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On the Optimal Boundary of a Three-Dimensional Singular Stochastic Control Problem Arising in Irreversible Investment*

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Abstract. This paper examines a Markovian model for the optimal irreversible investment problem of a firm aiming at minimizing total expected costs of production. We model market uncertainty and the cost of investment per unit of production capacity as two independent one-dimensional regular diffusions, and we consider a general convex running cost function. The optimization problem is set as a three-dimensional degenerate singular stochastic control problem.

We provide the optimal control as the solution of a Skorohod reflection problem at a suitable free-boundary surface. Such boundary arises from the analysis of a family of two-dimensional parameter-dependent optimal stopping problems and it is characterized in terms of the family of unique continuous solutions to parameter-dependent nonlinear integral equations of Fredholm type.

Key words: irreversible investment, singular stochastic control, optimal stopping, free-boundary problems, nonlinear integral equations.

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JEL classification: C02, C73, E22, D92.

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1 Introduction

In this paper we study a Markovian model for a firm's optimal irreversible investment problem. The firm aims at minimizing total expected costs of production when its running cost function depends on the uncertain condition of the economy as well as on on the installed production capacity, and the cost of investment per unit of production capacity is random. In mathematical terms, this amounts to solving the three-dimensional degenerate singular stochastic control problem

$$V(x,y,z) := \inf_{\nu} \mathbb{E}\bigg[\int_0^\infty e^{-rt} c(X_t^x, z+\nu_t)dt + \int_0^\infty e^{-rt} Y_t^y d\nu_t\bigg], \tag{1.1}$$

where the infimum is taken over a suitable set of nondecreasing admissible controls. Here X and Y are two independent one-dimensional diffusion processes modeling market uncertainty and the cost of investment per unit of production capacity, respectively. The control process ν_t is the cumulative investment made up to time t and c is a general convex cost function. We solve problem (1.1) by relying on the connection existing between singular stochastic control and optimal stopping (see, e.g., [1] and [26]). In fact, we provide the optimal investment strategy ν^* in terms of a free-boundary surface $(x,y) \mapsto z^*(x,y)$ that splits the state space into *action* and *inaction* regions. Such surface arises from an associated family of two-dimensional, infinite time-horizon optimal stopping problems and it is uniquely characterized through a family of continuous solutions to parameter-dependent, nonlinear integral equations of Fredholm type. To the best of our knowledge this is a new feature in the theory of singular stochastic control of multi-dimensional systems.

The connection between singular stochastic control and optimal stopping has been thoroughly studied in the literature. It turns out that under appropriate assumptions the derivative of V in the direction of the controlled state variable equals the value function of a suitable optimal stopping problem whose first optimal stopping time is $\tau^* = \inf\{t \geq 0 : \nu_t^* > 0\}$, with ν^* the optimal control (see, e.g., [26]). This feature was firstly noticed in [4] and then it was rigorously proved, via purely probabilistic arguments, in [26] in the case of a Brownian motion additively controlled by a nondecreasing process. Later on, this kind of link was established also for more complicated dynamics of the controlled diffusion (see, e.g., [1], [5], and [6]) and, recently, singular stochastic control problems with controls of bounded-variation were brought in contact with zero-sum optimal stopping games in [7] and [28].

In the mathematical economic literature singular stochastic control problems are often employed to model the irreversible (partially reversible) optimal investment problem of a firm operating in an uncertain environment (see [11], [13], [18], [19], [24], [29], [33], [39] and references therein, among many others). The monotone (bounded-variation) control represents in fact the cumulative investment (investment-disinvestment) policy of such firm its aim is maximizing total net expected profits or, alternatively, minimizing total expected costs. The optimal timing problem associated to the optimal investment one is then related to real options as pointed out by [32] and [37] among others.

Problems of stochastic irreversible (or partially reversible) investment have been tackled via a number of different approaches. Among others, these include dynamic programming techniques (see, e.g., [18], [24], [29] and [33]), stochastic first-order conditions and the Bank-El Karoui's Representation Theorem [2] (see, e.g., [3], [12], [19] and [39]).

Notice that due to the three-dimensional structure of our problem (1.1) a direct study of the associated Hamilton-Jacobi-Bellman equation with the aim of finding explicit smooth solutions (as in the two-dimensional problem of [33], among others) seems hard to apply. In fact, differently to, e.g., [33], in our case the linear part of the Hamilton-Jacobi-Bellman equation for the value function of problem (1.1) is a PDE (rather than a ODE) and it does not have a general solution.

On the other hand, arguing as in [19], we might tackle problem (1.1) by relying on a stochastic first-order conditions approach; that would allow us to characterize the unique optional solution l^* of the Bank-El Karoui representation problem (cf. [2]) as $l^*_t = z^*(X^x_t, Y^y_t)$, with z^* the free-boundary surface that splits the state space into action and inaction regions. However, the integral equation for the free-boundary which derives from the main result of [19] (i.e., [19, Th. 3.11]) cannot be found in our multi-dimensional setting. Therefore it seems very hard to obtain any information on the geometry of the free-boundary surface $z^*(x, y)$ by using only the characterization of the process l^*_t.

In this paper we study problem (1.1) by relying on the connection between singular stochastic control and optimal stopping and by combining techniques from probability and PDE theory. We show that the optimal control ν^* is the minimal effort needed to keep the (optimally controlled) state process above a free-boundary surface z^* whose level curves $z^*(x, y) = z$, $z \in \mathbb{R}^+$, are the free-boundaries $y^*(\,\cdot\,; z)$ of the parameter-dependent optimal stopping problems associated to the original singular control one. Under some further mild conditions, we characterize each optimal boundary $y^*(\,\cdot\,; z)$, $z \in \mathbb{R}^+$, as the unique continuous solution of nonlinear integral equation of Fredholm type (see our Theorem 4.10 below).

The issue of finding integral equations for the free-boundary of optimal stopping problems has been successfully addressed in a number of papers (cf. [35] for a survey). In the context of one-dimensional stochastic (ir)reversible investment problems on a finite time-horizon integral equations for the optimal boundaries have been obtained by an application of Peskir's local time-space calculus (see [11] and [13] and references therein for details). However, those arguments cannot be applied in our case since it seems quite hard to prove that the process $\{y^*(X^x_t; z),\, t \geq 0\}$ is a semimartingale for each given $z \in \mathbb{R}^+$ as it is required in [36, Th. 2.1]. On the other hand, multi-dimensional settings have been studied for instance in [35, Sec. 13] where a diffusion X was considered along with its running supremum S. Unlike [35, Sec. 13] here we deal with a genuine two dimensional diffusion (X, Y) with X and Y independent. This gives rise to a completely different analysis of the problem and new methods have been developed.

The paper is organized as follows. In Section 2 we set the stochastic irreversible investment problem. In Sections 3 and 4 we introduce the associated family of optimal stopping problems and we characterize its value functions and its optimal-boundaries. The form of the optimal control is provided in Section 5. Finally, some technical results are discussed in Appendix A.

2 The Stochastic Irreversible Investment Problem

In this section we set the stochastic irreversible investment problem object of our study. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the filtration generated by a two-dimensional Brownian motion $W = \{(W^1_t, W^2_t),\, t \geq 0\}$ and augmented with $\mathbb{P}$-null sets.

1. A real process $X = \{X_t,\, t \geq 0\}$ represents the uncertain status of the economy (typically, the demand of a good or, more generally, some indicator of macroeconomic conditions). We assume that X is a time-homogeneous Markov diffusion satisfying the stochastic differential equation (SDE)

$$dX_t = \mu_1(X_t)dt + \sigma_1(X_t)dW^1_t, \qquad X_0 = x, \tag{2.1}$$

   for some Borel functions μ_1 and σ_1 to be specified. To account for the dependence of X on its initial position we denote the solution of (2.1) by X^x.

2. A one-dimensional positive process $Y = \{Y_t,\, t \geq 0\}$ represents the cost of investment per unit of production capacity. We assume that Y evolves according to the SDE

$$dY_t = \mu_2(Y_t)dt + \sigma_2(Y_t)dW^2_t, \qquad Y_0 = y, \tag{2.2}$$

   for some Borel functions μ_2 and σ_2 to be specified as well. Again, to account for the dependence of Y on y, we denote the solution of (2.2) by Y^y.

3. A control process $\nu = \{\nu_t,\ t \geq 0\}$ describes an investment policy of the firm and ν_t is the cumulative investment made up to time t. We say that a control process ν is admissible if it belongs to the nonempty convex set

$$\mathcal{V} := \{\nu : \Omega \times \mathbb{R}^+ \mapsto \mathbb{R}^+ \,|\, t \mapsto \nu_t \text{ is càdlàg, nondecreasing, } \mathbb{F}\text{-adapted}\}. \tag{2.3}$$

   In the following we set $\nu_{0^-} = 0$, for every $\nu \in \mathcal{V}$.

4. A purely controlled process $Z = \{Z_t,\, t \geq 0\}$, represents the production capacity of the firm and it is defined by

$$Z_t := z + \nu_t, \qquad z \in \mathbb{R}^+. \tag{2.4}$$

   The process Z depends on its initial position z and on the control (investment) process ν, therefore we denote it by $Z^{z,\nu}$.

We assume that the uncontrolled diffusions X^x and Y^y have state-space $\mathcal{I}_1 = (\underline{x}, \overline{x}) \subseteq \mathbb{R}$ and $\mathcal{I}_2 = (\underline{y}, \overline{y}) \subseteq \mathbb{R}^+$, respectively, with $\underline{x}, \overline{x}, \underline{y}, \overline{y}$ natural boundary points. We recall that a boundary point ξ is natural for one of our diffusion processes if it is: non-entrance and non-exit. That is, ξ cannot be a starting point for the process and it cannot be reached in finite time (cf. for instance [9, Ch. 2, p. 15]). Moreover if such ξ is finite one also has $\mu_i(\xi) = \sigma_i(\xi) = 0$ with $i = 1$ if $\xi = \overline{x}$ (or $\xi = \underline{x}$) and with $i = 2$ if $\xi = \overline{y}$ (or $\xi = \underline{y}$). That is shown in Appendix A.1 for the sake of completeness.

We make the following

Assumption 2.1.

(i) The coefficients $\mu_i : \mathbb{R} \mapsto \mathbb{R}$, $\sigma_i : \mathbb{R} \mapsto \mathbb{R}^+$, $i = 1, 2$, *are such that*

$$|\mu_i(\zeta) - \mu_i(\zeta')| \;\le\; K_i|\zeta - \zeta'|, \quad |\sigma_i(\zeta) - \sigma_i(\zeta')| \;\le\; M_i|\zeta - \zeta'|^\gamma, \quad \forall \zeta, \zeta' \in \mathcal{I}_i,$$

for some $K_i > 0$, $M_i > 0$ *and* $\gamma \in [\frac{1}{2}, 1]$.

(ii) The diffusions X^x *and* Y^y *are nondegenerate, i.e.* $\sigma_i^2 > 0$ *in* $\mathcal{I}_i$, $i = 1, 2$.

Assumption 2.1 guarantees that

$$\int_{\zeta-\varepsilon_o}^{\zeta+\varepsilon_o} \frac{1 + |\mu_i(y)|}{|\sigma_i(y)|^2}\, dy < +\infty, \quad \text{for some } \varepsilon_o > 0 \text{ and every } \zeta \text{ in } \mathcal{I}_i \tag{2.5}$$

and hence both (2.1) and (2.2) have a weak solution that is unique in the sense of probability law (cf. [27, Ch. 5.5]). Such solutions do not explode in finite time due to the sublinear growth of the coefficients. On the other hand, Assumption 2.1-(i) also guarantees pathwise uniqueness for the solutions of (2.1) and (2.2) by the Yamada-Watanabe result (see [27, Ch. 5.2, Prop. 2.13] and [27, Ch. 5.3, Rem. 3.3], among others). Therefore, (2.1) and (2.2) have a unique strong solution due to [27, Ch. 5.3, Cor. 3.23] for any $x \in \mathcal{I}_1$ and $y \in \mathcal{I}_2$. Also, it follows from (2.5) that the diffusion processes X^x and Y^y are regular in $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively; that is, X^x (resp., Y^y) hits a point ζ (resp., ζ') with positive probability, for any x and ζ in $\mathcal{I}_1$ (resp., y and ζ' in $\mathcal{I}_2$). Hence the state spaces $\mathcal{I}_1$ and $\mathcal{I}_2$ cannot be decomposed into smaller sets from which X^x and Y^y could not exit (see, e.g., [40, Ch. V.7]). Finally, there exist continuous versions of X^x and Y^y and we shall always refer to those versions throughout this paper.

Assumption 2.1 implies the Yamada-Watanabe comparison criterion (see, e.g., [27, Ch. 5.2, Prop. 2.18]); i.e.,

$$x,\, x' \in \mathcal{I}_1\,, \quad x \le x' \Longrightarrow X^x_t \le X^{x'}_t, \quad \mathbb{P}\text{-a.s. } \forall t \ge 0. \tag{2.6}$$

Moreover, repeating arguments as in the proof of [27, Ch. 5.2, Prop. 2.13] one also finds

$$x_n \to x_0 \text{ in } \mathcal{I}_1 \text{ as } n \to \infty \implies X^{x_n}_t \xrightarrow{L^1} X^{x_0}_t \implies X^{x_n}_t \xrightarrow{\mathbb{P}} X^{x_0}_t, \quad \forall t \ge 0; \tag{2.7}$$

Analogously, for the unique solution of (2.2) one has

$$y,\, y' \in \mathcal{I}_2\,, \quad y \le y' \Longrightarrow Y^y_t \le Y^{y'}_t, \quad \mathbb{P}\text{-a.s. } \forall t \ge 0; \tag{2.8}$$

and

$$y_n \to y_0 \text{ in } \mathcal{I}_2 \text{ as } n \to \infty \implies Y^{y_n}_t \xrightarrow{L^1} Y^{y_0}_t \implies Y^{y_n}_t \xrightarrow{\mathbb{P}} Y^{y_0}_t, \quad \forall t \ge 0. \tag{2.9}$$

Standard estimates on the solution of SDEs with coefficients having sublinear growth imply that (cf., e.g., [30, Ch. 2.5, Cor. 12])

$$\mathbb{E}\Big[|X_t^x|^q\Big] \le \kappa_{0,q}(1+|x|^q)e^{\kappa_{1,q}t}, \qquad \mathbb{E}\Big[|Y_t^y|^q\Big] \le \theta_{0,q}(1+|y|^q)e^{\theta_{1,q}t}, \qquad t \ge 0, \tag{2.10}$$

for any $q \ge 0$, and for some $\kappa_{i,q} := \kappa_{i,q}(\mu_1, \sigma_1) > 0$ and $\theta_{i,q} := \theta_{i,q}(\mu_2, \sigma_2) > 0$, $i = 0, 1$.

Within this setting we consider a firm that incurs investment costs and a running cost $c(x, z)$ depending on the state of economy x and the production capacity z. The firm's total expected cost of production associated to an investment strategy $\nu \in \mathcal{V}$ is

$$\mathcal{J}_{x,y,z}(\nu) := \mathbb{E}\bigg[\int_0^\infty e^{-rt}c(X_t^x, Z_t^{z,\nu})dt + \int_0^\infty e^{-rt}Y_t^y d\nu_t\bigg], \tag{2.11}$$

for any $(x, y, z) \in \mathcal{I}_1 \times \mathcal{I}_2 \times \mathbb{R}^+$. Here r is a positive discount factor and the cost function $c : \overline{\mathcal{I}}_1 \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfies

Assumption 2.2.

(i) $c \in C^0(\overline{\mathcal{I}}_1 \times \mathbb{R}^+; \mathbb{R}^+)$, $c(x, \cdot) \in C^1(\mathbb{R}^+)$ *for every* $x \in \overline{\mathcal{I}}_1$, *and* $c_z \in C^\alpha(\overline{\mathcal{I}}_1 \times \mathbb{R}^+; \mathbb{R})$ *for some* $\alpha > 0$ *(that is,* c_z *is* α*-Hölder continuous).*

(ii) $c(x, \cdot)$ *is convex for all* $x \in \overline{\mathcal{I}}_1$ *and* $c_z(\cdot, z)$ *is nonincreasing for every* $z \in \mathbb{R}^+$.

(iii) c *and* c_z *satisfy a polynomial growth condition with respect to x; that is, there exist locally bounded functions* $\eta_o, \gamma_o : \mathbb{R}^+ \mapsto \mathbb{R}^+$, *and a constant* $\beta \ge 0$ *such that*

$$|c(x, z)| + |c_z(x, z)| \le \eta_o(z) + \gamma_o(z)|x|^\beta.$$

Throughout this paper we also make the following standard assumption that guarantees in particular finiteness for our problem (see Remark 2.4-(**3**) and Lemma 2.6 below)

Assumption 2.3. $r > \kappa_{1,\beta} \vee \theta_{1,1}$,

with $\kappa_{1,q}$ and $\theta_{1,q}$, $q \ge 0$, as in (2.10) and with β of Assumption 2.2-(*iii*).

Remark 2.4. **1.** *Any function* c *of the spread* $|x - z|$ *between capacity and demand in the form*

$$c(x, z) = K_0|x - z|^\delta, \quad K_0 \ge 0,\ \delta > 1, \tag{2.12}$$

satisfies Assumption 2.2. We observe that (2.12) *is a natural choice, e.g., in an energy market framework where x represents the demand net of renewables (thus having stochastic nature) and z the amount of conventional supply. Failing to meet the demand as well as an excess of supply generate costs for the energy provider.*
2. *The second part of Assumption 2.2-(ii) captures the negative impact on marginal costs due to an increase of demand. It is intuitive in* (2.12) *that an increase of z will produce a reduction (increase) of costs which is more significant the more the demand is above (below) the supply.*
3. *It follows from* (2.10)*, Assumption 2.2-(iii) and Assumption 2.3 that* c *and* c_z *satisfy the integrability conditions*

(a) $\mathbb{E}\bigg[\int_0^\infty e^{-rt}c(X_t^x, z)dt\bigg] < \infty, \quad \forall (x, z) \in \mathcal{I}_1 \times \mathbb{R}^+;$

(b) $\mathbb{E}\bigg[\int_0^\infty e^{-rt}|c_z(X_t^x, z)|dt\bigg] < \infty, \quad \forall (x, z) \in \mathcal{I}_1 \times \mathbb{R}^+.$

The firm's manager aims at picking an irreversible investment policy $\nu^* \in \mathcal{V}$ (cf. (2.3)) that minimizes the total expected cost (2.11). Therefore, by denoting the state space $\mathcal{O} := \mathcal{I}_1 \times \mathcal{I}_2 \times \mathbb{R}^+$, the firm's manager is faced with the optimal irreversible investment problem with value function

$$V(x, y, z) := \inf_{\nu \in \mathcal{V}} \mathcal{J}_{x,y,z}(\nu), \quad (x, y, z) \in \mathcal{O}. \tag{2.13}$$

Remark 2.5. *The form of our cost functional* (2.11) *does not allow a reduction of the dimensionality of problem* (2.13) *through an appropriate change of measure when Y is a discounted exponential martingale (e.g., a geometric Brownian motion). That could have been possible instead in the context of profit maximization problems with separable operating profit functions, as the Cobb-Douglas one.*

Notice that (2.10), Assumption 2.2-(*ii*) and Assumption 2.3 (cf. also Remark 2.4-(**3**)), together with the convexity of $c(x, \cdot)$ and the affine nature of $Z^{z,\nu}$ in the control variable lead to the following

Lemma 2.6. *The value function $V(x, y, z)$ of* (2.13) *is finite for all $(x, y, z) \in \mathcal{O}$ and such that $z \mapsto V(x, y, z)$ is convex.*

Remark 2.7. *If an optimal control ν^* exists, then it must be $J_{x,y,z}(\nu^*) \leq J_{x,y,z}(0)$ and hence*

$$\mathbb{E}\bigg[\int_0^\infty e^{-r\,t}c(X_t^x, z + \nu_t^*)dt\bigg] < +\infty. \tag{2.14}$$

Therefore, there is no loss of generality if we restrict the set of admissible controls to those in $\mathcal{V}$ which also fulfill (2.14)*.*

Problem (2.13) is a degenerate, three-dimensional, convex singular stochastic control problem of monotone follower type (see, e.g., [16], [26] and references therein). Moreover, if c is strictly convex, then $\mathcal{J}_{x,y,z}(\cdot)$ of (2.11) is strictly convex on $\mathcal{V}$ as well, and hence if a solution to (2.13) exists, it must be unique. Existence of a solution ν^* of convex (concave) singular stochastic control problems is a well known result in the literature (see, e.g., [27], [28] or [39]) and it usually relies on an application of (a suitable version of) Komlòs' Theorem.

Here we follow a different approach and in Section 5 we provide the optimal control ν^* in terms of the free-boundaries of a suitable family of optimal stopping problems that we start studying in the next section.

3 The Family of Associated Optimal Stopping Problems

In the literature on stochastic, irreversible investment problems (cf. [1], [11], [19], [29], [39], among many others), or more generally on singular stochastic control problems of monotone follower type (see, e.g., [3], [5], [16] and [26]), it is well known that a convex (concave) monotone control problem may be associated to a suitable family of optimal stopping problems, parametrized with respect to the state space of the controlled variable (see also [13], [18] and [28] in the case of a bounded variation control problem, whose associated optimal stopping problem is a Dynkin game).

We now introduce the family of optimal stopping problems that we expect to be associated to the singular control problem (2.13). Set

$$\mathcal{T} := \{\tau \in [0,\infty] \ \mathbb{F}\text{-stopping times}\},$$

and define

$$\Psi_{x,y,z}(\tau) := \mathbb{E}\bigg[\int_0^\tau e^{-rt} c_z(X^x_t, z)dt - e^{-r\tau} Y^y_\tau\bigg], \quad \tau \in \mathcal{T}, \quad (x,y) \in \mathcal{I}_1 \times \mathcal{I}_2, \quad z \in \mathbb{R}^+. \tag{3.1}$$

For any $z \in \mathbb{R}^+$ we consider the optimal stopping problem

$$v(x,y;z) := \sup_{\tau \in \mathcal{T}} \Psi_{x,y,z}(\tau), \qquad (x,y) \in \mathcal{I}_1 \times \mathcal{I}_2. \tag{3.2}$$

Notice that $\{v(x,y;z),\, z \in \mathbb{R}^+\}$ is a family of two-dimensional parameter-dependent optimal stopping problems.

The basic formal connections one expects between the singular stochastic control problem (2.13) and the optimal stopping problem (3.2) are the following (see, e.g., [1, Sec. 5]):

1. For fixed $(x,y,z) \in \mathcal{O}$ the first optimal stopping time τ^* of problem (3.2) can be defined in terms of the optimal control ν^* of problem (2.13) by[1]

$$\tau^* = \inf\{t \geq 0 : \nu^*_t > 0\}. \tag{3.3}$$

2. The value function V of (2.13) is differentiable with respect to z and

$$V_z(x,y,z) = v(x,y;z), \qquad (x,y,z) \in \mathcal{O}. \tag{3.4}$$

Remark 3.1. *The optimality of* τ^* *in* (3.3), *the existence of* V_z *and the equality* (3.4) *may be proved directly by suitably adapting to our setting the techniques employed in [1] or [26]. However we obtain these results as a byproduct of our verification theorem in Section 5.*

In the rest of the present section and in the next one, we fix $z \in \mathbb{R}^+$ and we study the optimal stopping problem (3.2). Denote its state space by $Q := \mathcal{I}_1 \times \mathcal{I}_2$. We introduce the following (cf. [27, Ch. 1, Def. 4.8])

[1] From the economic point of view, this means that a firm's manager who aims at optimally (irreversibly) investing may equivalently consider the problem of profitably exercising the investment option.

Definition 3.2. *A right-continuous stochastic process $\xi := \{\xi_t, t \geq 0\}$ is of class (D) if the family of random variables $\{\xi_\tau \mathbb{1}_{\{\tau<\infty\}}, \tau \in \mathcal{T}\}$ is uniformly integrable,*

and we make the following technical

Assumption 3.3. *The process $\{e^{-rt}Y^y_t, t \geq 0\}$ is an $(\mathcal{F}_t)$-supermartingale of class (D).*

Remark 3.4. 1. *The gain process $e^{-rt}Y^y_t$ is of class (D) if, e.g., $\mathbb{E}[\sup_{t\geq 0} e^{-rt}Y^y_t] < \infty$, a standard technical assumption in the general theory of optimal stopping (see, e.g, [35, Ch. I]).*
2. *Assumptions 2.3 and 3.3 imply that $\lim_{t\to\infty} e^{-rt}Y^y_t = 0$ $\mathbb{P}$-a.s. In fact, $\{e^{-rt}Y^y_t,\ t \geq 0\}$ is a positive $(\mathcal{F}_t)$-supermartingale with continuous paths (cf. also Assumption 2.1) and there always exists $\Xi := \lim_{t\to\infty} e^{-rt}Y^y_t \geq 0$ (cf. [27, Ch. 1, Problem 3.16]). Fatou's Lemma gives*

$$0 \leq \mathbb{E}[\Xi] = \mathbb{E}[\lim_{t\to\infty} e^{-rt}Y^y_t] \leq \liminf_{t\to\infty}\mathbb{E}[e^{-rt}Y^y_t]$$

and, estimates in (2.10) and Assumption 2.3 imply $\lim_{t\to\infty}\mathbb{E}[e^{-rt}Y^y_t] = 0$, hence $\mathbb{E}[\Xi] = 0$. Since $\Xi \geq 0$ $\mathbb{P}$-a.s., then $\lim_{t\to\infty} e^{-rt}Y^y_t = 0$ $\mathbb{P}$-a.s.

In light of Remark 3.4 from now on we will adopt the convention

$$e^{-r\tau}Y^y_\tau \mathbb{1}_{\{\tau=\infty\}} := \lim_{t\to\infty} e^{-rt}Y^y_t = 0, \quad a.s. \tag{3.5}$$

Also we set

$$e^{-r\tau}|f(X^x_\tau, Y^y_\tau)|\mathbb{1}_{\{\tau=\infty\}} := \limsup_{t\to\infty} e^{-rt}|f(X^x_t, Y^y_t)|, \quad a.s., \tag{3.6}$$

for any Borel-measurable function f.

The next lemma will be useful in what follows.

Lemma 3.5. *Under Assumptions 2.1, 2.3 and 3.3 it holds*

$$\mathbb{E}[e^{-r\tau}Y^y_\tau] = y + \mathbb{E}\left[\int_0^\tau e^{-rt}\big(\mu_2(Y^y_t) - rY^y_t\big)dt\right], \quad \textit{for } \tau \in \mathcal{T}. \tag{3.7}$$

Proof. The result holds for bounded stopping times $\tau_n := \tau \wedge n$, with $\tau \in \mathcal{T}$ and $n \in \mathbb{N}$, by Itô's formula and since the stochastic integral is a true martingale by Assumptions 2.1 and 2.3. Taking limits as $n \to \infty$ and using Assumptions 2.1, 2.3, 3.3 and dominated convergence one finds (3.7). □

In the rest of this section we aim at characterizing v of (3.2).

Proposition 3.6. *Under Assumptions 2.1, 2.2, 2.3 and 3.3 the following hold:*

1. v is such that

$$-y \leq v(x, y; z) \leq C(z)(1 + |x|^\beta + |y|), \qquad \forall (x, y) \in Q, \tag{3.8}$$

for a constant $C(z) > 0$ depending on z.

2. $v(\,\cdot\,,y;z)$ *is nonincreasing for every* $y \in \mathcal{I}_2$.

3. $v(x,\,\cdot\,;z)$ *is nonincreasing for every* $x \in \mathcal{I}_1$.

Proof. 1. The lower bound follows by taking $\tau = 0$ in (3.2). Assumptions 2.1, 2.2-(iii), 2.3, 3.3 and Lemma 3.5 guarantee the upper bound.

2. The fact that $x \mapsto c_z(x,z)$ is nonincreasing (cf. Assumption 2.2-(ii)) and (2.6) imply

$$v(x_2,y;z) - v(x_1,y;z) \le \sup_{\tau\in\mathcal{T}} \mathbb{E}\Big[\int_0^\tau e^{-rt}\big(c_z(X_t^{x_2},z) - c_z(X_t^{x_1},z)\big)dt\Big] \le 0, \quad \text{for } x_2 > x_1.$$

3. It follows from (2.8) and arguments as in point 2. □

Proposition 3.7. *Under Assumptions 2.1, 2.2, 2.3 and 3.3 the value function* $v(\,\cdot\,;z)$ *of the optimal stopping problem* (3.2) *is continuous on* Q.

Proof. Fix $z \in \mathbb{R}^+$ and let $\{(x_n,y_n),\, n\in\mathbb{N}\} \subset Q$ be a sequence converging to $(x,y) \in Q$. Take $\varepsilon > 0$ and let $\tau^\varepsilon := \tau^\varepsilon(x,y;z)$ be an ε-optimal stopping time for the optimal stopping problem with value function $v(x,y;z)$. Then we have

$$v(x,y;z) - v(x_n,y_n;z) \le \varepsilon + \mathbb{E}\Big[\int_0^{\tau^\varepsilon} e^{-rt}\big(c_z(X_t^{x},z) - c_z(X_t^{x_n},z)\big)dt - e^{-r\tau^\varepsilon}\big(Y^y_{\tau^\varepsilon} - Y^{y_n}_{\tau^\varepsilon}\big)\Big]. \tag{3.9}$$

Taking into account (2.7) and (2.9), Assumptions 2.2, 2.3 and 3.3, we can apply dominated convergence (in its weak version requiring only convergence in measure; see, e.g., [8, Ch. 2, Th. 2.8.5]) to the right hand side of the inequality above and get

$$\liminf_{n\to\infty} v(x_n,y_n;z) \ge v(x,y;z) - \varepsilon. \tag{3.10}$$

Similarly, taking ε-optimal stopping times $\tau_n^\varepsilon := \tau^\varepsilon(x_n,y_n;z)$ for the optimal stopping problem with value function $v(x_n,y_n;z)$, and using Lemma 3.5 we get

$$\begin{aligned}
v(x_n,y_n;z) - v(x,y;z) \le{}& \varepsilon + \mathbb{E}\Big[\int_0^{\tau_n^\varepsilon} e^{-rt}\big(c_z(X_t^{x_n},z) - c_z(X_t^{x},z)\big)dt - e^{-r\tau_n^\varepsilon}\big(Y^{y_n}_{\tau_n^\varepsilon} - Y^{y}_{\tau_n^\varepsilon}\big)\Big]\\
={}& \varepsilon + \mathbb{E}\Big[\int_0^{\tau_n^\varepsilon} e^{-rt}\big(c_z(X_t^{x_n},z) - c_z(X_t^{x},z)\big)dt\Big] - (y_n - y)\\
&+ \mathbb{E}\Big[\int_0^{\tau_n^\varepsilon} e^{-rt}\big[r\big(Y_t^{y_n} - Y_t^{y}\big) - \big(\mu_2(Y_t^{y_n}) - \mu_2(Y_t^{y})\big)\big]dt\Big]\\
\le{}& \varepsilon + \mathbb{E}\Big[\int_0^{\infty} e^{-rt}\big|c_z(X_t^{x_n},z) - c_z(X_t^{x},z)\big|dt\Big] + |y - y_n|\\
&+ C\,\mathbb{E}\Big[\int_0^{\infty} e^{-rt}\big|Y_t^{y_n} - Y_t^{y}\big|dt\Big],
\end{aligned} \tag{3.11}$$

for some $C > 0$ and where we have used Lipschitz continuity of μ_2 (cf. Assumption 2.1) in the last step. Recalling now (2.7) and (2.9), (2.10), Assumptions 2.2 and 2.3, we can apply again dominated convergence in its weak version (cf. [8, Ch. 2, Th. 2.8.5]) to the right hand side of the inequality above to obtain

$$\limsup_{n\to\infty} v(x_n, y_n; z) \leq v(x, y; z) + \varepsilon. \tag{3.12}$$

Now (3.10) and (3.12) imply continuity of $v(\,\cdot\,, \cdot\,; z)$ by arbitrariness of $\varepsilon > 0$. □

Remark 3.8. *Arguments similar to those used in the proof of Proposition 3.7 above may also be employed to show that $(x, y, z) \mapsto v(x, y; z)$ is continuous in $\mathcal{O}$.*

Since the state space $Q = \mathcal{I}_1 \times \mathcal{I}_2$ of the diffusion $\{(X_t^x, Y_t^y), t \geq 0\}$ may be unbounded, it is convenient for studying the variational inequality associated to our optimal stopping problem, to approximate problem (3.2) by a sequence of problems on bounded domains. Let $\{Q_n,\, n \in \mathbb{N}\}$ be a sequence of sets approximating Q, and we assume that

$$\begin{cases} Q_n \text{ is open, bounded and connected for every } n \in \mathbb{N}, \\ Q_n \subset Q \text{ for every } n \in \mathbb{N}, \\ \partial Q_n \in C^{2+\alpha} \text{ for some } \alpha > 0 \text{ depending on } n \in \mathbb{N}, \\ Q_n \subset Q_{n+1} \text{ for every } n \in \mathbb{N}, \\ \lim_{n\to\infty} Q_n := \bigcup_{n\geq 0} Q_n = Q. \end{cases} \tag{3.13}$$

Clearly it is always possible to find such a sequence of sets. The optimal stopping problem (3.2) is then localized as follows. Given $n \in \mathbb{N}$ define the stopping time

$$\tau_n = \tau_n(x, y; z) := \inf\{t \geq 0 \mid (X_t^x, Y_t^y) \notin Q_n\} \tag{3.14}$$

and notice that $\tau_\infty = \tau_\infty(x, y; z) := \inf\{t \geq 0 \mid (X_t^x, Y_t^y) \notin Q\} = \infty$ a.s., since we are assuming that the boundaries of the diffusions X^x and Y^y are natural, hence non attainable. Moreover, from the last of (3.13) we obtain

$$\tau_n \uparrow \tau_\infty = \infty \quad \mathbb{P}\text{-a.s., as } n \to \infty. \tag{3.15}$$

With τ_n as in (3.14), we can define the approximating optimal stopping problem

$$v_n(x, y; z) := \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\int_0^{\tau_n\wedge\tau} e^{-rt} c_z(X_t^x, z)dt - e^{-r(\tau_n\wedge\tau)} Y_{\tau_n\wedge\tau}^y\right], \quad (x, y) \in Q, \tag{3.16}$$

and prove the following

Proposition 3.9. *Let Assumptions 2.1, 2.2, 2.3 and 3.3 hold. Then*

1. $v_n(\,\cdot\,; z) \leq v_{n+1}(\,\cdot\,; z) \leq v(\,\cdot\,; z)$ *on Q for all $n \in \mathbb{N}$.*

2. $v_n(x, y; z) = -y$ *for $(x, y) \in Q \setminus Q_n$ and all $n \in \mathbb{N}$ (in particular for every $(x, y) \in \partial Q_n$, since Q_n is open).*

3. $v_n(x,y;z) \uparrow v(x,y;z)$ *as* $n \to \infty$ *for every* $(x,y) \in Q$.

4. If $\{v_n(\,\cdot\,;z),\, n \in \mathbb{N}\} \subset C^0(Q)$*, then* $v_n(\,\cdot\,;z)$ *converges to* $v(\,\cdot\,;z)$ *uniformly on all compact subsets* $\mathcal{K} \subset\subset Q$.

Proof. 1. It follows from (3.15) and by comparison of (3.16) with (3.2).

2. This claim follows from the definition of τ_n and of v_n (see (3.14) and (3.16), respectively).

3. For fixed $(x,y) \in Q$ denote by $\tau^\varepsilon := \tau^\varepsilon(x,y;z)$ an ε-optimal stopping time of $v(x,y;z)$, then

$$\begin{aligned} 0 &\le v(x,y;z) - v_n(x,y;z) \\ &\le \mathbb{E}\bigg[\int_{\tau_n\wedge\tau^\varepsilon}^{\tau^\varepsilon} e^{-rt} c_z(X_t^x, z)dt - \left(e^{-r\tau^\varepsilon} Y_{\tau^\varepsilon}^y - e^{-r\tau_n} Y_{\tau_n}^y\right) \mathbb{1}_{\{\tau_n < \tau^\varepsilon\}}\bigg] + \varepsilon, \end{aligned}$$

where the first inequality is due to 1 above. Now, the sequence of random variables $\{Z_n, n \in \mathbb{N}\}$ defined by

$$Z_n := \int_{\tau_n\wedge\tau^\varepsilon}^{\tau^\varepsilon} e^{-rt} c_z(X_t^x, z)dt - \left(e^{-r\tau^\varepsilon} Y_{\tau^\varepsilon}^y - e^{-r\tau_n} Y_{\tau_n}^y\right) \mathbb{1}_{\{\tau_n < \tau^\varepsilon\}}$$

is uniformly integrable due to Assumptions 2.2, 2.3 and 3.3, and $\lim_{n\to\infty} Z_n = 0$ $\mathbb{P}$-a.s., by Remark 3.4-(**2**) and (3.15). Then 3 follows from Vitali's convergence theorem and arbitrariness of ε.

4. Since $v(\,\cdot\,;z) \in C^0(Q)$, the claim follows from 1 and 3 above and by Dini's Lemma. □

Remark 3.10. *For each* $n \in \mathbb{N}$*, the continuity of* $v_n(\,\cdot\,;z)$ *can be proved by its definition* (3.16)*. However, we will obtain it as a byproduct of the characterization of* $v_n(\,\cdot\,;z)$ *as the solution of a suitable variational inequality.*

Denote by $\mathbb{L}$ the second order elliptic differential operator associated to the two-dimensional diffusion $\{(X_t, Y_t), t \ge 0\}$. Since X and Y are independent then $\mathbb{L} := \mathbb{L}_X + \mathbb{L}_Y$, with

$$\begin{aligned} \left(\mathbb{L}_X f\right)(x,y) &:= \frac{1}{2}(\sigma_1)^2(x)\frac{\partial^2}{\partial x^2} f(x,y) + \mu_1(x)\frac{\partial}{\partial x} f(x,y), \\ \left(\mathbb{L}_Y f\right)(x,y) &:= \frac{1}{2}(\sigma_2)^2(y)\frac{\partial^2}{\partial y^2} f(x,y) + \mu_2(y)\frac{\partial}{\partial y} f(x,y), \end{aligned}$$

for $f \in C_b^2(\overline{Q})$. Fix $n \in \mathbb{N}$ and $z \in \mathbb{R}^+$. From standard arguments we can formally associate the function $v_n(\,\cdot\,,\,\cdot\,;z)|_{Q_n}$ to the variational inequality (parametrized in z)

$$\max\Big\{\big(\mathbb{L} - r\big)u(x,y;z) + c_z(x,z), -u(x,y;z) - y\Big\} = 0, \quad (x,y) \in Q_n, \tag{3.17}$$

with boundary condition

$$u(x,y;z) = -y, \qquad (x,y) \in \partial Q_n. \tag{3.18}$$

Proposition 3.11. *Under Assumptions 2.1, 2.2, 2.3 and 3.3, for each* $n \in \mathbb{N}$ *and* $z \in \mathbb{R}^+$ *there exists a unique function* $u_n(\cdot\,;z) \in W^{2,p}(Q_n)$ *for all* $1 \le p < \infty$*, satisfying* (3.17) *a.e. in* Q_n *and the boundary condition* (3.18).

Proof. Since μ_i, σ_i, $i = 1, 2$ are bounded and continuous on Q_n, it suffices to apply [22, Ch. I, Th. 3.2 and Th. 3.4]. □

Remark 3.12. *Note that by well known Sobolev's inclusions (see for instance [10, Ch. 9, Cor. 9.15]), the space $W^{2,p}(Q_n)$ with $p \in (2, \infty)$ can be continuously embedded into $C^1(\overline{Q}_n)$. Hence, the boundary condition* (3.18) *is well-posed for functions in the class $W^{2,p}(Q_n)$, $p \in (2, \infty)$. In the following we shall always refer to the unique C^1 representative of elements of $W^{2,p}(Q_n)$.*

The function $u_n(\,\cdot\,; z)$ of Proposition 3.11 can be continuously extended outside Q_n by setting

$$u_n(x, y; z) = -y, \qquad (x, y) \in Q \setminus Q_n. \tag{3.19}$$

We denote such extension again by u_n with a slight abuse of notation.

Denote by $L^q_r(\mathbb{R}^+)$, $q \in [1, \infty)$, the L^q-spaces on $\mathbb{R}^+$ with respect to the measure $e^{-rs}ds$. We recall that X and Y are independent and make the following

Assumption 3.13. *For every $(x, y) \in \mathcal{I}_1 \times \mathcal{I}_2$ and $t \geq 0$ the laws of X^x_t and Y^y_t have densities $p_1(t, x, \,\cdot\,)$ and $p_2(t, y, \,\cdot\,)$, respectively. Moreover*

1) $(t, \zeta, \xi) \mapsto p_i(t, \zeta, \xi)$ is continuous on $(0, \infty) \times \mathcal{I}_i \times \mathcal{I}_i$, $i = 1, 2$;

2) For any compact set $\mathcal{K} \subset \mathcal{I}_1 \times \mathcal{I}_2$ there exists $q > 1$ (possibly depending on $\mathcal{K}$) such that $p_1(\,\cdot\,, x, \,\cdot\,)p_2(\,\cdot\,, y, \,\cdot\,) \in L^1_r(\mathbb{R}^+; L^q(\mathcal{K}))$, for all $(x, y) \in \mathcal{K}$.

Remark 3.14. *Assumption 3.13 is clearly satisfied in the benchmark case of X and Y given by two independent geometric Brownian motions. The literature on the existence and smoothness of densities for the probability laws of solutions of SDEs driven by Brownian motion is huge and it mainly relies on PDEs' and Malliavin Calculus' techniques (see, e.g., [21] and [34] as classical references on the topic). In general, the existence of a density for the law of a one-dimensional diffusion is guaranteed under some very mild assumptions (see, e.g., the recent paper [20]). Sufficient conditions on our (μ_i, σ_i), $i = 1, 2$, to obtain Gaussian bounds for the transition densities and their first derivatives may be found for instance in [21, Ch. 1, Th. 11]. One can also refer to, e.g., [15] and references therein for more recent generalizations under weaker assumptions.*

Let us define the continuation and stopping regions of our approximating optimal stopping problem (3.16) respectively by

$$\mathcal{C}^n_z := \{(x, y) \in Q \mid v_n(x, y; z) > -y\}, \quad \mathcal{A}^n_z := \{(x, y) \in Q \mid v_n(x, y; z) = -y\}. \tag{3.20}$$

We provide now a verification theorem linking v_n of (3.16) to u_n of Proposition 3.11.

Proposition 3.15. *Let Assumptions 2.1, 2.2, 2.3, 3.3 and 3.13 hold and let $n \in \mathbb{N}$. Then $v_n(\,\cdot\,; z) = u_n(\,\cdot\,; z)$ over Q_n. Moreover, the stopping time*

$$\tau^*_n(x, y; z) := \inf\big\{t \geq 0 \,|\, (X^x_t, Y^y_t) \notin \mathcal{C}^n_z\big\} \tag{3.21}$$

is optimal for problem (3.16).

Proof. Recall that u_n has been extended to Q in (3.19). If $(x,y) \in Q \setminus Q_n$, then the claim clearly follows from Proposition 3.9-(2). Assume $(x,y) \in Q_n$; since $u_n \in W^{2,p}(Q_n)$, by [23, Ch. 7.6] we can find a sequence $\{u_n^k(\cdot\,;z)\,, k \in \mathbb{N}\} \subset C^\infty(Q)$ such that $u_n^k(\cdot\,;z) \to u_n(\cdot\,;z)$ in $W^{2,p}(Q_n)$, $p \in [1,+\infty)$, as $k \to \infty$. Moreover, since u_n is continuous and $\overline{Q}_n$ is a compact, we have $u_n^k(\cdot\,;z) \to u_n(\cdot\,;z)$ uniformly on $\overline{Q}_n$ (cf. [23, Ch. 7.2, Lemma 7.1]).

Dynkin's formula yields for any bounded stopping time τ

$$u_n^k(x,y;z) = \mathbb{E}\left[e^{-r(\tau\wedge\tau_n)}u_n^k(X^x_{\tau\wedge\tau_n}, Y^y_{\tau\wedge\tau_n};z) - \int_0^{\tau\wedge\tau_n} e^{-rt}(\mathbb{L}-r)u_n^k(X^x_t, Y^y_t;z)\,dt\right]. \quad (3.22)$$

Then by localization arguments and using (3.6), (3.22) actually holds for any $\tau \in \mathcal{T}$. We claim (and we will prove it later) that taking limits as $k \to \infty$ in (3.22) leads to

$$u_n(x,y;z) = \mathbb{E}\left[e^{-r(\tau\wedge\tau_n)}u_n(X^x_{\tau\wedge\tau_n}, Y^y_{\tau\wedge\tau_n};z) - \int_0^{\tau\wedge\tau_n} e^{-rt}(\mathbb{L}-r)u_n(X^x_t, Y^y_t;z)\,dt\right], \ \forall \tau \in \mathcal{T}. \quad (3.23)$$

The right-hand side of (3.23) is well defined since Assumption 3.13 implies that the law of (X^x, Y^y) is absolutely continuous with respect to the Lebesgue measure and $(\mathbb{L}-r)u_n$ is defined up to a Lebesgue null-measure set. We now use the variational inequality (3.17) in (3.23) to obtain

$$u_n(x,y;z) \geq \mathbb{E}\left[-e^{-r(\tau\wedge\tau_n)}Y^y_{\tau\wedge\tau_n} + \int_0^{\tau\wedge\tau_n} e^{-rt}c_z(X^x_t, z)\,dt\right]. \quad (3.24)$$

Hence, by arbitrariness of τ, one has $u_n(x,y;z) \geq v_n(x,y;z)$.

To obtain the reverse inequality take

$$\tau := \inf\left\{t \geq 0 \,|\, u_n(X^x_t, Y^y_t;z) = -Y^y_t\right\} \quad (3.25)$$

in (3.23) and recall that $u_n = -y$ on $Q \setminus Q_n$, that $u_n \in C^0(\overline{Q}_n)$ (cf. Remark 3.12) and $\overline{Q}_n$ is bounded so that u_n is bounded in $\overline{Q}_n$ as well. It follows that

$$\begin{aligned} e^{-r(\tau\wedge\tau_n)}u_n(X^x_{\tau\wedge\tau_n}, Y^y_{\tau\wedge\tau_n};z) &= e^{-r(\tau\wedge\tau_n)}u_n(X^x_{\tau\wedge\tau_n}, Y^y_{\tau\wedge\tau_n};z)\mathbb{1}_{\{\tau\wedge\tau_n<\infty\}} \\ &= -\,e^{-r(\tau\wedge\tau_n)}Y^y_{\tau\wedge\tau_n}\mathbb{1}_{\{\tau\wedge\tau_n<\infty\}} = -e^{-r(\tau\wedge\tau_n)}Y^y_{\tau\wedge\tau_n} \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (3.26)$$

by (3.5) and (3.6). Moreover, by (3.17), we have $(\mathbb{L}_X - r)u_n = -c_z$ on the set $\{(x,y) \in Q_n \,|\, u_n(x,y;z) > -y\}$. Hence (3.23) and (3.26) give

$$u_n(x,y;z) = \mathbb{E}\left[-e^{-r(\tau\wedge\tau_n)}Y^y_{\tau\wedge\tau_n} + \int_0^{\tau\wedge\tau_n} e^{-rt}c_z(X^x_t, z)\,dt\right] \leq v_n(x,y;z). \quad (3.27)$$

Therefore, we conclude that $u_n = v_n$ on Q, and that the stopping time τ defined in (3.25) is optimal for problem (3.16) and coincides with the stopping time $\tau_n^*(x,y;z)$ defined in (3.21).

Now, to complete the proof we only need to show that (3.23) follows from (3.22) as $k \to \infty$. In fact, the term on the left-hand side of (3.22) converges pointwisely and the first term in the expectation on the right-hand side converges by uniform convergence. To check convergence of the integral term in the expectation on the right-hand side we take $q_n > 0$ as in Assumption 3.13-(2), p_n such that $\frac{1}{p_n} + \frac{1}{q_n} = 1$ and for simplicity denote $q := q_n$ and $p := p_n$. Then, by Hölder's inequality we have

$$\begin{aligned}
&\left| \mathbb{E}\left[\int_0^{\tau \wedge \tau_n} e^{-rt} (\mathbb{L} - r)(u_n^k - u_n)(X_t^x, Y_t^y; z)\, dt \right] \right| \\
&\leq \int_0^{\infty} e^{-rt} \left(\int_{Q_n} \left| (\mathbb{L} - r)(u_n^k - u_n)(\xi, \zeta; z) \right| p_1(t, x, \xi) p_2(t, y, \zeta) d\xi\, d\zeta \right) dt \qquad (3.28) \\
&\leq C_{M_1, M_2, r, n} \left\| u_n^k - u_n \right\|_{W^{2,p}(Q_n)}
\end{aligned}$$

where last inequality follows by Assumptions 2.1-(i) and 3.13-(2) with $C_{M_1,M_2,r,n} > 0$ depending on Q_n, r and $M_i := \sup_{\overline{Q}_n} \{|\mu_i| + |\sigma_i|\}$, $i = 1, 2$. Now, the right-hand side of (3.28) vanishes as $k \to \infty$ by definition of u_n^k. □

Lemma 3.16. *One has*

$$(\mathbb{L} - r) v_n(x, y) = ry - \mu_2(y), \qquad \textit{for a.e. } (x, y) \in \mathcal{A}_z^n \cap Q_n. \tag{3.29}$$

Proof. Recall that $v_n \equiv u_n$ and that $u_n(\,\cdot\,; z) \in W^{2,p}(Q_n)$ (cf.(3.19), Proposition 3.15, Proposition 3.11 and (3.19), respectively). Set $\bar{v}_n(x, y; z) := v_n(x, y; z) + y$, hence $\bar{v}_n \in C^1(Q_n)$ by Sobolev's embedding (see for instance [10, Ch. 9, Cor. 9.15]) and proving (3.29) amounts to showing that $(\mathbb{L} - r)\bar{v}_n = 0$ a.e. on $\mathcal{A}_z^n \cap Q_n$. Since $\bar{v}_n = 0$ over $\mathcal{A}_z^n$, it must also be $\nabla \bar{v}_n = 0$ over $\mathcal{A}_z^n \cap Q_n$. To complete the proof it thus remains to show that the Hessian matrix $D^2 \bar{v}_n$ is zero a.e. over $\mathcal{A}_z^n \cap Q_n$. This follows by [17, Cor. 1-(i), p. 84][2] with f therein defined by $f := \nabla \bar{v}_n$.
□

Proposition 3.17. *For every $(x, y) \in Q$ the following representation holds*

$$v_n(x, y; z) = \mathbb{E}\left[\int_0^{\tau_n} e^{-rt} \Big(c_z(X_t^x, z) \mathbb{1}_{\{(X_t^x, Y_t^y) \in \mathcal{C}_z^n\}} - (rY_t^y - \mu_2(Y_t^y)) \mathbb{1}_{\{(X_t^x, Y_t^y) \in \mathcal{A}_z^n\}} \Big)\, dt - e^{-r\tau_n} Y_{\tau_n}^y \right]. \tag{3.30}$$

Proof. Taking $\tau = \infty$ in (3.23) and considering (3.26) and Proposition 3.15, we get

$$v_n(x, y; z) = \mathbb{E}\left[-e^{-r\tau_n} Y_{\tau_n}^y - \int_0^{\tau_n} e^{-rt} (\mathbb{L} - r) v_n(X_t^x, Y_t^y; z)\, dt \right]. \tag{3.31}$$

It follows from Propositions 3.11, 3.15 and from Lemma 3.16 that

$$(\mathbb{L} - r) v_n(x, y; z) = c_z(x, z) \mathbb{1}_{\{(x,y) \in \mathcal{C}_z^n\}} - (ry - \mu_2(y)) \mathbb{1}_{\{(x,y) \in \mathcal{A}_z^n\}}, \quad \text{for a.e. } (x, y) \in Q_n, \tag{3.32}$$

and we have the claim by using (3.31) and Assumption 3.13 in (3.32). □

[2] It is worth noting that [17, Cor. 1-(i), p. 84] requires f to be Lipschitz continuous, which is not guaranteed for us. However Lipschitz continuity is only needed there to have existence a.e. of the gradient ∇f, which we have due to [17, Th. 1, p. 235] since $\nabla \bar{v}_n \in W^{1,p}(Q_n)$.

We now aim at proving a probabilistic representation of v similar to (3.30). The idea is to pass (3.30) to the limit as $n \uparrow \infty$ and use Proposition 3.9. For that we first define the continuation and stopping regions of problem (3.2) as

$$\mathcal{C}_z := \{(x,y) \in Q \mid v(x,y;z) > -y\}, \quad \mathcal{A}_z := \{(x,y) \in Q \mid v(x,y;z) = -y\}. \tag{3.33}$$

It is worth recalling that (3.1) and standard arguments based on exit times from small subsets of Q give the following inclusion

$$\mathcal{A}_z \subset L^-_z := \big\{(x,y) \in Q \,|\, c_z(x,z) \leq \mu_2(y) - ry\big\}. \tag{3.34}$$

We observe that since $v_n \leq v$ and $\{v_n, n \in \mathbb{N}\}$ is an increasing sequence then

$$\mathcal{C}^n_z \subset \mathcal{C}^{n+1}_z \subset \mathcal{C}_z, \quad \mathcal{A}^n_z \supset \mathcal{A}^{n+1}_z \supset \mathcal{A}_z, \quad \forall n \in \mathbb{N}. \tag{3.35}$$

On the other hand, the pointwise convergence $v_n \uparrow v$ (cf. Proposition 3.9) implies that if $(x_0, y_0) \in \mathcal{C}_z$, then $v(x_0,y_0)+y_0 \geq \varepsilon_0$ for some $\varepsilon_0 > 0$ and $v_n(x_0,y_0)+y_0 \geq \varepsilon_0/2$ for all $n \geq n_0$ and suitable $n_0 \in \mathbb{N}$. Hence we have

$$\lim_{n\to\infty} \mathcal{C}^n_z := \bigcup_{n\geq 0} \mathcal{C}^n_z = \mathcal{C}_z, \qquad \lim_{n\to\infty} \mathcal{A}^n_z := \bigcap_{n\geq 0} \mathcal{A}^n_z = \mathcal{A}_z \tag{3.36}$$

and the following representation result.

Theorem 3.18. *Under Assumptions 2.1, 2.2, 2.3, 3.3 and 3.13 the following representation holds for every* $(x,y) \in Q$*:*

$$v(x,y;z) = \mathbb{E}\bigg[\int_0^\infty e^{-rt}\Big(c_z(X^x_t,z)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{C}_z\}} - (rY^y_t - \mu_2(Y^y_t))\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{A}_z\}}\Big)dt\bigg]. \tag{3.37}$$

Proof. We study (3.30) in the limit as $n \uparrow \infty$. Observe that:

1. The left-hand side of (3.30) converges pointwisely to $v(x,y;z)$ by Proposition 3.9-(3);

2. $\{e^{-r\tau_n}Y^y_{\tau_n},\, n \in \mathbb{N}\}$ is a family of random variables uniformly integrable and converging a.s. to 0, due to (3.15) and to Assumptions 2.3 and 3.3 (see also the discussion in Remark 3.4-(**2**)). Hence $\lim_{n\to\infty} \mathbb{E}\,[e^{-r\tau_n}Y^y_{\tau_n}] = 0$, by Vitali's convergence Theorem;

3. From (3.35), one has

$$\begin{aligned}
&\bigg|\mathbb{E}\bigg[\int_0^{\tau_n} e^{-rt}c_z(X^x_t,z)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{C}_n\}}dt - \int_0^\infty e^{-rt}c_z(X^x_t,z)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{C}\}}dt\bigg]\bigg| \qquad (3.38)\\
&\leq \mathbb{E}\bigg[\int_0^\infty e^{-rt}|c_z(X^x_t,z)|\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{C}\setminus\mathcal{C}_n\}}dt\bigg] + \mathbb{E}\bigg[\int_{\tau_n}^\infty e^{-rt}|c_z(X^x_t,z)|\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{C}\}}dt\bigg].
\end{aligned}$$

The first term in the right-hand side of (3.38) converges to zero as $n \to \infty$ by dominated convergence and (3.36) (cf. Assumptions 2.2-(iii), 2.3 and Remark 2.4-(**3**)). Similarly, dominated convergence and (3.15) give

$$\lim_{n\to\infty} \mathbb{E}\bigg[\int_{\tau_n}^\infty e^{-rt}|c_z(X^x_t,z)|\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{C}\}}dt\bigg] = 0.$$

4. From (3.36) it follows that for a.e. $(t,\omega) \in \mathbb{R}^+ \times \Omega$

$$\lim_{n\to\infty} \mathbb{1}_{[0,\tau_n]}(t)e^{-rt}\Big[rY^y_t - \mu_2(Y^y_t)\Big]\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{A}^n_z\}} = e^{-rt}\Big[rY^y_t - \mu_2(Y^y_t)\Big]\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{A}_z\}}.$$

Moreover, due to Lipschitz-continuity of μ_2 (cf. Assumption 2.1),

$$\Big|e^{-rt}\Big[rY^y_t - \mu_2(Y^y_t)\Big]\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{A}^n_z\}}\Big| \;\le\; e^{-rt}\Big|rY^y_t - \mu_2(Y^y_t)\Big| \;\le\; e^{-rt}C_0(1+Y^y_t),$$

for some $C_0 > 0$ depending on y and r. The last expression of the inequality above is integrable in $\mathbb{R}^+ \times \Omega$ by (2.10) and by Assumption 2.3. Hence dominated convergence and (3.15) yield

$$\lim_{n\to\infty} \mathbb{E}\bigg[\int_0^{\tau_n} e^{-rt}\Big[rY^y_t - \mu_2(Y^y_t)\Big]\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{A}^n_z\}}dt\bigg] = \mathbb{E}\bigg[\int_0^{\infty} e^{-rt}\Big[rY^y_t - \mu_2(Y^y_t)\Big]\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{A}_z\}}dt\bigg].$$

Now taking $n \to \infty$ in (3.30) and using *1-4* above, (3.37) follows. □

Set

$$H(x,y;z) := c_z(x,z)\mathbb{1}_{\{(x,y)\in\mathcal{C}_z\}} - (ry - \mu_2(y))\mathbb{1}_{\{(x,y)\in\mathcal{A}_z\}} \tag{3.39}$$

so that (3.37) may be written as

$$v(x,y;z) = \mathbb{E}\bigg[\int_0^\infty e^{-rt}H(X^x_t,Y^y_t;z)dt\bigg]. \tag{3.40}$$

Due to (3.8) and Assumption 2.3, the strong Markov property and standard arguments based on conditional expectations applied to the representation formula (3.40) allow to verify that

$$\bigg\{e^{-rt}v(X^x_t,Y^y_t;z) + \int_0^t e^{-rs}H(X^x_s,Y^y_s;z)ds,\ t\ge 0\bigg\} \text{ is an } (\mathcal{F}_t)\text{-martingale}, \tag{3.41}$$

for all $(x,y) \in Q$.

By similar methods one can check that

$$\big|e^{-r\tau}v(X^x_\tau,Y^y_\tau;z)\big| \le \mathbb{E}\bigg[\int_0^\infty e^{-rt}\big|H(X^x_t,Y^y_t;z)\big|dt\bigg|\mathcal{F}_\tau\bigg], \quad \tau\in\mathcal{T}, \tag{3.42}$$

and hence

$$\text{the family } \big\{e^{-r\tau}v(X^x_\tau,Y^y_\tau;z),\ \tau\in\mathcal{T}\big\} \text{ is uniformly integrable.} \tag{3.43}$$

Now, recalling (3.34) and according to standard theory of optimal stopping (cf., e.g., [35, Th. 2.4]), the martingale property (3.41) gives

Theorem 3.19. *Fix $(x,y) \in Q$. Under Assumptions 2.1, 2.2, 2.3, 3.3 and 3.13, the process*

$$S := \bigg\{e^{-rt}v(X^x_t,Y^y_t;z) + \int_0^t e^{-rs}c_z(X^x_s,z)ds,\ t\ge 0\bigg\} \tag{3.44}$$

is an $(\mathcal{F}_t)$*-supermartingale and*

$$\mathbb{E}\bigg[e^{-r\tau}v(X^x_\tau, Y^y_\tau; z) + \int_0^\tau e^{-rs}c_z(X^x_s, z)ds\bigg] \le v(x, y; z)\,, \quad \forall \tau \in \mathcal{T}. \tag{3.45}$$

Moreover, the stopping time

$$\tau^* = \tau^*(x, y; z) := \inf\{\, t \ge 0 \mid v(X^x_t, Y^y_t; z) = -Y^y_t \,\} \tag{3.46}$$

is optimal for problem (3.2) *and the process* $\big\{S_{t\wedge\tau^*},\, t \ge 0\big\}$ *is an* $(\mathcal{F}_t)$*-martingale.*

Proof. The supermartingale property (3.44) easily follows from (3.41) and (3.34). Similarly, (3.45) is true for any $\sigma_n := \tau \wedge n$ with $\tau \in \mathcal{T}$ and $n \in \mathbb{N}$, i.e. (cf. (3.44))

$$\mathbb{E}[S_{\sigma_n}] \le S_0. \tag{3.47}$$

Then (3.45) is obtained by taking limits as $n \to \infty$ and by using dominated convergence, (3.43) and the fact that $S_{\sigma_n} \to S_\tau$ $\mathbb{P}$-a.s. by Proposition 3.7 and continuity of paths.

For the optimality of τ^* notice that (3.47) holds with equality if $\sigma_n = \tau^* \wedge n$ and, moreover, $\big(v(X^x_{\tau^*}, Y^y_{\tau^*}; z) + Y^y_{\tau^*}\big)\mathbb{1}_{\{\tau^* \le n\}} = 0$ $\mathbb{P}$-a.s. Hence one has

$$v(x, y; z) = \mathbb{E}\bigg[\int_0^{\tau^*\wedge n} e^{-rt}c_z(X^x_t, z)dt - \mathbb{1}_{\{\tau^*\le n\}}e^{-r\tau^*}Y^y_{\tau^*} + \mathbb{1}_{\{\tau^*>n\}}e^{-rn}v(X^x_n, Y^y_n; z)\bigg]. \tag{3.48}$$

Taking limits as $n \to \infty$ and using Assumptions 2.2, 2.3, 3.3, Proposition 3.6-(1), and dominated convergence one obtains

$$v(x, y; z) = \mathbb{E}\bigg[\int_0^{\tau^*} e^{-rt}c_z(X^x_t, z)dt - e^{-r\tau^*}Y^y_{\tau^*}\bigg], \tag{3.49}$$

hence optimality of τ^*. The martingale property of $\big\{S_{t\wedge\tau^*},\, t > 0\big\}$ easily follows from the results above. □

4 Characterization of the Optimal Boundary

In this section we will provide a characterization of the optimal boundaries of the family of optimal stopping problems (3.2). For that we define

$$y^*(x; z) := \inf\{y \in \mathcal{I}_2 \mid v(x, y; z) > -y\}, \quad (x, z) \in \mathcal{I}_1 \times \mathbb{R}^+, \tag{4.1}$$

with the convention $\inf \emptyset = \overline{y}$. Notice that under this convention $y^*(\,\cdot\,; z)$ takes values in $\overline{\mathcal{I}}_2$. We will show that under suitable conditions $y^*(\,\cdot\,; z)$ splits $\mathcal{I}_1 \times \mathcal{I}_2$ into $\mathcal{C}_z$ and $\mathcal{A}_z$ (cf. (3.33)). Moreover, we will characterize $y^*(\,\cdot\,; z)$ as the unique continuous solution of a nonlinear integral equation of Fredholm type.

Remark 4.1. *Integral equations for the optimal boundaries of one-dimensional optimal stopping problems on a finite time-horizon are often obtained by an application of the so-called local time space calculus (cf. [36]). In order to do so in our case we should prove that the process* $\{y^*(X^x_t; z),\, t \geq 0\}$ *is a semimartingale for each given* $z \in \mathbb{R}^+$ *as required in [36, Th. 2.1]. That seems an extremely hard task and we will follow a different approach mainly based on the results of Section 3 and probabilistic techniques.*

We now make the following

Assumption 4.2. *Assumptions 2.1, 2.2, 2.3, 3.3 and 3.13 hold. Moreover, the map* $y \mapsto ry - \mu_2(y)$ *is strictly increasing.*

Proposition 4.3. *Under Assumption 4.2 one has (cf.* (3.33)*)*

$$\mathcal{C}_z = \{(x,y) \in Q \mid y > y^*(x;z)\}, \quad \mathcal{A}_z = \{(x,y) \in Q \mid y \leq y^*(x;z)\}. \tag{4.2}$$

Proof. It suffices to show that $y \mapsto v(x,y;z) + y$ is nondecreasing for each $x \in \mathcal{I}_1$, $z \in \mathbb{R}^+$. Set $\bar{u} := v + y$, take y_1 and y_2 in $\mathcal{I}_2$ such that $y_2 > y_1$ and set $\tau_1 := \inf\{t \geq 0 \mid (X^x_t, Y^{y_1}_t) \notin \mathcal{C}_z\}$, which is optimal for $v(x,y_1;z)$. From Lemma 3.5 and the superharmonic characterization of Theorem 3.19 we obtain

$$\begin{aligned}\bar{v}(x,y_2;z) - \bar{v}(x,y_1;z) \geq\ & \mathbb{E}\Big[e^{-r\tau_1}\big(\bar{v}(X^x_{\tau_1}, Y^{y_2}_{\tau_1}; z) - \bar{v}(X^x_{\tau_1}, Y^{y_1}_{\tau_1}; z)\big)\Big] \\ & + \mathbb{E}\Big[\int_0^{\tau_1} e^{-rt}\Big(r\big(Y^{y_2}_t - Y^{y_1}_t\big) - \big(\mu_2(Y^{y_2}_t) - \mu_2(Y^{y_1}_t)\big)\Big)dt\Big] \\ \geq\ & \mathbb{E}\Big[e^{-r\tau_1}\big(\bar{v}(X^x_{\tau_1}, Y^{y_2}_{\tau_1}; z) - \bar{v}(X^x_{\tau_1}, Y^{y_1}_{\tau_1}; z)\big)\Big],\end{aligned} \tag{4.3}$$

where the last inequality follows by (2.8) and Assumption 4.2. Note that the last expression in (4.3) is well defined thanks to Assumption 3.3 and (3.43). Moreover, since $\bar{v} \geq 0$ it holds

$$\mathbb{E}\Big[e^{-r\tau_1}\big(\bar{v}(X^x_{\tau_1}, Y^{y_2}_{\tau_1}; z) - \bar{v}(X^x_{\tau_1}, Y^{y_1}_{\tau_1}; z)\big)\Big] \geq -\mathbb{E}\Big[e^{-r\tau_1}\bar{v}(X^x_{\tau_1}, Y^{y_1}_{\tau_1}, z)\Big]. \tag{4.4}$$

By Assumption 2.3, Proposition 3.6-(1) and since $\mathbb{1}_{\{\tau_1 \leq n\}} e^{-r\tau_1}\bar{v}(X^x_{\tau_1}, Y^{y_1}_{\tau_1}; z) = 0$ $\mathbb{P}$-a.s., Fatou's Lemma gives

$$\begin{aligned}\mathbb{E}\Big[e^{-r\tau_1}\bar{v}(X^x_{\tau_1}, Y^{y_1}_{\tau_1}; z)\Big] &= \mathbb{E}\Big[\liminf_{n\to\infty} e^{-r(\tau_1\wedge n)}\bar{v}(X^x_{\tau_1\wedge n}, Y^{y_1}_{\tau_1\wedge n}; z)\Big] \\ &\leq \liminf_{n\to\infty} \mathbb{E}\left[e^{-rn}\bar{v}(X^x_n, Y^{y_1}_n; z)\mathbb{1}_{\{\tau_1 > n\}}\right] = 0\end{aligned} \tag{4.5}$$

Now (4.3), (4.4) and (4.5) imply that $y \mapsto \bar{v}(x,y;z)$ is increasing and therefore (4.2) holds. □

Notice that (3.37) and (4.2) imply

$$v(x,y;z) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(c_z(X^x_t, z)\mathbb{1}_{\{Y^y_t > y^*(X^x_t;z)\}} - (rY^y_t - \mu_2(Y^y_t))\mathbb{1}_{\{Y^y_t \leq y^*(X^x_t;z)\}}\Big)dt\Big]. \tag{4.6}$$

Under Assumption 3.13, (4.6) can also be expressed in a purely analytical way as

$$\begin{aligned} v(x,y;z) = &\int_0^\infty e^{-rt}\Big[\int_{\underline{x}}^{\overline{x}} p_1(t,x,\xi)c_z(\xi,z)\Big(\int_{y^*(\xi;z)}^{\overline{y}} p_2(t,y,\eta)d\eta\Big)d\xi\Big]dt \\ &- \int_0^\infty e^{-rt}\Big[\int_{\underline{x}}^{\overline{x}} p_1(t,x,\xi)\Big(\int_{\underline{y}}^{y^*(\xi;z)} (r\eta-\mu_2(\eta))p_2(t,y,\eta)d\eta\Big)d\xi\Big]dt, \end{aligned} \tag{4.7}$$

for any $(x,y,z)\in\mathcal{O}$.

Proposition 4.4. *Under Assumption 4.2 one has*

1. *the function $y^*(\,\cdot\,;z)$ is nondecreasing and right-continuous for any $z\in\mathbb{R}^+$;*
2. *the function $y^*(x;\,\cdot\,)$ is nonincreasing and left-continuous for any $x\in\mathcal{I}_1$;*

Proof. Claims 1 and 2 follow by adapting arguments from the proof of [25, Prop. 2.2] and by using our Proposition 3.6-(2)-(3), and Proposition 3.7. □

It follows from Propositions 4.3 and 4.4-(1) that the regions $\mathcal{C}_z$ and $\mathcal{A}_z$ are connected for every $z\in\mathbb{R}^+$, and the optimal stopping time $\tau^*(x,y;z)$ defined in (3.46) can be written as

$$\tau^*(x,y;z) = \inf\big\{t\ge 0\,|\,Y^y_t \le y^*(X^x_t;z)\big\}. \tag{4.8}$$

Thanks to the representation (4.6) or (4.7), under the following further assumptions we can prove the C^1-regularity of the function v.

Assumption 4.5. *The functions $p_1(t,\cdot,\xi)$ and $p_2(t,\cdot,\eta)$ are differentiable for each $(t,\xi)\in\mathbb{R}^+\times\mathcal{I}_1$ and each $(t,\eta)\in\mathbb{R}^+\times\mathcal{I}_2$, respectively. Moreover, denoting by p_i', $i=1,2$ the partial derivative of p_i with respect to the second variable, it holds*

1) *$x\mapsto p_1'(t,x,\xi)$ is continuous in $\mathcal{I}_1$ for all $(t,\xi)\in\mathbb{R}^+\times\mathcal{I}_1$ and, for any $(x,y,z)\in\mathcal{O}$, there exists $\delta>0$ such that $\sup_{\zeta\in[x-\delta,x+\delta]}\big|p_1'(t,\zeta,\xi)\big|\le\psi_1(t,\xi;\delta)$ for some ψ_1 such that*

$$(t,\xi,\eta)\mapsto e^{-rt}\psi_1(t,\xi;\delta)p_2(t,y,\eta)\big(c_z(\xi,z)+\eta\big) \text{ is in } L^1(\mathbb{R}^+\times\mathcal{I}_1\times\mathcal{I}_2); \tag{4.9}$$

2) *$y\mapsto p_2'(t,y,\eta)$ is continuous in $\mathcal{I}_2$ for all $(t,\eta)\in\mathbb{R}^+\times\mathcal{I}_2$ and, for any $(x,y,z)\in\mathcal{O}$, there exists $\delta>0$ such that $\sup_{\zeta\in[y-\delta,y+\delta]}\big|p_2'(t,\zeta,\eta)\big|\le\psi_2(t,\eta;\delta)$ for some ψ_2 such that*

$$(t,\xi,\eta)\mapsto e^{-rt}\psi_2(t,\eta;\delta)p_1(t,x,\xi)\big(c_z(\xi,z)+\eta\big) \text{ is in } L^1(\mathbb{R}^+\times\mathcal{I}_1\times\mathcal{I}_2). \tag{4.10}$$

Proposition 4.6. *Under Assumptions 4.2 and 4.5 one has $v(\,\cdot\,;z)\in C^1(Q)$ for every $z\in\mathbb{R}^+$.*

Proof. The proof follows by (4.7), by Assumption 4.5 and standard dominated convergence arguments. □

Proposition 4.6 above states in particular the so-called *smooth-fit* condition across the free-boundary, i.e. the continuity of $v_x(\,\cdot\,;z)$ and $v_y(\,\cdot\,;z)$ at $\partial\mathcal{A}_z$. With the aim of characterizing the boundary $y^*(\,\cdot\,;z)$ as unique *continuous* solution of a (parametric) integral equation we make the following additional

Assumption 4.7. *The drift coefficient* μ_2 *is continuously differentiable in* $\mathcal{I}_2$ *and* $\frac{\partial \mu_2}{\partial y} < r$. *Moreover,* $(\mu_2, \sigma_2) \in C^{1+\delta}(\mathcal{I}_2)$, *for some* $\delta > 0$.

Proposition 4.8. *Under Assumptions 4.2, 4.5 and 4.7, the function* $y^*(\,\cdot\,; z) : \mathcal{I}_1 \to \overline{\mathcal{I}}_2$ *is continuous.*

Proof. We know that the function $y^*(\,\cdot\,; z)$ is nondecreasing and right-continuous by Proposition 4.4-(1). Hence it suffices to show that it is also left-continuous. Arguing by contradiction, we assume that there exists $x_0 \in \mathcal{I}_1$ such that $y^*(x_0-; z) := \lim_{x \uparrow x_0} y^*(x; z) < y^*(x_0; z)$. Then, there also exist $y_0 \in \mathcal{I}_2$ and $\varepsilon > 0$ such that

$$\Sigma_z := (x_0 - \varepsilon, x_0) \times (y_0 - \varepsilon, y_0 + \varepsilon) \subset \mathcal{C}_z, \qquad \{x_0\} \times (y_0 - \varepsilon, y_0 + \varepsilon) \subset \mathcal{A}_z.$$

Notice that, by standard arguments on free-boundary problems and optimal stopping (cf. for instance [35, Ch. 3, Sec. 7]), one has that $v(\,\cdot\,; z) \in C^2(\mathcal{C}_z)$ and solves

$$\frac{1}{2}\sigma_1^2(x) v_{xx}(x, y; z) = -\mu_1(x) v_x(x, y; z) - (\mathbb{L}_Y - r) v(x, y; z) - c_z(x, z), \qquad (x, y) \in \mathcal{C}_z. \tag{4.11}$$

On the other hand, since $(\mu_2, \sigma_2) \in C^{1+\delta}(\mathcal{I}_2)$, regularity results on uniformly elliptic partial differential equations (cf. for instance [23, Ch. 6, Th. 6.17]) imply that one actually has $v_y(\,\cdot\,; z) \in C^{2+\delta}(\mathcal{C}_z)$. Hence we can differentiate (4.11) with respect to y to find

$$\frac{1}{2}\sigma_1^2(x) (v_y)_{xx}(x, y; z) = -\mu_1(x) (v_y)_x(x, y; z) - (\mathcal{R} - r) v_y(x, y; z), \qquad (x, y) \in \mathcal{C}_z, \tag{4.12}$$

where

$$(\mathcal{R} f)(x, y) := \frac{1}{2}\sigma_2^2(y) f_{yy}(x, y) + \Big[\frac{\partial \sigma_2^2}{\partial y}(y) + \mu_2(y)\Big] f_y(x, y) + \frac{\partial \mu_2}{\partial y}(y) f(x, y), \qquad f \in C_b^2(Q).$$

Take now $y_1, y_2 \in (y_0 - \varepsilon, y_0 + \varepsilon)$ with $y_1 < y_2$ and set

$$F_\phi(x; y_1, y_2, z) := -\int_{y_1}^{y_2} v_{xx}(x, y; z) \phi'(y) dy, \qquad x \in (x_0 - \varepsilon, x_0), \tag{4.13}$$

where ϕ is real-valued, arbitrarily chosen and such that

$$\phi \in C_c^\infty(y_1, y_2), \qquad \phi \geq 0, \qquad \int_{y_1}^{y_2} \phi(y) dy > 0.$$

From now on we will write $F_\phi(x)$ instead of $F_\phi(x; y_1, y_2, z)$ to simplify the notation. Multiply both sides of (4.12) by $2\phi(y)/\sigma_1^2(x)$ and integrate by parts with respect to $y \in (y_1, y_2)$; it follows

$$\begin{aligned} F_\phi(x) &= -\int_{y_1}^{y_2} \frac{1}{\sigma_1^2(x)} \Big[\mu_1(x) v_{xy}(x, y; z) + (\mathcal{R} - r) v_y(x, y; z)\Big] \phi(y) dy \\ &= \frac{\mu_1(x)}{\sigma_1^2(x)} \int_{y_1}^{y_2} v_x(x, y; z) \phi'(y) dy + \frac{1}{\sigma_1^2(x)} \int_{y_1}^{y_2} v(x, y; z) \frac{\partial}{\partial y} (\mathcal{R} - r)^* \phi(y) dy, \end{aligned} \tag{4.14}$$

for every $x \in (x_0 - \varepsilon, x_0)$, with $(\mathcal{R} - r)^*$ denoting the adjoint of $(\mathcal{R} - r)$. Now, recalling Proposition 4.6 and the definition of $\mathcal{C}_z$ and $\mathcal{A}_z$ one also has

$$\begin{cases} v(x_0, y; z) = -y, & \forall y \in [y_1, y_2], \\ v_x(x_0, y; z) = 0, & \forall y \in [y_1, y_2], \\ v_y(x_0, y; z) = -1, & \forall y \in [y_1, y_2], \end{cases} \tag{4.15}$$

and thus, taking limits in (4.14), one obtains

$$\begin{aligned} \lim_{x \uparrow x_0} F_\phi(x) &= -\frac{1}{\sigma_1^2(x_0)} \int_{y_1}^{y_2} y \frac{\partial}{\partial y} (\mathcal{R} - r)^* \phi(y) dy = \frac{1}{\sigma_1^2(x_0)} \int_{y_1}^{y_2} [(\mathcal{R} - r)1] \phi(y) dy \\ &= \frac{1}{\sigma_1^2(x_0)} \int_{y_1}^{y_2} \Big(\frac{\partial}{\partial y} \mu_2(y) - r \Big) \phi(y) dy \ < \ 0, \end{aligned} \tag{4.16}$$

where the last inequality follows from Assumption 4.7. Since F_ϕ is clearly continuous in $(x_0 - \varepsilon, x_0)$, we see from (4.16) that it must be $F_\phi < 0$ in a left neighborhood of x_0 and, without any loss of generality, we assume that $F_\phi < 0$ in $(x_0 - \varepsilon, x_0)$. Recalling (4.13), we have for each $\delta \in (0, \varepsilon)$

$$\begin{aligned} 0 > \int_{x_0 - \delta}^{x_0} F_\phi(x) dx &= - \int_{x_0 - \delta}^{x_0} \int_{y_1}^{y_2} v_{xx}(x, y; z) \phi'(y) dy\, dx \\ &= - \int_{y_1}^{y_2} [v_x(x_0, y; z) - v_x(x_0 - \delta, y; z)] \phi'(y) dy \\ &= \int_{y_1}^{y_2} v_x(x_0 - \delta, y; z) \phi'(y) dy = - \int_{y_1}^{y_2} v_{xy}(x_0 - \delta, y; z) \phi(y) dy, \end{aligned}$$

by (4.15) and Fubini-Tonelli's theorem. This implies that $v_{xy}(\,\cdot\,; z) > 0$ in Σ_z by arbitrariness of ϕ and δ and hence the function $x \mapsto v_y(x, y; z)$ is strictly increasing in $(x_0 - \varepsilon, x_0)$ for any $y \in [y_1, y_2]$. It then follows from the last of (4.15)

$$v_y(\,\cdot\,; z) < -1 \quad \text{in } \Sigma_z \subset \mathcal{C}_z. \tag{4.17}$$

On the other hand, $v_y(\,\cdot\,; z)$ solves (4.12) subject to the boundary condition $v_y(\,\cdot\,; z) = -1$ on $\partial\mathcal{C}_z$ by Proposition 4.6. Therefore it admits the standard Feynman-Kac representation (see, e.g., [27, Ch. 5, Sec. 7.B])

$$v_y(x, y; z) = \mathbb{E}\Big[- e^{\int_0^{\tau_{\mathcal{C}_z}} \left(\frac{\partial}{\partial y} \mu_2(\tilde{Y}_t^y) - r \right) dt} \Big], \tag{4.18}$$

where $\tau_{\mathcal{C}_z} := \inf\{t \geq 0 \mid (X_t^x, \tilde{Y}_t^y) \notin \mathcal{C}_z\}$, and with $\tilde{Y}^y$ solving

$$\begin{cases} d\tilde{Y}_t^y = \Big[\frac{\partial \sigma_2^2}{\partial y} (\tilde{Y}_t^y) + \mu_2(\tilde{Y}_t^y) \Big] dt + \sigma_2(\tilde{Y}_t^y) dW_t^2, & t > 0, \\ \tilde{Y}_0^y = y. \end{cases}$$

Since $r > \frac{\partial \mu_2}{\partial y}$ by Assumption 4.7, (4.18) implies $v_y(\,\cdot\,; z) > -1$ in $\mathcal{C}_z$, contradicting (4.17) and concluding the proof. □

In order to find an upper bound for $y^*(\,\cdot\,;z)$ we now denote

$$F(x,y;z) := c_z(x,z) - \mu_2(y) + ry, \qquad (x,y) \in \overline{Q}, \tag{4.19}$$

and define

$$\vartheta(x;z) := \inf\{y \in \mathcal{I}_2 \,|\, F(x,y;z) > 0\} \in \overline{\mathcal{I}}_2, \quad x \in \mathcal{I}_1, \tag{4.20}$$

with the convention $\inf \emptyset = \overline{y}$. Then by Proposition 4.3 and by (3.34), we have

$$y^*(\,\cdot\,;z) \le \vartheta(\,\cdot\,;z). \tag{4.21}$$

Lemma 4.9. *Under Assumption 4.2 and 4.7, the function $\vartheta(\,\cdot\,;z)$ is nondecreasing and continuous. Moreover, if $\vartheta(x;z) \in \mathcal{I}_2$ then $\vartheta(x;z)$ is the unique solution to the equation $F(x,\cdot;z) = 0$ in $\mathcal{I}_2$. Finally one has*

$$\big\{(x,y) \in Q \mid c_z(x,z) - \mu_2(y) + ry < 0\big\} = \{(x,y) \in Q \mid y < \vartheta(x;z)\}. \tag{4.22}$$

Proof. Since $x \mapsto F(x,y;z)$ is nonincreasing (cf. Assumption 2.2-(ii)) and $y \mapsto F(x,y;z)$ is increasing by Assumption 4.7 and $(x,y) \mapsto F(x,y;z)$ it is not hard to see that $\vartheta(\cdot;z)$ is nondecreasing and right-continuous.

The definition of $\vartheta(\cdot;z)$ and the continuity of F guarantee that if $\vartheta(x;z) \in \mathcal{I}_2$ then $\vartheta(x;z)$ solves $F(x,\cdot;z) = 0$ in $\mathcal{I}_2$. Assumption 4.7 then implies that $\vartheta(x;z)$ is actually the unique solution of such equation.

Let us now show that $\vartheta(\,\cdot\,;z)$ is continuous. Take x_0 such that $\vartheta(x_0;z) > \underline{y}$ and assume that $\vartheta(x_0-;z) < \vartheta(x_0;z)$. Take a sequence $\{x_n\,,\, n \in \mathbb{N}\} \subset \mathcal{I}_1$ increasing and such that $x_n \uparrow x_0$. One has $F(x_n, \vartheta(x_n;z);z) \ge 0$ for all $n \in \mathbb{N}$ and hence in the limit one finds $F(x_0, \vartheta(x_0-;z);z) \ge 0 \ge F(x_0, \vartheta(x_0;z);z)$ which implies $\vartheta(x_0-;z) \ge \vartheta(x_0;z)$ since $y \mapsto F(x,y;z)$ is increasing.

Clearly (4.22) follows from the previous properties. □

Consider now the class of functions

$$\mathcal{M}_z := \{f : \mathcal{I}_1 \to \overline{\mathcal{I}}_2, \text{ continuous, nondecreasing and dominated from above by } \vartheta(\,\cdot\,;z)\},$$

and define

$$\mathcal{D}_f := \{x \in \mathcal{I}_1 \mid f(x) \in \mathcal{I}_2\}, \qquad f \in \mathcal{M}_z.$$

Clearly $\mathcal{M}_z$ is nonempty as $\vartheta(\,\cdot\,;z) \in \mathcal{M}_z$ by Lemma 4.9, and $\mathcal{D}_f$ is an open sub-interval (possibly empty) of $\mathcal{I}_1$. We set

$$\underline{x}_f := \inf\{x \in \mathcal{I}_1 \mid f(x) > \underline{y}\}, \qquad \overline{x}_f := \sup\{x \in \mathcal{I}_1 \mid f(x) < \overline{y}\}, \tag{4.23}$$

with the conventions $\inf \emptyset = \overline{x}$, $\sup \emptyset = \underline{x}$. Notice that by monotonicity of any arbitrary $f \in \mathcal{M}_z$ we have $f \equiv \underline{y}$ on $(\underline{x}, \underline{x}_f)$ (if the latter is nonempty) and, analogously, $f \equiv \overline{y}$ on $(\overline{x}_f, \overline{x})$ (if the latter is nonempty). Given a function $\hat{y}(\,\cdot\,;z) \in \mathcal{M}_z$, we set

$$\widehat{H}(x,y;z) := c_z(x,z)\mathbb{1}_{\{y > \hat{y}(x;z)\}} - \big(ry - \mu_2(y)\big)\mathbb{1}_{\{y \le \hat{y}(x;z)\}} \tag{4.24}$$

and define

$$w(x,y;z) := \mathbb{E}\left[\int_0^\infty e^{-rt}\widehat{H}(X_t^x, Y_t^y; z)dt\right]. \tag{4.25}$$

Notice that

$$\big|w(x,y;z)\big| \le C(z)\big(1+|x|^\beta+|y|\big), \qquad \text{for } (x,y)\in Q, \tag{4.26}$$

by Assumptions 2.1, 2.2, 2.3 (cf. also (3.8)). Moreover, as in (3.41) and (4.2) one can verify that

$$\left\{e^{-rt}w(X_t^x, Y_t^y; z) + \int_0^t e^{-rs}\widehat{H}(X_s^x, Y_s^y; z)ds,\ t\ge 0\right\} \text{ is an } (\mathcal{F}_t)\text{-martingale} \tag{4.27}$$

and that

$$\text{the family } \left\{e^{-r\tau}w(X_\tau^x, Y_\tau^y; z)\,,\, \tau\in\mathcal{T}\right\} \text{ is uniformly integrable.} \tag{4.28}$$

To simplify notation from now on we set

$$\hat{x} := \overline{x}_{\hat{y}(\cdot\,;z)}, \qquad \check{x} := \underline{x}_{\hat{y}(\cdot\,;z)}, \qquad \hat{\mathcal{D}}_z := \mathcal{D}_{\hat{y}(\cdot\,;z)} \tag{4.29}$$

and

$$x^* := \overline{x}_{y^*(\cdot\,;z)}, \qquad x_* := \underline{x}_{y^*(\cdot\,;z)}, \qquad \mathcal{D}_z^* := \mathcal{D}_{y^*(\cdot\,;z)}. \tag{4.30}$$

We can now state the main result of this section. We use arguments inspired by [35, Sec. 25] and references therein.

Theorem 4.10. *Let Assumptions 4.2, 4.5 and 4.7 hold. Assume that $\mathcal{C}_z \neq \emptyset$ and $\mathcal{A}_z \neq \emptyset$. Then $y^*(\,\cdot\,;z)$ is the unique nontrivial solution within the class $\mathcal{M}_z$ of the equation*

$$\begin{aligned}-y(x;z) = &\int_0^\infty e^{-rt}\left[\int_{\underline{x}}^{\overline{x}} p_1(t,x,\xi)c_z(\xi,z)\left(\int_{y(\xi;z)}^{\overline{y}} p_2(t,y(x;z),\eta)d\eta\right)d\xi\right]dt \\ &-\int_0^\infty e^{-rt}\left[\int_{\underline{x}}^{\overline{x}} p_1(t,x,\xi)\left(\int_{\underline{y}}^{y(\xi;z)}(r\eta-\mu_2(\eta))p_2(t,y(x;z),\eta)d\eta\right)d\xi\right]dt;\end{aligned} \tag{4.31}$$

that is, $y^*(\,\cdot\,;z)$ is the unique function $y(\,\cdot\,;z)\in\mathcal{M}_z$ with $\mathcal{D}_{y(\cdot;z)}\neq\emptyset$ and such that (4.31) *holds for each $x\in\mathcal{D}_{y(\cdot;z)}$.*

Proof. *Existence.* First of all we observe that $y^*(\,\cdot\,;z)\in\mathcal{M}_z$ by Propositions 4.4, 4.8 and (4.21). The fact that $y^*(\,\cdot\,;z)$ solves (4.31) for each $x\in\mathcal{D}_z^*$ follows by evaluating both sides of (4.6) at points of the boundary $(x, y^*(x;z))\in\partial\mathcal{A}_z$, which yields

$$\begin{aligned}-y^*(x;z) \;=\; &\int_0^\infty e^{-rt}\mathbb{E}\Big[c_z(X_t^x,z)\mathbb{1}_{\{Y_t^{y^*(x;z)}>y^*(X_t^x;z)\}}\Big]dt \\ &-\int_0^\infty e^{-rt}\mathbb{E}\Big[(rY_t^{y^*(x;z)}-\mu_2(Y_t^{y^*(x;z)}))\mathbb{1}_{\{Y_t^{y^*(x;z)}\le y^*(X_t^x;z)\}}\Big]dt.\end{aligned} \tag{4.32}$$

From (4.32) and by Assumption 3.13, we see that $y^*(\,\cdot\,;z)$ solves (4.31).

Uniqueness. Let $\hat{y}(\,\cdot\,;z) \in \mathcal{M}_z$ be a nontrivial solution of (4.31) and recall (4.29) and (4.30). We need to show that $\hat{y}(\,\cdot\,;z) \equiv y^*(\,\cdot\,;z)$.

Step 1. Here we show that $\hat{y}(\,\cdot\,;z) \geq y^*(\,\cdot\,;z)$.

Case (i): $\mathcal{D}^*_z \cap \hat{\mathcal{D}}_z \neq \emptyset$. Assume by contradiction that $\hat{y}(x;z) < y^*(x;z)$ for some $x \in \mathcal{D}^*_z \cap \hat{\mathcal{D}}_z$, take $y < \hat{y}(x;z)$ and set $\sigma = \sigma(x,y,z) := \inf\big\{t \geq 0 \,|\, Y^y_t \geq y^*(X^x_t;z)\big\}$.

Then from (3.41) and (4.27) it follows (up to localization arguments as in the proofs of Theorem 3.19 and Lemma A.1) that

$$\mathbb{E}\left[e^{-r\sigma} v(X^x_\sigma, Y^y_\sigma; z)\right] = v(x,y;z) + \mathbb{E}\left[\int_0^\sigma e^{-rt}\big(rY^y_t - \mu_2(Y^y_t)\big)\,dt\right], \tag{4.33}$$

$$\mathbb{E}\left[e^{-r\sigma} w(X^x_\sigma, Y^y_\sigma; z)\right] = w(x,y;z) - \mathbb{E}\left[\int_0^\sigma e^{-rt}\hat{H}(X^x_t, Y^y_t; z)\,dt\right]. \tag{4.34}$$

Lemma A.1 in Appendix A ensures that $v \geq w$ everywhere and that $w(x,y;z) = v(x,y;z) = -y$ since $y < \hat{y}(x;c) < y^*(x;z)$ (cf. (A-3)). Then subtracting (4.34) from (4.33) one has

$$\begin{aligned} 0 &\leq \mathbb{E}\left[\int_0^\sigma e^{-rt}\left[\big(rY^y_t - \mu_2(Y^y_t)\big) + \hat{H}(X^x_t, Y^y_t; z)\right]\,dt\right] \\ &= \mathbb{E}\left[\int_0^\sigma e^{-rt}\left[c_z(X^x_t, z) - \big(\mu_2(Y^y_t) - rY^y_t\big)\right]\mathbb{1}_{\{\hat{y}(X^x_t;z) < Y^y_t < y^*(X^x_t;z)\}}\,dt\right]. \end{aligned} \tag{4.35}$$

Notice that the continuity of trajectories of (X^x, Y^y) and the continuity of $y^*(\,\cdot\,;z)$ give $\sigma > 0$ $\mathbb{P}$-a.s. Moreover, from the continuity of $y^*(\,\cdot\,;z)$ and $\hat{y}(\,\cdot\,;z)$ one gets that the set $\big\{(x,y) \in Q \mid \hat{y}(x;z) < y < y^*(x;z)\big\}$ is open and not empty. These facts, combined with the fact that $y^*(\,\cdot\,;z) \leq \vartheta(\,\cdot\,;z)$ and with (4.22), imply that the last expression in (4.35) must be strictly negative and we reach a contradiction. Therefore $\hat{y}(\,\cdot\,;z) \geq y^*(\,\cdot\,;z)$ on $\mathcal{D}^*_z \cap \hat{\mathcal{D}}_z$. Since $\mathcal{D}^*_z \cap \hat{\mathcal{D}}_z = (\check{x} \vee x_*\,,\, \hat{x} \wedge x^*)$, this leads to $\check{x} \leq x_*$ and $\hat{x} \leq x^*$ by monotonicity and continuity of $\hat{y}(\,\cdot\,;z)$ and $y^*(\,\cdot\,;z)$, hence $\mathcal{D}^*_z \cap \hat{\mathcal{D}}_z = (x_*\,,\,\hat{x})$. Outside $\mathcal{D}^*_z \cap \hat{\mathcal{D}}_z$ we then have $y^*(x;z) = \underline{y} \leq \hat{y}(x;z)$ for $x \leq x_*$ and $\hat{y}(x;z) = \overline{y} \geq y^*(x;z)$ for $x \geq \hat{x}$ and the claim follows.

Case (ii): $\mathcal{D}^*_z \cap \hat{\mathcal{D}}_z = \emptyset$. By monotonicity of $y^*(\,\cdot\,;z)$ and $\hat{y}(\,\cdot\,;z)$ one has either $\hat{x} \leq x_*$ or $\check{x} \geq x^*$. If $\hat{x} \leq x_*$ then $\hat{y}(\,\cdot\,;z) \geq y^*(\,\cdot\,;z)$ on $\mathcal{I}_1$; if $\check{x} \geq x^*$ we can use the same arguments as above to find $\check{x} = \overline{x}$ which contradicts the assumption that $\hat{\mathcal{D}}_z \neq \emptyset$.

Step 2. Here we show that $\hat{y}(\,\cdot\,;z) \leq y^*(\,\cdot\,;z)$. Assume, by contradiction, that there exists $x \in \mathcal{I}_1$ such that $\hat{y}(x;z) > y^*(x;z)$. Take $y \in (y^*(x;z)\,,\,\hat{y}(x;z))$ and consider the stopping time $\tau^* = \tau^*(x,y;z) := \inf\{t \geq 0 \mid Y^y_t \leq y^*(X^x_t;z)\}$. This is the first optimal stopping time for the problem (3.2), as it is the first entry time in the stopping region $\mathcal{A}_z$ (cf. (3.46) and (4.2)). As in *Step* 1 above, (3.41) and (4.27) give

$$\mathbb{E}\left[e^{-r\tau^*} v(X^x_{\tau^*}, Y^y_{\tau^*}; z)\right] = v(x,y;z) - \mathbb{E}\left[\int_0^{\tau^*} e^{-rt} c_z(X^x_t, z)\,dt\right], \tag{4.36}$$

$$\mathbb{E}\left[e^{-r\tau^*} w(X^x_{\tau^*}, Y^y_{\tau^*}; z)\right] = w(x,y;z) - \mathbb{E}\left[\int_0^{\tau^*} e^{-rt}\widehat{H}(X^x_t, Y^y_t; z)\,dt\right]. \tag{4.37}$$

By using (3.43) and a localization argument as in the proof of Theorem 3.19, we obtain $\mathbb{E}\big[e^{-r\tau^*}v(X^x_{\tau^*}, Y^y_{\tau^*}; z)\big] = -\mathbb{E}\big[e^{-r\tau^*}Y^y_{\tau^*}\big]$. On the other hand, we know from *Step* 1 above that $\hat{y}(\,\cdot\,; z) \geq y^*(\,\cdot\,; z)$, hence $\mathbb{E}\big[e^{-r\tau^*}w(X^x_{\tau^*}, Y^y_{\tau^*}; z)\big] = -\mathbb{E}\big[e^{-r\tau^*}Y^y_{\tau^*}\big]$ by (A-1), (A-3), the fact that $\overline{y}$ is a natural boundary point and by localization arguments as in the proof of Lemma A.1. Taking also into account that $v \geq w$ (cf. Lemma A.1) and subtracting (4.37) from (4.36) we obtain

$$\begin{aligned} 0 &\geq \mathbb{E}\left[\int_0^{\tau^*} e^{-rt}\left(\widehat{H}(X^x_t, Y^y_t; z) - c_z(X^x_t, z)\right)dt\right] \\ &= -\mathbb{E}\left[\int_0^{\tau^*} e^{-rt}\left(c_z(X^x_t, z) + (rY^y_t - \mu_2(Y^y_t))\right)\mathbb{1}_{\{y^*(X^x_t;z) < Y^y_t < \hat{y}(X^x_t;z)\}}\,dt\right]. \end{aligned} \tag{4.38}$$

Now $\tau^* > 0$ $\mathbb{P}$-a.s. by continuity of trajectories of (X^x, Y^y) and of $y^*(\,\cdot\,; z)$. Moreover the set $\big\{(x, y) \in Q \,|\, y^*(x; z) < y < \hat{y}(x; z)\big\}$ is open in Q and not empty, by continuity of $y^*(\,\cdot\,; z)$ and $\hat{y}(\,\cdot\,; z)$. Since by assumption $\hat{y}(\,\cdot\,; z) \leq \vartheta(\,\cdot\,; z)$, these facts together with (4.22) imply that the last term in (4.38) must be strictly positive thus leading to a contradiction. Hence $\hat{y}(\,\cdot\,; z) \leq y^*(\,\cdot\,; z)$. □

Remark 4.11. *Expressions similar to* (4.7) *and* (4.31) *for the value function of optimal stopping problems and their free-boundaries have already been proved in the context of numerous examples with one dimensional diffusions and finite time-horizon (cf. [35] for a survey). However, to the best of our knowledge, in the context of infinite time-horizon and genuine 2-dimensional diffusions* (4.7) *and* (4.31) *are a novelty in the literature on their own.*

Regarding the assumptions $\mathcal{C}_z \neq \emptyset$ and $\mathcal{A}_z \neq \emptyset$ in Theorem 4.10, we provide the following characterization.

Proposition 4.12. *1. The continuation set $\mathcal{C}_z$ is not empty if and only if the set*

$$L^+_z := \big\{(x, y) \in Q \,|\, c_z(x, z) - \mu_2(y) + ry > 0\big\} \tag{4.39}$$

is not empty.

2. The stopping set $\mathcal{A}_z$ is not empty if and only if

$$\lim_{x \uparrow \overline{x}} \mathbb{E}\left[\int_0^\infty e^{-rt} c_z(X^x_t, z)dt\right] < -\underline{y}. \tag{4.40}$$

Proof. For the first claim notice that $L^+_z \subset \mathcal{C}_z$ (cf. also (3.34)) so that $L^+_z \neq \emptyset \Rightarrow \mathcal{C}_z \neq \emptyset$. To prove the reverse implication it suffices to observe that, by using (3.7) into (3.1), if $L^+_z = \emptyset$ then any stopping rule would produce a payoff smaller or equal than the one of immediate stopping and therefore $\mathcal{C}_z = \emptyset$.

For the second claim we observe that

$$\mathcal{A}_z = \emptyset \iff \mathcal{C}_z = Q \iff \tau^* = +\infty \ \mathbb{P}-a.s.\ \forall (x, y) \in Q \iff v(x, y; z) > -y \quad \forall (x, y) \in Q.$$

Hence $\mathcal{A}_z = \emptyset$ if and only if

$$v(x,y;z) = \mathbb{E}\left[\int_0^\infty e^{-rt} c_z(X_t^x, z)dt\right] > -y \quad \forall (x,y) \in Q. \tag{4.41}$$

Then (4.40) implies that $\mathcal{A}_z \neq \emptyset$. Conversely, if $\mathcal{A}_z \neq \emptyset$, then there exists a point $(x,y) \in Q$ such that stopping at once is more profitable than (for instance) never stopping. For such a point

$$0 = y + v(x,y;z) \geq y + \mathbb{E}\left[\int_0^\infty e^{-rt} c_z(X_t^x, z)dt\right]. \tag{4.42}$$

Since $y > \underline{y}$ and $c_z(\,\cdot\,, z)$ is nonincreasing (cf. Assumption 2.2-(ii)), then (4.40) must hold. □

In principle Theorem 4.10 fully characterizes the optimal boundary of problem (3.2), but it has the drawback that the region $\mathcal{D}_z^* = (x_*, x^*)$, with x_* and x^* as in (4.30), is defined implicitly. For the purpose of numerical evaluation of (4.31) it would be helpful to know $\mathcal{D}_z^*$ in advance rather than computing it at the same time as $y^*(\,\cdot\,; z)$. Recall (4.23) and define

$$\theta_* := \underline{x}_{\vartheta(\cdot;z)} = \inf\big\{x \in \mathcal{I}_1 \,|\, \vartheta(x;z) > \underline{y}\big\}, \quad \theta^* := \overline{x}_{\vartheta(\cdot;z)} = \sup\big\{x \in \mathcal{I}_1 \,\mid\, \vartheta(x;z) < \overline{y}\big\}, \tag{4.43}$$

with the convention $\inf \emptyset = \overline{x}$, $\sup \emptyset = \underline{x}$. Since $y^*(\,\cdot\,;z) \leq \vartheta(\,\cdot\,;z)$, we have $x_* \geq \theta_*$ and $x^* \geq \theta^*$. To characterize x_* we will make use of the following algebraic equation

$$-\underline{y} = \int_0^\infty e^{-rt}\Big(\int_{\underline{x}}^x p_1(t,x;\xi)c_z(\xi,z)d\xi - r\underline{y}\int_x^{\overline{x}} p_1(t,x,\xi)d\xi\Big)dt. \tag{4.44}$$

Similarly, if $\overline{y} < +\infty$, a characterization of x^* will be given in terms of the algebraic equation

$$-\overline{y} = \int_0^\infty e^{-rt}\Big(\int_{\underline{x}}^x p_1(t,x;\xi)c_z(\xi,z)d\xi - r\overline{y}\int_x^{\overline{x}} p_1(t,x,\xi)d\xi\Big)dt. \tag{4.45}$$

Proposition 4.13. *Let Assumptions 4.2, 4.5, 4.7 hold. Let $\mathcal{C}_z \neq \emptyset$ and $\mathcal{A}_z \neq \emptyset$. Then*

1. *$x_* \in \mathcal{I}_1$ if and only if (4.44) has a unique solution $\tilde{x} \in (\theta_*, \overline{x})$. Moreover $x_* = \tilde{x}$ and if such solution does not exist then $x_* = \underline{x}$.*

2. *If $\overline{y} < +\infty$, then $x^* \in \mathcal{I}_1$ if and only if (4.45) has a unique solution $\tilde{x}' \in (\theta^*, \overline{x})$. Moreover $x^* = \tilde{x}'$ and if such solution does not exist, then $x^* = \overline{x}$.*

3. *If $\overline{y} = +\infty$ and there exists $\lambda > 0$ such that $r - \frac{\partial \mu_2}{\partial y} \geq \lambda$ on $\mathcal{I}_2$, then $x^* = \overline{x}$.*

Proof. 1. Existence and uniqueness of a solution of (4.44) $(\theta_*, \overline{x})$ is discussed in Appendix A.2.

Proof of $\Rightarrow$. Take a sequence $\{x_n,\, n \in \mathbb{N}\} \subset \mathcal{I}_1$ such that $x_n \downarrow x_*$ and notice that by Theorem 4.10 we have for every $n \in \mathbb{N}$

$$\begin{aligned}-y^*(x_n;z) = &\int_0^\infty e^{-rt}\left[\int_{\underline{x}}^{\overline{x}} p_1(t,x_n,\xi)c_z(\xi,z)\left(\int_{y^*(\xi;z)}^{\overline{y}} p_2(t,y^*(x_n;z),\eta)d\eta\right)d\xi\right]dt \\ &- \int_0^\infty e^{-rt}\left[\int_{\underline{x}}^{\overline{x}} p_1(t,x_n,\xi)\left(\int_{\underline{y}}^{y^*(\xi;z)} (r\eta - \mu_2(\eta))p_2(t,y^*(x_n;z),\eta)d\eta\right)d\xi\right]dt.\end{aligned} \tag{4.46}$$

We aim to take limits of (4.46) as $n \uparrow \infty$. For the left hand-side of (4.46) we have $y^*(x_n; z) \downarrow \underline{y}$, by continuity of $y^*(\,\cdot\,; z)$ and definition of x_*. On the other hand, taking into account that $y^*(\,\cdot\,; z) = \underline{y}$ for $\xi \leq x_*$, the first term of the right-hand side of (4.46) can be written as

$$\begin{aligned}
&\int_0^\infty e^{-rt}\Big[\int_{\underline{x}}^{\overline{x}} p_1(t, x_n, \xi) c_z(\xi, z)\Big(\int_{y^*(\xi;z)}^{\overline{y}} p_2(t, y^*(x_n; z), \eta) d\eta\Big) d\xi\Big] dt \qquad (4.47)\\
&= \int_0^\infty e^{-rt}\Big[\int_{\underline{x}}^{x_*} p_1(t, x_n, \xi) c_z(\xi, z) d\xi \\
&\qquad + \int_{x_*}^{\overline{x}} p_1(t, x_n, \xi) c_z(\xi, z)\Big(\int_{\underline{y}}^{\overline{y}} \mathbb{1}_{\{\eta > y^*(\xi;z)\}} p_2(t, y^*(x_n; z), \eta) d\eta\Big) d\xi\Big] dt.
\end{aligned}$$

Now notice that:

(i) for any $t > 0$ the sequence of probability measures with densities $\{p_1(t, x_n, \xi),\, n \in \mathbb{N}\}$ on $\mathcal{I}_1$ converges pointwisely to $p_1(t, x_*, \xi) d\xi$ by Assumption 3.13;

(ii) for any given and fixed $t > 0$ and $z \in \mathbb{R}^+$ the sequence of probability measures with densities $\{p_2(t, y^*(x_n; z), \eta),\, n \in \mathbb{N}\}$ on $\mathcal{I}_2$ converges weakly to the Dirac's delta measure $\delta_{\underline{y}}(\eta)$, due to #8 of [31, Ch. II, Sec. 3] (see also (A-9) in Appendix A);

(iii) for every $\xi > x_*$, the function $\mathcal{I}_2 \to \mathbb{R}$, $\eta \mapsto c_z(\xi, z)\mathbb{1}_{\{\eta > y^*(\xi;z)\}} \equiv 0$ $\delta_{\underline{y}}$-a.e.

Then, taking into account *(i)-(iii)* we can apply Portmanteau Theorem to the integral with respect to $d\eta$ in the right hand side of (4.47) and dominated convergence to the one with respect to $d\xi$ to obtain

$$\lim_{n \to +\infty} \int_{\underline{x}}^{\overline{x}} p_1(t, x_n, \xi) c_z(\xi, z)\Big(\int_{y^*(\xi;z)}^{\overline{y}} p_2(t, y^*(x_n; z), \eta) d\eta\Big) d\xi = \int_{\underline{x}}^{x_*} p_1(t, x_*, \xi) c_z(\xi, z) d\xi$$

Finally, a further application of dominated convergence to the integral with respect to dt, gives

$$\begin{aligned}
&\lim_{n \to +\infty} \int_0^\infty e^{-rt}\left[\int_{\underline{x}}^{\overline{x}} p_1(t, x_n, \xi) c_z(\xi, z)\Big(\int_{y^*(\xi;z)}^{\overline{y}} p_2(t, y^*(x_n; z), \eta) d\eta\Big) d\xi\right] dt \\
&= \int_0^\infty e^{-rt}\left[\int_{\underline{x}}^{x_*} p_1(t, x_*, \xi) c_z(\xi, z) d\xi\right] dt.
\end{aligned}$$

Similar arguments can be applied to the second term of the right-hand side of (4.46). In fact for $\xi > x_*$ the map $\eta \mapsto (r\eta - \mu_2(\eta))\mathbb{1}_{\{\eta \leq y^*(\xi;z)\}}$ is bounded on $\overline{\mathcal{I}}_2$ and it is continuous at $\underline{y}$. Moreover $(r\eta - \mu_2(\eta))\mathbb{1}_{\{\eta \leq y^*(\xi;z)\}} = r\underline{y} - \mu_2(\underline{y})$, $\delta_{\underline{y}}$-a.e.

Proof of $\Leftarrow$. Assume now that $\theta_* < \overline{x}$ and that $\tilde{x} \in (\theta_*, \overline{x})$ uniquely solves (4.44). It is proven in Appendix A, Section A.2, that $\tilde{x}$ is the optimal boundary of the one-dimensional optimal stopping problem

$$\underline{v}(x; z) := \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-rt} c_z(X_t^x, z) dt - \underline{y} e^{-r\tau}\right], \qquad (4.48)$$

and hence that $\underline{\mathcal{A}}_z := \{x \in \mathcal{I}_1 \,|\, \underline{v}(x; z) = -\underline{y}\} = \{x \in \mathcal{I}_1 \,|\, x \geq \tilde{x}\}$. By arguments as in the proof of Proposition 3.7 we have $\underline{v}(x; z) = \lim_{y \downarrow \underline{y}} v(x, y; z)$. Moreover $0 < \underline{v}(x; z) + \underline{y} \leq v(x, y; z) + y$

for all $(x, y) \in (\underline{x}, \tilde{x}) \times \mathcal{I}_2$ by monotonicity of $y \mapsto v(x, y; z) + y$ (cf. Proposition 4.3), and hence $x_* \geq \tilde{x} > \underline{x}$. Also $x_* < \overline{x}$, since otherwise $\mathcal{A}_z = \emptyset$ thus contradicting the assumption $\mathcal{A}_z \neq \emptyset$. Therefore $x_* \in \mathcal{I}_1$ and hence by the arguments of the first part of this proof x_* solves (4.44). Since such solution is unique it must be $\tilde{x} = x_*$.

2. The proof of this second claim works thanks to arguments similar to the ones employed for the first one. One has to consider, in place of (4.48), the optimal stopping problem

$$\overline{v}(x; z) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[\int_0^\tau e^{-rt} c_z(X_t^x, z) dt - \overline{y} e^{-r\tau} \right].$$

3. The further assumption guarantees that $\vartheta(\,\cdot\,; z) < +\infty$ on $\mathcal{I}_1$ and the claim follows. □

Remark 4.14. *Despite their rather involved definition x_* and x^* have a quite clear probabilistic interpretation. In fact, they are the free-boundaries of the optimal stopping problems*

$$\underline{v}(x; z) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[\int_0^\tau e^{-rt} c_z(X_t^x, z) dt - \underline{y} e^{-r\tau} \right], \quad \overline{v}(x; z) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[\int_0^\tau e^{-rt} c_z(X_t^x, z) dt - \overline{y} e^{-r\tau} \right],$$

respectively, with $\underline{v}(\,\cdot\,; z) = \lim_{y \downarrow \underline{y}} v(\,\cdot\,, y; z)$ and $\overline{v}(\,\cdot\,; z) = \lim_{y \uparrow \overline{y}} v(\,\cdot\,, y; z)$.

5 The Optimal Control

In this section we characterize the optimal control ν^* of (2.13) by showing that it is optimal to exert the minimal effort needed to reflect the (optimally controlled) state process Z^{z,ν^*} at a (random) boundary intimately connected to y^* of Theorem 4.10.

5.1 The action/inaction regions

Define

$$\mathcal{C} := \{(x, y, z) \in \mathcal{O} \mid v(x, y; z) > -y\} \quad \text{and} \quad \mathcal{A} := \{(x, y, z) \in \mathcal{O} \mid v(x, y; z) = -y\}. \tag{5.1}$$

The sets $\mathcal{C}$ and $\mathcal{A}$ are respectively the candidate inaction region and the candidate action region for the control problem (2.13).

Remark 5.1. *We notice that the formal connection* (3.4) *yields*

$$\mathcal{C} = \{(x, y, z) \in \mathcal{O} \mid V_z(x, y, z) > -y\}, \quad \mathcal{A} = \{(x, y, z) \in \mathcal{O} \mid V_z(x, y, z) = -y\}. \tag{5.2}$$

Intuitively, $\mathcal{A}$ is the region in which it is optimal to invest immediately, whereas $\mathcal{C}$ is the region in which it is profitable to delay the investment option.

Throughout this section all the assumptions made so far will be standing assumptions, i.e. Assumptions 2.1, 2.2, 2.3, 3.3, 3.13, 4.2, 4.5 and 4.7 hold and we will not repeat them in the statement of the next results.

It immediately follows from the fact that $c_z(x, \cdot)$ is nondecreasing for each $x \in \mathcal{I}_1$ that

Proposition 5.2. *The function* $z \mapsto v(x, y; z)$ *is nondecreasing for every* $(x, y) \in Q$.

The nondecreasing property of $z \mapsto v(x, y; z)$ implies that for fixed $(x, y) \in Q$ the region $\mathcal{A}$ is below $\mathcal{C}$, and we define the boundary between these two regions by

$$z^*(x, y) := \inf\{z \in \mathbb{R}^+ \mid v(x, y; z) > -y\}, \tag{5.3}$$

with the convention $\inf \emptyset = \infty$. Then (5.1) can be equivalently written as

$$\mathcal{C} = \{(x, y, z) \in \mathcal{O} \mid z > z^*(x, y)\}, \quad \mathcal{A} = \{(x, y, z) \in \mathcal{O} \mid z \leq z^*(x, y)\}. \tag{5.4}$$

We can also easily observe from (4.1) and (5.3) and from the nondecreasing property of $z \mapsto v(x, y; z)$ and of $y \mapsto v(x, y; z) + y$ (cf. Proposition 5.2 and Proposition 4.3, respectively) that

$$z \ > \ z^*(x, y) \iff v(x, y; z) \ > \ -y \iff y \ > \ y^*(x; z), \quad (x, y, z) \in \mathcal{O}. \tag{5.5}$$

Hence for any $x \in \mathcal{I}_1$, z^* of (5.3) can be seen as the pseudo-inverse of the nonincreasing (cf. Proposition 4.4) function $z \mapsto y^*(x; z)$; that is,

$$z^*(x, y) = \inf\{z \in \mathbb{R}^+ \mid y > y^*(x; z)\}, \quad (x, y) \in Q. \tag{5.6}$$

It thus follows that the characterization of y^* of Theorem 4.10 is actually equivalent to a complete characterization of z^* thanks to (5.6).

Set

$$\overline{z}(x, y) := \inf\{z \in \mathbb{R}^+ \mid c_z(x, z) - \mu_2(y) + ry > 0\}, \quad (x, y) \in Q,$$

with the usual convention $\inf \emptyset = \infty$, and recall $\vartheta(x; z)$ of Lemma 4.9. Then the nondecreasing property of $z \mapsto c_z(x, z) - \mu_2(y) + ry$ and of $y \mapsto c_z(x, z) - \mu_2(y) + ry$ (cf. Assumption 2.2 and Assumption 4.2, respectively) implies that

$$z \ > \ \overline{z}(x, y) \iff c_z(x, z) - \mu_2(y) + ry > 0 \iff y \ > \ \vartheta(x; z), \quad (x, y, z) \in \mathcal{O},$$

and therefore that

$$\overline{z}(x, y) = \inf\{z \in \mathbb{R}^+ \mid y > \vartheta(x; z)\}. \tag{5.7}$$

Proposition 5.3. *One has*

1. $z^* \leq \overline{z}$ *over* Q.
2. $z^*(\,\cdot\,, y)$ *is nondecreasing for each* $y \in \mathcal{I}_2$ *and* $z^*(x, \,\cdot\,)$ *is nonincreasing for each* $x \in \mathcal{I}_1$.
3. $z^*(\,\cdot\,, y)$ *is right-continuous for each* $y \in \mathcal{I}_2$ *and* $z^*(x, \,\cdot\,)$ *is left-continuous for each* $x \in \mathcal{I}_1$.
4. $(x, y) \mapsto z^*(x, y)$ *is upper-semicontinuous.*

Proof. *1.* It follows by (5.6), (5.7) and (3.34).

2. The first claim follows from the fact that $v(\,\cdot\,, y; z)$ is nonincreasing for each $y \in \mathcal{I}_2$, $z \in \mathbb{R}^+$, by Proposition 3.6; the fact that $y \mapsto v(x, y; z) + y$ is nondecreasing for each $x \in \mathcal{I}_1$, $z \in \mathbb{R}^+$ (cf. proof of Proposition 4.3) implies the second one.

3. The proof of these two properties follows from the fact that $v(\cdot)$ is continuous by Proposition 3.7 and Remark 3.8, and from point 2 above by using arguments as those employed in [25, Prop. 2.2].

4. Notice that by (5.5) one has

$$\{(x, y) \in \mathcal{I}_1 \times \mathcal{I}_2 : z > z^*(x, y)\} = \{(x, y) \in \mathcal{I}_1 \times \mathcal{I}_2 : v(x, y; z) > -y\}, \tag{5.8}$$

for any $z \in \mathbb{R}^+$. The set on the right-hand side above is open since it is the preimage of an open set via the continuous mapping $(x, y) \mapsto v(x, y; z) + y$ (cf. Proposition 3.7). Hence the set on the left-hand side of (5.8) is open as well and thus $(x, y) \mapsto z^*(x, y)$ is upper-semicontinuous. □

Now Proposition 5.3 and the following

Assumption 5.4. $\lim_{z \uparrow \infty} c_z(x, z) = \infty$ *for every* $x \in \mathcal{I}_1$

imply

Proposition 5.5. *Under Assumption 5.4,* $\overline{z}$ *is finite on* Q*.*

Then, thanks to Proposition 5.3-(1) one also has

Corollary 5.6. z^* *is finite on* Q*.*

The topological characterization of the regions $\mathcal{C}$ and $\mathcal{A}$ is given in the following

Proposition 5.7. *$\mathcal{C}$ is open and $\mathcal{A}$ is closed. Moreover, under Assumption 5.4, they are connected.*

Proof. The fact that $\mathcal{C}$ is open and $\mathcal{A}$ is closed follows from (5.1) and Remark 3.8. Corollary 5.6 and (5.4) imply the second part of the claim. □

5.2 Optimal Control: a Verification Theorem

The results obtained in Section 3 on the optimal stopping problem (3.2) (especially the superharmonic characterization of Theorem 3.19) allow us to provide the expression of the optimal control ν^* of problem (2.13) in terms of the boundary z^* of (5.3). Moreover, as a byproduct, we will also show (see Corollary 5.10 below) that the connection (3.4) holds true with V as in (2.13) and v as in (3.2).

Recall (3.2) and define the functions

$$\Phi(x, z) := \mathbb{E}\bigg[\int_0^\infty e^{-rt} c(X_t^x, z) dt\bigg], \quad (x, z) \in \mathcal{I}_1 \times \mathbb{R}^+, \tag{5.9}$$

$$\varphi(x,z) := \frac{\partial}{\partial z}\Phi(x,z) = \mathbb{E}\bigg[\int_0^\infty e^{-rt} c_z(X_t^x, z)dt\bigg], \quad (x,z) \in \mathcal{I}_1 \times \mathbb{R}^+, \tag{5.10}$$

and

$$U(x,y,z) := \Phi(x,z) - \int_z^\infty (v(x,y;q) - \varphi(x,q))dq, \quad (x,y,z) \in \mathcal{O}. \tag{5.11}$$

Notice that $v(x,y;z) \geq \varphi(x,z)$ for every $(x,y,z) \in \mathcal{O}$, and therefore function U in (5.11) above is well-defined (but, a priori, it may be equal to $-\infty$).

Introduce the nondecreasing process

$$\nu_t^* := \sup_{0 \leq s \leq t} [z^*(X_s^x, Y_s^y) - z]^+, \quad t \geq 0, \qquad \nu_{0-}^* = 0, \tag{5.12}$$

with $z^*(x,y)$ as in (5.3).

Proposition 5.8. *Under Assumption 5.4 the process ν^* of* (5.12) *is an admissible control.*

Proof. Recall the set of admissible controls $\mathcal{V}$ of (2.3). Clearly ν^* is a.s. finite thanks to Corollary 5.6. To prove that $\nu^* \in \mathcal{V}$ it remains to show that: *i)* $t \mapsto \nu_t^*$ is right-continuous with left-limits; *ii)* ν^* is $(\mathcal{F}_t)$-adapted.

We start by proving *i)*. Clearly, $t \mapsto \nu_t^*$ admits left-limit at any point since it is nondecreasing. To show that ν^* has right-continuous sample paths, first notice that

$$\limsup_{s \downarrow t} z^*(X_s^x, Y_s^y) \leq z^*(X_t^x, Y_t^y) \tag{5.13}$$

by upper-semicontinuity of z^* (cf. Proposition 5.3) and continuity of $(X_\cdot^x, Y_\cdot^y)$. Moreover, from (5.12) and (5.13) we obtain

$$\begin{aligned}\lim_{s \downarrow t} \nu_s^* &= \nu_t^* \vee \lim_{s \downarrow t} \sup_{t < u \leq s} [z^*(X_u^x, Y_u^y) - z]^+ \\ &= \nu_t^* \vee \limsup_{s \downarrow t} [z^*(X_s^x, Y_s^y) - z]^+ \leq \nu_t^* \vee [z^*(X_t^x, Y_t^y) - z]^+ = \nu_t^*. \end{aligned} \tag{5.14}$$

Since $\lim_{s \downarrow t} \nu_s^* \geq \nu_t^*$ by monotonicity of $t \mapsto \nu_t^*$, then (5.14) implies right continuity.

As for *ii)* the process $z^*(X^x, Y^y)$ is progressively measurable since it is the composition of the the Borel-measurable function z^* (which is upper semicontinuous by Proposition 5.3) with the progressively measurable process (X^x, Y^y). Therefore ν^* is progressively measurable by [14, Th. IV.33, part (a)], hence adapted and *ii)* above holds. □

Theorem 5.9. *Let Assumption 5.4 hold. Fix $(x,y,z) \in \mathcal{O}$ and take $\Phi(x,z)$, $\varphi(x,z)$ and $U(x,z)$ as in* (5.9), (5.10) *and* (5.11), *respectively. Then one has $U(x,y,z) = V(x,y,z)$ and ν^* as in* (5.12) *is optimal for the singular control problem* (2.13).

It clearly follows from Theorem 5.9 the following

Corollary 5.10. *The identity* (3.4) *holds true.*

The proof of Theorem 5.9 is inspired by the arguments developed in [1] and [16]. It is based on a probabilistic verification argument relying on the superharmonic characterization of v described in Theorem 3.19.

Proof of Theorem 5.9. For $\nu \in \mathcal{V}$ define its right-continuous inverse (cf. [38, Ch. 0, Sec. 4])

$$\tau^\nu(\xi) := \inf\{t \geq 0 \mid \nu_t > \xi\}, \qquad \xi \geq 0. \tag{5.15}$$

The process $\tau^\nu := \{\tau^\nu(\xi),\ \xi \geq 0\}$ has increasing, right-continuous sample paths and hence it admits left-limits

$$\tau^\nu_-(\xi) := \inf\{t \geq 0 \mid \nu_t \geq \xi\}, \qquad \xi \geq 0. \tag{5.16}$$

The set of points $\xi \in \mathbb{R}^+$ at which $\tau^\nu(\xi)(\omega) \neq \tau^\nu_-(\xi)(\omega)$ is a.s. countable for a.e. $\omega \in \Omega$.

Since ν is right-continuous and $\tau^\nu(\xi)$ is the first entry time of an open set, it is an $(\mathcal{F}_{t+})$-stopping time for any given and fixed $\xi \geq 0$. However $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous (cf. Section 2) and hence $\tau^\nu(\xi)$ is an $(\mathcal{F}_t)$-stopping time. Moreover, $\tau^\nu_-(\xi)$ is the first entry time of the right-continuous process ν into a closed set and hence it is an $(\mathcal{F}_t)$-stopping time as well for any $\xi \geq 0$. It then follows by (3.45) that

$$v(x,y;q) \geq \mathbb{E}\bigg[e^{-r\tau^\nu(\xi)}v(X^x_{\tau^\nu(\xi)}, Y^y_{\tau^\nu(\xi)}; q) + \int_0^{\tau^\nu(\xi)} e^{-rs}c_z(X^x_s, q)ds\bigg], \tag{5.17}$$

for any $\xi \geq 0$ and $(x,y,q) \in \mathcal{O}$. Then for any $(x,y,z) \in \mathcal{O}$, taking $\xi = q - z$, $q \geq z$ in (5.17) and recalling (5.9), (5.10) and (5.11) we obtain

$$\begin{aligned}
U(x,y,z) - \Phi(x,z) \leq & -\int_z^\infty \bigg(\mathbb{E}\bigg[e^{-r\tau^\nu(q-z)}v(X^x_{\tau^\nu(q-z)}, Y^y_{\tau^\nu(q-z)}; q) + \\
& + \int_0^{\tau^\nu(q-z)} e^{-rs}c_z(X^x_s, q)ds\bigg]\bigg)dq + \int_z^\infty \mathbb{E}\bigg[\int_0^\infty e^{-rs}c_z(X^x_s, q)ds\bigg]dq \\
\leq & \int_z^\infty \mathbb{E}\bigg[e^{-r\tau^\nu(q-z)}Y^y_{\tau^\nu(q-z)}\bigg]dq - \int_z^\infty \mathbb{E}\bigg[\int_0^{\tau^\nu(q-z)} e^{-rs}c_z(X^x_s, q)ds\bigg]dq \\
& + \int_z^\infty \mathbb{E}\bigg[\int_0^\infty e^{-rs}c_z(X^x_s, q)ds\bigg]dq,
\end{aligned} \tag{5.18}$$

where we have used that $v(\,\cdot\,,\zeta;\,\cdot\,) \geq -\zeta$ (cf. Proposition 3.6) in the second inequality. We now claim (and we will prove it later) that we can apply Fubini-Tonelli's Theorem in the last expression of (5.18) to obtain

$$\begin{aligned}
U(x,y,z) - \Phi(x,z) \leq & \mathbb{E}\bigg[\int_z^\infty e^{-r\tau^\nu(q-z)}Y^y_{\tau^\nu(q-z)}dq - \int_z^\infty \bigg(\int_0^{\tau^\nu(q-z)} e^{-rs}c_z(X^x_s, q)ds\bigg)dq\bigg] \\
& + \mathbb{E}\bigg[\int_z^\infty \bigg(\int_0^\infty e^{-rs}c_z(X^x_s, q)ds\bigg)dq\bigg].
\end{aligned} \tag{5.19}$$

The change of variable formula of [38, Ch. 0, Prop. 4.9] (see also [1, eq. (4.7)]) implies

$$\int_z^\infty e^{-r\tau^\nu(q-z)}Y^y_{\tau^\nu(q-z)}dq = \int_0^\infty e^{-rs}Y^y_s d\nu_s. \tag{5.20}$$

Moreover $\tau^\nu(q-z) < s$ if and only if $\nu_s > q-z$, $s \geq 0$ and therefore from (5.19) and (5.20) we obtain

$$\begin{aligned} U(x,y,z) - \Phi(x,z) &\leq \mathbb{E}\bigg[\int_0^\infty e^{-rs} Y^y_s d\nu_s + \int_z^\infty \Big(\int_{\tau^\nu(q-z)}^\infty e^{-rs} c_z(X^x_s,q) ds\Big) dq\bigg] \\ &= \mathbb{E}\bigg[\int_0^\infty e^{-rs} Y^y_s d\nu_s + \int_z^\infty \Big(\int_0^\infty e^{-rs} c_z(X^x_s,q) \mathbf{1}_{\{\nu_s > q-z\}} ds\Big) dq\bigg] \\ &= \mathbb{E}\bigg[\int_0^\infty e^{-rs} Y^y_s d\nu_s + \int_0^\infty e^{-rs} \Big(\int_z^{z+\nu_s} c_z(X^x_s,q) dq\Big) ds\bigg] \\ &= \mathbb{E}\bigg[\int_0^\infty e^{-rs} Y^y_s d\nu_s + \int_0^\infty e^{-rs} \Big[c(X^x_s, Z^{z,\nu}_s) - c(X^x_s, z)\Big] ds\bigg] \\ &= \mathcal{J}_{x,y,z}(\nu) - \Phi(x,z). \end{aligned} \tag{5.21}$$

Since $\nu \in \mathcal{V}$ is arbitrary it follows

$$U(x,y,z) \leq V(x,y,z). \tag{5.22}$$

Now we want to show that picking ν^* as in (5.12) in the arguments above all the inequalities become equalities due to (3.46). First notice that (3.46), (5.1) and (5.4) give

$$\tau^*(x,y;q) = \inf\{t \geq 0 \mid z^*(X^x_t, Y^y_t) \geq q\}. \tag{5.23}$$

Then fix $z \in \mathbb{R}^+$, take $t \geq 0$ arbitrary and note that by (5.16) and (5.23) we have $\mathbb{P}$-a.s. the equivalences

$$\begin{aligned} &\tau^{\nu^*}_-(q-z) \leq t \iff \nu^*_t \geq q-z \iff \sup_{0\leq s\leq t}[z^*(X^x_s, Y^y_s) - z]^+ \geq q-z \\ &\iff z^*(X^x_\theta, Y^y_\theta) \geq q \;\text{ for some } \theta \in [0,t] \iff \tau^*(x,y;q) \leq t. \end{aligned}$$

So we can conclude that $\tau^{\nu^*}_-(q-z) = \tau^*(x,y;q)$ $\mathbb{P}$-a.s. and for a.e. $q \geq z$. However, by (5.15) and (5.16) we also have $\tau^{\nu^*}_-(q-z) = \tau^{\nu^*}(q-z)$ $\mathbb{P}$-a.s. and for a.e. $q \geq z$; hence

$$\tau^{\nu^*}(q-z) = \tau^*(x,y;q) \;\mathbb{P}\text{-a.s. and for a.e. } q \geq z. \tag{5.24}$$

Now take $\nu = \nu^*$ and $\xi = q - z$ in order to obtain equality in (5.17) by Theorem 3.19 and (5.24). Optimality of $\tau^* = \tau^{\nu^*}$ (cf. (5.24)) also gives equality in (5.18); then we can interchange the integrals and argue as in (5.19) and (5.21) to obtain $U(x,y,z) = \mathcal{J}_{x,y,z}(\nu^*)$. Then $U = V$ on $\mathcal{O}$ by (5.22) and ν^* is optimal.

To conclude the proof we need to show that we could actually interchange the order of integration in (5.18) to get (5.19). Clearly

$$\int_z^\infty \mathbb{E}\Big[e^{-r\tau^\nu(q-z)} Y^y_{\tau^\nu(q-z)}\Big] dq = \mathbb{E}\bigg[\int_z^\infty e^{-r\tau^\nu(q-z)} Y^y_{\tau^\nu(q-z)} dq\bigg],$$

by Tonelli's Theorem since Y^y has positive sample paths. Therefore we have only to show that

$$\mathbb{E}\bigg[\int_z^\infty \Big(\int_{\tau^\nu(q-z)}^\infty e^{-rs} |c_z(X^x_s,q)| ds\Big) dq\bigg] < \infty. \tag{5.25}$$

Define

$$q_s^* := \inf\{q \in \mathbb{R} : c_z(X_s^x, q) > 0\},$$

which exists unique since $c(x,\cdot)$ is convex. Now, recall that $\tau^\nu(q-z) < s$ if and only if $\nu_s > q-z$, $s \geq 0$; then Tonelli's Theorem, Remark 2.7 and the fact that $c \geq 0$ give

$$\begin{aligned}
&\mathbb{E}\Big[\int_z^\infty \Big(\int_{\tau^\nu(q-z)}^\infty e^{-rs}|c_z(X_s^x,q)|ds\Big)dq\Big] = \mathbb{E}\Big[\int_z^\infty \Big(\int_0^\infty e^{-rs}|c_z(X_s^x,q)|\mathbb{1}_{\{\tau^\nu(q-z)<s\}}ds\Big)dq\Big]\\
&= \mathbb{E}\Big[\int_0^\infty e^{-rs}\Big(\int_z^{z+\nu_s}|c_z(X_s^x,q)|dq\Big)ds\Big] = \mathbb{E}\Big[\int_0^\infty e^{-rs}\Big(\int_{(z+\nu_s)\wedge q_s^*}^{z+\nu_s} c_z(X_s^x,q)dq\Big)ds\Big]\\
&\qquad -\mathbb{E}\Big[\int_0^\infty e^{-rs}\Big(\int_z^{(z+\nu_s)\wedge q_s^*} c_z(X_s^x,q)dq\Big)ds\Big]\\
&\leq \mathbb{E}\Big[\int_0^\infty e^{-rs}c(X_s^x,z)ds + \int_0^\infty e^{-rs}c(X_s^x,z+\nu_s)ds\Big] < \infty.
\end{aligned}$$

□

A Appendix

Lemma A.1. *Let Assumptions 4.2, 4.5, 4.7 hold and assume that $\mathcal{C}_z \neq \emptyset$ and $\mathcal{A}_z \neq \emptyset$. Let $\hat{y}(\,\cdot\,;z) : \mathcal{I}_1 \to \overline{\mathcal{I}}_2$ be a nontrivial solution of* (4.31) *and take w as in* (4.25). *Then $v(\,\cdot\,;z) \geq w(\,\cdot\,;z)$ on Q.*

Proof. Recall the notation introduced in (4.29).

Step 1. Since $\hat{y}(\,\cdot\,;z)$ is a nontrivial solution of (4.31), i.e. of (4.32), it is easy to see that w of (4.25) verifies

$$w(x,\hat{y}(x;z);z) = -\hat{y}(x;z), \qquad \forall x \in \hat{\mathcal{D}}_z, \tag{A-1}$$

and therefore

$$w(x,\hat{y}(x;z);z) \leq v(x,\hat{y}(x;z);z), \qquad \forall x \in \hat{\mathcal{D}}_z, \tag{A-2}$$

Step 2. Here we show that

$$w(x,y;z) = -y, \quad \forall y < \hat{y}(x;z),\ \forall x \in \hat{\mathcal{D}}_z \cup [\hat{x},\overline{x}). \tag{A-3}$$

which implies

$$w(x,y;z) \leq v(x,y;z), \quad \forall y < \hat{y}(x;z),\ x \in \hat{\mathcal{D}}_z \cup [\hat{x},\overline{x}).$$

Take $x \in \hat{\mathcal{D}}_z \cup [\hat{x},\overline{x})$, $y < \hat{y}(x;z)$ and define $\sigma = \sigma(x,y;z) := \inf\big\{t \geq 0 \,|\, Y_t^y \geq \hat{y}(X_t^x;z)\big\}$. By definition of $\hat{y}(\,\cdot\,;z)$ and σ we have

$$\widehat{H}(X_t^x,Y_t^y;z) = -\big(rY_t^y - \mu_2(Y_t^y)\big), \quad \forall t \leq \sigma,\ \mathbb{P}\text{-a.s.} \tag{A-4}$$

Then using the martingale property (4.27) up to the stopping time $\sigma \wedge n$, $n \in \mathbb{N}$, it follows by (A-4) that

$$w(x,y;z) = \mathbb{E}\Big[- e^{-r\sigma} Y^y_\sigma \mathbb{1}_{\{\sigma \le n\}} + e^{-rn} w(X^x_n, Y^y_n; z)\mathbb{1}_{\{\sigma > n\}} - \int_0^{\sigma\wedge n} e^{-rt}\big(rY^y_t - \mu_2(Y^y_t)\big)dt\Big]. \tag{A-5}$$

Assumption 2.3, (4.24) and the bound (4.26), give in the limit as $n \to \infty$

$$w(x,y;z) = \mathbb{E}\Big[- e^{-r\sigma} Y^y_\sigma - \int_0^{\sigma} e^{-rt}\big(rY^y_t - \mu_2(Y^y_t)\big)dt\Big] = y, \tag{A-6}$$

where the last equality follows by Lemma 3.5. Hence (A-3) is proved.

Step 3. Here we prove that

$$w(x,y;z) \le v(x,y;z), \qquad \forall y > \hat{y}(x;z),\ \forall x \in (\underline{x}, \check{x}] \cup \hat{\mathcal{D}}_z. \tag{A-7}$$

Take $x \in (\underline{x}, \check{x}] \cup \hat{\mathcal{D}}_z$ and $y > \hat{y}(x;z)$ and consider the stopping time

$$\tau = \tau(x,y;z) := \inf\big\{t \ge 0 \mid Y^y_t \le \hat{y}(X^x_t; z)\big\}.$$

By definitions of $\hat{y}(\,\cdot\,;z)$ and τ and by using the same localization argument as in *Step 2* above, we obtain

$$w(x,y;z) = \mathbb{E}\left[-e^{-r\tau} Y^y_\tau + \int_0^{\tau} e^{-rs} c_z(X^x_t, z)dt\right] \le v(x,y;z). \tag{A-8}$$

Step 4. Now Lemma A.1 follows by (A-1), (A-3) and (A-7). □

A.1 Further properties of natural boundaries

Here we show that $\mu_2(\underline{y}) = \sigma_2(\underline{y}) = 0$. The same holds for $\overline{y}$ if it is finite. Analogously, μ_1 and σ_1 are zero at $\underline{x}$ and $\overline{x}$ whenever the latter are finite. For the proof we rely on #8 of [31, Ch. II, Sec. 3, p. 32] that guarantees

$$\lim_{y \downarrow \underline{y}} \mathbb{E}\Big[\int_0^\infty e^{-rt} f(Y^y_t)dt\Big] = \frac{1}{r} f(\underline{y}) \qquad \text{for any } f \in C_b(\mathbb{R}); \tag{A-9}$$

that is, the family of probability measures on $\mathcal{I}_2$ with densities $\{p_2(t,y,\cdot), y \in \mathcal{I}_2\}$, $t > 0$, (cf. Assumption 3.13) converges weakly to the Dirac's delta measure $\delta_{\underline{y}}(\cdot)$, for any $t > 0$, when $y \downarrow \underline{y}$.

Case 1. If $\mathcal{I}_2$ is bounded, an application of Dynkin's formula to any $g \in C^2_b(\mathbb{R})$ leads to

$$g(y) = -\mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(\frac{1}{2}\sigma_2^2(Y^y_t)g''(Y^y_t) + \mu_2(Y^y_t)g'(Y^y_t) - rg(Y^y_t)\Big)dt\Big]. \tag{A-10}$$

Then taking limits as $y \downarrow \underline{y}$, noting that μ_2 and σ_2 are bounded and continuous and by applying (A-9) we get

$$\frac{1}{2}\sigma_2^2(\underline{y})g''(\underline{y}) + \mu_2(\underline{y})g'(\underline{y}) = 0, \tag{A-11}$$

and since g is arbitrary it must be $\mu_2(\underline{y}) = \sigma_2(\underline{y}) = 0$.

Case 2. If $\mathcal{I}_2$ is unbounded (i.e. if $\mathcal{I}_2 = (\underline{y}, \infty)$) we approximate (μ_2, σ_2) by continuous bounded functions (μ_2^n, σ_2^n) such that $\mu_2^n = \mu_2$ and $\sigma_2^n = \sigma_2$ on $[\underline{y}, n \vee \underline{y}]$ with $\mu_2^n(y) \to \mu_2(y)$ and $\sigma_2^n(y) \to \sigma_2(y)$ as $n \to \infty$ pointwise on $\mathcal{I}_2$. For $y \in (\underline{y}, n \vee \underline{y})$ the associated diffusion with coefficients μ_2^n and σ_2^n, denoted by $Y^{y,n}$, coincides with Y^y up to the first exit time from $(\underline{y}, n \vee \underline{y})$ by uniqueness of the solution of (2.2); moreover, $\underline{y}$ is a natural boundary for $Y^{y,n}$ as well. Repeating arguments as in *Case 1* above we get $\mu_2^n(\underline{y}) = \sigma_2^n(\underline{y}) = 0$ for all $n \in \mathbb{N}$, thus $\mu_2(\underline{y}) = \sigma_2(\underline{y}) = 0$.

A.2 Discussion on Problem (4.48)

Problem (4.48) is standard in the optimal stopping literature (cf. for instance [35] for methods of solution) and hence we only sketch arguments leading to its main properties. It is easy to see that $x \mapsto \underline{v}(x; z)$ is nonincreasing and hence there exists $b_* \in \overline{\mathcal{I}}_1$ such that $\underline{\mathcal{A}}_z = [b_*, \overline{x})$, where the boundary value $\overline{x}$ cannot be included as otherwise $\mathcal{A}_z = \emptyset$ thus contradicting the assumption of Proposition 4.13. It is possible to show that $\underline{v}(\,\cdot\,; z) \in C^1(\mathcal{I}_1)$, $\underline{v}_{xx}(\,\cdot\,; z)$ is locally bounded at b_* and hence that the probabilistic representation

$$\underline{v}(x; z) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(c_z(X_t^x; z)\mathbb{1}_{\{X_t^x < b_*\}} - r\underline{y}\mathbb{1}_{\{X_t^x \geq b_*\}}\Big)dt\Big] \tag{A-12}$$

holds by Itô-Tanaka formula. Since (A-12) holds for any $x \in \mathcal{I}_1$, then if $b_* \in \mathcal{I}_1$ by evaluating (A-12) for $x = b_*$, one easily finds that b_* solves (4.44). Arguments similar to (but simpler than) those employed in the proof of Theorem 4.10 show that (4.44) admits a unique solution in $(\theta_*, \overline{x})$ and therefore it must be $\tilde{x} = b_*$. On the other hand, if $b_* = \underline{x}$, repeating arguments as those of the proof of Theorem 4.10, *Step 2*, one can show that $\tilde{x} = b_*$, thus concluding.

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