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A Non Convex Singular Stochastic Control Problem and its Related Optimal Stopping Boundaries*

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Abstract. We show that the equivalence between certain problems of singular stochastic control (SSC) and related questions of optimal stopping known for convex performance criteria (see, for example, Karatzas and Shreve (1984)) continues to hold in a non convex problem provided a related discretionary stopping time is introduced. Our problem is one of storage and consumption for electricity, a partially storable commodity with both positive and negative prices in some markets, and has similarities to the finite fuel monotone follower problem. In particular we consider a non convex infinite time horizon SSC problem whose state consists of an uncontrolled diffusion representing a real-valued commodity price, and a controlled increasing bounded process representing an inventory. We analyse the geometry of the action and inaction regions by characterising the related optimal stopping boundaries.

Keywords: finite-fuel singular stochastic control; optimal stopping; free-boundary; smooth-fit; Hamilton-Jacobi-Bellman equation; irreversible investment.

MSC2010 subsject classification: 91B70, 93E20, 60G40, 49L20.

JEL classification: C02, C61, E22, D92.

1 Introduction and Problem Formulation

It is well known that convexity of the performance criterion suffices to link certain singular stochastic control problems to related problems of optimal stopping (cf. [15], [23] and [25], among others). In this paper we investigate the connection with optimal stopping for a non convex, infinite time-horizon, two-dimensional, degenerate singular stochastic control problem motivated by a storage-consumption problem for electricity. The non convexity arises because the Ornstein-Uhlenbeck electricity price model allows for both positive and negative prices.

We model the purchase of electricity over time at a stochastic real-valued spot price $(X_t)_{t\geq 0}$ for storage in a battery (for example, the battery of an electric vehicle). The battery must be full at a random terminal time, any deficit being met by a less efficient charging method so that the terminal spot price is weighted by a convex function Φ of the undersupply. We show in Appendix A that this optimisation problem is equivalent to solving the following problem.

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Letting $\lambda > 0$ and $c \in [0, 1]$ be constants, $\{\nu : \nu \in \mathcal{S}_c\}$ a set of admissible bounded increasing controls, $(X^x_t)_{t\geq 0}$ a continuous strong Markov process starting from $x \in \mathbb{R}$ at time zero and $C^{c,\nu}_t$ a process representing the inventory level at time t:

$$C^{c,\nu}_t = c + \nu_t, \quad t \geq 0, \tag{1.1}$$

the problem is to find

$$U(x, c) := \inf_{\nu\in\mathcal{S}_c} \mathcal{J}_{x,c}(\nu), \tag{1.2}$$

with

$$\mathcal{J}_{x,c}(\nu) := \mathsf{E}\bigg[\int_0^\infty e^{-\lambda s}\lambda X^x_s \Phi(C^{c,\nu}_s)ds + \int_0^\infty e^{-\lambda s} X^x_s \, d\nu_s\bigg], \tag{1.3}$$

and the minimising control policy ν^*.

Since electricity spot prices typically exhibit seasonality, mean reversion and volatility, the standard approach in the literature is to model such prices through a mean reverting process (see, e.g., [20] or [30] and references therein). Such prices may additionally take negative values at times due to the requirement to balance supply and demand in an electrical power system. Here we assume that X follows a standard time-homogeneous Ornstein-Uhlenbeck process[1] with positive volatility σ, positive adjustment rate θ and positive asymptotic (or equilibrium) value μ. On a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$, with $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by a one-dimensional standard Brownian motion $(B_t)_{t\geq 0}$ and augmented by P-null sets, we therefore take X^x as the unique strong solution of

$$\begin{cases} dX^x_t = \theta(\mu - X^x_t)dt + \sigma dB_t, \quad t > 0, \\ X^x_0 = x \in \mathbb{R}. \end{cases} \tag{1.4}$$

Since it is not possible to store electricity at large scales we assume that the amount of energy in the inventory is bounded above by 1 (this resembles a so-called *finite fuel* constraint, see for example [15]): for any initial level $c \in [0, 1]$ the set of admissible controls is

$$\begin{aligned}\mathcal{S}_c := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+, (\nu_t(\omega))_{t\geq 0} \text{ is nondecreasing, left-continuous, adapted} \\ \text{with } c + \nu_t \leq 1 \ \forall t \geq 0,\ \nu_0 = 0 \ \ \mathsf{P}-\text{a.s.}\},\end{aligned}$$

and ν_t represents the cumulative amount of energy purchased up to time t. From now on we make the following standing Assumption on the running cost factor $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$.

Assumption 1.1. *Φ lies in $C^2(\mathbb{R})$ and is decreasing and strictly convex with $\Phi(1) = 0$.*

We note that we do not cover with Assumption 1.1 the case when Φ is linear: the solution in this case follows as a corollary to Sections 2 and 3 below.

With these specifications problem (1.2) shares common features with the class of finite fuel, singular stochastic control problems of monotone follower type (see, e.g., [5], [11], [15], [16], [25] and [26] as classical references on finite fuel monotone follower problems). Such problems, with finite or infinite fuel and a running cost (profit) which is convex (concave) in the control variable, have been well studied for over 30 years (see, e.g., [1], [2], [4], [10], [15], [16], [22], [23], [25] and [26], among many others). Remarkably it turns out that convexity (or concavity), together with other more technical conditions, is sufficient to prove that such singular stochastic control problems are equivalent to related problems of optimal stopping; moreover the optimally controlled state process is the solution of a Skorohod reflection problem at the free boundary of the latter (see, e.g., [10], [15], [23], [25] and [26]).

[1] See Appendix B for general facts on the Ornstein-Uhlenbeck process.

In our case the factor Φ appearing in the running cost is strictly convex, the marginal cost $e^{-\lambda s}X^x_s\,d\nu_s$ of exercising control is linear in the control variable, and the set of admissible controls $\mathcal{S}_c$ (cf. (1.5)) is convex. However the Ornstein-Uhlenbeck process X^x of (1.4) can assume negative values with positive probability and is also a factor of the running cost so that the total expected cost functional (1.3) is not convex in the control variable. Therefore the connection between singular stochastic control and optimal stopping as addressed in [15], [23] and [25], among others, is no longer guaranteed for problem (1.2). To the best of our knowledge non convex singular stochastic control problems have not received significant attention so far and the behaviour of their optimal policies has not yet been thoroughly investigated. In this paper we show that while the optimal policy in problem (1.2) exhibits an interesting non-standard behaviour, it may nevertheless still be linked to related optimal stopping problems through the introduction of a discretionary stopping time.

We now briefly summarise the main findings that will be discussed and proved in detail in Sections 1.1, 2, 3 and 4. We begin in Section 1.1 by restating the problem (1.2) as a singular stochastic control problem with discretionary stopping (SSCDS, Eq. 1.8). To the best of our knowledge SSCDS problems were originally introduced in [12] in 1994. In that paper the authors aimed at minimising a quadratic cost depending on a Brownian motion linearly controlled by a bounded variation process, with a constant cost of exercising control. The case of finite fuel SSCDS was considered in 2000 in [28] where a terminal quadratic cost at the time of discretionary stopping was also included. A detailed analysis of singular control problems with discretionary stopping via variational inequalities may be found in [32] and [33].

Our SSCDS problem (1.8) exhibits three regimes depending on the sign of the function

$$k(c) := \lambda + \theta + \lambda\,\Phi'(c) \tag{1.5}$$

over $c \in [0,1]$. As shown in Section 2.1, when the abovementioned classical link to optimal stopping holds the function k appears in the running cost of the related optimal stopping problem (see Eq. 2.4). Since $c \mapsto k(c)$ is strictly increasing by the strict convexity of Φ (cf. Assumption 1.1) define $\hat{c} \in \mathbb{R}$ as the unique solution of

$$k(\hat{c}) = 0 \tag{1.6}$$

should one exist, in which case $\hat{c}$ may belong to $[0,1]$ or not depending on the choice of Φ and on the value of the parameters of the model.

In Section 2 we study the case in which $k(c) > 0$ for all $c \in [0,1]$ (and hence $\hat{c} < 0$, if it exists). We show that although problem (1.2) is non convex, the optimal control policy behaves as that of a convex finite fuel singular stochastic control problem of monotone follower type (cf., e.g., [15], [25] and [26]). That is, the optimal control ν^* is the minimal increase in inventory to keep the (optimally) controlled state variable inside the closure of the continuation region of an associated optimal stopping problem up to the time at which the inventory is full. Moreover, the directional derivative U_c of the control problem's value function coincides with the value function of the associated optimal stopping problem. In this case the discretionary stopping time in the equivalent formulation (1.8) plays no role at it is formally P-a.s. infinite.

On the other hand, in Section 3 we assume $k(c) < 0$ for all $c \in [0,1]$ (and hence $\hat{c} > 1$). In this case we find that the discretionary stopping feature in problem (1.8) dominates as the optimal control policy ν is a.s. zero. Equivalently, in the original problem (1.2) it is optimal to do nothing up to the first hitting time of X at a repelling barrier (in the language of [28]) and then to immediately fill the inventory. In particular the classical connection between SSC and optimal stopping observed to hold in the previous case breaks down for these parameter values and, to the best of our knowledge, this is a rare example of such an effect in the literature on SSC problems.

In Section 4 we perform a detailed heuristic analysis of the case in which $\hat{c}$ exists in $[0,1]$, showing that the solution to problem (1.8) involves exercising both singular stochastic control and discretionary stopping in nontrivial ways. Equivalently, both reflecting and repelling boundaries coexist in problem (1.2). A rigorous study of this case may require the development of new techniques and it is left as an interesting open problem. Finally, Appendix B collects some well known facts on the Ornstein-Uhlenbeck process used for X.

Before concluding this Section we observe that problem (1.2) may also fit in the economic literature as an irreversible investment problem with stochastic investment cost. It is well known that in the presence of a convex cost criterion (or concave profit) the optimal (stochastic) irreversible investment policy consists in keeping the production capacity at or above a certain reference level ℓ (see, e.g., [8], [14] and [34]; cf. also [3] among others for the case of stochastic investment costs) which has been recently characterized in [18] and [35] and referred to as *base capacity*. The index ℓ_t describes the desirable level of capacity at time t. If the firm has capacity $C_t > \ell_t$, then it faces excess capacity and should wait. If the capacity is below ℓ_t, then it should invest $\nu_t = \ell_t - C_t$ in order to reach the level ℓ_t.

Our analysis shows that in the presence of non convex costs it is not always optimal to invest just enough to keep the capacity at or above a base capacity level. In fact, for a suitable choice of the parameters ($\hat{c} < 0$) the optimal investment policy is of a purely dichotomous *bang-bang* type: either do not invest, or go to full capacity. On the other hand, for a different choice of the parameters ($\hat{c} > 1$) a base capacity policy is optimal regardless of the non convexity of the total expected costs. To the best of our knowledge this result is a novelty also in the mathematical economics literature on irreversible investment under uncertainty.

1.1 A Problem with Discretionary Stopping

In this Section we establish the equivalence between problem (1.2) and a finite-fuel type singular stochastic control problem with discretionary stopping (cf. [12] and [28] as classical references on this topic). For this it is useful to observe that, for fixed $x \in \mathbb{R}$, the process $(X^x_t)_{t\geq 0}$ and the processes $(I^x_t)_{t\geq 0}$, $(J^x_t)_{t\geq 0}$ defined by

$$I^x_t := \int_0^t e^{-\lambda s}\lambda X^x_s \Phi(C^{c,\nu}_s)ds \qquad \text{and} \qquad J^x_t := \int_{[0,t)} e^{-\lambda s} X^x_s d\nu_s \tag{1.7}$$

respectively are uniformly bounded in $L^2(\Omega, \mathsf{P})$ and hence uniformly integrable, which can be easily verified by standard properties of the Ornstein-Uhlenbeck process (1.4) (see Appendix B), Assumption 1.1, the finite fuel type condition $C^{c,\nu}_t = c + \nu_t \leq 1 \;\forall\, t \geq 0$ and an integration by parts.

Proposition 1.2. *Recall U from* (1.2)*. Then one has $U \equiv \hat{U}$ with*

$$\hat{U}(x,c) = \inf_{\nu\in\mathcal{S}_c,\,\tau\geq 0} \mathsf{E}\bigg[\int_0^\tau e^{-\lambda s}\lambda X^x_s\Phi(C^{c,\nu}_s)ds + \int_{[0,\tau)} e^{-\lambda s}X^x_s\,d\nu_s + e^{-\lambda\tau}X^x_\tau(1 - C^{c,\nu}_\tau)\bigg] \tag{1.8}$$

for $(x,c) \in \mathbb{R}\times[0,1]$ and where τ must be a P-a.s. finite stopping time.

Proof. Fix $(x,c) \in \mathbb{R}\times[0,1]$. Take in (1.8) a sequence of deterministic stopping times $(t_n)_{n\in\mathbb{N}}$ such that $t_n \uparrow \infty$ as $n \uparrow \infty$ and use uniform integrability, path continuity of X^x, I^x, J^x (cf. (1.7)) and that $\lim_{n\uparrow\infty} \mathsf{E}[e^{-\lambda t_n}X^x_{t_n}(1 - C^{c,\nu}_{t_n})] = 0$, to obtain $\hat{U} \leq U$ in the limit as $n \to \infty$. To show the reverse inequality, for any admissible $\nu \in \mathcal{S}_c$ and any stopping time $\tau \geq 0$ set

$$\hat{\nu}_t := \begin{cases} \nu_t, & t \leq \tau, \\ 1 - c, & t > \tau. \end{cases} \tag{1.9}$$

The control $\hat{\nu}$ is admissible and then from the definition of U (cf. (1.2)) it follows that

$$U(x,c) \leq \mathcal{J}_{x,c}(\hat{\nu}) = \mathsf{E}\bigg[\int_0^\tau e^{-\lambda s}\lambda X_s^x \Phi(C_s^{c,\nu})ds + \int_{[0,\tau)} e^{-\lambda s} X_s^x\, d\nu_s + e^{-\lambda\tau} X_\tau^x (1 - C_\tau^{c,\nu})\bigg].$$

Since the previous inequality holds for any admissible ν and any P-a.s. finite stopping time $\tau \geq 0$ we conclude that $U \leq \hat{U}$, hence $U \equiv \hat{U}$. □

Note. 1. Although the value functions (1.2) and (1.8) are equal, the existence of an optimal policy in problem (1.2) does not necessarily imply the existence of an optimal policy in (1.8). For example, in many cases (including convex or concave SSC problems) it turns out that P-a.s. finite stopping times are not optimal in (1.8) and one should formally take $\tau = +\infty$ a.s., hence the infimum in (1.8) is not attained.

2. At first sight problem (1.8) may appear to suffer from non-uniqueness of solutions since both the second (cost of control) and third (stopping cost) terms are linear in X. Taking the right-open interval $[0,\tau)$ in the second term addresses this issue. Henceforward it will be understood that integrals with respect to $(\nu_t)_{t\geq 0}$ are taken over right-open intervals and the usual integral notation will be used.

3. We will show that depending on the problem parameters, three proper regimes of optimal control with discretionary stopping arise in the formulation (1.8) and these now follow in Sections 2, 3 and 4.

2 The Case $\hat{c} < 0$

In this Section we address problem (1.2) assuming that $k(c) > 0$ (cf. (1.5)) for all $c \in [0,1]$; that is, $\hat{c} < 0$. The method employed is that of [23], solving a related optimal stopping problem and integrating its value function. We then verify that this yields both U and the optimal control η^*.

2.1 The Associated Optimal Stopping Problem

In the infinite time horizon optimal stopping problem

$$v(x;c) := \sup_{\sigma\geq 0} \mathsf{E}\bigg[- e^{-\lambda\sigma} X_\sigma^x + \int_0^\sigma e^{-\lambda s}\lambda\, X_s^x \Phi'(c)ds\bigg], \tag{2.1}$$

let σ be a P-a.s. finite stopping time. Problem (2.1) is parametric in c and it is the optimal stopping problem that we expect to be naturally associated (in the sense of [15], [23] or [25], among others) to control problem (1.2). Integrating by parts in (2.1) and noting that the martingale $(\int_0^t e^{-\lambda s}\sigma dB_s)_{t\geq 0}$ is uniformly integrable we can write

$$u(x;c) := v(x;c) + x = \sup_{\sigma\geq 0} \mathsf{E}\bigg[\int_0^\sigma e^{-\lambda s}\left[k(c)X_s^x - \theta\mu\right] ds\bigg], \tag{2.2}$$

with $k(c)$ as in (1.5). For each $c \in [0,1]$ we define the continuation and stopping regions of problem (2.2) by

$$\mathcal{C}_c := \{x : u(x;c) > 0\} \quad \text{and} \quad \mathcal{S}_c := \{x : u(x;c) = 0\}, \tag{2.3}$$

respectively. From standard arguments based on exit times from small balls one notes that $\mathcal{S}_c \subset \{x : x \leq \frac{\theta\mu}{k(c)}\}$ as it is never optimal to stop immediately in its complement $\{x : x > \frac{\theta\mu}{k(c)}\}$. Since $x \mapsto u(x;c)$ is increasing, $\mathcal{S}_c$ lies below $\mathcal{C}_c$ and we also expect the optimal stopping strategy to be of threshold type.

Now, for any given $c \in [0,1]$ and $\beta(c) \in \mathbb{R}$ we define the hitting time $\sigma_\beta(x,c) := \inf\{t \geq 0 : X^x_t \leq \beta(c)\}$. For simplicity we set $\sigma_\beta(x,c) = \sigma_\beta$. A natural candidate value function for problem (2.2) is of the form

$$u^\beta(x;c) = \begin{cases} \mathsf{E}\Big[\int_0^{\sigma_\beta} e^{-\lambda s}\,[k(c)X^x_s - \theta\mu]\,ds\Big], & x > \beta(c), \\ 0, & x \leq \beta(c). \end{cases} \tag{2.4}$$

To simplify notation set

$$G(x;c) := \mathsf{E}\Big[\int_0^{\infty} e^{-\lambda s}\,[k(c)X^x_s - \theta\mu]\,ds\Big]. \tag{2.5}$$

An application of Fubini's theorem, (B-1) and some simple algebra leads to

Lemma 2.1. *For all* $(x,c) \in \mathbb{R} \times [0,1]$ *one has*

$$G(x;c) = \frac{\mu(k(c)-\theta)}{\lambda} + \frac{k(c)(x-\mu)}{\lambda+\theta}, \tag{2.6}$$

$$G_x(x;c) = \frac{k(c)}{\lambda+\theta}. \tag{2.7}$$

Define the infinitesimal generator of the diffusion X^x by

$$\mathbb{L}_X f\,(x) := \frac{1}{2}\sigma^2 f''(x) + \theta(\mu - x)f'(x), \quad \text{for } f \in C^2_b(\mathbb{R}) \text{ and } x \in \mathbb{R}. \tag{2.8}$$

The analytical expression for u^β is provided in

Lemma 2.2. *For* u^β *as in* (2.4) *it holds*

$$u^\beta(x;c) = \begin{cases} G(x;c) - \dfrac{G(\beta(c);c)}{\phi_\lambda(\beta(c))}\phi_\lambda(x), & x > \beta(c) \\ 0, & x \leq \beta(c) \end{cases} \tag{2.9}$$

with G *as in* (2.6) *and with* ϕ_λ *the strictly decreasing fundamental solution of* $(\mathbb{L}_X - \lambda)f = 0$ *(cf.* (B-2) *in Appendix).*

Proof. From (2.4) and (2.6) we have that for all $x > \beta(c)$

$$\begin{aligned} u^\beta(x;c) &= \mathsf{E}\Big[\int_0^{\infty} e^{-\lambda s}\,[k(c)X^x_s - \theta\mu]\,ds\Big] - \mathsf{E}\Big[\int_{\sigma_\beta}^{\infty} e^{-\lambda s}\,[k(c)X^x_s - \theta\mu]\,ds\Big] \\ &= G(x;c) - \mathsf{E}\Big[\mathsf{E}\Big[\int_{\sigma_\beta}^{\infty} e^{-\lambda s}\,[k(c)X^x_s - \theta\mu]\,ds\Big|\mathcal{F}_{\sigma_\beta}\Big]\Big] \\ &= G(x;c) - \mathsf{E}\Big[e^{-\lambda\sigma_\beta}G(X^x_{\sigma_\beta};c)\Big], \end{aligned} \tag{2.10}$$

by the strong Markov property. Notice that $|G(x;c)| \leq L(c)(1+|x|)$ for some $L(c) > 0$ and recall that $(e^{-\lambda t}|X^x_t|)_{t\geq 0}$ is uniformly integrable. Since X^x is positively recurrent we have $e^{-\lambda\sigma_\beta}G(X^x_{\sigma_\beta};c) = e^{-\lambda\sigma_\beta}G(\beta(c);c)$, P-a.s. for $-\infty < \beta(c) < \infty$, and it follows that

$$u^\beta(x;c) = G(x;c) - G(\beta(c);c)\mathsf{E}_x[e^{-\lambda\sigma_\beta}] = G(x;c) - G(\beta(c);c)\frac{\phi_\lambda(x)}{\phi_\lambda(\beta(c))}, \tag{2.11}$$

where the last equality is due to well known properties of hitting times that we summarise in Appendix B for completeness (cf. (B-5)). $\square$

The candidate optimal boundary $\beta_*(c)$ is found by imposing the familiar *principle of smooth fit*, i.e. the continuity of the first derivative u^β_x at the boundary β_*, that is

$$G_x(\beta_*(c);c) - \frac{G(\beta_*(c);c)}{\phi_\lambda(\beta_*(c))}\phi'_\lambda(\beta_*(c)) = 0. \tag{2.12}$$

Proposition 2.3. *Define*

$$x_0(c) := -\frac{\theta\mu\Phi'(c)}{k(c)} > 0. \tag{2.13}$$

For each given $c \in [0,1]$ *there exists a unique solution* $\beta_*(c) \in (-\infty, x_0(c))$ *of* (2.12). *Moreover,* $\beta_* \in C^1([0,1])$ *and is strictly decreasing.*

Proof. Since we seek finite valued solutions of (2.12) and $\phi_\lambda(x) > 0$ for all $x \in (-\infty,+\infty)$ we may consider the equivalent problem of finding $x \in \mathbb{R}$ such that $H(x;c) = 0$, where

$$H(x;c) := G_x(x;c)\phi_\lambda(x) - G(x;c)\phi'_\lambda(x). \tag{2.14}$$

We first notice that $G(x_0(c);c) = 0$ (cf. (2.6) and (2.13)) and since $k(c) > 0$, then *(i)* $G(x;c) > 0$ for $x > x_0(c)$, *(ii)* $G(x;c) < 0$ for $x < x_0(c)$ and *(iii)* $G_x(x;c) > 0$ for all x. Hence

$$H(x_0(c);c) = G_x(x_0(c);c)\phi_\lambda(x_0(c)) > 0. \tag{2.15}$$

Recall also that ϕ_λ is strictly convex (cf. (B-2) and (B-4) in Appendix B), then it easily follows by (2.6) and (2.14) that

$$H_x(x;c) = -G(x;c)\phi''_\lambda(x) > 0, \qquad \text{for } x < x_0(c). \tag{2.16}$$

Moreover, $H(x;c) > 0$ for all $x \geq x_0(c)$ and so if $\beta_*(c)$ exists such that $H(\beta_*(c);c) = 0$ then $\beta_*(c) < x_0(c)$. Differentiation of (2.16) with respect to x gives

$$H_{xx}(x;c) = -G_x(x;c)\phi''_\lambda(x) - G(x;c)\phi'''_\lambda(x) < 0, \qquad \text{for } x < x_0(c),$$

which implies that $x \mapsto H(x;c)$ is continuous, strictly increasing and strictly concave on $(-\infty, x_0(c))$. Hence, by (2.15) there exists a unique $\beta_*(c) < x_0(c)$ solving $H(\beta_*(c);c) = 0$ (and equivalently (2.12)). Since $H_x(\beta_*(c);c) > 0$ for all $c \in [0,1]$ (cf. (2.16)), then $\beta_* \in C^1([0,1])$ from the implicit function theorem with

$$\beta'_*(c) = -\frac{H_c(\beta_*(c);c)}{H_x(\beta_*(c);c)}, \quad c \in [0,1]. \tag{2.17}$$

We now show that $c \mapsto \beta_*(c)$ is strictly decreasing. A direct study of the sign of the right hand side of (2.17) seems non-trivial so we use a different trick. It is not hard to verify from (2.13) that $c \mapsto x_0(c)$ is strictly decreasing since $c \mapsto \Phi'(c)$ is strictly increasing. Setting $\bar{x} := \beta_*(c)$ in (2.12), straightforward calculations give

$$\frac{\phi'_\lambda(\bar{x})}{\phi_\lambda(\bar{x})} = \frac{G_x(\bar{x};c)}{G(\bar{x};c)} = \frac{k(c)}{\bar{x}k(c) + \mu\theta\Phi'(c)} = \frac{1}{\bar{x} - x_0(c)}$$

so that $c \mapsto \frac{G_x(\bar{x};c)}{G(\bar{x};c)}$ is strictly decreasing. Since $c \mapsto x_0(c)$ is continuous it is always possible to pick $c' > c$ sufficiently close to c so that $\bar{x} < x_0(c') < x_0(c)$ (hence $G(\bar{x};c') < 0$) and one finds

$$\frac{G_x(\bar{x};c')}{G(\bar{x};c')} < \frac{\phi'_\lambda(\bar{x})}{\phi_\lambda(\bar{x})} \tag{2.18}$$

and therefore $H(\bar{x};c') > 0$. It follows that $\beta_*(c') < \beta_*(c)$, since $x \mapsto H(x;c)$ is increasing for $x < x_0(c')$. Then $c \mapsto \beta_*(c)$ is a strictly decreasing map. □

We verify the optimality of β_* in the next

Theorem 2.4. *The boundary β_* of Proposition 2.3 is optimal for (2.2) in the sense that*

$$\sigma^* = \inf\{t \geq 0 : X_t^x \leq \beta_*(c)\} \tag{2.19}$$

is an optimal stopping time and $u^{\beta_*} \equiv u$.

Proof. The candidate value function u^{β_*} (cf. (2.9)) is such that $u^{\beta_*}(\cdot, c) \in C^1(\mathbb{R})$ by Proposition 2.3 and

$$u_{xx}^{\beta_*}(x;c) = \begin{cases} -\frac{G(\beta_*(c);c)}{\phi_\lambda(\beta_*(c))}\phi_\lambda''(x), & x > \beta_*(c), \\ 0, & x < \beta_*(c). \end{cases} \tag{2.20}$$

Then $u^{\beta_*}(\cdot, c)$ is convex which implies that it is also nonnegative, since $u_x^{\beta_*}(\beta_*(c), c) = u^{\beta_*}(\beta_*(c), c) = 0$ by (2.9) and (2.12).

It is easily checked that

$$(\mathbb{L}_X - \lambda)u^{\beta_*}(x;c) = \begin{cases} \theta\mu - k(c)x, & x > \beta_*(c), \\ 0, & x \leq \beta_*(c). \end{cases} \tag{2.21}$$

We claim (and will prove later) that

$$\beta_*(c) < \frac{\theta\mu}{k(c)} =: \hat{x}_0(c) \tag{2.22}$$

so that $(\mathbb{L}_X - \lambda)u^{\beta_*}(x;c) \leq \theta\mu - k(c)x$ for all $x \in \mathbb{R}$.

Fix $(x, c) \in \mathbb{R} \times [0, 1]$. Take now $R > 0$ such that $\beta_*(c) \in (-R, R)$ and define $\tau_R := \inf\{t \geq 0 : X_t^x \notin (-R, R)\}$. By convexity of $u^{\beta_*}(\cdot, c)$, Ito-Tanaka's formula (see, [27], Chapter 3, Section 3.6 D, among others) gives

$$\mathsf{E}\left[e^{-\lambda(\tau_R\wedge\tau)}u^{\beta_*}(X_{\tau_R\wedge\tau}^x, c)\right] \leq u^{\beta_*}(x, c) + \mathsf{E}\left[\int_0^{\tau_R\wedge\tau} e^{-\lambda s}[\theta\mu - k(c)X_s^x]ds\right], \tag{2.23}$$

for an arbitrary stopping time $\tau \geq 0$ (possibly passing to the limit of a sequence of bounded stopping times). The contribution of the local time at β_* equals zero because the principle of smooth fit holds (cf. (2.12)) and the term involving the stochastic integral vanishes as it is a uniformly integrable martingale. Now $\tau_R \wedge \tau \uparrow \tau$ as $R \uparrow \infty$ and the integral inside the expectation on the right hand side of (2.23) is uniformly integrable. Then taking limits as $R \uparrow \infty$ and using that $u^{\beta_*} \geq 0$ we obtain

$$u^{\beta_*}(x;c) \geq \mathsf{E}\left[\int_0^{\tau} e^{-\lambda s}[k(c)X_s^x - \theta\mu]ds\right].$$

Since τ is arbitrary we can take the supremum over all stopping times to obtain $u^{\beta_*} \geq u$.

To prove the reverse inequality we take $\tau = \sigma^*$ to have strict inequality in (2.23). Then we notice that $0 \leq u^{\beta_*}(x, c) \leq |G(\beta_*(c); c)| + |G(x; c)|$ for $x > \beta_*(c)$ so that recurrence of X^x implies that

$$\left(e^{-\lambda\tau}u^{\beta_*}(X_\tau^x, c)\right)_{\tau\geq 0} \text{ is uniformly integrable and } e^{-\lambda\sigma^*}u^{\beta_*}(X_{\sigma^*}^x; c) = e^{-\lambda\,\sigma^*}u^{\beta_*}(\beta_*(c), c). \tag{2.24}$$

Therefore

$$\lim_{R\to\infty} \mathsf{E}\left[e^{-\lambda\,(\tau_R\wedge\sigma^*)}u^{\beta_*}(X_{\tau_R\wedge\sigma^*}^x, c)\right] = \mathsf{E}\left[e^{-\lambda\sigma^*}u^{\beta_*}(\beta_*(c), c)\right] = 0, \tag{2.25}$$

and in the limit we find $u^{\beta_*} = u$.

To conclude the proof we only need to show that (2.22) holds true. Set $\hat{x}_0 = \hat{x}_0(c)$ for simplicity. We have

$$\frac{H(\hat{x}_0; c)}{\phi_\lambda(\hat{x}_0)} = \frac{k(c)}{\lambda + \theta} - \frac{\theta\mu(k(c) - \theta)}{\lambda(\lambda + \theta)} \frac{\phi'_\lambda(\hat{x}_0)}{\phi_\lambda(\hat{x}_0)} \tag{2.26}$$

by (2.6), (2.14) and (2.13); since $\big(\mathbb{L}_X - \lambda\big)\phi_\lambda = 0$ and $\phi''_\lambda > 0$ we also have

$$\theta(\mu - \hat{x}_0)\phi'_\lambda(\hat{x}_0) - \lambda\phi_\lambda(\hat{x}_0) < 0. \tag{2.27}$$

It is clear that if $k(c) \geq \theta$ then the right hand side of (2.26) is strictly positive and $\beta_*(c) < \hat{x}_0(c)$. On the other hand, if $k(c) < \theta$ then $\mu - \hat{x}_0 < 0$ and from (2.27) we get

$$\frac{\phi'(\hat{x}_0)}{\phi(\hat{x}_0)} > \frac{\lambda}{\theta\mu}\left(\frac{k(c)}{k(c) - \theta}\right). \tag{2.28}$$

Now plugging (2.28) into the right hand side of (2.26) we find $H(\hat{x}_0; c)/\phi_\lambda(\hat{x}_0) > 0$ so that again $\beta_*(c) < \hat{x}_0(c)$. □

Remark 2.5. *In the case when $\hat{c} = 0$ (cf. (1.6)) one only has $\beta_* \in C^1((0, 1])$, as in fact $\lim_{c \downarrow \hat{c}} \beta_*(c) = +\infty$ along with its derivative. For $c = \hat{c}$ the optimal stopping time for problem (2.2) is $\sigma^* = 0$ for any $x \in \mathbb{R}$.*

2.2 Solution to the Stochastic Control Problem

In this Section we aim at providing a solution to the singular stochastic control problem (1.2) by starting from the solution of the optimal stopping problem (2.2) and guessing that the classical connection to singular stochastic control holds.

By Proposition 2.3 we know that $c \mapsto \beta_*(c)$ is strictly decreasing and so has a strictly decreasing inverse. We define this inverse

$$g_*(x) := \beta_*^{-1}(x), \qquad x \in \mathbb{R}. \tag{2.29}$$

Obviously $g_* : \mathbb{R} \to [0, 1]$ is continuous. Moreover, since $\beta_* \in C^1$ and $\beta'_* < 0$ (cf. again Proposition 2.3), then $g_* \in C^1$ on $\mathbb{R}$ except at points $x = \beta_*(1)$ and $x = \beta_*(0)$; however g'_* exists almost everywhere and is bounded.

Define the function

$$F(x, c) := -\int_c^1 v(x; y)dy = x(1 - c) - \int_c^1 u(x; y)dy. \tag{2.30}$$

We expect that $F(x, c) = U(x, c)$ for all $(x, c) \in \mathbb{R} \times [0, 1]$, with U as defined in (1.2).

Proposition 2.6. *The function $F(x, c)$ in (2.30) is such that $x \mapsto F(x, c)$ is concave, $F \in C^{2,1}(\mathbb{R} \times [0, 1])$ and the following bounds hold*

$$\big|F(x, c)\big| + \big|F_c(x, c)\big| \leq C_1(1 + |x|), \quad \big|F_x(x, c)\big| + \big|F_{xx}(x, c)\big| \leq C_2 \tag{2.31}$$

for $(x, c) \in \mathbb{R} \times [0, 1]$ and some positive constants C_1 and C_2.

Proof. Recall (2.9) and that $u^{\beta_*} \equiv u$ (cf. Theorem 2.4). Concavity of F as in (2.30) easily follows by observing that $x \mapsto u(x; c)$ is convex (cf. again Theorem 2.4). It is also easy to verify from (2.6) and (2.9) that u is of the form $u(x; c) = A(c)P(x) + B(c)$ for suitable continuous functions A, B and P, so that $(x, c) \mapsto F(x, c)$ is continuous on $\mathbb{R} \times [0, 1]$ and $c \mapsto F_c(x, c)$ is continuous on $[0, 1]$ as well. From the definition of u^{β_*} (cf. (2.9)), (2.12), (2.20) and continuity of β_* it is straightforward to verify that for $x \in K \subset \mathbb{R}$ with K bounded, $|u_x|$ and $|u_{xx}|$ are at least bounded by a function $Q_K(c) \in L^1(0,1)$. It follows that evaluating F_x and F_{xx} one can take derivatives inside the integral in (2.30) to obtain

$$F_x(x,c) = (1-c) - \int_c^1 u_x(x;y)dy = (1-c) - \int_{g_*(x)\vee c}^1 u_x(x;y)dy \tag{2.32}$$

and

$$F_{xx}(x,c) = -\int_c^1 u_{xx}(x;y)dy = -\int_{g_*(x)\vee c}^1 u_{xx}(x;y)dy. \tag{2.33}$$

Therefore $F \in C^{2,1}$ by (2.9), (2.12), (2.20) and continuity of $g_*(\cdot)$ (cf. (2.29)). In particular for F_{xx} we obtain

$$F_{xx}(x,c) = \phi''_\lambda(x) \int_{g_*(x)\vee c}^1 \frac{G(\beta_*(y),y)}{\phi_\lambda(\beta_*(y))}dy \tag{2.34}$$

due to the affine nature of $x \mapsto G(x,c)$ (cf. (2.6)).

Recall now that $\phi_\lambda(x)$ and all its derivatives approach zero as $x \to \infty$ and that $g_*(x) = 1$ for $x < \beta_*(1)$ and $g_*(x) = 0$ for $x > \beta_*(0)$. Then bounds (2.31) follow from (2.9), (2.30), (2.32) and (2.33). □

From standard theory of stochastic control (e.g., see [19], Chapter VIII), we expect that the value function U of (1.2) identifies with an appropriate solution w to the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\{-\mathbb{L}_X w + \lambda w - \lambda x\Phi(c), -w_c - x\} = 0 \quad \text{for all } (x,c) \in \mathbb{R} \times [0,1]. \tag{2.35}$$

Recall Proposition 2.6.

Proposition 2.7. *For all $(x,c) \in \mathbb{R} \times [0,1]$ we have that F is a classical solution of*

$$\max\{-\mathbb{L}_X F + \lambda F - \lambda x\Phi(c), -F_c - x\} = 0. \tag{2.36}$$

Proof. First we observe that (2.6) and (2.30) give

$$F(x;c) = \mu\Phi(c) + (x-\mu)\frac{\lambda\Phi(c)}{\lambda+\theta} + \phi_\lambda(x)\int_c^1 \frac{G(\beta_*(y);y)}{\phi_\lambda(\beta_*(y))}dy \quad \text{for all } c > g_*(x) \tag{2.37}$$

For any fixed $c \in [0,1]$ and $x \in \mathbb{R}$ such that $F_c(x;c) > -x$, i.e. $c > g_*(x)$ (cf. (2.30)), one has

$$(\mathbb{L}_X - \lambda)F(x;c) = -\lambda\Phi(c)x$$

by (2.37). On the other hand, for arbitrary $(x,c) \in \mathbb{R} \times [0,1]$ we notice that

$$(\mathbb{L}_X - \lambda)F(x;c) = (1-c)(\theta\mu - (\lambda+\theta)x) - \int_c^1 (\mathbb{L}_X - \lambda)u(x;y)dy$$

by (2.32) and (2.33). Now, recalling (2.21) one has

$$\int_c^1 (\mathbb{L}_X - \lambda)u(x;y)dy \leq \int_c^1 [\theta\mu - k(y)x]dy = [\theta\mu - (\lambda+\theta)x](1-c) + \lambda\Phi(c)x,$$

since $\theta\mu - k(c)x \geq 0$ when $F_c(x;c) = -x$, i.e. $c < g_*(x)$, by (2.22). Then

$$(\mathbb{L}_X - \lambda)F(x;c) \geq -\lambda\Phi(c)x \quad \text{for all } (x,c) \in \mathbb{R} \times [0,1].$$

□

We now aim at providing a candidate optimal control policy ν^* for problem (1.2). Let $(x, c) \in \mathbb{R} \times [0,1]$ and consider the process

$$\nu^*_t = \Big[g_*\big(\inf_{0 \le s \le t} X^x_s\big) - c\Big]^+, \qquad t > 0, \quad \nu^*_0 = 0, \tag{2.38}$$

with g_* as in (2.29).

Proposition 2.8. *The process ν^* of* (2.38) *is an admissbile control.*

Proof. Fix $\omega \in \Omega$ and recall (1.5). By definition $t \mapsto \nu^*_t(\omega)$ is clearly increasing and such that $C^{c,\nu^*}_t(\omega) \le 1$, for any $t \ge 0$, since $0 \le g_*(x) \le 1$, $x \in \mathbb{R}$. The map $x \mapsto g_*(x)$ is continuous, then $t \mapsto \nu^*_t(\omega)$ is continuous, apart from a possible initial jump at $t = 0$, by continuity of paths $t \mapsto X^x_t(\omega)$.

To prove that $\nu^* \in \mathcal{S}_c$ it thus remains to show that ν^* is $(\mathcal{F}_t)$-adapted. To this end, first notice that continuity of $g_*(\cdot)$ also implies its Borel measurability and hence progressive measurability of the process $g_*(X^x)$. Then ν^* is progressively measurable since $g_*\big(\inf_{0 \le s \le t} X^x_s\big) = \sup_{0 \le s \le t} g_*(X^x_s)$, by monotonicity of g_*, and by [13], Theorem IV.33. Hence ν^* is $(\mathcal{F}_t)$-adapted. $\square$

To show optimality of ν^* we introduce the *action* and *inaction* sets

$$\mathcal{C} := \big\{(x,c)\,:\, F_c(x;c) > -x\big\} \quad \text{and} \quad \mathcal{S} := \big\{(x,c)\,:\, F_c(x;c) = -x \tag{2.39}$$

respectively, with $(x, c) \in \mathbb{R} \times [0,1]$. Recalling that $F_c = u$ makes clear their connection to the sets defined in (2.3).

Proposition 2.9. *Let $C^*_t := C^{c,\nu^*}_t = c + \nu^*_t$, with ν^* as in* (2.38)*. Then ν^* solves the Skorohod problem*

1. $(C^*_t, X^x_t) \in \overline{\mathcal{C}}$, P*-almost surely, for each $t > 0$;*
2. $\displaystyle\int_0^T e^{-\lambda t}\mathbb{1}_{\{(C^*_t, X^x_t) \in \mathcal{C}\}} d\nu^*_t = 0$ *almost surely, for all $T \ge 0$,*

where $\overline{\mathcal{C}} := \{(x,c) : c \ge g_*(x)\}$ denotes the closure of the inaction region $\mathcal{C}$ (cf. (2.39)*).*

Proof. The result is somewhat standard (see, e.g., [27], p. 210 and [36] as classical references on the topic). We provide here its proof for completeness.

By monotonicity of g_* we have

$$C^*_t = c + \nu^*_t = c + \Big[g_*\big(\inf_{0 \le s \le t} X^x_s\big) - c\Big]^+ \ge g_*(X^x_t) \wedge 1 = g_*(X^x_t),$$

since $0 \le g_* \le 1$. Hence 1. follows.

To prove 2. fix $\omega \in \Omega$ and suppose that for some $t > 0$ we have $(C^*_t(\omega), X^x_t(\omega)) \in \mathcal{C}$, i.e. $C^*_t(\omega) > g_*(X^x_t(\omega))$. We distinguish two cases. In the case that $g_*\left(\inf_{0 \le u \le t} X^x_u(\omega)\right) \ge c$, we have $g_*\left(\inf_{0 \le u \le t} X^x_u(\omega)\right) = C^*_t(\omega) > g_*(X^x_t(\omega))$ and then by monotonicity of g_* we have $\inf_{0 \le u \le t} X^x_u(\omega) < X^x_t(\omega)$. By continuity of $t \mapsto X^x_t(\omega)$ we deduce that $r \mapsto \inf_{0 \le u \le r} X^x_u(\omega)$ is constant in the interval $r \in [t, t+\epsilon(\omega))$ for some $\epsilon(\omega) > 0$. In the case that $g_*\left(\inf_{0 \le u \le t} X^x_u(\omega)\right) < c$, we have $c = C^*_t(\omega) > g_*(X^x_t(\omega))$ and then again by monotonicity and continuity of g_*, continuity of $X^x_t(\omega)$, there exists $\epsilon(\omega) > 0$ such that $c > g_*\left(\inf_{0 \le u \le t+\epsilon(\omega)} X^x_u(\omega)\right)$ and so $\nu^*_r(\omega) = 0$ for all $r \in [0, t+\epsilon(\omega))$.

Summarising, we have shown that if $(C^*_t(\omega), X^x_t(\omega)) \in \mathcal{C}$ then ν^* is constant in a right (stochastic) neighbourhood of t, establishing the second part. $\square$

Theorem 2.10. *The control* ν^* *defined in* (2.38) *is optimal for problem* (1.2) *and* $F \equiv U$ *(cf.* (2.30)*).*

Proof. The proof is based on a verification argument and, as usual, divides into two steps.

Step 1. Fix $(x, c) \in \mathbb{R} \times [0, 1]$ and take $R > 0$. Set $\tau_R := \inf\{t \geq 0 : X^x_t \notin (-R, R)\}$, take an admissible control ν, and recall the regularity results for F of Proposition 2.6. Then we can use Ito's formula in its classical form up to the stopping time $\tau_R \wedge T$, for some $T > 0$, to obtain

$$\begin{aligned} F(x;c) =& \mathsf{E}\left[e^{-\lambda(\tau_R\wedge T)}F(X^x_{\tau_R\wedge T}, C^{c,\nu}_{\tau_R\wedge T})\right] - \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}(\mathbb{L}_X - \lambda)F(X^x_s, C^{c,\nu}_s)ds\right] \\ &- \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}F_c(X^x_s, C^{c,\nu}_s)d\nu_s\right] \\ &- \mathsf{E}\left[\sum_{0\leq s<\tau_R\wedge T} e^{-\lambda s}\left(F(X^x_s, C^{c,\nu}_{s+}) - F(X^x_s, C^{c,\nu}_s) - F_c(X^x_s, C^{c,\nu}_s)\Delta\nu_s\right)\right] \end{aligned}$$

where $\Delta\nu_s := \nu_{s+} - \nu_s$ and the expectation of the stochastic integral vanishes since F_x is bounded on $(x, c) \in [-R, R] \times [0, 1]$.

Now, recalling that any $\nu \in \mathcal{S}_c$ can be decomposed into the sum of its continuous part and of its pure jump part, i.e. $d\nu = d\nu^{cont} + \Delta\nu$, one has (see [19], Chapter 8, Section VIII.4, Theorem 4.1 at pp. 301-302)

$$\begin{aligned} F(x;c) =& \mathsf{E}\left[e^{-\lambda(\tau_R\wedge T)}F(X^x_{\tau_R\wedge T}, C^{c,\nu}_{\tau_R\wedge T})\right] - \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}(\mathbb{L}_X - \lambda)F(X^x_s, C^{c,\nu}_s)ds\right] \\ &- \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}F_c(X^x_s, C^{c,\nu}_s)d\nu^{cont}_s - \sum_{0\leq s<\tau_R\wedge T} e^{-\lambda s}\left(F(X^x_s, C^{c,\nu}_{s+}) - F(X^x_s, C^{c,\nu}_s)\right)\right]. \end{aligned}$$

Since F satisfies the HJB equation (2.36) (cf. Proposition 2.7) and by noticing that

$$F(X^x_s, C^{c,\nu}_{s+}) - F(X^x_s, C^{c,\nu}_s) = \int_0^{\Delta\nu_s} F_c(X^x_s, C^{c,\nu}_s + u)du, \tag{2.40}$$

we obtain

$$\begin{aligned} F(x;c) \leq& \mathsf{E}\left[e^{-\lambda(\tau_R\wedge T)}F(X^x_{\tau_R\wedge T}, C^{c,\nu}_{\tau_R\wedge T})\right] + \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}\lambda X^x_s\Phi(C^{c,\nu}_s)ds\right] \\ &+ \mathsf{E}\left[\int_0^{\tau_R\wedge T} e^{-\lambda s}X^x_s d\nu^{cont}_s\right] + \mathsf{E}\left[\sum_{0\leq s<\tau_R\wedge T} e^{-\lambda s}X^x_s\Delta\nu_s\right] \\ =& \mathsf{E}\left[e^{-\lambda(\tau_R\wedge T)}F(X^x_{\tau_R\wedge T}, C^{c,\nu}_{\tau_R\wedge T}) + \int_0^{\tau_R\wedge T} e^{-\lambda s}\lambda X^x_s\Phi(C^{c,\nu}_s)ds + \int_0^{\tau_R\wedge T} e^{-\lambda s}X^x_s d\nu_s\right]. \end{aligned} \tag{2.41}$$

When taking limits as $R \to \infty$ we have $\tau_R \wedge T \to T$, P-a.s. The integral terms in the last expression on the right hand side of (2.41) are uniformly integrable (cf. (1.7)) and F has sub-linear growth (cf. (2.31)). Then we also take limits as $T \uparrow \infty$ and it follows that

$$F(x;c) \leq \mathsf{E}\left[\int_0^\infty e^{-\lambda s}\lambda X^x_s\Phi(C^{c,\nu}_s)ds + \int_0^\infty e^{-\lambda s}X^x_s d\nu_s\right], \tag{2.42}$$

due to the fact that $\lim_{T\to\infty} \mathsf{E}[e^{-\lambda T}F(X^x_T, C^{c,\nu}_T)] = 0$. Since the latter holds for all admissible ν we have $F(x;c) \leq U(x;c)$.

Step 2. If $c = 1$ then $F(x, 1) = U(x, 1) = 0$. Take then $c \in [0, 1)$, C^* as in Proposition 2.9 and define $\rho := \inf\{t \geq 0 \,:\, \nu^*_t = 1 - c\}$. We can repeat the arguments of Step 1. using Ito's formula with τ_R replaced by $\tau_R \wedge \rho$ to find

$$\begin{aligned}
F(x;c) =& \mathsf{E}\left[e^{-\lambda\,(\tau_R\wedge\rho)}F(X^x_{\tau_R\wedge\rho}, C^*_{\tau_R\wedge\rho})\right] - \mathsf{E}\Big[\int_0^{\tau_R\wedge\rho} e^{-\lambda s}(\mathbb{L}_X - \lambda)F(X^x_s, C^*_s)ds\Big] \\
&- \mathsf{E}\Big[\int_0^{\tau_R\wedge\rho} e^{-\lambda s}F_c(X^x_s, C^*_s)d\nu^{*,cont}_s\Big] \\
&- \mathsf{E}\left[\sum_{0\leq s<\tau_R\wedge\rho} e^{-\lambda s}\left(F(X^x_s, C^*_{s+}) - F(X^x_s, C^*_s)\right)\right].
\end{aligned}$$

If we now recall Proposition 2.7, Proposition 2.9 and (2.40), then from the above we obtain

$$F(x;c) = \mathsf{E}\Big[e^{-\lambda\,(\tau_R\wedge\rho)}F(X^x_{\tau_R\wedge\rho}, C^*_{\tau_R\wedge\rho}) + \int_0^{\tau_R\wedge\rho} e^{-\lambda s}\lambda X^x_s \Phi(C^*_s)ds + \int_0^{\tau_R\wedge\rho} e^{-\lambda s}X^x_s d\nu^*_s\Big] \tag{2.43}$$

As $R \to \infty$, again $\tau_R \to \infty$, clearly $\tau_R \wedge \rho \to \rho$, P-a.s. and $\mathsf{E}\left[e^{-\lambda(\tau_R\wedge\rho)}F(X^x_{\tau_R\wedge\rho}, C^*_{\tau_R\wedge\rho})\right] \to 0$. Moreover, we also notice that since $d\nu^*_s \equiv 0$ and $\Phi(C^*_s) \equiv 0$ for $s > \rho$ the integrals in the last expression of (2.43) may be extended beyond ρ up to $+\infty$ so as to obtain

$$F(x;c) = \mathsf{E}\Big[\int_0^\infty e^{-\lambda s}\lambda X^x_s \Phi(C^*_s)ds + \int_0^\infty e^{-\lambda s}X^x_s d\nu^*_s\Big] = \mathcal{J}_{x;c}(\nu^*). \tag{2.44}$$

Then $F \equiv U$ and ν^* is optimal. □

3 The Case $\hat{c} > 1$

Throughout this Section we consider problem (1.2) in the case when $k(c) < 0$ for all $c \in [0, 1]$ (cf. (1.5)); that is, $\hat{c} > 1$. This turns out to be different from Section 2 above and the usual link between singular stochastic control and optimal stopping analysed in [15], [23], [25], among others, breaks down. Instead in the formulation (1.8) the optimal policy never increases the inventory level before exercising discretionary stopping, which occurs at a $\mathsf{P}-$a.s. finite stopping time. The discretionary stopping boundary is then a repelling boundary in the solution to (1.2) which, when reached, causes the inventory to be immediately filled and so the solution is bang-bang. We characterise this free boundary algebraically. We also discuss the breakdown of the principle of smooth fit for the value function of the control problem: namely, the second order mixed derivative U_{cx} is not continuous across the optimal boundary.

Since initial inspection of the problem suggests that the classical connection with optimal stopping might not hold in this case, to solve the optimisation problem (1.2) for U it is now convenient to tackle directly the Hamilton-Jacobi-Bellman equation that one expects to be associated with U by the dynamic programming principle. For this we need a guess regarding the shape of the action and inaction regions. Observe that total inaction produces an overall cost equal to

$$\lambda\Phi(c)\int_0^\infty e^{-\lambda s}\mathsf{E}\left[X^x_s\right]ds \tag{3.1}$$

(cf. (1.2)). If alternatively at time zero we increase the inventory by a small fixed amount Δ^0 and then do nothing for the remaining time, the cost of control is $x\Delta^0$ and approximating

$\Phi(c+C_s) \sim \Phi(c) + \Phi'(c)\Delta^0$ the overall cost is

$$\begin{aligned} &\lambda\Phi(c)\int_0^\infty e^{-\lambda s}\mathsf{E}\left[X_s^x\right]ds + \Delta^0\lambda\Phi'(c)\int_0^\infty e^{-\lambda s}\mathsf{E}\big[X_s^x\big]ds + x\Delta^0 \\ &= \lambda\Phi(c)\int_0^\infty e^{-\lambda s}\mathsf{E}\left[X_s^x\right]ds + \frac{\Delta^0}{\lambda+\theta}\big(k(c)x + \theta\mu\Phi'(c)\big), \end{aligned} \tag{3.2}$$

recalling that $\mathsf{E}[X_s^x] = \mu + (x-\mu)e^{-\theta s}$ (cf. (B-1)) to obtain the second term. Now comparing (3.1) and (3.2) we observe that when $x > -\theta\mu\Phi'(c)/k(c)$, the second term in (3.2) is negative, hence a small increment of inventory reduces the overall cost. On the other hand, when $x < -\theta\mu\Phi'(c)/k(c)$ increasing the inventory by a small amount increases the overall cost. This suggests that we should expect the inaction region to lie below the action region.

Moreover, since the curve $c \mapsto -\theta\mu\Phi'(c)/k(c)$ is strictly decreasing as Φ is strictly convex, small increments of the inventory for $x > -\theta\mu\Phi'(c)/k(c)$ keep the state process (X, C) inside the profitable region $x > -\theta\mu\Phi'(c)/k(c)$. It follows that infinitesimal increments due to a possible reflecting boundary as in Section 2 do not seem to lead to an optimal strategy.

For each $c \in [0, 1]$, there should be a point $\gamma_*(c) \in \mathbb{R}$ such that the action region is given by $\big\{x : x \geq \gamma_*(c)\big\}$. From the considerations above and a natural symmetry of the present setting with respect to the one treated in the previous Section 2 we also argue that $c \mapsto \gamma_*(c)$ should be decreasing, thus implying that once the couple (X, C) enters the action region, the optimal strategy is to immediately fill the inventory. Finally we also observe that it is not difficult to show from (1.2) that $x \mapsto U(x, c)$ has at most sub-linear growth. Indeed integrating by parts the cost term $\int_0^\infty e^{-\lambda s}X_s^x\,d\nu_s$ and noting that the martingale $M_t := \int_0^t \sigma e^{-\lambda s}\nu_s dB_s$ is uniformly integrable, we can write for any $\nu \in \mathcal{S}_c$

$$\begin{aligned} \mathcal{J}_{x,c}(\nu) &\leq \mathsf{E}\bigg[\int_0^\infty e^{-\lambda s}\Big(\lambda|X_s^x|\Phi(C_s^{c,\nu}) + |\nu_s|[\lambda|X_s^x| + \theta(\mu + |X_s^x|)]\Big)ds\bigg] \\ &\leq K(c)(1+|x|), \end{aligned}$$

for some suitable $K(c) > 0$, by (B-1), Assumption 1.1 and the fact that any admissible ν is nonnegative and uniformly bounded.

Now the differential problem for U in the spirit of our Proposition 2.7 amounts to finding a couple (W, γ) solving the following system

$$\begin{cases} \mathbb{L}_X W(x,c) - \lambda W(x,c) = -\lambda\, x\, \Phi(c), & \text{for } x < \gamma(c),\ c \in [0,1], \\ W_c(x,c) \geq -x, & \text{for } (x,c) \in \mathbb{R}\times[0,1], \\ W(x,c) = x(1-c), & \text{for } x \geq \gamma(c),\ c \in [0,1], \\ W_x(\gamma(c),c) = (1-c), & \text{for } c \in [0,1]. \end{cases} \tag{3.3}$$

Conditions (3.3) might be seen as the minimal ones under which we may hope to perform a verification theorem to show that $W = U$. It is natural to replace the third condition of (3.3) by $W_c(\gamma(c), c) = -\gamma(c)$; we will verify a posteriori that this condition does indeed hold. More interestingly, there are a number of examples in singular stochastic control where instead of the last condition (or together with it) one imposes the principle of smooth fit giving in this case $W_{cx}(\gamma(c), c) = -1$ (see for instance [17] and [31]). As mentioned above we will check that this condition breaks down thus preventing verification of the usual connection to optimal stopping.

The solution of (3.3) is provided in the next

Theorem 3.1. *Let ψ_λ be the increasing fundamental solution of $(\mathbb{L}_X - \lambda)f = 0$ (cf. (B-3) in Appendix) and define*

$$\overline{x}_0(c) := \frac{\theta\mu\Phi(c)}{\zeta(c)}, \tag{3.4}$$

where $\zeta(c) := (\lambda+\theta)(1-c) - \lambda\Phi(c) = \int_c^1 k(y)dy < 0$. Then, there exists a unique couple (W,γ) solving (3.3), with $W \in W^{2,1,\infty}_{loc}(\mathbb{R}\times(0,1))$ and $W_c(\gamma(c),c) = -1$. The function γ is decreasing, $\gamma \in C^1([0,1])$ and for each given $c \in [0,1]$ it is the unique solution in $[\overline{x}_0(c), +\infty)$ of

$$\frac{\psi_\lambda(x)}{\psi'_\lambda(x)} = x - \overline{x}_0(c). \tag{3.5}$$

For any $c \in [0,1]$, the function W may be expressed in terms of γ as

$$W(x,c) = \begin{cases} \frac{\psi_\lambda(x)}{\psi_\lambda(\gamma(c))}\Big[\gamma(c)(1-c) - \lambda\,\Phi(c)\left(\frac{\gamma(c)-\mu}{\lambda+\theta} + \frac{\mu}{\lambda}\right)\Big] & \\ \qquad\qquad +\lambda\,\Phi(c)\left[\frac{x-\mu}{\lambda+\theta} + \frac{\mu}{\lambda}\right], & \text{for } x < \gamma(c) \\ x(1-c), & \text{for } x \geq \gamma(c). \end{cases} \tag{3.6}$$

Moreover, one has $W_{cx}(\gamma(c),c) < -1$, i.e. the map $x \mapsto W_c(x,c)$ is not C^1 across the boundary γ.

Proof. The proof will be carried out in several steps.

1. From standard ODE theory we expect that a solution of the first equation in (3.3) should be given by

$$W(x,c) = A(c)\psi_\lambda(x) + B(c)\phi_\lambda(x) + \lambda\,\Phi(c)\left[\frac{x-\mu}{\lambda+\theta} + \frac{\mu}{\lambda}\right], \tag{3.7}$$

with ϕ_λ and ψ_λ as in (B-2) and (B-3), respectively. Observe that for $x > \gamma(c)$ one should have $W(x,c) = x(1-c)$ and therefore the sublinear growth is fulfilled as $x \to +\infty$; however, as $x \to -\infty$ one has that $\phi_\lambda(x) \to +\infty$ with a superlinear trend. Since we are trying to identify U, it is then natural to set $B(c) \equiv 0$. Imposing the third and fourth conditions stated in (3.3) for $x = \gamma(c)$ we find

$$A(c)\psi_\lambda(\gamma(c)) = \Big[\gamma(c)(1-c) - \lambda\,\Phi(c)\left(\frac{\gamma(c)-\mu}{\lambda+\theta} + \frac{\mu}{\lambda}\right)\Big] \tag{3.8}$$

and

$$A(c)\psi'_\lambda(\gamma(c)) = (1-c) - \frac{\lambda\Phi(c)}{\lambda+\theta}. \tag{3.9}$$

Taking the ratio between the two we find that $\gamma(c)$ should solve (3.5) and since $\psi_\lambda/\psi'_\lambda > 0$ a possible solution must be in the set $[\overline{x}_0(c), +\infty)$. As we are looking for finite-valued solutions of (3.5), we may equivalently consider the problem of finding $x \in [\overline{x}_0(c), +\infty)$ such that $\overline{H}(x,c) = 0$ with

$$\overline{H}(x,c) := \psi_\lambda(x)\Big[(1-c) - \frac{\lambda\Phi(c)}{\lambda+\theta}\Big] - \psi'_\lambda(x)\Big[x(1-c) - \lambda\,\Phi(c)\left(\frac{x-\mu}{\lambda+\theta} + \frac{\mu}{\lambda}\right)\Big]. \tag{3.10}$$

From direct calculation and recalling that $\psi'_\lambda > 0$ and $\psi''_\lambda > 0$ (cf. (B-3) and (B-4)) we find that $\overline{H}_x(x,c) > 0$ and $\overline{H}_{xx}(x,c) > 0$ on $x \in [\overline{x}_0(c), +\infty)$; moreover, since $\overline{H}(\overline{x}_0(c), c) < 0$ then there exists a unique $\gamma(c)$ solving (3.5). Now, from (3.5), (3.8) and (3.9) we can equivalently set

$$\begin{aligned} A(c) :=& \frac{1}{\psi_\lambda(\gamma(c))}\Big[\gamma(c)(1-c) - \lambda\,\Phi(c)\left(\frac{\gamma(c)-\mu}{\lambda+\theta} + \frac{\mu}{\lambda}\right)\Big] \\ =& \frac{1}{\psi'_\lambda(\gamma(c))}\Big[(1-c) - \frac{\lambda\Phi(c)}{\lambda+\theta}\Big] \end{aligned} \tag{3.11}$$

and (3.6) follows by extending W to be $x(1-c)$ for $x > \gamma(c)$.

2. Using (3.5) and (3.6) it is easy to check that $W(\gamma(c), c) = \gamma(c)(1-c)$ and $W_x(\gamma(c), c) = (1-c)$.

3. In order to establish the monotonicity of γ we study the derivative with respect to c of the map $c \mapsto x - \overline{x}_0(c)$. From direct derivation and simple algebra we obtain

$$\frac{d}{d\,c}(x - \overline{x}_0(c)) = -\frac{d}{d\,c}\overline{x}_0(c) = -\frac{\theta\mu(\lambda+\theta)[\Phi'(c)(1-c)+\Phi(c)]}{\zeta^2(c)} > 0, \tag{3.12}$$

where the last inequality holds since $-\Phi(c) = \int_c^1 \Phi'(y)dy > \Phi'(c)(1-c)$ by strict convexity of Φ. Now (3.12) guarantees that $c \mapsto x - \overline{x}_0(c)$ is increasing and then the implicit function theorem and arguments similar to those that led to (2.18) in the proof of Proposition 2.3 allow us to conclude that $\gamma \in C^1([0,1])$ and is decreasing.

4. We aim now at proving the second condition in (3.3). Recalling that W has been extended to be $x(1-c)$ for $x \geq \gamma(c)$ then the result is trivial in that region. Consider only $x < \gamma(c)$. From (3.6) we can write

$$\begin{aligned} W(x,c) = x(1-c) &- \Big[x(1-c) - \lambda\,\Phi(c)\left(\frac{x-\mu}{\lambda+\theta} + \frac{\mu}{\lambda}\right)\Big] \\ &+ \frac{\psi_\lambda(x)}{\psi_\lambda(\gamma(c))}\Big[\gamma(c)(1-c) - \lambda\,\Phi(c)\left(\frac{\gamma(c)-\mu}{\lambda+\theta} + \frac{\mu}{\lambda}\right)\Big] \end{aligned} \tag{3.13}$$

and since γ is differentiable, recalling (2.6) and rearranging terms

$$\begin{aligned} W_c(x,c) =& -x + G(x,c) - \frac{\psi_\lambda(x)}{\psi_\lambda(\gamma(c))}G(\gamma(c),c) \\ &+ \frac{\psi_\lambda(x)}{\psi_\lambda(\gamma(c))}\gamma'(c)\Big[(1-c) - \frac{\lambda\Phi(c)}{\lambda+\theta}\Big]\Big[1 - \frac{\psi'_\lambda(\gamma(c))}{\psi_\lambda(\gamma(c))}(\gamma(c) - \overline{x}_0(c))\Big] \\ =& -x + G(x,c) - \frac{\psi_\lambda(x)}{\psi_\lambda(\gamma(c))}G(\gamma(c),c), \end{aligned} \tag{3.14}$$

where the last equality follows since γ solves (3.5). Notice that as a by-product of (3.14) we have that $W_c(\gamma(c), c) = -\gamma(c)$ as well. Now differentiating (3.14) with respect to x and then taking $x = \gamma(c)$ gives

$$W_{cx}(\gamma(c),c) + 1 = \frac{k(c)}{\lambda+\theta} - \frac{\psi'_\lambda(\gamma(c))}{\psi_\lambda(\gamma(c))}G(\gamma(c),c) \tag{3.15}$$

and hence from (2.6) and (3.5) and some simple algebra

$$\begin{aligned} W_{cx}(\gamma(c),c) + 1 =& -\frac{k(c)}{(\lambda+\theta)}\frac{1}{(\gamma(c) - \overline{x}_0(c))}\Big[\frac{\mu\theta\Phi'(c)}{k(c)} + \overline{x}_0(c)\Big] \\ =& -\frac{\theta\mu}{\zeta(c)}\frac{1}{(\gamma(c) - \overline{x}_0(c))}[\Phi'(c)(1-c) + \Phi(c)]. \end{aligned} \tag{3.16}$$

Since $\gamma(c) > \overline{x}_0(c)$, $\zeta(c) < 0$ and $\Phi'(c)(1-c) + \Phi(c) < 0$, by convexity of Φ, we conclude that

$$W_{cx}(\gamma(c), c) + 1 < 0, \qquad c \in [0,1]. \tag{3.17}$$

For $x < \gamma(c)$ we can differentiate with respect to c and x the first equation in (3.3), set $\bar{u}(x,c) := W_{xc}(x,c) + 1$ and find

$$\mathbb{L}_X \bar{u}(x,c) - (\lambda+\theta)\bar{u}(x,c) = -k(c) > 0, \quad \text{for } c \in [0,1] \text{ and } x < \gamma(c), \tag{3.18}$$

with boundary condition $\bar{u}(\gamma(c),c) = W_{xc}(\gamma(c),c)+1 < 0$. Taking $\sigma_\gamma := \inf\big\{t \geq 0 \,:\, X^x_t \geq \gamma(c)\big\}$ and using Ito's formula we find

$$\bar{u}(x,c) = \mathsf{E}\left[e^{-(\lambda+\theta)\sigma_\gamma}\bar{u}\big(X^x_{\sigma_\gamma}, c\big) + k(c)\int_0^{\sigma_\gamma} e^{-(\lambda+\theta)s}ds\right], \quad \text{for } c \in [0,1] \text{ and } x < \gamma(c). \tag{3.19}$$

It follows from (3.14) that $\bar{u}(\,\cdot\,,c)$ is bounded for $x < \gamma(c)$ and hence $e^{-(\lambda+\theta)\sigma_\gamma}\bar{u}\big(X^x_{\sigma_\gamma},c\big) = e^{-(\lambda+\theta)\sigma_\gamma}\bar{u}\big(\gamma(c),c\big)$, P-a.s. Moreover, $k(c) < 0$ and (3.17) imply that the right-hand side of (3.19) is strictly negative. It follows that $W_{xc}(x,c)+1 < 0$ for all $x < \gamma(c)$ and hence $x \mapsto W_c(x,c)+x$ is decreasing. Since $W_c(\gamma(c),c)+\gamma(c) = 0$ by (3.14), then we can conclude $W_c(x,c)+x \geq 0$ for all $(x,c) \in \mathbb{R}\times[0,1]$. □

Remark 3.2. *Equation* (3.17) *shows that the map* $x \mapsto W_c(x,c)$ *is not* C^1 *across the boundary* γ. *This might be seen as an indication that the classical connection between singular control and optimal stopping (in the sense, e.g., of [15], [23], [25]) breaks down in this example. In fact if* W_c *were the value function of a suitable optimal stopping problem we would expect the principle of smooth fit to hold (cf. Proposition 2.3 above). It is worth noting that in the literature on singular stochastic control, continuity of* W_{cx} *is usually verified (cf. for instance [17] and [31]).*

From now, since γ solving (3.5) is the unique candidate optimal boundary, we set $\gamma_* := \gamma$.

Proposition 3.3. *The function* W *of Theorem 3.1 solves*

$$\max\big\{-\big(\mathbb{L}_X - \lambda\big)W(x,c) - \lambda\Phi(c)\,x\,,\, -W_c(x,c) - x\big\} = 0 \quad \textit{for } (x,c) \in \mathbb{R}\times[0,1] \tag{3.20}$$

in the almost everywhere sense. Moreover, $W(x,1) = U(x,1) = 0$.

Proof. The boundary condition at $c = 1$ obviously follows from (3.6). Since W solves (3.3) then it also solves (3.20) in $x < \gamma_*(c)$, $c \in [0,1]$. It thus remains to prove that $\big(\mathbb{L}_X - \lambda\big)W(x,c) \geq -\lambda\Phi(c)\,x$ for $x > \gamma_*(c)$. Notice that since $W(x,c) = x(1-c)$ in that region then $\big(\mathbb{L}_X - \lambda\big)W(x,c) = (1-c)\big[\theta\mu - (\lambda+\theta)x\big]$. Set

$$\tilde{x}(c) := \frac{(1-c)\theta\mu}{\zeta(c)}, \tag{3.21}$$

where again $\zeta(c) := (\lambda+\theta)(1-c) - \lambda\Phi(c) = \int_c^1 k(y)dy$, and observe that $(1-c)\big[\theta\mu - (\lambda+\theta)x\big] \geq -\lambda\Phi(c)\,x$ for all $x \geq \tilde{x}(c)$. Then to conclude we need only to verify that $\gamma_*(c) > \tilde{x}(c)$ for $c \in [0,1]$. It suffices to prove that $\overline{H}(\tilde{x}(c),c) < 0$ (cf. (3.10)) and the result will follow since $\overline{H}(\cdot,c)$ is strictly increasing and such that $\overline{H}(\gamma_*(c),c) = 0$.

Fix $c \in [0,1]$ and denote $\tilde{x} := \tilde{x}(c)$ and $\overline{x}_0 := \overline{x}_0(c)$ for simplicity. Then some simple algebra gives

$$\frac{\psi_\lambda(\tilde{x})}{\psi'_\lambda(\tilde{x})} - (\tilde{x} - \overline{x}_0) = \frac{\psi_\lambda(\tilde{x})}{\psi'_\lambda(\tilde{x})} - \frac{\theta\mu}{\zeta(c)}(1 - c - \Phi(c)) = \frac{\psi_\lambda(\tilde{x})}{\psi'_\lambda(\tilde{x})} - \frac{\tilde{x}}{(1-c)}\left[(1-c) - \Phi(c)\right], \tag{3.22}$$

where the last equality follows from (3.21). Since $\psi''_\lambda > 0$ and ψ_λ solves $\big(\mathbb{L}_X - \lambda\big)\psi_\lambda = 0$ we easily obtain

$$\frac{\psi_\lambda(\tilde{x})}{\psi'_\lambda(\tilde{x})} > \frac{\theta(\mu - \tilde{x})}{\lambda} \tag{3.23}$$

and from the right hand side of (3.22) also

$$\begin{aligned}\frac{\psi_\lambda(\tilde{x})}{\psi'_\lambda(\tilde{x})} - (\tilde{x} - \overline{x}_0) >& \frac{\theta(\mu - \tilde{x})}{\lambda} - \frac{\tilde{x}\,[\lambda(1-c) - \lambda\Phi(c)]}{\lambda(1-c)} \\ =& \frac{\big(\theta\mu - (\lambda+\theta)\tilde{x}\big)(1-c) + \lambda\Phi(c)\tilde{x}}{\lambda(1-c)} = 0.\end{aligned} \tag{3.24}$$

The inequality above implies $\overline{H}(\tilde{x}(c), c) < 0$ and hence $\gamma_*(c) > \tilde{x}(c)$. Hence $\big(\mathbb{L}_X - \lambda\big)W(x,c) \geq -\lambda\Phi(c)\,x$ for $x > \gamma_*(c)$.

□

Introduce the stopping time

$$\tau_* := \inf\big\{t \geq 0 \,:\, X^x_t \geq \gamma_*(c)\big\}, \tag{3.25}$$

and for any $c \in [0,1)$ define the admissible control strategy

$$\nu^*_t := \begin{cases} 0, & t \leq \tau_*, \\ (1-c), & t > \tau_*. \end{cases} \tag{3.26}$$

Then one has the following

Theorem 3.4. *The admissible control ν^* of* (3.26) *is optimal for problem* (1.2) *and $W \equiv U$.*

Proof. The proof employs arguments similar to those used in the proof of Theorem 2.10. We recall the regularity of W by Theorem 3.1 and notice that $\big|W(x,c)\big| \leq K(1+|x|)$ for a suitable $K > 0$. Then an application of Ito's formula in the weak version of [19], Chapter 8, Section VIII.4, Theorem 4.1, easily gives $W(x,c) \leq U(x,c)$ for all $(x,c) \in \mathbb{R}\times[0,1]$ (cf. also arguments in step 1 of the proof of Theorem 2.10).

On the other hand, taking $C^*_t := C^{c,\nu^*}_t = c + \nu^*_t$, $c \in [0,1)$, with ν^* as in (3.26), and applying Ito's formula again (possibly using localisation arguments as in the proof of Theorem 2.10) we find

$$\begin{aligned}W(x,c) =& \mathsf{E}\left[e^{-\lambda\tau_*}W(X^x_{\tau_*}, C^*_{\tau_*}) + \int_0^{\tau_*} e^{-\lambda s}\lambda\, X^x_s\,\Phi(C^*_s)ds\right] \\ &- \mathsf{E}\left[\int_0^{\tau_*} e^{-\lambda s}W_c(X^x_s, C^*_s)d\,\nu^{*,cont}_s\right] \\ &- \mathsf{E}\left[\sum_{0\leq s<\tau_*} e^{-\lambda s}\big(W(X^x_s, C^*_{s+}) - W(X^x_s, C^*_{s+})\big)\right].\end{aligned} \tag{3.27}$$

Since $(X^x_s, C^*_s) = (X^x_s, c)$ for $s \leq \tau_*$, then the third and fourth term on the right hand side of (3.27) equal zero, whereas for the first term we have from (3.3) and (3.26)

$$\begin{aligned}\mathsf{E}\left[e^{-\lambda\tau_*}W(X^x_{\tau_*}, c + \nu^*_{\tau_*})\right] =& \mathsf{E}\left[e^{-\lambda\tau_*}W(X^x_{\tau_*}, c)\right] \\ =& \mathsf{E}\left[e^{-\lambda\tau_*}X^x_{\tau_*}(1-c)\right] = \mathsf{E}\left[\int_0^\infty e^{-\lambda s}X^x_s d\,C^*_s\right].\end{aligned} \tag{3.28}$$

For the second term on the right hand side of (3.27) it holds

$$\mathsf{E}\left[\int_0^{\tau_*} e^{-\lambda s}\lambda\, X^x_s\, \Phi(c+\nu^*_s)ds\right] = \mathsf{E}\left[\int_0^{\infty} e^{-\lambda s}\lambda\, X^x_s\, \Phi(c+\nu^*_s)ds\right], \tag{3.29}$$

since $\Phi(1) = 0$ by Assumption 1.1. Now, (3.27), (3.28) and (3.29) give $W(x,c) = U(x,c)$, and C^* is optimal. □

4 Heuristic Discussion on the Case $\hat{c} \in (0,1)$

In this Section we provide a detailed heuristic discussion on the case $\hat{c} \in [0,1]$. A rigorous mathematical analysis of this case is a challenging problem which deserves further investigation and will be addressed in future research.

When $\hat{c} \in (0,1)$ the function $k(\cdot)$ of (1.5) changes its sign over $[0,1]$ and we expect to observe a mixture of the problems studied in Sections 2 and 3. In this sense the original problem (1.2) turns into a genuine non convex singular stochastic control problem with discretionary stopping through Proposition 1.2. Our preliminary analysis (also based on numerical experiments) shows evidence that neither pure reflecting strategies as in Section 2 nor pure repelling strtegies as in Section 3 would attain the infimum in (1.8).

It is important to notice that for $c \in (\hat{c},1]$ the optimisation problem is exactly the same as the one studied in Section 2 except that the penalty function should be re-scaled by setting $\hat{\Phi}(\,\cdot\,) := \Phi(\hat{c} + \,\cdot\,)$ as the running cost in (1.3). Then the solution must be formally the same, i.e. the optimal strategy is the minimal increase in inventory to keep the process (X,C) in the interior of an inaction region delimited by a C^1 decreasing boundary $c \mapsto \hat{\beta}(c)$. Clearly $\hat{\beta}$ is different from β_* of Proposition 2.3 due to the rescaled running cost, but the solution method follows the same steps as those described in Section 2.

Analysing the case $c \in [0,\hat{c}]$ turns out to be more challenging and we can only provide heuristic arguments. The same discussion as the one introducing our study in Section 3 suggests that there should exists a boundary $c \mapsto \hat{\gamma}(c)$ such that the inaction region lies below this boundary. The results of Section 3 also support the idea that $\hat{\gamma}$ should be decreasing and therefore act as a repelling boundary.

The coexistence of reflecting and repelling boundaries is not a novelty but determining their geometry is a challenging matter. Our guessed optimal strategy can be summarised by distinguishing three regimes depending on the initial state (x_0,c_0) of the two dimensional controlled state process (X^{x_0},C^{c_0}):

(a) If $x_0 > \hat{\beta}(c_0)$ wait until $X^{x_0}_t$ hits the reflecting boundary $\hat{\beta}(c_0)$; then perform the minimal increase in inventory to keep (X^{x_0},C^{c_0}) inside the closure of the set $\big\{(x,c) \in \mathbb{R}\times[0,1] : x > \hat{\beta}(c)\big\}$ until the inventory is full.

(b) If $x_0 < \hat{\gamma}(c_0)$ do nothing until the hitting time $\hat{\tau}$ at which $X^{x_0}_{\hat{\tau}} = \hat{\gamma}(c_0)$. Then increase the inventory by the minimal amount Δ_0 (by a jump) such that either the inventory is filled, or take $X^{x_0}_{\hat{\tau}} = \hat{\beta}(c_0+\Delta_0)$ and then continue as in (a). Roughly speaking this amounts to pushing (X^{x_0},C^{c_0}) into the continuation region of the setting (a).

(c) If $\hat{\gamma}(c_0) \le x_0 \le \hat{\beta}(c_0)$ then proceed as in (b) and increase the inventory by the minimal amount Δ_0 until either the inventory is filled, or take $x_0 = \hat{\beta}(c_0+\Delta_0)$ and then continue as in (a).

We now cast the free boundary problem associated to our guess for the optimal policy in problem (1.8) (or equivalently (1.2)) based on the discussion above and leave it as an open problem for future study.

The problem amounts to finding a triple of functions $(W,\hat\beta,\hat\gamma)$ such that $W \in W^{2,1,\infty}(\mathbb{R}\times(0,1))$ with $|W(x,c)| \le L(1+|x|)$ for $(x,c)\in\mathbb{R}\times[0,1]$ and for some $L>0$; $c\mapsto\hat\beta(c)$ and $c\mapsto\hat\gamma(c)$ are strictly decreasing (possibly C^1) curves and $\hat g := \hat\beta^{-1}$ denotes the inverse function of $\hat\beta$; moreover the functions W, $\hat\beta$ and $\hat\gamma$ solve

$$\begin{cases} (\mathbb{L}_X-\lambda)W(x,c) = -\lambda x\Phi(c) & \text{if } x>\hat\beta(c) \text{ or } x<\hat\gamma(c) \text{ for } c\in[0,1];\\ W_c(x,c)\ge -x & \text{for } (x,c)\in\mathbb{R}\times[0,1];\\ W_c(x,c) = -x \;\text{ and }\; W_{cx}(x,c)=-1 & \text{for } x=\hat\beta(c) \text{ and } c\in[0,1];\\ W(x,c) = x\big(\hat g(x)-c\big)+W(x,\hat g(x)) & \text{for } x=\hat\gamma(c) \text{ and } c\in[0,1];\\ W_x(x,c) = \big(\hat g(x)-c\big)+W_x(x,\hat g(x)) & \text{for } x=\hat\gamma(c) \text{ and } c\in[0,1];\\ W(x,1)=0 & \text{for } x\in\mathbb{R}. \end{cases} \tag{4.1}$$

The fourth and fifth equations in (4.1), representing the boundary conditions at the repelling boundary in (1.2) (equivalently, at the discretionary stopping boundary in (1.8)), describe mathematically the effect of the optimal control in (b) and (c) of the above list. These equations couple the two boundaries $\hat\beta$ and $\hat\gamma$ in a non-trivial way and substantially increase the difficulty of problem (4.1) compared to Sections 2 and 3.

A A Problem of Storage and Consumption

A problem naturally arising in the analysis of power systems is the optimal recharging of electricity storage. We consider the point of view of an agent that commits to fully charging an electrical battery on or before a randomly occurring time $\tau>0$ of demand. At any time $t>0$ prior to the arrival of the demand the agent may increase the stored energy level C_t (within the limits of its capacity, which is one unit) by buying electricity at the spot price X_t. Several specifications of the spot price dynamics can be considered. We take $(X_t)_{t\ge0}$ as a continuous, strong Markov process adapted to a filtration $(\mathcal{F}_t)_{t\ge0}$ on a complete probability space $(\Omega,\mathcal{F},\mathsf{P})$.

If the battery is not full at time τ then it is filled by a less efficent method so that the terminal spot price is weighted by a strictly convex function Φ, and so is equal to $\Psi(X_\tau,C_\tau)=X_\tau\Phi(C_\tau)$ with $\Phi(1)=0$ (cf. Assumption 1.1). We suppose that the level of stored energy can only be increased (that is, electricity may be bought from the grid but not sold to the grid) and the process $C_t=c+\nu_t$ follows the dynamics (1.1) with $\nu\in\mathcal{S}_c$ (cf. (1.5)). For simplicity and with no loss of generality (as will be seen) we assume that costs are discounted at the rate $r=0$.

The aim of the agent is to minimise the future expected costs by optimally increasing the level of stored energy within its limited capacity. The agent then faces the optimisation problem with random maturity

$$\inf_{\nu\in\mathcal{S}_c}\mathsf{E}\Big[\int_0^\tau X_t d\nu_t + X_\tau\Phi(C_\tau)\Big]. \tag{A-1}$$

Various specifications for the law of τ are clearly possible. Here we consider only the case of τ independent of the filtration $(\mathcal{F}_t)_{t\ge0}$ and distributed according to an exponential law with parameter $\lambda>0$; that is,

$$\mathsf{P}\big(\tau>t\big)=e^{-\lambda t}. \tag{A-2}$$

This choice effectively models the demand as completely unpredictable.

By the assumption of independence of τ and (X, C), for any ν we easily obtain

$$\mathsf{E}\Big[X_\tau \Phi(C_\tau)\Big] = \mathsf{E}\Big[\int_0^\infty \lambda e^{-\lambda t} X_t \Phi(C_t) dt\Big] \tag{A-3}$$

and

$$\begin{aligned}\mathsf{E}\Big[\int_0^\tau X_t d\nu_t\Big] =& \mathsf{E}\Big[\int_0^\infty \lambda e^{-\lambda s}\Big(\int_0^s X_t d\nu_t\Big) ds\Big] \\ =& \mathsf{E}\Big[\int_0^\infty \Big(\int_t^\infty \lambda e^{-\lambda s} ds\Big) X_t d\nu_t\Big] = \mathsf{E}\Big[\int_0^\infty e^{-\lambda t} X_t d\nu_t\Big],\end{aligned} \tag{A-4}$$

where the integrals were exchanged by an application of Fubini's theorem. It then follows that problem (A-1) may be rewritten as in (1.2) and (1.3).

B Facts on the Ornstein-Uhlenbeck Process

Recall the Ornstein-Uhlenbeck process X of (1.4). It is well known that X is a positively recurrent Gaussian process (cf., e.g., [6], Appendix 1, Section 24, pp. 136-137) with state space $\mathbb{R}$ and that (1.4) admits the explicit solution

$$X_t^x = \mu + (x-\mu)e^{-\theta t} + \int_0^t \sigma e^{\theta(s-t)} dB_s. \tag{B-1}$$

We introduced its infinitesimal generator $\mathbb{L}_X$ in (2.8); the characteristic equation $\mathbb{L}_X u = \lambda u$, $\lambda > 0$, admits the two linearly independent, positive solutions (cf. [21], p. 280)

$$\phi_\lambda(x) := e^{\frac{\theta(x-\mu)^2}{2\sigma^2}} D_{-\frac{\lambda}{\theta}}\Big(\frac{(x-\mu)}{\sigma}\sqrt{2\theta}\Big) \tag{B-2}$$

and

$$\psi_\lambda(x) := e^{\frac{\theta(x-\mu)^2}{2\sigma^2}} D_{-\frac{\lambda}{\theta}}\Big(-\frac{(x-\mu)}{\sigma}\sqrt{2\theta}\Big), \tag{B-3}$$

which are strictly decreasing and strictly increasing, respectively. In both (B-2) and (B-3) D_α is the cylinder function of order α (see [7], Chapter VIII, among others) and it is also worth recalling that (see, e.g., [7], Chapter VIII, Section 8.3, eq. (3) at page 119)

$$D_\alpha(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\frac{t^2}{2} - xt} dt, \quad \mathrm{Re}(\alpha) < 0, \tag{B-4}$$

where $\Gamma(\cdot)$ is Euler's Gamma function.

We denote by P_x the probability measure on $(\Omega, \mathcal{F})$ induced by the process $(X_t^x)_{t\geq 0}$, i.e. such that $\mathbb{P}_x(\,\cdot\,) = \mathbb{P}(\,\cdot\,|X(0) = x)$, $x \in \mathbb{R}$, and by $\mathsf{E}_x[\,\cdot\,]$ the expectation under this measure. Then, it is a well known result on one-dimensional regular diffusion processes (see, e.g., [6], Chapter I, Section 10) that

$$\mathsf{E}_x[e^{-\lambda \tau_y}] = \begin{cases} \dfrac{\phi_\lambda(x)}{\phi_\lambda(y)}, & x \geq y, \\ \\ \dfrac{\psi_\lambda(x)}{\psi_\lambda(y)}, & x \leq y, \end{cases} \tag{B-5}$$

with ϕ_λ and ψ_λ as in (B-2) and (B-3) and $\tau_y := \inf\{t \geq 0 : X_t^x = y\}$ the hitting time of X^x at level $y \in \mathbb{R}$. Due to the recurrence property of the Ornstein-Uhlenbeck process X one has $\tau_y < \infty$ P_x-a.s. for any $x, y \in \mathbb{R}$.

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References

[1] Alvarez, L.H.R. (1999). *A Class of Solvable Singular Stochastic Control Problems*, Stoch. Stoch. Rep. 67, pp. 83–122.

[2] Alvarez, L.H.R. (1999). *Singular Stochastic Control, Linear Diffusions, and Optimal Stopping: a Class of Solvable Problems*, SIAM J. Control Optim. 39(6), pp. 1697–1710.

[3] Baldursson, F.M., Karatzas, I. (1997). *Irreversible Investment and Industry Equilibrium*, Finance Stoch. 1, pp. 69–89.

[4] Bank, P. (2005). *Optimal Control under a Dynamic Fuel Constraint*, SIAM J. Control Optim. 44, pp. 1529–1541.

[5] Beneš, V.E., Shepp, L.A., Witsenhausen, H.S. (1980). *Some Solvable Stochastic Control Problems*, Stochastics 4, pp. 39–83.

[6] Borodin, A.N., Salminen, P. (2002). *Handbook of Brownian Motion-Facts and Formulae* 2nd edition. Birkhäuser.

[7] Bateman, H. (1981). *Higher Trascendental Functions*, Volume II. McGraw-Hill Book Company.

[8] Bertola, G. (1998). *Irreversible Investment*, Res. Econ. 52, pp. 3–37.

[9] Blanchet-Scalliet, C., El Karoui, N., Jeanblanc, M., Martellini, L. (2008). *Optimal investment decisions when time-horizon is uncertain*, J. Math. Econom. 44, pp. 1100–1113.

[10] Chiarolla, M.B., Haussmann, U.G. (1994). *The Free-Boundary of the Monotone Follower*, SIAM J. Control Optim. 32, pp. 690–727.

[11] Chow, P.L., Menaldi, J.L., Robin, M. (1985). *Additive Control of Stochastic Linear Systems with Finite Horizon*, SIAM J. Control Optim. 23, pp. 858–899.

[12] Davis, M., Zervos, M. (1994). *A Problem of Singular Stochastic Control with Discretionary Stopping*, Ann. Appl. Probab. 4(1), pp. 226–240.

[13] Dellacherie, C., Meyer, P. (1978). *Probabilities and Potential*, Chapters I–IV, North-Holland Mathematics Studies 29.

[14] Dixit, A.K., Pindyck, R.S. (1994). *Investment under Uncertainty*, Princeton University Press, Princeton.

[15] El Karoui, N., Karatzas, I. (1988). *Probabilistic Aspects of Finite-fuel, Reflected Follower Problems*. Acta Applicandae Math. 11, pp. 223–258.

[16] El Karoui, N., Karatzas, I. (1991). *A New Approach to the Skorohod Problem and its Applications*. Stoch. Stoch. Rep. 34, pp. 57–82.

[17] Federico, S., Pham, H. (2013). *Characterization of the Optimal Boundaries in Reversible Investment Problems*, **arXiv**:1203.0895.

[18] Ferrari, G. (2012). *On an Integral Equation for the Free-Boundary of Stochastic, Irreversible Investment Problems*, **arXiv**:1211.0412v1. Forthcoming on Ann. Appl. Probab.

[19] Fleming, W.H., Soner, H.M. (2005). *Controlled Markov Processes and Viscosity Solutions*, 2nd Edition. Springer.

[20] Geman, H., Roncoroni, A. (2006). *Understanding the Fine Structure of Electricity Prices*, J. Business 79(3), pp. 1225-1261.

[21] Jeanblanc, M., Yor, M., Chesney, M. (2009). *Mathematical Methods for Financial Markets*, Springer.

[22] Karatzas, I. (1983). *A Class of Singular Stochastic Control Problems*, Adv. Appl. Probability 15, pp. 225–254.

[23] Karatzas, I., Shreve, S.E. (1984). *Connections between Optimal Stopping and Singular Stochastic Control I. Monotone Follower Problems*, SIAM J. Control Optim. 22, pp. 856–877.

[24] Karatzas, I., Shreve, S.E. (1985). *Connections between Optimal Stopping and Singular Stochastic Control II. Reflected Follower Problems*, SIAM J. Control Optim. 23, pp. 433–451.

[25] Karatzas, I. (1985). *Probabilistic Aspects of Finite-fuel Stochastic Control*, Proc. Nat'l. Acad. Sci. U.S.A., 82(17), pp. 5579–5581.

[26] Karatzas, I., Shreve, S.E. (1986). *Equivalent Models for Finite-fuel Stochastic Control*, Stochastics 18, pp. 245–276.

[27] Karatzas, I., Shreve, S.E. (1998). *Brownian Motion and Stochastic Calculus* 2nd Edition. Springer.

[28] Karatzas, I., Ocone, D., Wang, H., Zervos, M. (2000). *Finite-Fuel Singular Control with Discretionary Stopping*, Stoch. Stoch. Rep. 71, pp. 1–50.

[29] Kobila, T. Ø. (1993). *A Class of Solvable Stochastic Investment Problems Involving Singular Controls*, Stoch. Stoch. Rep. 43, pp. 29–63.

[30] Lucia, J., Schwartz, E.S. (2002). *Electricity Prices and Power Derivatives: Evidence from the Nordic Power Exchange*, Rev. Derivatives Res. 5(1), pp. 5-50.

[31] Merhi, A., Zervos, M. (2007). *A Model for Reversible Investment Capacity Expansion*, SIAM J. Control Optim. 46(3), pp. 839–876.

[32] Morimoto, H. (2003). *Variational Inequalities for Combined Control and Stopping*, SIAM J. Control Optim. 42, pp. 686–708.

[33] Morimoto, H. (2010). *A Singular Control Problem with Discretionary Stopping for Geometric Brownian Motion*, SIAM J. Control Optim. 48(6), pp. 3781–3804.

[34] Pindyck, R.S. (1988). *Irreversible Investment, Capacity Choice, and the Value of the Firm*, Am. Econ. Rev. 78, pp. 969–985.

[35] Riedel, F., Su, X, (2011). *On Irreversible Investment*, Finance Stoch. 15(4), pp. 607–633.

[36] Skorohod, A.V. (1961). *Stochastic equations for Diffusion Processes in Bounded Region*, Theory Probab. Appl. 6, pp. 264–274.