University of Bielefeld

Dissertation

Geometric reduction theory

Henning Niesdroy

supervised by
Prof. Dr. Kai-Uwe Bux

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ABSTRACT

In 2013 Prof. Dr. Bux, Prof. Dr. Köhl and Dr. Witzel published an article on higher finiteness properties of reductive arithmetic groups in positive characteristic ([BKW13]). An essential tool in their work is the transmission of algebraic reduction theory into pure geometry:

Given a semisimple linear algebraic group $\mathbf{G}$ and a finite product of local function fields k_f, they created a reduction theory on the building accociated to $\mathbf{G}(k_f)$. Later on Prof. Dr. Bux asked, whether they created the *right* kind of *geometric reduction theory* and if there is a universally valid reduction theory on arbitrary CAT(0)-spaces. As an intermediate step reaching this highly ambitious aim he asked for a second example:

In this work we create an analogous reduction theory on the product of a symmetric space and a building associated to $\mathbf{G}(k_n)$, where k_n is a finite product of local number fields.

ACKNOWLEDGEMENTS

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INTRODUCTION

Consider $\mathrm{SL}_n(\mathbb{R})$, the symmetric space $X_\infty := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ and the action of $\mathrm{SL}_n(\mathbb{Z})$ on X_∞. Reduction theory describes a fundamental domain in X_∞.

Generalizing reduction theory to arithmetic groups, Godement found an adelic formulation treating all places simultaneously [God64]: Let k be a global number field, $\mathbf{G}$ be a semisimple group (think of SL_n), and let $\mathbb{A}$ be the ring of adeles of k. Then $\mathbf{G}(k)$ is discrete in $\mathbf{G}(\mathbb{A})$ and Godement found a coarse fundamental domain for the action of $\mathbf{G}(k)$ on $\mathbf{G}(\mathbb{A})$. Later, Behr and Harder transferred this to the case when k is not a global number field, but a function field [Har69].

In 2012 Bux-Köhl-Witzel gave a geometric reformulation of Behr-Harder for the S-arithmetic case (see [BKW13]). They introduced a so called *reduction datum* explained in Section 1.1.

Now the natural question arises, whether this geometric reformulation can be transferred back to the case of a number field.

To gain an insight on what a reduction datum is we present the simplest number field case we can think of: Let $\mathbf{G}$ be SL_2 and let S, a finite set of places, contain the absolute real valuation only, i.e. consider $\mathrm{SL}_2(\mathbb{Z})$ acting on the hyperbolic upper half plane $\mathbb{H}^2$ by Möbius transformation. In that case there exists a constant $r \in \mathbb{R}$ such that $\mathbb{H}^2$ is covered by horoballs of height r centered at the rational points at infinity of $\mathbb{H}^2$, i.e. each point $x \in \mathbb{H}^2$ is contained in at least one $\mathrm{SL}_2(\mathbb{Z})$-translate of some chosen horoball of height r, see Figure 1.

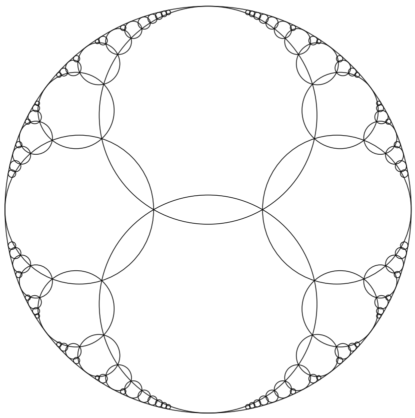


Figure 1: Horoballs of height r covering $\mathbb{H}^2$ allow assignments of any point $x \in \mathbb{H}^2$ to some set of rational vertices at infinity of $\mathbb{H}^2$.

Hence, using the constant r, we can assign to each $x \in \mathbb{H}^2$ a set of rational vertices, namely the centers of those horoballs of height r that contain x. We call the constant $r \in \mathbb{R}$ a *lower reduction bound*. On the other hand we may sufficiently increase the height of those horoballs, lets say to height R, such that each point $x \in \mathbb{H}^2$ is contained in at most one $\mathrm{SL}_2(\mathbb{Z})$-translate of some chosen horoball of height R, see Figure 2.

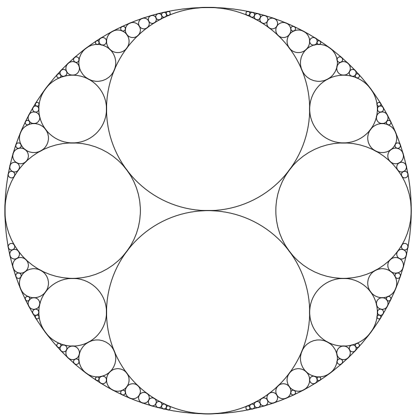


Figure 2: Horoballs of height R not quite covering $\mathbb{H}^2$ allow assignments of most but not all points $x \in \mathbb{H}^2$ to a unique rational vertex at infinity of $\mathbb{H}^2$.

Hence, using the constant R, we may not assign to each $x \in \mathbb{H}^2$ some rational vertex, but if we can, we may do uniquely. In that case we call the constant $R \in \mathbb{R}$ an *upper reduction bound*.

Currently we can not be precise on what a reduction datum is, because the rank of this example is to small. But roughly speaking a reduction datum determines a lower reduction bound and an upper reduction bound on some CAT(0)-space X_S depending on $\mathbf{G}$ and a finite set of places S, i.e. it assigns to most points in X_S a unique rational simplex in its boundary.

More figuratively spoken, one could describe a reduction datum as follows:

> *Imagine you are somewhere on earth and it is completely dark outside. All you see are the stars far far away. A reduction datum tells you which stars you are close to.*

In this work we proof an analogous statement of [BKW13, Theorem 1.9], i.e. a generalization of the previous example.

CHAPTER 1

STATEMENT

During the next three sections we introduce the reader to the topic of this thesis and recapitulate the groundwork that has already been laid. More specifically we do the following:

In Section 1.1 we explain what a reduction datum is. After setting some basic notation we illustrate the geometry discussed in this work and, in the very end, state the main theorem of geometric reduction theory. In Section 1.2 we restate important statements of algebraic reduction theory as presented in [God64]. In Section 1.3 we give a short overview of the essential steps taken in this thesis. Moreover we try to illustrate the close relation between geometric reduction theory and algebraic reduction theory.

1.1 Geometric reduction theory

Let k be a global number field and $\mathbf{G}$ be a linear algebraic, connected, semisimple, k-isotropic k-group. Replacing $\mathbf{G}$ by its Weil restriction from k to $\mathbb{Q}$ we may assume $k = \mathbb{Q}$ without loss of generality (see for example [Wei61, Section 1.3]) which we do throughout this work.

For any finite set of places S including the archimedean place ∞ we consider the ring of <u>S-adeles</u>

$$\mathbb{A}_S = \mathbb{R} \times \prod_{p \in S \setminus \{\infty\}} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p.$$

$\mathbb{A}_{S'} \hookrightarrow \mathbb{A}_S$ is a directed system indexed by the family of finite sets of places ordered by inclusion. The ring of <u>adeles</u> $\mathbb{A}$ is by definition its direct limit.

Now we fix a finite set of places S including the archimedean place ∞. For a non-archimedean place $p \in S$ we define X_p to be the Bruhat-Tits building associated to $\mathbf{G}(\mathbb{Q}_p)$. For the archimedean place we denote by X_∞ the symmetric space associated to $\mathbf{G}(\mathbb{R})$. The product space

$$X_S = X_\infty \times \prod_{p \in S \setminus \{\infty\}} X_p$$

is a CAT(0)-space by [AB08, Theorem 11.16]. The group $\mathbf{G}(\mathbb{A})$ acts isometrically on X_S (components not in S act trivially) and hence on its visual boundary $\partial(X_S)$. Likewise $\mathbf{G}(\mathbb{A}_S)$, $\mathbf{G}(\mathbb{Q})$ and an S-arithmetic lattice Γ act on X_S and $\partial(X_S)$.

Denote by Δ the spherical building for $\mathbf{G}(\mathbb{Q})$ with set of chambers $\mathcal{C}(\Delta)$ and set of vertices $\mathcal{V}(\Delta)$. The idea of a reduction datum is to assign to *most* points $x \in X_S$ a unique simplex in Δ that is *close* to x:

To have a notion of *distance* from points in X_S to points in Δ we assume to have an isometric embedding of Δ into $\partial(X_S)$. Moreover we suppose to have a family $\{h_v : X_S \to \mathbb{R} \mid v \in \mathcal{V}(\Delta)\}$ of rescaled Busemann functions with h_v centered at v for all $v \in \mathcal{V}(\Delta)$, i.e.

$$h_v(x) = \lim_{t\to\infty} (t - d(x, \gamma_v(t)))$$

where $\gamma_v \colon X_S \to \mathbb{R}$ is a geodesic with visual endpoint $\gamma_v(\infty) = v$ for all $v \in \mathcal{V}(\Delta)$. Furthermore we assume to have a family $\mathcal{A}$ of totally geodesic flat subspaces in X_S (think of an apartment system) such that:

(A) For each chamber $c \in \mathcal{C}(\Delta)$ the union of all flat subspaces $\Sigma' \in \mathcal{A}$ with c in their visual boundary covers X_S.

For a chamber $c \in \mathcal{C}(\Delta)$ we consider the function

$$h_c(x) := \max\{h_v(x) \mid v \in \mathcal{V}(c)\}.$$

Given some $\Sigma' \in \mathcal{A}$ containing c in its visual boundary and a real parameter $s \in \mathbb{R}$, we consider the closed and convex set

$$Y_{\Sigma',c}(s) := \{x \in \Sigma' \mid h_c(x) \leqslant s\}.$$

We denote the union over all those subsets by

$$Y_c(s) := \bigcup_{\substack{\Sigma' \in \mathcal{A} \\ c \subset \partial(\Sigma')}} Y_{\Sigma',c}(s) \overset{(A)}{=} \{x \in X_S \mid h_c(x) \leqslant s\}.$$

Because $Y_{\Sigma',c}(s)$ is closed and convex in Σ' there is a closest point projection

$$\mathrm{pr}^s_{\Sigma',c} : \Sigma' \to Y_{\Sigma',c}(s),$$

see Figure 1.1. By Condition (A) we may choose for each $x \in X_S$ some $\Sigma' \in \mathcal{A}$ containing x and hence consider $\mathrm{pr}^s_{\Sigma',c}(x)$ for each $s \in \mathbb{R}$. We require the following condition to be satisfied:

(B) For any $c \in \mathcal{C}(\Delta)$ and $v \in \mathcal{V}(c)$ the values $h_v(\mathrm{pr}^s_{\Sigma',c}(x)) =: b^s_{c,v}(x)$ are independent of the flat Σ' containing x.

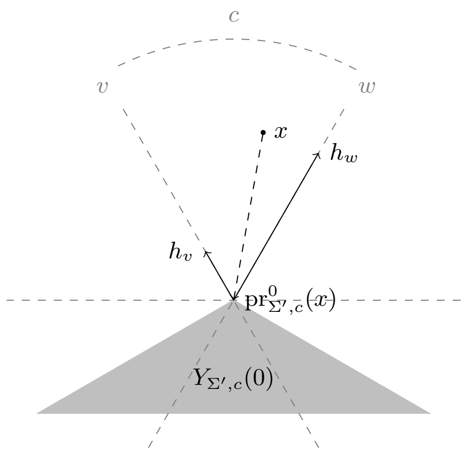


Figure 1.1: Closest point projection on the set of points with height less or equal to zero for all Busemann functions centered at some vertex in c.

With the above setup we may define for each $x \in X_S$ and $c \in \mathcal{C}(\Delta)$ the set

$$\sigma_s(x,c) := \left\{ v \in \mathcal{V}(c) \;\middle|\; b^s_{c,v}(x) = s \right\},$$

i.e. we collect those vertices $v \in \mathcal{V}(c)$ such that the inequality $h_v(\mathrm{pr}^s_{\Sigma',c}(x)) \leqslant s$ is sharp, see Figure 1.2.

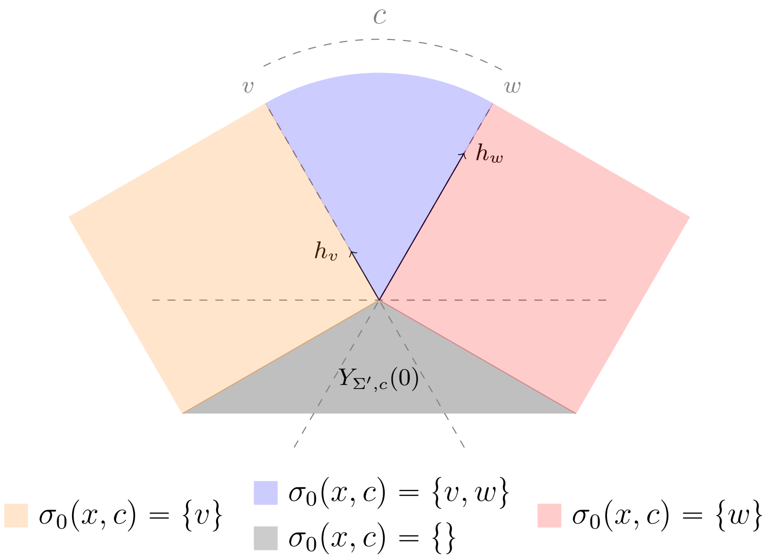


Figure 1.2: Apartment separated into the sets of points that are 0-*close* to either all vertices of c, only one vertex of c or non of its vertices.

We say that a chamber $c \in \mathcal{C}(\Delta)$ s-reduces $x \in X_S$ if $\sigma_s(x,c) = \mathcal{V}(c)$. For a chamber $c \in \mathcal{C}(\Delta)$ and a flat $\Sigma' \in \mathcal{A}$ with c in its visual boundary we consider the set

$$N_{\Sigma',c}(s) := \{x \in \Sigma' \mid x \text{ is } s\text{-reduced by } c\},$$

and call it a thin Minkowski cone (above c of height s). Moreover we consider the union of all thin Minkowski cones above c of height s

$$N_c(s) := \bigcup_{\substack{\Sigma' \in \mathcal{A} \\ c \subset \partial(\Sigma')}} N_{\Sigma',c}(s) \overset{(A),(B)}{=} \{x \in X_S \mid x \text{ is } s\text{-reduced by } c\},$$

and call it a thick Minkowski cone (above c of height s). See Figure 1.3 to get an impression of what these sets are. It is drawn in case of $\mathbf{G} = \mathrm{SL}_2$ with S containing the absolute real value only.

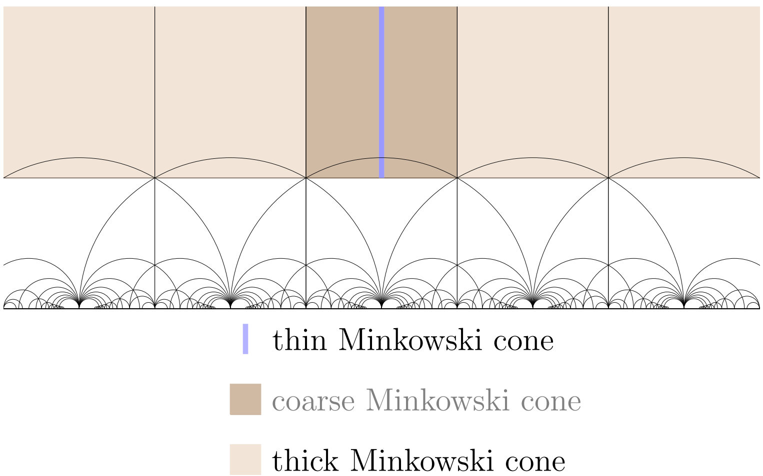


Figure 1.3: Minkowski cones for $\mathbf{G} = \mathrm{SL}_2$.

Now one might ask why we allow the family of Busemann functions to be rescaled. The reason stems from the following. We want the family of thin Minkowski cones above some chamber $c \in \mathcal{C}(\Delta)$, and hence the corresponding family of thick Minkowski cones, to behave like a filtration in $s \in \mathbb{R}$, i.e. we want to avoid a situation as presented in Figure 1.4. Instead we ask for a situation as presented in Figure 1.5. Formally we require the following condition to be satisfied:

(C) For $c \in \mathcal{C}(\Delta)$ and $\Sigma' \in \mathcal{A}$ with c in their visual boundary we have

  (i) $N_{\Sigma',c}(s_2) \subset N_{\Sigma',c}(s_1)$ if and only if $s_1 < s_2$;

  (ii) $\Sigma' = \bigcup_{s \in \mathbb{R}} N_{\Sigma',c}(s)$.

In case of (unit speed) Busemann functions condition (C) fails for example if Δ corresponds to E_6, E_7, E_8 or D_n.

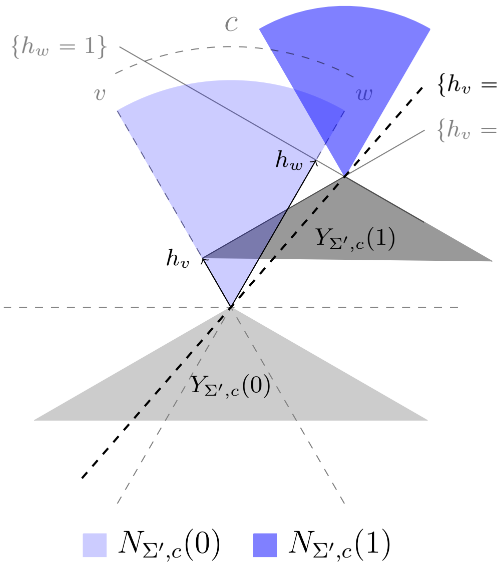


Figure 1.4: (Thin) Minkowski cones of height 0 and 1 without adjusted velocity of Busemann functions.

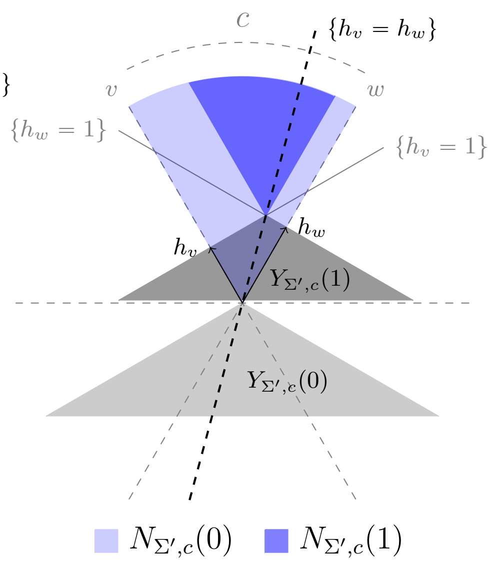


Figure 1.5: (Thin) Minkowski cones of height 0 and 1 with adjusted velocity of Busemann functions.

Now we define a reduction datum, analogue to how it was introduced in [BKW13]: A reduction datum consists of a family of flat subspaces $\mathcal{A}$ in X_S satisfiying condition (A) together with a family $\{h_v : X_S \to \mathbb{R} \mid v \in \mathcal{V}(\Delta)\}$ of rescaled Busemann functions satisfying condition (B) and (C) and two real constants $r < R$ such that:

> Each point $x \in X_S$ is r-reduced by some chamber $c \in \mathcal{C}(\Delta)$ and for each such chamber the set $\sigma_R(x, c)$ is contained in *any* chamber $c' \in \mathcal{C}(\Delta)$ that r-reduces x.

By [BKW13, Observation 1.7] we may alternatively formulate the above condition as follows:

> X_S is covered by the family of thick Minkowski cones of height r, i.e. $X_S = \cup_{c \in \mathcal{C}(\Delta)} N_c(r)$. Moreover $x \in N_c(r) \cap N_{c'}(r)$ implies $\sigma_R(x, c) = \sigma_R(x, c')$ for all $x \in X_S$ and $c, c' \in \mathcal{C}(\Delta)$,

see Figure 1.6. We call the constant $r \in \mathbb{R}$ a lower reduction bound and the related constant $R \in \mathbb{R}$ an upper r-reduction bound.

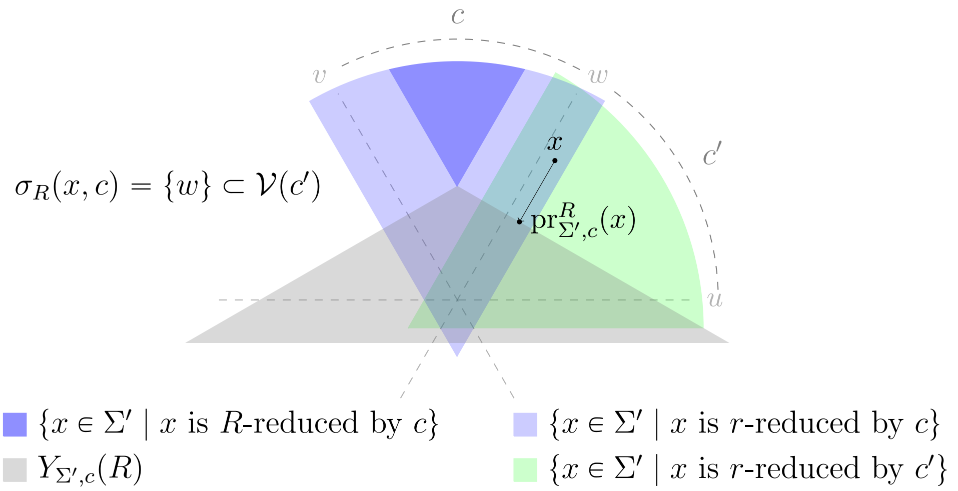


Figure 1.6: Minkowski cones with non-empty intersection and a common simplex at both their boundaries.

A reduction datum is called Γ-invariant if for each $\gamma \in \Gamma$, each vertex $v \in \mathcal{V}(\Delta)$ and each point $x \in X_S$ we have $h_{\gamma \cdot v}(\gamma \cdot x) = h_v(x)$. A Γ-invariant reduction datum is called Γ-cocompact if for each $s \geqslant r$ the set

$$\begin{aligned} Y_s &:= \left(\bigcup_{c \in \mathcal{C}(\Delta)} N_c(r) \backslash Y_c(s) \right)^C \\ &= \{x \in X_S \mid h_c(x) \leqslant s \text{ for all } c \in \mathcal{C}(\Delta)\ r\text{-reducing } x\} \end{aligned}$$

is relatively compact modulo the action of Γ, see Figure 1.7. Furthermore we say that a subset $B \subset X_S$ can be uniformly r-reduced if there is a chamber $c \in \mathcal{C}(\Delta)$ that r-reduces all points of B simultaneously. Given a non-negative real number d we call a reduction datum d-uniform if every subset $B \subset X_S$ of diameter at most d can be uniformly r-reduced. With these notations we may formulate the main theorem of geometric reduction theory as follows:

Theorem 1.1.1. *For every diameter d, there is a d-uniform, Γ-invariant and Γ-cocompact reduction datum on X_S.*

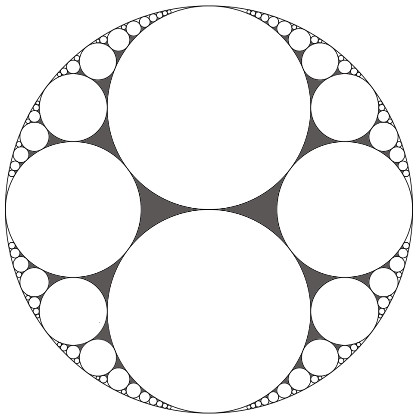


Figure 1.7: Y_R for $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}^2$ (marked gray).

The purpose of a reduction datum

For the sake of a better understanding of a reduction datum, we highlight the relation between the precise definition and the idea given in the intruduction. Moreover, due to a hint of Prof. Dr. Hoffmann concerning the work of James Arthur (see the subsection below), we state an almost analogous version of Theorem 1.1.1 formulated in the language of characteristic functions.

Given a reduction datum with constants $r, R \in \mathbb{R}$ the thick Minkowski cones $N_c(r)$ and $N_c(R)$ differ only by some compact action, i.e. there exists a compact set $\mathrm{K} \subset \mathbf{G}(\mathbb{A}_S)$ depending on c with $N_c(R) \subset N_c(r) \subset \mathrm{K} \cdot N_c(R)$, see Figure 1.6. However the informations they contribute are significantly different. On the one hand the constant r enabels us to assign to each $x \in X_S$ a non-empty set of rational chambers, namely those chambers such that their thick Minkowski cones of height r contain x. However the assignement does not necessarily attach unique chambers. On the other hand the same procedure with R instead of r does not assign to each $x \in X_S$ some chamber, but if it does, it does uniquely. The uniqueness follows because the definition of a reduction datum implies $N_c(R) \cap N_{c'}(R) = \varnothing$ for $c \neq c'$.

Now the actual purpose of a reduction datum is to do something in between these two properties. It does not assign to all points a unique chamber, but to *most* points a unique simplex, namely the simplex $\sigma_R(x, c)$ for some $c \in \mathcal{C}(\Delta)$ with $x \in N_c(r)$. (In that case we regard $\sigma_R(x, c)$ as the set of vertices $v \in \mathcal{V}(c)$ such that x is *far* away from the wall of $N_c(r)$, that is opposite to v, see Figure 1.8.) Now *most* means all points that are contained in some thick Minkowski cone $N_c(r)$ such that $\sigma_R(x,c)$ is non-empty, i.e. to all points $x \in X_S \backslash \mathring{Y}_R$ with

$$\mathring{Y}_s := \{x \in X_S \mid h_c(x) < s \text{ for all } c \in \mathcal{C}(\Delta) \text{ } r\text{-reducing } x\}\,.$$

We say *most*, because $\mathring{Y}_R$ is relatively compact modulo the action of Γ.

Next let τ be an arbitrary simplex in Δ and denote by A_τ the set of points that are assigned to τ, i.e.

$$A_\tau := \{x \in X_S \mid \exists\, c \in \mathcal{C}(\Delta) \text{ with } x \in N_c(r) \text{ and } \sigma_R(x,c) = \tau\}\,.$$

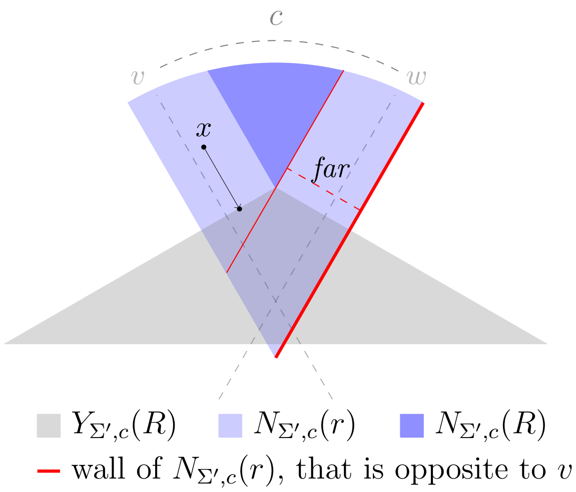


Figure 1.8: Minkowski cones of height r respectively R with identical boundary but associated walls, that are far apart from each other.

Now let $\mathbb{1}_{A_\tau}$ (resp. $\mathbb{1}_{\mathring{Y}_R}$) denote the characteristic funtion of A_τ (resp. $\mathring{Y}_R$). The following statement more or less resembles Theorem 1.1.1:

Lemma 1.1.2. *The sum* $\sum_\tau \mathbb{1}_{A_\tau} + \mathbb{1}_{\mathring{Y}_R}$ *equals* 1.

The work of James Arthur

While proofreading the complete work, Prof. Dr. Hoffmann perceived the similarity between Theorem 1.1.1 and [Art78, Lemma 6.4]. Therein Arthur discribes a $\mathbf{G}(\mathbb{Q})$-invariant partition of $\mathbf{G}(\mathbb{A})$ into disjoint subsets indexed by the parabolic subgroups of $\mathbf{G}$, i.e. by the simplices of Δ. Restricted to $\mathbf{G}(\mathbb{A}_S)$ this partition can be visualized in X_S:

The maximal compact subgroups of $\mathbf{G}(\mathbb{A}_S)$ are in one-to-one-correspondence with the set of vertices in X_S, which are tuples $(x_p)_{p\in S}$, consisting of points x_p of the symmetric space at archimedean places p and vertices x_p at non-archimedean places. Namely the stabilizer of a vertex in X_S is a maximal compact subgroup in $\mathbf{G}(\mathbb{A}_S)$ and vice versa. Moreover, there are only finitely many conjugacy classes of maximal compact subgroups. The set of these classes is in one-to-one-correspondence with the set of types of vertices.

Arthur has an independent definition of what is called Busemann function in the geometric context, but it works only on a conjugacy class of special maximal compact subgroups. Now choose a special maximal compact subgroup K in $\mathbf{G}(\mathbb{A}_S)$ and let $* \in X_S$ denote the stabilized vertex. The embedding $\mathbf{G}(\mathbb{A}_S)/K \hookrightarrow X_S,\ g \cdot K \mapsto g \cdot *$ covers only one $\mathbf{G}(\mathbb{A}_S)$-orbit of vertices. Arthur's partition of $\mathbf{G}(\mathbb{A}_S)$ can now be seen as a partition of that image in X_S.

A priori, it is not clear how to interpolate this partition all over X_S, not even if it is possible. Theorem 1.1.1, respectively Lemma 1.1.2, states that it is possible and the proof shows how to do it.

1.2 Algebraic reduction theory

In Section 1.2 we give an overview on the history of algebraic reduction theory from its origin until the work of Siegel in 1959. In Section 1.2 we present the state of algebraic reduction theory in 1964.

The history of algebraic reduction theory

There are different opinions about the origin of (algebraic) reduction theory but the author sees the start in AD 1801. That year Gauß published his famous work *Disquisitiones arithmeticae*. Therein he examined positive definite quadratic forms in two variables with real entries and determinant 1. For the corresponding action of $\mathrm{SL}_2(\mathbb{Z})$ he discovered a coarse fundamental domain (see [Gau06, p. 135]). In today's language this means a coarse fundamental domain for the action of the arithmetic group $\mathrm{SL}_2(\mathbb{Z})$ on its associated symmetric space $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

We visualize this fundamental domain by means of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{H}^2$ by Möbius transformation. Therewith one may identify the connected component of the set of positive definite quadratic forms in two variables with determinant 1 with $\mathbb{H}^2$ equivariently. Hence the coarse fundamental domain appears as one in $\mathbb{H}^2$, see Figure 1.9.

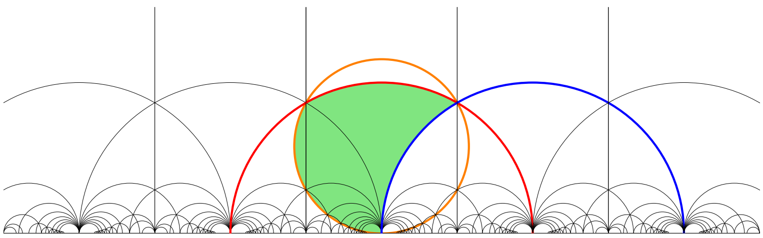


Figure 1.9: Coarse fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}^2$ by Gauß.

Unsurprisingly the name *reduction theory* goes back to Gauß as well. By *formas reductas* he denoted those quadratic forms contained in the fundamental domain.

The theory was developed further by Hermite in 1850. In *Lettres de M. Ch. Hermite à M. Jacobi* he discovered a coarse fundamental domain for the action of $\mathrm{SL}_n(\mathbb{Z})$ on the set of positive definite quadratic forms in n variables with real entries and determinant 1 (see [Her50, p. 272ff]). In other words he discovered a coarse fundamental domain for the action of the arithmetic group $\mathrm{SL}_n(\mathbb{Z})$ on its associated symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. For the special case of $n = 2$ that fundamental domain is visualized in Figure 1.10.

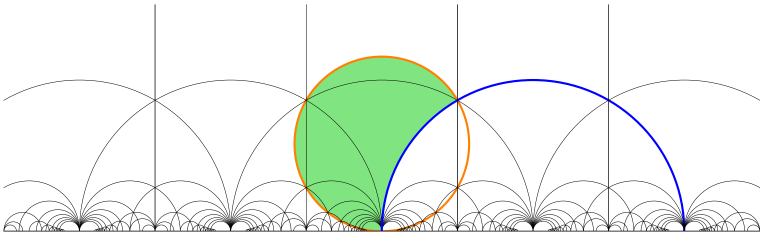


Figure 1.10: Coarse fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}^2$ by Hermite.

The next step is due to Minkowski. In 1910, one year after he died, his work *Geometrie der Zahlen* was published. Therein he proved that one may sharpen the inequalities, Hermite had used, to describe a smaller fundamental domain (see [Min10, p. 198]). Moreover Minkowski proved the new fundamental domain to be of finite volume. In Figure 1.11 we illustrate this fundamental domain for the special case of $n = 2$.

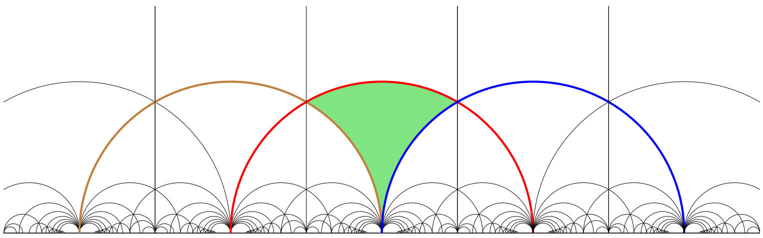


Figure 1.11: Fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}^2$ by Minkowski.

Half a century later, in 1959, Siegel has weakend the conditions further. In [Sie59, §13 − 18] he examined non-degenerate, indefinite quadratic forms in $p + q$ variables. More precisely he examined the action of $\mathrm{O}(p, q)(\mathbb{Z})$ on a specially constructed *action space* and determined a coarse fundamental domain. Today we would say that he determined a coarse fundamental domain for the action of $\mathrm{O}(p, q)(\mathbb{Z})$ on its associated symmetric space $\mathrm{O}(p, q)(\mathbb{R})/\left(\mathrm{O}(p)(\mathbb{R}) \times \mathrm{O}(q)(\mathbb{R})\right)$.

The state of algebraic reduction theory in 1964

In the early sixties of the twentieth century Borel and Harish-Chandra proved results on fundamental domains for linear algebraic groups defined over $\mathbb{Q}$ (see [Bor62] and [BHC62]). About the same time Mostow and Tamagawa proved similar results for special cases, but the methods they used were simpler (see [MT62]). Later, Godement and Weil found out how these methods can be modified in order to prove with their help the remaining cases. In this section we summarize essential results as they were presented in [God64]. However we remark that in [God64] the group $\mathbf{G}(\mathbb{A})$ acts from the right while in this work it acts from the left. This causes minor differences in the formulation.

The apartments in Δ correspond to maximal $\mathbb{Q}$-split tori in $\mathbf{G}$, its simplices to parabolic $\mathbb{Q}$-subgroups ordered by reverse inclusion. The stabilizer of a chamber $c \in \mathcal{C}(\Delta)$, denoted by $\mathbf{P}_c$, is a minimal $\mathbb{Q}$-parabolic subgroup, the stabilizer of a vertex $v \in \mathcal{V}(\Delta)$, denoted by $\mathbf{P}_v$, is a maximal $\mathbb{Q}$-parabolic subgroup.

Choose a chamber $\mathfrak{c} \in \mathcal{C}(\Delta)$ and call it standard chamber. Moreover call its vertices $\mathfrak{v}, \mathfrak{w}, \ldots \in \mathcal{V}(\mathfrak{c})$ standard vertices. Let $\mathbf{T}_{\max} \leqslant \mathbf{P}_\mathfrak{c}$ be a maximal torus in $\mathbf{P}_\mathfrak{c}$ defined over $\mathbb{Q}$. It does not need to split over $\mathbb{Q}$ nor $\mathbb{Q}_p$. Therefore we moreover consider a maximal $\mathbb{Q}$-split torus $\mathbf{T} \leqslant \mathbf{T}_{\max}$ and maximal $\mathbb{Q}_p$-split tori $\mathbf{T} \leqslant \mathbf{T}_p \leqslant \mathbf{T}_{\max}$ for $p \neq \infty$. For a linear algebraic group $\mathbf{H}$ we denote by $X(\mathbf{H})$ its set of $\mathbb{Q}$-characters. Following [God64, p. 257-08] there is a set of simple roots (simple system) $\{\varphi^\mathfrak{c}_\mathfrak{v} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\} \subset X(\mathbf{T})$ belonging to the panels of $\mathfrak{c}$. Since

$$X(\mathbf{T}) \otimes \mathbb{Q} = X(\mathbf{P}_\mathfrak{c}) \otimes \mathbb{Q}, \tag{1.1}$$

([Har69, p. 47]) we may regard each $\varphi^\mathfrak{c}_\mathfrak{v}$ as an element of $X(\mathbf{P}_\mathfrak{c}) \otimes \mathbb{Q}$. Moreover $\{\varphi^\mathfrak{c}_\mathfrak{v} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\}$ is a basis for $X(\mathbf{T}) \otimes \mathbb{Q}$, see Figure 1.12.
Let $\|-\|\colon \mathbb{A}^\times \to \mathbb{R}$ denote the idele-norm, i.e. $\|-\| = \prod_p |-|_p$ where $|-|_p$ denotes the standard norm on $\mathbb{Q}_p$. For a linear algebraic group $\mathbf{H}$ we define $\mathbf{H}(\mathbb{A})^\circ$ to be the subgroup of elements $h \in \mathbf{H}(\mathbb{A})$ with $\|\chi(h)\| = 1$ for all $\chi \in X(\mathbf{H})$. By [God64, §8] there are compact subsets

$$E \subset \mathbf{G}(\mathbb{A}) \quad \text{with } \mathbf{G}(\mathbb{A}) \quad = \mathbf{P}_\mathfrak{c}(\mathbb{A}) \cdot E \text{ and} \tag{1.2}$$
$$F \subset \mathbf{P}_\mathfrak{c}(\mathbb{A})^\circ \text{ with } \mathbf{P}_\mathfrak{c}(\mathbb{A})^\circ = \mathbf{P}_\mathfrak{c}(\mathbb{Q}) \cdot F. \tag{1.3}$$

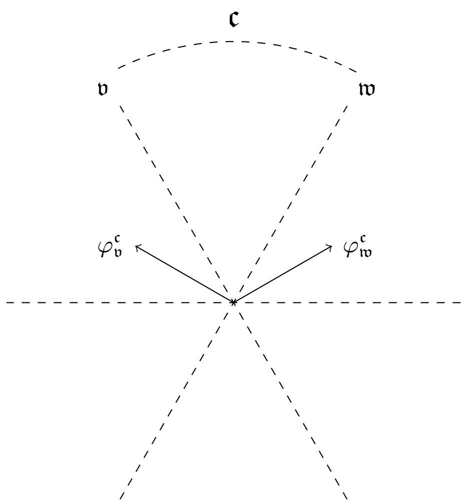


Figure 1.12: Set of simple roots (simple system) for $\mathbf{G} = \mathrm{SL}_3$.

Let $\mathbf{T}_\infty$ denote the group of elements $t = (t_p)_p \in \mathbf{T}(\mathbb{A})$ with $t_p = 1$ for all $p \neq \infty$ and its connected component by $\mathbf{T}_\infty^+$. For a positive real number $s \in \mathbb{R}_{>0}$ we consider the subset $\mathbf{T}_\infty^+(s)$ of elements $t \in \mathbf{T}_\infty^+$ with $\|\varphi_{\mathfrak{v}}^{\mathfrak{c}}(t)\| = \varphi_{\mathfrak{v}}^{\mathfrak{c}}(t) \geqslant s$ for all $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$. With the above notation we may extract from [God64, §10] the following theorem:

Theorem 1.2.1. *There is a constant $C_1 > 0$ such that*

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{Q}) \cdot F \cdot \mathbf{T}_\infty^+(C_1) \cdot E.$$

Moreover [God64, LEMME 3] can be rephrased as follows:

Theorem 1.2.2. *For any positive constant C_1 there is another positive constant C_2 such that the following holds:*

> *Let $f, f' \in F$, $e, e' \in E$, $t, t' \in \mathbf{T}_\infty^+(C_1)$ and $q \in \mathbf{G}(\mathbb{Q})$ with $q \cdot fte = f't'e'$. If $\varphi_{\mathfrak{v}}^{\mathfrak{c}}(t) \geqslant C_2$ for some $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$, then we have $q \in \mathbf{P}_{\mathfrak{v}}(\mathbb{Q})$.*

Furthermore we know from [God64, p. 11f] that the equations

$$\mathbf{P}_{\mathfrak{c}}(\mathbb{A}) = \mathbf{P}_{\mathfrak{c}}(\mathbb{A})^\circ \cdot \mathbf{T}(\mathbb{A}) \text{ and} \tag{1.4}$$
$$\mathbf{T}(\mathbb{A}) = \mathbf{T}(\mathbb{A})^\circ \cdot \mathbf{T}_\infty^+ \tag{1.5}$$

hold. Therefore the following lemma is valid:

Lemma 1.2.3. $\mathbf{G}(\mathbb{A}) = \mathbf{P}_{\mathfrak{c}}(\mathbb{Q}) \cdot F \cdot \mathbf{T}_\infty^+ \cdot E.$

Proof.

$$\begin{aligned}\mathbf{G}(\mathbb{A}) &\overset{(1.2)}{=} \mathbf{P}_{\mathfrak{c}}(\mathbb{A}) \cdot E \\ &\overset{(1.4)}{=} \mathbf{P}_{\mathfrak{c}}(\mathbb{A})^\circ \cdot \mathbf{T}(\mathbb{A}) \cdot E \\ &\overset{(1.5)}{=} \mathbf{P}_{\mathfrak{c}}(\mathbb{A})^\circ \cdot \mathbf{T}(\mathbb{A})^\circ \cdot \mathbf{T}_\infty^+ \cdot E \\ &\overset{(1.1)}{=} \mathbf{P}_{\mathfrak{c}}(\mathbb{A})^\circ \cdot \mathbf{T}_\infty^+ \cdot E \\ &\overset{(1.3)}{=} \mathbf{P}_{\mathfrak{c}}(\mathbb{Q}) \cdot F \cdot \mathbf{T}_\infty^+ \cdot E.\end{aligned}$$

□

The following lemma illustrates the relation between Γ, $\mathbf{P}_{\mathfrak{c}}(\mathbb{Q})$ and $\mathbf{G}(\mathbb{Q})$:

Lemma 1.2.4 ([God64, THÉORÈME 11])**.** *The set* $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}_{\mathfrak{c}}(\mathbb{Q})$ *of double cosets is finite.*

Let V be a finite dimensional vector space that is, as algebraic variety, defined over $\mathbb{Q}$. Now $V(\mathbb{A}) = V(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}$ and $\mathrm{GL}(V)(\mathbb{A})$ identifies canonically with $\mathrm{GL}(V(\mathbb{A}))$ (see [God64, p. 2]). A non-zero element $x \in V(\mathbb{A})$ is called primitive, if there exists an element $g \in \mathrm{GL}(V(\mathbb{A}))$ with $g(x) \in V(\mathbb{Q})$. We denote by $V(\mathbb{A})'$ the set of primitive elements. By definition $\mathrm{GL}(V(\mathbb{A}))$ acts on $V(\mathbb{A})'$.

Lemma 1.2.5 ([God64, p. 2])**.** *There exists a map* $\|| - \|| : V(\mathbb{A})' \to \mathbb{R}_{>0}$ *(called height), that is compatible with the idele norm, i.e.* $\||a \cdot x\|| = \|a\| \cdot \||x\||$ *for all* $a \in \mathbb{A}^\times$ *and* $x \in V(\mathbb{A})'$.

1.3 Strategy of the proof

This section provides a preview of the most important steps that we take in this work to prove Theorem 1.1.1.

For any place p we choose a maximal compact subgroup K_p in $\mathbf{G}(\mathbb{Q}_p)$ respectively $\mathbf{G}(\mathbb{R})$. For $p \in S$ there is a unique point $*_p \in X_p$, respectively $*_\infty \in X_\infty$, with stabilizer K_p. Next we let $\mathfrak{X}_p$ denote the $\mathbf{G}(\mathbb{Q}_p)$-orbit of $*_p$, respectively the $\mathbf{G}(\mathbb{R})$-orbit of $*_\infty$, i.e.

$$\begin{aligned}\mathfrak{X}_p &= \mathbf{G}(\mathbb{Q}_p) \cdot *_p \cong \mathbf{G}(\mathbb{Q}_p)/K_p; \\ \mathfrak{X}_\infty &= \mathbf{G}(\mathbb{R}) \cdot *_\infty \cong \mathbf{G}(\mathbb{R})/K_\infty.\end{aligned}$$

For non-archimedean places $p \in S$ there are only finitely many $\mathbf{G}(\mathbb{Q}_p)$-orbits of vertices in X_p. Hence there is a uniform upper bound for the distance of any point in X_p to $\mathfrak{X}_p$, i.e. $\mathfrak{X}_p$ and X_p are in finite hausdorff distance.

For the archimedean place $p = \infty$ it is even easier. Because $\mathbf{G}(\mathbb{R})$ acts transitively on X_∞, the orbit space $\mathfrak{X}_\infty$ and X_∞ are not only in finite hausdorff distance, they are even equal.

By definition the compact subgroup $K := \prod_{p \in S} K_p \times \prod_{p \notin S} \mathbf{G}(\mathbb{Z}_p)$ is the stabilizer of $* := \prod_{p \in S} *_p \in X_S$ in $\mathbf{G}(\mathbb{A}_S)$. Now we *pretend* $\mathbf{G}(\mathbb{A}_S)/K_S$ to be X_S: Define

$$\mathfrak{X}_S := \prod_{p \in S} \mathfrak{X}_p = \mathbf{G}(\mathbb{A}_S) \cdot * \cong \mathbf{G}(\mathbb{A}_S)/K_S.$$

Since S is finite, $\mathfrak{X}_S$ and X_S are in finite hausdorff distance.

As seen in Section 1.1, geometric reduction theory on X_S is about Busemann functions on X_S. Thanks to what we have done so far, we may reduce the problem to the subset $\mathfrak{X}_S$: Busemann functions on X_S are determined by their values on $\mathfrak{X}_S$. Hence we may even look for functions on $\mathbf{G}(\mathbb{A}_S)$ that are invariant under multiplication by K from the right. Since the resulting Busemann functions shall be centered at the vertices of Δ we have only little choice:

In Section 2.1 we detect characters $\varpi_\mathfrak{v} : \mathbf{P}_\mathfrak{v} \to \mathrm{GL}_1$ for each $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ and *enlarge* these to so called scaling functions on $\mathbf{G}(\mathbb{A})$. Initially we push them forward $\mathbf{G}(\mathbb{Q})$-invariantly to all remaining vertices of Δ. Afterwards we make them invariant under multiplication by K from the right. It turns out that those scaling functions are restrictions of Busemann functions on X_S centered at the vertices of Δ.

With this family of Busemann functions we prove Theorem 1.1.1. That is where algebraic reduction theory comes into play. Using a duality between the roots $\{\varphi_\mathfrak{v}^\mathfrak{c} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\}$ and the weights $\{\varpi_\mathfrak{v} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\}$ we can make it a problem on roots. Rescaling the Busemann functions such that condition (C) holds, we may apply Theorem 1.2.1 and Theorem 1.2.2. Eventually the existence of a lower reduction bound r is a consequence of Theorem 1.2.1 and the existence of an upper reduction bound R is a consequence of Theorem 1.2.2.

CHAPTER 2

PRELIMINARY

In the coming five sections we do the main work of this thesis. We follow the strategy as presented in Section 1.3, or more precisely we do the following:

In Section 2.1 we establish so called scaling functions $\mathcal{S}_v\colon \mathbf{G}(\mathbb{A}) \to \mathbb{R}$ for each vertex $v \in \mathcal{V}(\Delta)$. These we use to reformulate the main statements of algebraic reduction theory in Section 2.2. In particular a first connection between geometric reduction theory and algebraic reduction theory will become clear. In Section 2.3 we present a set of flat subspaces in X_S that satisfies condition (A). We use these flat subspaces in Section 2.4 to explain that the logarithm of each scaling function restricted to $\mathbf{G}(\mathbb{A}_S)$ is the restriction of some Busemann function on X_S. After modifying those Busemann functions we prove almost analogous but purely geometric statements of Theorem 1.2.1 and Theorem 1.2.2 in Section 2.5.

2.1 Scaling functions, roots and weights

Given a parabolic subgroup $\mathbf{P} \leqslant \mathbf{G}$ and a character $\varpi\colon \mathbf{P} \to \mathrm{GL}_1$ we call a function $\mathcal{S}_\varpi\colon \mathbf{G}(\mathbb{A}) \to \mathbb{R}$ a <u>scaling function for ϖ</u> if

$$\mathcal{S}_\varpi(p \cdot g) = \|\varpi(p)\| \cdot \mathcal{S}_\varpi(g)$$

for all $p \in \mathbf{P}(\mathbb{A})$ and $g \in \mathbf{G}(\mathbb{A})$. Purpose of this section is to prove the existence of non-trivial weights $\varpi_v\colon \mathbf{P}_v \to \mathrm{GL}_1$, almost dual roots $\varphi_v^c\colon \mathbf{P}_c \to \mathrm{GL}_1$ and corresponding scaling functions for all $c \in \mathcal{C}(\Delta)$ and $v \in \mathcal{V}(c)$ where $\mathcal{V}(c)$ denotes the set of vertices of c. Moreover we want the family of scaling functions to be $\mathbf{G}(\mathbb{Q})$-invariant, i.e. $\mathcal{S}_{\varpi_{q\cdot v}}(q \cdot g) = \mathcal{S}_{\varpi_v}(g)$ and $\mathcal{S}_{\varphi_{q\cdot v}^{q\cdot c}}(q \cdot g) = \mathcal{S}_{\varphi_v^c}(g)$ for all $q \in \mathbf{G}(\mathbb{Q})$, $g \in \mathbf{G}(\mathbb{A})$, $c \in \mathcal{C}(\Delta)$ and $v \in \mathcal{V}(c)$. Additionally each scaling function shall be invariant under multiplication by K from the right. In the first subsection we deal with the standard vertices and establish standard roots, weights and scaling functions. During the subsection afterwards we take those standard functions and push them forward $\mathbf{G}(\mathbb{Q})$-invariantly to all remaining vertices of Δ.

Scaling functions related to standard vertices

Fix a standard vertex $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$. Denote by $\mathfrak{p}$ the Lie algebra of $\mathbf{P}_\mathfrak{v}$ and by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$. We consider the $\binom{\dim(\mathfrak{g})}{\dim(\mathfrak{p})}$-dimensional vector space $V_\mathfrak{g} := \wedge^{\dim(\mathfrak{p})}\mathfrak{g}$ and its one-dimensional subspace $V_\mathfrak{p} := \wedge^{\dim(\mathfrak{p})}\mathfrak{p}$. As $\mathbf{G}$ acts on $V_\mathfrak{g}$ via the adjoint representation $(\wedge^{\dim(\mathfrak{p})}\operatorname{Ad}) =: \rho$ so does $\mathbf{P}_\mathfrak{v}$ on $V_\mathfrak{p}$. Because $V_\mathfrak{p}$ is one-dimensional, there is a (non-trivial) weight $\varpi_\mathfrak{v} \colon \mathbf{P}_\mathfrak{v} \to \mathrm{GL}_1$ with

$$\rho(p)(\mathfrak{a}) = \varpi_\mathfrak{v}(p) \cdot \mathfrak{a} \tag{2.1}$$

for all $p \in \mathbf{P}_\mathfrak{v}$ and $\mathfrak{a} \in V_\mathfrak{p}$. Next we proof $\varphi_\mathfrak{v}^\mathfrak{c} \in X(\mathbf{P}_\mathfrak{c})$: We call a root $\lambda \in X(\mathbf{T})$ positive if $\mathfrak{g}_\lambda \subseteq \mathfrak{u}_\mathfrak{c}$. Now $\mathfrak{u}_\mathfrak{c}$ decomposes into $\mathfrak{u}_\mathfrak{c} = \bigoplus_{\lambda\in\Phi} \mathfrak{g}_\lambda$, where Φ is the set of positive roots. Moreover Φ is partially ordered via $\phi \preccurlyeq \psi$ if and only if there exists $\lambda \in \Phi$ with $\psi = \phi + \lambda$. For $\varphi_\mathfrak{v}^\mathfrak{c} \in X(\mathbf{T})$ set

$$\begin{aligned} \Phi' &:= \{\lambda \in \Psi \mid \lambda \succcurlyeq \varphi_\mathfrak{v}^\mathfrak{c}\}\,; \\ \Phi'' &:= \{\lambda \in \Psi \mid \lambda \succ \varphi_\mathfrak{v}^\mathfrak{c}\}\,. \end{aligned}$$

Now $\bigoplus_{\lambda\in\Phi'} \mathfrak{g}_\lambda =: \mathfrak{u}'$ and $\bigoplus_{\lambda\in\Phi''} \mathfrak{g}_\lambda =: \mathfrak{u}''$ are ideals in $\mathfrak{p}_\mathfrak{c}$ and $\varphi_\mathfrak{v}^\mathfrak{c}(p) = \det(\operatorname{Ad}_{\mathfrak{u}'/\mathfrak{u}''})(p) \in X(\mathbf{P}_\mathfrak{c})$. So far we have

$$\begin{aligned} \text{roots} \quad &\{\varphi_\mathfrak{w}^\mathfrak{c} \in X(\mathbf{P}_\mathfrak{c}) \mid \mathfrak{w} \in \mathcal{V}(\mathfrak{c})\} \ \text{ and} \\ \text{weights} \quad &\{\varpi_\mathfrak{w} \in X(\mathbf{P}_\mathfrak{v}) \mid \mathfrak{w} \in \mathcal{V}(\mathfrak{c})\}\,. \end{aligned}$$

We claim those roots and weights to be *almost dual*. To explain what we mean by almost dual we consider $X(\mathbf{P}_\mathfrak{c}) \otimes \mathbb{Q}$ as $\mathbb{Q}$-vector space, i.e. additively, and endow it with an inner product $\langle -, - \rangle$ that is invariant under the action of the Weyl group $W = N(T)/Z(T)$. Depending on the context, we regard $\{\varphi_\mathfrak{v}^\mathfrak{c} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\}$ either as a set of characters or as a set of reflections. Moreover we regard W as generated by those reflections.

Now set $\varpi_\mathfrak{v}^\mathfrak{c} \colon \mathbf{P}_\mathfrak{c} \to \mathrm{GL}_1$ to be the restriction of $\varpi_\mathfrak{v}$ onto $\mathbf{P}_\mathfrak{c}$. Now there are rational numbers $c_{\mathfrak{v}\mathfrak{w}} \in \mathbb{Q}$ and $n_{\mathfrak{v}\mathfrak{w}} \in \mathbb{Q}$ such that

$$\varphi_\mathfrak{v}^\mathfrak{c} = \prod_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} (\varpi_\mathfrak{w}^\mathfrak{c})^{c_{\mathfrak{v}\mathfrak{w}}} \quad \text{and} \quad \varpi_\mathfrak{v}^\mathfrak{c} = \prod_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} (\varphi_\mathfrak{w}^\mathfrak{c})^{n_{\mathfrak{v}\mathfrak{w}}} \tag{2.2}$$

(compare for example [BKW13, 11.4]).

Moreover we have

$$\begin{aligned} 0 &\leqslant n_{\mathfrak{v}\mathfrak{w}} && \text{for all } \mathfrak{v}, \mathfrak{w}; \\ 0 &< n_{\mathfrak{v}\mathfrak{v}} && \text{for all } \mathfrak{v}; \\ 0 &< c_{\mathfrak{v}\mathfrak{v}} && \text{for all } \mathfrak{v}; \\ \langle \varpi_{\mathfrak{w}}^{\mathfrak{c}}, \varphi_{\mathfrak{v}}^{\mathfrak{c}} \rangle &= 0 && \text{if } \mathfrak{w} \neq \mathfrak{v}; \\ \langle \varpi_{\mathfrak{w}}^{\mathfrak{c}}, \varphi_{\mathfrak{v}}^{\mathfrak{c}} \rangle &> 0 && \text{if } \mathfrak{w} = \mathfrak{v}; \\ \langle \varphi_{\mathfrak{w}}^{\mathfrak{c}}, \varphi_{\mathfrak{v}}^{\mathfrak{c}} \rangle &\leqslant 0 && \text{if } \mathfrak{w} \neq \mathfrak{v}; \\ \langle \varpi_{\mathfrak{w}}^{\mathfrak{c}}, \varpi_{\mathfrak{v}}^{\mathfrak{c}} \rangle &\geqslant 0 && \text{if } \mathfrak{w} \neq \mathfrak{v}. \end{aligned} \tag{2.3}$$

That is the reason to call the bases almost dual, see Figure 2.1.

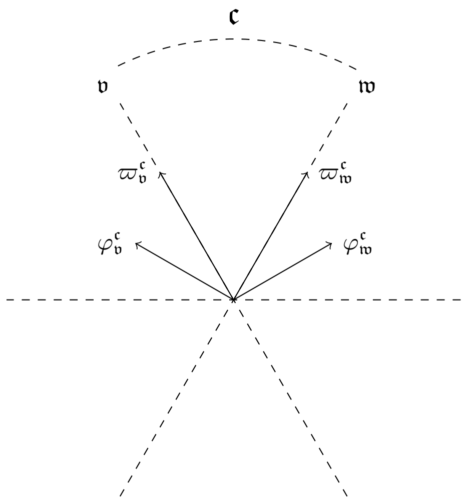


Figure 2.1: Set of simple roots and corresponding set of weights.

To construct scaling functions for each weight, we apply Lemma 1.2.5 and use the fact that $\varpi_{\mathfrak{v}}$ occurs as a weight of some representation: Choose some primitive $x \in V_{\mathfrak{p}}(\mathbb{A})$ and set

$$\begin{aligned} \mathcal{S}^1_{\varpi_{\mathfrak{v}}} \colon \mathbf{G}(\mathbb{A}) &\longrightarrow \mathbb{R}_{>0}, \\ g &\longmapsto \|\!|\rho(g^{-1})(x)|\!\|^{-1}. \end{aligned}$$

Lemma 2.1.1. *$\mathcal{S}^1_{\varpi_{\mathfrak{v}}}$ is a scaling function for $\varpi_{\mathfrak{v}}$.*

Proof. For arbitrary $p \in \mathbf{P}_{\mathfrak{v}}(\mathbb{A})$ and $g \in \mathbf{G}(\mathbb{A})$ we have

$$\begin{aligned} \mathcal{S}^1_{\varpi_{\mathfrak{v}}}(p \cdot g) &= \|\!|\rho(g^{-1} \cdot p^{-1})(x)|\!\|^{-1} \\ &\overset{(2.1)}{=} \|\!|\rho(g^{-1})(\varpi_{\mathfrak{v}}(p^{-1}) \cdot x)|\!\|^{-1} \\ &\overset{1.2.5}{=} \|\varpi_{\mathfrak{v}}(p)\| \cdot \|\!|\rho(g^{-1}) \cdot x|\!\|^{-1} \\ &= \|\varpi_{\mathfrak{v}}(p)\| \cdot \mathcal{S}^1_{\varpi_{\mathfrak{v}}}(g). \end{aligned}$$

□

Up to now $\mathcal{S}^1_{\varpi_\mathfrak{v}}$ does not need to be invariant under right multiplication by K. Therefore we average over K, i.e. we choose some Haar-measure $d\omega$ on $\mathbf{G}(\mathbb{A})$ and set

$$\begin{aligned}\mathcal{S}^2_{\varpi_\mathfrak{v}} : \mathbf{G}(\mathbb{A}) &\longrightarrow \mathbb{R}_+,\\ g &\longmapsto \int_{k\in K} \mathcal{S}^1_{\varpi_\mathfrak{v}}(g\cdot k)\ d\omega.\end{aligned} \tag{2.4}$$

By Lemma 2.1.1 $\mathcal{S}^2_{\varpi_\mathfrak{v}}$ is a scaling function for $\varpi_\mathfrak{v}$ that is invariant under right multiplication by K. However, it turns out that $\mathcal{S}^2_{\varpi_\mathfrak{v}}$ is still not good enough for our purpose. Each $g \in \mathbf{G}(\mathbb{A})$ splits into $g = g_S \cdot g_{\overline{S}}$ with

$$(g_S)_p := \begin{cases} g_p & \text{if } p \in S,\\ 1 & \text{if } p \notin S,\end{cases} \quad \text{and} \quad (g_{\overline{S}})_p := \begin{cases} 1 & \text{if } p \in S,\\ g_p & \text{if } p \notin S.\end{cases}$$

In Section 2.4 we need scaling functions to respect this decomposition, i.e. we need $\mathcal{S}_{\varpi_\mathfrak{v}}(g_S \cdot g_{\overline{S}}) = \mathcal{S}_{\varpi_\mathfrak{v}}(g_S)\cdot\mathcal{S}_{\varpi_\mathfrak{v}}(g_{\overline{S}})$ for all $g \in \mathbf{G}(\mathbb{A})$. Therefore we set

$$\begin{aligned}\mathcal{S}_{\varpi_\mathfrak{v}} : \mathbf{G}(\mathbb{A}) &\longrightarrow \mathbb{R}_+;\\ g &\longmapsto \left(\frac{\mathcal{S}^2_{\varpi_\mathfrak{v}}(g_S)}{\mathcal{S}^2_{\varpi_\mathfrak{v}}(1)}\right)\cdot\left(\frac{\mathcal{S}^2_{\varpi_\mathfrak{v}}(g_{\overline{S}})}{\mathcal{S}^2_{\varpi_\mathfrak{v}}(1)}\right).\end{aligned} \tag{2.5}$$

Theorem 2.1.2. *$\mathcal{S}_{\varpi_\mathfrak{v}}$ satisfies the following conditions:*

(i) $\mathcal{S}_{\varpi_\mathfrak{v}}(p\cdot g) = \|\varpi_\mathfrak{v}(p)\|\cdot\mathcal{S}_{\varpi_\mathfrak{v}}(g)$ *for all* $p \in \mathbf{P}_\mathfrak{v}(\mathbb{A}), g \in \mathbf{G}(\mathbb{A})$*;*

(ii) $\mathcal{S}_{\varpi_\mathfrak{v}}(g\cdot k) = \mathcal{S}_{\varpi_\mathfrak{v}}(g)$ *for all* $g \in \mathbf{G}(\mathbb{A}), k \in K$*;*

(iii) $\mathcal{S}_{\varpi_\mathfrak{v}}(1) = 1$*;*

(iv) $\mathcal{S}_{\varpi_\mathfrak{v}}(p) = \|\varpi_\mathfrak{v}(p)\|$ *for all* $p \in \mathbf{P}_\mathfrak{v}(\mathbb{A})$*;*

(v) $\mathcal{S}_{\varpi_\mathfrak{v}}(g_S\cdot g_{\overline{S}}) = \mathcal{S}_{\varpi_\mathfrak{v}}(g_S)\cdot\mathcal{S}_{\varpi_\mathfrak{v}}(g_{\overline{S}})$ *for all* $g \in \mathbf{G}(\mathbb{A})$*.*

Proof. By Lemma 2.1.1, equation (2.4) and equation (2.5), $\mathcal{S}_{\varpi_\mathfrak{v}}$ clearly satisfies (i), (ii), (iii) and (v). Condition (iv) is a direct consequence of (i) and (iii). □

Next we deal with the roots. Due to equation (2.2) we set

$$\mathcal{S}_{\varphi^\mathfrak{c}_\mathfrak{v}} := \prod_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} (\mathcal{S}_{\varpi_\mathfrak{w}})^{c_{\mathfrak{v}\mathfrak{w}}}. \tag{2.6}$$

We obtain an analogous statement for the standard roots as for the standard weights:

Lemma 2.1.3. $\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}$ *satisfies the following conditions:*

(i) $\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(p \cdot g) = \|\varphi_{\mathfrak{v}}^{\mathfrak{c}}(p)\| \cdot \mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g)$ *for all* $p \in \mathbf{P}_{\mathfrak{v}}(\mathbb{A}), g \in \mathbf{G}(\mathbb{A})$*;*

(ii) $\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g \cdot k) = \mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g)$ *for all* $g \in \mathbf{G}(\mathbb{A}), k \in K$*;*

(iii) $\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(1) = 1$*;*

(iv) $\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(p) = \|\varphi_{\mathfrak{v}}^{\mathfrak{c}}(p)\|$ *for all* $p \in \mathbf{P}_{\mathfrak{v}}(\mathbb{A})$*;*

(v) $\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g_S \cdot g_{\overline{S}}) = \mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g_S) \cdot \mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g_{\overline{S}})$ *for all* $g \in \mathbf{G}(\mathbb{A})$.

Proof. This follows from Theorem 2.1.2 and the equations (2.2) and (2.6). □

Scaling functions related to arbitrary vertices

Until now we have roots, weights and corresponding scaling functions only for standard vertices. The idea to continue is to transfer them $\mathbf{G}(\mathbb{Q})$-invariantly to all remaining vertices of Δ:

$\mathbf{G}(\mathbb{Q})$ acts typepreserving and chamber-transitive on Δ. Hence, given a chamber $c \in \mathcal{C}(\Delta)$ and a vertex $v \in \mathcal{V}(c)$, there exists an element $q \in \mathbf{G}(\mathbb{Q})$ and a unique $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$, such that $c = q \cdot \mathfrak{c}$ (and $\mathbf{P}_c(\mathbb{A}) = q \cdot \mathbf{P}_{\mathfrak{c}}(\mathbb{A}) \cdot q^{-1}$) and $v = q \cdot \mathfrak{v}$. For such quadruple $(c, v, q, \mathfrak{v})$ we define $\varphi_v^c \in X(\mathbf{P}_c)$ via

$$\begin{aligned} \varphi_v^c : \mathbf{P}_c(\mathbb{A}) &\longrightarrow \mathrm{GL}_1(\mathbb{A}), \\ p &\longmapsto \varphi_{\mathfrak{v}}^{\mathfrak{c}}(q^{-1}pq). \end{aligned}$$

To prove that φ_v^c is well defined pick two elements $q, \hat{q} \in \mathbf{G}(\mathbb{Q})$ with $q \cdot \mathfrak{c} = \hat{q} \cdot \mathfrak{c}$, i.e. $q^{-1}\hat{q} \in \mathbf{P}_{\mathfrak{c}}(\mathbb{Q})$ and some $p \in \mathbf{P}_c(\mathbb{Q})$. Now

$$\varphi_{\mathfrak{v}}^{\mathfrak{c}}(q^{-1}pq) = \varphi_{\mathfrak{v}}^{\mathfrak{c}}(\hat{q}^{-1}q) \cdot \varphi_{\mathfrak{v}}^{\mathfrak{c}}(q^{-1}pq) \cdot \varphi_{\mathfrak{v}}^{\mathfrak{c}}(q^{-1}\hat{q}) = \varphi_{\mathfrak{v}}^{\mathfrak{c}}(\hat{q}^{-1}p\hat{q}).$$

Hence φ_v^c is well defined. Accordingly we define

$$\begin{aligned} \varpi_v : \mathbf{P}_v(\mathbb{A}) &\longrightarrow \mathrm{GL}_1(\mathbb{A}), \\ p &\longmapsto \varpi_{\mathfrak{v}}(q^{-1}pq). \end{aligned} \tag{2.7}$$

By an analogous calculation one proves ϖ_v to be well defined. Now set

$$\begin{aligned} \mathcal{S}_{\varpi_v} : \mathbf{G}(\mathbb{A}) &\longrightarrow \mathbb{R}_+ \\ g &\longmapsto \mathcal{S}_{\varpi_{\mathfrak{v}}}(q^{-1} \cdot g). \end{aligned} \tag{2.8}$$

Well-definedness follows obviously from Theorem 2.1.2 (i) and $\|\mathbb{Q}^{\times}\| = 1$.

Lemma 2.1.4. *Given $v \in \mathcal{V}(\Delta)$, $p \in \mathbf{P}_v(\mathbb{A})$, $g \in \mathbf{G}(\mathbb{A})$, $q' \in \mathbf{G}(\mathbb{Q})$ and $k \in K$ the following assertions hold:*

(i) $\mathcal{S}_{\varpi_{q' \cdot v}}(q' \cdot g) = \mathcal{S}_{\varpi_v}(g)$;

(ii) $\mathcal{S}_{\varpi_v}(p \cdot g) = \|\varpi_v(p)\| \cdot \mathcal{S}_{\varpi_v}(g)$;

(iii) $\mathcal{S}_{\varpi_v}(g \cdot k) = \mathcal{S}_{\varpi_v}(g)$.

Proof. Claim (i), i.e. the $\mathbf{G}(\mathbb{Q})$-invariance, holds by definition (2.8). Now claim (ii) results from (2.7) and Theorem 2.1.2 (i). The invariance under right multiplication by K, i.e. claim (iii), follows from Theorem 2.1.2 (ii). □

Analogously to equation (2.6) we define

$$\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}} := \prod_{\substack{w \in \mathcal{V}(c) \\ w = q \cdot \mathfrak{w}}} (\mathcal{S}_{\varpi_w})^{c_{\mathfrak{v}\mathfrak{w}}}. \tag{2.9}$$

Alternatively we could write

$$\begin{aligned} \mathcal{S}_{\varphi_v^c} : \mathbf{G}(\mathbb{A}) &\longrightarrow \mathbb{R}_+, \\ g &\longmapsto \mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(q^{-1} \cdot g). \end{aligned} \tag{2.10}$$

Again we recieve an analogous statement for the roots as for the weights:

Lemma 2.1.5. *Given a chamber $c \in \mathcal{C}(\Delta)$, some vertex $v \in \mathcal{V}(c)$ and arbitrary $p \in \mathbf{P}_v(\mathbb{A})$, $g \in \mathbf{G}(\mathbb{A})$, $q' \in \mathbf{G}(\mathbb{Q})$ and $k \in K$ the following assertions hold:*

(i) $\mathcal{S}_{\varphi_{q' \cdot v}^{q' \cdot c}}(q' \cdot g) = \mathcal{S}_{\varphi_v^c}(g)$;

(ii) $\mathcal{S}_{\varphi_v^c}(p \cdot g) = \|\varphi_v^c(p)\| \cdot \mathcal{S}_{\varphi_v}(g)$;

(iii) $\mathcal{S}_{\varphi_v^c}(g \cdot k) = \mathcal{S}_{\varphi_v^c}(g)$.

We skip the proof because it resembles the proof of Lemma 2.1.4.

2.2 Algebraic reduction theory in terms of scaling functions

In this section we recast Theorem 1.2.1 and Theorem 1.2.2 in terms of scaling functions. Thereafter we prove a compactness criterion. Because we want to phrase these results additively, we apply the logarithm first: For each chamber $c \in \mathcal{C}(\Delta)$ and each vertex $v \in \mathcal{V}(c)$ we consider

$$\log \circ \mathcal{S}_{\varpi_v} : \mathbf{G}(\mathbb{A}) \to \mathbb{R} \text{ and}$$
$$\log \circ \mathcal{S}_{\varphi_v^c} : \mathbf{G}(\mathbb{A}) \to \mathbb{R}.$$

Because $E \subset \mathbf{G}(\mathbb{A})$ is compact the following constants exist:

$$\begin{aligned} c_{\min,1} &:= \min\left\{\log(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(e)), \log(\mathcal{S}_{\varpi_{\mathfrak{v}}}(e)) \mid e \in E,\ \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\}; \\ c_{\max,1} &:= \max\left\{\log(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(e)), \log(\mathcal{S}_{\varpi_{\mathfrak{v}}}(e)) \mid e \in E,\ \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\}. \end{aligned} \tag{2.11}$$

Lemma 2.2.1. *By Lemma 1.2.3* $\mathbf{G}(\mathbb{A}) = \mathbf{P}_{\mathfrak{c}}(\mathbb{Q}) \cdot F \cdot \mathbf{T}_\infty^+ \cdot E$. *Given* $g \in \mathbf{G}(\mathbb{A})$ *with* $g = p \cdot f \cdot t \cdot e$ *the following assertions hold for each* $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ *and all* $c_\mp \in \mathbb{R}$*:*

(i) If $\log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(g)\right) \geqslant c_-$ *we have* $\varpi_{\mathfrak{v}}(t) \geqslant \exp(c_- - c_{\max,1})$.

(ii) If $\log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(g)\right) \leqslant c_+$ *we have* $\varpi_{\mathfrak{v}}(t) \leqslant \exp(c_+ - c_{\min 1,})$.

(iii) If $\log\left(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g)\right) \geqslant c_-$ *we have* $\varphi_{\mathfrak{v}}^{\mathfrak{c}}(t) \geqslant \exp(c_- - c_{\max,1})$.

(iv) If $\log\left(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g)\right) \leqslant c_+$ *we have* $\varphi_{\mathfrak{v}}^{\mathfrak{c}}(t) \leqslant \exp(c_+ - c_{\min,1})$.

Proof. The calculations for (i), (ii) (iii) and (iv) are all analogue. Therefore we only present the first:

$$\begin{aligned} c_- &\leqslant \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(g)\right) \\ &= \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(p \cdot f \cdot t \cdot e)\right) \\ &\overset{2.1.2(i)}{=} \log\left(\|\varpi_{\mathfrak{v}}(p \cdot f \cdot t)\|\right) + \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(e)\right) \\ &\overset{\substack{\|\mathbb{Q}^\times\|=1 \\ F \subset \mathbf{P}(\mathbb{A})^\circ}}{=} \log\left(\|\varpi_{\mathfrak{v}}(t)\|\right) + \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(e)\right) \\ &\leqslant \log\left(\|\varpi_{\mathfrak{v}}(t)\|\right) + c_{\max,1}. \end{aligned}$$

□

Applying Lemma 2.2.1 we may formulate the main theorems of Section 1.2 in terms of scaling functions. In this language Theorem 1.2.1 reads as follows:

Theorem 2.2.2. *There is a constant* $\hat{r} \in \mathbb{R}$ *such that for each* $g \in \mathbf{G}(\mathbb{A})$ *there exists a chamber* $c \in \mathcal{C}(\Delta)$ *with* $\log(\mathcal{S}_{\varphi_v^c}(g)) \geqslant \hat{r}$ *for all* $v \in \mathcal{V}(c)$*.*

Proof. By Theorem 1.2.1 there is a positive real number $C_1 > 0$ such that $\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{Q}) \cdot F \cdot \mathbf{T}_\infty^+(C_1) \cdot E$. With respect to this decomposition we write $g = q \cdot f \cdot t \cdot e$ for each $g \in \mathbf{G}(\mathbb{A})$. For such decomposition choose $c = q \cdot \mathfrak{c}$. Then we have:

$$\begin{aligned} \log(\mathcal{S}_{\varphi_v^c}(g)) &= \log(\mathcal{S}_{\varphi_v^c}(q \cdot f \cdot t \cdot e)) \\ &\overset{2.1.5(i)}{=} \log(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(f \cdot t \cdot e)) \\ &\overset{2.1.3(i)}{=} \log(\|\varphi_{\mathfrak{v}}^{\mathfrak{c}}(t)\|) + \log(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(e)) \\ &\geqslant \log(C_1) + c_{\min,1} \end{aligned}$$

for all $v = q \cdot \mathfrak{v} \in \mathcal{V}(c)$. Now set $\hat{r} := \log(C_1) + c_{\min,1}$. □

Moreover Theorem 1.2.2 translates to:

Theorem 2.2.3. *For any constant* $\hat{r} \in \mathbb{R}$ *there is another constant* $\hat{R} \in \mathbb{R}$ *such that the following holds:*

Let $c, c' \in \mathcal{C}(\Delta)$*,* $v \in \mathcal{V}(c)$ *and* $g \in \mathbf{G}(\mathbb{A})$ *with*

$$\begin{aligned} \log(\mathcal{S}_{\varphi_w^c}(g)) &\geqslant \hat{r} \text{ for all } w \in \mathcal{V}(c); \\ \log(\mathcal{S}_{\varphi_v^c}(g)) &\geqslant \hat{R}; \\ \log(\mathcal{S}_{\varphi_{w'}^{c'}}(g)) &\geqslant \hat{r} \text{ for all } w' \in \mathcal{V}(c'). \end{aligned}$$

Then $v \in \mathcal{V}(c')$*.*

Proof. Let $\hat{r} \in \mathbb{R}$ be an arbitrary constant. Now set $C_1 := \exp(\hat{r} - c_{\max,1})$ and apply Theorem 1.2.2 to obtain a corresponding constant $C_2 \in \mathbb{R}_+$. Next define $\hat{R} := \log(C_2) + c_{\max,1}$.

Now let $c, c' \in \mathcal{C}(\Delta)$, $v \in \mathcal{V}(c)$ and $g \in \mathbf{G}(\mathbb{A})$ satisfy the condition above. Moreover choose $q, q' \in \mathbf{G}(\mathbb{Q})$ and $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ with $c = q \cdot \mathfrak{c}$, $c' = q' \cdot \mathfrak{c}$ and $v = q \cdot \mathfrak{v}$. In that case we have

$$\begin{aligned} \log(\mathcal{S}_{\varphi_{\mathfrak{w}}^{\mathfrak{c}}}(q^{-1} \cdot g)) &\overset{(2.8)}{=} \log(\mathcal{S}_{\varphi_w^c}(g)) \geqslant \hat{r} \text{ for all } w = q \cdot \mathfrak{w} \in \mathcal{V}(c); \\ \log(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(q^{-1} \cdot g)) &\overset{(2.8)}{=} \log(\mathcal{S}_{\varphi_v^c}(g)) \geqslant \hat{R}; \\ \log(\mathcal{S}_{\varphi_{\mathfrak{w}}^{\mathfrak{c}}}(q'^{-1} \cdot g)) &\overset{(2.8)}{=} \log(\mathcal{S}_{\varphi_{w'}^{c'}}(g)) \geqslant \hat{r} \text{ for all } w' = q' \cdot \mathfrak{w} \in \mathcal{V}(c'). \end{aligned} \tag{2.12}$$

By Lemma 1.2.3 we may write

$$q^{-1} \cdot g = p \cdot f \cdot t \cdot e \quad \text{and} \quad q'^{-1} \cdot g = p' \cdot f' \cdot t' \cdot e' \tag{2.13}$$

with $p, p' \in \mathbf{P}_{\mathfrak{c}}(\mathbb{Q})$, $f, f' \in F$, $t, t' \in \mathbf{T}_{\infty}^{+}$ and $e, e' \in E$. We deduce from equation (2.12) and Lemma 2.2.1 that

$$\begin{aligned}
\varphi_{\mathfrak{w}}^{\mathfrak{c}}(t) &\geqslant C_1 \text{ for all } \mathfrak{w} \in \mathcal{V}(\mathfrak{c});\\
\varphi_{\mathfrak{v}}^{\mathfrak{c}}(t) &\geqslant C_2;\\
\varphi_{\mathfrak{w}}^{\mathfrak{c}}(t') &\geqslant C_1 \text{ for all } \mathfrak{w} \in \mathcal{V}(\mathfrak{c})
\end{aligned}$$

and hence $t, t' \in \mathbf{T}_{\infty}^{+}(C_1)$. By equation (2.13) we moreover have

$$\big(p^{-1} \cdot (q^{-1} \cdot q') \cdot p'\big) \cdot f't'e' = fte.$$

Thus Theorem 1.2.2 implies $(p^{-1} \cdot (q^{-1} \cdot q') \cdot p') \in \mathbf{P}_{\mathfrak{v}}(\mathbb{Q})$. Since $p, p' \in \mathbf{P}_{\mathfrak{c}}(\mathbb{Q})$ we even have $(q^{-1} \cdot q') \in \mathbf{P}_{\mathfrak{v}}(\mathbb{Q})$. Therefore

$$v = q \cdot \mathfrak{v} = q \cdot (q^{-1} \cdot q') \cdot \mathfrak{v} = q' \cdot \mathfrak{v} \in \mathcal{V}(q' \cdot \mathfrak{c}) = \mathcal{V}(c').$$

□

Theorem 2.2.4 (Compactness Criterion). *Given a subset $H \subset \mathbf{G}(\mathbb{A})$ the following assertions are equivalent:*

(i) H is relatively compact modulo $\mathbf{G}(\mathbb{Q})$.

(ii) There exist real constants $c_-, c_+ \in \mathbb{R}$ such that for each $h \in H$ there exists a chamber $c \in \mathcal{C}(\Delta)$ with

$$c_- \leqslant \log\big(\mathcal{S}_{\varphi_v^c}(h)\big) \;\; \textit{and} \;\; \log\left(\mathcal{S}_{\varpi_v}(h)\right) \leqslant c_+$$

for all $v \in \mathcal{V}(c)$.

(iii) There exist real constants $\tilde{c}_-, \tilde{c}_+ \in \mathbb{R}$ such that for each $h \in H$ there exists a chamber $c \in \mathcal{C}(\Delta)$ with

$$\tilde{c}_- \leqslant \log\left(\mathcal{S}_{\varpi_v}(h)\right) \leqslant \tilde{c}_+$$

for all $v \in \mathcal{V}(c)$.

(iv) There exist real constants $\hat{c}_-, \hat{c}_+ \in \mathbb{R}$ such that for each $h \in H$ there exists a chamber $c \in \mathcal{C}(\Delta)$ with

$$\hat{c}_- \leqslant \log\big(\mathcal{S}_{\varphi_v^c}(h)\big) \leqslant \hat{c}_+$$

for all $v \in \mathcal{V}(c)$.

Proof. (*i*) ⇒ (*ii*): Let $B \subset \mathbf{G}(\mathbb{A})$ be compact with $H \subset \mathbf{G}(\mathbb{Q}) \cdot B$. Define

$$\begin{aligned} c_- &:= \min\left\{\min\{\log(\mathcal{S}_{\varpi_{\mathfrak{v}}}(b)), \log(\mathcal{S}_{\varphi^{\mathfrak{c}}_{\mathfrak{v}}}(b))\} \mid b \in B,\ \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\}; \\ c_+ &:= \max\left\{\max\{\log(\mathcal{S}_{\varpi_{\mathfrak{v}}}(b)), \log(\mathcal{S}_{\varphi^{\mathfrak{c}}_{\mathfrak{v}}}(b))\} \mid b \in B,\ \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\}. \end{aligned}$$

Given an $h = q \cdot b \in H$ choose $c := q \cdot \mathfrak{c}$. Then we have

$$\begin{aligned} c_- &\leqslant \log(\mathcal{S}_{\varphi^{\mathfrak{c}}_{\mathfrak{v}}}(b)) \overset{2.1.5(i)}{=} \log(\mathcal{S}_{\varphi^{c}_{v}}(h)); \\ c_+ &\geqslant \log(\mathcal{S}_{\varpi_{\mathfrak{v}}}(b)) \overset{2.1.4(i)}{=} \log(\mathcal{S}_{\varpi_{v}}(h)). \end{aligned}$$

(*ii*) ⇒ (*iii*): Let $c_-, c_+ \in \mathbb{R}$ be as in the above theorem and $h \in H$. By assumption there is a chamber $c \in \mathcal{C}(\Delta)$ with

$$c_- \leqslant \log\left(\mathcal{S}_{\varphi^{c}_{v}}(h)\right) \text{ and } \log\left(\mathcal{S}_{\varpi_v}(h)\right) \leqslant c_+$$

for all $v \in \mathcal{V}(c)$. Now choose $q \in \mathbf{G}(\mathbb{Q})$ with $c = q \cdot \mathfrak{c}$ and write $q^{-1} \cdot h = p \cdot f \cdot t \cdot e$ with respect to the decomposition of $\mathbf{G}(\mathbb{A})$ in Lemma 1.2.3. Then we have

$$\begin{aligned} & \log\left(\mathcal{S}_{\varpi_v}(h)\right) \\ \overset{(2.8)}{=} \; & \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(p \cdot f \cdot t \cdot e)\right) \\ \overset{2.1.2(i)}{=} \; & \log(\|\varpi_{\mathfrak{v}}(p \cdot f \cdot t)\|) + \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(e)\right) \\ \overset{(2.2)}{=} \; & \sum_{\mathfrak{w} \in \mathcal{V}(\mathfrak{c})} n_{\mathfrak{v}\mathfrak{w}} \cdot \log\left(\|\varphi^{\mathfrak{c}}_{\mathfrak{w}}(p \cdot f \cdot t)\|\right) + \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(e)\right) \\ \overset{\substack{(2.11) \\ n_{\mathfrak{v}\mathfrak{w}} \geqslant 0}}{\geqslant} \; & \sum_{\mathfrak{w} \in \mathcal{V}(\mathfrak{c})} n_{\mathfrak{v}\mathfrak{w}} \cdot \left(\log(\|\varphi^{\mathfrak{c}}_{\mathfrak{w}}(p \cdot f \cdot t)\|) + \log\left(\mathcal{S}_{\varphi^{\mathfrak{c}}_{\mathfrak{w}}}(e)\right) - c_{\max,1}\right) + c_{\min,1} \\ \overset{\substack{2.1.5 \\ (i),(ii)}}{\geqslant} \; & \sum_{\mathfrak{w} \in \mathcal{V}(\mathfrak{c})} n_{\mathfrak{v}\mathfrak{w}} \cdot \left(\log\left(\mathcal{S}_{\varphi^{q \cdot \mathfrak{c}}_{q \cdot \mathfrak{w}}}(h)\right) - c_{\max,1}\right) + c_{\min,1} \\ \overset{n_{\mathfrak{v}\mathfrak{w}} \geqslant 0}{\geqslant} \; & \sum_{\mathfrak{w} \in \mathcal{V}(\mathfrak{c})} n_{\mathfrak{v}\mathfrak{w}} \cdot (c_- - c_{\max,1}) + c_{\min,1} \\ =: \; & \tilde{c}_-. \end{aligned}$$

Last set $\tilde{c}_+ := c_+$.

(*iii*) ⇒ (*iv*): Let $\tilde{c}_-, \tilde{c}_+ \in \mathbb{R}$ be as in the theorem and $h \in H$. By assumption there is a chamber $c \in \mathcal{C}(\Delta)$ with

$$\tilde{c}_- \leqslant \log(\mathcal{S}_{\varpi_v}(h)) \leqslant \tilde{c}_+$$

for all $v \in \mathcal{V}(c)$. Now choose $q \in \mathbf{G}(\mathbb{Q})$ with $c = q \cdot \mathfrak{c}$. For $v = q \cdot \mathfrak{v}$ we have

$$\begin{aligned}
&\log\left(\mathcal{S}_{\varphi_v^c}(h)\right) \\
\overset{2.1.5(i)}{=}\; &\log\left(\mathcal{S}_{\varphi_\mathfrak{v}^\mathfrak{c}}(q^{-1}\cdot h)\right) \\
\overset{(2.6)}{=}\; &\sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} c_{\mathfrak{vw}} \cdot \log\left(\mathcal{S}_{\varpi_\mathfrak{w}}(q^{-1}\cdot h)\right) \\
=\; &\sum_{\substack{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})\\ c_{\mathfrak{vw}}\geqslant 0}} c_{\mathfrak{vw}} \cdot \log\left(\mathcal{S}_{\varpi_\mathfrak{w}}(q^{-1}\cdot h)\right) + \sum_{\substack{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})\\ c_{\mathfrak{vw}}< 0}} c_{\mathfrak{vw}} \cdot \log\left(\mathcal{S}_{\varpi_\mathfrak{w}}(q^{-1}\cdot h)\right).
\end{aligned}$$

Using the condition we conclude that

$$\begin{aligned}
\hat{c}_- \;:=\; &\sum_{\substack{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})\\ c_{\mathfrak{vw}}\geqslant 0}} c_{\mathfrak{vw}} \cdot \tilde{c}_- + \sum_{\substack{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})\\ c_{\mathfrak{vw}}< 0}} c_{\mathfrak{vw}} \cdot \tilde{c}_+ \\
\leqslant\; &\log\left(\mathcal{S}_{\varphi_v^c}(h)\right) \\
\leqslant\; &\sum_{\substack{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})\\ c_{\mathfrak{vw}}\geqslant 0}} c_{\mathfrak{vw}} \cdot \tilde{c}_+ + \sum_{\substack{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})\\ c_{\mathfrak{vw}}< 0}} c_{\mathfrak{vw}} \cdot \tilde{c}_- \\
=:\; &\hat{c}_+
\end{aligned}$$

for all $v \in \mathcal{V}(c)$.

(iv) $\Rightarrow$ *(i)*: Given $\hat{c}_-, \hat{c}_+ \in \mathbb{R}$ as in the theorem we define

$$B := \left\{ t \in \mathbf{T}_\infty^+ \;\middle|\; \begin{array}{c} \exp(\hat{c}_- - c_{\max,1}) \leqslant \varphi_\mathfrak{v}^\mathfrak{c}(t) \leqslant \exp(\hat{c}_+ - c_{\min,1}) \\ \text{for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c}) \end{array} \right\}.$$

The set B is compact because $\mathbf{T}_\infty^+$ is of $\mathbb{Q}$-rank $|\mathcal{V}(\mathfrak{c})|$. We show that H is contained in $\mathbf{G}(\mathbb{Q}) \cdot F \cdot B \cdot E$:

Let $h \in H$. By assumption there is a chamber $c = q \cdot \mathfrak{c} \in \mathcal{V}(\Delta)$ with

$$\hat{c}_- \leqslant \log\left(\mathcal{S}_{\varphi_v^c}\right)(h) \overset{(2.8)}{=} \log\left(\mathcal{S}_{\varphi_\mathfrak{v}^\mathfrak{c}}\right)(q^{-1}\cdot h) \leqslant \hat{c}_+$$

for all $v \in \mathcal{V}(c)$. With respect to the segmentation of $\mathbf{G}(\mathbb{A})$ in Lemma 1.2.3 we write $q^{-1} \cdot h = p \cdot f \cdot t \cdot e$ and conclude from Lemma 2.2.1 that

$$\exp(\hat{c}_- - c_{\max,1}) \leqslant \varphi_\mathfrak{v}^\mathfrak{c}(t) \leqslant \exp(\hat{c}_+ - c_{\min,1})$$

for all $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$, i.e. $t \in B$. Hence

$$h = q \cdot p \cdot f \cdot t \cdot e \in \mathbf{G}(\mathbb{Q}) \cdot F \cdot B \cdot E.$$

Since E and F are compact the proof is done. □

2.3 The set of apartments

In this section we search for a family $\mathcal{A}$ of flat subspaces in $X_S = \prod_{p \in S} X_p$ that satisfies condition (A). We deal with this problem for the archimedean place and the non-archimedean places individually. Finally we prove $\mathcal{A}$ to serve as an appropriate substitution of an apartment system on X_S. This is going to be a decisive step to show that condition (B) is true.

Theorem 2.3.1 ([Bor69, Theorem 11.4]). *Let $\mathbf{U}$ denote the unipotent radical of $\mathbf{P}_\mathfrak{c}$ and $Z(\mathbf{T})$ the centralizer of $\mathbf{T}$ in $\mathbf{G}$. Then $\mathbf{P}_\mathfrak{c}$ can be written as semi-direct product $\mathbf{P}_\mathfrak{c} = \mathbf{U} \rtimes Z(\mathbf{T})$. If $\mathbf{M}$ denotes the connected component of $\left(\bigcap_{\chi \in X(Z(\mathbf{T}))} \ker \chi\right)$, we moreover have $Z(\mathbf{T}) = \mathbf{M} \cdot \mathbf{T}$. Finally $\mathbf{P}_\mathfrak{c} = \mathbf{U} \rtimes (\mathbf{M} \cdot \mathbf{T})$.*

Given a non-archimedean place $p \in S$, the space X_p is a building and thus carries an apartment system $\mathcal{A}_p$. To rewrite $\mathcal{A}_p$ let Σ_p denote the apartment corresponding to the maximal $\mathbb{Q}_p$-split torus $\mathbf{T}_p$ in $\mathbf{P}_\mathfrak{c}$. We assume $*_p \in \Sigma_p$ without loss of generality. Since $\mathbf{G}(\mathbb{Q}_p)$ acts strongly transitive on X_p the translates of Σ_p under the action of $\mathbf{P}_\mathfrak{c}(\mathbb{Q}_p)$ cover X_p. Applying Theorem 2.3.1 we even know that the translates of Σ_p under the action of $\mathbf{U}(\mathbb{Q}_p)\mathbf{M}(\mathbb{Q}_p)$ cover X_p, since $\mathbf{T}(\mathbb{Q}_p)$, as a subgroup of $\mathbf{T}_p(\mathbb{Q}_p)$, stabilizes Σ_p. For simplicity we denote this set of translates by $\mathcal{A}_p^\mathfrak{c}$ (compare [BKW13, p. 42]). Now $\mathcal{A}_p = \bigcup_{q \in \mathbf{G}(\mathbb{Q})} q \cdot \mathcal{A}_p^\mathfrak{c}$.

We want to deal with the archimedean case similarly. However X_∞ is no building, it carries no apartment system, $\mathbf{G}(\mathbb{R})$ does not act strongly transitive. In order to obtain a substitution we need the following lemma:

Lemma 2.3.2 (Corollary of [BS73, Proposition 1.5]). *The following assertions hold:*

(i) $\mathbf{P}_\mathfrak{c}(\mathbb{R})$ acts transitively on X_∞.

(ii) Let $x_\infty = u_\infty m_\infty t_\infty \cdot *_\infty \in X_\infty$ with $u_\infty \in \mathbf{U}(\mathbb{R})$, $m_\infty \in \mathbf{M}(\mathbb{R})$ and $t_\infty \in \mathbf{T}(\mathbb{R})^\circ$. Then t_∞ is unique.

Now set $\Sigma_\infty := \mathbf{T}(\mathbb{R})^\circ \cdot *_\infty$. It is flat and totally geodesic in X_∞, provided $\mathfrak{k}_\infty \perp \mathfrak{p}_\mathfrak{c}$. Due to Lemma 2.3.2 (i) its $\mathbf{U}(\mathbb{R})\mathbf{M}(\mathbb{R})$-translates cover X_∞. Therefore we define

$$\mathcal{A}_\infty^\mathfrak{c} := \{u_\infty m_\infty \cdot \Sigma_\infty \mid u_\infty \in \mathbf{U}(\mathbb{R}),\ m_\infty \in \mathbf{M}(\mathbb{R})\}.$$

The idea is to use $\mathcal{A}_\infty^\mathfrak{c}$ as an archimedean analogue of $\mathcal{A}_p^\mathfrak{c}$. Therefore we set $\mathcal{A}_\infty := \cup_{q \in \mathbf{G}(\mathbb{Q})} q \cdot \mathcal{A}_\infty^\mathfrak{c}$. Slightly abusing notation we call its elements archimedean apartments.

So far we have an apartment system on each factor of X_S. To obtain an apartment system on X_S we define the standard apartment $\Sigma := \prod_{p \in S} \Sigma_p$, obtain $* \in \Sigma$, and set

$$\mathcal{A}^{\mathfrak{c}} := \{um \cdot \Sigma \mid u \in \mathbf{U}(\mathbb{A}_S), m \in \mathbf{M}(\mathbb{A}_S)\}\,. \tag{2.14}$$

For an arbitrary chamber $c = q \cdot \mathfrak{c} \in \mathcal{C}(\Delta)$ we set $\mathcal{A}^c := q \cdot \mathcal{A}^{\mathfrak{c}}$. We remark that for any chamber $c \in \mathcal{C}(\Delta)$ the flat subspaces of $\mathcal{A}^c$ cover X_S. Now set

$$\mathcal{A} := \bigcup_{c \in \mathcal{C}(\Delta)} \mathcal{A}^c. \tag{2.15}$$

Up to now we did not embed Δ into $\partial(X_S)$. Hence we can not say $c \in \mathcal{C}(\Delta)$ to be in the visual boundary of any apartment in $\mathcal{A}^c$. Though, in Corollary 2.4.8, we embed Δ into $\partial(X_S)$ such that $\mathcal{A}^c$ contains only apartments that contain c in their visual boundary. This is going to prove that condition (A) holds.

Theorem 2.3.3. *Given a chamber $c \in \mathcal{C}(\Delta)$ and two apartments $\Sigma', \Sigma'' \in \mathcal{A}^c$ there is an isometry $i : \Sigma' \to \Sigma''$ fixing the intersection.*

Proof. The apartments Σ' and Σ'' are products of flat subspaces, i.e. $\Sigma' = \prod_{p \in S} \Sigma'_p$ and $\Sigma'' = \prod_{p \in S} \Sigma''_p$. Therefore we may consider this problem for the archimedean place and the non-archimedean places individually:

For a non-archimedean place p the space X_p is a building and the subspaces Σ'_p and Σ''_p are apartments therein. Hence there is an isometry of coxeter complexes $i_p : \Sigma'_p \to \Sigma''_p$ fixing the intersection.

For the archimedean place ∞ the space X_∞ is no building. Therefore there is no abstract isometry from Σ'_∞ to Σ''_∞ just by definition. Though, by definition of $\mathcal{A}^c$, there are elements $q \in \mathbf{G}(\mathbb{Q})$ and $u'_\infty m'_\infty$, $u''_\infty m''_\infty \in \mathbf{U}(\mathbb{R})\mathbf{M}(\mathbb{R})$ such that

$$\begin{aligned} \Sigma'_\infty &= q \cdot u'_\infty m'_\infty \cdot \Sigma_\infty \text{ and} \\ \Sigma''_\infty &= q \cdot u''_\infty m''_\infty \cdot \Sigma_\infty. \end{aligned}$$

Thus $q \cdot u_\infty m_\infty \cdot q^{-1} := q \cdot (u''_\infty m''_\infty)(u'_\infty m'_\infty)^{-1} \cdot q^{-1}$ identifies Σ'_∞ with Σ''_∞ isometrically. Moreover it fixes the intersection: Given $q \cdot u'_\infty m'_\infty \cdot (t'_\infty \cdot *_\infty) = q \cdot u''_\infty m''_\infty \cdot (t''_\infty \cdot *_\infty) \in \Sigma'_\infty \cap \Sigma''_\infty$ we know by Lemma 2.3.2 (ii) that $t'_\infty = t''_\infty$. Hence we have

$$\begin{aligned} q \cdot u_\infty m_\infty \cdot q^{-1} \cdot q \cdot u'_\infty m'_\infty \cdot (t'_\infty \cdot *_\infty) &= q \cdot u''_\infty m''_\infty \cdot (t'_\infty \cdot *_\infty) \\ &= q \cdot u''_\infty m''_\infty \cdot (t''_\infty \cdot *_\infty) \\ &= q \cdot u'_\infty m'_\infty \cdot (t'_\infty \cdot *_\infty). \end{aligned}$$

□

2.4 Busemann functions

In this section we construct a family of Γ-invariant rescaled Busemann functions on X_S, which is in one-to-one correspondence with the set of vertices $\mathcal{V}(\Delta)$, that is closely related to the family of scaling functions discussed in Section 2.1. Moreover we embed Δ isometrically into the visual boundary of X_S such that each vertex $v \in \mathcal{V}(\Delta)$ is the center of some Busemann function and vice versa. The basic idea of the approach is the following:

In Section 1.3 we have seen that $\mathfrak{X}_S$ and X_S are in finite hausdorff-distance. Hence Busemann functions on X_S are determined by their values on $\mathfrak{X}_S$. Moreover $\mathfrak{X}_S \cong \mathbf{G}(\mathbb{A}_S)/K$. For this reason we have established a family of scaling functions $\{\mathcal{S}_{\varpi_v} : \mathbf{G}(\mathbb{A}) \to \mathbb{R}_+ \mid v \in \mathcal{V}(\Delta)\}$ in Section 2.1, such that each is invariant under multiplication by K from the right. In the following subsection we are going to prove the standard scaling functions to be restrictions of Busemann functions centered at the standard vertices. During the subsection thereafter we deal with all remaining vertices. It turns out that the family of Busemann functions, in contrast to the family of scaling functions, is Γ-invariant, but not $\mathbf{G}(\mathbb{Q})$-invariant.

Busemann functions related to standard vertices

For each standard vertex $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ define

$$\begin{aligned} \overline{\log \circ \mathcal{S}_{\varpi_{\mathfrak{v}}}} \colon \mathbf{G}(\mathbb{A}_S)/K &\longrightarrow \mathbb{R}, \\ g \cdot K &\longmapsto \log(\mathcal{S}_{\varpi_{\mathfrak{v}}}(g)). \end{aligned} \tag{2.16}$$

Lemma 2.4.1 (compare [BKW13, Lemma 12.1]). *Given a standard vertex $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ there exists an affine function $\overline{h_{\mathfrak{v}}} \colon \Sigma \to \mathbb{R}$ that agrees with $\overline{\log \circ \mathcal{S}_{\mathfrak{v}}}$ on $\Sigma \cap \mathfrak{X}_S$.*

Proof. First we remark that

$$\begin{aligned} \Sigma \cap \mathfrak{X}_S &= \Sigma_\infty \cap \mathfrak{X}_\infty \times \prod_{p \in S\setminus\{\infty\}} \Sigma_p \cap \mathfrak{X}_p \\ &= \mathbf{T}(\mathbb{R}) \cdot *_\infty \times \prod_{p \in S\setminus\{\infty\}} \mathbf{T}_p(\mathbb{Q}_p) \cdot *_p \\ &= \left(\mathbf{T}(\mathbb{R}) \times \prod_{p \in S\setminus\{\infty\}} \mathbf{T}_p(\mathbb{Q}_p) \right) \cdot *. \end{aligned}$$

Now let $p \in \mathbf{P}_{\mathfrak{c}}(\mathbb{A}_S)$ be arbitrary. For each $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ we obtain

$$\overline{\log \circ \mathcal{S}_{\varpi_{\mathfrak{v}}}}(p \cdot K) \overset{(2.16)}{=} \log(\mathcal{S}_{\varpi_{\mathfrak{v}}}(p)) \overset{2.1.2(iv)}{=} \log(\|\varpi_{\mathfrak{v}}(p)\|). \tag{2.17}$$

Considering this statement just for $p = (t_p)_{p\in S} \in \mathbf{T}(\mathbb{R}) \times \prod_{p\in S\setminus\{\infty\}} \mathbf{T}_p(\mathbb{Q}_p)$ the claim follows. □

Affine functions are restrictions of Busemann functions up to rescaling, hence there is a geodesic, possibly of non-unit speed, defining $\overline{h}_{\mathfrak{v}}$. Furthermore Σ is flat in X_S, and hence geodesics in Σ are geodesics in X_S. Therefore we may consider $\overline{h}_{\mathfrak{v}}$ as a rescaled Busemann function on X_S.

Theorem 2.4.2. *$\overline{h}_{\mathfrak{v}} \colon X_S \to \mathbb{R}$ agrees with $\overline{\log \circ \mathcal{S}_{\varpi_{\mathfrak{v}}}}$ on $\mathfrak{X}_S$.*

Proof. By Section 2.3 we know that X_S is coverd by $\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$-translates of Σ. Hence

$$\mathfrak{X}_S = \mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S) \cdot (\Sigma \cap \mathfrak{X}_S)\,.$$

By Theorem 2.1.2 (i) and the definition of $\mathbf{U}$ and $\mathbf{M}$ we know that $\mathcal{S}_{\varpi_{\mathfrak{v}}}$ is invariant under left multiplication by $\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$ and hence, by (2.16),

$$\overline{\log \circ \mathcal{S}_{\varpi_{\mathfrak{v}}}}(um \cdot g \cdot K) = \overline{\log \circ \mathcal{S}_{\varpi_{\mathfrak{v}}}}(g \cdot K), \tag{2.18}$$

for all $um \in \mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$ and all $g \in \mathbf{G}(\mathbb{A}_S)$. If we can prove the rescaled Busemann function $\overline{h}_{\mathfrak{v}}$ to be invariant under left multiplication by $\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$ as well, the claim follows from Lemma 2.4.1.

Let $um \in \mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$ be arbitrary. By (2.17) the center of $\overline{h}_{\mathfrak{v}}$ coincides with the visual endpoint of the weight $\varpi_{\mathfrak{v}}$. As such it is fixed by $\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$. Since Busemann functions are determined by their center and the value of one point it suffices to prove equality of $\overline{h}_{\mathfrak{v}}(-)$ and $\overline{h}_{\mathfrak{v}}(um-)$ at one point. To determine such point we remark that $X_S = \prod_{p\in S} X_p$ is a product of CAT(0)-spaces. Therefore the rescaled Busemann function $\overline{h}_{\mathfrak{v}}$ decomposes into a sum of rescaled Busemann functions, i.e. $\overline{h}_{\mathfrak{v}} = \sum_{p\in S} \overline{h}_{\mathfrak{v},p}$. For the archimedean place we have $\overline{h}_{\mathfrak{v},\infty}(x_\infty) = \overline{h}_{\mathfrak{v},\infty}(u_\infty m_\infty \cdot x_\infty)$ for all $x_\infty \in X_\infty$ by [Leu95, Lemma 1.3]. Considering a non-archimedean place $p \neq \infty$ we know that Σ_p and $u_p m_p \cdot \Sigma_p$ are apartments in X_p that share a sector. Hence there exists a point $\mathfrak{a}_p \in \Sigma_p \cap \mathfrak{X}_p$ with $u_p m_p \cdot \mathfrak{a}_p \in (\Sigma_p \cap u_p m_p \cdot \Sigma_p \cap \mathfrak{X}_p)$. Now define $\overline{um} \in \mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$ and $\mathfrak{a} \in \mathfrak{X}_S$ by

$$\overline{u_p m_p} = \begin{cases} 1 & \text{if } p = \infty, \\ u_p m_p & \text{if } p \neq \infty, \end{cases} \qquad \text{and} \quad \mathfrak{a}_p = \begin{cases} *_\infty & \text{if } p = \infty, \\ \mathfrak{a}_p & \text{if } p \neq \infty. \end{cases}$$

We conclude that

$$\begin{aligned}
\overline{h}_{\mathfrak{v}}(um\cdot\mathfrak{a}) &= \sum_{p\in S}\overline{h}_{\mathfrak{v},p}(u_pm_p\cdot\mathfrak{a}_p)\\
&= \overline{h}_{\mathfrak{v},\infty}(u_\infty m_\infty\cdot\mathfrak{a}_\infty)+\sum_{p\in S\setminus\{\infty\}}\overline{h}_{\mathfrak{v},p}(u_pm_p\cdot\mathfrak{a}_p)\\
&= \overline{h}_{\mathfrak{v},\infty}(\mathfrak{a}_\infty)+\sum_{p\in S\setminus\{\infty\}}\overline{h}_{\mathfrak{v},p}(u_pm_p\cdot\mathfrak{a}_p)\\
&= \overline{h}_{\mathfrak{v}}(\overline{um}\cdot\mathfrak{a})\\
&\overset{\overline{um}\cdot\mathfrak{a}\in\Sigma\cap\mathfrak{X}_S}{\underset{2.4.1}{=}} \overline{\log\circ\mathcal{S}_{\varpi_{\mathfrak{v}}}}(\overline{um}\cdot\mathfrak{a})\\
&\overset{(2.18)}{=} \overline{\log\circ\mathcal{S}_{\varpi_{\mathfrak{v}}}}(\mathfrak{a})\\
&\overset{\mathfrak{a}\in\Sigma\cap\mathfrak{X}_S}{\underset{2.4.1}{=}} \overline{h}_{\mathfrak{v}}(\mathfrak{a}).
\end{aligned}$$

□

Busemann functions related to arbitrary vertices

In this subsection we want to create a family of Busemann functions such that *each* rational vertex is the center of one and only one Busemann function. Moreover this family shall be closely related to the family of scaling functions discussed in Section 2.1.

At first glance it seems plausible to use the chamber transitive action of $\mathbf{G}(\mathbb{Q})$ on Δ to transfer the standard Busemann functions $\mathbf{G}(\mathbb{Q})$-invariantly to all remaining vertices. Though a second glance reveals this procedure to cause troubles: To analyze a potential connection between a family of Busemann functions on X_S and the family of scaling functions on $\mathbf{G}(\mathbb{A})$, we would like to use $\mathbf{G}(\mathbb{A}_S)/K \cong \mathfrak{X}_S$ evenly distributed in X_S. Now $\mathbf{G}(\mathbb{Q}) \nless \mathbf{G}(\mathbb{A}_S)$. Hence the best we can do is to use $\mathbf{G}(\mathbb{Q})\cap\mathbf{G}(\mathbb{A}_S) = \Gamma$ to transfer the standard Busemann functions. Now another problem occurs. Transferring the standard vertices using Γ only, we might not reach all rational vertices. However, by Lemma 1.2.4, there is a finite set of representatives $\{q_1,\dots,q_m\}$ for the set of double cosets $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}_{\mathfrak{c}}(\mathbb{Q})$. Since $\mathbf{P}_{\mathfrak{c}}(\mathbb{Q})$ fixes $\mathfrak{c}$ we therefore have for each vertex $v\in\mathcal{V}(\Delta)$ some $\gamma\in\Gamma$, some $q_i\in\{q_1,\dots,q_m\}$ and a unique $\mathfrak{v}\in\mathcal{V}(\mathfrak{c})$ such that $v=\gamma\cdot q_i\cdot\mathfrak{v}$. Thus we define

$$\overline{h}_{\gamma,q_i,\mathfrak{v}}(x) := \overline{h}_{\mathfrak{v}}\left(q_i^{-1}\cdot\gamma^{-1}\cdot x\right) = \overline{h}_{\mathfrak{v}}\left((q_i^{-1}\cdot\gamma^{-1})_S\cdot x\right) \tag{2.19}$$

for all $x\in X_S$. This definition causes $\overline{h}_{\gamma,q_i,\mathfrak{v}}$ to be centered at the visual endpoint of $\varpi_{\gamma\cdot q_i\cdot\mathfrak{v}}$ and the family $\left\{\overline{h}_{\gamma,q_i,\mathfrak{v}}\;\middle|\;\gamma\in\Gamma, q_i\in\{q_1,\dots,q_m\},\mathfrak{v}\in\mathcal{V}(\mathfrak{c})\right\}$

to be Γ-invariant in the sense that

$$\overline{h}_{\gamma'\cdot\gamma,q_i,\mathfrak{v}}(\gamma'\cdot x) = \overline{h}_{\gamma,q_i,\mathfrak{v}}(x) \tag{2.20}$$

for all $\gamma, \gamma' \in \Gamma$, $q_i \in \{q_1, ..., q_m\}$, $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ and $x \in X_S$. Now we encounter a problem: Given two descriptions $\gamma' \cdot q_i \cdot \mathfrak{v} = \gamma \cdot q_j \cdot \mathfrak{v}$ of the same vertex, the rescaled Busemann functions $\overline{h}_{\gamma',q_i,\mathfrak{v}}$ and $\overline{h}_{\gamma,q_j,\mathfrak{v}}$ have the same speed and a common center but do not need to coincide:

Lemma 2.4.3. *Let $\gamma, \gamma' \in \Gamma$, $q_i, q_j \in \{q_1, ..., q_m\}$ and $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ such that $\gamma' \cdot q_i \cdot \mathfrak{v} = \gamma \cdot q_j \cdot \mathfrak{v}$. Then*

$$\overline{h}_{\gamma',q_i,\mathfrak{v}} - \overline{h}_{\gamma,q_j,\mathfrak{v}} = \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right).$$

Proof. Since Busemann functions are determined by their center and the value of one point, it suffices to prove the difference at $* \in \mathfrak{X}_S$:

$$\begin{aligned}
&\overline{h}_{\gamma',q_i,\mathfrak{v}}(*)\\
\overset{(2.19)}{=}\;& \overline{h}_{\mathfrak{v}}\left((q_i^{-1}\gamma'^{-1})_S \cdot *\right)\\
\overset{2.4.2}{=}\;& \overline{\log\circ\mathcal{S}_{\varpi_{\mathfrak{v}}}}\left((q_i^{-1}\gamma'^{-1})_S \cdot K\right)\\
\overset{(2.16)}{=}\;& \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}\left((q_i^{-1}\gamma'^{-1})_S\right)\right)\\
\overset{\|\mathbb{Q}^\times\|=1}{=}\;& \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}\left((q_i^{-1}\gamma'^{-1})_S\right)\right) + \log\left(\|\varpi_{\mathfrak{v}}(q_j^{-1}\gamma^{-1}\gamma' q_i)\|\right)\\
=\;& \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}\left((q_i^{-1}\gamma'^{-1})_S\right)\right) + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_S)\|\right)\\
&\qquad + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right)\\
\overset{2.1.2(i)}{=}\;& \log(\mathcal{S}_{\varpi_{\mathfrak{v}}}((q_j^{-1}\gamma^{-1})_S) + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right)\\
\overset{(2.16)}{=}\;& \overline{\log\circ\mathcal{S}_{\varpi_{\mathfrak{v}}}}((q_j^{-1}\gamma^{-1})_S \cdot K) + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right)\\
\overset{2.4.2}{=}\;& \overline{h}_{\mathfrak{v}}((q_j^{-1}\gamma^{-1})_S \cdot *) + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right)\\
\overset{(2.19)}{=}\;& \overline{h}_{\gamma,q_j,\mathfrak{v}}(*) + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right).
\end{aligned} \tag{2.21}$$

□

Fortunately we may prove that the constant $\log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right)$ does not depend on a particular choice of γ respectively γ', i.e. given $\gamma_1, \gamma_2, \gamma \in \Gamma$, $q_i, q_j \in \{q_1, ..., q_m\}$ and $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ with $\gamma_1 \cdot q_i \cdot \mathfrak{v} = \gamma_2 \cdot q_i \cdot \mathfrak{v} = \gamma \cdot q_j \cdot \mathfrak{v}$ we have

$$\log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma_1 q_i)_{\overline{S}})\|\right) = \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma_2 q_i)_{\overline{S}})\|\right). \tag{2.22}$$

To prove equation (2.22) we need the following lemma. Therefore we remark that

$$\begin{aligned}
\mathbb{Z}_S &= \{q \in \mathbb{Q} \mid |q|_p \leqslant 1 \text{ for all } p \notin S\};\\
\mathbb{Z}_S^\times &= \{q \in \mathbb{Q} \mid |q|_p = 1 \text{ for all } p \notin S\}.
\end{aligned}$$

Lemma 2.4.4. *Let* $\mathbf{H}$ *be a linear algebraic group,* $\varpi \in X(\mathbf{H})$ *and* Γ' *an* S*-arithmetic lattice in* $\mathbf{H}$*. For* $\gamma \in \Gamma'$ *we have* $\|(\varpi(\gamma))_{\overline{S}}\| = \|(\varpi(\gamma))_S\| = 1$.

Proof. Since $\gamma \in \Gamma'$ we have $\varpi(\gamma)^n \in \mathbb{Z}_S^\times$ for some $n \in \mathbb{N}$. Therefore

$$\|(\varpi(\gamma))_{\overline{S}}\| = \prod_{p \notin S} |\varpi(\gamma)|_p = 1.$$

Moreover we have

$$\begin{aligned} \|(\varpi(\gamma))_S\| &\overset{\|\mathbb{Q}^\times\|=1}{=} \|(\varpi(\gamma))_S\| \cdot \|\varpi(\gamma)\|^{-1} \\ &= \|(\varpi(\gamma))_S\| \cdot \|(\varpi(\gamma))_S\|^{-1} \cdot \|(\varpi(\gamma))_{\overline{S}}\|^{-1} \\ &= \|(\varpi(\gamma))_{\overline{S}}^{-1}\| \\ &= 1. \end{aligned}$$

□

Corollary 2.4.5. *Given* $\gamma_1, \gamma_2, \gamma \in \Gamma,\ q_i, q_j \in \{q_1, ..., q_m\}$ *and* $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ *with* $\gamma_1 \cdot q_i \cdot \mathfrak{v} = \gamma_2 \cdot q_i \cdot \mathfrak{v} = \gamma \cdot q_j \cdot \mathfrak{v}$ *we have*

$$\log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma_1 q_i)_{\overline{S}})\|\right) = \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma_2 q_i)_{\overline{S}})\|\right),$$

i.e. equation (2.22) *holds.*

Proof. By assumption $\gamma_1^{-1}\gamma_2 \in \mathbf{P}_{q_i\cdot\mathfrak{v}}(\mathcal{O}_S)$ and hence we deduce from Lemma 2.4.4 that

$$\begin{aligned} &\log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma_1 q_i)_{\overline{S}})\|\right) \\ \overset{2.4.4}{=}\ &\log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma_1 q_i)_{\overline{S}})\|\right) + \log(\|\varpi_{q_i\cdot\mathfrak{v}}((\gamma_1^{-1}\gamma_2)_{\overline{S}})\|) \\ \overset{(2.7)}{=}\ &\log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma_1 q_i)_{\overline{S}})\|\right) + \log(\|\varpi_{\mathfrak{v}}((q_i^{-1}\gamma_1^{-1}\gamma_2 q_i)_{\overline{S}})\|) \\ =\ &\log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma_2 q_i)_{\overline{S}})\|\right). \end{aligned}$$

□

Now we may adjust the 0-level of all Busemann functions centered at the same vertex to one another. We do the same trick as Leuzinger did in [Leu95, Section 3], i.e. we choose a representative in each Γ-conjugacy class of maximal parabolic $\mathbb{Q}$-subgroups.

Given $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ and $i \in \{1, ..., m\}$ we define $k(i,\mathfrak{v}) \in \{1, ..., m\}$ to be the *smallest* index such that there exists some $\gamma_{i,\mathfrak{v}} \in \Gamma$ with $q_i \cdot \mathfrak{v} = \gamma_{i,\mathfrak{v}} q_{k(i,\mathfrak{v})} \cdot \mathfrak{v}$. By Corollary 2.4.5 we may set

$$s(i,\mathfrak{v}) := \log\left(\|\varpi_{\mathfrak{v}}((q_i^{-1} \cdot \gamma_{i,\mathfrak{v}} \cdot q_{k(i,\mathfrak{v})})_{\overline{S}})\|\right). \tag{2.23}$$

It turns out that it suffices to adapt the 0-level of some Busemann function $\overline{h}_{\gamma,q_i,\mathfrak{v}}$ by the constant $s(i,\mathfrak{v})$, i.e. independently of γ. We define

$$\tilde{h}_{\gamma,q_i,\mathfrak{v}} := \overline{h}_{\gamma,q_i,\mathfrak{v}} + s(i,\mathfrak{v}). \tag{2.24}$$

Lemma 2.4.6. *If $\gamma' \cdot q_i \cdot \mathfrak{v} = \gamma \cdot q_j \cdot \mathfrak{v}$ we have $\tilde{h}_{\gamma',q_i,\mathfrak{v}} = \tilde{h}_{\gamma,q_j,\mathfrak{v}}$.*

Proof. By assumption we have $k(i,\mathfrak{v}) = k(j,\mathfrak{v}) =: k$. More precisely there exists $\gamma_{i,\mathfrak{v}} \in \Gamma$ such that

$$\begin{aligned} \gamma_{i,\mathfrak{v}} \cdot q_k \cdot \mathfrak{v} &= q_i \cdot \mathfrak{v}; \\ \gamma^{-1} \cdot \gamma' \cdot \gamma_{i,\mathfrak{v}} \cdot q_k \cdot \mathfrak{v} &= q_j \cdot \mathfrak{v}. \end{aligned}$$

Hence we may set $\gamma_{j,\mathfrak{v}} := \gamma^{-1} \cdot \gamma' \cdot \gamma_{i,\mathfrak{v}}$ and obtain

$$\begin{aligned} &\tilde{h}_{\gamma',q_i,\mathfrak{v}} \\ \overset{(2.24)}{=}\ &\overline{h}_{\gamma',q_i,\mathfrak{v}} + s(i,\mathfrak{v}) \\ \overset{2.4.3}{=}\ &\overline{h}_{\gamma,q_j,\mathfrak{v}} + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right) \\ &\qquad + s(i,\mathfrak{v}) \\ \overset{(2.24)}{=}\ &\tilde{h}_{\gamma,q_j,\mathfrak{v}} - s(j,\mathfrak{v}) \\ &\qquad + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right) \\ &\qquad + s(i,\mathfrak{v}) \\ \overset{(2.23)}{=}\ &\tilde{h}_{\gamma,q_j,\mathfrak{v}} + \log\left(\|\varpi_{\mathfrak{v}}((q_k^{-1}\gamma_{j,\mathfrak{v}}^{-1} q_j)_{\overline{S}})\|\right) \\ &\qquad + \log\left(\|\varpi_{\mathfrak{v}}((q_j^{-1}\gamma^{-1}\gamma' q_i)_{\overline{S}})\|\right) \\ &\qquad + \log\left(\|\varpi_{\mathfrak{v}}((q_i^{-1}\gamma_{i,\mathfrak{v}} q_k)_{\overline{S}})\|\right) \\ =\ &\tilde{h}_{\gamma,q_j,\mathfrak{v}} + \log\left(\|\varpi_{\mathfrak{v}}((q_k^{-1}\gamma_{j,\mathfrak{v}}^{-1} q_j \cdot q_j^{-1}\gamma^{-1}\gamma' q_i \cdot q_i^{-1}\gamma_{i,\mathfrak{v}} q_k)_{\overline{S}}\|\right) \\ =\ &\tilde{h}_{\gamma,q_j,\mathfrak{v}} + \log\left(\|\varpi_{\mathfrak{v}}(1)\|\right). \end{aligned}$$

□

Given an arbitrary vertex $v \in \mathcal{V}(\Delta)$ and some description $v = \gamma \cdot q_i \cdot \mathfrak{v}$ we use Lemma 2.4.6 to define

$$\tilde{h}_v := \tilde{h}_{\gamma,q_i,\mathfrak{v}}. \tag{2.25}$$

Next we do two things at the same time: We investigate the relation between $\left\{\tilde{h}_v \,\middle|\, v \in \mathcal{V}(\Delta)\right\}$ and $\{\mathcal{S}_v \mid v \in \Delta\}$ and, simultaneously, prove the announced assertion the family of Busemann functions not to be $\mathbf{G}(\mathbb{Q})$-invariant. For simplicity we assume $q_1 = 1$ from now on.

Theorem 2.4.7. *Given a vertex* $v = \gamma' \cdot q_i \cdot \mathfrak{v}$, *some* $\gamma \in \Gamma$, $p \in \mathbf{P}_v(\mathbb{A})$, $g \in \mathbf{G}(\mathbb{A})$ *and* $x \in X_S$ *the following assertions hold:*

(i) $\tilde{h}_{\gamma \cdot v}(\gamma \cdot x) = \tilde{h}_v(x)$;

(ii) $\tilde{h}_v(g \cdot *) = \log\left(\mathcal{S}_{\varpi_v}(g)\right) - \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right)$;

(iii) $\tilde{h}_\mathfrak{v}(g \cdot *) = \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}(g_S)\right)$;

(iv) $\tilde{h}_v(p \cdot x) = \log\left(\|\varpi_v(p_S)\|\right) + \tilde{h}_v(x)$;

(v) $\tilde{h}_{q_i \cdot \mathfrak{v}}(q_i \cdot x) = \tilde{h}_\mathfrak{v}(x) + s(i, \mathfrak{v})$.

Proof. Claim (i) holds by (2.20), (2.24) and (2.25). We proof assertion (ii) by direct calculation:

$$\begin{aligned}
&\tilde{h}_v\left(g \cdot *\right) \\
\overset{(2.25)}{=}\ & \tilde{h}_{\gamma, q_i, \mathfrak{v}}\left(g \cdot *\right) \\
\overset{(2.24)}{=}\ & \overline{h}_{\gamma, q_i, \mathfrak{v}}\left(g \cdot *\right) + s(i, \mathfrak{v}) \\
\overset{(2.19)}{=}\ & \overline{h}_\mathfrak{v}\left(\left(q_i^{-1} \cdot \gamma^{-1} \cdot g\right)_S \cdot *\right) + s(i, \mathfrak{v}) \\
\overset{2.4.2}{=}\ & \overline{\log \circ \mathcal{S}_{\varpi_\mathfrak{v}}}\left(\left(q_i^{-1} \cdot \gamma^{-1} \cdot g\right)_S \cdot K\right) + s(i, \mathfrak{v}) \\
\overset{(2.16)}{=}\ & \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_i^{-1} \cdot \gamma^{-1} \cdot g\right)_S\right)\right) + s(i, \mathfrak{v}) \\
\overset{(2.23)}{=}\ & \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_i^{-1} \cdot \gamma^{-1} \cdot g\right)_S\right)\right) + \log\left(\left\|\varpi_\mathfrak{v}\left(\left(q_{k(i,\mathfrak{v})}^{-1} \gamma_{i,\mathfrak{v}}^{-1} q_i\right)_S\right)\right\|\right) \\
\overset{2.1.2(i)}{=}\ & \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_{k(i,\mathfrak{v})}^{-1} \gamma_{i,\mathfrak{v}}^{-1} \gamma^{-1} \cdot g\right)_S\right)\right) + \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right) \\
& \qquad - \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right) \\
\overset{2.1.2(v)}{=}\ & \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(q_{k(i,\mathfrak{v})}^{-1} \cdot \left(\gamma_{i,\mathfrak{v}}^{-1} \gamma^{-1} \cdot g\right)_S\right)\right) - \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right) \\
\overset{2.1.2(ii)}{=}\ & \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(q_{k(i,\mathfrak{v})}^{-1} \gamma_{i,\mathfrak{v}}^{-1} \gamma^{-1} \cdot g_S\right)\right) - \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right) \\
\overset{2.1.4(i)}{=}\ & \log\left(\mathcal{S}_{\varpi_v}(g_S)\right) - \log\left(\mathcal{S}_{\varpi_\mathfrak{v}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right).
\end{aligned}$$

Now (iii) follows from (ii), Theorem 2.1.2 (iii) and the assumption $q_1 = 1$. To prove (iv) we remark that both sides, the left hand side and the right hand side of the equation, are rescaled Busemann functions of the same speed with a common center. Hence it suffices to prove the equation for a singular x, e.g. $x = *$. Now (iv) is a direct consequence of (ii) and

Theorem 2.1.2 (iv). To prove (v) we may equivalently prove $\tilde{h}_{q_i\cdot\mathfrak{v}}(y) = \tilde{h}_{\mathfrak{v}}(q_i^{-1}\cdot y) + s(i,\mathfrak{v})$ for all $y \in X_S$. As before it suffices to prove the assertion for $y = *$:

$$
\begin{aligned}
&\tilde{h}_{q_i\cdot\mathfrak{v}}(*)\\
\overset{2.4.7(ii)}{=}\;& \log\left(\mathcal{S}_{\varpi_{q_i\cdot\mathfrak{v}}}(1)\right) - \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}\left(\left(q^{-1}_{k(i,\mathfrak{v})}\right)_{\overline{S}}\right)\right)\\
\overset{(2.8)}{=}\;& \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}((q_i^{-1})_S\cdot(q_i^{-1})_{\overline{S}})\right) - \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}\left(\left(q^{-1}_{k(i,\mathfrak{v})}\right)_{\overline{S}}\right)\right)\\
\overset{2.1.2(v)}{=}\;& \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}((q_i^{-1})_S)\right) + \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}((q_i^{-1})_{\overline{S}})\right) - \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}\left(\left(q^{-1}_{k(i,\mathfrak{v})}\right)_{\overline{S}}\right)\right)\\
\overset{\substack{2.1.2\\(i),(ii)}}{=}\;& \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}((q_i^{-1})_S)\right) + \log\left(\left\|\varpi_{\mathfrak{v}}\left(\left(q_i^{-1}\gamma_{1,\mathfrak{v}}q_{k(i,\mathfrak{v})}\right)_{\overline{S}}\right)\right\|\right)\\
\overset{\substack{2.4.7(iii)\\(2.23)}}{=}\;& \tilde{h}_{\mathfrak{v}}(q_i\cdot *) + s(i,\mathfrak{v}).
\end{aligned}
$$

□

Corollary 2.4.8. *Let $v \in \mathcal{V}(\Delta)$ and $c \in \mathcal{C}(\Delta)$. The following assertions hold:*

(i) Let $e_v \in \partial(X_S)$ denote the visual endpoint of ϖ_v, i.e. the center of $\tilde{h}_v$. The map $v \mapsto e_v$ induces an isometric embedding of Δ into $\partial(X_S)$.

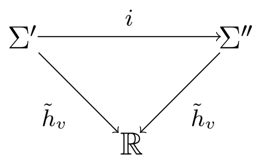
(ii) Each $\Sigma' \in \mathcal{A}^c$ contains c in its visual boundary. Hence, by equation (2.15), condition (A) is true.

Proof. Given a spherical apartment of Δ, the distance between two of its vertices is given by the angles between the corresponding weights. Thus claim (i) holds on each spherical apartment by definition of e_v. The assertion now follows from the fact that any two points in Δ are contained in a common apartment. Now (ii) follows from (i).

□

We finish this section with a result which we need to prove that the family of rescaled Busemann functions satisfies condition (B).

Lemma 2.4.9. *Consider $c \in \mathcal{C}(\Delta)$, $v \in \mathcal{V}(c)$ and two apartments $\Sigma', \Sigma'' \in \mathcal{A}^c$ with non-empty intersection. The isometry $i : \Sigma' \to \Sigma''$ from Theorem 2.3.3 makes the following diagramm commutative:*



Proof. By assumption Σ' and Σ'' share a point. Hence, by Lemma 2.4.8, they share a sector defining c. By Theorem 2.3.3 the isometry $i : \Sigma' \to \Sigma''$ fixis that sector and thus the chamber c. Because Busemann functions are determined by their center and the value of one point the claim follows. □

2.5 From algebraic reduction theory to pure geometry

The main goal of this section is to proof Theorem 2.5.7. It provides an algebraic description of the purely geometrically defined Minkowski cones introduced in Section 1.1. In the course of this section we are going to repeat the definition.

The current family of rescaled Busemann functions $\left\{ \tilde{h}_v \;\middle|\; v \in \mathcal{V}(\Delta) \right\}$ might not satisfy condition (C). Therefore we need to rescale each $\tilde{h}_v$. As illustrated in Figure 1.5 there are lots of different possibilities changing the speed such that condition (C) is satisfied but there is one significant that behaves particularly well: For an arbitrary chamber $c \in \mathcal{C}(\Delta)$ and an arbitrary vertex $v \in \mathcal{V}(c)$ we define

$$\mu_v^c := \sum_{\substack{w \in \mathcal{V}(c) \\ w = q \cdot \mathfrak{w}}} c_{\mathfrak{v}\mathfrak{w}} \cdot \tilde{h}_w : X_S \to \mathbb{R}.$$

For each $q_i \in \{q_1, ..., q_m\}$ we moreover set

$$c(i,\mathfrak{v}) := \sum_{\mathfrak{w} \in \mathcal{V}(\mathfrak{c})} c_{\mathfrak{v}\mathfrak{w}} \cdot s(i, \mathfrak{w}).$$

We obtain a statement analogous to Theorem 2.4.7:

Lemma 2.5.1. *Given a chamber $c = \gamma' \cdot q_i \cdot \mathfrak{c}$, a vertex $v = \gamma' \cdot q_i \cdot \mathfrak{v}$, some $\gamma \in \Gamma$, $p \in \mathbf{P}_c(\mathbb{A})$, $g \in \mathbf{G}(\mathbb{A})$ and $x, y \in X_S$ the following assertions hold:*

(i) $\mu_{\gamma \cdot v}^{\gamma \cdot c}(\gamma \cdot x) = \mu_v^c(x)$;

(ii) $\mu_v^c(g \cdot *) = \log(\mathcal{S}_{\varphi_v^c}(g_S)) - \log\left(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right)$;

(iii) $\mu_{\mathfrak{v}}^{\mathfrak{c}}(g \cdot *) = \log(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}(g_S))$;

(iv) $\mu_v^c(p \cdot x) = \log(\|\varphi_v^c(p_S)\|) + \mu_v^c(x)$;

(v) $\mu_{q_i \cdot \mathfrak{v}}^{q_i \cdot \mathfrak{c}}(q_i \cdot x) = \mu_{\mathfrak{v}}^{\mathfrak{c}}(x) + c(i, \mathfrak{v})$;

(vi) There is a constant $a_\mu \in \mathbb{R}_{\geqslant 0}$*, independent of* $x, y \in X_S$*,* $c \in \mathcal{C}(\Delta)$ *and* $v \in \mathcal{V}(c)$*, such that*

$$\mu_v^c(y) - a_\mu \cdot d_{X_S}(x,y) \leqslant \mu_v^c(x) \leqslant \mu_v^c(y) + a_\mu \cdot d_{X_S}(x,y).$$

Proof. The assertions (i) to (v) follow directly from equations (2.2), (2.9) and Theorem 2.4.7. The existence of a constant a_μ as in (vi) and its independence from x and y follows, since μ_v^c is the sum of rescaled Busemann functions. The independence from $c \in \mathcal{C}(\Delta)$ and $v \in \mathcal{V}(c)$ holds, because the speed of all rescaled Busemann functions depends on the length of the finite number of standard roots $\{\varphi_{\mathfrak{w}}^{\mathfrak{c}} \mid \mathfrak{w} \in \mathcal{V}(\Delta)\}$ only, as (iv) proves. □

Next we define $x_1 := t_1 \cdot *$ with $t_1 \in \mathbf{T}(\mathbb{A}_S)$ such that $\mu_{\mathfrak{v}}^{\mathfrak{c}}(x_1) = 1$ for all $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$.

Lemma 2.5.2. *For* $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ *the constant* $s_{\mathfrak{v}} := (\tilde{h}_{\mathfrak{v}}(x_1))^{-1} = \left(\sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} n_{\mathfrak{v}\mathfrak{w}}\right)^{-1}$ *is strictly positive, compare Figure 2.2.*

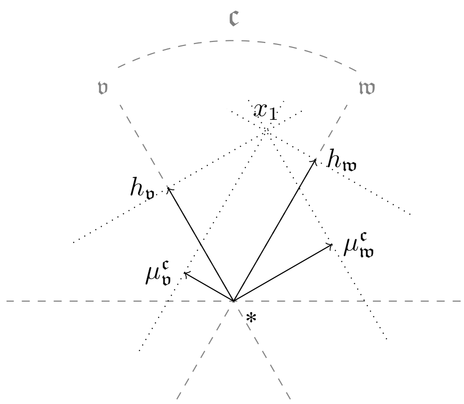


Figure 2.2: Rescaled Busemann functions, related to the set of simple roots, and rescaled Busemann functions, related to the corresponding set of weights, such that x_1 is in each level of height 1.

Proof. Calculation proves the claim.

$$
\begin{aligned}
s_{\mathfrak{v}}^{-1} &= \tilde{h}_{\mathfrak{v}}(t_1 \cdot *) \\
&\overset{2.4.7(iii)}{=} \log\left(\mathcal{S}_{\varpi_{\mathfrak{v}}}(t_1)\right) \\
&\overset{2.1.2(iv)}{=} \log\left(\|\varpi_{\mathfrak{v}}(t_1)\|\right) \\
&\overset{(2.2)}{=} \log\left(\prod_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} \|(\varphi^{\mathfrak{c}}_{\mathfrak{w}}(t_1))\|^{n_{\mathfrak{v}\mathfrak{w}}}\right) \\
&\overset{2.1.3(iv)}{=} \log\left(\prod_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} (\mathcal{S}_{\varphi^{\mathfrak{c}}_{\mathfrak{w}}}(t_1))^{n_{\mathfrak{v}\mathfrak{w}}}\right) \\
&\overset{2.5.1(iii)}{=} \sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} n_{\mathfrak{v}\mathfrak{w}} \cdot \mu^{\mathfrak{c}}_{\mathfrak{w}}(t_1 \cdot *) \\
&= \sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} n_{\mathfrak{v}\mathfrak{w}} \\
&\overset{(2.3)}{>} 0.
\end{aligned}
$$

□

For an arbitrary vertex $v = \gamma \cdot q_i \cdot \mathfrak{v} \in \mathcal{V}(\Delta)$ define

$$h_v := s_{\mathfrak{v}} \cdot \tilde{h}_v. \tag{2.26}$$

This causes $h_{\mathfrak{v}}(x_1) = 1$ for all $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$. In the course of this section we prove that $\{h_v \mid v \in \mathcal{V}(\Delta)\}$ satisfies condition (B) and (C).

Next repeat some notation already given in Section 1.1: For $c \in \mathcal{C}(\Delta)$ let

$$h_c(x) := \max\left\{h_v(x) \mid v \in \mathcal{V}(c)\right\}.$$

Moreover, for an apartment $\Sigma' \in \mathcal{A}^c$ and a real parameter $s \in \mathbb{R}$, we set

$$Y_{\Sigma',c}(s) := \left\{x \in \Sigma' \mid h_c(x) \leqslant s\right\}. \tag{2.27}$$

It is a closed and convex subset of Σ'. Hence there is a closest point projection

$$\mathrm{pr}^s_{\Sigma',c} : \Sigma' \to Y_{\Sigma',c}(s).$$

Given some point $x \in \Sigma'$ its projection point $\mathrm{pr}^s_{\Sigma',c}(x)$ is defined by

(I) $\mathrm{pr}^s_{\Sigma',c}(x) \in Y_{\Sigma',c}(s)$;

(II) $d(x, \mathrm{pr}^s_{\Sigma',c}(x)) = d(x, Y_{\Sigma',c}(s))$.

Given two apartments $\Sigma', \Sigma'' \in \mathcal{A}^c$ the isometry $i : \Sigma' \to \Sigma''$ of Lemma 2.4.9 identifies $Y_{\Sigma',c}(s)$ with $Y_{\Sigma'',c}(s)$. Hence

$$i \circ \mathrm{pr}^s_{\Sigma',c} = \mathrm{pr}^s_{\Sigma'',c} \circ i. \tag{2.28}$$

Lemma 2.5.3. *Let $c \in \mathcal{C}(\Delta)$, $x \in X_S$ and $\Sigma', \Sigma'' \in \mathcal{A}^c$ both containing x. Then*

$$h_v(\mathrm{pr}_{\Sigma',c}(x)) = h_v(\mathrm{pr}_{\Sigma'',c}(x)),$$

i.e. condition (B) is satisfied.

Proof. Because x is in the intersection of Σ' and Σ'' it is fixed by i. Hence

$$\begin{aligned} h_v(\mathrm{pr}_{\Sigma',c}(x)) &\overset{2.4.9}{=} h_v(i \circ \mathrm{pr}_{\Sigma',c}(x)) \\ &\overset{(2.28)}{=} h_v(\mathrm{pr}_{\Sigma'',c} \circ i(x)) \\ &= h_v(\mathrm{pr}_{\Sigma'',c}(x)). \end{aligned}$$

□

By Corollary 2.4.8 (ii) condition (A) is true, i.e. for each chamber $c \in \mathcal{C}(\Delta)$ the set of apartments $\mathcal{A}^c$ covers X_S. Given a point $x \in X_S$ we may therefore choose an apartment $\Sigma' \in \mathcal{A}^c$ containing it. By Lemma 2.5.3 the value $h_v(\mathrm{pr}^s_{\Sigma',c}(x)) =: b^s_{c,v}(x)$ is independent of that choice. Thus we may define:

$$\sigma_s(x,c) := \left\{ v \in \mathcal{V}(\Delta) \;\middle|\; b^s_{c,v}(x) = s \right\}. \tag{2.29}$$

We say that a point $x \in X_S$ is <u>s-reduced</u> by $c \in \mathcal{C}(\Delta)$ if $\sigma_s(x,c) = \mathcal{V}(c)$. See again Figure 1.2. A subset $B \subset X_S$ is <u>uniformly s-reduced</u> if there is a chamber $c \in \mathcal{C}(\Delta)$ that s-reduces all points of B simultaneously. For an apartment $\Sigma' \in \mathcal{A}^c$ we set

$$N_{\Sigma',c}(s) := \{x \in \Sigma' \mid x \text{ is } s\text{-reduced by } c\} \tag{2.30}$$

and call it a <u>thin Minkowski cone</u> (above c of height s). It is the normal cone above $Y_{\Sigma',c}(s)$, i.e. it consists precisely of those points in Σ' whose projection point lies in the tip of $Y_{\Sigma',c}(s)$. Moreover we define

$$N_c(s) := \{x \in X_S \mid x \text{ is } s\text{-reduced by } c\} = \bigcup_{\Sigma' \in \mathcal{A}^c} N_{\Sigma',c}(s) \tag{2.31}$$

and call it a <u>thick Minkowski cone</u> (above c of height s), see Figure 1.3.

Next we set

$$Z_{\Sigma',c}(s) := \{x \in \Sigma' \mid \mu_v^c(x) \geqslant s \text{ for all } v \in \mathcal{V}(c)\}\,; \tag{2.32}$$

$$Z_c(s) := \{x \in X_S \mid \mu_v^c(x) \geqslant s \text{ for all } v \in \mathcal{V}(c)\} = \bigcup_{\Sigma' \in \mathcal{A}^c} Z_{\Sigma',c}(s). \tag{2.33}$$

By Theorem 2.5.1 (iv) one could loosely call $Z_{\Sigma,\mathfrak{c}}(s)$ a *geometric analogue* of $\mathbf{T}_\infty^+(s)$ discussed in Theorem 1.2.1.

To translate algebraic reduction theory into geometric reduction theory we want both families, $\{N_c(s) \mid s \in \mathbb{R}\}$ and $\{Z_c(s) \mid s \in \mathbb{R}\}$, to be *nested filtrations*, i.e. given $c \in \mathcal{C}(\Delta)$ and $s \in \mathbb{R}$ we require the existence of $s_1, s_2 \in \mathbb{R}$ with

$$Z_c(s_1) \subset N_c(s) \subset Z_c(s_2); \tag{2.34}$$

$$N_c(s_1) \subset Z_c(s) \subset N_c(s_2), \tag{2.35}$$

compare [BKW13, Observation 12.7]. Obviously $Z_c(s)$ is a filtration in s and hence equation (2.34) holds. Now (2.35) holds if and only if $N_c(s)$ is a filtration in s, i.e. if and only if condition (C) holds. Though it is not clear that $N_c(s)$ is a filtration in s and hence equation (2.35) requires a proof. That is where the particular change of speed, see equation (2.26), comes into play. Instead of proving nestedness we even prove equality, i.e. we prove $N_c(s)$ to be equal to $Z_c(s)$ for all $c \in \mathcal{C}(\Delta)$ and all $s \in \mathbb{R}$: Consider

$$\Sigma_0 := \Sigma \cap \left(\bigcap_{\mathfrak{v} \in \mathcal{V}(\mathfrak{c})} \ker\left(\mu_{\mathfrak{v}}^{\mathfrak{c}}\right) \right) = \Sigma \cap \left(\bigcap_{\mathfrak{v} \in \mathcal{V}(\mathfrak{c})} \ker\left(\tilde{h}_{\mathfrak{v}}\right) \right). \tag{2.36}$$

It is the tip of $Y_{\Sigma,\mathfrak{c}}(0)$, respectively $N_{\Sigma,\mathfrak{c}}(0)$.

Lemma 2.5.4. *For each $s \in \mathbb{R}$ and $x \in \Sigma$ we have*

(i) $Y_{\Sigma,\mathfrak{c}}(s) = Y_{\Sigma,\mathfrak{c}}(0) + s \cdot x_1$;

(ii) $\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(x + s \cdot x_1) = \mathrm{pr}^0_{\Sigma,\mathfrak{c}}(x) + s \cdot x_1$.

If we additionally consider some $um \in \mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$ we recieve

(iii) $Y_{um\cdot\Sigma,\mathfrak{c}}(s) = um \cdot Y_{\Sigma,\mathfrak{c}}(s)$;

(iv) $\mathrm{pr}^s_{um\cdot\Sigma,\mathfrak{c}}(um \cdot x) = um \cdot \mathrm{pr}^s_{\Sigma,\mathfrak{c}}(x)$.

Now let $c \in \mathcal{C}(\Delta)$, $\Sigma' \in \mathcal{A}^c$ *and* $\gamma \in \Gamma$ *be arbitrary. Then*

(v) $Y_{\gamma\cdot\Sigma',\gamma\cdot c}(s) = \gamma \cdot Y_{\Sigma',c}(s)$;

(vi) $\mathrm{pr}^s_{\gamma\cdot\Sigma',\gamma\cdot c}(\gamma \cdot x) = \gamma \cdot \mathrm{pr}^s_{\Sigma',c}(x)$.

Proof. Claim (i) follows from the definition of x_1 and the linearity of Busemann functions. Assertion (ii) is a direct consequence of (i).

Assertion (iii) results from Theorem 2.4.7 (iv) and the definition of $\mathbf{U}$ and $\mathbf{M}$. Now (iv) follows from (iii).

We deduce assertion (v) from the Γ-invariance of the family of Busemann functions $\{h_v \mid v \in \mathcal{V}(\Delta)\}$ (see Theorem 2.4.7 (i)). Now (vi) results from (v). □

By Lemma 2.5.1 (iii) and Lemma 2.1.3 (iii), respectively Theorem 2.4.7 (iii) and Theorem 2.1.2 (iii) we have $* \in \Sigma_0$. Considering $*$ as the origin of Σ those $\mu^{\mathfrak{c}}_{\mathfrak{v}}$ and $\tilde{h}_{\mathfrak{v}}$ for $\mathfrak{v} \in \mathcal{V}(\mathfrak{c})$ become linear functions on Σ. Regarding $\{\varphi^{\mathfrak{c}}_{\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\}$ and $\{\varpi^{\mathfrak{c}}_{\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\}$ as elements of the $\mathbb{Q}$-vector space $X(\mathbf{P}_{\mathfrak{c}}) \otimes \mathbb{Q}$ defining reflections, compare Section 2.1, we may recast Theorem 2.4.7 (iv) respectively Lemma 2.5.1 (iv) as follows:

$$\begin{aligned} h_{\mathfrak{v}}(t \cdot *) &= s_{\mathfrak{v}} \cdot \langle \varpi^{\mathfrak{c}}_{\mathfrak{v}}, t \rangle \text{ and} \\ \mu^{\mathfrak{c}}_{\mathfrak{v}}(t \cdot *) &= \langle \varphi^{\mathfrak{c}}_{\mathfrak{v}}, t \rangle \end{aligned} \tag{2.37}$$

for all $t \in \mathbf{T}(\mathbb{A}_S)$. Hence we have

$$\Sigma_0 = \bigcap_{\mathfrak{v} \in \mathcal{V}(\mathfrak{c})} (\varphi^{\mathfrak{c}}_{\mathfrak{v}})^{\perp} = \bigcap_{\mathfrak{w} \in \mathcal{V}(\mathfrak{c})} (\varpi^{\mathfrak{c}}_{\mathfrak{w}})^{\perp} \tag{2.38}$$

and thus Σ decomposes into

$$\begin{aligned} \Sigma &= \Sigma_0 \oplus \operatorname{span}\{\varphi^{\mathfrak{c}}_{\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\} \\ &= \Sigma_0 \oplus \operatorname{span}\{\varpi_{\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\}. \end{aligned} \tag{2.39}$$

Theorem 2.5.5. *$N_{\Sigma,\mathfrak{c}}(s) = Z_{\Sigma,\mathfrak{c}}(s)$ for all $s \in \mathbb{R}$ (see Figure 2.3).*

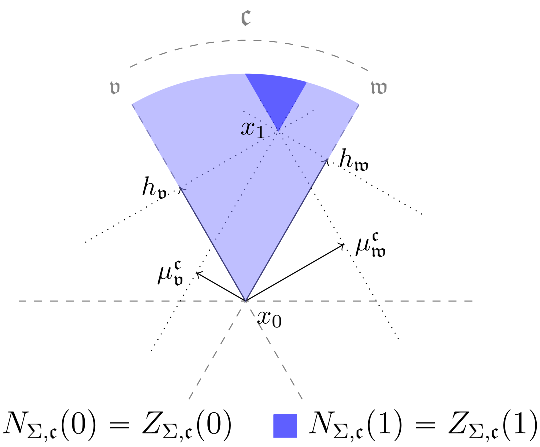


Figure 2.3: Changing the speed of all Busemann functions causes the Minkowski cone $N_{\Sigma,\mathfrak{c}}(s)$ to conincide with $Z_{\Sigma,\mathfrak{c}}(s)$ for all $s \in \mathbb{R}$.

Proof. First we prove the special case of $s = 0$:

$$
\begin{aligned}
& Y_{\Sigma,\mathfrak{c}}(0) \\
\overset{(2.27)}{=} \; & \{x \in \Sigma \mid h_{\mathfrak{w}}(x) \leqslant 0 \text{ for all } \mathfrak{w} \in \mathcal{V}(\mathfrak{c})\} \\
\overset{\substack{(2.36)\\(2.37)\\(2.39)}}{=} \; & \left\{ \left(\sum_{\mathfrak{v} \in \mathcal{V}(\mathfrak{c})} a_{\mathfrak{v}} \cdot \varphi^{\mathfrak{c}}_{\mathfrak{v}} \right) + x_0 \;\middle|\; \begin{array}{c} \left\langle \varpi_{\mathfrak{w}}, \sum_{\mathfrak{v} \in \mathcal{V}(\mathfrak{c})} a_{\mathfrak{v}} \cdot \varphi^{\mathfrak{c}}_{\mathfrak{v}} \right\rangle \leqslant 0 \\ \text{for all } \mathfrak{w} \in \mathcal{V}(\mathfrak{c}),\ x_0 \in \Sigma_0 \end{array} \right\} \\
\overset{(2.3)}{=} \; & \left\{ \left(\sum_{\mathfrak{v} \in \mathcal{V}(\mathfrak{c})} a_{\mathfrak{v}} \cdot \varphi^{\mathfrak{c}}_{\mathfrak{v}} \right) + x_0 \;\middle|\; \begin{array}{c} a_{\mathfrak{v}} \leqslant 0 \text{ for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c}), \\ x_0 \in \Sigma_0 \end{array} \right\}.
\end{aligned}
\tag{2.40}
$$

$$
\begin{aligned}
& N_{\Sigma,\mathfrak{c}}(0) \\
\overset{(2.30)}{=} \; & \{x \in \Sigma \mid h_{\mathfrak{w}}(\mathrm{pr}^0_{\Sigma,\mathfrak{c}}(x)) = 0 \text{ for all } \mathfrak{w} \in \mathcal{V}(\mathfrak{c})\} \\
\overset{\substack{(2.36)\\(2.39)}}{=} \; & \left\{ \left(\sum_{\mathfrak{w} \in \mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}} \right) + x_0 \;\middle|\; \begin{array}{c} \mathrm{pr}^0_{\Sigma,\mathfrak{c}} \left(\sum_{\mathfrak{w} \in \mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}} + x_0 \right) \in \Sigma_0, \\ x_0 \in \Sigma_0 \end{array} \right\}
\end{aligned}
$$

$$\overset{(2.38)}{=} \left\{ \left(\sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}} \right) + x_0 \;\middle|\; \mathrm{pr}^0_{\Sigma,\mathfrak{c}} \left(\sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}} \right) \in \Sigma_0,\ x_0 \in \Sigma_0 \right\}$$

$$\overset{(\star)}{=} \left\{ \left(\sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}} \right) + x_0 \;\middle|\; \begin{array}{c} Y_{\Sigma,\mathfrak{c}}(0) \subset \left\{ \left\langle \sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}}, - \right\rangle \leqslant 0 \right\}, \\ x_0 \in \Sigma_0 \end{array} \right\}$$

$$\overset{(2.40)}{=} \left\{ \left(\sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}} \right) + x_0 \;\middle|\; \begin{array}{c} \left\langle \sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}}, \sum_{\mathfrak{v}\in\mathcal{V}(\mathfrak{c})} a_{\mathfrak{v}} \cdot \varphi^{\mathfrak{c}}_{\mathfrak{v}} \right\rangle \leqslant 0 \\ \text{for all } a_{\mathfrak{v}} \leqslant 0 \text{ and } \mathfrak{v} \in \mathcal{V}(\mathfrak{c}),\ x_0 \in \Sigma_0 \end{array} \right\}$$

$$\overset{(2.3)}{=} \left\{ \left(\sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}} \right) + x_0 \;\middle|\; \begin{array}{c} \left\langle \sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}} \cdot \varpi_{\mathfrak{w}}, \varphi^{\mathfrak{c}}_{\mathfrak{v}} \right\rangle \geqslant 0 \\ \text{for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c}),\ x_0 \in \Sigma_0 \end{array} \right\}$$

$$\overset{\substack{(2.36)\\(2.37)\\(2.39)}}{=} \{x \in \Sigma \mid \mu^{\mathfrak{c}}_{\mathfrak{v}}(x) \geqslant 0 \text{ for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\}$$

$$\overset{(2.33)}{=} Z_{\Sigma,\mathfrak{c}}(0).$$

($\star$) Figure 2.4 shows a point projected onto the tip of $Y_{\Sigma,\mathfrak{c}}(0)$. Figure 2.5 shows a point not projected onto the tip of $Y_{\Sigma,\mathfrak{c}}(0)$.

The general case for arbitrary $s \in \mathbb{R}$ results from the special case:

$$\begin{aligned} & N_{\Sigma,\mathfrak{c}}(s) \\ \overset{(2.30)}{=}\ & \left\{x \in \Sigma \;\middle|\; h_{\mathfrak{v}}(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(x)) = s \text{ for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\} \\ \overset{(2.26)}{=}\ & \left\{x \in \Sigma \;\middle|\; h_{\mathfrak{v}}(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(x)) = h_{\mathfrak{v}}(s \cdot x_1) \text{ for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\} \\ \overset{2.5.4(ii)}{=}\ & \left\{x \in \Sigma \;\middle|\; h_{\mathfrak{v}}(\mathrm{pr}^0_{\Sigma,\mathfrak{c}}(x - s \cdot x_1)) = 0 \text{ for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\} \\ =\ & s \cdot x_1 + \left\{x \in \Sigma \;\middle|\; h_{\mathfrak{v}}(\mathrm{pr}^0_{\Sigma,\mathfrak{c}}(x)) = 0 \text{ for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\} \\ \overset{(2.30)}{=}\ & s \cdot x_1 + N_{\Sigma,\mathfrak{c}}(0) \\ \overset{\substack{\text{special}\\\text{case}}}{=}\ & s \cdot x_1 + Z_{\Sigma,\mathfrak{c}}(0) \\ \overset{\text{Def} x_1}{=}\ & Z_{\Sigma,\mathfrak{c}}(s). \end{aligned}$$

□

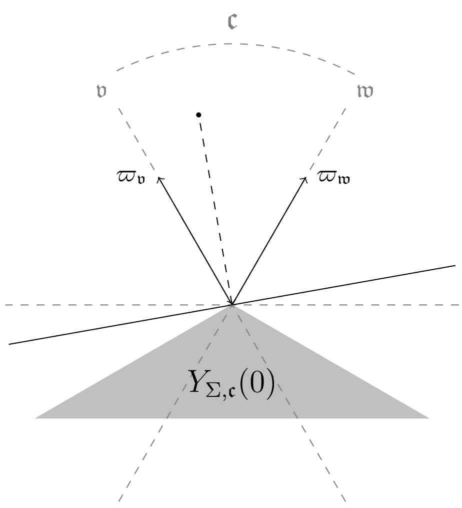


Figure 2.4: A point that is projected onto the tip of $Y_{\Sigma,\mathfrak{c}}(0)$.

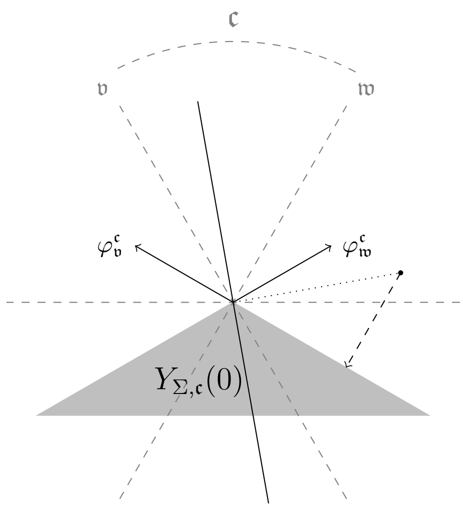


Figure 2.5: A point that is not projected onto the tip of $Y_{\Sigma,\mathfrak{c}}(0)$.

To deal with the general case of an arbitrary chamber $c \in \mathcal{C}(\Delta)$ we have to choose for each $q_i \in \{q_1, ..., q_m\}$ a vector $z_i \in \Sigma$ such that

$$h_{\mathfrak{v}}(z_i) = s_{\mathfrak{v}} \cdot s(i, \mathfrak{v}) \text{ for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c}). \tag{2.41}$$

Direct calculation shows that

$$\mu_{\mathfrak{v}}^{\mathfrak{c}}(z_i) = c(i, \mathfrak{v}) \text{ for all } \mathfrak{v} \in \mathcal{V}(\mathfrak{c}). \tag{2.42}$$

Lemma 2.5.6. *For all $\gamma \in \Gamma,\ q_i \in \{q_1, ..., q_m\},\ um \in \mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S),\ x \in X_S$ and $s \in \mathbb{R}$ the following holds:*

(i) $Z_{\gamma \cdot q_i \cdot um \cdot \Sigma, \gamma \cdot q_i \cdot \mathfrak{c}}(s) = \gamma \cdot q_i \cdot um \cdot (Z_{\Sigma,\mathfrak{c}}(s) - z_i)$;

(ii) $Y_{\gamma \cdot q_i \cdot um \cdot \Sigma, \gamma \cdot q_i \cdot \mathfrak{c}}(s) = \gamma \cdot q_i \cdot um \cdot (Y_{\Sigma,\mathfrak{c}}(s) - z_i)$;

(iii) $\mathrm{pr}^s_{\gamma \cdot q_i \cdot um \cdot \Sigma, \gamma \cdot q_i \cdot \mathfrak{c}}(\gamma \cdot q_i \cdot um \cdot x) = \gamma \cdot q_i \cdot um \cdot (\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(x + z_i) - z_i)$;

(iv) $N_{\gamma \cdot q_i \cdot um \cdot \Sigma, \gamma \cdot q_i \cdot \mathfrak{c}}(s) = \gamma \cdot q_i \cdot um \cdot (N_{\Sigma,\mathfrak{c}}(s) - z_i)$.

Proof. Assertion (i) essentially follows from Lemma 2.5.1:

$$\begin{aligned}
& Z_{\gamma\cdot q_i\cdot um\cdot\Sigma,\gamma\cdot q_i\cdot\mathfrak{c}}(s) \\
\overset{(2.32)}{=} & \{x \in \gamma\cdot q_i\cdot um\cdot\Sigma \mid \mu^{\gamma\cdot q_i\cdot\mathfrak{c}}_{\gamma\cdot q_i\cdot\mathfrak{v}}(x) \geqslant s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} \\
= & \{\gamma\cdot q_i\cdot um\cdot y \mid y\in\Sigma,\ \mu^{\gamma\cdot q_i\cdot\mathfrak{c}}_{\gamma\cdot q_i\cdot\mathfrak{v}}(\gamma\cdot q_i\cdot um\cdot y) \geqslant s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} \\
\overset{\substack{2.5.1\\(i),(v)}}{=} & \{\gamma\cdot q_i\cdot um\cdot y \mid y\in\Sigma,\ \mu^{\mathfrak{c}}_{\mathfrak{v}}(y) \geqslant s - c(i,\mathfrak{v}) \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} \\
\overset{(2.42)}{=} & \gamma\cdot q_i\cdot um\cdot\{y\in\Sigma \mid \mu^{\mathfrak{c}}_{\mathfrak{v}}(y+z_i) \geqslant s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} \\
= & \gamma\cdot q_i\cdot um\cdot(\{y\in\Sigma \mid \mu^{\mathfrak{c}}_{\mathfrak{v}}(y) \geqslant s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} - z_i) \\
\overset{(2.32)}{=} & \gamma\cdot q_i\cdot um\cdot(Z_{\Sigma,\mathfrak{c}}(s) - z_i).
\end{aligned}$$

Assertion (ii) follows similarly from Theorem 2.4.7:

$$\begin{aligned}
& Y_{\gamma\cdot q_i\cdot um\cdot\Sigma,\gamma\cdot q_i\cdot\mathfrak{c}}(s) \\
\overset{(2.27)}{=} & \{x \in \gamma\cdot q_i\cdot um\cdot\Sigma \mid h_{\gamma\cdot q_i\cdot\mathfrak{v}}(x) \leqslant s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} \\
= & \{\gamma\cdot q_i\cdot um\cdot y \mid y\in\Sigma,\ h_{\gamma\cdot q_i\mathfrak{v}}(\gamma\cdot q_i\cdot um\cdot y) \leqslant s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} \\
\overset{\substack{2.4.7(i),(v)\\(2.26)}}{=} & \{\gamma\cdot q_i\cdot um\cdot y \mid y\in\Sigma,\ h_{\mathfrak{v}}(y) \leqslant s - s_{\mathfrak{v}}\cdot s(i,\mathfrak{v}) \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} \\
\overset{(2.41)}{=} & \gamma\cdot q_i\cdot um\cdot\{y\in\Sigma \mid h_{\mathfrak{v}}(y+z_i) \leqslant s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} \\
= & \gamma\cdot q_i\cdot um\cdot(\{y\in\Sigma \mid h_{\mathfrak{v}}(y) \leqslant s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\} - z_i) \\
\overset{(2.27)}{=} & \gamma\cdot q_i\cdot um\cdot(Y_{\Sigma,\mathfrak{c}}(s) - z_i).
\end{aligned}$$

Assertion (iii) follows from assertion (ii). We have to prove that each point on the right hand side satisfies the conditions (I) and (II) from the left hand side. Since (I) is obvious we only present the calculation for (II):

$$\begin{aligned}
& d(\gamma\cdot q_i\cdot um\cdot x, \gamma\cdot q_i\cdot um\cdot(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(x+z_i) - z_i)) \\
= & d(x+z_i, \mathrm{pr}^s_{\Sigma,\mathfrak{c}}(x+z_i)) \\
= & d(x+z_i, Y_{\Sigma,\mathfrak{c}}(s)) \\
= & d(\gamma\cdot q_i\cdot um\cdot x, \gamma\cdot q_i\cdot um\cdot(Y_{\Sigma,\mathfrak{c}}(s) - z_i)) \\
\overset{2.5.6(ii)}{=} & d(\gamma\cdot q_i\cdot um\cdot x, Y_{\gamma\cdot q_i\cdot um\cdot\Sigma,\gamma\cdot q_i\cdot\mathfrak{c}}(s)).
\end{aligned}$$

Assertion (iv) is a consequence of Theorem 2.4.7 and assertion (iii):

$$
\begin{aligned}
& N_{\gamma\cdot q_i\cdot um\cdot\Sigma,\gamma\cdot q_i\cdot\mathfrak{c}}(s) \\
\overset{(2.30)}{=}\ & \left\{ x\in\gamma\cdot q_i\cdot um\cdot\Sigma \;\middle|\; \begin{array}{c} h_{\gamma\cdot q_i\cdot\mathfrak{v}}(\mathrm{pr}^s_{\gamma\cdot q_i\cdot um\cdot\Sigma,\gamma\cdot q_i\cdot\mathfrak{c}}(x)) = s \\ \text{for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c}) \end{array} \right\} \\
=\ & \left\{ \gamma\cdot q_i\cdot um\cdot y \;\middle|\; \begin{array}{c} h_{\gamma\cdot q_i\cdot\mathfrak{v}}(\mathrm{pr}^s_{\gamma\cdot q_i\cdot um\cdot\Sigma,\gamma\cdot q_i\cdot\mathfrak{c}}(\gamma\cdot q_i\cdot um\cdot y)) = s \\ \text{for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c}),\ y\in\Sigma \end{array} \right\} \\
\overset{2.5.6(iii)}{=}\ & \left\{ \gamma\cdot q_i\cdot um\cdot y \;\middle|\; \begin{array}{c} h_{\gamma\cdot q_i\cdot\mathfrak{v}}(\gamma\cdot q_i\cdot um\cdot(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(y+z_i)-z_i)) = s \\ \text{for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c}),\ y\in\Sigma \end{array} \right\} \\
\overset{\substack{2.4.7(i),(v)\\(2.26)}}{=}\ & \left\{ \gamma\cdot q_i\cdot um\cdot y \;\middle|\; \begin{array}{c} h_{\mathfrak{v}}(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(y+z_i)-z_i) = s - s_{\mathfrak{v}}\cdot s(i,\mathfrak{v}) \\ \text{for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c}),\ y\in\Sigma \end{array} \right\} \\
\overset{(2.41)}{=}\ & \gamma\cdot q_i\cdot um\cdot\left\{ y\in\Sigma \;\middle|\; h_{\mathfrak{v}}(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(y+z_i)) = s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\right\} \\
=\ & \gamma\cdot q_i\cdot um\cdot\left(\left\{ y\in\Sigma \;\middle|\; h_{\mathfrak{v}}(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(y)) = s \text{ for all } \mathfrak{v}\in\mathcal{V}(\mathfrak{c})\right\} - z_i\right) \\
\overset{(2.30)}{=}\ & \gamma\cdot q_i\cdot um\cdot(N_{\Sigma,\mathfrak{c}}(s) - z_i).
\end{aligned}
$$

□

Theorem 2.5.7. *For all $c\in\mathcal{C}(\Delta)$ and all $s\in\mathbb{R}$ we have $Z_c(s) = N_c(s)$.*

Proof. Given a chamber $c = \gamma\cdot q_i\cdot\mathfrak{c}$ we have

$$
\begin{aligned}
N_c(s) &\overset{(2.31)}{=} \bigcup_{\Sigma'\in\mathcal{A}^c} N_{\Sigma',c}(s) \\
&\overset{(2.14)}{=} \bigcup_{um\in\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)} N_{\gamma\cdot q_i\cdot um\cdot\Sigma,\gamma\cdot q_i\cdot\mathfrak{c}}(s) \\
&\overset{2.5.6(iv)}{=} \bigcup_{um\in\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)} \gamma\cdot q_i\cdot um\cdot(N_{\Sigma,\mathfrak{c}}(s) - z_i) \\
&\overset{2.5.5}{=} \bigcup_{um\in\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)} \gamma\cdot q_i\cdot um\cdot(Z_{\Sigma,\mathfrak{c}}(s) - z_i) \\
&\overset{2.5.6(i)}{=} \bigcup_{um\in\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)} Z_{\gamma\cdot q_i\cdot um\cdot\Sigma,\gamma\cdot q_i\cdot\mathfrak{c}}(s) \\
&\overset{(2.14)}{=} \bigcup_{\Sigma'\in\gamma\cdot q_i\cdot\mathcal{A}^{\mathfrak{c}}} Z_{\Sigma',c}(s) \\
&\overset{(2.33)}{=} Z_c(s).
\end{aligned}
$$

□

Since $Z_{\Sigma',c}(s)$ is a filtration in s for all $c \in \mathcal{C}(\Delta)$ and all $\Sigma' \in \mathcal{A}^c$, Theorem 2.5.7 demonstrates that $\{h_v \mid v \in \mathcal{V}(\Delta)\}$ fulfills condition (C). We go on to the last lemma of this section. It partially generalizes Theorem 2.5.5 and is proven similarly.

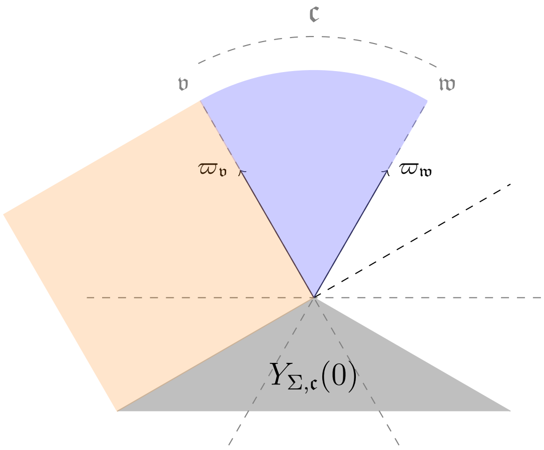
Lemma 2.5.8. *For each $x \in X_S$, $c \in \mathcal{C}(\Delta)$ and $s \in \mathbb{R}$ the following holds:*

$$\textit{If } v \in \sigma_s(x,c) \textit{ we have } \mu_v^c(x) \geqslant s.$$

Proof. We consider the special case $c = \mathfrak{c}$ first. By Section 2.3 we know that $\mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S) \cdot \Sigma = X_S$. Hence, given an $x \in X_S$ we have $x = um \cdot y$ for some $um \in \mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$ and $y \in \Sigma$. We conclude that

$$\begin{aligned}
&\mathfrak{v} \in \sigma_s(x,\mathfrak{c})\\
\overset{(2.29)}{\Leftrightarrow}\quad & h_{\mathfrak{v}}\left(\mathrm{pr}^s_{um\cdot\Sigma,\mathfrak{c}}(um\cdot y)\right) = s\\
\overset{\substack{2.4.7(iv)\\2.5.4(iv)}}{\Leftrightarrow}\quad & h_{\mathfrak{v}}\left(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(y)\right) = s\\
\overset{\substack{2.5.4(ii)\\(2.26)}}{\Leftrightarrow}\quad & h_{\mathfrak{v}}\left(\mathrm{pr}^0_{\Sigma,\mathfrak{c}}(y - s\cdot x_1)\right) = 0\\
\overset{(2.37)}{\Leftrightarrow}\quad & \left\langle \varpi_{\mathfrak{v}}, \mathrm{pr}^0_{\Sigma,\mathfrak{c}}(y - s\cdot x_1)\right\rangle = 0\\
\overset{(\star)}{\Rightarrow}\quad & y - s\cdot x_1 = \sum_{\mathfrak{w}\in\mathcal{V}(\mathfrak{c})} b_{\mathfrak{w}}\cdot\varpi_{\mathfrak{w}} + x_0 \text{ with } b_{\mathfrak{v}} \geqslant 0,\ x_0 \in \Sigma_0\\
\overset{\substack{(2.3)\\(2.37)}}{\Leftrightarrow}\quad & \mu^{\mathfrak{c}}_{\mathfrak{v}}(y - s\cdot x_1) \geqslant 0\\
\Leftrightarrow\quad & \mu^{\mathfrak{c}}_{\mathfrak{v}}(y) \geqslant s\\
\overset{2.5.1(iv)}{\Leftrightarrow}\quad & \mu^{\mathfrak{c}}_{\mathfrak{v}}(x) \geqslant s.
\end{aligned}$$

$(\star)$: See Figure 2.6.



$\left\{h_{\mathfrak{v}}(\mathrm{pr}^0_{\Sigma,\mathfrak{c}}(x)) = 0\right\} = \{x \mid \mathfrak{v} \in \sigma_0(x, \mathfrak{c})\}$

Figure 2.6: The set of points in Σ that are 0-*close* to $\mathfrak{v}$.

Now consider an arbitrary chamber $c = \gamma \cdot q_i \cdot \mathfrak{c}$ with $v = \gamma \cdot q_i \cdot \mathfrak{v}$. Moreover write $x = \gamma \cdot q_i \cdot um \cdot y$ with $um \in \mathbf{U}(\mathbb{A}_S)\mathbf{M}(\mathbb{A}_S)$ and $y \in \Sigma$.

$$
\begin{aligned}
& v \in \sigma_s(x, c) \\
\Leftrightarrow\quad & \gamma \cdot q_i \cdot \mathfrak{v} \in \sigma_s(\gamma \cdot q_i \cdot um \cdot y, \gamma \cdot q_i \cdot \mathfrak{c}) \\
\overset{(2.29)}{\Leftrightarrow}\quad & h_{\gamma \cdot q_i \cdot \mathfrak{v}}(\mathrm{pr}^s_{\gamma \cdot q_i \cdot um \cdot \Sigma, \gamma \cdot q_i \cdot \mathfrak{c}}(\gamma \cdot q_i \cdot um \cdot y)) = s \\
\overset{2.5.6(iii)}{\Leftrightarrow}\quad & h_{\gamma \cdot q_i \cdot \mathfrak{v}}(\gamma \cdot q_i \cdot um \cdot (\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(y + z_i) - z_i)) = s \\
\overset{\substack{2.4.7(i),(v)\\(2.26)}}{\Leftrightarrow}\quad & h_{\mathfrak{v}}(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(y + z_i) - z_i) + s_{\mathfrak{v}} \cdot s(i, \mathfrak{v}) = s \\
\overset{(2.41)}{\Leftrightarrow}\quad & h_{\mathfrak{v}}(\mathrm{pr}^s_{\Sigma,\mathfrak{c}}(y + z_i)) = s \\
\overset{(2.29)}{\Leftrightarrow}\quad & \mathfrak{v} \in \sigma_s(y + z_i, \mathfrak{c}) \\
\overset{\substack{\text{special}\\\text{case}}}{\Rightarrow}\quad & \mu^{\mathfrak{c}}_{\mathfrak{v}}(y + z_i) \geqslant s \\
\overset{(2.42)}{\Leftrightarrow}\quad & \mu^{\mathfrak{c}}_{\mathfrak{v}}(y) + c(i, \mathfrak{v}) \geqslant s \\
\overset{\substack{2.5.1(i),\\(iv),(v)}}{\Leftrightarrow}\quad & \mu^{\gamma \cdot q_i \cdot \mathfrak{c}}_{\gamma \cdot q_i \cdot \mathfrak{v}}(\gamma \cdot q_i \cdot um \cdot y) \geqslant s \\
\Leftrightarrow\quad & \mu^c_v(x) \geqslant s.
\end{aligned}
$$

□

CHAPTER 3

PROOF AND CONCLUSION

3.1 Proof

In this section we prove Theorem 1.1.1.

Theorem 3.1.1. *Consider the* Γ*-invariant family* $\{h_v \mid v \in \mathcal{V}(\Delta)\}$ *of rescaled Busemann functions. For this family and an arbitrary* $d \in \mathbb{R}_+$ *there is a constant* $r \in \mathbb{R}$*, which we call a* <u>*lower d-reduction bound*</u>*, such that the following holds:*

> *Every ball of diameter at most* d *is uniformly* r*-reduced by some chamber* c*.*

Moreover, given any lower reduction bound $r \in \mathbb{R}$*, there is corresponding constant* $R \in \mathbb{R}$*, which we call an* <u>*upper r-reduction bound*</u>*, such that:*

> *For any chamber* c *that* r*-reduces* x*, the simplex* $\sigma_R(x, c)$ *is contained in any chamber* c' *that* r*-reduces* x*.*

Proof. To prove the first part, we have to argue the existence of a constant $r \in \mathbb{R}$, such that each ball of diameter at most d is contained in $N_c(r)$ for some chamber $c \in \mathcal{C}(\Delta)$. By Theorem 2.5.7 we may equivalently prove each ball of diameter at most d to be contained in $Z_c(r)$ for some chamber $c \in \mathcal{C}(\Delta)$. This last assertion results from Lemma 2.5.1 and Theorem 2.2.2:

First we apply Lemma 2.5.1 (vi) to reduce the problem to balls of diameter $d = 0$, i.e. to points in X_S, see Figure 3.1. Because $\mathfrak{X}_S$ and X_S are in finite hausdorff-distance we may apply the same argument again to reduce the problem to points in $\mathfrak{X}_S$. Next recall that $\mathfrak{X}_S \cong \mathbf{G}(\mathbb{A}_S)/K$. Now Theorem 2.2.2 comes into play: There exists a constant $\hat{r} \in \mathbb{R}$ such that given an arbitrary $g \in \mathbf{G}(\mathbb{A})$ there exists a chamber $c \in \mathcal{C}(\Delta)$ with $\log(\mathcal{S}_{\varphi_v^c}(g)) \geqslant \hat{r}$ for all $v \in \mathcal{V}(c)$. Now we set

$$c_{\min,2} := \min\left\{\log\left(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right) \,\middle|\, i \in \{1, \dots, m\}, \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\},$$
$$c_{\max,2} := \max\left\{\log\left(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right) \,\middle|\, i \in \{1, \dots, m\}, \mathfrak{v} \in \mathcal{V}(\mathfrak{c})\right\},$$

and choose some $r \leqslant \hat{r} - c_{\max,2}$.

For $g \in \mathbf{G}(\mathbb{A}_S)$ Lemma 2.5.1 (ii) implies

$$\mu_v^c(g \cdot *) = \log(\mathcal{S}_{\varphi_v^c}(g)) - \log\left(\mathcal{S}_{\varphi_{\mathfrak{v}}^{\mathfrak{c}}}\left(\left(q_{k(i,\mathfrak{v})}^{-1}\right)_{\overline{S}}\right)\right) \geqslant r.$$

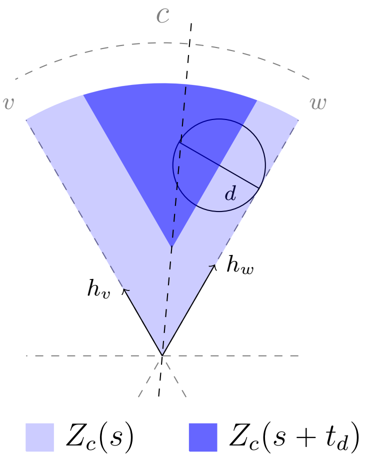


Figure 3.1: Minkowski cones of height $s + t_d$ contain each ball of radius d whose center is located in the corresponding cone of height s.

To prove the second assertion we let $r \in \mathbb{R}$ be any lower reduction bound. Next we choose some $\hat{r} \leqslant r + c_{\min,2}$ and consider a corresponding constant $\hat{R}$ that satisfies the condition in Theorem 2.2.3. Last we choose some $R \geqslant \hat{R} - c_{\min,2}$. With this choice the assertion holds:

Let $g \in \mathbf{G}(\mathbb{A}_S)$ and $g \cdot * \in X_S$ be r-reduced by two chambers $c, c' \in \mathcal{C}(\Delta)$, i.e.

$$\begin{aligned} & g \cdot * \in N_c(r) \cap N_{c'}(r) \\ \overset{2.5.7}{\Leftrightarrow} \quad & g \cdot * \in Z_c(r) \cap Z_{c'}(r) \\ \overset{(2.33)}{\Leftrightarrow} \quad & \begin{cases} \mu_w^c(g \cdot *) \geqslant r \text{ for all } w \in \mathcal{V}(c), \\ \mu_{w'}^{c'}(g \cdot *) \geqslant r \text{ for all } w' \in \mathcal{V}(c'), \end{cases} \\ \overset{2.5.1(ii)}{\Rightarrow} \quad & \begin{cases} \log(\mathcal{S}_{\varphi_w^c}(g)) \geqslant \hat{r} \text{ for all } w \in \mathcal{V}(c), \\ \log(\mathcal{S}_{\varphi_{w'}^{c'}}(g)) \geqslant \hat{r} \text{ for all } w' \in \mathcal{V}(c'). \end{cases} \end{aligned} \tag{3.1}$$

Given $v \in \mathcal{V}(c)$ with $v \in \sigma_R(g \cdot *, c)$ we moreover have

$$v \in \sigma_R(g \cdot *, c) \overset{2.5.8}{\Rightarrow} \mu_v^c(g \cdot *) \geqslant R \overset{2.5.1(ii)}{\Rightarrow} \log(\mathcal{S}_{\varphi_v^c}(g)) \geqslant \hat{R}.$$

By equation (3.1) and Theorem 2.2.3 we have $v \in \mathcal{V}(c')$. □

We have just proven the existence of a Γ-invariant, d-uniform reduction datum. It remains to verify the Γ-cocompactness. Therefore we investigate the following commutative diagram:

$$\begin{array}{ccc} \mathbf{G}(\mathbb{A}_S) & \xhookrightarrow{\iota} & \mathbf{G}(\mathbb{A}) \\ \Big\downarrow{\scriptstyle \pi_S} & & \Big\downarrow{\scriptstyle \pi} \\ \Gamma\backslash\mathbf{G}(\mathbb{A}_S) & \xhookrightarrow[\bar{\iota}]{} & \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) \end{array}$$

We start to examine the projections π and π_S: Because $\mathbf{G}(\mathbb{Q})$ is discret in $\mathbf{G}(\mathbb{A})$, the map π is a covering map with fiber $\mathbf{G}(\mathbb{Q})$. Hence, given some $g \in \mathbf{G}(\mathbb{A})$, there exists a neighbourhood $U_{\pi(g)}$ of $\pi(g)$ such that

$$\pi^{-1}(U_{\pi(g)}) = \dot{\bigcup_{q \in \mathbf{G}(\mathbb{Q})}} q \cdot E_g = \mathbf{G}(\mathbb{Q}) \cdot E_g$$

and $\pi \colon E_g \to U_{\pi(g)}$ is a homeomorphism. Since $\mathbf{G}(\mathbb{A})$, and hence $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$, is locally compact, we may assume E_g and $U_{\pi(g)}$ to be compact without loss of generality.

Lemma 3.1.2. *A subset $H \subset \mathbf{G}(\mathbb{A})$ is (relatively) compact modulo $\mathbf{G}(\mathbb{Q})$ if and only if $\pi(H) \subset \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ is (relatively) compact.*

Proof. Assume $\pi(H)$ to be compact. In that case there exist finitely many elements $h_1, ..., h_n \in H$ such that

$$\pi(H) \subset \bigcup_{i=1}^{n} U_{\pi(h_i)}$$

with $U_{\pi(h_i)}$ as above and hence

$$H \subset \pi^{-1}(\pi(H)) \subset \mathbf{G}(\mathbb{Q}) \cdot \left(\bigcup_{i=1}^{n} E_{h_i} \right).$$

The other direction is clear, since the image of compact sets is compact. □

Since Γ is discrete in $\mathbf{G}(\mathbb{A}_S)$ one equivalently proves the analogous lemma:

Lemma 3.1.3. *A subset $H_S \subset \mathbf{G}(\mathbb{A}_S)$ is (relatively) compact modulo Γ if and only if $\pi_S(H_S) \subset \Gamma\backslash\mathbf{G}(\mathbb{A}_S)$ is (relatively) compact.*

Next we investigate the inclusions ι and $\bar{\iota}$:

Lemma 3.1.4. *$\iota\colon \mathbf{G}(\mathbb{A}_S) \hookrightarrow \mathbf{G}(\mathbb{A})$ and $\bar{\iota}\colon \Gamma\backslash\mathbf{G}(\mathbb{A}_S) \hookrightarrow \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ are injective and open.*

Proof. Obviously both maps are injective. Moreover $\iota\colon \mathbf{G}(\mathbb{A}_S) \hookrightarrow \mathbf{G}(\mathbb{A})$ is open because $\mathbf{G}(\mathbb{Z}_p)$ is open in $\mathbf{G}(\mathbb{Q}_p)$ for all $p \notin S$. To prove that $\bar{\iota}\colon \Gamma\backslash\mathbf{G}(\mathbb{A}_S) \hookrightarrow \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ is open choose some open set $U \subset \Gamma\backslash\mathbf{G}(\mathbb{A}_S)$. By definition of the quotient topology $\bar{\iota}(U)$ is open in $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ if and only if its preimage $\pi^{-1}(\bar{\iota}(U))$ is open in $\mathbf{G}(\mathbb{A})$. That the preimage is in fact open follows because $\iota\colon \mathbf{G}(\mathbb{A}_S) \hookrightarrow \mathbf{G}(\mathbb{A})$ is open:

$$\pi^{-1}(\bar{\iota}(U)) = \pi^{-1}(\bar{\iota}(\pi_S(\pi_S^{-1}(U)))) = \pi^{-1}(\pi(\iota(\pi_S^{-1}(U)))) = \mathbf{G}(\mathbb{Q}) \cdot \iota(\pi_S^{-1}(U)).$$

□

Lemma 3.1.5. *Let X, Y be topological spaces and $f\colon X \hookrightarrow Y$ an injective open map. A subset $Z \subset X$ is compact if and only if its image $f(Z) \subset Y$ is compact.*

Proof. Let $Z \subset X$ such that $f(Z) \subset Y$ is compact. Now choose an open covering $\{U_i \mid i \in I\}$ of Z. Since $f\colon X \hookrightarrow Y$ is open $\{f(U_i) \mid i \in I\}$ is an open covering of $f(Z)$. Hence there is a finite subcovering $\{f(U_i) \mid 1 \leqslant i \leqslant n\}$. The claim now follows from the injectivity of f:

$$Z = f^{-1}(f(Z)) \subset f^{-1}\left(\bigcup_{i=1}^{n} f(U_i)\right) = \bigcup_{i=1}^{n} f^{-1}(f(U_i)) = \bigcup_{i=1}^{n} U_i.$$

□

Corollary 3.1.6. *A subset $H_S \subset \mathbf{G}(\mathbb{A}_S)$ is (relatively) compact modulo Γ if and only if its image $\iota(H_S) \subset \mathbf{G}(\mathbb{A})$ is (relatively) compact modulo $\mathbf{G}(\mathbb{Q})$.*

Proof. This results from Lemma 3.1.2, Lemma 3.1.3, Lemma 3.1.4 and Lemma 3.1.5. □

Theorem 3.1.7. *Given a lower reduction bound r and some $s \geqslant r$ the set*

$$Y_s := \{x \in X_S \mid h_c(x) \leqslant s \text{ for all } c \in \mathcal{C}(\Delta) \text{ that } r\text{-reduce } x\}$$

is Γ-invariant and relatively compact modulo Γ.

Proof. The set Y_s is Γ-invariant due to Lemma 2.5.6 (iv) and the Γ-invariance of the family $\{h_v \mid v \in \mathcal{V}(\Delta)\}$. It remains to show that Y_s is relatively compact modulo Γ:

Because $\mathfrak{X}_S$ and X_S are in finite hausdorff-distance, it is enough to prove the Γ-invariant set $Y_s \cap \mathfrak{X}_S$ to be relatively compact modulo Γ. Alternatively we may, since $\mathfrak{X}_S \cong \mathbf{G}(\mathbb{A}_S)/K$, search for a Γ-invariant subset $H_S \subset \mathbf{G}(\mathbb{A}_S)$ with $H_S \cdot * = Y_s \cap \mathfrak{X}_S$, that is relatively compact modulo Γ. By Corollary 3.1.6 it even suffices to show such H_S to be relatively compact in $\mathbf{G}(\mathbb{A})$ modulo the action of $\mathbf{G}(\mathbb{Q})$:

Consider the Γ-invariant set $H_S := \{h \in \mathbf{G}(\mathbb{A}_S) \mid h \cdot * \in Y_s\}$ and pick an arbitrary $h \in H_S$. Since r is a lower reduction bound there is a chamber $c \in \mathcal{C}(\Delta)$ that r-reduces $h \cdot *$, i.e. $h \cdot * \in N_c(r)$. We deduce from Theorem 2.5.7 and the assumption that

$$r \leqslant \mu_v^c(h \cdot *) \text{ and } h_v(h \cdot *) \leqslant s$$

for all $v \in \mathcal{V}(c)$. Because $s_{\mathfrak{v}}$ is positive by Lemma 2.5.2, we conclude that

$$\begin{aligned} \log(\mathcal{S}_{\varphi_v^c}(h)) &\overset{2.5.1(ii)}{\geqslant} r + c_{\min,2}, \\ \log(\mathcal{S}_{\varpi_v}(h)) &\overset{\substack{2.4.7(ii)\\(2.26)}}{\leqslant} s/s_{\mathfrak{v}} + c_{\max,2} \end{aligned}$$

for $v = q \cdot \mathfrak{v} \in \mathcal{V}(c)$. The compactness criterion, Theorem 2.2.4, implies H_S to be relatively compact in $\mathbf{G}(\mathbb{A})$ modulo the action of $\mathbf{G}(\mathbb{Q})$. $\square$

3.2 Conclusion

In this work we have, as announced in the abstract, created a reduction theory on X_S analogously to like it was done in [BKW13]. Though we cannot consider it as an intermediate step to some universally valid geometric reduction theory as we hoped for. Indeed the final goal to create a universaly valid reduction theory on CAT(0)-spaces someday, seems to have become impossible, or is at least brought into even more remote future:

We percieved that one may not assume the Busemann functions to be of unit speed without loss of generality. Counterexamples (E_6, E_7, E_8 and D_n) prove that one must adapt the velocity of all Busemann functions in order to satisfy condition (C). Unfortunatelly that adaption highly depends on the structure of Δ at infinity of X_S. Given an arbitrary CAT(0)-space there is no such structure.

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3.1 Minkowski cones of height $s + t_d$ contain each ball of radius d whose center is located in the corresponding cone of height s. . 52

LIST OF SYMBOLS

$\mathbb{A}$	ring of adeles 1
$\mathbb{A}_S$	ring of S-adeles 1
$\mathcal{A}$	apartment system in X 29
$\mathcal{A}^c$	apartments in X 29
$\mathcal{A}^{\mathfrak{c}}$	apartments in X 29
$\mathcal{A}_\infty$	*apartment system* in X_∞ 28
$\mathcal{A}^{\mathfrak{c}}_\infty$	*apartments* in X_∞ 28
$\mathcal{A}_p$	apartment system in X_p 28
$\mathcal{A}^{\mathfrak{c}}_p$	set of apartments in X_p 28
$b^s_{c,v}$	function from X to $\mathcal{V}(\Delta)$ 41
$c_{\max,1}$	constant 23
$c_{\max,2}$	constant 51
$c_{\min,1}$	constant 23
$c_{\min,2}$	constant 51
$c(i,\mathfrak{v})$	constant 38
$c_{\mathfrak{v}\mathfrak{w}}$	constant 18
$\mathfrak{c}$	standard chamber of Δ 12
$\mathcal{C}(\Delta)$	chambers of Δ 2
Δ	rational building in $\partial(X_S)$ 2
$\partial(X_S)$	visual boundary of X_S 2
E	compact subset of $\mathbf{G}(\mathbb{A})$ 12
F	compact subset of $\mathbf{P}_{\mathfrak{c}}(\mathbb{A})^\circ$ 12
$\mathbf{G}$	linear algebraic group 1
Γ	S-arithmetic lattice 2
$\mathbf{H}(\mathbb{A})^\circ$	subgroup of $\mathbf{H}(\mathbb{A})$ 12
h_c	maximum of Busemann functions 2, 40
h_v	(general) Busemann function 2, 40

$\overline{h}_{\gamma,q_i,\mathfrak{v}}$	(general) Busemann function 32
$\overline{h}_{\mathfrak{v}}$	standard Busemann function 30
$\tilde{h}_{\gamma,q_i,\mathfrak{v}}$	(general) Busemann function 35
$\tilde{h}_{v}$	(general) Busemann function 35
K	compact subgroup of $\mathbf{G}(\mathbb{A})$ 15
$k(i,\mathfrak{v})$	*smallest* index in $\{1,...,m\}$ with ... 34
$\overline{\log \circ \mathcal{S}_{\varpi_\mathfrak{v}}}$	function on $\mathbf{G}(\mathbb{A}_S)/K$ 30
$\mathbf{M}$	almost Levi factor of $\mathbf{P}_\mathfrak{c}$ 28
μ_v^c	affine linear function 38
$n_{\mathfrak{v}\mathfrak{w}}$	normal cone of $Y_{\Sigma',c}(s)$ 18
$N_c(s)$	normal cone 4, 41
$N_{\Sigma',c}(s)$	normal cone of $Y_{\Sigma',c}(s)$ 4, 41
$\mathbf{P}_c$	minimal $\mathbb{Q}$-parabolic subgroup 12
$\mathbf{P}_v$	maximal $\mathbb{Q}$-parabolic subgroup 12
$\mathrm{pr}^s_{\Sigma',c}$	projection to $Y_{\Sigma',c}$ 2, 40
φ_v^c	(general) root 21
$\varphi_\mathfrak{v}^\mathfrak{c}$	standard root 12
ϖ_v	(general) weight 21
$\varpi_\mathfrak{v}$	standard weight 18
$\varpi_\mathfrak{v}^\mathfrak{c}$	standard weight restricted to $\mathbf{P}_\mathfrak{c}$ 18
$q_1,\dots,q_m$	representatives for $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ 32
$s(i,\mathfrak{v})$	constant 34
$s_\mathfrak{v}$	constant 39
$\mathcal{S}_{\varphi_\mathfrak{v}^\mathfrak{c}}$	scaling function for $\varphi_\mathfrak{v}^\mathfrak{c}$ 20
$\mathcal{S}_{\varphi_v^c}$	scaling function for φ_v^c 22
$\mathcal{S}^1_{\varpi_\mathfrak{v}}$	scaling function for $\varpi_\mathfrak{v}$ 19
$\mathcal{S}^2_{\varpi_\mathfrak{v}}$	scaling function for $\varpi_\mathfrak{v}$ 20
$\mathcal{S}_{\varpi_\mathfrak{v}}$	scaling function for $\varpi_\mathfrak{v}$ 20
$\mathcal{S}_{\varpi_v}$	scaling function for ϖ_v 21
Σ	standard apartment in X 29
Σ_∞	(standard) flat subspace in X_∞ 28
Σ_p	(standard) apartment in X_p 28
Σ_0	tip of $Y_{\Sigma,c}$ 42

$\sigma_s(x,c)$	subset of $\mathcal{V}(c)$ 3, 41
$\mathbf{T}$	maximal $\mathbb{Q}$-split torus in $\mathbf{P}_{\mathfrak{c}}$ 12
$\mathbf{T}_\infty$	real part of $\mathbf{T}(\mathbb{A})$ 13
$\mathbf{T}_\infty^+$	connected component of $\mathbf{T}_\infty$ 13
$\mathbf{T}_\infty^+(s)$	positive chamber in $\mathbf{T}_\infty^+$ 13
$\mathbf{T}_p$	maximal $\mathbb{Q}_p$-split torus in $\mathbf{P}_{\mathfrak{c}}$ 12
$\mathbf{T}_{\max}$	maximal torus in $\mathbf{P}_{\mathfrak{c}}$ 12
$\mathbf{U}$	unipotant radical of $\mathbf{P}_{\mathfrak{c}}$ 28
$\mathcal{V}(c)$	vertices of c 17
$\mathcal{V}(\Delta)$	vertices of Δ 2
$\mathfrak{v}$, $\mathfrak{w}$,...	standard vertices of Δ 12
W	Weyl group 18
X_S	CAT(0)-space 1
X_∞	symmetric space associated to $\mathbf{G}(\mathbb{R})$ 1
X_p	Bruhat-Tits building associated to $\mathbf{G}(\mathbb{Q}_p)$ 1
$\mathfrak{X}_S$	$\mathbf{G}(\mathbb{A}_S)$-orbit of $*$ 15
$\mathfrak{X}_\infty$	$\mathbf{G}(\mathbb{Q}_p)$-orbit of $*_\infty$ 14
$\mathfrak{X}_p$	$\mathbf{G}(\mathbb{Q}_p)$-orbit of $*_p$ 14
$X(\mathbf{H})$	$\mathbb{Q}$-characters of $\mathbf{H}$ 12
x_1	vector in Σ 39
Y_s	relatively compact modulo Γ 54
$Y_c(s)$	convex subset of Σ 2
$Y_{\Sigma',c}(s)$	convex subset of Σ' 2, 40
$Z_c(s)$	normal cone 42
$Z_{\Sigma',c}(s)$	normal cone of $Y_{\Sigma',c}(s)$ 42
z_i	vector in Σ 46
$\|-\|$	idele norm on $\mathbb{A}^\times$ 12
$\||-\||$	map on $V(\mathbb{A})'$ compatible with $\|-\|$ 14
$*$	point in X_S 15
$*_\infty$	point in X_∞ 14
$*_p$	point in X_p 14

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