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Microeconomic foundations of representative agent models by means of ultraproducts *

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Abstract

This paper applies microeconomic foundations to construct a "representative agent". Herzberg [12] constructed a representative utility function for finite-dimensional social decision problems and since the decision problems of macroeconomic theory are typically infinite-dimensional, Herzberg's original result is insufficient for many applications. We therefore generalise his result by allowing the social alternatives to belong to a general reflexive Banach space and provide sufficient conditions for our new results to be satisfied in economic applications.

Key words: Microfoundation; representative agent; social choice; reflexive Banach space; convex optimisation; ultrafilter; bounded ultrapower; nonstandard analysis.
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1 Introduction

Modern macroeconomic theory looks for microeconomic foundations, namely consumers and firms, and it considers how these microeconomic entities in an economy make their decisions and then how these many individuals' choices give rise to economy-wide "macroeconomic" outcomes (see Gillman [11] and Mankiw [16]).

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Since in realistic models, consumers and firms are heterogeneous, an accurate and comprehensive mathematical description of the aggregate behaviour is typically an intractable problem. One solution is to assume the existence of a "representative agent" in a loosely defined sense.

Indeed, in mathematical models of macroeconomic theory, the assumption of a "representative agent" is ubiquitous (see [13]), but the search for a rigorous justification has so far been unsuccessful and was ultimately abandoned until very recently. Herzberg [12] constructed a representative utility function for finite-dimensional social decision problems, based on an bounded ultrapower construction over the reals, with respect to the ultrafilter induced by the underlying social choice function (via the Kirman–Sondermann [14] correspondence). However, since the decision problems of macroeconomic theory are typically infinite-dimensional, Herzberg's original result is insufficient for many applications. We therefore generalise his result by allowing the social alternatives to belong to a general reflexive Banach space W.

In this paper, we suppose that individuals have "cardinal utilities", i.e. for every individual, there is an utility function which induces his (or her) preference ordering. Furthermore, the aggregation of individual preferences is assumed to result from a social welfare function satisfying all Arrovian rationality axioms (Unanimity, Independence and No Dictatorship).

Arrow [3] has shown that if the set of individuals is finite and the set of alternatives is at least three, then Arrow's axioms are inconsistent (Arrow's Impossibility Theorem). Fishburn [9] has shown that if the set of individuals is infinite and the set of alternatives is at least three, then Arrow's axioms are consistent (Fishburn's Possibility Theorem). Kirman and Sondermann [14] [Theorem 1; Proposition 2] have shown that the collection of decisive coalitions generated by a social welfare function satisfying all Arrovian rationality axioms, is always a non-principal ultrafilter. This is only possible if the set of individuals is infinite.

We also suppose that the set of alternatives is a compact non-empty convex subset of a given reflexive Banach space W (Herzberg [12] has supposed that the set of alternatives is a finite-dimensional vector space). Furthermore, we assume that the set of admissible utility functions are parametrised and the parameter set is a compact subset of a given Banach space X. For our results, the set of parametrised admissible utility functions contains only continuous and strictly concave functions.

Using a nonstandard enlargement of the superstructure over $(X \oplus W) \cup \mathbb{R}$, obtained by a bounded ultrapower construction with respect to the non-principal ultrafilter, we prove that there exists for every utility profile, some $\mathcal{D}$-socially acceptable utility function (Theorem 1) and then we use this theorem for existence of a representative utility function (Theorem 2). These results depend on certain regularity features of the so-called admissible utility functions (Axiom (A5)). We provide sufficient conditions

(Corollary 2 and Corollary 3) for these regularities to be satisfied in diverse economic applications.

2 The Model and Formulation

We are concerned with a social decision problem, therefore we need a model for introducing population, alternatives and utility functions. We use the following model:

Let N be a set of **individuals**[1] and $\mathcal{C}$ a set of **alternatives**. We fix some subset $\mathcal{D}$ of the power-set of N. We call the elements of $\mathcal{D}$, **potentially decisive coalitions**. Also for generalising Herzberg's [12] results, we will have to assume that $\mathcal{C}$ is a compact non-empty convex subset of a given reflexive Banach space W (with norm $\| \cdot \|_W$).

We fix some class $\mathcal{M}$ of functions from $\mathcal{C}$ to $\mathbb{R}$. The elements of $\mathcal{M}$ are called **admissible utility functions**. Every individual's utility function, u_i, belongs to $\mathcal{M}$. Elements of $\mathcal{M}^N$ will be called **utility profiles** and $\underline{u} = \langle u_i \rangle_{i \in N} \in \mathcal{M}^N$.

We now employ some social choice theory notations and formulations from Kirman and Sondermann [14].

Definition 1 *A relation $P \subseteq \mathcal{C} \times \mathcal{C}$ is called a **weak order** if and only if P is "asymmetric" and "negatively transitive". $\mathcal{P}$ denotes the set of all weak orders on $\mathcal{C}$. For all $x, y \in \mathcal{C}$ and $\underline{P} = \langle P_i \rangle_{i \in N} \in \mathcal{P}^N$, we define*

$$C(x, y, \underline{P}) := \{i \in N \,:\, xP_iy\}.$$

Definition 2 *For $x \in \mathcal{C}$ and $P \in \mathcal{P}$, x will be called **P-maximal** if and only if for all $y \in \mathcal{C} \backslash \{x\}$, we have xPy.*

Definition 3 *For $u : \mathcal{C} \longrightarrow \mathbb{R}$ and $P \in \mathcal{P}$, we say that u is a **utility representation** of P if and only if for all $x, y \in \mathcal{C}$,*

$$u(x) > u(y) \Leftrightarrow xPy.$$

Remark 1 *We denote $P^u \in \mathcal{P}$ as a notation that utility function u, induces the preference P. Similarly, given an N-sequence $\underline{u} = \langle u_i \rangle_{i \in N}$ of functions from $\mathcal{C}$ to $\mathbb{R}$, we define*

$$\underline{P}^{\underline{u}} := \langle P^{u_i} \rangle_{i \in N} \in \mathcal{P}^N.$$

We say that the utility profile $\underline{u}$ induces the preference profile $\underline{P}^{\underline{u}}$.

[1] Subsets of N are called coalitions.

Definition 4 *A* ***social welfare function*** *is a map* $\sigma : \mathcal{P}^N \longrightarrow \mathcal{P}$.

According to our notations, Arrow's rationality axioms for σ will be formulated as follows:

(A1) (Unanimity Preservation): For all $x, y \in \mathcal{C}$ and $\underline{P} \in \mathcal{P}^N$, if $C(x, y, \underline{P}) = N$ then $x\sigma(\underline{P})y$.

(A2) (Independence of Irrelevant Alternatives): For all $x, y \in \mathcal{C}$ and $\underline{P}, \underline{P}' \in \mathcal{P}^N$, if $C(x, y, \underline{P}) = C(x, y, \underline{P}')$ and $C(y, x, \underline{P}) = C(y, x, \underline{P}')$, then

$$x\sigma(\underline{P})y \Leftrightarrow x\sigma(\underline{P}')y, \qquad y\sigma(\underline{P})x \Leftrightarrow y\sigma(\underline{P}')x.$$

(A3) (No Dictatorship): There is no $i_0 \in N$ such that for all $x, y \in \mathcal{C}$ and $\underline{P} \in \mathcal{P}^N$,

$$xP_{i_0}y \Rightarrow x\sigma(\underline{P})y.$$

Definition 5 *We say that a coalition $C \subseteq N$ is* ***σ-decisive*** *if and only if for all $x, y \in \mathcal{C}$ and $\underline{P} \in \mathcal{P}^N$ one has $x\sigma(\underline{P})y$ whenever xP_iy for all $i \in C$ and yP_jx for all $j \in N\backslash C$. The set of σ-decisive coalitions is denoted by $\mathcal{D}_\sigma$.*

For the following, recall that a *filter* on N is a set $\mathcal{D}$ of subsets of N satisfying:

1. $\emptyset \notin \mathcal{D}$.
2. If $D_1, D_2 \in \mathcal{D}$ then $D_1 \cap D_2 \in \mathcal{D}$.
3. If $D_1 \in \mathcal{D}$ and $D_1 \subseteq D_2$ then $D_2 \in \mathcal{D}$.

An *ultrafilter* on N is a filter $\mathcal{D}$ on N which is maximal with respect to inclusion, i.e., it is a filter $\mathcal{D}$ for which any other filter $\mathcal{D}'$ on N satisfying $\mathcal{D} \subset \mathcal{D}'$, actually satisfies $\mathcal{D} = \mathcal{D}'$.[2] In other words, an ultrafilter on N is the collection of all subsets of N which have μ-measure 1 for some μ, which is a finitely additive $\{0, 1\}$-valued measure on 2^N. An ultrafilter is *non-principal* if the intersection of all its members is empty. Otherwise it is called *principal*, and one can show that the intersection has exactly one element (in our interpretation a **dictator**).

Kirman and Sondermann [14] have shown that a social welfare function σ satisfies all Arrovian rationality axioms ((A1), (A2) and (A3)) if and only if $\mathcal{D}_\sigma$ is a non-principal ultrafilter. We therefore require that $\mathcal{D}$ is a non-principal ultrafilter on N and this is only possible if N is infinite[3]. Thus, we impose the following axiom:

[2] One can show that a filter $\mathcal{D}$ is an ultrafilter if and only if for all $D \subseteq N$, either $D \in \mathcal{D}$ or $N\backslash D \in \mathcal{D}$ (e.g. see Bell and Slomson [4]).

[3] If N is finite, then every ultrafilter on it is principal (for more details see Bell and Slomson [4]).

(A4) $\mathcal{D}$ is a non-principal ultrafilter on N (and therefore N is infinite).

Parametrisations are ubiquitous in macroeconomics, motivating our next assumption:

(A5) Let Z be a compact subset of a given Banach space X (with norm $\|\cdot\|_X$). There exists a continuous function $v : Z \times \mathcal{C} \longrightarrow \mathbb{R}$ such that for every $z \in Z$, $v(z, \cdot)$ is strictly concave and[4]

$$\mathcal{M} \subseteq \{\, v(z, \cdot) \quad : \quad z \in Z \,\}.$$

(In other words, given any $\underline{u} \in \mathcal{M}^N$, there is an N-sequence $\langle z_i \rangle_{i \in N} \in (Z)^N$ such that $u_i = v(z_i, \cdot)$ for every $i \in N$.)

Remark 2 *For all $u \in \mathcal{M}$, u attains its unique global maximum on $\mathcal{C}$.*

Proof. We are concerned with the maximization problem

$$\sup_{r \in \mathcal{C}} v(z, r). \tag{1}$$

This problem is dual of the minimization problem in Ekeland and Temam [8] [Chapter 2 (p. 34)]. Then according to (A5) (continuity and concavity), the problem (1) has a solution for each $z \in Z$, and this solution is unique since the function $v(z, \cdot)$ is also assumed to be strictly concave over $\mathcal{C}$ (in (A5)). (See Proposition 1.2. (p. 35) in Ekeland and Temam [8].)

□

3 Main Results

In this section after introducing two important definitions, we will show the existence of $\mathcal{D}$-socially acceptable and representative utility functions. Afterwards, we will provide conditions for previous regularities to be satisfied in economic applications.

Definition 6 *An admissible utility function $\varphi : \mathcal{C} \longrightarrow \mathbb{R}$ is said to be $\mathcal{D}$-*socially acceptable *for $\underline{u}$ if and only if there exists some $\tilde{x} \in \mathcal{C}$ with $\varphi(\tilde{x})=$ sup φ such that for every $y \in \mathcal{C} \backslash \{\tilde{x}\}$, the coalition of i with $u_i(\tilde{x}) > u_i(y)$ is decisive.*

Definition 7 *An admissible utility function $\varphi : \mathcal{C} \longrightarrow \mathbb{R}$ is called* ***σ-representative*** *of $\underline{P} \in \mathcal{P}^N$ if and only if there exists some $\tilde{x} \in \mathcal{C}$ with $\varphi(\tilde{x})=$ sup φ and any such $\tilde{x}$ is also $\sigma(\underline{P})$-maximal.*

[4]Properness is obvious, since v goes to real numbers.

Theorem 1 *Assuming (A4) and (A5), there exists for every* $\underline{u} \in \mathcal{M}^N$ *some* $\mathcal{D}$*-socially acceptable utility function.*

The proof, can be found in section 6, contains nonstandard functional analysis. An informal discussion of the proof methodology can be found in section 4.

Theorem 2 *Suppose* σ *satisfies (A1), (A2) and (A3). Then:*

1. $\mathcal{D} := \mathcal{D}_\sigma$ *satisfies (A4).*
2. *If, in addition,* $\mathcal{M}$ *satisfies (A5), then there exists for every* $\underline{u} \in \mathcal{M}^N$ *some admissible utility function which is* σ*-representative of the preference profile* $\underline{P}^{\underline{u}}$ *induced by* $\underline{u}$*.*

For the proof of Theorem 2, we show that the maximiser of this representative utility function maximises the $\mathcal{D}_\sigma$-socially acceptable utility function, $\mathcal{D}_\sigma$ being the ultrafilter of σ-decisive coalitions. In other words, the maximiser of this representative utility function is the optimal alternative according to the social preference relation. The complete proof can be found in section 6.

Consider a function u, defined by

$$\forall c \in \mathcal{C} \quad , \qquad u(c) = \max_{y \in Y} f(y, c), \tag{2}$$

where Y is some set (called choice set) and $f : Y \times \mathcal{C} \longrightarrow \mathbb{R}$ is some function, interpreted as felicity function of some agent depending on the social parameter c and his (or her) choice y.

Lemma 1 *Suppose* Y *is a compact non-empty convex subset of* X *and* $f : Y \times \mathcal{C} \longrightarrow \mathbb{R}$ *is continuous and strictly concave. Then* u *is continuous and strictly concave.*

We prove Lemma 1 using convex analysis (in section 6). Lemma 1 immediately implies the following corollary:

Corollary 1 *Suppose* Y *is a compact non-empty convex subset of* X *and* $f : Y \times \mathcal{C} \longrightarrow \mathbb{R}$ *is continuous and strictly concave. Then the function* u *defined by (2) satisfies Axiom (A5).*

Corollary 2 *Suppose* Y *is a compact non-empty convex subset of* X *and* $g : Z \times Y \times \mathcal{C} \longrightarrow \mathbb{R}$ *is continuous and* $g(z, \cdot)$ *is strictly concave for all* $z \in Z$*. Let*

$$\mathcal{M} \subseteq \{ \max_{y \in Y} g(z, y, \cdot) \quad : \quad z \in Z \},$$

and $\mathcal{D}$ *be a non-principal ultrafilter on* N*. Then there exists for every* $\underline{u} \in \mathcal{M}^N$ *some* $\mathcal{D}$*-socially acceptable utility function.*

Corollary 3 *Suppose σ satisfies (A1), (A2) and (A3) and let Y, g and $\mathcal{M}$ be as in Corollary 2. Then there exists for every $\underline{u} \in \mathcal{M}^N$ some admissible utility function which is σ-representative of the preference profile $\underline{P}^{\underline{u}}$ induced by $\underline{u}$.*

4 Proof methodology

For the proof of Theorem 1, we make a superstructure over $(X \oplus W) \cup \mathbb{R}$, i.e. $V((X \oplus W) \cup \mathbb{R})$, where $X \oplus W$ is a Banach space which is established by algebraic direct sum of two Banach spaces X and W with norm $\| x \oplus w\|_\infty = \max\{\| x\|_X, \| w\|_W\}$.[5]

We make the superstructure $V((X \oplus W) \cup \mathbb{R})$ by iterating the power-set operator countably many times. Then we construct a bounded ultrapower of $V((X \oplus W) \cup \mathbb{R})$ by collecting the equivalence classes of sequences in $V((X \oplus W) \cup \mathbb{R})$, that are bounded in the superstructure hierarchy, using the non-principal ultrafilter $\mathcal{D}$ on N. Afterwards we embed this bounded ultrapower into the superstructure $V(^*(X \oplus W) \cup {}^*\mathbb{R})$ in such a way that this embedding satisfies extension and transfer principle. We work on this nonstandard universe, which consists of equivalence classes of sequences of superstructure elements with respect to equivalence relation of "almost sure agreement" according to $\mathcal{D}$. On the *image of $X \oplus W$, a standard operator with respect to the canonical topology on $X \oplus W$ is definable. We shall verify that the standard part of the $\mathcal{D}$-equivalence class of a utility profile is the parameter of a $\mathcal{D}$-socially acceptable utility function. The proof of this assures the continuity and S-continuity arguments with features of the bounded ultrapower construction.

5 Discussion

The traditional reason for using the assumption of the "representative agent" in mathematical models of macroeconomic theory is that it provides microeconomic foundations for aggregate behaviour. So far there have been few attempts at rigorously justifying this assumption. Our contribution combines Arrovian aggregation theory (on an infinite electorate) with structural assumptions on the individual optimisation problem. We adopt the hypothesis that the social planner's goal is maximising the social welfare function. After aggregating individual preferences, we have shown that there is a representative utility function by proving that the maximiser of this representative utility function is the optimal alternative according to the social preference relation. We have also provided sufficient conditions

[5]One can show that two norms $\| x \oplus w\|_p = (\| x\|_X^p + \| w\|_W^p)^{1/p}$ for $1 \leq p < \infty$ and $\| x \oplus w\|_\infty = \max\{\| x\|_X, \| w\|_W\}$ for $p = \infty$, are equivalent on $\mathbb{R}$.

for these results to be satisfied in economic applications.

6 Proofs

Proof of Theorem 1. Fix an arbitrary $\underline{u} \in \mathcal{M}^N$ and by (A5), let $\langle z_i \rangle_{i \in N} \in (Z)^N$ be such that $u_i = v(z_i, \cdot)$ for every $i \in N$.

The ultrapower construction can easily be adapted to construct an embedding

$$^* : V\left((X \oplus W) \cup \mathbb{R}\right) \longrightarrow V(^*(X \oplus W) \cup {}^*\mathbb{R}),$$

where $^*(X \oplus W)$ is a Banach space over $^*\mathbb{R}$ satisfying:
extension: $^*(X \oplus W)$ and $^*\mathbb{R}$ are proper extensions of $X \oplus W$ and $\mathbb{R}$, respectively, and $^*x = x$ for all $x \in (X \oplus W) \cup \mathbb{R}$,
and
transfer: If $\Phi(v_1, \cdots, v_n)$ is an $\in$-formula with bounded quantifiers and n free variables, then for all $A_1, \cdots, A_n \in V\left((X \oplus W) \cup \mathbb{R}\right)$,

$$V\left((X \oplus W) \cup \mathbb{R}\right) \models \Phi[A_1, \cdots, A_n] \quad \Leftrightarrow \quad V(^*(X \oplus W) \cup {}^*\mathbb{R}) \models \Phi[^*A_1, \cdots, {}^*A_n].$$

(See e.g. Albeverio, Fenstad, Høegh-Krohn and Lindstrøm [1])

For the rest of the proof, we work in the resulting nonstandard universe. We have to construct some parameter $\tilde{z}$ such that $v(\tilde{z}, \cdot)$ is $\mathcal{D}$-socially acceptable.

Put $\bar{z} := [\langle z_i \rangle_{i \in N}]_{\mathcal{D}} \in {}^*Z$. Since Z is a compact, then every element of $^*Z$ is nearstandard (Anderson [2] [Chapter 2; Theorem 2.3.2.])[6] and let $\tilde{z} := {}^\circ\bar{z}$. Applying the transfer principle of nonstandard analysis to Remark 2, we learn that $^*v(\bar{z}, \cdot)$ attains its unique global *maximum in some $\bar{x} \in {}^*\mathcal{C}$.

Consider now the map

$$w : Z \longrightarrow \mathcal{C},$$

which assigns to each $z \in Z$ the unique $x = w(z) \in \mathcal{C}$ such that

$$x \in \arg\sup_{r \in \mathcal{C}} v(z, r).$$

(Existence and uniqueness follow from Remark 2.)
By transfer principle,

$$^*w : {}^*Z \longrightarrow {}^*\mathcal{C},$$

hence $^*w(\bar{z}) \in {}^*\mathcal{C}$ and since $\mathcal{C}$ is a compact, every element of $^*\mathcal{C}$ is nearstandard (see again Anderson [2] [Chapter 2; Theorem 2.3.2.]) and therefore $\bar{x} = {}^*w(\bar{z})$ is nearstandard. We put $\tilde{x} := {}^\circ\bar{x}$.

[6] We only used the easy part of this theorem, which does not require saturations.

Due to (A5), v is continuous and hence ${}^*v$ is S-continuous (Albeverio, Fenstad, Høegh-Krohn and Lindstrøm [1]). Therefore, we have for all $y \in \mathcal{C}$,

$$v(\tilde{z}, y) - v(\tilde{z}, \tilde{x}) \simeq {}^*v(\bar{z}, y) - {}^*v(\bar{z}, \tilde{x}) \simeq {}^*v(\bar{z}, y) - {}^*v(\bar{z}, \bar{x}). \tag{3}$$

The right-hand side of Equation (3) is a non-positive hyperreal (since $\bar{x}$ is a global *maximum of ${}^*v(\bar{z}, \cdot)$), so the standard part is non-positive, but the standard part is exactly $v(\tilde{z}, y) - v(\tilde{z}, \tilde{x})$; then

$$v(\tilde{z}, y) - v(\tilde{z}, \tilde{x}) \leq 0 \quad ; \qquad \forall y \in \mathcal{C}.$$

Since we have a unique global maximum (according to Remark 2), thus

$$v(\tilde{z}, y) - v(\tilde{z}, \tilde{x}) < 0 \quad ; \qquad \text{for all standard } y \neq \tilde{x}. \tag{4}$$

The rest of the proof is basically as in Herzberg [12]. In order to verify that $v(\tilde{z}, \cdot)$ is $\mathcal{D}$-socially acceptable, we still need to show that for every $y \in \mathcal{C} \backslash \{\tilde{x}\}$, the set of all $i \in N$ with $u_i(\tilde{x}) > u_i(y)$ is decisive (i.e. $\in \mathcal{D}$). Define a function f by $f(h) := v(h, \tilde{x}) - v(h, y)$ for all $h \in Z$, whence

$$\{i \in N \,:\, u_i(\tilde{x}) > u_i(y)\} = \{i \in N \,:\, f(z_i) > 0\}. \tag{5}$$

Due to the construction of the nonstandard embedding $*$ via the bounded ultrapower (with respect to $\mathcal{D}$) of the superstructure $V\left((X \oplus W) \cup \mathbb{R}\right)$, one has

$$\{i \in N \,:\, u_i(\tilde{x}) > u_i(y)\} \in \mathcal{D} \Leftrightarrow {}^*f(\bar{z}) > 0. \tag{6}$$

However, by applying the transfer principle to the defining equation for f and due to S-continuity of ${}^*v$, we get

$${}^*f(\bar{z}) = {}^*v(\bar{z}, \tilde{x}) - {}^*v(\bar{z}, y) \simeq {}^*v(\bar{z}, \bar{x}) - {}^*v(\bar{z}, y).$$

The standard part of the right-hand side is strictly positive (by inequality (4)) and therefore ${}^\circ({}^*f(\bar{z})) > 0$. Hence ${}^*f(\bar{z}) > 0$ and by equivalence (6) we have

$$\{i \in N \,:\, u_i(\tilde{x}) > u_i(y)\} \in \mathcal{D}.$$

□

Proof of Theorem 2. (1) Kirman and Sondermann [14] [Theorem 1; Proposition 2] have shown that $\mathcal{D} = \mathcal{D}_\sigma$ is a non-principal ultrafilter whenever σ satisfies (A1), (A2) and (A3).

(2) According to Theorem 1, for an arbitrary $\underline{u} \in \mathcal{M}^N$ there exist some $\varphi \in \mathcal{M}$ and $\tilde{x} \in \mathcal{C}$ such that $\varphi(\tilde{x})=$ sup φ and for every $y \in \mathcal{C}\backslash\{\tilde{x}\}$

$$\{i \in N \,:\, u_i(\tilde{x}) > u_i(y)\} \in \mathcal{D}_\sigma.$$

Therefore, for an arbitrary $y \in \mathcal{C}\backslash\{\tilde{x}\}$,

$$\{i \in N \,:\, \tilde{x}P_i^{\underline{u}}y\} \supseteq \{i \in N \,:\, u_i(\tilde{x}) > u_i(y)\} \in \mathcal{D}_\sigma.$$

Since $\mathcal{D}_\sigma$ is an ultrafilter, hence

$$\{i \in N \,:\, \tilde{x}P_i^{\underline{u}}y\} \in \mathcal{D}_\sigma,$$

and this implies $\tilde{x}\sigma(\underline{P^u})y$ (Kirman and Sondermann [14] [Theorem 1(i)]). The proof is complete since y was an arbitrary element of $\mathcal{C}\backslash\{\tilde{x}\}$.

□

Proof of Lemma 1. Since f is strictly concave, then for every $y', y'' \in Y$ and every $c', c'' \in \mathcal{C}$ and $\lambda \in [0,1]$ we have

$$f\left(\lambda y' + (1-\lambda)y'', \lambda c' + (1-\lambda)c''\right) > \lambda f(y', c') + (1-\lambda)f(y'', c'').$$

Taking the maximum of both sides over $y', y'' \in Y$ and using the convexity of Y, we obtain

$$u\left(\lambda c' + (1-\lambda)c''\right) > \lambda u(c') + (1-\lambda)u(c''),$$

and therefore u is strictly concave.

Since Y is a compact set and f is continuous, the extreme value theorem implies the existence of maximum on Y. Let $\{c_n\}$ be a sequence in $\mathcal{C}$ converging to c and $\{y_n\} \in \arg\max_n f(y_n, c_n)$ be a sequence in Y. Since Y is a compact set, there exists a convergent subsequence $\{y_{n_k}\}$ which converges to some $y \in Y$. If it is shown that $y \in \arg\max_y f(y, c)$, then

$$\lim_{k\to\infty} u(c_{n_k}) = \lim_{k\to\infty} f(y_{n_k}, c_{n_k}) = f(y, c) = u(c),$$

which would prove the continuity of u.
Suppose that $y \notin \arg\max_y f(y, c)$, i.e. there exists an $\hat{y} \in Y$ such that $f(\hat{y}, c) > f(y, c)$. By the continuity of f, we get

$$\lim_{j\to\infty} f(\hat{y}, c_{n_j}) = f(\hat{y}, c) > f(y, c) = \lim_{j\to\infty} f(y_{n_j}, c_{n_j}).$$

This implies that for sufficiently large j,

$$f(\hat{y}, c_{n_j}) > f(y_{n_j}, c_{n_j}),$$

which would mean y_{n_j} is not a maximiser and this is a contradiction of $\{y_n\} \in \arg\max_n f(y_n, c_n)$.

□

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