nash equilibrium *if for all* $i \in \{1, \dots, n\}$, $(\hat{x}^i, \hat{C}^i) \in \mathcal{B}_{w^i}$ *and* $U^i(\hat{x}^i, \hat{C}^i, \hat{C}^{-i}) = V^i(\hat{C}^{-i})$.

While this equilibrium notion does not limit the ability of any agent to optimize against given strategies of the others, it does limit the extent of dynamic interaction that can take

[4]that is, a time of increase for $C_*(\omega, \cdot)$.

place. Although agents do react to the evolving exogenous uncertainty, they take the contribution processes of others as given and do not react to deviations from announced (equilibrium) play. Therefore, one might term such an equilibrium as one in *precommitment strategies*. Unfortunately there are serious conceptual difficulties in defining a related game with more explicit feedback strategies as argued by Back and Paulsen (2009), which is why we consider simple Nash equilibria here. Besides these conceptual problems, the choice of open-loop strategies can be justified at the modeling stage if agents are not able to observe the opponents' investments in the public good.

As in the social planner's case we shall first characterize solutions of the best reply problems (4.1) by means of a stochastic Kuhn-Tucker approach. The next Proposition accomplishes this. Its proof may be obtained by adopting arguments similar to those employed to prove Proposition 3.3, and therefore we omit its proof for the sake of brevity.

Proposition 4.2. *Let $\hat{C}^{-i}$ be given and Assumptions 1 and 2 hold. Then $(\hat{x}^i, \hat{C}^i) \in \mathcal{B}_{w^i}$ attains $V^i(\hat{C}^{-i})$ (cf. (4.1)) if and only if there exists a Lagrange multiplier $\lambda^i > 0$ such that for any stopping time $\tau \in \mathcal{T}$ the following first order conditions hold true*

$$\begin{cases} E\Big[\int_0^T \psi_x(t)\hat{x}^i(t)dt + \int_0^T \psi_c(t)d\hat{C}^i(t)\Big] = w^i, \\ E\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds} u_c^i(\hat{x}^i(t), \hat{C}(t))\,dt\Big|\mathscr{F}_\tau\Big] \le \lambda^i \psi_c(\tau), \quad P-a.s., \\ E\Big[\int_0^T \Big(E\Big[\int_t^T e^{-\int_0^s r(u)\,du} u_c^i(\hat{x}^i(s), \hat{C}(s))\,ds\Big|\mathscr{F}_t\Big] - \lambda^i\psi_c(t)\Big)d\hat{C}^i(t)\Big] = 0, \\ e^{-\int_0^\tau r(u)\,du} u_x^i(\hat{x}^i(\tau), \hat{C}(\tau)) \le \lambda^i\psi_x(\tau), \quad P-a.s. \text{ with equality whenever } \hat{x}^i(\tau) > 0. \end{cases} \tag{4.2}$$

Again, the Inada conditions (cf. Assumption 1.i.) imply that the fourth one of (4.2) is always binding. Hence, we may equivalently rewrite (4.2) as

$$\begin{cases} E\Big[\int_0^T \psi_x(t)\hat{x}^i(t)dt + \int_0^T \psi_c(t)d\hat{C}^i(t)\Big] = w^i, \\ E\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds} h^i(\lambda^i e^{\int_0^t r(s)\,ds}\psi_x(t), \hat{C}(t))\,dt\Big|\mathscr{F}_\tau\Big] \le \lambda^i \psi_c(\tau), \quad P-a.s., \\ E\Big[\int_0^T \Big(E\Big[\int_t^T e^{-\int_0^s r(u)\,du} h^i(\lambda^i e^{\int_0^s r(u)\,du}\psi_x(s), \hat{C}(s))\,ds\Big|\mathscr{F}_t\Big] - \lambda^i\psi_c(t)\Big)d\hat{C}^i(t)\Big] = 0, \\ e^{-\int_0^\tau r(u)\,du} u_x^i(\hat{x}^i(\tau), \hat{C}(\tau)) = \lambda^i\psi_x(\tau), \quad P-a.s., \end{cases} \tag{4.3}$$

where again $h^i(\psi, c) := u_c^i(g^i(\psi, c), c)$ with $g^i(\cdot, c)$ the inverse of $u_x^i(\cdot, c)$.

A more explicit characterization of the Nash equilibrium in the spirit of that shown in Theorem 3.4 for the social planner's optimal policy seems difficult to obtain in the generality of our public good contribution game. In fact the first order conditions in (4.3) are mutually interconnected and this leads to a daunting (infinite-dimensional) fixed point problem. However, when all the agents have the same utility function and the same initial wealth, each player is left with the same optimization problem (cf. (4.1)). In such a symmetric setting our approach enables us to show that the Nash equilibrium is triggered again (as in the social planner problem) by a single signal process ℓ^* which is uniquely determined through a stochastic backward equation. That is claimed in the following theorem. Its proof can be obtained by employing arguments completely similar to those used to prove Theorem 3.4 and therefore it is omitted.

Theorem 4.3. *Let Assumption 2 hold and assume that all the agents have the same initial wealth $w^i = w$ and the same utility function $u^i \equiv u$, for some utility function u satifying Assumption 1 and some $w > 0$. Let $g(\cdot, c)$ denote the inverse of $u_x(\cdot, c)$, as well as $h(\psi, c) := u_c(g(\psi, c), c)$ for any $\psi, c > 0$. Then, for a suitable Lagrange multiplier $\lambda > 0$, the Nash equilibrium of the public good contribution game (4.1) is given by*

$$\begin{cases} C^i_*(t) = (\sup_{0\leq u\leq t} \ell^*(u)) \vee 0, & i = 1, \dots, n, \\ x^i_*(t) = g\Big(\lambda\psi_x(t), C_*(t)\Big), & i = 1, \dots, n, \end{cases} \tag{4.4}$$

where the optional, upper right-continuous process ℓ^* uniquely solves the stochastic backward equation

$$E\bigg[\int_\tau^T e^{-\int_0^t r(s)\,ds} h\Big(\lambda e^{\int_0^t r(u)du}\psi_x(t), n \sup_{\tau\leq u\leq t} \ell^*(u)\Big)\,dt\bigg|\mathscr{F}_\tau\bigg] = \lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}} \tag{4.5}$$

for any stopping time $\tau \in [0, T]$, P-a.s.

5 Explicit Results and the Free Rider Effect

In this section we explicitly solve the backward equations (3.11) and (4.5) in a symmetric homogeneous setting so to find the social planner optimal policy and the Nash equilibrium strategy for the public good contribution game (cf. Theorem 3.4 and Theorem 4.4). Our model has utilities $u^i(x, c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$, $i = 1, \dots, n$, for some $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta < 1$, and prices $\psi_x(t) = e^{-rt}\mathcal{E}_x(t)$, $\psi_c(t) = e^{-rt}\mathcal{E}_c(t)$, for some exponential Lévy processes[5] $\mathcal{E}_c$ and $\mathcal{E}_x$ (including the important special case of geometric Brownian motion) and a constant interest rate $r > 0$. Moreover, $w^i = w$ for all $i = 1, \dots, n$. This setting will be kept throughout this section.

Remarkably, our approach will also enable us to study in Section 5.3 the role played by uncertainty and irreversibility of public good contributions in the so called free rider effect.

5.1 Explicit Results for a Symmetric Economy with Cobb-Douglas Utility

Proposition 5.1. *Assume $\gamma^i = \frac{1}{n}$ for every $i = 1, \dots, n$ and define the processes*

$$\gamma(t) := \frac{1}{A}\bigg[\Big(\frac{\alpha+\beta}{\alpha}\Big)\mathcal{E}_x(t) \inf_{0\leq s\leq t}\Big(\mathcal{E}_c^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}}(s)\mathcal{E}_x^{\frac{\alpha\beta}{1-\alpha-\beta}}(s)\Big)\bigg]^{-\frac{1}{1-\alpha}}, \tag{5.1}$$

[5] We refer to Bertoin (1996), among others, for a detailed introduction to Lévy processes. Notice also that the martingale property of Assumption 2 is without loss of generality in this case, one just has to correct r by the Lévy exponents of $\mathcal{E}_x$ and $\mathcal{E}_c$, respectively. In any case, the martingale property of $\mathcal{E}_x$ and $\mathcal{E}_c$ is not needed in the proof of the following results.

$$\theta(t) := \sup_{0\le s\le t}\left(\mathcal{E}_c^{-\frac{1-\alpha}{1-\alpha-\beta}}(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)\right), \tag{5.2}$$

and the constants

$$l_0 := \frac{nw}{\mathbb{E}\Big[\int_0^\infty \psi_x(t)\gamma(t)dt + \int_0^\infty \psi_c(t)d\theta(t)\Big]} \tag{5.3}$$

and

$$A := E\Big[\int_0^\infty \delta e^{-ru}\inf_{0\le s\le u}\left(\mathcal{E}_c(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(u-s)\right)du\Big] \tag{5.4}$$

with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\alpha-1}}$.

Then, if l_0 *and* A *are finite*[6], *the social planner's optimal solution is such that*

$$C_*(t) = l_0\theta(t) \tag{5.5}$$

and

$$x_*^i(t) = \frac{1}{n}l_0\gamma(t), \quad i = 1, \dots, n, \tag{5.6}$$

with

$$\lambda = \frac{1}{n^\alpha}A^{1-\alpha}l_0^{\alpha+\beta-1}.$$

Proof. By Theorem 3.4 to find the social planner optimal policy it suffices to solve backward equation (3.11).

Recall that $h^i(\psi, c) = u_c^i(g^i(\psi, c), c)$, where $g^i(\cdot, c)$ is the inverse of $u_x^i(\cdot, c)$. For any $\lambda > 0$, simple algebra leads to $h^i(\frac{\lambda}{\gamma^i}e^{rt}\psi_x(t), C(t)) = \delta(n\lambda\mathcal{E}_x(t))^{\frac{\alpha}{\alpha-1}}C^{\frac{\alpha+\beta-1}{1-\alpha}}(t)$ with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\alpha-1}}$. Set $C_*(t) = \sup_{0\le s\le t} l^*(s)\vee 0$ for some progressively measurable process l^* to be found and then (3.11) becomes

$$E\Big[\int_\tau^\infty \delta e^{-rs}(n\lambda\mathcal{E}_x(s))^{\frac{\alpha}{\alpha-1}}\Big(\sup_{\tau\le u\le s} l^*(u)\Big)^{\frac{\alpha+\beta-1}{1-\alpha}}ds\Big|\mathscr{F}_\tau\Big] = \lambda e^{-r\tau}\mathcal{E}_c(\tau),$$

i.e.,

$$E\Big[\int_0^\infty \delta e^{-ru}(n\lambda)^{\frac{\alpha}{\alpha-1}}\frac{\mathcal{E}_x^{\frac{\alpha}{\alpha-1}}(u+\tau)}{\mathcal{E}_c(\tau)}\inf_{0\le s\le u}\Big(l^{*\frac{\alpha+\beta-1}{1-\alpha}}(s+\tau)\Big)du\Big|\mathscr{F}_\tau\Big] = \lambda. \tag{5.7}$$

Make now the *ansatz* $l^*(t) := l_0\mathcal{E}_c^{\frac{1-\alpha}{\alpha+\beta-1}}(t)\mathcal{E}_x^{\frac{\alpha}{\alpha+\beta-1}}(t)$ for some constant l_0, and use independence and stationarity of Lévy increments to rewrite (5.7) as

$$\frac{1}{n^{\frac{\alpha}{1-\alpha}}}l_0^{\frac{\alpha+\beta-1}{1-\alpha}}E\Big[\int_0^\infty \delta e^{-ru}\inf_{0\le s\le u}\Big(\mathcal{E}_c(s)\mathcal{E}_x^{\frac{\alpha}{\alpha-1}}(u-s)\Big)du\Big] = \lambda^{\frac{1}{1-\alpha}}.$$

By setting $A := E[\int_0^\infty \delta e^{-ru}\inf_{0\le s\le u}\Big(\mathcal{E}_c(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(u-s)\Big)du]$ (cf. (5.4)) and by solving the previous equation for λ one easily obtains

$$\lambda := \frac{1}{n^\alpha}A^{1-\alpha}l_0^{\alpha+\beta-1}.$$

[6]Notice that l_0 of (5.3) and A of (5.4) are finite under some further specifications on the discount factor as it is shown in the proof of Proposition 5.4 in Section 5.3 below.

On the other hand, $x_*^i(t) = [n\lambda \left(\frac{\alpha+\beta}{\alpha}\right) \mathcal{E}_x(t) C_*^{-\beta}(t)]^{\frac{1}{\alpha-1}}$ by (3.8) and therefore

$$x_*^i(t) = (n\lambda)^{-\frac{1}{1-\alpha}} \left[\left(\frac{\alpha+\beta}{\alpha}\right) \mathcal{E}_x(t) l_0^{-\beta} \inf_{0\le s\le t} \left(\mathcal{E}_c^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}}(s) \mathcal{E}_x^{\frac{\alpha\beta}{1-\alpha-\beta}}(s) \right) \right]^{-\frac{1}{1-\alpha}};$$

that is,

$$x_*^i(t) = \frac{1}{n} l_0 \gamma(t) \tag{5.8}$$

with $\gamma(t)$ as in (5.1).

To determine l_0 we make use of the budget constraint $\mathbb{E}[\int_0^\infty \psi_x(t) x_*(t) dt + \int_0^\infty \psi_c(t) dC_*(t)] = nw$. In fact, recalling that $x_* := \sum_{i=1}^n x_*^i$, from (5.8) we find

$$l_0 E\left[\int_0^\infty \psi_x(t)\gamma(t) dt + \int_0^\infty \psi_c(t) d\theta(t) \right] = nw, \tag{5.9}$$

since $C_*(t) = \sup_{0\le s\le t} l^*(s) = l_0 \sup_{0\le s\le t} (\mathcal{E}_c^{-\frac{1-\alpha}{1-\alpha-\beta}}(s) \mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)) = l_0\theta(t)$ with $\theta(t)$ as in (5.2). By solving (5.9) for l_0 (5.3) follows. □

Remark 5.2.

i. As usual in stochastic control problems of the monotone follower type (see, e.g., the seminal paper by Karatzas and Shreve (1984)), the optimal aggregated public good level $C_*(t)$ (cf. (5.1) and (5.5)) is a singular process since it increases only on the boundary l^*, i.e. on a set of zero Lebesgue measure.

ii. The form of the optimal social planner's policy found in Proposition 5.1 exhibits a behaviour which is typical for a Cobb-Douglas utility function. Indeed the ratio $\frac{x_*^i(t)}{w}$ is independent of n, since $\frac{x_*^i(t)}{w} = \gamma(t)(E[\int_0^\infty \psi_x(t)\gamma(t)dt + \int_0^\infty \psi_c(t) d\theta(t)])^{-1}$, whereas $C_*(t) \sim n$ and $\lambda \sim n^{-(1-\beta)}$.

5.2 Explicit Results for a Symmetric Game with Cobb-Douglas Utility

In this section we explicitly solve the best reply problems (4.1). The proof of the following reasult employs arguments similar to those used for the proof of Proposition 5.1, and therefore it is given in Appendix A, Section A.3, for the sake of completeness.

Proposition 5.3. *Define the processes*

$$\gamma(t) := \frac{1}{A}\left[\left(\frac{\alpha+\beta}{\alpha}\right) \mathcal{E}_x(t) \inf_{0\le s\le t} \left(\mathcal{E}_c^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}}(s) \mathcal{E}_x^{\frac{\alpha\beta}{1-\alpha-\beta}}(s) \right) \right]^{-\frac{1}{1-\alpha}}, \tag{5.10}$$

$$\theta(t) := \sup_{0\le s\le t} \left(\mathcal{E}_c^{-\frac{1-\alpha}{1-\alpha-\beta}}(s) \mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s) \right), \tag{5.11}$$

and the constants

$$\kappa := \frac{w}{\mathbb{E}\left[\int_0^\infty \psi_x(t)\gamma(t) dt + \frac{1}{n}\int_0^\infty \psi_c(t) d\theta(t) \right]} \tag{5.12}$$

and

$$A := E\left[\int_0^\infty \delta e^{-ru} \inf_{0\le s\le u}\left(\mathcal{E}_c(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(u-s)\right)du\right] \tag{5.13}$$

with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\alpha-1}}$.

Then, if κ and A are finite[7]*, the symmetric Nash equilibrium of game (4.1) is given by*

$$\hat{C}^i(t) = \frac{\kappa}{n}\theta(t), \quad i = 1, \dots, n, \tag{5.14}$$

$$\hat{x}^i(t) = \kappa\gamma(t), \quad i = 1, \dots, n, \tag{5.15}$$

with

$$\lambda^i = A^{1-\alpha}\kappa^{\alpha+\beta-1}, \quad i = 1, \dots, n.$$

5.3 The Free Rider Effect

In economics, the free rider problem occurs when those who benefit from resources, goods, or services do not pay for them, which results in either an under-provision of those goods or services, or in an overuse or degradation of a common property resource. The free rider problem is common among public goods, because of their *non-excludability* - once provided it is for everybody - and *non-rivalry* - the consume of the good by an agent does not reduce the amount available to others. We refer to Cornes and Sandler (1996) or Laffont (1988) for more details.

Thanks to the results obntained in Sections 5.1 and 5.2, we are now able to explicitly evaluate the free rider effect for our symmetric economy with Cobb-Douglas utilities and Lévy uncertainty. Let x^i_* be the optimal private consumption in the social planner's problem (cf. (5.6)), and let $\hat{x}^i$ denote the Nash equilibrium private consumption (cf. (5.15)). Then one has

$$\begin{aligned} x^i_*(t) &= \frac{w\gamma(t)}{E\left[\int_0^\infty \psi_x(t)\gamma(t)dt + \int_0^\infty \psi_c(t)d\theta(t)\right]} \\ &\le \frac{w\gamma(t)}{E\left[\int_0^\infty \psi_x(t)\gamma(t)dt + \frac{1}{n}\int_0^\infty \psi_c(t)d\theta(t)\right]} = \hat{x}^i(t), \end{aligned}$$

with equality for $n = 1$. It follows that in a strategic context each agent spends more for the private consumption than what would be suggested by the social planner. On the other hand, we have $\kappa \le l_0$ (with κ as in (5.12), l_0 as in (5.3) and equality if $n = 1$) which implies that the social planner's optimal cumulative contribution into the public good (5.5) is bigger than the corresponding Nash equilibrium counterpart (5.14). That is, our model shows a free rider effect.

The evaluation of the free rider effect can be made even more explicit in a Black-Scholes setting and with the public good taken as a numéraire.

Proposition 5.4. *Let C_* be the optimal aggregated public good contribution for the social planner problem (cf. (5.5)) and let $\hat{C}$ denote its Nash-equilibrium value (cf. (5.14)). Assume*

[7] In the proof of Proposition 5.4 it is shown that κ of (5.12) and A of (5.13) are finite if the discount factor r is sufficiently big.

$\psi_c(t) = e^{-rt}$ and $\psi_x(t) = e^{-rt}\mathcal{E}_x(t) \equiv e^{-rt+\sigma W(t)}$, $\sigma > 0$, for a one-dimensional Brownian motion W and for some r such that $\sqrt{2r} > \frac{\sigma\alpha}{1-\alpha-\beta}$[8]. Then, for any $n \geq 1$ one has

$$\frac{\hat{C}(t)}{C_*(t)} = \frac{\kappa}{l_0} = \frac{\alpha+\beta}{n\alpha+\beta} \leq 1, \tag{5.16}$$

where κ and l_0 are as in (5.12) and (5.3), respectively.

Proof. From (5.5) and (5.14) it easily follows that

$$\frac{\hat{C}(t)}{C_*(t)} = \frac{\kappa}{l_0} = \frac{E\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt + \int_0^\infty e^{-rt}d\theta(t)\Big]}{E\Big[n\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt + \int_0^\infty e^{-rt}d\theta(t)\Big]}$$

$$= \frac{E\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt + r\int_0^\infty e^{-rt}\theta(t)dt\Big]}{E\Big[n\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt + r\int_0^\infty e^{-rt}\theta(t)dt\Big]}, \tag{5.17}$$

with $\gamma(t)$ and $\theta(t)$ as in (5.10) and (5.11), respectively. Then, in order to obtain (5.16), we need to evaluate

$$E\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt\Big] \qquad \text{and} \qquad E\Big[\int_0^\infty re^{-rt}\theta(t)dt\Big].$$

We have

$$E\Big[\int_0^\infty re^{-rt}\theta(t)dt\Big] = E\Big[\int_0^\infty re^{-rt}\sup_{0\leq s\leq t}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)\,dt\Big]$$

$$=E\Big[e^{-\frac{\sigma\alpha}{1-\alpha-\beta}\inf_{0\leq s\leq\tau_r}W(s)}\Big] = E\Big[e^{\frac{\sigma\alpha}{1-\alpha-\beta}\sup_{0\leq s\leq\tau_r}(-W(s))}\Big] \tag{5.18}$$

$$=\frac{\sqrt{2r}}{\sqrt{2r}-\frac{\sigma\alpha}{1-\alpha-\beta}}, \tag{5.19}$$

where τ_r is an independent exponentially distributed random time and where the last equality follows from $\sup_{0\leq s\leq\tau_r}(-W(s)) \sim Exp(\sqrt{2r})$ (cf., e.g., Bertoin (1996), Chapter VII). On the other hand, recall γ as in (5.10) and exploit that $W(t) - \sup_{0\leq u\leq t}W(u)$ is independent of $\sup_{0\leq u\leq t}W(u)$ (i.e. Excursion Theory for Lévy processes) and that $W(t) - \sup_{0\leq u\leq t}W(u)$ has

[8] Notice that the martingale property of Assumption 2 is without loss of generality in this case, one just has to correct r by $\frac{1}{2}\sigma^2$, i.e. by the Laplace exponents of $\sigma\frac{W(t)}{t}$.

the same distribution as $\inf_{0\leq u\leq t} W(u)$ (i.e. Duality Theorem) to find

$$
\begin{aligned}
&E\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt\Big]\\
=&\frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} E\Big[\int_0^\infty re^{-rt}e^{-\frac{\sigma\alpha}{1-\alpha}W(t)}e^{-\frac{\sigma\alpha\beta}{(1-\alpha)(1-\alpha-\beta)}\inf_{0\leq u\leq t}W(u)}dt\Big]\\
=&\frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} E\Big[e^{\frac{\sigma\alpha}{1-\alpha}(-W(\tau_r))}e^{\frac{\sigma\alpha\beta}{(1-\alpha)(1-\alpha-\beta)}\sup_{0\leq u\leq\tau_r}(-W(u))}\Big]\\
=&\frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} E\Big[e^{\frac{\sigma\alpha}{1-\alpha}[\tilde{W}(\tau_r))-\sup_{0\leq u\leq\tau_r}\tilde{W}(u)]}e^{\frac{\sigma\alpha}{1-\alpha-\beta}\sup_{0\leq u\leq\tau_r}\tilde{W}(u)}\Big]\\
=&\frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} E\Big[e^{\frac{\sigma\alpha}{1-\alpha}\inf_{0\leq u\leq\tau_r}\tilde{W}(u)}\Big]E\Big[e^{\frac{\sigma\alpha}{1-\alpha-\beta}\sup_{0\leq u\leq\tau_r}\tilde{W}(u)}\Big]\\
=&\frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} E\Big[e^{-\frac{\sigma\alpha}{1-\alpha}\sup_{0\leq u\leq\tau_r}W(u)}\Big]E\Big[e^{\frac{\sigma\alpha}{1-\alpha-\beta}\sup_{0\leq u\leq\tau_r}\tilde{W}(u)}\Big]\\
=&\frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} \left[\frac{\sqrt{2r}}{\sqrt{2r}+\frac{\sigma\alpha}{1-\alpha}}\right]\left[\frac{\sqrt{2r}}{\sqrt{2r}-\frac{\sigma\alpha}{1-\alpha-\beta}}\right],
\end{aligned}
$$

where we have defined the new Brownian motion $\tilde{W} := -W$ and where we have used once more $\sup_{0\leq s\leq\tau_r} W(s) \sim \sup_{0\leq s\leq\tau_r}\tilde{W}(s) \sim Exp(\sqrt{2r})$. Again, if τ_r is an independent exponentially distributed random time one has

$$
\begin{aligned}
A &= E\Big[\int_0^\infty \delta\, e^{-rt}\inf_{0\leq s\leq t}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(t-s)\,dt\Big] = \frac{\delta}{r}E\Big[e^{-\frac{\sigma\alpha}{1-\alpha}\sup_{0\leq s\leq\tau_r}W(\tau_r-s)}\Big]\\
&= \frac{\delta}{r}E\Big[e^{-\frac{\sigma\alpha}{1-\alpha}\sup_{0\leq s'\leq\tau_r}W(s')}\Big] = \frac{\delta}{r}\left[\frac{\sqrt{2r}}{\sqrt{2r}+\frac{\sigma\alpha}{1-\alpha}}\right]
\end{aligned}
$$

with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}}$ and because $\sup_{0\leq s\leq\tau_r}(-W(s)) \sim Exp(\sqrt{2r})$. Therefore

$$
E\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt\Big] = \frac{\alpha}{\beta}\left[\frac{\sqrt{2r}}{\sqrt{2r}-\frac{\sigma\alpha}{1-\alpha-\beta}}\right]. \tag{5.20}
$$

Finally, by plugging (5.18) and (5.20) into (5.17), some simple algebra leads to (5.16). □

We observe that the ratio $\hat{C}/C_*$, the underprovision of the public good due to free-riding, does not depend on σ, the volatility of the Brownian motion W. Thus, in our model

Corollary 5.5. *The degree of free-riding does not depend on the level of uncertainty.*

This seems to be in contrast to the finding of other different models from the economic literature in which it is shown that uncertainty may have some effect on the free rider effect (cf. Austen-Smith (1980), Eichberger and Kelsey (1999) and Ewald and Wang (2010), among others). Moreover, we show that also irreversibility of public good provisions do not have any effect on free-riding. These two results represent the main economically interesting conclusions of our paper.

We now evaluate the role that irreversibility of the public good contribution has in the free rider effect. To do so we compare the ratio (5.16) with the analogous one we shall obtain by assuming instead perfect reversibility of C; i.e., by assuming that each agent can adjust contribution in the public good freely at every point of time.

Proposition 5.6. *Assume perfect reversibility of the public good contribution. Denote by $C_\star$ the optimal aggregated public good contribution made by the social planner and by $\widetilde{C}$ its Nash equilibrium value. Then, under the same hypotheses of Proposition (5.4), one has*

$$\frac{\widetilde{C}(t)}{C_\star(t)} = \frac{\alpha+\beta}{n\alpha+\beta} \tag{5.21}$$

for any $n \geq 1$.

Proof. Under perfect reversibility of the public good contribution, the optimal investment criterion is to equate the marginal operating profit with the user cost of capital (see, e.g., Jorgenson (1963)). Hence the first-order conditions for optimality in the social planner's problem read

$$\begin{cases} \dfrac{\alpha}{\alpha+\beta}(x^i_\star)^{\alpha-1}(t)C^\beta_\star(t) = \lambda_\star n\mathcal{E}_x(t), \\[2ex] \dfrac{\beta}{\alpha+\beta}(x^i_\star)^{\alpha}(t)C^{\beta-1}_\star(t) = \lambda_\star r, \\[2ex] E\left[\displaystyle\int_0^\infty \psi_x(t)\sum_{i=1}^n x^i_\star(t)\,dt + r\int_0^\infty e^{-rt}C_\star(t)\,dt\right] = nw, \end{cases} \tag{5.22}$$

whereas for the Nash equilibrium they are

$$\begin{cases} \dfrac{\alpha}{\alpha+\beta}(\widetilde{x}^i)^{\alpha-1}(t)\widetilde{C}^\beta(t) = \widetilde{\lambda}\mathcal{E}_x(t), \\[2ex] \dfrac{\beta}{\alpha+\beta}(\widetilde{x}^i)^{\alpha}(t)\widetilde{C}^{\beta-1}(t) = \widetilde{\lambda} r, \\[2ex] E\left[\displaystyle\int_0^\infty \psi_x(t)\widetilde{x}^i(t)\,dt + r\int_0^\infty e^{-rt}\frac{1}{n}\widetilde{C}(t)\,dt\right] = w. \end{cases} \tag{5.23}$$

By solving systems (5.22) and (5.23) one easily obtains

$$\begin{cases} x^i_\star(t) = \left(\dfrac{r\alpha}{\beta}\right)\left[\lambda_\star\left(\dfrac{\alpha+\beta}{\alpha}\right)\left(\dfrac{r\alpha}{\beta}\right)^{1-\alpha}\right]^{-\frac{1}{1-\alpha-\beta}}(n\mathcal{E}_x(t))^{-\frac{(1-\beta)}{1-\alpha-\beta}}, \\[2ex] C_\star(t) = \left[\lambda_\star\left(\dfrac{\alpha+\beta}{\alpha}\right)\left(\dfrac{r\alpha}{\beta}\right)^{1-\alpha}\right]^{-\frac{1}{1-\alpha-\beta}}(n\mathcal{E}_x(t))^{-\frac{\alpha}{1-\alpha-\beta}}, \\[2ex] \lambda_\star^{-\frac{1}{1-\alpha-\beta}} = \dfrac{nw}{rn^{-\frac{\alpha}{1-\alpha-\beta}}\left(\frac{\alpha+\beta}{\beta}\right)\left[\left(\frac{r\alpha}{\beta}\right)^{1-\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)\right]^{-\frac{1}{1-\alpha-\beta}}E\left[\int_0^\infty e^{-rt}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(t)\,dt\right]}, \end{cases}$$

and

$$\begin{cases} \widetilde{x}^i(t) = \left(\frac{r\alpha}{\beta}\right)\left[\lambda_\star\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{r\alpha}{\beta}\right)^{1-\alpha}\right]^{-\frac{1}{1-\alpha-\beta}} \mathcal{E}_x^{-\frac{(1-\beta)}{1-\alpha-\beta}}(t), \\ \widetilde{C}(t) = \left[\lambda_\star\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{r\alpha}{\beta}\right)^{1-\alpha}\right]^{-\frac{1}{1-\alpha-\beta}} \mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(t), \\ \widetilde{\lambda}^{-\frac{1}{1-\alpha-\beta}} = \dfrac{nw}{rn^{-\frac{\alpha}{1-\alpha-\beta}}\left(\frac{n\alpha+\beta}{n\beta}\right)\left[\left(\frac{r\alpha}{\beta}\right)^{1-\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha-\beta}}\right]^{-\frac{1}{1-\alpha-\beta}} E\left[\int_0^\infty e^{-rt}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(t)\,dt\right]}, \end{cases}$$

with $E\left[\int_0^\infty e^{-rt}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(t)\,dt\right] < \infty$ since $\sqrt{2r} > \frac{\sigma\alpha}{1-\alpha-\beta}$. Then (5.21) easily follows. □

Corollary 5.7. *For any $n \geq 1$ one has*

$$\frac{\hat{C}(t)}{C_*(t)} = \frac{\widetilde{C}(t)}{C_\star(t)} \leq 1.$$

That is, irreversibility of the public good contributions does not influence the degree of free-riding.

In conclusion, we have shown that in our model, for a symmetric economy with Cobb-Douglas utilities, the degree of underprovision of the public good due to free-riding does not depend on irreversibility of the public good contributions or the level of uncertainty, when the latter is given by an exogenous one-dimensional Brownian motion. This interesting conclusion shelds new light on the old economic problem of public good contribution showing that irreversibility and uncertainty not necessarily mitigate the degree of free riding.

A Some Proofs and Technical Results

A.1 On the Proof of Proposition 3.3

In this section we prove Proposition 3.3. The proof is a generalization of Theorem 3.2 in Bank and Riedel (2001) to the case of a multivariate optimal consumption problem with both monotone and absolutely continuous controls. Sufficiency easily follows from concavity of the utility functions u^i, $i = 1, \dots, n$. On the other hand, the next Lemma accomplishes the proof of the necessity part. Necessity is proved by linearizing the original problem (3.1) around its optimal solution $(\underline{x}_*, \underline{C}_*)$, by showing that $(\underline{x}_*, \underline{C}_*)$ solves the linearized problem as well and that it satisfies some flat-off conditions as those of (3.7).

Recall the notation $x := \sum_{i=1}^n x^i$ and $C := \sum_{i=1}^n C^i$.

Lemma A.1. *Let Assumptions 1 and 2 hold and $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$ be optimal for problem (3.1) and set*

$$\Psi_*(t) := E\left[\int_t^T e^{-\int_0^s r(u)\,du}\sum_{i=1}^n \gamma^i\, u_c^i(x_*^i(s), C_*(s))\,ds \Big| \mathscr{F}_t\right]. \tag{A-1}$$

Then $(\underline{x}_*, \underline{C}_*)$

i. solves the linear optimization problem

$$\sup_{(\underline{x},\underline{C})\in\mathcal{B}_w} E\bigg[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t),C_*(t))x^i(t)dt+\int_0^T \Psi_*(t)dC(t)\bigg]; \tag{A-2}$$

ii. satisfies

$$\begin{cases} \Big(e^{-\int_0^t r(s)ds}\gamma^i u_x^i(x_*^i(t),C_*(t))-M\psi_x(t)\Big)\hat{x}^i(t)=0, \quad i=1,\dots,n, \\ E\bigg[\int_0^T \Big(\Psi_*(t)-M\psi_c(t)\Big)dC_*(t)\bigg]=0, \end{cases} \tag{A-3}$$

with

$$\begin{aligned} M := P-\operatorname{ess\,sup}\sup_{t\in[0,T]}\bigg[\max\bigg\{&\frac{e^{-\int_0^t r(s)ds}\gamma^1 u_x^1(x_*^1(t),C_*(t))}{\psi_x(t)},\dots \\ &\dots,\frac{e^{-\int_0^t r(s)ds}\gamma^n u_x^n(x_*^n(t),C_*(t))}{\psi_x(t)},\frac{\Psi_*(t)}{\psi_c(t)}\bigg\}\bigg]. \end{aligned} \tag{A-4}$$

Proof. The proof splits into two steps.

Step 1. Let $(\underline{x}_*,\underline{C}_*)\in\mathcal{B}_w$ be optimal for problem (3.1). For $(\underline{x},\underline{C})\in\mathcal{B}_w$ and $\epsilon\in[0,1]$, define the admissible strategy $(\underline{x}_\epsilon,\underline{C}_\epsilon)$ with $\underline{x}_\epsilon(t):=\epsilon\underline{x}(t)+(1-\epsilon)\underline{x}_*(t)$ and such that $C_\epsilon(t)=\epsilon C(t)+(1-\epsilon)C_*(t)$. Notice that $\underline{x}_\epsilon(t)$ and $C_\epsilon(t)$ converge to $\underline{x}_*(t)$ and $C_*(t)$, respectively, a.s. for $t\in[0,T]$ when $\epsilon\downarrow 0$. Now, optimality of $(\underline{x}_*,\underline{C}_*)$, concavity of u^i and an application of Fubini's Theorem allow us to write

$$\begin{aligned} 0 \;\geq\;& \frac{1}{\epsilon}[U_{SP}(\underline{x}_\epsilon,\underline{C}_\epsilon)-U_{SP}(\underline{x}_*,\underline{C}_*)] \\ \geq\;& E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n\gamma^i u_x^i(x_\epsilon^i(t),C_\epsilon(t))(x^i(t)-x_*^i(t))\,dt\bigg] \qquad \text{(A-5)} \\ &+E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n\gamma^i u_c^i(x_\epsilon^i(t),C_\epsilon(t))(C(t)-C_*(t))\,dt\bigg] \qquad \text{(A-6)} \\ =\;& E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n\gamma^i u_x^i(x_\epsilon^i(t),C_\epsilon(t))(x^i(t)-x_*^i(t))\,dt\bigg] \\ &+E\bigg[\int_0^T \Phi_\epsilon(t)(dC(t)-dC_*(t))\bigg], \end{aligned} \tag{A-7}$$

where $\Phi_\epsilon(t):=\int_t^T e^{-\int_0^s r(u)\,du}\sum_{i=1}^n\gamma^i\,u_c^i(x_\epsilon^i(s),C_\epsilon(s))\,ds$. One has

$$\begin{aligned} &\liminf_{\epsilon\downarrow 0} E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n\gamma^i u_x^i(x_\epsilon^i(t),C_\epsilon(t))x^i(t)dt\bigg] \\ &\geq E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n\gamma^i u_x^i(x_*^i(t),C_*(t))x^i(t)dt\bigg], \end{aligned}$$

and

$$\liminf_{\epsilon\downarrow 0} E\bigg[\int_0^T \Phi_\epsilon(t)dC(t)\bigg] \geq E\bigg[\int_0^T \Phi_*(t)dC(t)\bigg],$$

with $\Phi_* := \Phi_0$, by Fatou's Lemma. We now claim (and we prove it later) that

$$\lim_{\epsilon\downarrow 0} E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t))x_*^i(t)dt\bigg]$$
$$= E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))x_*^i(t)dt\bigg], \tag{A-8}$$

and

$$\lim_{\epsilon\downarrow 0} E\bigg[\int_0^T \Phi_\epsilon(t)dC_*(t)\bigg] = E\bigg[\int_0^T \Phi_*(t)dC_*(t)\bigg]. \tag{A-9}$$

Hence from (A-5)

$$E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))x^i(t)\,dt\bigg] + E\bigg[\int_0^T \Phi_*(t)dC(t)\bigg]$$
$$\leq E\bigg[\int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))x_*^i(t)\,dt\bigg] + E\bigg[\int_0^T \Phi_*(t)dC_*(t)\bigg].$$

and by replacing Φ_* with its optional projection Ψ_* as defined in (A-1) (cf. Jacod Jacod (1979), Theorem 1.33) it follows that $(\underline{x}_*, \underline{C}_*)$ is optimal for problem (A-2) as well.

To conclude the proof we must prove (A-8) and (A-9). To prove (A-8) it suffices to show that the family $(\Gamma_\epsilon^1)_{\epsilon\in[0,\frac{1}{2}]}$ given by

$$\Gamma_\epsilon^1(t) := e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t))x_*^i(t)$$

is $P\otimes dt$-uniformly integrable. Concavity of u^i and the fact that $x_\epsilon^i(t) \geq \frac{1}{2}x_*^i(t)$ a.s. for $\epsilon \in [0, \frac{1}{2}]$ and every $t \in [0, T]$ lead to

$$\Gamma_\epsilon^1(t) \leq 2e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t))x_\epsilon^i(t) \leq 2e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u^i(x_\epsilon^i(t), C_\epsilon(t)). \tag{A-10}$$

The last term in the right-hand side of (A-10) is $P \otimes dt$-uniformly integrable by Assumption 1.iii. Then (A-8) holds by Vitali's Convergence Theorem.

As for (A-9) note that by Fubini's Theorem

$$\int_0^T \Phi_\epsilon(t)dC_*(t) = \int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_c^i(x_\epsilon^i(t), C_\epsilon(t))C_*(t)dt.$$

Hence, to have (A-9) it suffices to show that the family

$$\Gamma_\epsilon^2(t) := e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_c^i(x_\epsilon^i(t), C_\epsilon(t))C_*(t)$$

is $P \otimes dt$-uniformly integrable, but this follows by employing arguments similar to those used for $(\Gamma^1_\epsilon)_{\epsilon\in[0,\frac{1}{2}]}$.

Step 2. We now show that the flat-off conditions (A-3) hold for any solution $(\underline{\hat{x}}, \underline{\hat{C}})$ of the linear problem (A-2). Then, by Step 1, they also hold for $(\underline{x}_*, \underline{C}_*)$.

Notice that for every $(\underline{x}, \underline{C}) \in \mathcal{B}_w$ one has

$$E\Big[\int_0^T \sum_{i=1}^n e^{-\int_0^t r(s)ds}\gamma^i u^i_x(x^i_*(t), C_*(t))x^i(t)dt + \int_0^T \Psi_*(t)dC(t)\Big]$$
$$\le ME\Big[\int_0^T \sum_{i=1}^n \psi_x(t)x^i(t)dt + \int_0^T \psi_c(t)dC(t)\Big] = Mw \qquad \text{(A-11)}$$

by definition of M (cf. (A-4)). Obviously, if $(\underline{x}, \underline{C})$ satisfies (A-3) we then have equality in (A-11). On the other hand, if

$$\sup_{(\underline{x},\underline{C})\in\mathcal{B}_w} E\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u^i_x(x^i_*(t), C_*(t))x^i(t)dt + \int_0^T \Psi_*(t)dC(t)\Big] = Mw, \qquad \text{(A-12)}$$

then equality holds through (A-11) and we obtain (A-3).

It therefore remains to prove (A-12). To this end take $K < M$ and define the stopping times

$$\begin{cases} \tau^i_K := \inf\{t \in [0,T) : e^{-\int_0^t r(s)ds}\gamma^i u^i_x(x^i_*(t), C_*(t)) \le K\psi_x(t)\} \wedge T, \quad i = 1, \dots, n, \\ \sigma_K := \inf\{t \in [0,T) : \Psi_*(t) > K\psi_c(t)\} \wedge T, \end{cases}$$

together with the investment strategies

$$x^i_K(t) := \alpha \mathbb{1}_{[0,\tau^i_K]}(t), \qquad C_K(t) := \alpha \mathbb{1}_{[\sigma_K,T]}(t),$$

for some α such that $E[\int_0^T \sum_{i=1}^n \psi_x(t)x^i_K(t)dt + \int_0^T \psi_c(t)dC_K(t)] = w$. We then have

$$\begin{aligned} Mw &\ge \sup_{(\underline{x},\underline{C})\in\mathcal{B}_w} E\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u^i_x(x^i_*(t), C_*(t))x^i(t)dt + \int_0^T \Psi_*(t)dC(t)\Big] \\ &\ge E\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u^i_x(x^i_*(t), C_*(t))x^i_K(t)dt + \int_0^T \Psi_*(t)dC_K(t)\Big] \\ &\ge K\,E\Big[\int_0^T \sum_{i=1}^n \psi_x(t)x^i_K(t)dt + \alpha\psi_c(\sigma_K)\mathbb{1}_{\{\sigma_K<T\}}\Big] \\ &\ge K\,E\Big[\int_0^T \sum_{i=1}^n \psi_x(t)x^i_K(t)dt + \int_0^T \psi_c(t)dC_K(t)\Big] = Kw, \end{aligned}$$

which yields (A-12) by letting $K \uparrow M$. □

We are now able to prove Proposition 3.3.

Proof of Proposition 3.3

Proof. Sufficiency follows from concavity of utility function u^i, $i = 1, \dots, n$, (cf. Assumption 1). Indeed, for $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$ satisfying (3.7) and for $(\underline{x}, \underline{C})$ any other admissible policy we may write

$$\begin{aligned} U_{SP}(\underline{x}_*, \underline{C}_*) - U_{SP}(\underline{x}, \underline{C}) \geq & E\Big[\int_0^T e^{-\int_0^t r(s)ds} \sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))(x_*^i(t) - x^i(t))dt\Big] \\ & + E\Big[\int_0^T e^{-\int_0^t r(s)ds} \sum_{i=1}^n \gamma^i u_c^i(x_*^i(t), C_*(t))(C_*(t) - C(t))dt\Big] \\ = & E\Big[\int_0^T e^{-\int_0^t r(s)ds} \sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))(x_*^i(t) - x^i(t))dt\Big] \\ & + E\Big[\int_0^T \Big(\int_t^T e^{-\int_0^s r(u)\,du} \sum_{i=1}^n \gamma^i u_c^i(x_*^i(s), C_*(s))\,ds\Big)(dC_*(t) - dC(t))\Big] \\ \geq & \lambda(w - w) = 0, \end{aligned}$$

where (3.7) and Fubini's Theorem lead to the second inequality, whereas the last one is implied by the first and the fourth of (3.7) and by the budget constraint. Finally, Lemma A.1 yields the proof of the necessary part. □

A.2 Proposition A.2

Proposition A.2. *Let Assumptions 1 and 2 hold, and define for every $i = 1, \dots, n$ $g^i(\cdot, c)$ as the inverse of $u_x^i(\cdot, c)$, as well as $h^i(\psi, c) := u_c^i(g(\psi, c), c)$ for any $\psi, c > 0$. Then there exists an optional process l^* which solves the backward stochastic equation*

$$E\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i h^i\Big(\frac{\lambda}{\gamma^i} e^{\int_0^t r(u)du} \psi_x(t), \sup_{\tau \leq u \leq t} l^*(u)\Big)\,dt \Big| \mathscr{F}_\tau\Big] = \lambda \psi_c(\tau) \mathbb{1}_{\{\tau < T\}} \tag{A-13}$$

for any $\tau \in \mathcal{T}$. Moreover, l^* has upper right-continuous sample paths and then it is unique up to indistinguishability.

Proof. Recall that the mapping $c \mapsto h^i(\psi, c)$ is the composition of $c \mapsto g^i(\psi, c)$ and $c \mapsto u_c^i(x, c)$, and hence it is continuous, strictly decreasing and satisfies the Inada conditions

$$\lim_{c \downarrow 0} h^i(\psi, c) = +\infty, \qquad \lim_{c \uparrow \infty} h^i(\psi, c) = 0. \tag{A-14}$$

These properties are inherited by the function $\sum_{i=1}^n \gamma^i h^i(\psi, \cdot)$, being $\gamma^i > 0$, $i = 1, \dots, n$. Moreover, for any given $\lambda > 0$ the process $\lambda \psi_c(t) \mathbb{1}_{\{t < T\}}$ vanishes at T, it is of class (D) and lower semicontinuous in expectation, by Assumption 2. Thanks to Inada conditions (A-14), we can suitably apply Theorem 3 of Bank and El Karoui (2004), as in the example in Section 3.1 therein, to have existence of an optional signal process l^* solving (A-13). Then, adopting arguments as those in the proof of Theorem 1 in Bank and Küchler (2007) one can show that such l^* is upper right-continuous and therefore it is unique up to indistinguishability by Bank and El Karoui (2004), Theorem 1, and Meyer's optional section theorem (see, e.g., Dellacherie and Meyer (1978), Theorem IV.86). □

A.3 Proof of Proposition 5.3

Due to Theorem 4.4, to find the Nash equilibrium strategy of the public good contribution game (4.1) in our homogeneous and symmetric setting it suffices to solve backward equation (4.5).

Recall that $h^i(\psi, c) := u^i_c(g^i(\psi, c), c)$ with $g^i(\cdot, c)$ the inverse of $u^i_x(\cdot, c)$. For any $\lambda^i > 0$, straightforward computations lead to $h^i(\lambda^i e^{rt}\psi_x(t), C(t)) = \delta(\lambda^i \mathcal{E}_x(t))^{\frac{\alpha}{\alpha-1}} C^{\frac{\alpha+\beta-1}{1-\alpha}}(t)$, with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\alpha-1}}$. Set $C^i_*(t) = \sup_{0\le s\le t} \ell^*(s) \vee 0$ for some progressively measurable process ℓ^* solving

$$E\left[\int_\tau^\infty \delta e^{-rs}(\lambda^i \mathcal{E}_x(s))^{\frac{\alpha}{\alpha-1}} \left(n \sup_{\tau\le u\le s} \ell^*(u)\right)^{\frac{\alpha+\beta-1}{1-\alpha}} ds \middle| \mathscr{F}_\tau\right] = \lambda^i e^{-r\tau}\mathcal{E}_c(\tau),$$

i.e.,

$$E\left[\int_0^\infty \delta e^{-ru}(\lambda^i)^{\frac{\alpha}{\alpha-1}} \frac{\mathcal{E}_x^{\frac{\alpha}{\alpha-1}}(u+\tau)}{\mathcal{E}_c(\tau)} \inf_{0\le s\le u}\left(n\ell^{*\frac{\alpha+\beta-1}{1-\alpha}}(s+\tau)\right) du \middle| \mathscr{F}_\tau\right] = \lambda^i. \tag{A-15}$$

Now take $\ell^*(t) := \frac{\kappa}{n}\mathcal{E}_c^{\frac{1-\alpha}{\alpha+\beta-1}}(t)\mathcal{E}_x^{\frac{\alpha}{\alpha+\beta-1}}(t)$ for some constant κ and use independence and stationarity of Lévy increments to rewrite (A-15) as

$$\kappa^{\frac{\alpha+\beta-1}{1-\alpha}} E\left[\int_0^\infty \delta e^{-ru} \inf_{0\le s\le u}\left(\mathcal{E}_c(s)\mathcal{E}_x^{\frac{\alpha}{\alpha-1}}(u-s)\right) du\right] = (\lambda^i)^{\frac{1}{1-\alpha}}. \tag{A-16}$$

By defining $A := E[\int_0^\infty \delta e^{-ru} \inf_{0\le s\le u}\left(\mathcal{E}_c(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(u-s)\right) du]$ (cf. (5.13)), and by solving (A-16) for λ^i one obtains

$$\lambda^i := A^{1-\alpha}\kappa^{\alpha+\beta-1}.$$

But now $x^i_*(t) = [\lambda^i \left(\frac{\alpha+\beta}{\alpha}\right)\mathcal{E}_x(t)C_*^{-\beta}(t)]^{\frac{1}{\alpha-1}}$, and therefore

$$x^i_*(t) = (\lambda^i)^{-\frac{1}{1-\alpha}}\left[\left(\frac{\alpha+\beta}{\alpha}\right)\mathcal{E}_x(t)\kappa^{-\beta} \inf_{0\le s\le t}\left(\mathcal{E}_c^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}}(s)\mathcal{E}_x^{\frac{\alpha\beta}{1-\alpha-\beta}}(s)\right)\right]^{-\frac{1}{1-\alpha}};$$

that is,

$$x^i_*(t) = \kappa\gamma(t) \tag{A-17}$$

with $\gamma(t)$ as in (5.10).

To determine κ we use the budget constraint $\mathbb{E}[\int_0^\infty \psi_x(t)x^i_*(t)dt + \int_0^\infty \psi_c(t)dC^i_*(t)] = w$. Indeed, by (A-17) we have

$$\kappa E\left[\int_0^\infty \psi_x(t)\gamma(t)dt + \frac{1}{n}\int_0^\infty \psi_c(t)d\theta(t)\right] = w, \tag{A-18}$$

since $C^i_*(t) = \sup_{0\le s\le t}\ell^*(s) = \frac{\kappa}{n}\sup_{0\le s\le t}(\mathcal{E}_c^{-\frac{1-\alpha}{1-\alpha-\beta}}(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)) = \kappa\theta(t)$ with $\theta(t)$ as in (5.11). Now the result follows by solving (A-18) for κ.

References

Austen-Smith, D. (1980). Individual contributions to public goods. *Econ. Letters 5*, 359–361.

Back, K. and D. Paulsen (2009). Open loop equilibria and perfect competition in option exercise games. *Rev. Financ. Stud. 22*(11), 4531–4552.

Bag, P. K. and S. Roy (2011). On sequential and simultaneous contributions under incomplete information. *Int. J. Game Theory 40*(1), 1–27.

Balder, E.J. (1990). New sequential compactness results for spaces of scalarly integrable functions. *J. Math. Anal. Appl. 151*(1), 1–16.

Bank, P. (2005). Optimal control under a dynamic fuel constraint. *SIAM J. Control Optim. 44*(4), 1529–1541.

Bank, P. and N. El Karoui (2004). A stochastic representation theorem with applications to optimization and obstacle problems. *Ann. Probab. 32*, 1030–1067.

Bank, P. and H. Föllmer (2003). American options, multi-armed bandits, and optimal consumption plans: A unifying view. In R. Carmona, E. Çinlar, I. Ekeland, E. Jouini, J. Scheinkman, and N. Touzi (Eds.), *Paris-Princeton Lectures on Mathematical Finance 2002*, Volume 1814 of *Lecture Notes in Math.*, pp. 1–42. Berlin: Springer.

Bank, P. and C. Küchler (2007). On Gittin's index theorem in continuous time. *Stochastic Process. Appl. 117*, 1357–1371.

Bank, P. and F. Riedel (2001). Optimal consumption choice with intertemporal substitution. *Ann. Appl. Probab. 11*, 750–788.

Bank, P. and F. Riedel (2003). Optimal dynamic choice of durable and perishable goods. Discussion Paper 28/2003, Bonn Graduate School of Economics.

Battaglini, M., S. Nunnari, and T. Palfrey (2014). Dynamic Free Riding with Irreversible Investments *Am. Econ. Rev. 104*(9), 2858–2871.

Bergstrom, T., L. Blume, and H. Varian (1986). On the private provision of public goods. *J. Public Econ. 29*, 25–49.

Bertoin, J. (1996). *Lévy Processes.* Cambridge: Cambridge University Press.

Billingsley, P. (1986). *Probability and Measures.* New York: John Wiley and Sons, Inc.

Chiarolla, M. B. and G. Ferrari (2014). Identifying the free boundary of a stochastic irreversible investment problem via the Bank-El Karoui representation theorem. *SIAM J. Control Optim. 52*(2), 1048–1070.

Chiarolla, M. B., G. Ferrari, and F. Riedel (2013). Generalized Kuhn-Tucker conditions for N-firm stochastic irreversible investment under limited resources. *SIAM J. Control Optim. 51*(5), 3863–3885.

Chiarolla, M. B. and U. G. Haussmann (1994). The Free Boundary of the Monotone Follower. *SIAM J. Control Optim. 32*(3), 690–727.

Cornes, R. and T. Sandler (1996). *The Theory of Externalities, Public Goods and Club Goods* (2nd ed.). Cambridge: Cambridge University Press.

Dellacherie, C. and P.-A. Meyer (1978). *Probabilities and Potential*, Volume 29 of *Mathematics Studies*. Amsterdam New York Oxford: North-Holland.

Eichberger, J. and D. Kelsey (1999). Free riders do not like uncertainty. Economic Series 9912, University of Saarland.

El Karoui, N. and I. Karatzas (1991). A new approach to the Skorohod problem, and its applications. *Stoch. Stoch. Rep. 34*, 57–82.

Ewald, C.O. and Wang, W.-K. (2010). Dynamic voluntary provision of public goods with uncertainty: A stochastic differential game model. *Decis. Econ. Finance 33*, 97–116.

Ferrari, G. (2015). On an integral equation for the free boundary of stochastic irreversible investment problems. *Ann. Appl. Probab. 25*(1), 150–176.

Fershtman, C. and S. Nitzan (1991). Dynamic voluntary provision of public goods. *Eur. Econ. Rev. 35*, 1057–1067.

Gradstein, M., S. Nitzan, and S. Slutsky (1993). Private provision of public goods under price uncertainty. *Soc. Choice Welfare 10*, 371–382.

Groves, T. and J. Ledyard (1977). Optimal allocation of public goods: A solution to the 'free rider' problem. *Econometrica 45*(4), 783–809.

Jacod, J. (1979). *Calcul Stochastique et Problèmes de Martingales*, Volume 714 of *Lecture Notes in Math.* Berlin Heidelberg New York: Springer.

Jorgenson, D. W. (1963). Capital theory and investment behavior. *Am. Econ. Rev. 53*, 247–257.

Kabanov, Y. (1999). Hedging and liquidation under transaction costs in currency markets. *Finance Stoch. 3*, 237–248.

Karatzas, I. (1981). The monotone follower problem in stochastic decision theory. *Appl. Math. Optim. 7*, 175–189.

Karatzas, I. and S. E. Shreve (1984). Connections between optimal stopping and singular stochastic control I. monotone follower problems. *SIAM J. Control Optim. 22*, 856–877.

Komlós, J. (1967). A generalization of the problem of Steinhaus. *Acta Math. Acad. Sci. Hungar. 18*, 217–229.

Laffont, J.-J. (1988). *Fundamentals of Public Economics*. Cambridge, Mass.: MIT Press.

Levhari, D. and L. J. Mirman (1980). The great fish war: An example using Nash-Cournot solution. *Bell J. Econ. 11*, 322–334.

Lockwood, B. and J. Thomas (2002). Gradualism and irreversibility. *Rev. Econ. Stud. 69*, 339–356.

Marx, L. M. and S. A. Matthews (2000). Dynamic voluntary contributions to a public project. *Rev. Econ. Stud. 67*, 327–358.

Matthews, S. A. (2011). Achievable outcomes of dynamic contribution games. PIER Working Paper 11 016, University of Pennsylvania.

Olson, M. (1965). *The Logic of Collective Action.* Cambridge, Mass.: Harvard University Press.

Palfrey, T. R. and H. Rosenthal (1984a). Participation and the provision of discrete public goods: A strategic analysis. *J. Public Econ. 24*, 171–193.

Palfrey, T. R. and H. Rosenthal (1984b). Testing game-theoretic models of free riding: New evidence on probability bias and learning. In T. Palfrey (Ed.), *Laboratory Research in Political Economy*, pp. 239–267. Ann Arbor: University of Michigan Press.

Riedel, F. and X. Su (2011). On irreversible investment. *Finance Stoch. 15*(4), 607–633.

Samuelson, P. (1954). The pure theory of public expenditure. *Rev. Econ. Stat. 36*, 387–389.

Sandler, T., F. Sterbenz, and J. Posnett (1987). Free riding and uncertainty. *Eur. Econ. Rev. 31*, 1605–1617.

Steg, J.-H. (2012). Irreversible investment in oligopoly. *Finance Stoch. 16*(2), 207–224.

Varian, H. R. (1994). Sequential contributions to public goods. *J. Public Econ. 53*, 165–186.

Wirl, F. (1996). Dynamic voluntary provision of public goods: Extension to nonlinear strategies. *Eur. J. Polit. Econ. 12*, 555–560.

Wirl, F. (2007). Do multiple Nash equilibria in Markov strategies mitigate the tragedy of the commons? *J. Econ. Dyn. Control 31*, 3723–3740.

Yeung, D. W. K. and L. A. Petrosyan (2013). Subgame consistent cooperative provision of public goods. *Dyn. Games Appl. 3*(3), 419–442.