Corrigendum and addendum to "Universal algebra for general aggregation theory"*

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Abstract

In a recent paper [*Journal of Logic and Computation*, forthcoming (2014); `doi:10.1093/logcom/ext009`], it was claimed that a universal algebraic approach to general aggregation theory based on MV-homomorphisms could even cover linear probabilistic opinion pooling. This is not so, however. The reason is that there are no non-trivial homomorphisms from direct powers of the standard MV-algebra onto the standard MV-algebra itself; an analytic proof thereof is given in this note.

Key words: propositional attitude aggregation; probabilistic opinion pooling; fuzzy logic; standard MV-algebra; homomorphism; Cauchy's functional equation; projection; direct product

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In the original paper [10] which this erratum corrects, a universal algebraic approach to the theory of aggregating propositional attitudes (or general aggregation theory) – recently introduced by Dietrich and List [5, 6, 7, 8] – was discussed. This approach is based on the theory of C.C. Chang's MV-algebras [3, 4], treating propositional-attitude functions as MV-homomorphisms. In that paper the opinion was expressed, without presenting a proof, that even linear probabilistic opinion pooling could be subsumed under this theory.

This is not so, however, since there is no straightforward reduction of the category of probability measures to that of MV-homomorphisms. For example, an MV-homomorphism h from a Boolean algebra $\mathcal{A}$ to the standard MV-algebra $[0,1]$ would have to satisfy $h(A) = h(\underbrace{A \oplus \cdots \oplus A}_{n \text{ times}}) = \min\{nh(A), 1\}$ for all $A \in \mathcal{A}$ and positive integers n, whence the image of h must actually be (a subset of and hence) equal to $\{0, 1\}$.

Also, there are no non-trivial homomorphisms from direct powers of the standard MV-algebra onto the standard MV-algebra itself (as we shall show at the end of this paper). Linear averaging, as a map from $[0,1]^n$ (the set of n-element fuzzy sets) to $[0,1]$, is never an MV-homomorphism except when it is a mere projection. More precisely, let n be a positive integer and $[0,1]^n$ the direct product of n copies of the standard MV-algebra $[0,1]$. A *projection* from $[0,1]^n$ to $[0,1]$ is a map π such that there exists some $k < n$ such that $\pi(x_0, \ldots, x_{n-1}) = x_k$ of all $x \in [0,1]^n$. Then we have the following

Fact. *The only MV-homomorphisms from* $[0,1]^n$ *to* $[0,1]$ *are the projections.*

(We shall give a new analytic proof below; algebraic proofs, based on deep representation and classification theorems for MV-algebras, are also known.[1])

This Fact implies that the corollary [10, Corollary 4.3] which we introduced as a version of McConway's characterisation of linear opinion pools [11, Theorem 3.3] is in fact an impossibility theorem. Although the corollary is not technically incorrect, it does not convey the full information and was seriously misinterpreted in the original paper. A more exact characterisation result is the following (using the terminology of the original paper):

Let F be a rational, universal, Paretian and strongly systematisable aggregator, let the algebra of truth values be the standard MV-algebra $[0,1]$*, and suppose the electorate N is finite. Then the decision criterion of F is a projection from* $[0,1]^N$ *to* $[0,1]$.

Now, in the original paper, this Corollary read "linear map" instead of 'projection'. While the proof of the linearity was essentially correct, it overlooked the fact that there are no non-trivial homomorphisms from $[0,1]^N$ to $[0,1]$; in this setting a linear map is always a projection.

In sum, $[0,1]$ as an MV-algebraic object is structurally very different from $[0,1]$ as a set of probabilities. This means that the standard MV-algebra should be regarded as a set of degrees of 'imprecision' or 'vagueness' that cannot be reduced to probabilities and *a fortiori* not to probabilistic credences. Hence, our aggregation theory based on MV-homomorphisms [10] covers the aggregation of fuzzy propositional attitudes, but not probabilistic opinion pooling.

[1] In particular, after this proof was written down, the author contacted Professor Daniele Mundici who has independently produced a more elegant algebraic proof and kindly provided additional relevant references. A similar result is stated in Flaminio, Godo and Kroupa [9, Theorem 3.3].

However, finitely-additive probability measures on MV-algebras are known as *states* and have also been studied by numerous authors, following the seminal paper by Mundici [12]. An overview of related literature can be found in Flaminio, Godo and Kroupa [9]. One may hope that ultimately an algebraic framework for aggregation theory which does encompass probabilistic opinion pooling may be developed, even though the concept of homomorphy on which our approach was based is not suitable for this purpose.

Finally we come to the

Proof of the Fact. Let $f : [0,1]^n \to [0,1]$ be an MV-homomorphism. Then

$$f(x_0, \ldots, x_{n-1}) \oplus f(y_0, \ldots, y_{n-1}) = f(x_0 \oplus y_0, \ldots, x_{n-1} \oplus y_{n-1})$$

for all $x, y \in [0,1]^n$. Hence (by the definition of $\oplus$ in the standard MV-algebra, viz. $x \oplus y = \min\{x+y, 1\}$ for all $x, y \in [0,1]$ and the componentwise definition of $\oplus$ in the direct power $[0,1]^n$) one has for all $x_0, y_0, \ldots, x_{n-1}, y_{n-1} \in [0,1]$ with $x_i + y_i \leq 1$ for all $i < n$,

$$\min\{f(x_0, \ldots, x_{n-1}) + f(y_0, \ldots, y_{n-1}), 1\} = f(x_0 + y_0, \ldots, x_{n-1} + y_{n-1}). \tag{1}$$

Define $w_i = x_i + y_i$ (≤ 1) for all $i < n$.

Now, if $w_i < 1$ for all $i < n$, we must have $f(w) < 1$ and thus even

$$f(x_0, \ldots, x_{n-1}) + f(y_0, \ldots, y_{n-1}) = f(x_0 + y_0, \ldots, x_{n-1} + y_{n-1}). \tag{2}$$

(For suppose $f(w) = 1$ while $w_i < 1$ for all $i < n$. Then, $f(1-w) = 0$ while $1 - w_i > 0$ for all $i < n$. Since f preserves $\leq$, defined componentwise, [2] it follows that $f = 0$ on $[0, \min_i w_i]^n$. In particular, there exists some $N \in \mathbf{N}$ such that $f(1/N, \ldots, 1/N) = 0$. But then, since f is a homomorphism and $\oplus$ was defined componentwise, $f(1) = Nf(1/N, \ldots, 1/N)) = 0 \neq 1$, a contradiction to homomorphy.)

One can now emulate McConway's [11, Proof of Theorem 3.3] original argument for the characterisation of linear opinion pools: An iterated application of the preceding equation yields for all $z_0, \ldots, z_{n-1} \in [0,1)$,

$$\begin{aligned} f(z_0, \ldots, z_{n-1}) &= f(z_0, 0, \ldots, 0) + f(0, z_1, \ldots, z_{n-1}) \\ &= f(z_0, 0, \ldots, 0) + f(0, z_1, 0, \ldots, 0) + f(0, 0, z_2, \ldots, z_{n-1}) \\ &= \sum_{i=0}^{n-1} f\left(\underbrace{0, \ldots, 0}_{i}, z_i, \underbrace{0, \ldots, 0}_{n-i-1}\right). \end{aligned}$$

Hence, definining f_i by

$$f_i(z) = f\left(\underbrace{0, \ldots, 0}_{i}, z, \underbrace{0, \ldots, 0}_{n-i-1}\right) \tag{3}$$

[2] To see that f preserves $\leq$ thus defined, note the following: For all $i < n$ and $x_i, y_i \in [0,1]$,

$$x_i \leq y_i \Leftrightarrow \neg x_i \oplus y_i = 1,$$

hence if $\leq$ is defined componentwise,

$$x \leq y \Leftrightarrow \neg x \oplus y = 1.$$

Since f is a homomorphism and thus commutes with $\neg$ and $\oplus$, this equivalence can be used to establish that $f(x) \leq f(y)$ whenever $x \leq y$.

for every $z \in [0,1)$ and each $i < n$, we obtain

$$f(z_0, \ldots, z_{n-1}) = \sum_{i=0}^{n-1} f_i(z_i)$$

for all $z_0, \ldots, z_{n-1} \in [0,1)$. Moreover, the combination of Equations (2) and (3) also yields $f_i(x+y) = f_i(x) + f_i(y)$ for all $x, y \in [0,1)$ with $x + y < 1$ and each $i < n$.

Therefore, every f_i satisfies Cauchy's functional equation. Also, the range of every f_i is by definition contained in the range of f and thus in $[0,1]$, whence $f_i(x)$ is nonnegative for all $x \in [0,1]$ and every $i < n$. Hence there exists for every $i < n$ some α_i such that $f_i(x) = \alpha_i x$ for all $x \in [0,1)$ (cf. Aczél [1, 2]), and this α_i must be nonnegative. Thus, $f(z_0, \ldots, z_{n-1}) = \sum_{i=0}^{n-1} \alpha_i z_i$ for all $z \in [0,1)^n$. Hence, f is continuous in 0. But, as a homomorphism, f satisfies $f(1-x) = 1 - f(x)$ for all $x \in [0,1]^n$ and therefore f is continuous in $1 = (1, \ldots, 1)$, too. It follows that $f(z_0, \ldots, z_{n-1}) = \sum_{i=0}^{n-1} \alpha_i z_i$ even for all $z \in [0,1]^n$. Again since f is a homomorphism, $1 = f(1) = \sum_{i=0}^{n-1} \alpha_i$.

However, there must be some $k < n$ such that $\alpha_i = \delta_{ik}$ for all $i < n$. For, suppose otherwise. Then we would have $\alpha_j \in (0,1)$ for some $j < n$. Let $d = \frac{1}{2\alpha_j}$ and $x = (\delta_{ij} d)_{i=0}^{n-1}$. Then, $x \oplus x = (\delta_{ij})_{i=0}^{n-1}$ and hence $f(x \oplus x) = \alpha_j < 1$, but $f(x) = d\alpha_j = \frac{1}{2}$ and thus $f(x) \oplus f(x) = 1$. Hence $f(x) \oplus f(x) \neq f(x \oplus x)$ whence f would not be a homomorphism, contradiction.

Hence, in sum, there exists some $k < n$ such that $f(x) = x_k$ for all $x \in [0,1]^n$.

Conversely, the componentwise definition of $\neg$ and $\oplus$ in the direct power $[0,1]^n$ implies immediately that the projections are MV-homomorphisms. $\square$

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