Representation Zeta Functions of Special Linear Groups

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La Luna piena minchionò la Lucciola:
- Sarà l'effetto de l'economia,
ma quel lume che porti è debboluccio...
- Sì, - disse quella - ma la luce è mia!
Trilussa, "La Lucciola", Acqua e Vino.

Contents

Introduction

Background and motivation

Let G be a group. For $n \in \mathbb{N}$, we denote by $r_n(G)$ the number of isomorphism classes of n-dimensional irreducible complex representations of G. When G is a topological or an algebraic group, it is tacitly understood that representations enumerated by $r_n(G)$ are continuous or rational, respectively. Furthermore, throughout this work, G is (representation) *rigid*, i.e. $r_n(G)$ is finite for all $n \in \mathbb{N}$.

Character degrees and conjugacy classes have been studied in depth in finite group theory (see [16, 30] and references therein). In [25], Liebeck and Shalev take an asymptotic point of view to the problem: they focus on character degrees of finite groups H of Lie type as $|H|$ tends to infinity. In a similar flavour but for an infinite group G, *representation growth* is concerned with the arithmetic properties of the sequence $r_n(G)$ as n tends to infinity. This interest is inspired also by the investigations in the area of subgroup growth, which studies the distribution of finite-index subgroups in G (see for instance, [10, 27]).

The function $r_n(G)$ as n varies in $\mathbb{N}$ is called the representation growth function of G. If the sequence $R_N(G) = \sum_{n=1}^{N} r_n(G)$, $N \in \mathbb{N}$, is bounded by a polynomial in N, the group G is said to have *polynomial representation growth* (PRG). The representation growth of a group with PRG can be studied by means of the *representation zeta function*, namely, the Dirichlet series

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s},$$

where s is a complex variable. It is well known that the *abscissa of convergence* $\alpha(G)$ of the series $\zeta_G(s)$, that is, the infimum of all $\alpha \in \mathbb{R}$ such that $\zeta_G(s)$ converges on the complex half-plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha\}$, gives the precise degree of polynomial growth: $\alpha(G)$ is the smallest value such that $R_N(G) = O(1 + N^{\alpha(G)+\varepsilon})$ for every $\varepsilon \in \mathbb{R}_{>0}$ (see [9, Theorem 4.20]).

The first crucial steps in the description of the representation growth of arithmetic groups were made by Larsen and Lubotzky in [22]. In this context one typically studies also the representation growth of p-adic analytic groups. A compact p-adic analytic group G is rigid if and only if it is FAb, that is, if every open subgroup has finite abelianization (see [5, Proposition 2]). Making use of model-theory, Jaikin-Zapirain proved in [18] that the representation zeta function of a FAb compact p-adic analytic group $(p > 2)$[1] is a rational function in p^{-s}. This means that the coefficients of the Dirichlet generating function satisfy a linear recurrence relation.

Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic 0, maximal ideal $\mathfrak{p}$ and residue field characteristic p. Let also G be a linear algebraic group defined over $\mathbb{Z}$. The principal m-th congruence subgroup is $\mathrm{G}^m(\mathfrak{o}) = \ker_{\mathfrak{o}} \sigma_m \leq \mathrm{G}(\mathfrak{o})$ where

$$\sigma_m : \mathrm{G}(\mathfrak{o}) \to \mathrm{G}(\mathfrak{o}/\mathfrak{p}^m)$$

[1]When $p = 2$, the rationality holds for uniformly powerful groups and it is conjectured to hold generally, as for $p \neq 2$.

is the reduction mod $\mathfrak{p}^m$. Key examples of FAb compact p-adic analytic groups are the special linear groups $\mathrm{SL}_n(\mathfrak{o})$ and their principal congruence subgroups $\mathrm{SL}_n^m(\mathfrak{o})$.

The arithmetic groups whose representation growth has been typically studied are arithmetic subgroups of semisimple algebraic groups H defined over number fields. More precisely, these are groups Γ which are commensurable to $\mathrm{H}(\mathcal{O}_S)$, where H is a connected, simply connected semisimple algebraic group defined over a number field k and $\mathcal{O}_S$ is the the ring of S-integers in k for a finite set S of places of k including all the archimedean ones. Let Γ be of this form. Lubotzky and Martin showed that Γ has PRG if and only if it has the congruence subgroup property CSP (see [26]). According to a result of Larsen and Lubotzky (see [22, Proposition 1.3]), when Γ has the CSP, the representation zeta function of Γ admits an Euler product decomposition. For instance, when $\Gamma = \mathrm{H}(\mathcal{O}_S)$, the Euler product decomposition is

$$\zeta_\Gamma(s) = \zeta_{\mathrm{H}(\mathbb{C})}(s)^{|k:\mathbb{Q}|} \cdot \prod_{v \notin S} \zeta_{\mathrm{H}(\mathcal{O}_v)}(s)\,. \tag{0.1}$$

Here, the first factor enumerates the *rational* irreducible representations of the group $\mathrm{H}(\mathbb{C})$ and has been studied by Witten in [36]. Larsen and Lubotzky have computed its abscissa of convergence in [22, Theorem 5.1]. By $\mathcal{O}_v$ we denote the ring of integers in the completion k_v of k at the non-archimedean place v. The factors indexed by $v \notin S$ are representation zeta functions of virtually pro-p groups counting irreducible representations with *finite image* (i.e. continuous irreducible representations); in [22, Theorem 8.1], $1/15$ is established as a lower bound for their abscissa.

For what concerns the global abscissa of convergence $\alpha(\Gamma)$, Avni proves in [1] that arithmetic groups with CSP have rational abscissa of convergence. Larsen and Lubotzky made the following conjecture.

Conjecture 0.1 (Larsen and Lubotzky [22, Conjecture 1.5]). *Let H be a higher-rank semisimple group. Then, for any two irreducible lattices* Γ_1 *and* Γ_2 *in H,* $\alpha(\Gamma_1) = \alpha(\Gamma_2)$.

In [2, Theorem 1.2] Avni, Klopsch, Onn and Voll prove a variant of Larsen and Lubotzky conjecture for higher-rank semisimple groups in characteristic 0 assuming that both $\alpha(\Gamma_1)$ and $\alpha(\Gamma_2)$ are finite. In [3], the same authors introduce the use of p-adic integrals in the study of representation growth of compact p-adic analytic groups. In particular they relate the representation zeta function to a generalized Igusa zeta function of the type described in [35]. In doing so, they prove that representation zeta functions of generic members of families of p-adic analytic pro-p groups obtained from a perfect Lie lattice (e.g. the principal congruence subgroups $\mathrm{SL}_h^m(\mathfrak{o})$ ($h \in \mathbb{N}$) for almost all $m \in \mathbb{N}$) satisfy functional equations (see [3, Theorem A]). Using p-adic integration, they compute explicit formulas for the representation zeta function for almost all of the principal congruence subgroups of $\mathrm{SL}_3(\mathfrak{o})$ and $\mathrm{SU}_3(\mathfrak{O}, \mathfrak{o})$, where $\mathfrak{O}$ is an unramified quadratic extension of $\mathfrak{o}$. Using approximative Clifford theory they are able to deduce from these formulae the abscissae of convergence of arithmetic groups of type A_2 establishing Larsen and Lubotzky's conjecture for groups of type A_2. The same authors in [4] classify the similarity classes of 3×3 matrices in $\mathfrak{gl}_3(\mathfrak{o})$ and $\mathfrak{gu}_3(\mathfrak{o})$ and obtain again the explicit formulae in [3] avoiding p-adic integration. Using again Clifford theory, they then deduce explicit formulae for the representation zeta functions of $\mathrm{SL}_3(\mathfrak{o})$ and of $\mathrm{SU}_3(\mathfrak{o})$.

By computing the representation zeta function of the principal congruence subgroups of $\mathrm{SL}_4(\mathfrak{o})$, the present work marks the beginning of an analogous line of investigation for arithmetic groups of type A_3.

Main results and techniques

Main results. Let G be a linear algebraic group defined over $\mathbb{Z}$ with Lie algebra $\mathbf{g} = \mathrm{Lie}(\mathrm{G})$. Let k be a number field with ring of integers $\mathcal{O}$. Let $\mathfrak{p} \lhd \mathcal{O}$ be a non-zero prime ideal such that the reduction σ_r is surjective for all $r \in \mathbb{N}$. By Hensel's lemma this happens for all but finitely many prime ideals of $\mathcal{O}$ (see [21, Chapter II, Proposition 4.1]). Let π be a uniformizer for $\mathfrak{p}$ and identify the residue field $\mathcal{O}_\mathfrak{p}/\mathfrak{p}$ with $\mathbb{F}_q$. For convenience of notation we shall set $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$ and $\mathfrak{o}_r = \mathfrak{o}/\mathfrak{p}^r$.

Definition 0.2. Let $r \in \mathbb{N}$ and $a \in \mathbf{g}(\mathfrak{o}_r)$. We define the *(group-)shadow*

$$\mathrm{Sh}_{\mathrm{G}(\mathfrak{o}_r)}(a) \leq \mathrm{G}(\mathbb{F}_q)$$

of a to be the reduction mod $\mathfrak{p}$ of the group stabilizer of a for the adjoint action of $\mathrm{G}(\mathfrak{o}_r)$ on $\mathbf{g}(\mathfrak{o}_r)$. Analogously, the *Lie-shadow*

$$\mathrm{Sh}_{\mathbf{g}(\mathfrak{o}_r)}(a) \leq \mathbf{g}(\mathbb{F}_q)$$

of a is the reduction mod $\mathfrak{p}$ of the Lie centralizer of a.

Remark 0.3. Definition 0.2 borrows from [4, Definition 2.2]. The crucial difference here is that [4, Definition 2.2] also associates a conjugacy class of such shadows to each adjoint orbit in $\mathbf{g}(\mathfrak{o}_r)$. Instead we shall work with single elements or we shall consider isomorphism types of shadows, thus obtaining a coarser invariant than the one described in [4].

The first main result concerns adjoint orbits in $\mathbf{g}(\mathfrak{o}_r)$.

Theorem A. *Let $r \in \mathbb{N}$ and $a \in \mathbf{g}(\mathfrak{o}_r)$. Assume that $\mathbf{g}(\mathfrak{o})$ admits a non-degenerate invariant symmetric form and that a has a lift to $\mathbf{g}(\mathfrak{o}_{r+1})$ with the same shadow. Then the set of $\mathrm{G}(\mathfrak{o}_{r+1})$-adjoint orbits in $\mathbf{g}(\mathfrak{o}_{r+1})$ containing a lift of the element a is in 1-1 correspondence with the set of orbits for the co-adjoint action of $\mathrm{Sh}_{\mathrm{G}(\mathfrak{o}_r)}(a)$ on*

$$\mathrm{Hom}_{\mathbb{F}_q}(\mathrm{Sh}_{\mathbf{g}(\mathfrak{o}_r)}(a), \mathbb{F}_q).$$

In case $\mathrm{G} = \mathrm{GL}_n$, $\mathbf{g}(\mathfrak{o})$ is equipped with the form $\mathrm{tr}(XY)$ and $r = 2$, Theorem A is [19, Theorem 1]. Indeed, as proved in [19, Lemma 6] for any $n \times n$ matrix over $\mathbb{F}_q$ there is an $n \times n$ matrix over $\mathfrak{o}_2$ with the same shadow. In Section 5.5.2 we prove that this is not true in general, namely we prove (non-constructively) that there are levels $r \in \mathbb{N}$ and elements a in $\mathfrak{sl}_4(\mathfrak{o}_r)$ that do not admit lifts with the same shadow. For completeness, in (5.46), we exhibit an example in $\mathfrak{sl}_4(\mathbb{Z}/27\mathbb{Z})$. With the further hypothesis of the existence of a lift with the same shadow, the proof of Theorem A closely follows the strategies adopted by Jambor and Plesken.

The second main result is an explicit formula for the representation zeta function of congruence subgroups $\mathrm{SL}_4^m(\mathfrak{o})$, where m is permissible, i.e. such that $\mathfrak{p}^m\,\mathfrak{sl}_4(\mathfrak{o})$ is saturable and potent (see Definitions 1.1 and 1.4). [3, Proposition 2.3] ensures that almost all non-negative integers are permissible (which specific ones depending on $\mathfrak{o}$). The same result also implies that all non-negative integers are permissible when $\mathfrak{o}$ is unramified over $\mathbb{Z}_p$.

Theorem B. *Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic 0 whose residue field has cardinality q and characteristic not equal to 2. Then, for all permissible m,*

$$\zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(s) = q^{15m}\frac{\mathcal{F}(q, q^{-s})}{\mathcal{G}(q, q^{-s})}$$

where

$$\begin{aligned}\mathcal{F}(q,t) = qt^{18} \\ &- \left(q^7 + q^6 + q^5 + q^4 - q^3 - q^2 - q\right)t^{15}\end{aligned}$$

$$
\begin{aligned}
&+ \big(q^8 - 2\,q^5 - q^3 + q^2\big)t^{14}\\
&+ \big(q^9 + 2\,q^8 + 2\,q^7 - 2\,q^5 - 4\,q^4 - 2\,q^3 - q^2 + 2\,q + 1\big)t^{13}\\
&- \big(q^{10} + q^9 + q^8 - 2\,q^7 - 2\,q^6 - 2\,q^5 + 2\,q^3 + q^2 + q\big)t^{12}\\
&+ \big(q^8 + 2\,q^6 + q^4 - q^3 - q^2 - q\big)t^{11}\\
&+ \big(q^8 + q^7 - 2\,q^4 + q\big)t^{10}\\
&- \big(2\,q^{10} + q^9 + q^8 - q^7 - 3\,q^6 - 2\,q^5 - 3\,q^4 - q^3 + q^2 + q + 2\big)t^9\\
&+ \big(q^9 - 2\,q^6 + q^3 + q^2\big)t^8\\
&- \big(q^9 + q^8 + q^7 - q^6 - 2\,q^4 - q^2\big)t^7\\
&- \big(q^9 + q^8 + 2\,q^7 - 2\,q^5 - 2\,q^4 - 2\,q^3 + q^2 + q + 1\big)t^6\\
&+ \big(q^{10} + 2\,q^9 - q^8 - 2\,q^7 - 4\,q^6 - 2\,q^5 + 2\,q^3 + 2\,q^2 + q\big)t^5\\
&+ \big(q^8 - q^7 - 2\,q^5 + q^2\big)t^4\\
&+ \big(q^9 + q^8 + q^7 - q^6 - q^5 - q^4 - q^3\big)t^3\\
&+ q^9
\end{aligned}
$$

$$\mathcal{G}(q,t) = q^9\big(1-qt^3\big)\big(1-qt^4\big)\big(1-q^2t^5\big)\big(1-q^3t^6\big).$$

Remark 0.4. The palindromic symmetry of $\mathcal{F}(q,t)$ in Theorem B implies that $\zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(s)$ satisfies to the functional equation of [3, Theorem A], e.g. when $m \in \mathbb{N}$ is permissible for $\mathbb{Z}_p$:

$$\zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(s)_{|q\to q^{-1}} = q^{-15\cdot m} \cdot \zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(s).$$

Simple substitutions reveal that $\zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(-2) = 0$, in accordance with [13]; while $\mathcal{F}(1,t) = \mathcal{G}(1,t)$.

In [33] T. Rossmann introduces the topological representation zeta function of a torsion-free free nilpotent group. Following his approach one may also define a topological representation zeta function attached to $\zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(s)$. It is indeed possible to compare the properties of this function with the properties of the topological representation zeta function of nilpotent groups proved in [33]; the only caveat here is that, in order to account for the differences in the application of the Kirillov orbit method in the two cases (compare [3, Proposition 3.1, Corollary 3.7] and [34, Theorem 2.6]), one substitutes s with $s-2$. With this in mind one computes

$$\zeta^{\mathrm{top}}_{\mathrm{SL}_4^m}(s) = \frac{8\big(15\,s^3 + 26\,s^2 + 11\,s - 1\big)(s+2)}{(5\,s-2)(4\,s-1)(3\,s-1)(2\,s-1)}, \tag{0.2}$$

from which it follows that

$$\zeta^{\mathrm{top}}_{\mathrm{SL}_4^m}(s-2) = \frac{8\big(15\,s^3 - 64\,s^2 + 87\,s - 39\big)s}{(5\,s-12)(4\,s-9)(3\,s-7)(2\,s-5)}. \tag{0.3}$$

One sees that, analogously to [33, Proposition 4.5], its limit as $s \to \infty$ is 1 and that, analogously to [33, Proposition 4.8], all its poles are rational and smaller than 15. The substitution of s with $s-2$ also makes sure that $\zeta^{\mathrm{top}}_{\mathrm{SL}_4^m}$ vanishes at 0 and its zeroes have real part between 0 and 14 (see [33, Question 7.4, Question 7.5]).

Organization of this work. We start off in Chapter 1 with a quick introduction to the main techniques on which our investigation builds. These include the Kirollov orbit method and the Poincaré series of a matrix of linear forms. Chapter 2 introduces our version of the similarity class invariant called the *shadow*. We use it to generalize results of Jambor and Plesken (see [19]) and obtain Theorem A.

Chapter 3 focuses on particularly interesting examples of Lie rings for which results in Chapter 2 hold, namely $\mathfrak{sl}_h(\mathfrak{o})$ for $h \in \mathbb{N}$. Provided it admits a shadow-preserving lift, we manage to quantitatively classify the lifts of a traceless matrix over a finite quotient of $\mathfrak{o}$ according to the isomorphism type of its shadow. We apply this result by computing the representation zeta function of almost all principal congruence subgroups of $\mathrm{SL}_3(\mathfrak{o})$ for $q > 2$ and $3 \nmid q$, thus obtaining again the formula in [3]. Our approach resembles closely the one in [4], however we classify only the conjugacy classes of $\mathfrak{sl}_3(\mathfrak{o}/\mathfrak{p}^r)$ $(r \in \mathbb{N})$ having non-minimal dimensional centralizer and we do it according to the isomorphism type of their shadow rather than according to the conjugacy class of their shadow.

Owing to the restriction imposed on the matrices to lift (presence of shadow-hereditary lifts), the method used to compute the representation zeta function of $\mathrm{SL}_3^m(\mathfrak{o})$ ($m \in \mathbb{N}$ permissible) cannot be followed to compute the representation zeta function of $\mathrm{SL}_4^m(\mathfrak{o})$. For this reason, Chapter 4 is devoted to adapting the methods in Chapter 3. In doing so we obtain in Theorem 4.20 a streamlined formula for the Poincaré series of semisimple Lie rings whose commutator matrix has smooth and irreducible rank loci. By the theory of sheets of classical Lie algebras (see [29]), examples of such rings are $\mathfrak{sl}_4(\mathfrak{o})$ and $\mathfrak{sl}_5(\mathfrak{o})$ but not $\mathfrak{sl}_6(\mathfrak{o})$ (see also Section 4.3.1). This formula, although not explicit, already allows for the computation the abscissa of convergence and may, in the future, be used to treat several other examples beside $\mathfrak{sl}_4(\mathfrak{o})$, such as $\mathfrak{sl}_5(\mathfrak{o})$ and $\mathfrak{so}_7(\mathfrak{o})$ (cf. [29, Table 3] for the latter). Chapter 5 contains the computation of the representation zeta function of $\mathrm{SL}_4^m(\mathfrak{o})$. This uses the results of Chapter 4 combined with an analysis of the conjugacy classes in $\mathfrak{sl}_4(\mathbb{F}_q)$. The latter relies on the theory of sheets of classical Lie algebras in [7, 6, 29] and on classification results for the centralizers in $\mathrm{SL}_4(\mathbb{F}_q)$ (see [24]).

The present work does not treat outer forms; however, with an argument similar to the one found in [3, Section 6.2], it might be possible to adapt the results herein to principal congruence subgroups of $\mathrm{SU}_3(\mathfrak{O}, \mathfrak{o})$, where $\mathfrak{O}$ is an unramified quadratic extension of $\mathfrak{o}$.

Notation. We denote with $\mathbb{N}$ the set of the positive integers $\{1, 2, \dots\}$, while $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ are the natural numbers. Analogously, for $n \in \mathbb{N}$ we set $[n] = \{1, \dots, n\}$ and $[n]_0 = \{0, \dots, n\}$. In this work, p is a rational prime. The field of p-adic numbers is denoted by $\mathbb{Q}_p$ and the ring of p-adic integers by $\mathbb{Z}_p$. More generally, we denote with k a number field with ring of integers $\mathcal{O}$. Fixed non-zero a prime ideal $\mathfrak{p} \triangleleft \mathcal{O}$ we set $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$ and denote by q the cardinality of the residue field $\mathbb{F}_q = \mathcal{O}/\mathfrak{p}$. The field of fractions of $\mathfrak{o}$ is denoted with $\mathfrak{k}$ and ν denotes the $\mathfrak{p}$-adic valuation both on $\mathfrak{o}$ and $\mathfrak{k}$.

As conventional, the multiplicative group of a field $\mathbf{K}$ is $\mathbf{K}^*$. We extend this notation to non-trivial $\mathfrak{o}$-modules as follows. Given such a module M, we write $M^* = M \smallsetminus \mathfrak{p}M$. For the trivial $\mathfrak{o}$-module we set $\{0\}^* = \{0\}$. The Pontryagin dual of a compact abelian group $\mathfrak{a}$ is

$$\widehat{\mathfrak{a}} = \mathrm{Irr}(\mathfrak{a}) = \mathrm{Hom}_{\mathbb{Z}}^{\mathrm{cont}}(\mathfrak{a}, \mathbb{C}^*).$$

By analogy, we write $\widehat{G} = \mathrm{Irr}(G)$ for the collection of continuous, irreducible complex characters of a profinite group G.

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CHAPTER 1

Background

1.1. p-adic analytic pro-p groups and the Kirillov orbit method

In this section we introduce the fundamental tools from p-adic Lie theory. Among these, the Kirillov orbit method allows us to describe fully the irreducible representations of a potent and saturable pro-p group. We refer to [11], [20] and [3, Section 2.1] for a more detailed introduction to saturable pro-p groups and their Lie theory.

1.1.1. Potent and saturable groups. The original notion of saturability was introduced by Lazard making use of valuation maps (see [23, 20]). In [11], González-Sánchez characterized these groups by means of potent filtrations.

Definition 1.1. Let G be a finitely generated pro-p group. We say that G is *saturable* when it is *torsion-free* and it admits a *potent filtration*: a descending series G_i $(i \in \mathbb{N})$ of normal subgroups of G such that:

(1) $G = G_1$,
(2) $\bigcap_{i\in\mathbb{N}} G_i = 1$,
(3) $[G_i, G] \subseteq G_{i+1}$ and $[G_i,_{p-1} G] \subseteq G_{i+1}^p$ for all $i \in \mathbb{N}$.

Here, $[G_i,_{p-1} G]$ is the left-normed iterated commutator with one occurrence of G_i and $p-1$ occurrences of G.

Examples of saturable pro-p groups are uniformly powerful pro-p groups. More generally, if we denote with $\gamma_p(G)$ the p-th term of the lower central series of G and with $\Phi(G)$ the Frattini subgroup of G, every torsion-free finitely generated pro-p group with $\gamma_p(G) \subseteq \Phi(G)^p$ is saturable. Another relevant class of examples comes from torsion-free p-adic analytic pro-p groups of dimension less than p which indeed are always saturable as shown in [14].

1.1.1.1. *Lie theory.* Consider a saturable pro-p group G. As explained in [14], it is possible to associate with it a saturable $\mathbb{Z}_p$-Lie lattice $\mathfrak{g} = \log(G)$, which coincides with G as a topological space. In case we are given a saturable $\mathbb{Z}_p$-Lie lattice $\mathfrak{g}$ first, we can recover a saturable group $G = \exp(\mathfrak{g})$ defining a group multiplication on $\mathfrak{g}$ by means of the Hausdorff series.

Our aim is to compute representation zeta functions and for this purpose we need to consider rigid groups. In [5] it was proved that a p-adic analytic group is rigid if and only if it is FAb, which means, every open subgroup has finite abelianization. The following characterizes FAb groups among saturable pro-p groups.

Proposition 1.2 ([3, Proposition 2.1]). *Let G be a saturable pro-p group, and let $\mathfrak{g} = \log(G)$ be the saturable $\mathbb{Z}_p$-Lie lattice associated with it. Then the following are equivalent:*

(1) G is FAb.
(2) G has finite abelianization $G/[G,G]$.
(3) $\mathfrak{g}$ has finite abelianization $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$.
(4) $\mathbb{Q}_p \otimes \mathfrak{g}$ is a perfect $\mathbb{Q}_p$-Lie algebra.

Remark 1.3. A compact p-adic analytic pro-p group is FAb if and only if it has an open FAb saturable pro-p subgroup (see [3, Section 2.1]).

1.1.1.2. *Potent groups.* The version of the Kirillov orbit method that we are going to deploy in Section 1.1.2 applies to *potent* and saturable pro-p groups and $\mathbb{Z}_p$-Lie lattices. We recall the following definition:

Definition 1.4. We say that a $\mathbb{Z}_p$-Lie lattice $\mathfrak{g}$ is *potent* when $\gamma_{p-1}(\mathfrak{g}) \subseteq p\mathfrak{g}$ for $p > 2$ and $\gamma_2(\mathfrak{g}) \subseteq 4\mathfrak{g}$ for $p = 2$. Analogoulsy, a pro-p group G is *potent* when $\gamma_{p-1}(G) \subseteq G^p$ for $p > 2$ and $\gamma_2(G) \subseteq G^4$ for $p = 2$.

Remark 1.5. If G is a saturable group and $\mathfrak{g} = \log(G)$ is the $\mathbb{Z}_p$-Lie lattice associated with it, then G is potent if and only if $\mathfrak{g}$ is potent.

Examples of potent groups are saturable pro-p groups of dimension less that p. Even more, [14, Theorem A] ensures that every torsion-free p-adic analytic pro-p group of dimension less that p is potent and saturable.

By [26, Lemma 2.2], the abscissa of convergence of the representation zeta function of a group can be read off from the representations zeta function of a finite index subgroup. This means that given a pro-p group G containing a finite index subgroup H that is potent and saturable, it is possible to apply the Kirillov orbit method to H in order to determine $\alpha(G)$. By Remark 1.5 one can consider Lie rings rather than groups. For this reason, we record here a definition and result from [3].

Definition 1.6. Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic 0 and residue field characteristic p, and let $\mathfrak{g}$ be an $\mathfrak{o}$-Lie lattice. For $m \in \mathbb{N}_0$, let $\mathfrak{g}^m = \mathfrak{p}^m\mathfrak{g}$, where $\mathfrak{p}$ denotes the prime ideal in $\mathfrak{o}$. We call m *permissible* for $\mathfrak{g}$ when $\mathfrak{g}^m$ is potent and saturable as a $\mathbb{Z}_p$-Lie lattice.

Given an $\mathfrak{o}$-Lie lattice $\mathfrak{g}$, [3, Proposition 2.3] shows that almost all non-negative integers are permissible.

Proposition 1.7 ([3, Proposition 2.3]). *Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic 0 and residue field characteristic p, and let $\mathfrak{g}$ be an $\mathfrak{o}$-Lie lattice. Let $m \in \mathbb{N}_0$ and let $e = e(\mathfrak{o}, \mathbb{Z}_p)$ be the absolute ramification index of $\mathfrak{o}$.*

If $m > e\cdot(p-1)^{-1}$, then $\mathfrak{g}^m$ is saturable. Moreover, if $p > 2$ and $m \geq e\cdot(p-2)^{-1}$, then $\mathfrak{g}^m$ is potent. If $p = 2$ and $m \geq 2e$, then $\mathfrak{g}^m$ is potent.

Remark 1.8. If $e(\mathfrak{o}, \mathbb{Z}_p) = 1$ then for $p > 2$ every $m \geq 1$ is permissible for every $\mathfrak{o}$-Lie lattice $\mathfrak{g}$, and similarly, for $p = 2$ every $m \geq 2$ is permissible.

1.1.2. Kirillov orbit method. First developed by Howe in [15] in the realm of compact p-adic analytic groups and applied to the study of representation zeta functions of FAb compact p-adic analytic groups by Jaikin-Zapirain in [18], the Kirillov orbit method is a powerful tool that completely describes the irreducible representations of a group in terms of co-adjoint orbits in an $\mathbb{Z}_p$-Lie lattice associated with the group. The version that we shall employ works with potent and saturable pro-p groups and it is due to Gonzalez-Sanchez (see [12] for a more exhaustive description). As we wish to work with rigid groups, we restrict ourselves to FAb potent and saturable pro-p groups in accordance with Proposition 1.2.

Let G be a FAb potent and saturable pro-p group and let $\mathfrak{g} = \log(G)$. We consider the Pontryagin dual of the compact abelian group $(\mathfrak{g}, +)$

$$\mathrm{Irr}(\mathfrak{g}) = \widehat{\mathfrak{g}} = \mathrm{Hom}_{\mathbb{Z}}^{\mathrm{cont}}(\mathfrak{g}, \mathbb{C}^*),$$

i.e. the group $\mathrm{Hom}_{\mathbb{Z}}^{\mathrm{cont}}(\mathfrak{g}, \mathbb{C}^*) = \mathrm{Hom}_{\mathbb{Z}}^{\mathrm{cont}}(\mathfrak{g}, \mu_{p^\infty})$ of continuous complex characters of the additive group $\mathfrak{g}$, where $\mu_{p^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p$ is the group of complex roots of unity of order a power of p. With each $\omega \in \mathrm{Irr}(\mathfrak{g})$ we associate a biadditive bilinear form

$$\begin{aligned} b_\omega :\ \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mu_{p^\infty} \\ (x, y) &\longmapsto \omega([x, y]). \end{aligned}$$

We define the radical of the bilinear form b_ω as

$$\mathrm{Rad}(\omega) = \mathrm{Rad}(b_\omega) = \{x \in \mathfrak{g} \mid \forall\, y \in \mathfrak{g} \,:\, b_\omega(x, y) = 1\}.$$

From [12, Corollary 2.13] and [18, Theorem 5.2] it follows that

$$\zeta_G(s) = \sum_{\omega \in \mathrm{Irr}\,(\mathfrak{g})} |\mathfrak{g} : \mathrm{Rad}(\omega)|^{-\frac{s+2}{2}}. \tag{1.1}$$

1.2. Commutator matrix and Poincaré series

We give a short summary of some facts in [3, Section 2.2, Section 3.1]. Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic 0, with maximal ideal $\mathfrak{p} = \pi\,\mathfrak{o}$, field of fractions $\mathfrak{k}$ and residue field $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$ of characteristic p. Let $\mathfrak{g}$ be an $\mathfrak{o}$-Lie lattice such that $\mathfrak{k} \otimes_{\mathfrak{o}} \mathfrak{g}$ is perfect with $\dim_{\mathfrak{k}}(\mathfrak{k} \otimes_{\mathfrak{o}} \mathfrak{g}) = d$. The following lemma explains how to conveniently sort irreducible representations of the Lie lattice $\mathfrak{g}$.

Lemma 1.9 ([3, Lemma 2.4]). *The dual of an $\mathfrak{o}$-Lie lattice $\mathfrak{g}$ can be written as a disjoint union:*

$$\widehat{\mathfrak{g}} = \dot{\bigcup_{r \in \mathbb{N}_0}} \mathrm{Irr}_r(\mathfrak{g}), \ \textit{where}\ \mathrm{Irr}_r(\mathfrak{g}) \cong \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}, \mathfrak{o}/\mathfrak{p}^r)^*.$$

For $r \in \mathbb{N}_0$, an element of $\omega \in \mathrm{Irr}_r(\mathfrak{g})$ is said to have level $\mathrm{lev}(\omega) = r$. Proposition 1.7 ensures that for every sufficiently large $m \in \mathbb{N}_0$, the m-th congruence sublattice $\mathfrak{g}^m$ corresponds to a FAb potent and saturable pro-p group $G^m = \exp(\mathfrak{g}^m)$, on which the Kirillov orbit method outlined in Section 1.1.2 can be applied. In this case the sets $\mathrm{Irr}_r(\mathfrak{g}^m)$ are G^m-invariant and therefore each irreducible representation of G^m corresponds to a co-adjoint orbit $\mathcal{C} \subseteq \mathrm{Irr}_r(\mathfrak{g}^m)$ for some level $r \in \mathbb{N}_0$.

Thanks to this categorization of irreducible representations, we shall rephrase the problem of counting representations in a counting problem involving a matrix of linear forms with coefficients in $\mathfrak{o}$.

1.2.1. Commutator matrix. We choose an $\mathfrak{o}$-basis $\mathcal{B} = \{b_1, \dots, b_d\}$ for the $\mathfrak{o}$-Lie ring $\mathfrak{g}$. For any $b_i, b_j \in \mathcal{B}$, there are $\lambda^1_{i,j}, \dots, \lambda^d_{i,j} \in \mathfrak{o}$ such that

$$[b_i, b_j] = \sum_{h=1}^{d} \lambda^h_{i,j} b_h.$$

The coefficients $\lambda^h_{i,j}$ for $i, j, h = 1, \dots, d$ are called the *structure constants* of $\mathfrak{g}$ with respect to $\mathcal{B}$. By means of them we define the *commutator matrix* of $\mathfrak{g}$ as

$$\mathcal{R}_{\mathcal{B}}(\mathbf{Y}) = \left(\sum_{h=1}^{d} \lambda^h_{i,j} Y_h \right)_{i,j} \in \mathrm{Mat}_d(\mathfrak{o}[\mathbf{Y}]) \tag{1.2}$$

with variables $\mathbf{Y} = (Y_1, \dots, Y_d)$.

We consider now $\mathbf{w} \in W(\mathfrak{o}) = (\mathfrak{o}^d)^*$; the matrix $\mathcal{R}_{\mathcal{B}}(\mathbf{w})$ is an antisymmetric $d \times d$ matrix. Therefore its elementary divisors can be arranged in $n = \lfloor d/2 \rfloor$ pairs $(\mathfrak{p}^{a_1}, \mathfrak{p}^{a_1}), \dots, (\mathfrak{p}^{a_n}, \mathfrak{p}^{a_n})$ for $0 \le a_1 \le \dots \le a_n \in (\mathbb{N}_0 \cup \{\infty\})$ together with $\mathfrak{p}^\infty = \{0\}$ if d is odd. We define

$$\nu(\mathcal{R}_{\mathcal{B}}(\mathbf{w})) = (a_1, \dots, a_n).$$

For $r \in \mathbb{N}$, let

$$W_r(\mathfrak{o}) = (W(\mathfrak{o}) + (\mathfrak{p}^r)^{(d)})/(\mathfrak{p}^r)^{(d)} = ((\mathfrak{o}/\mathfrak{p}^r)^d)^*. \tag{1.3}$$

Let $\overline{\mathbf{w}} = \sigma_r(\mathbf{w})$, be the valuation of the matrix $\mathcal{R}_{\mathcal{B}}(\overline{\mathbf{w}}) = \sigma_r(\mathcal{R}_{\mathcal{B}}(\mathbf{w}))$ is defined as

$$\nu(\mathcal{R}_{\mathcal{B}}(\overline{\mathbf{w}}))) = (\min\{a_i, r\})_{i=1,\dots,n} \in \{0, 1, \dots r\}^n.$$

We work with the congruence sublattices $\mathfrak{g}^m$ of $\mathfrak{g}$. Since $\mathcal{B}$ is an $\mathfrak{o}$-basis for $\mathfrak{g}$, it follows that $\pi^m\mathcal{B}$ is an $\mathfrak{o}$-basis for $\mathfrak{g}^m$. We can therefore define a coordinate system

$$\mathfrak{g}^m \longrightarrow \mathfrak{o}^d\ , \qquad z = \textstyle\sum_{i=1}^d z_i(\pi^m b_i) \longmapsto \mathbf{z} = (z_1, \dots, z_d).$$

Definition 1.10. We define

$$\mathcal{B}^\vee = \{b_1^\vee, \dots, b_d^\vee\} \subseteq \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}, \mathfrak{o})$$

by $b_i^\vee(b_j) = \delta_{i,j}$ for all $i, j \in \{1, \dots, d\}$. It is a standard computation to see that $\mathcal{B}^\vee$ is an $\mathfrak{o}$-basis for $\mathfrak{g}^\vee = \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}, \mathfrak{o})$. Therefore we call $\mathcal{B}^\vee$ the dual basis to $\mathcal{B}$.

We define a coordinate system on $\mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}^m, \mathfrak{o})$ by shifting the dual basis $\mathcal{B}^\vee$:

$$\mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}^m, \mathfrak{o}) \longrightarrow \mathfrak{o}^d\ ,\ w = \textstyle\sum_{i=1}^d w_i(\pi^{-m} b_i^\vee) \longmapsto \mathbf{w} = (w_1, \dots, w_d).$$

Since $\mathcal{B}^\vee$ is the dual basis of $\mathcal{B}$, we have that $w(z) = \mathbf{w} \cdot \mathbf{z}$ for z and w as above.

Definition 1.11. Let $r \in \mathbb{N}$. We say that $w \in \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}^m, \mathfrak{o})$ is a *representative* of $\omega \in \mathrm{Irr}_r(\mathfrak{g}^m)$ when ω is the image of w in the natural surjection

$$\mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}^m, \mathfrak{o}) \to \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}^m, \mathfrak{o}/\mathfrak{p}^r) \cong \mathrm{Irr}_r(\mathfrak{g}^m),$$

where $\mathrm{Irr}_r(\mathfrak{g}^m)$ is defined as in Lemma 1.9. We see now how a representative w of $\omega \in \mathrm{Irr}_r(\mathfrak{g}^m)$ can be used to compute $\mathrm{Rad}(\omega)$.

Definition 1.12. Let $m, r \in \mathbb{N}_0$. Consider $\omega \in \mathrm{Irr}_r(\mathfrak{g}^m)$ and let $w \in \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}^m, \mathfrak{o})^*$ represent ω. We define

$$\begin{aligned} \mathrm{Rad}(w) &= \{x \in \mathfrak{g} \mid \forall\, y \in \mathfrak{g}\ :\ w([x,y]) = 0\} \\ \mathrm{Rad}_r(w) &= \{x \in \mathfrak{g} \mid \forall\, y \in \mathfrak{g}\ :\ w([x,y]) \equiv 0 \bmod \mathfrak{p}^r\}. \end{aligned}$$

It is clear from the discussion above that if $\omega \in \mathrm{Irr}_r(\mathfrak{g}^m)$ is represented by w then $z \in \mathrm{Rad}(\omega)$ if and only if $z \in \mathrm{Rad}_r(w)$. Expressing this in coordinates we can highlight the link between the Kirillov orbit method and the commutator matrix.

Lemma 1.13. *Let $\omega \in \mathrm{Irr}_r(\mathfrak{g}^m)$ and let $w \in \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}^m, \mathfrak{o})^*$ be one ot its representatives. Let $\mathbf{w}$ be the coordinates of w in the $\mathfrak{o}$-basis $\mathcal{B}^\vee$ and let $r \in \mathbb{N}_0$. Then for every $z \in \mathfrak{g}^m$ with $\mathcal{B}$-coordinates $\mathbf{z} \in \mathfrak{o}^d$ we have*

$$\begin{aligned} z \in \mathrm{Rad}(w) &\iff \mathbf{z} \cdot \mathcal{R}_{\mathcal{B}}(\mathbf{w}) = 0, \\ z \in \mathrm{Rad}_r(w) &\iff \mathbf{z} \cdot \pi^m\, \mathcal{R}_{\mathcal{B}}(\mathbf{w}) \equiv 0 \mod \mathfrak{p}^r. \end{aligned}$$

Proof. The first double implication follows immediately from the definition of commutator-matrix, indeed for all $x, y \in \mathfrak{g}^m$ we have $w([x,y]) = \pi^m \mathbf{x}\, \mathcal{R}_{\mathcal{B}}(\mathbf{w})\, \mathbf{y}^{\mathrm{t}}$, where $\mathbf{x}$ and $\mathbf{y}$ are the coordinates of x and y in the basis $\pi^m\mathcal{B}$. The second double implication is [3, Lemma 3.3]. ☐

1.2.2. Poincaré series. We briefly recall the definition of Poincaré series associated with a matrix of linear forms and its relation with the representation zeta function of G^m for each permissible $m \in \mathbb{N}$ as expressed in [3, Section 3]. We borrow the notation from [34, Section 3.1].

Let $\mathcal{R} \in \mathrm{Mat}_e(\mathfrak{o}[\mathbf{Y}])$ be an antisymmetric matrix of linear forms in f variables. Set $n = \lfloor e/2 \rfloor$ and let $I = \{i_1, \dots, i_\ell\}_< \subseteq [n-1]_0$. We impose $i_0 = 0$ and $i_{\ell+1} = n$ and we write

$$\mu_j = i_{j+1} - i_j$$

with $j \in [\ell]_0$. For $\mathbf{r}_I = (r_1, \dots, r_\ell) \in \mathbb{N}^{|I|}$, we set $N = \sum_{j=1}^{\ell} r_j$ and. We define

$$N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R}) = \{\mathbf{w} \in W_N(\mathfrak{o}) \mid \nu(\mathcal{R}(\mathbf{w})) = (\underbrace{0, \dots, 0}_{\mu_\ell}, \underbrace{r_\ell, \dots, r_\ell}_{\mu_{\ell-1}}, \dots, \underbrace{N, \dots, N}_{\mu_0}) \in \mathbb{N}_0^n\}$$

and

$$\mathcal{P}_{\mathcal{R}}(s) = \sum_{\substack{I \subseteq [n-1]_0 \\ I = \{i_1, \dots, i_\ell\}_<}} \sum_{\mathbf{r}_I \in \mathbb{N}^{|I|}} |N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R})| \, q^{-s\sum_{j=1}^{\ell} r_j(n-i_j)}.$$

Let $\mathfrak{g}$ be as defined at the beginning of Section 1.2 and $\mathcal{B}$ be as in Section 1.2.1. Let $\mathcal{R} = \mathcal{R}_{\mathcal{B}}$ (in particular we have $e = f = d$). We set

$$\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathfrak{g}) = |N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R})| \tag{1.4}$$

This is clearly well defined as changing basis for $\mathfrak{g}$ results in a linear invertible substitution of variables in the linear forms constituting the entries of $\mathcal{R}$. As a consequence we can define

$$\mathcal{P}_{\mathfrak{g}}(s) = \sum_{\substack{I \subseteq [n-1]_0 \\ I = \{i_1, \dots, i_\ell\}_<}} \sum_{\mathbf{r}_I \in \mathbb{N}^{|I|}} \mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathfrak{g}) \, q^{-s\sum_{j=1}^{\ell} r_j(n-i_j)}. \tag{1.5}$$

As it will be useful in in the following chapter, we record the following.

Remark 1.14. Let $w \in N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R})$, then the definition of $N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R})$ entails that $\mathrm{rk}_{\mathbb{F}_q} \sigma(\mathcal{R}(w)) = n - i_\ell$.

The following illustrates the relation between the representation zeta function and the Poincaré series.

Proposition 1.15 ([3, Proposition 3.1]). *Let $\mathfrak{g}$ be as defined at the beginning of Section 1.2. For all m that are permissible for $\mathfrak{g}$ we have:*

$$\zeta_{G^m}(s) = q^{d \cdot m} \, \mathcal{P}_{\mathfrak{g}}(s+2).$$

1.3. Hensel's lemma

Throughout this section let k be a number field with ring of integers $\mathcal{O}$. Let $\mathfrak{p}$ be a non-zero prime in $\mathcal{O}$ and $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$ denote the localization of $\mathcal{O}$ at $\mathfrak{p}$. Let also n, m be integers such that $0 \leq m \leq n$.

Definition 1.16. Let h, k be integers such that $0 \leq h \leq k$. Let $\mathbf{f} = (f_1, \dots, f_h)$ be a system of h polynomials in the k variables $\mathbf{X} = (X_1, \dots, X_k)$. The matrix

$$\mathrm{M}_{\mathbf{f}}(\mathbf{X}) = \left(\frac{\partial f_i}{\partial X_j}\right)_{i,j} \in \mathrm{Mat}_{h,k}(\mathfrak{o}[\mathbf{X}])$$

is called the *Jacobian matrix* of $\mathbf{f}$.

The following proposition is a special case of [8, III.4.5 Corollary 3].

LEMMA 1.17 (Hensel's Lemma). *Let* $\mathbf{f} = (f_{m+1}, \ldots, f_n)$ *be a system of* $n-m$ *elements of* $\mathfrak{o}[X_1, \ldots, X_n]$ *and let* $\mathrm{J}_{\mathbf{f}}(\mathbf{X})$ *denote the minor of* $\mathrm{M}_{\mathbf{f}}(\mathbf{X})$ *consisting of the columns of index* j *such that* $m+1 \leq j \leq n$. *Let* $r \in \mathbb{N}$ *and* $\mathbf{a} \in \mathfrak{o}^n$ *be such that* $\mathrm{M}_{\mathbf{f}}(\mathbf{a})$ *is invertible in* $\mathfrak{o}$ *and* $\mathbf{f}(\mathbf{a}) \equiv 0 \mod (\mathfrak{p}^r)^{(n-m)}$. *Then there are* $n-m$ *formal power series without constant term* ϕ_i $(m+1 \leq i \leq n)$ *in* $\mathfrak{o}[\![X_1, \ldots, X_n]\!]$ *such that for all* $\mathbf{t} = (t_1, \ldots, t_n) \in (\mathfrak{p}^r)^{(m)}$,

$$f_i(a_1 + t_1, \ldots, a_m + t_m, a_{m+1} + \phi_{m+1}(\mathbf{t}), \ldots, a_n + \phi_n(\mathbf{t})) = 0 \textit{ for } m+1 \leq i \leq n.$$

Since a smooth m-dimensional irreducible affine subscheme of the n-dimensional affine space is locally defined by the vanishing of $n-m$ coordinate functions, the following is a direct consequence of Lemma 1.17.

PROPOSITION 1.18. *Let* S *be a smooth irreducible affine* m*-dimensional subscheme of the* n*-dimensional affine space over* k *with good reduction modulo* $\mathfrak{p}$. *Let* $r \in \mathbb{N}$ *and let* $a \in S(\mathfrak{o}/\mathfrak{p}^r)$. *Then*

$$\#\{x \in S(\mathfrak{o}/\mathfrak{p}^{r+1}) \mid \exists \widehat{x} \in S(\mathfrak{o}) \textit{ s.t. } \widehat{x} \equiv x \bmod \mathfrak{p}^{r+1} \textit{ and } \widehat{x} \equiv a \bmod \mathfrak{p}^r\} = q^{n-m}.$$

In other words, the point a *has exactly* q^{n-m} *lifts among the* $\mathfrak{o}/\mathfrak{p}^{r+1}$*-rational points of* S *that lift to* $\mathfrak{o}$*-rational points of* S.

CHAPTER 2

Adjoint orbits in Lie rings

Let G be a linear algebraic group defined over $\mathbb{Z}$ with Lie algebra $\mathbf{g} = \mathrm{Lie}(\mathrm{G})$. Let k be a number field with ring of integers $\mathcal{O}$. Let $\mathfrak{p} \lhd \mathcal{O}$ be a non-zero prime ideal such that the reduction mod $\mathfrak{p}^r$

$$\mathrm{G}(\mathcal{O}_\mathfrak{p}) \to \mathrm{G}(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^r)$$

is surjective for all $r \in \mathbb{N}$. By Hensel's lemma this happens for all but finitely many prime ideals of $\mathcal{O}$ (see [21, Chapter II, Proposition 4.1]). Let π be a uniformizer for $\mathfrak{p}$ and $\mathbb{F}_q$ be the residue field $\mathfrak{o}/\mathfrak{p}$. We set $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$ and $\mathfrak{o}_r = \mathfrak{o}/\mathfrak{p}^r$. We shall also denote $G = \mathrm{G}(\mathfrak{o})$ and $\mathfrak{g} = \mathbf{g}(\mathfrak{o})$; analogously, for all $r \in \mathbb{N}$, $G_r = \mathrm{G}(\mathfrak{o}_r)$ and $\mathfrak{g}_r = \mathbf{g}(\mathfrak{o}_r)$.

Let $r, t \in \mathbb{N}$ be positive integers with $r < t$ and let $R = \mathfrak{o}_t$ or $R = \mathfrak{o}$, we define $\sigma_r : R \to \mathfrak{o}_r$ to be be the reduction modulo $\mathfrak{p}^r$. For simplicity, $\sigma = \sigma_1$ and $\bar{\mathfrak{g}} = \mathfrak{g}_1$ and $\bar{G} = G_1$. With a little abuse, the maps induced by σ_r and σ on G_t and $\mathfrak{g}_t$ will also be called σ_r and σ.

We consider the adjoint representation of G. Since, for some $n \in \mathbb{N}$, the linear algebraic group G is a closed subscheme of the algebraic group GL_n, it is a well known fact that the adjoint action of $\mathrm{G}(R) \leq \mathrm{GL}_n(R)$ on $\mathbf{g}(R) \subseteq \mathfrak{gl}_n(R)$ is given by the conjugation by elements in the group (see [28, Example 7.13]). In other words, for all $g \in \mathrm{G}(R)$ and all $a \in \mathbf{g}(R)$ the following is true:

$$\mathrm{Ad}(g)(a) = gag^{-1}.$$

We define

$$\mathrm{C}_{G_r}(a) = \{g \in G_r \mid gag^{-1} = a\}$$
$$\mathrm{C}_{\mathfrak{g}_r}(a) = \{x \in \mathfrak{g}_r \mid [x, a] = a\}.$$

2.1. Shadows

Fix $r \in \mathbb{N}$, for the rest of this chapter we set $\sigma_r : \mathfrak{o}_{r+1} \to \mathfrak{o}_r$. Given a point $a \in \mathfrak{g}_r$, we want to describe the orbits in $\mathfrak{g}_{r+1}$ for the action of G_{r+1} that lie above a, i.e. G_{r+1}-orbits having non-trivial intersection with $\sigma_r^{-1}(a) \subseteq \mathfrak{g}_{r+1}$.

DEFINITION 2.1. The *group-shadow* of a is

$$\mathrm{Sh}_{G_r}(a) = \sigma(\mathrm{C}_{G_r}(a)) \leq \bar{G}$$

and analogously, *the Lie-shadow* of a is

$$\mathrm{Sh}_{\mathfrak{g}_r}(a) = \sigma(\mathrm{C}_{\mathfrak{g}_r}(a)) \leq \bar{\mathfrak{g}}.$$

Since $\mathrm{C}_{G_r}(a)$ acts on $\mathrm{C}_{\mathfrak{g}_r}(a)$ by conjugation, the group $\mathrm{Sh}_{G_r}(a)$ acts on $\mathrm{Sh}_{\mathfrak{g}_r}(a)$ by conjugation. We denote with $\mathrm{Sh}_{\mathfrak{g}_r}(a)^\vee = \mathrm{Hom}_{\mathbb{F}_q}(\mathrm{Sh}_{\mathfrak{g}_r}(a), \mathbb{F}_q)$ the dual module.

Let $b \in \sigma_r^{-1}(a)$ and let $\widetilde{\mathcal{C}}$ be its G_{r+1}-orbit. Then $\widetilde{\mathcal{C}} \cap \sigma_r^{-1}(a)$ is completely determined by the action of

$$\widetilde{S} = \sigma_r^{-1}(\mathrm{C}_{G_r}(a))$$

on $\sigma_r^{-1}(a)$. Indeed, let $g \in G_{r+1}$ be such that $g.b \in \sigma_r^{-1}(a)$ then $\sigma_r(g.b) = \sigma_r(g).a = a$, which means $g \in \widetilde{S}$. Therefore we may restrict to the action of $\widetilde{S}$ on $\sigma_r^{-1}(a)$.

2.2. The action of the kernel

Following [19] we proceed in two steps: first we consider the orbits for the action of the normal subgroup $N = \ker \sigma_r \trianglelefteq \widetilde{S}$ and then we act on them with the factor group $\widetilde{S}/N$. We describe now the orbits in $\sigma_r^{-1}(a)$. The following is analogous to [19, Lemma 5].

LEMMA 2.2. *Let $b \in \sigma_r^{-1}(a)$, and let $\delta_b : \pi^r \mathfrak{g}_{r+1} \to \pi^r \mathfrak{g}_{r+1}$ be defined by $x \mapsto [x, b]$. Then there is a 1-1 correspondence between* $\operatorname{coker} \delta_b$ *and the N-orbits in $\sigma_r^{-1}(a)$.*

PROOF. Since $\sigma_r(b) = a$, we have $\sigma_r^{-1}(a) = \{b + \pi^r z \mid z \in \mathfrak{g}_{r+1}\}$. Given $b + \pi^r z \in \sigma_r^{-1}(a)$ and $1 + \pi^r y \in N$ $(z, y \in \mathfrak{g}_{r+1})$, the conjugation happens as follows:

$$(1 + \pi^r y)(b + \pi^r z)(1 - \pi^r y) = b + \pi^r (z + yb - by).$$

Since $y, z \in \mathfrak{g}_{r+1}$ are arbitrary, this means that any two elements of $\sigma_r^{-1}(a)$, say $b + \pi^r z$ and $b + \pi^r z'$, are conjugate if and only if $\pi^r z$ and $\pi^r z'$ represent the same element in $\operatorname{coker} \delta_b$. In other words, we can associate each orbit in $\sigma_r^{-1}(a)/N$ with one and only one element of $\operatorname{coker} \delta_b$. □

The $\mathfrak{o}_{r+1}$-module $\pi^r \mathfrak{g}_{r+1}$ can be viewed as a $\mathbb{F}_q$-vector space because $\mathfrak{p}$ acts trivially on it.

LEMMA 2.3. *Let b, δ_b as in Lemma 2.2 and define $\delta_a : \pi^{r-1}\mathfrak{g}_r \to \pi^{r-1}\mathfrak{g}_r$ by $x \mapsto [x, a]$. Then* $\operatorname{coker} \delta_b \cong \operatorname{coker} \delta_a$ *as $\mathbb{F}_q$-vector spaces.*

PROOF. First we observe that, as we did for $\pi^r \mathfrak{g}_{r+1}$, also $\pi^{r-1}\mathfrak{g}_r$ may be viewed as a $\mathbb{F}_q$-vector space. Now, the map

$$\begin{aligned} \varphi_r :\ \pi^r \mathfrak{g}_{r+1} &\longrightarrow \pi^{r-1}\mathfrak{g}_r \\ \pi^r x &\longmapsto \pi^{r-1}\sigma_r(x) \end{aligned}$$

induces an isomorphism of $\mathbb{F}_q$-vector spaces $\operatorname{coker} \delta_b \cong \operatorname{coker} \delta_a$. □

Lemma 2.3 allows us to substitute $\operatorname{coker} \delta_b$ with $\operatorname{coker} \delta_a$ on which $\widetilde{S}/N$ acts with the action induced by the bijection φ_r in Lemma 2.3. The next section is devoted to finding an explicit description of this action.

2.3. Action of the factor group

Let a, b, δ_a, δ_b be as in Lemmata 2.2 and 2.3. First of all we observe that we may replace the action of $\widetilde{S}/N$ on $\operatorname{coker} \delta_b$ by the action of $\widetilde{S}/\widetilde{N}$ on $\operatorname{coker} \delta_b$, where $\widetilde{N} = \ker \sigma \trianglelefteq \widetilde{S}$. Indeed $1 + \mathfrak{p}B$ acts trivially on $\pi^r \mathfrak{g}_{r+1}$, and $\widetilde{S}/\widetilde{N} = \mathrm{Sh}_{G_r}(a)$ by definition of $\widetilde{S}$.

DEFINITION 2.4. The centralizer $\mathrm{C}_{G_r}(a)$ acts naturally by conjugation on $\pi^{r-1}A$. Since $1 + \mathfrak{p}A$ is in the kernel of this action, this action induces an action of $\mathrm{Sh}_{G_r}(a)$ on $\pi^{r-1}A$; explicitly, an element $c \in \mathrm{Sh}_{G_r}(a)$ acts on $\pi^{r-1}A$ by conjugating by any of its lifts to $\mathrm{C}_{G_r}(a)$. We call this the action of $\mathrm{Sh}_{G_r}(a)$ on $\pi^{r-1}A$ by *conjugation by lifts*.

Analogously to the approach of [19, Section 2.2], the key to understanding the action of $\mathrm{Sh}_{G_r}(a)$ on $\operatorname{coker} \delta_a$ is to find a lift b of a with the same shadow. What we mean is made precise in the following definitions:

DEFINITION 2.5. Let $r \in \mathbb{N}$. We say that $b \in \mathfrak{g}_{r+1}$ is *shadow-preserving lift* of a when $\sigma_r(b) = a$ and $\mathrm{Sh}_{G_{r+1}}(b) = \mathrm{Sh}_{G_r}(a)$.

DEFINITION 2.6. We say that a group-shadow S is *hereditary* if, for every $r \in \mathbb{N}$, every $x \in \mathfrak{g}_r$ such that $\mathrm{Sh}_{G_r}(x) = S$ admits a shadow-preserving lift. If every shadow of $\mathfrak{g}$ is hereditary, we say that $\mathfrak{g}$ is *shadow-hereditary.*

EXAMPLE 2.7. By [4, Lemma 6.4], the Lie ring $\mathfrak{sl}_3(\mathfrak{o})$ is shadow-hereditary. In Section 5.5.2, however, we shall see that $\mathfrak{sl}_4(\mathfrak{o})$ is not shadow-hereditary.

By definition of conjugation by lifts, $\pi^{r-1}\mathfrak{g}_r$ and $\bar{\mathfrak{g}}$ are isomorphic as $\mathrm{Sh}_{G_r}(a)$-modules. The next lemma shows that the action of $\mathrm{Sh}_{G_r}(a)$ on $\pi^{r-1}\mathfrak{g}_r$ by conjugation by lifts is indeed what induces the action of $\mathrm{Sh}_{G_r}(a)$ on $\operatorname{coker}\delta_a$.

LEMMA 2.8. *Assume that the element a admits a shadow-preserving lift $b \in \mathfrak{g}_{r+1}$. Then the action of* $\mathrm{Sh}_{G_r}(a)$ *on* $\operatorname{coker}\delta_a$ *induced by the bijection in Lemma 2.3 is the linear action induced by the conjugation in $\pi^{r-1}\mathfrak{g}_r$ by lifts of elements in* $\mathrm{Sh}_{G_r}(a)$.

PROOF. Consider $c \in \mathrm{Sh}_{G_r}(a)$. This element acts on $\operatorname{coker}\delta_b$ conjugating by any of its lifts to G_{r+1}. Since b has the same shadow as a, we can choose $\tilde{c} \in \mathrm{C}_{G_{r+1}}(b)$ lifting c.

In order to see how $\tilde{c}$ acts on $\operatorname{coker}\delta_b$, first we see how it acts on an arbitrary lift of a:

$$\tilde{c}(b + \pi^r x)\tilde{c}^{-1} = b + \pi^r \tilde{c}x\tilde{c}^{-1}.$$

The equation above implies that the representative of $\operatorname{coker}\delta_b$ that we need to add to b in order to obtain $\tilde{c}(b+\pi^r x)\tilde{c}^{-1}$ is $\pi^r\tilde{c}x\tilde{c}^{-1}$, which means that $\mathrm{Sh}_{\mathfrak{g}_r}(a)$ is acting by conjugation on $\operatorname{coker}\delta_b$. Now the action on $\operatorname{coker}\delta_a$ is obtained via the map φ_r in the proof Lemma 2.3. Under this identification of $\operatorname{coker}\delta_a$ and $\operatorname{coker}\delta_b$, the representative $\pi^r\tilde{c}x\tilde{c}^{-1}$ maps onto the representative $\sigma_r(c)\sigma_r(x)\sigma_r(c)^{-1}$; and this describes the action induced by the action of $\mathrm{Sh}_{G_r}(a)$ on $\pi^{r-1}A$ by conjugation by lifts. □

2.4. Intrinsic description of the orbits

So far we have established a 1-1 correspondence between the G_{r+1}-orbits in $\mathfrak{g}_{r+1}$ intersecting $\sigma_r^{-1}(a)$ non-trivially and $\mathrm{Sh}_{G_r}(a)$-orbits in $\operatorname{coker}\delta_a$. Now we replace $\operatorname{coker}\delta_a$ with $\mathrm{Sh}_{\mathfrak{g}_r}(a)^\vee$.

We begin by replacing $\operatorname{coker}\delta_a$ with $(\ker\delta_a)^\vee = \mathrm{Hom}_{\mathbb{F}_q}(\ker\delta_a, \mathbb{F}_q)$. The action of $\mathrm{Sh}_{G_r}(a)$ on $\ker\delta_a$ will be the one induced by the conjugation by lifts of $\mathrm{Sh}_{G_r}(a)$ on $\pi^{r-1}\mathfrak{g}_r$ described in Definition 2.4. From now onwards we assume that $\mathfrak{g}$ admits a non-degenerate invariant symmetric form.

EXAMPLE 2.9. The assumption of the existence of a non-degenerate symmetric invariant bilinear form might seem rather obscure at first. However Cartan's criterion for semisimplicity (see for instance [17, Section III.4]) ensures that when G is semisimple, Lie(G)($\mathbb{C}$) admits such a form. Excluding finitely many primes, this remains valid for Lie(G)($\mathfrak{o}$).

LEMMA 2.10. *Let $C = \mathrm{Sh}_{\mathfrak{g}_r}(a)$. Then* $(\ker\delta_a)^\vee$ *and* $\operatorname{coker}\delta_a$ *are isomorphic as $\mathbb{F}_qC$-modules, where C acts by conjugation by lifts on* $\ker\delta_a$.

PROOF. We follow the proof of [19, Lemma 8]. We assume that $\delta_a : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$. With this assumption it is possible to consider the dual map of δ_a, namely $\delta_a^{\mathrm{t}} : \bar{\mathfrak{g}}^\vee \longrightarrow \bar{\mathfrak{g}}^\vee$. Its kernel $\ker\delta_a^{\mathrm{t}}$ is a $\mathbb{F}_qC$-module in a natural way by the dual of the conjugation by C. We consider the dual module $(\ker\delta_a^{\mathrm{t}})^\vee$ and we prove that there is an $\mathbb{F}_qC$-module isomorphism between $\operatorname{coker}\delta_a$ and $(\ker\delta_a^{\mathrm{t}})^\vee$. Indeed, the evaluation

$$\begin{aligned} \alpha_1 :\quad \operatorname{coker}\delta_a &\longrightarrow (\ker\delta_a^{\mathrm{t}})^\vee \\ x + \operatorname{im}\delta_a &\longmapsto (\psi \mapsto \psi(x)) \end{aligned}$$

is an isomorphism of $\mathbb{F}_q$-vector spaces and one checks that it is also an $\mathbb{F}_qC$-module homomorphism.

The second step consists in proving that $\ker\delta_a \cong \ker\delta_a^{\mathrm{t}}$ as $\mathbb{F}_qC$-modules. The bilinear form $\overline{\kappa} : \bar{\mathfrak{g}} \times \bar{\mathfrak{g}} \to \mathbb{F}_q$ induced by κ is non-degenerate, hence it induces a $\mathbb{F}_q$-vector space isomorphism:

$$\begin{aligned} \alpha_2 :\ \ker\delta_a &\longrightarrow \ker\delta_a^{\mathrm{t}} \\ x &\longmapsto (y \mapsto \overline{\kappa}(y,x)). \end{aligned}$$

Since $\overline{\kappa}$ is invariant, α_2 is an $\mathbb{F}_qC$-module isomorphism. □

REMARK 2.11. Under the identification of $\pi^{r-1}\mathfrak{g}_r$ with $\bar{\mathfrak{g}}$, $\ker\delta_a$ corresponds to $\mathrm{Sh}_{\mathfrak{g}_r}(a)$. Indeed the identification is given by the isomorphism $\varphi : \pi^{r-1}\mathfrak{g}_r \to \bar{\mathfrak{g}}$ defined by $\pi^{r-1}x \mapsto \sigma(x)$. It thus suffices to prove that

$$\operatorname{im}\varphi_{|\ker\delta_a} = \mathrm{Sh}_{\mathfrak{g}_r}(a)\,.$$

Let $x \in \mathrm{C}_{\mathfrak{g}_r}(a)$, and $\bar{x} = \sigma(x) \in \mathrm{Sh}_{\mathfrak{g}_r}(a)$. By definition, $\pi^{r-1}x \in \ker\delta_a$. Thus $\varphi(\pi^{r-1}x) = \bar{x}$ and we conclude.

For convenience of notation we set

$$\gamma = (\alpha_2^{-1})^{\mathrm{t}} \circ \alpha_1 : \operatorname{coker}\delta_a \to \mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee}, \tag{2.1}$$

where α_1 and α_2 are as in the proof of Lemma 2.10 and $(\alpha_2^{-1})^{\mathrm{t}}$ is the dual map to $\alpha_2^{-1} : \ker\delta_a^{\mathrm{t}} \to \ker\delta_a$.

2.5. Adjoint orbits

We are now able to prove Theorem A:

THEOREM A. *Assume that $\mathfrak{g}$ admits a non-degenerate invariant symmetric form and that $a \in \mathfrak{g}_r$ has a shadow-preserving lift in the sense of Definition 2.5. Then the set of G_{r+1}-orbits in $\mathfrak{g}_{r+1}$ for the action by conjugation intersecting $\sigma_r^{-1}(a)$ non-trivially is in 1-1 correspondence with the set of $\mathrm{Sh}_{G_r}(a)$-orbits in $\mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee}$.*

PROOF. Let $b \in \mathfrak{g}_{r+1}$ be a shadow-preserving lift of a. Let c be an element of a $\mathrm{Sh}_{G_r}(a)$-orbit on $\mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee}$. Let γ be the isomorphism in (2.1) and let $\pi^{r-1}x_c + \operatorname{im}\delta_a = \gamma^{-1}(c)$. In the isomorphism of Lemma 2.3, the corresponding element is $\pi^r x_c + \operatorname{im}\delta_b \in \operatorname{coker}\delta_b$. Then the G_{r+1}-conjucacy classes in $\mathfrak{g}_{r+1}$ intersecting $\sigma_r^{-1}(a)$ are represented by the elements $b + \pi^r x_c$ where c runs over a system of representatives of the orbits $\mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee} / \mathrm{Sh}_{G_r}(a)$. □

2.6. Centralizer and shadow of a lift

Given $a \in \mathfrak{g}_r$ and a similarity class $\widetilde{\mathcal{C}} \subseteq \mathfrak{g}_{r+1}$ lying above a, we would like to compute $\mathrm{Sh}_{G_{r+1}}(x)$ and $\mathrm{Sh}_{\mathfrak{g}_{r+1}}(x)$ for $x \in \widetilde{\mathcal{C}}$ in order to be able to reiterate the process and describe the orbits of the action of G_{r+1} on $\mathfrak{g}_{r+1}$ lying above x. In this section we see that it is even possible to compute $\mathrm{C}_{G_{r+1}}(x)$.

As showed in Remark 2.11, $\mathrm{Sh}_{G_r}(a)$ acts on $\mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee}$, hence $\mathrm{C}_{G_r}(a)$ acts on $\mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee}$ in the following way: let $c \in \mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee}$ and $g \in \mathrm{C}_{G_r}(a)$, we define

$$g.c = \sigma(g).c.$$

This last action is crucial to understanding $\mathrm{C}_{G_{r+1}}(x)$ as the following explains.

THEOREM 2.12. *Assume that $\mathfrak{g}$ admits a non-degenerate invariant symmetric form and that $a \in \mathfrak{g}_r$ admits a shadow-preserving lift. Let $x \in \mathfrak{g}_{r+1}$ be a lift of $a \in \mathfrak{g}_r$, and let the orbit of x for the action of G_{r+1} be represented by the orbit of $c \in \mathrm{Sh}_{\mathfrak{g}_r}(a)$ in* *the* 1-1 *correspondence of Theorem A. Then* $\mathrm{C}_{G_{r+1}}(x)$ *is an extension of* $\mathrm{Sh}_{\mathfrak{g}_r}(a)$ *by* $\mathrm{Stab}_{\mathrm{C}_{G_r}(a)}(c)$*.*

PROOF. We consider N as in Section 2.2. Let $H = N \cap \mathrm{C}_{G_{r+1}}(x)$. Then

$$\frac{\mathrm{C}_{G_{r+1}}(x)}{H} \cong \sigma_r(\mathrm{C}_{G_{r+1}}(x)) = \mathrm{Stab}_{\mathrm{C}_{G_r}(a)}(c).$$

We choose $b \in \sigma_r^{-1}(a)$ with the same shadow as a. Then

$$x = b + \pi^r x_c,$$

where $\pi^r x_c$ is a representative of $\gamma^{-1}(c) \in \operatorname{coker} \delta_a$ (as explained in Remark 2.11 and Section 2.5).

Recall from the proof of Lemma 2.2 that an element $1+\pi^r y \in N$ acts as follows:

$$(1 + \pi^r y)(b + \pi^r x_c)(1 - \pi^r y) = b + \pi^r(x_c + [y, b]).$$

Hence $1 + \pi^r y$ fixes x if and only if $y \in \mathrm{C}_{\mathfrak{g}_{r+1}}(b)$. By the choice of b we have then

$$H = 1 + \pi^r \,\mathrm{C}_{\mathfrak{g}_{r+1}}(b)$$

where $\pi^r \,\mathrm{C}_{\mathfrak{g}_{r+1}}(b) \cong \mathrm{Sh}_{\mathfrak{g}_r}(a)$. □

Looking at how we described the elements of $\mathrm{C}_{G_{r+1}}(x)$ in the proof of Theorem 2.12 and reducing them modulo $\mathfrak{p}$ we can determine $\mathrm{Sh}_{G_{r+1}}(x)$.

COROLLARY 2.13. *Let* $\mathfrak{g}$ *admit a non-degenerate invariant symmetric form. Assume that* $a \in \mathfrak{g}_r$ *admits a shadow-preserving lift. Let* $x \in \mathfrak{g}_{r+1}$ *be a lift of* $a \in \mathfrak{g}_r$*, and let the orbit of* x *for the action of* G_{r+1} *be represented by the orbit of* $c \in \mathrm{Sh}_{\mathfrak{g}_r}(a)^\vee$ *in the* 1-1 *correspondence of Theorem A. Then* $\mathrm{Sh}_{G_{r+1}}(x) = \mathrm{Stab}_{\mathrm{Sh}_{G_r}(a)}(c)$*.*

The following proposition is useful in the computation of the shadow of $\mathrm{Sh}_{G_{r+1}}(x)$.

LEMMA 2.14. *Let* $\mathfrak{s} = \mathrm{Sh}_{\mathfrak{g}_r}(a)$ *and* $S = \mathrm{Sh}_{G_r}(a)$*. Let* $\mathcal{B}_\mathfrak{s}$ *be an* $\mathbb{F}_q$*-basis for* $\mathfrak{s}$ *and* $\mathcal{R}_\mathfrak{s}$ *be the commutator matrix of* $\mathfrak{s}$ *with respect to* $\mathcal{B}_\mathfrak{s}$*. Let* $\omega \in \mathfrak{s}^\vee$ *and* $\mathbf{w}$ *be its coordinates with respect to the dual basis* $\mathcal{B}_\mathfrak{s}^\vee$*. Then,*

$$y \in \mathrm{Lie}(\mathrm{Stab}_S(\omega)) \iff \mathbf{y} \in \ker_{\mathbb{F}_q} \mathcal{R}_\mathfrak{s}((\mathbf{w})),$$

where $\mathbf{y}$ *denote the coordinates of* y *with respect to* $\mathcal{B}_\mathfrak{s}$*.*

PROOF. By definition of co-adjoint action,

$$\mathrm{Lie}(\mathrm{Stab}_S(\omega)) = \mathrm{Rad}(b_\omega) = \{y \in S \mid \omega([y, v]) = 0 \,\forall\, v \in S\}.$$

The matrix of the bilinear form $b_\omega(\cdot,\cdot) = \omega([\cdot,\cdot])$ is, by definition of commutator matrix, $\mathcal{R}_\mathfrak{s}(\mathbf{w})$, and we conclude. □

CHAPTER 3

Special linear groups

In this chapter we prove quantitative statements about the number of lifts of elements of the Lie rings attached to the special linear groups. Thanks to the fact that $\mathfrak{sl}_3(\mathfrak{o})$ is shadow-hereditary, we shall also be able to apply these results in order to recompute the representation zeta function of $\mathrm{SL}_3^m(\mathfrak{o})$ for $q > 2$, $3 \nmid q$ and permissble m (see [3, Theorem E]). We keep the notation established at the beginning of Chapter 2, but applied to the specific case $\mathrm{G} = \mathrm{SL}_h$. In particular $\mathfrak{g} = \mathfrak{sl}_h(\mathfrak{o})$ admits a non-degenerate invariant symmetric form for almost all non-zero prime ideals $\mathfrak{p}$, viz. the normalized Killing form. Let $\mathfrak{p}$ be such a prime ideal.

3.1. Number of lifts

In case a shadow-preserving lift is available, Theorem 2.12 gives us a method for computing the number of lifts of a point $a \in \mathfrak{g}_r$ that have a prescribed shadow.

Definition 3.1. Let $a \in \mathfrak{g}_r$ such that $\mathrm{Sh}_{G_r}(a) = S$ is hereditary. Let $b \in \sigma_r^{-1}(a)$ have shadow $\mathrm{Sh}_{G_{r+1}}(b) = T$. We define $a_{S,T}$ and $c_{S,T}$ as the number of similarity classes with shadow isomorphic to T that lie above a and the number of lifts of a with shadow isomorphic to T, respectively.

Proposition 3.7 explains why Definition 3.1 does not depend on a and b but only on the isomorphism type of the shadows S and T. The following definition and Lemma 3.3 are needed.

Definition 3.2. Let $r \in \mathbb{N}$. Given a group-shadow S, we define

$$\mathrm{As}(S) = \mathrm{Span}(S) \cap \bar{\mathfrak{g}},$$

where $\mathrm{Span}(S)$ is the additive span of S when considered as a subset of $\mathrm{Mat}_h(\mathbb{F}_q)$.

Let $a \in \mathfrak{g}_r$ with $\mathrm{Sh}_{G_r}(a) = S$. The following shows that $\mathrm{Sh}_{\mathfrak{g}_r}(a)$ only depends on S and not directly on a.

Lemma 3.3 ([4, Lemma 2.3]). *Assume $q > 2$. Let $a \in \mathfrak{g}_r$ with $\mathrm{Sh}_{G_r}(a) = S$, then* $\mathrm{Sh}_{\mathfrak{g}_r}(a) = \mathrm{As}(S)$.

We assume henceforth and for the rest of the chapter that $q > 2$. Lemma 3.3 legitimates the following definitions:

Definition 3.4. For all $r \in \mathbb{N}$, we choose a transversal set for the collection of all isomorphism classes of group-shadows of elements in $\mathfrak{g}_r$ and we denote it with $\mathfrak{Sh}(\mathfrak{g}_r)$ and call its members *isomorphism types* of shadows of level r. We choose a transversal set for the collection of all group-shadows of all $\mathfrak{g}_t$ ($t \in \mathbb{N}$). We denote this set with

$$\mathfrak{Sh}(\mathfrak{g})$$

and call its elements isomorphism types of shadows. In what follows we shall indicate isomorphism types of shadows (of level r) with boldface roman capitals, e.g. $\mathbf{S}$. Let $\mathbf{S} \in \mathfrak{Sh}(\mathfrak{g}_r)$ and $\mathbf{T} \in \mathfrak{Sh}(\mathfrak{g}_{r+1})$. Definitions 3.1 and 3.2 allow us to write $a_{\mathbf{S},\mathbf{T}}$, $c_{\mathbf{S},\mathbf{T}}$ and $\mathrm{As}(\mathbf{S})$ because $\mathbf{S} = \mathrm{Sh}_{G_r}(a)$, for some $a \in \mathfrak{g}_r$ and $\mathbf{T} = \mathrm{Sh}_{G_{r+1}}(x)$, for some $x \in \mathfrak{g}_{r+1}$.

DEFINITION 3.5. Let $r \in \mathbb{N}$ and $\mathbf{S} \in \mathfrak{Sh}(\mathfrak{g}_r)$. We define

$$d_{\mathbf{S}} = \dim_{\mathbb{F}_q} \mathrm{As}(\mathbf{S})\,.$$

Notice that if $a \in \mathfrak{g}_r$ and $\mathrm{Sh}_{G_r}(a) \cong \mathbf{S}$, then $d_{\mathbf{S}} = \dim_{\mathbb{F}_q} \mathrm{Sh}_{\mathfrak{g}_r}(a)$ by Lemma 3.3. The number $d_{\mathbf{S}}$ is called the *dimension* of $\mathbf{S}$.

DEFINITION 3.6. Let $r \in \mathbb{N}$, $\mathbf{S} \in \mathfrak{Sh}(\mathfrak{g}_r)$ and $\mathbf{T} \in \mathfrak{Sh}(\mathfrak{g}_{r+1})$. We define

$$\Lambda(\mathbf{S}, \mathbf{T}) = |\{c \in \mathrm{As}(\mathbf{S})^{\vee} \mid \mathrm{Stab}_{\mathbf{S}}\,(c) \cong \mathbf{T}\}|.$$

Let $a \in \mathfrak{g}_r$ with $\mathrm{Sh}_{G_r}(a) = S$. Assume that a admits a shadow-preserving lift and let $\mathbf{T} \in \mathfrak{Sh}(\mathfrak{g}_{r+1})$. From Lemma 3.3 it follows that

$$\Lambda(\mathbf{S}, \mathbf{T}) = |\{c \in \mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee} \mid \mathrm{Stab}_{\mathbf{S}}\,(c) \cong \mathbf{T}\}|.$$

PROPOSITION 3.7. *Let $\mathbf{S}, \mathbf{T} \in \mathfrak{Sh}(\mathfrak{g})$. Let $r \in \mathbb{N}$ and $a \in \mathfrak{g}_r$ with $\mathrm{Sh}_{G_r}(a) \cong \mathbf{S}$. Assume further that $a \in \mathfrak{g}_r$ admits a shadow-preserving lift. Then the number $c_{\mathbf{S},\mathbf{T}}$ of lifts of a with shadow isomorphic to $\mathbf{T}$ is equal to*

$$q^{d-d_{\mathbf{S}}}\, \Lambda(\mathbf{S}, \mathbf{T}).$$

PROOF. Let $b \in \mathfrak{g}_{r+1}$ be a lift of a such that $\mathrm{Sh}_{\mathfrak{g}_{r+1}}(b) \cong \mathbf{T}$. Let $\mathcal{C}$ be the G_r-orbit of a and let $\widetilde{\mathcal{C}}$ be the G_{r+1}-orbit of b. By Definition 3.1,

$$c_{\mathbf{S},\mathbf{T}} = \frac{|\widetilde{\mathcal{C}}|}{|\mathcal{C}|}\, a_{\mathbf{S},\mathbf{T}} = \frac{|G_{r+1}|}{|G_r|}\, \frac{|\mathrm{C}_{G_r}\,(a)|}{|\mathrm{C}_{G_{r+1}}\,(b)|}\, a_{\mathbf{S},\mathbf{T}}.$$

By Theorem 2.12 we have that

$$|\mathrm{C}_{G_{r+1}}\,(b)| = |\mathrm{Sh}_{\mathfrak{g}_r}(a)|\, |\mathrm{Stab}_{\mathrm{C}_{G_r}(a)}(c)|,$$

where $c \in \widetilde{\mathcal{C}}' \subseteq \mathrm{Sh}_{\mathfrak{g}_r}(a)^{\vee}$, the orbit that represents $\widetilde{\mathcal{C}}$ in the 1-1 correspondence of Theorem A. In accordance with the definition of the action of $\mathrm{C}_{G_r}\,(a)$ on $\mathrm{Sh}_{\mathfrak{g}_r}(a)$ in Section 2.6, we have that

$$\frac{|\mathrm{C}_{G_r}\,(a)|}{|\mathrm{Stab}_{\mathrm{C}_{G_r}(a)}(c)|} = |\widetilde{\mathcal{C}}'|.$$

By Lemma 3.3 and Definition 3.5, $|\mathrm{Sh}_{\mathfrak{g}_r}(a)| = q^{d_{\mathbf{S}}}$, while $\frac{|G_{r+1}|}{|G_r|} = q^{\dim_{\mathbb{F}_q} \mathfrak{g}}$. Therefore we have

$$c_{\mathbf{S},\mathbf{T}} = q^{d-d_{\mathbf{S}}}\, |\widetilde{\mathcal{C}}'|\, a_{\mathbf{S},\mathbf{T}}.$$

Observing that, by Theorem A,

$$|\widetilde{\mathcal{C}}'|\, a_{\mathbf{S},\mathbf{T}} = \Lambda(\mathbf{S}, \mathbf{T}),$$

we conclude. □

3.2. The Poincaré series of $\mathfrak{sl}_3(\mathfrak{o})$

When $\mathfrak{g}$ is shadow-hereditary (cf. Definition 2.6) Proposition 3.7 can be used iteratively. As proved in [4, Lemma 6.4] (see also Example 2.7), the Lie ring $\mathfrak{sl}_3(\mathfrak{o})$ is shadow-hereditary. In this case Section 3.1 gives a direct way of computing the Poincaré series. As a result we obtain the representation zeta function of $\mathrm{SL}_3^m(\mathfrak{o})$ when $q > 2$ and $3 \nmid q$. Our approach resembles closely the one in [4], however we classify only the conjugacy classes of $\mathfrak{sl}_3(\mathfrak{o}/\mathfrak{p}^r)$ $(r \in \mathbb{N})$ having non-minimal dimensional centralizer and we do it according to the isomorphism type of their shadow rather than according to the conjugacy class of their shadow. Throughout the rest of this chapter $\mathrm{G} = \mathrm{SL}_3$ (hence $d = 8$ and $n = 4$). The normalized Killing form described in [3, Section 6.1] is non-degenerate for $3 \nmid q$. We assume from now on that $3 \nmid q$ and we denote with κ the non-degenerate form.

3.2.1. Poincaré series with shadows. First of all we rephrase the summation defining the Poincaré series so that it fits the language of shadows introduced in Chapter 2. We shall need some notation: let $\mathbf{S}$ be an isomorphism type of shadows. Recall that in Definition 3.5 we defined $d_{\mathbf{S}} = \dim_{\mathbb{F}_q} \mathrm{As}(\mathbf{S})$. For $\mathbf{S} \in \mathfrak{Sh}(\mathfrak{sl}_3(\mathfrak{o}))$ we define

$$\delta(\mathbf{S}) = \frac{1}{2}(d - d_{\mathbf{S}}) = n - \left\lfloor \frac{1}{2} d_{\mathbf{S}} \right\rfloor .$$

Definition 3.8. A *decreasing sequence of shadows* is a set of isomorphism types of shadows

$$\{\mathbf{S}_1, \dots, \mathbf{S}_\ell\}$$

such that for $0 < i < j \leq t$ we have $d_{\mathbf{S}_i} > d_{\mathbf{S}_j}$. The set of all decreasing sequences of shadows is denoted with $\mathcal{D}$.

Definition 3.9. Let $\mathcal{I} = \{\mathbf{S}_1, \dots, \mathbf{S}_\ell\} \in \mathcal{D}$ and $\mathbf{r}_{\mathcal{I}} = (r_{\mathbf{S}_1}, \dots, r_{\mathbf{S}_\ell}) \in \mathbb{N}^{\mathcal{I}}$. Let $N = \sum_{\mathbf{S} \in \mathcal{I}} r_{\mathbf{S}}$ and $W_N(\mathfrak{o})$ be as in (1.3). We define

$$\mathcal{N}_{\mathcal{I},\mathbf{r}_{\mathcal{I}}}(\mathfrak{sl}_3(\mathfrak{o})) = \left\{ x \in W_N(\mathfrak{o}) \;\middle|\; \forall \mathbf{S}_i \in \mathcal{I} \, \forall r \in \left] \sum_{j \leq i} r_{\mathbf{S}_j}, \sum_{j \leq i+1} r_{\mathbf{S}_j} \right] : \mathrm{Sh}_{\mathrm{SL}_3(\mathfrak{o}_t)}(\sigma_r(x)) \cong \mathbf{S}_i \right\}.$$

3.2.1.1. *Rank loci and shadows.* Let $\mathcal{R}$ be the commutator matrix of $\mathfrak{sl}_3(\mathfrak{o})$ with respect to an $\mathfrak{o}$-basis $\mathcal{B}$. Let $k \in \{0, \dots, n\}$, the rank-$2k$ locus of $\mathcal{R}$ is

$$\mathrm{L}^{2k}_{\mathcal{R}}(\mathfrak{o}) = \{\mathbf{x} \in \mathfrak{o}^d \mid \mathrm{rk}_{\mathfrak{o}} \mathcal{R}(\mathbf{x}) = 2k\}.$$

In order to use the results from Chapter 2 we need to establish a correspondence between the rank loci of $\mathcal{R}$ and the loci of constant centralizer dimension, i.e.

$$\mathrm{X}^{d-2k}_{\mathfrak{sl}_3(\mathfrak{o})}(\mathfrak{o}) = \{x \in \mathfrak{sl}_3(\mathfrak{o}) \mid \mathrm{rk}_{\mathfrak{o}} \, \mathrm{C}_{\mathfrak{sl}_3(\mathfrak{o})}(x) = d - 2k\}.$$

for $2k \leq d$. Let $r \in \mathbb{N}$, we introduce some notation: the choice of the $\mathfrak{o}$-basis $\mathcal{B}$ for $\mathfrak{sl}_3(\mathfrak{o})$ determines coordinate systems

$$\iota : \mathfrak{sl}_3(\mathfrak{o}) \to \mathfrak{o}^d$$
$$\iota_r : \mathfrak{sl}_3(\mathfrak{o}_r) \to (\mathfrak{o}_r)^d.$$

We write $\bar{\iota} = \iota_1$ and we denote with η the dual of ι. The proof of [3, Lemma 2.4] provides us with an isomorphism

$$\eta_r : W_r(\mathfrak{o}) \to \mathrm{Irr}_r(\mathfrak{sl}_3(\mathfrak{o})).$$

We set

$$\mathcal{R}^r = (\sigma_r(g_{ij}))_{i,j=1,\dots,8}$$

for the reduction mod $\mathfrak{p}^r$ of $\mathcal{R}$. We denote with λ be the isomorphism from $\mathfrak{sl}_3(\mathfrak{o})$ to $\mathfrak{sl}_3(\mathfrak{o})^\vee = \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{sl}_3(\mathfrak{o}), \mathfrak{o})$ defined by the normalized Killing form κ. Let

$$\lambda_r : \mathfrak{sl}_3(\mathfrak{o}_r) \to \mathrm{Irr}_r(\mathfrak{sl}_3(\mathfrak{o})) \cong \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{sl}_3(\mathfrak{o}), \mathfrak{o}_r)$$

be the $\mathfrak{o}_r$-modules isomorphism induced by λ. We set $\xi_r = \eta_r^{-1} \circ \lambda_r$ and $\xi = \eta^{-1} \circ \lambda$.

We mimic the argument in [3, Section 5]. Let $x \in \mathfrak{sl}_3(\mathfrak{o})$, we have

$$\begin{aligned}
\mathrm{Rad}(\kappa(x, \cdot)) &= \{y \in \mathfrak{sl}_3(\mathfrak{o}) \mid \forall z \in \mathfrak{sl}_3(\mathfrak{o}) \, : \, \kappa(x, [y, z]) = 0\} \\
&= \{y \in \mathfrak{sl}_3(\mathfrak{o}) \mid \forall z \in \mathfrak{sl}_3(\mathfrak{o}) \, : \, \kappa([x, y], z) = 0\} \\
&= \{y \in \mathfrak{sl}_3(\mathfrak{o}) \mid [x, y] = 0\} \\
&= \mathrm{C}_{\mathfrak{sl}_3(\mathfrak{o})}(x) .
\end{aligned}$$

It follows, by Lemma 1.13, that $\xi\,\mathrm{X}^{d-2k}_{\mathfrak{sl}_3(\mathfrak{o})}(\mathfrak{o}) = \mathrm{L}^{2k}_{\mathcal{R}}(\mathfrak{o})$. Let $a \in \mathfrak{sl}_3(\mathfrak{o}_r)$, the compatibility of ξ with the reduction mod $\mathfrak{p}^r$ and the observations before Lemma 1.13 imply

$$\dim_{\mathbb{F}_q} \sigma(\ker_{\mathfrak{o}_r} \mathcal{R}^r(\xi_r(a))) = \dim_{\mathbb{F}_q} \mathrm{Sh}_{\mathfrak{sl}_3(\mathfrak{o}_r)}(a)\,. \tag{3.1}$$

3.2.1.2. *Poincaré series with shadows.* Definition 3.9 allows us to rewrite the Poincaré series of $\mathfrak{sl}_3(\mathfrak{o})$: for $I = \{i_1, \dots, i_\ell\}_< \subseteq [n-1]_0$, we define

$$\mathcal{D}_I = \left\{ \{\mathbf{S}_1, \dots, \mathbf{S}_\ell\} \in \mathcal{D} \;\middle|\; \left\lfloor \frac{d_{\mathbf{S}_j}}{2} \right\rfloor = i_j \;\forall j \in \{1, \dots, \ell\} \right\}.$$

Now set $\mathbf{r}_{\mathcal{I}} = \mathbf{r}_I$ for all $\mathcal{I} \in \mathcal{D}_I$. It follows from the definition of $\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathfrak{sl}_3(\mathfrak{o}))$ (see (1.4)) and from (3.1) that

$$\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathfrak{sl}_3(\mathfrak{o})) = \sum_{\mathcal{I} \in \mathcal{D}_I} |\mathcal{N}_{\mathcal{I},\mathbf{r}_{\mathcal{I}}}(\mathfrak{sl}_3(\mathfrak{o}))|.$$

With (1.5), this implies

$$\mathcal{P}_{\mathfrak{sl}_3(\mathfrak{o})}(s) = \sum_{\mathcal{I} \in \mathcal{D}} \sum_{\mathbf{r}_{\mathcal{I}} \in \mathbb{N}^{\mathcal{I}}} |\mathcal{N}_{\mathcal{I},\mathbf{r}_{\mathcal{I}}}(\mathfrak{sl}_3(\mathfrak{o}))| q^{-s \sum_{\mathbf{S} \in \mathcal{I}} r_{\mathbf{S}} \cdot \delta(\mathbf{S})}. \tag{3.2}$$

3.2.2. A multiplicative formula for the Poincaré series. We shall now use the results in Section 3.1 to compute the coefficients of the Poincaré series (3.2).

LEMMA 3.10. *Consider $\mathcal{I} = \{\mathbf{S}_1, \dots, \mathbf{S}_\ell\} \in \mathcal{D}$. Let $\mathbf{r}_{\mathcal{I}} = (r_{\mathbf{S}_1}, \dots, r_{\mathbf{S}_\ell}) \in \mathbb{N}^{\mathcal{I}}$. Let $\mathbf{S}_0 = \mathrm{SL}_3(\mathbb{F}_q)$ and $\mathcal{N}_{\mathcal{I},\mathbf{r}_{\mathcal{I}}}(\mathfrak{sl}_3(\mathfrak{o}))$ be as in Definition 3.9. Then*

$$|\mathcal{N}_{\mathcal{I},\mathbf{r}_{\mathcal{I}}}(\mathfrak{sl}_3(\mathfrak{o}))| = \prod_{\mathbf{S}_i \in \mathcal{I}} \left(\Lambda(\mathbf{S}_{i-1}, \mathbf{S}_i) \cdot q^{d - d_{\mathbf{S}_{i-1}}} \right) \cdot \prod_{\mathbf{S} \in \mathcal{I}} \left(\Lambda(\mathbf{S}, \mathbf{S}) \cdot q^{d - d_{\mathbf{S}}} \right)^{r_{\mathbf{S}} - 1}.$$

PROOF. From the definition of $\mathcal{N}_{\mathcal{I},\mathbf{r}_{\mathcal{I}}}(\mathfrak{sl}_3(\mathfrak{o}))$ (Definition 3.9) we have that

$$|\mathcal{N}_{\mathcal{I},\mathbf{r}_{\mathcal{I}}}(\mathfrak{sl}_3(\mathfrak{o}))| = \Lambda(\mathrm{SL}_3(\mathbb{F}_q), \mathbf{S}_1) \cdot \prod_{\mathbf{S}_i \in \mathcal{I} \smallsetminus \{\mathbf{S}_\ell\}} c_{\mathbf{S}_i, \mathbf{S}_{i+1}} \cdot \prod_{\mathbf{S} \in \mathcal{I}} {c_{\mathbf{S},\mathbf{S}}}^{r_{\mathbf{S}} - 1}.$$

Now it suffices to apply Proposition 3.7 to the equation above. □

REMARK 3.11. Let $\mathbf{S} \in \mathfrak{Sh}(\mathfrak{sl}_3(\mathfrak{o}))$ and $\mathfrak{s} = \mathrm{As}(\mathbf{S})$. Let $\mathcal{B}_{\mathfrak{s}}$ be an $\mathfrak{o}$-basis for $\mathfrak{s}$ and let $\mathcal{R}_{\mathfrak{s}}$ be the commutator matrix of $\mathfrak{s}$ with respect to $\mathcal{B}_{\mathfrak{s}}$. Consider the fixed points

$$\mathrm{Triv}_{\mathbf{S}}(\mathfrak{s}^\vee) = \{\omega \in \mathfrak{s}^\vee \mid g.\omega = \omega \;\forall g \in \mathbf{S}\} \subseteq \mathfrak{s}^\vee$$

for the action of $\mathbf{S}$ on $\mathfrak{s}^\vee$. Thanks to Lemma 1.13 we know that $\mathrm{Triv}_{\mathbf{S}}(\mathfrak{s}^\vee)$ is the set of elements for which $\mathcal{R}_{\mathfrak{s}}$ has rank 0, and therefore it is an $\mathbb{F}_q$-vector space of dimension $z_{\mathbf{S}} \in \mathbb{N}$, say. This implies

$$\Lambda(\mathbf{S}, \mathbf{S}) = |\mathrm{Triv}_{\mathbf{S}}(\mathfrak{s}^\vee)| = q^{z_{\mathbf{S}}}.$$

DEFINITION 3.12. Let $\mathcal{I}$ and $\mathbf{r}_{\mathcal{I}}$ be as in Lemma 3.10. We define

$$\mathrm{f}_{\mathcal{I}}(q) = q^{-(d - d_{\mathbf{S}_\ell}) - \sum_{\mathbf{S} \in \mathcal{I}} z_{\mathbf{S}}} \cdot \prod_{\mathbf{S}_i \in \mathcal{I}} \Lambda(\mathbf{S}_{i-1}, \mathbf{S}_i).$$

Remark 3.11 allows us to restate Lemma 3.10 as follows.

LEMMA 3.13. *Let $\mathcal{I}$ and $\mathbf{r}_{\mathcal{I}}$ be as in Lemma 3.10. Then*

$$|\mathcal{N}_{\mathcal{I},\mathbf{r}_{\mathcal{I}}}(\mathfrak{sl}_3(\mathfrak{o}))| = \mathrm{f}_{\mathcal{I}}(q) \cdot \prod_{\mathbf{S} \in \mathcal{I}} \left(q^{d - d_{\mathbf{S}} + z_{\mathbf{S}}} \right)^{r_{\mathbf{S}}}.$$

PROOF. According to Remark 3.11 we can write the equality of Lemma 3.10 as

$$|\mathcal{N}_{\mathcal{I},\mathbf{r}_\mathcal{I}}(\mathfrak{sl}_3(\mathfrak{o}))| = \prod_{\mathbf{S}_i\in\mathcal{I}} \Lambda(\mathbf{S}_{i-1},\mathbf{S}_i)\cdot q^{d-d_{\mathbf{S}_{i-1}}}\cdot \prod_{\mathbf{S}\in\mathcal{I}} \left(q^{d-d_\mathbf{S}+z_\mathbf{S}}\right)^{r_\mathbf{S}-1}.$$

It remains to compute the telescopic sum $\sum_{\mathbf{S}_i\in\mathcal{I}}(d_{\mathbf{S}_i}-d_{\mathbf{S}_{i-1}}) = d_{\mathbf{S}_\ell}-d_{\mathbf{S}_0} = -(d-d_{\mathbf{S}_\ell})$. □

We define

$$\mathbf{gp}\,(X) = \frac{X}{1-X}.$$

Lemma 3.13 and (3.2) imply the following:

$$\mathcal{P}_{\mathfrak{sl}_3(\mathfrak{o})}(s) = \sum_{\mathcal{I}\in\mathcal{D}} \mathrm{f}_\mathcal{I}(q)\cdot\prod_{\mathbf{S}\in\mathcal{I}} \mathbf{gp}\left(q^{d-d_\mathbf{S}+z_\mathbf{S}-s\cdot\delta(\mathbf{S})}\right). \tag{3.3}$$

3.3. The representation zeta function of $\mathrm{SL}_3^m(\mathfrak{o})$

Let $r\in\mathbb{N}$. We subdivide the elements of $\mathfrak{sl}_3(\mathfrak{o}_r)$ according to their shadow dimension: we say that $a\in\mathfrak{sl}_3(\mathfrak{o}_r)$ is *regular* if $\dim_{\mathbb{F}_q}\mathrm{Sh}_{\mathfrak{sl}_3(\mathfrak{o}_r)}(a)=2$ and that a is *subregular* if $\dim_{\mathbb{F}_q}\mathrm{Sh}_{\mathfrak{sl}_3(\mathfrak{o}_r)}(a)=4$. A little thought unveils that except for $0\in\mathfrak{sl}_3(\mathfrak{o}_r)$, whose shadow is $\mathrm{SL}_3(\mathbb{F}_q)$, elements of $\mathfrak{sl}_3(\mathfrak{o}_r)$ are either regular or subregular.

Consider a regular element $a\in\mathfrak{sl}_3(\mathfrak{o}_r)$ on level $r\in\mathbb{N}$. The action of $\mathrm{Sh}_{\mathrm{SL}_3(\mathfrak{o}_r)}(a)$ on $\mathrm{Sh}_{\mathfrak{sl}_3(\mathfrak{o}_r)}(a)^\vee$ is trivial. For this reason we do not need to distinguish regular elements according to their shadow and, for all $\mathbf{S}\in\mathfrak{Sh}(\mathfrak{sl}_3(\mathfrak{o}_r))$, we define

$$\Lambda(\mathbf{S},\mathbf{R}) = \sum_{\substack{\mathbf{T}\in\mathfrak{Sh}(\mathfrak{sl}_3(\mathfrak{o}_{r+1}))\\ d_\mathbf{T}=2}} \Lambda(\mathbf{S},\mathbf{T}). \tag{3.4}$$

3.3.1. Subregular elements. We start by considering the situation at level $r=1$. That is to say, we look at orbits for the action of $\mathrm{SL}_3(\mathbb{F}_q)$ on $\mathfrak{sl}_3(\mathbb{F}_q)$. An analysis of the Frobenius rational forms in $\mathfrak{sl}_3(\mathbb{F}_q)$ reveals that the possible minimal polynomials of a subregular element are

$$m_\alpha = (X-\alpha)(X-2\alpha),$$

where $\alpha\in\mathbb{F}_q$. In what follows we operate a case distinction depending on whether α is zero or not.

3.3.1.1. *Subregular semisimple.* Let $a\in\mathfrak{sl}_3(\mathbb{F}_q)$ have minimal polynomial

$$m_\alpha = (X-\alpha)(X-2\alpha)$$

for $\alpha\in\mathbb{F}_q^\times$. Since the factors of m_α are linear and distinct, a is semisimple and diagonalizable, we observe that $\mathrm{Sh}_{\mathrm{SL}_3(\mathbb{F}_q)}(a) = \mathrm{C}_{\mathrm{SL}_3(\mathbb{F}_q)}(a)\cong\mathrm{GL}_2(\mathbb{F}_q)$. Let $\mathbf{L}$ be the isomorphism type of the shadow of these elements. The orbit of a has cardinality

$$\frac{|\mathrm{SL}_3(\mathbb{F}_q)|}{|\mathrm{GL}_2(\mathbb{F}_q)|} = q^2(q^2+q+1).$$

Semisimple subregular elements form as many orbits as the possible different minimal polynomials m_α with $\alpha\neq 0$, i.e. $q-1$. Therefore there are

$$\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{L}) = q^5-q^2 \tag{3.5}$$

subregular semisimple elements in total.

Moreover, the $\mathrm{Sh}_{\mathrm{SL}_3(\mathbb{F}_q)}(a)$-action on $\mathrm{Sh}_{\mathfrak{sl}_3(\mathbb{F}_q)}(a)^\vee$ is the adjoint action of $\mathrm{GL}_2(\mathbb{F}_q)$ on $\mathfrak{gl}_2(\mathbb{F}_q)$ and as a consequence

$$\begin{aligned} &d_\mathbf{L}=4,\ z_\mathbf{L}=1\\ &\Lambda(\mathbf{L},\mathbf{R}) = q\cdot(q^3-1). \end{aligned} \tag{3.6}$$

TABLE 3.1. Lifting rules for $\mathrm{SL}_3^m(\mathfrak{o})$. **R** stands for any regular isomorphism type of shadows

$\mathbf{S}$	$d_{\mathbf{S}}$	$z_{\mathbf{S}}$	$\delta(\mathbf{S})$	$\mathbf{T}$	$\Lambda(\mathbf{S},\mathbf{T})$
$\mathrm{SL}_3(\mathbb{F}_q)$	8	0	0	$\mathbf{L}$	(q^5-q^2)
				$\mathbf{J}$	(q^4+q^3-q-1)
				$\mathbf{R}$	$q\cdot(q-1)\cdot(q^6+q^5+q^4-q^2-2q-1)$
$\mathbf{L}$	4	1	2	$\mathbf{R}$	$q\cdot(q^3-1)$
$\mathbf{J}$	4	1	2	$\mathbf{R}$	$q\cdot(q^3-1)$
$\mathbf{R}$	2	2	3	n.a.	n.a.

3.3.1.2. *Subregular nilpotent elements.* All subregular elements that are not semisimple have minimal polynomial X^2 i.e. they are nilpotent. Let $a\in\mathfrak{sl}_3(\mathbb{F}_q)$ be such an element, and let

$$\mathbf{J}=\left\{M=\begin{pmatrix}m_{11}&m_{12}&m_{13}\\0&m_{11}&0\\0&m_{32}&m_{33}\end{pmatrix}\;\middle|\;M\in\mathrm{SL}_3(\mathbb{F}_q)\right\}.$$

Then $\mathrm{Sh}_{\mathrm{SL}_3(\mathbb{F}_q)}(a)\cong\mathbf{J}$. We choose a basis for $\mathrm{As}(\mathbf{J})$:

$$e_0=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\ e_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},\ e_2=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\ e_3=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}.$$

The basis $\mathcal{B}=\{e_0,\dots,e_3\}$ allows us to compute the commutator matrix

$$\mathcal{R}_{\mathcal{B}}(X_0,\dots,X_3)=\begin{pmatrix}0&0&0&0\\0&0&3X_2&-3X_3\\0&-3X_2&0&X_0\\0&3X_3&-X_0&0\end{pmatrix}.$$

By Lemma 1.13 (as we assumend $3\nmid q$) there are q elements of $\mathrm{As}(\mathbf{J})^\vee$ on which $\mathbf{J}$ acts trivially. This gives us

$$\begin{aligned}&d_{\mathbf{J}}=4,\ z_{\mathbf{J}}=1\\&\Lambda(\mathbf{J},\mathbf{R})=q\cdot(q^3-1).\end{aligned}\tag{3.7}$$

The centralizer of a subregular nilpotent element has cardinality $(q-1)q^3$, therefore

$$\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{J})=q^4+q^3-q-1.\tag{3.8}$$

Finally, as a nonzero element in $\mathfrak{sl}_3(\mathfrak{o})$ is either regular or subregular, the previous computations also yield the number of regular elements at level 1:

$$\begin{aligned}\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{R})&=q^8-1-\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{J})-\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{L})\\&=q\cdot(q-1)\cdot(q^6+q^5+q^4-q^2-2q-1).\end{aligned}\tag{3.9}$$

Table 3.1 gives an overview of the results in equations (3.5) to (3.9) (see also [4, Table 2.2]). In principle we would still need to complete the investigation for shadows appearing only at higher levels; however, since a lift of a subregular element is either regular or preserves the shadow, Table 3.1 actually describes the situation for all levels.

3.3.2. Representation zeta function. We can now compute the right-hand side of (3.3). Using(3.3) we shall then be able to determine the Poincaré series of $\mathfrak{sl}_3(\mathfrak{o})$ when $q > 2$ and $3 \nmid q$.

First of all we work out the possible non-empty decreasing sequences of shadows for $\mathfrak{sl}_3(\mathfrak{o})$: these are $\{\mathbf{L}\}$, $\{\mathbf{J}\}$ and all $\{\mathbf{S}\}$, $\{\mathbf{L},\mathbf{S}\}$ and $\{\mathbf{J},\mathbf{S}\}$ where $\mathbf{S}$ is a regular isomorphism type of shadow. For each decreasing sequence $\mathcal{I}$ we shall now compute the product of geometric progression associated with it and the coefficient $\mathrm{f}_{\mathcal{I}}(q)$. To do this it is convenient to make a distinction based on whether a decreasing sequence contains a 4-dimensional shadow or not. We keep the convention of not distinguishing among isomorphism types of regular shadows and, for all $\mathbf{S} \in \mathfrak{Sh}(\mathfrak{sl}_3(\mathfrak{o}))$, we define

$$\mathrm{f}_{\{\mathbf{R}\}}(q) = \sum_{\substack{\mathbf{T}\in\mathfrak{Sh}(\mathfrak{sl}_3(\mathfrak{o}))\\ d_{\mathbf{T}}=2}} \mathrm{f}_{\{\mathbf{T}\}}(q)$$
$$\mathrm{f}_{\{\mathbf{S},\mathbf{R}\}}(q) = \sum_{\substack{\mathbf{T}\in\mathfrak{Sh}(\mathfrak{sl}_3(\mathfrak{o}))\\ d_{\mathbf{T}}=2}} \mathrm{f}_{\{\mathbf{S},\mathbf{T}\}}(q)\,.$$

3.3.2.1. *Decreasing sequences containing a subregular shadow.* We collect all the summands corresponding to decreasing sequences that feature a 4-dimensional shadow. Let

$$\mathcal{D}_{sub} = \{\{\mathbf{L}\},\{\mathbf{J}\},\{\mathbf{L},\mathbf{T}\},\{\mathbf{J},\mathbf{T}\}\}_{\substack{\mathbf{T}\in\mathfrak{Sh}(\mathfrak{sl}_3(\mathfrak{o}))\\ d_{\mathbf{T}}=2}}$$

be the set containing all of these decreasing sequences. With the help of Table 3.1, a quick computation yields

$$\begin{aligned}
\mathrm{f}_{\{\mathbf{L}\}}(q) &= q^{-5}\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{L}) = q^{-5}(q^5-q^2)\\
\mathrm{f}_{\{\mathbf{J}\}}(q) &= q^{-5}\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{J}) = q^{-5}(q^4+q^3-q-1)\\
\mathrm{f}_{\{\mathbf{L},\mathbf{R}\}}(q) &= q^{-9}\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{L})\Lambda(\mathbf{L},\mathbf{R}) = q^{-9}(q^9-2q^6+q^3)\\
\mathrm{f}_{\{\mathbf{J},\mathbf{R}\}}(q) &= q^{-9}\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{J})\Lambda(\mathbf{J},\mathbf{R}) = q^{-9}(q^8+q^7-2q^5-2q^4+q^2+q).
\end{aligned}$$

We compute the following part of the summation in (3.3):

$$\begin{aligned}
\mathcal{P}_{sub}(s) = {}&\big(\mathrm{f}_{\{\mathbf{L}\}}(q)+\mathrm{f}_{\{\mathbf{J}\}}(q)\big)\cdot\frac{q^{5-2s}}{1-q^{5-2s}}\\
&+\big(\mathrm{f}_{\{\mathbf{L},\mathbf{R}\}}(q)+\mathrm{f}_{\{\mathbf{J},\mathbf{R}\}}(q)\big)\cdot\frac{q^{13-5s}}{(1-q^{8-3s})(1-q^{5-2s})}.
\end{aligned}\tag{3.10}$$

3.3.2.2. *The regular shadow.* The last non-empty decreasing sequences remaining are the ones containing only one regular shadow. By reading Table 3.1 we compute the summand in (3.3):

$$\begin{aligned}
\mathcal{P}_{reg}(s) = \mathrm{f}_{\{\mathbf{R}\}}(q)\,\frac{q^{8-3s}}{1-q^{8-3s}} &= q^{-8}\Lambda(\mathrm{SL}_3(\mathbb{F}_q),\mathbf{R})(q)\frac{q^{8-3s}}{1-q^{8-3s}}\\
&= q^{-8}(q^8-q^5-q^4-q^3+q^2+q)\frac{q^{8-3s}}{1-q^{8-3s}}.
\end{aligned}\tag{3.11}$$

The empty shadow sequence gives rise to the summand 1, hence by (3.3)

$$\begin{aligned}
&\mathcal{P}_{\mathfrak{sl}_3(\mathfrak{o})}(s) = 1+\mathcal{P}_{sub}(s)+\mathcal{P}_{reg}(s)\\
&= \frac{q^{10}+\big(q^9+q^8-q^7-q^6-q^5\big)q^{3\,s}-\big(q^{10}+q^9+q^8-q^7-q^6\big)q^{2\,s}+q^{5\,s+5}}{(q^8-q^{3\,s})(q^5-q^{2\,s})q^5}.
\end{aligned}\tag{3.12}$$

Operating the substitution in Proposition 1.15 we deduce the following special case of [3, Theorem E].

THEOREM 3.14. *Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic* 0 *whose residue field has cardinality $q > 2$ and characteristic $p \neq 3$. Then for all permissible m,*

$$\zeta_{\mathrm{SL}_3^m(\mathfrak{o})}(s) = q^{8m} \frac{1 + u(q)q^{-3-2s} + u(q^{-1})q^{-2-3s} + q^{-5-5s}}{(1 - q^{1-2s})(1 - q^{2-3s})}$$

where $u(X) = X^3 + X^2 - X - 1 - X^{-1}$.

Notice that [3, Theorem E] also describes the representation zeta function of $\mathrm{SU}_3^1(\mathfrak{o})$.

CHAPTER 4

Reduction to the Lie algebra over the finite field

We keep the notation conventions established at the beginning of Chapter 2. In particular, recall that G is a linear algebraic group defined over $\mathbb{Z}$ with Lie algebra $\mathbf{g} = \mathrm{Lie}(\mathrm{G})$. In Chapter 2 we imposed that $\mathfrak{g} = \mathbf{g}(\mathfrak{o})$ admitted a non-degenerate invariant symmetric bilinear form (see Section 2.4). This assumption remains valid throughout this chapter, let κ be such bilinear form. Recall that $d = \mathrm{rk}_{\mathfrak{o}}\, \mathfrak{g} = \dim \mathrm{G}$ and that $n = \lfloor d/2 \rfloor$.

Chapter 3 uses the fact that $\mathfrak{sl}_3(\mathfrak{o})$ is shadow-hereditary to derive a method for computing $\mathcal{P}_{\mathfrak{sl}_3(\mathfrak{o})}$. In Section 5.5.2, however, we shall see that $\mathfrak{sl}_4(\mathfrak{o})$ is not shadow-hereditary, it follows that the approach adopted for $\mathfrak{sl}_3(\mathfrak{o})$ cannot be followed for $\mathfrak{sl}_4(\mathfrak{o})$. In the present chapter we see that in some cases this obstacle can be removed by restricting to decreasing sequences of shadows of elements in $\bar{\mathfrak{g}}$ (i.e. to sequences of centralizers of elements of $\bar{\mathfrak{g}}$ with decreasing dimension).

4.1. Notation

Before proceeding it is useful to introduce some terminology.

4.1.1. Commutator matrices and rank-varieties. Let $\mathfrak{k} = \mathrm{Frac}(\mathfrak{o})$. We fix an $\mathfrak{o}$-basis $\mathcal{B}$ for $\mathfrak{g}$ and for the rest of the chapter we denote with $\mathcal{R}$ the commutator matrix of $\mathfrak{g}$ with respect to $\mathcal{B}$.

Definition 4.1. For $2i \leq d$, let $P_i \subseteq \mathfrak{o}[\mathbf{Y}]$ be the ideal generated by the $2i \times 2i$ Pfaffians of $\mathcal{R}$. We write

$$\mathrm{V}_{\mathcal{R}}^{2i} = \mathrm{Spec}\left(\mathfrak{o}[\mathbf{Y}]/P_i\right).$$

The *rank-$2i$ locus* $\mathrm{L}_{\mathcal{R}}^{2i}$ of $\mathcal{R}$ is the scheme-theoretic complement of $\mathrm{V}_{\mathcal{R}}^{2(i-1)}$ as a closed subscheme of $\mathrm{V}_{\mathcal{R}}^{2i}$.

Lemma 4.2. *Let $\mathcal{B}'$ be another $\mathfrak{o}$-basis for $\mathfrak{g}$, and let S be the basis-change matrix from $\mathcal{B}$ to $\mathcal{B}'$. Then, for all $\mathbf{v} \in \mathfrak{o}^d$,*

$$S^{\mathrm{t}}\, \mathcal{R}'(\mathbf{v}) S = \mathcal{R}(\mathbf{v}S^{-\mathrm{t}}),$$

where $\mathcal{R}'$ the commutator matrix of $\mathfrak{g}$ with respect to $\mathcal{B}'$.

Proof. Let $\mathbf{v} = (v_1, \dots, v_d) \in \mathfrak{o}^d$. Let also $\mathcal{B}'^{\vee} = \{{b'_1}^{\vee}, \dots, {b'_d}^{\vee}\}$ be the dual basis of $\mathcal{B}'$. The matrix $\mathcal{R}'(\mathbf{v})$ is the matrix of the bilinear form b_ω defined in Section 1.1.2 where $\omega = \sum_{i=1}^{d} v_i {b'_i}^{\vee}$. Since S is the basis change from $\mathcal{B}$ to $\mathcal{B}'$, $\mathbf{v}S^{-\mathrm{t}}$ expresses the coordinates of ω with respect to $\mathcal{B}$. It follows that $\mathcal{R}(\mathbf{v}S^{-\mathrm{t}})$ is the matrix of b_ω with respect to $\mathcal{B}$. Hence the equality with $S^{\mathrm{t}}\, \mathcal{R}'(\mathbf{v})S$. □

Definition 4.3. We say that $\mathfrak{g}$ has *smooth rank loci* if for all $2i \leq d$ the rank-$2i$ locus $\mathrm{L}_{\mathcal{R}}^{2i}$ is smooth over $\mathfrak{k}$ and has good reduction mod $\mathfrak{p}$. We say that $\mathfrak{g}$ has *smooth and irreducible rank loci* if for all $2i \leq d$ the rank-$2i$ locus $\mathrm{L}_{\mathcal{R}}^{2i}$ is smooth and irreducible over $\mathfrak{k}$ and has good reduction mod $\mathfrak{p}$.

By Lemma 4.2, changing the basis of $\mathfrak{g}$ results in a linear invertible substitution of variables in the equations defining the rank loci of $\mathcal{R}$. Therefore Definition 4.3 does not depend on the choice of the basis $\mathcal{B}$.

EXAMPLE 4.4. Whenever the normalized Killing form is non-degenerate (i.e. for almost all non-zero prime ideals $\mathfrak{p}$ of $\mathcal{O}$), $\mathfrak{sl}_h(\mathcal{O}_\mathfrak{p})$ has smooth rank loci. These are irreducible for $h \leq 5$ but not for $h = 6$ (see Section 4.3.1 for the details).

DEFINITION 4.5. Let $r \in \mathbb{N}$. A *choice of rank-preserving lifts* of level r is a function $\varphi : \mathbb{F}_q^d \to \mathfrak{o}_r^d$ such that, for all $2k \leq d$ and all $\mathbf{x} \in \mathrm{L}_\mathcal{R}^{2k}(\mathbb{F}_q)$, $\varphi(\mathbf{x}) \in \mathrm{L}_\mathcal{R}^{2k}(\mathfrak{o}_r)$.

DEFINITION 4.6. Assume that $\mathfrak{g}$ has smooth and irreducible rank loci. By Hensel's lemma (cf. Lemma 1.17), for all $r \in \mathbb{N}$, there is a choice of rank-preserving lifts $\varphi : \mathbb{F}_q^d \to \mathfrak{o}_r^d$ such that for all $\mathbf{x} \in \mathbb{F}_q^d$, $\varphi(\mathbf{x})$ is also a smooth point of its rank locus. We say that such φ is a *smooth choice* of rank-preserving lifts of level r.

DEFINITION 4.7. Let $I = \{i_1, \ldots, i_\ell\}_< \subseteq [n-1]_0$. Assume that $\mathfrak{g}$ has smooth rank loci and let φ be a smooth choice of rank-preserving lifts of level 2 in the sense of Definition 4.6.

We define $F_{I,\varphi}(\mathcal{R})$ as the set of $(\overline{\mathbf{v}}_1, \ldots, \overline{\mathbf{v}}_\ell) \in ((\mathbb{F}_q{}^d)^*)^\ell$ such that, for all $j = 1, \ldots, \ell$,

$$\sum_{k \geq j} \overline{\mathbf{v}}_k \in \mathrm{L}_\mathcal{R}^{2(n-i_j)}(\mathbb{F}_q) \tag{4.1}$$

and

$$\varphi\left(\sum_{k \geq j} \overline{\mathbf{v}}_k\right) + \pi \overline{\mathbf{v}}_{j-1} \in \mathrm{L}_\mathcal{R}^{2(n-i_{j-1})}(\mathfrak{o}_2)\,. \tag{4.2}$$

4.2. Poincaré series for Lie rings with smooth and irreducible rank loci

Let I be as in Definition 4.7. We assume henceforth that $\mathfrak{g}$ has smooth and irreducible rank loci. Let φ be a smooth choice of rank-preserving lifts of level 2. The main objective of the current section is to define a surjective function $\theta_{I,\mathbf{r}_I,\varphi} : N_{I,\mathbf{r}_I}^\mathfrak{o}(\mathcal{R}) \to F_{I,\varphi}(\mathcal{R})$ (cf. Proposition 4.14). This will allow us to translate the problem of determining the cardinality of $N_{I,\mathbf{r}_I}^\mathfrak{o}(\mathcal{R})$ to a problem in the Lie algebra over the finite field.

4.2.1. Rank loci and centralizers. Let $\mathcal{L} = \mathbf{g}(\mathbb{C})$. In order to use the results from Chapter 2 we need to establish a correspondence between the rank loci of $\mathcal{R}$ and the loci of constant centralizer dimension, i.e.

$$\mathrm{X}_\mathcal{L}^{d-2k}(\mathfrak{o}) = \{x \in \mathfrak{g} \mid \mathrm{rk}_\mathfrak{o}\, \mathrm{C}_\mathfrak{g}\,(x) = d - 2k\}.$$

for $2k \leq d$. The argument is entirely analogous to the one for $\mathfrak{sl}_3(\mathfrak{o})$ in Section 3.2.1.1. Let $r \in \mathbb{N}$ throughout this section.

DEFINITION 4.8. The choice of an $\mathfrak{o}$-basis for $\mathfrak{g}$ determines coordinate systems

$$\iota : \mathfrak{g} \to \mathfrak{o}^d$$

$$\iota_r : \mathfrak{g}_r \to (\mathfrak{o}_r)^d.$$

We write $\bar{\iota} = \iota_1$.

Fix a coordinate system on $\mathfrak{g}$. The proof of [3, Lemma 2.4] provides us with an isomorphism

$$\eta_r : W_r(\mathfrak{o}) \to \mathrm{Irr}_r(\mathfrak{g}).$$

We also denote with η the dual of ι.

Notation 4.9. Let $n_1, n_2 \in \mathbb{N}$ and

$$\mathcal{R} = (g_{ij})_{i,j=1,\dots n_1} \in \mathrm{Mat}_{n_1}(\mathfrak{o}[Y_1, \dots, Y_{n_2}])\,.$$

We write

$$\mathcal{R}^r = (\sigma_r(g_{ij}))_{i,j=1,\dots n_1}$$

for the reduction mod $\mathfrak{p}^r$ of $\mathcal{R}$. When $r = 1$ we write $\overline{\mathcal{R}} = \mathcal{R}^1$.

We denote with λ be the isomorphism from $\mathfrak{g}$ to $\mathfrak{g}^\vee = \mathrm{Hom}_\mathfrak{o}(\mathfrak{g}, \mathfrak{o})$ defined by the invariant non-degenerate symmetric form κ. Let

$$\lambda_r : \mathfrak{g}_r \to \mathrm{Irr}_r(\mathfrak{g}) \cong \mathrm{Hom}_\mathfrak{o}(\mathfrak{g}, \mathfrak{o}_r)$$

be the $\mathfrak{o}_r$-modules isomorphism induced by λ. Let $\xi_r = \eta_r^{-1} \circ \lambda_r$ for $r \in \mathbb{N}$ and $\xi = \eta^{-1} \circ \lambda$.

Remark 4.10. By Definition 4.8, the following diagrams commute

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \xi\ } & (\mathfrak{o})^d \\ \downarrow{\scriptstyle \sigma_r} & & \downarrow{\scriptstyle \sigma_r} \\ \mathfrak{g}_r & \xrightarrow[\ \xi_r\ ]{} & (\mathfrak{o}_r)^d, \end{array} \qquad 1$$

for all $r \in \mathbb{N}$.

We are now able to prove the analogous of (3.1).

Lemma 4.11. *Let $x \in \mathfrak{g}_r$. Then, $\xi\,\mathrm{X}_\mathcal{L}^{d-2k}(\mathfrak{o}) = \mathrm{L}_\mathcal{R}^{2k}(\mathfrak{o})$ and*

$$\dim_{\mathbb{F}_q} \sigma(\ker_{\mathfrak{o}_r} \mathcal{R}^r(\xi_r(x))) = \dim_{\mathbb{F}_q} \mathrm{Sh}_{\mathfrak{g}_r}(x)\,.$$

Proof. We mimic the argument in [3, Section 5]. Let $x \in \mathfrak{g}$. Then

$$\begin{aligned} \mathrm{Rad}(\kappa(x,\cdot)) &= \{y \in \mathfrak{g} \mid \forall z \in \mathfrak{g}\ :\ \kappa(x,[y,z]) = 0\} \\ &= \{y \in \mathfrak{g} \mid \forall z \in \mathfrak{g}\ :\ \kappa([x,y],z) = 0\} \\ &= \{y \in \mathfrak{g} \mid [x,y] = 0\} \\ &= \mathrm{C}_\mathfrak{g}\,(x)\,. \end{aligned}$$

Definition 4.1 and Lemma 1.13 imply that $\xi\,\mathrm{X}_\mathcal{L}^{d-2k}(\mathfrak{o}) = \mathrm{L}_\mathcal{R}^{2k}(\mathfrak{o})$. The compatibility of ξ with the reduction mod $\mathfrak{p}^r$ (cf. Remark 4.10) and the observations before Lemma 1.13 suffice to conclude. □

4.2.2. The surjective function $\theta_{I,\mathbf{r}_I,\varphi}$.

The following lemma and subsequent definitions are needed in the proof of Proposition 4.14.

Lemma 4.12. *Let $r \geq 2$. Let $\mathbf{y} \in \mathfrak{o}_2^d$ and $\mathbf{z} \in \mathfrak{o}_r^d$. Assume that*

$$\mathrm{Sh}_{\mathfrak{g}_2}\big(\xi_2^{-1}(\mathbf{y})\big) = \mathrm{Sh}_{\mathfrak{g}_r}\big(\xi_r^{-1}(\mathbf{z})\big)\,.$$

Then, for all $\mathbf{u} \in \mathbb{F}_q^d$,

$$\mathrm{Sh}_{\mathfrak{g}_2}\big(\xi_2^{-1}(\mathbf{y} + \pi\mathbf{u})\big) \cong \mathrm{Sh}_{\mathfrak{g}_r}\big(\xi_r^{-1}(\mathbf{z} + \pi^{r-1}\mathbf{u})\big)\,.$$

Proof. Temporarily set $y = \xi_2^{-1}(\mathbf{y})$, $z = \xi_r^{-1}(\mathbf{z})$ and $u = \xi_1^{-1}(\mathbf{u})$. Let δ_y and δ_z be defined as in Lemma 2.3. By Corollary 2.13 the shadows of $y + \pi u$ and $z + \pi^{r-1} u$ depend on the stabilizers of $u + \mathrm{im}\,\delta_y$ and $u + \mathrm{im}\,\delta_x$ under the action of $\mathrm{Sh}_{\mathfrak{g}_2}(y)$ and $\mathrm{Sh}_{\mathfrak{g}_r}(z)$. Since the shadow of y and z are equal, $\mathrm{coker}\,\delta_y$ and $\mathrm{coker}\,\delta_z$ are isomorphic under a $\mathrm{Sh}_{\mathfrak{g}_2}(y)$-invariant isomorphism. This suffices to conclude. □

DEFINITION 4.13. We say that an antisymmetric $2n \times 2n$ matrix M over a ring $\mathfrak{o}$ is in block form if, for some $x_1, \ldots, x_n \in \mathfrak{o}$,

$$M = \mathrm{B}_{2n}(x_1, \ldots, x_n) = \begin{pmatrix} \mathrm{Bl}(x_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathrm{Bl}(x_n) \end{pmatrix},$$

where

$$\mathrm{Bl}(x) = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}.$$

In what follows we set $I = \{i_1, \ldots, i_\ell\}_< \subseteq [n-1]_0$. For $j = 0, \ldots, \ell+1$, we define

$$d_j = \begin{cases} 2i_j & \text{if } d \text{ is even} \\ 2i_j + 1 & \text{if } d \text{ is odd.} \end{cases} \tag{4.3}$$

We also set

$$f_j = \dim \mathrm{L}_{\mathcal{R}}^{2(n-i_j)} \; (j = 1, \ldots, \ell).$$

PROPOSITION 4.14. *Let $\mathbf{r}_I = (r_1, \ldots, r_\ell) \in \mathbb{N}^{|I|}$ and let $N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R})$ be defined as in Section 1.2.2. Assume that $\mathfrak{g}$ admits a non-degenerate invariant symmetric bilinear form κ. Assume further that $\mathfrak{g}$ has smooth and irreducible rank loci and let φ be a smooth choice of rank-preserving lifts of level 2. Then there is a surjective function*

$$\theta_{I,\mathbf{r}_I,\varphi} : N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R}) \to F_{I,\varphi}(\mathcal{R}),$$

whose fibres have cardinality

$$\prod_{k=1}^{\ell} q^{f_k(r_k-1)}.$$

PROOF. Recall that we defined $N = \sum_{j=1}^{\ell} r_j$. We set $I_j = \{i_1, \ldots, i_j\} \subseteq [n-1]_0$, $\mathbf{r}_{I_j} = \{r_1, \ldots, r_j\}$ and $N_j = \sum_{k=1}^{j} r_k$.

To define $\theta_{I,\mathbf{r}_I,\varphi}$, we start with an element $\mathbf{w} \in N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R})$ and construct a sequence of d-tuples $\mathbf{w}^\ell, \ldots, \mathbf{w}^1 \in \mathfrak{o}_N^d$ such that, for all $j = 1, \ldots, \ell$,

$$\sigma_{N-N_j}(\mathbf{w}^j) \in N^{\mathfrak{o}}_{I_j,\mathbf{r}_{I_j}}(\mathcal{R}). \tag{4.4}$$

This is done recursively as follows. Set $\mathbf{w}^\ell = \mathbf{w}$. Now let $k \in \{1, \ldots, \ell\}$ and assume that we have defined $\mathbf{w}^\ell, \ldots, \mathbf{w}^k$ such that property (4.4) holds. Thanks to [12, Lemma 3.2],

$$\mathfrak{r}_k = \mathrm{Rad}\, \eta_{r_k} \sigma_{r_k}(\mathbf{w}^k)$$

is a sub-Lie ring of $\mathfrak{g}$ of finite index. Since (4.4) holds for $\mathbf{w}^k$ and by Lemma 4.2, we can choose an $\mathfrak{o}$-basis

$$\mathcal{C}_k = \{\pi^{r_k} b_1^k, \ldots, \pi^{r_k} b^k_{d-d_k}, b^k_{d-d_k+1}, \ldots, b^k_d\}$$

for $\mathfrak{r}_k$ such that $\mathcal{B}_k = \{b_1^k, \ldots, b_d^k\}$ is an $\mathfrak{o}$-basis for $\mathfrak{g}$ and for all lifts $\widehat{\mathbf{w}}^k \in \mathfrak{o}^d$ of $\mathbf{w}^k$

$$\mathcal{R}_k\left(\varepsilon_k(\widehat{\mathbf{w}}^k)\right) = \begin{pmatrix} \mathrm{B}_{d-d_k}(1, \ldots, 1) & 0 \\ 0 & \mathrm{M}\left(\varepsilon_k(\widehat{\mathbf{w}}^k)\right) \end{pmatrix}, \tag{4.5}$$

where $\mathcal{R}_k$ is the commutator matrix of $\mathfrak{g}$ with respect to $\mathcal{B}_k$, $\mathrm{M}(\mathbf{Y})$ is a $d_k \times d_k$-matrix of linear forms with entries in $\mathfrak{o}[Y_1, \ldots, Y_d]$ and ε_k is the basis change from $\mathcal{B}$ to $\mathcal{B}_k$.

Let $\mathcal{H}_k = \{b_{d-d_k+1}, \ldots, b_d\}$ and $\mathfrak{h}_k = \mathrm{Span}\, \mathcal{H}_k$. For all $r \in \mathbb{N}$, let $\varepsilon_k^r : {\mathfrak{o}_r}^d \to {\mathfrak{o}_r}^d$ be the isomorphism induced by ε_k. Set $\mathbf{x}^k = \varepsilon_k^N(\mathbf{w}^k)$. Since $\mathfrak{g}$ has smooth rank loci and thanks to (4.4), $\sigma_{r_k}(\mathbf{w}^k)$ is a smooth point of $\mathrm{L}_{\mathcal{R}}^{d-d_k}(\mathfrak{o}_{r_k})$ and so is

$\sigma_{r_k}(\mathbf{x}^k) = \varepsilon_k^{r_k}(\sigma_{r_k}(\mathbf{w}^k))$ for $\mathrm{L}_{\mathcal{R}_k}^{d-d_k}(\mathfrak{o}_{r_k})$. Hensel's lemma implies that there is a lift $\widehat{\mathbf{y}}^k \in \mathfrak{o}^d$ of $\sigma_{r_k}(\mathbf{x}^k)$ such that

$$\mathfrak{h}_k = \ker \mathcal{R}_k(\widehat{\mathbf{y}}^k).$$

It follows that $\mathfrak{h}_k$ is a sub-Lie ring of $\mathfrak{g}$. Hence, for all lifts $\widehat{\mathbf{x}}^k \in \mathfrak{o}^d$ of $\mathbf{x}^k$

$$\mathrm{M}(\widehat{\mathbf{x}}^k) = \mathcal{R}_{\mathcal{H}_k}(\widehat{x}^k_{d-d_k+z_k+1}, \ldots, \widehat{x}^k_d), \tag{4.6}$$

where $\mathcal{R}_{\mathcal{H}_k}$ is the commutator matrix of $\mathfrak{h}_k$ with respect to $\mathcal{H}_k$ and z_k is the number of variables that do not appear in $\mathcal{R}_{\mathcal{H}_k}$. Since we know the valuation of the matrix $\mathcal{R}_k\left(\varepsilon_k(\widehat{\mathbf{w}}^k)\right)$, the submatrix $\mathcal{R}_{\mathcal{H}_k}$ is such that

$$\mathcal{R}_{\mathcal{H}_k}(\mathbf{z}) \equiv 0 \mod \mathfrak{p} \iff \mathbf{z} \equiv 0 \mod \mathfrak{p}.$$

This in particular implies that on the open set $\{\widehat{x} \in \mathfrak{o}^d \mid \sigma_k(\widehat{\mathbf{x}}) = \mathbf{x}^k\}$ the rank locus $\mathrm{L}_{\mathcal{R}}^{d-d_k}(\mathfrak{o})$ is described by the vanishing of $d_k - z_k$ coordinate-functions. By the irreducibility of the rank loci, this implies that $d - d_k + z_{k=f_k}$. By (4.4), for $f_k < h \leq d$, all of the x_h^k are multiples of π^{r_k}. Therefore we can define

$$\begin{aligned} \mathbf{x}^{k-1} &= (x_1^k, \ldots, x_{f_k}^k, \pi^{-r_k} x_{f_k+1}^k, \ldots, \pi^{-r_k} x_d^k) \\ \mathbf{w}^{k-1} &= (\varepsilon_k^{N_k})^{-1}(\mathbf{x}^{k-1}). \end{aligned} \tag{4.7}$$

It follows from the construction that

$$\sigma_{N-N_{k-1}}(\mathbf{w}^{k-1}) \in N^{\mathfrak{o}}_{I_{k-1},\mathbf{r}_{I_{k-1}}}(\mathcal{R}).$$

We shall now use the sequence $\mathbf{w}^1, \ldots, \mathbf{w}^\ell$ to define a sequence $\theta_{I,\mathbf{r}_I,\varphi}(\mathbf{w}) \in F_{I,\varphi}(\mathcal{R})$. We define

$$\mathbf{u}_j = \begin{cases} \mathbf{w}^\ell & \text{for } j = \ell \\ \mathbf{w}^j - \mathbf{w}^{j+1} & \text{for } j = 1, \ldots, \ell-1, \end{cases}$$

and, for $j = 1, \ldots, \ell$,

$$\overline{\mathbf{u}}_j = \sigma(\mathbf{u}_j).$$

The function $\theta_{I,\mathbf{r}_I,\varphi}$ is then defined by

$$\theta_{I,\mathbf{r}_I,\varphi}(\mathbf{w}) = (\overline{\mathbf{u}}_1, \ldots, \overline{\mathbf{u}}_\ell).$$

It remains to prove that $\theta_{I,\mathbf{r}_I,\varphi}$ is well defined and surjective. We start by proving that it is well defined. Indeed the sequence $(\mathbf{w}^1, \ldots, \mathbf{w}^\ell)$ depends on the choices of $\mathcal{B}_1, \ldots, \mathcal{B}_{\ell-1}$. However, Lemma 1.13 and the definition of $\mathfrak{h}_j$ imply that, for all $j = 1, \ldots, \ell$,

$$\sigma_{r_j}(\mathfrak{h}_j) = \ker \mathcal{R}^{r_j}(\sigma_{r_j}(\mathbf{w}^j)),$$

It follows that there is no ambiguity in the definition of $\overline{\mathbf{u}}^1, \ldots, \overline{\mathbf{u}}^\ell$.

To see that $\theta_{I,\mathbf{r}_I,\varphi}(\mathbf{w}) \in F_{I,\varphi}(\mathcal{R})$ for all $\mathbf{w} \in N^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathcal{R})$ we argue as follows: for all $j = 1, \ldots, \ell$,

$$\sum_{j \leq k < \ell} \overline{\mathbf{u}}_k = \sum_{j \leq k < \ell} \sigma(\mathbf{u}_k) = \sum_{j \leq k < \ell} \sigma(\mathbf{w}^k - \mathbf{w}^{k+1} + \mathbf{w}^\ell) = \sigma(\mathbf{w}^j).$$

Moreover, by (4.4), $\sigma_{N-N_j}(\mathbf{w}^j) \in N^{\mathfrak{o}}_{I_j,\mathbf{r}_{I_j}}(\mathcal{R})$. Thus, by Remark 1.14, it follows that

$$\mathrm{rk}\, \overline{\mathcal{R}} \left(\sum_{k \geq j} \overline{\mathbf{u}}_k \right) = 2(n - i_j).$$

It remains to verify (4.2). Fix $j \in \{1, \dots, \ell\}$. The isomorphisms η and λ are compatible with the reduction modulo powers of $\mathfrak{p}$, thus

$$\mathrm{Sh}_{\bar{\mathfrak{g}}}\left(\xi_1^{-1}\left(\sum_{k\geq j}\overline{\mathbf{u}}_k\right)\right) = \mathrm{Sh}_{\mathfrak{g}_2}\left(\xi_2^{-1}\varphi\left(\sum_{k\geq j}\overline{\mathbf{u}}_k\right)\right) = \mathrm{Sh}_{\mathfrak{g}_N}\left(\xi_N^{-1}(\mathbf{w}^j)\right),$$

because we are computing the shadow of two lifts of $\sum_{k\geq j}\overline{\mathbf{u}}_k$. Therefore by Lemmata 4.11 and 4.12, for all $j = 1, \dots, \ell$,

$$\varphi\left(\sum_{k\geq j}\overline{\mathbf{u}}_k\right) + \pi\overline{\mathbf{u}}_{j-1} \in \mathrm{L}_{\mathcal{R}}^{2(n-i_{j-1})}(\mathfrak{o}_2)\,.$$

This concludes the proof of the fact that $\theta_{I,\mathbf{r}_I,\varphi}(\mathbf{w}) \in F_{I,\varphi}(\mathcal{R})$.

We now show that $\theta_{I,\mathbf{r}_I,\varphi}$ is surjective. Let $(\overline{\mathbf{v}}_1, \dots, \overline{\mathbf{v}}_\ell) \in F_{I,\varphi}(\mathcal{R})$. Since φ is a smooth choice, for all $\overline{\mathbf{v}}_j$ ($j = 1, \dots, \ell$), Hensel's lemma provides us with a lift $\mathbf{v}_j$ such that

$$\varphi(\overline{\mathbf{v}}_j) = \varphi\sigma(\mathbf{v}_j) = \sigma_2(\mathbf{v}_j). \tag{4.8}$$

The sequence $(\mathbf{w}^1, \dots, \mathbf{w}^j)$ defined by

$$\mathbf{w}^j = \sum_{k\geq j}\mathbf{v}_j + \sum_{k<j}\left(\prod_{h=k+1}^{j}\pi^{r_h}\right)\mathbf{v}_k$$

has the property (4.4), and the element $\mathbf{w} = \mathbf{w}^\ell \in N_{I,\mathbf{r}_I}^{\mathfrak{o}}(\mathcal{R})$ is a preimage of $(\overline{\mathbf{v}}_1, \dots, \overline{\mathbf{v}}_\ell)$ by (4.8).

The statement on the cardinality of the fibres is a consequence of the construction of the sequence $\mathbf{w}^1, \dots, \mathbf{w}^\ell$ in (4.7). □

4.2.3. Loci of constant centralizer dimension. In what follows we seek to establish a dual notion to $F_{I,\varphi}(\mathcal{R})$. The commutator matrix $\mathcal{R}$ is not relevant to the definition anymore, so we keep our argument coordinate-free and replace $\mathfrak{o}^d$ with $\mathfrak{g}$ and $\mathfrak{o}_r^d$ with $\mathfrak{g}_r$ for all $r \in \mathbb{N}$. We need to define the analogous notion of rank-preserving lift.

DEFINITION 4.15. A *choice of shadow-preserving lifts* of level $r \in \mathbb{N}$ is a function $\psi : \bar{\mathfrak{g}} \to \mathfrak{g}_r$ such that: for all $x \in \bar{\mathfrak{g}}$, $\mathrm{Sh}_{G_r}(\psi(x)) = \mathrm{Sh}_{\bar{G}}(x)$.

When $\mathfrak{g}$ admits a non-degenerate invariant symmetric form and has smooth rank loci, we can define the analogous of a smooth choice of rank-preserving lifts.

DEFINITION 4.16. Assume that $\mathfrak{g}$ has smooth rank loci. Let $r \in \mathbb{N}$ and φ be a smooth choice of rank-preserving lifts of level r as in Definition 4.6. Lemma 4.11 provides us with a choice of shadow-preserving lifts of level r, namely

$$\psi = \xi_r^{-1} \circ \varphi \circ \xi_1.$$

A choice of shadow-preserving lifts obtained this way is called a *smooth choice*.

REMARK 4.17. In particular we see that when $\mathfrak{g}$ has smooth rank loci (and admits a non-degenerate invariant symmetric bilinear form), we can always find a smooth choice of shadow-preserving lifts of level 2.

DEFINITION 4.18. Assume that $\mathfrak{g}$ has smooth rank loci and let ψ be a choice of shadow-preserving lifts of level 2 (which exists by Remark 4.17). We define $C_{I,\psi}(\mathfrak{g})$

as the set of $(\overline{v}_1, \dots, \overline{v}_\ell) \in (\bar{\mathfrak{g}} \smallsetminus \{0\})^\ell$ such that, for all $j = 1, \dots, \ell$,

$$\dim_{\mathbb{F}_q} \mathrm{Sh}_{\bar{\mathfrak{g}}} \left(\sum_{k \geq j} v_k \right) = d_j \tag{4.9}$$

$$\dim_{\mathbb{F}_q} \mathrm{Sh}_{\mathfrak{g}_2} \left(\psi \left(\sum_{k \geq j} \overline{v}_k \right) + \pi \overline{v}_{j-1} \right) = d_{j-1}. \tag{4.10}$$

Let $r \in \mathbb{N}$, $a \in \mathfrak{g}_r$, $y \in \mathfrak{g}_{r+1}$ and $x = b + \pi^{r+1} y \in \mathfrak{g}_{r+1}$ be a lift of a. By Theorem A and Theorem 2.12, $\mathrm{Sh}_{\mathfrak{g}_{r+1}}(x)$ depends only on the class of y in $\operatorname{coker} \delta_a$. Therefore $|C_{I,\psi}(\mathfrak{g})|$ does not depend on the choice of shadow-preserving lifts and the following is a good definition.

Definition 4.19. Assume that $\mathfrak{g}$ has smooth and irreducible rank loci, and let ψ be a smooth choice of shadow-preserving lifts of level 2. We define

$$\mathrm{g}_{\mathfrak{g},I}(q) = |C_{I,\psi}(\mathfrak{g})| \cdot \prod_{i=1}^{\ell} q^{-f_i}.$$

Let $\mathcal{L} = \mathbf{g}(\mathbb{C})$. Lemma 4.11 and (4.3) have the consequence that, for all $i \in I$,

$$f_i = \dim_{\mathbb{C}} \mathrm{X}_{\mathcal{L}}^{d_i}(\mathbb{C}), \tag{4.11}$$

where

$$\mathrm{X}_{\mathcal{L}}^{2k}(\mathbb{C}) = \{x \in \mathcal{L} \mid \dim_{\mathbb{C}} \mathrm{C}_{\mathcal{L}}(x) = 2k\}.$$

The rest of this section is devoted to proving the following theorem.

Theorem 4.20. *Let* $\mathrm{rk}_{\mathfrak{o}} \mathfrak{g} = d$ *and* $n = \lfloor d/2 \rfloor$. *Assume that* $\mathfrak{g}$ *admits a non-degenerate invariant symmetric bilinear form. Assume further that* $\mathfrak{g}$ *has smooth and irreducible rank loci and let* ψ *be a smooth choice of shadow-preserving lifts of level* 2. *Then*

$$\mathcal{P}_{\mathfrak{g}}(s) = \sum_{\substack{I \subseteq [n-1]_0 \\ I = \{i_1, \dots, i_\ell\}_<}} \mathrm{g}_{\mathfrak{g},I}(q) \cdot \prod_{j=1}^{\ell} \mathbf{gp}\left(q^{f_j - s(n - i_j)} \right). \tag{4.12}$$

4.2.3.1. *Proof of Theorem 4.20.* Let $I = \{i_1, \dots, i_\ell\} \subseteq [n-1]_0$ and let φ be a smooth chice of rank-preserving lifts of level 2. Let $\psi = \xi_2^{-1} \circ \varphi \circ \xi_1$. By Definitions 4.7 and 4.18 and Lemma 4.11, the function

$$\begin{aligned} \xi_{seq} : \quad C_{I,\psi}(\mathfrak{g}) &\longrightarrow F_{I,\varphi}(\mathcal{R}) \\ (v_1, \dots, v_\ell) &\longmapsto (\xi_1(v_1), \dots, \xi_1(v_\ell)) \end{aligned} \tag{4.13}$$

is well defined and surjective. It is also injective as ξ_1 is.

Definition 4.21. Let $\mathbf{r}_I = (r_1, \dots, r_\ell) \in \mathbb{N}^{|I|}$. Let $N = \sum_{k=1}^{\ell} r_k$ and $N_j = \sum_{k=j}^{\ell} r_k$, for $j = 1, \dots, \ell$. We define

$$M_{I,\mathbf{r}_I}^{\mathfrak{o}}(\mathfrak{g}) = \left\{ x \in \mathfrak{g}_N \mid \dim_{\mathbb{F}_q} \mathrm{Sh}_{\mathfrak{g}_{N_j}}(x) = d_j \; \forall j = 1, \dots, \ell \right\}.$$

Remark 4.22. In the notation of Definition 4.21. By Remark 4.10 and Lemma 4.11,

$$M_{I,\mathbf{r}_I}^{\mathfrak{o}}(\mathfrak{g}) = \xi_N^{-1}(N_{I,\mathbf{r}_I}^{\mathfrak{o}}(\mathcal{R})).$$

DEFINITION 4.23. In the notation of Definition 4.21, let φ be a smooth choice of rank-preserving lifts of level 2 and let $\psi = \xi_2^{-1} \circ \varphi \circ \xi_1$. Let $\theta_{I,\mathbf{r}_I,\varphi}$ be as in Proposition 4.14. We define

$$\theta_{I,\mathbf{r}_I,\psi} : M^{\circ}_{I,\mathbf{r}_I}(\mathfrak{g}) \to C_{I,\psi}(\mathfrak{g})$$

by $\theta_{I,\mathbf{r}_I,\psi} = \xi_{seq}^{-1} \circ \theta_{I,\mathbf{r}_I,\varphi} \circ \xi_N$.

By Proposition 4.14 and Remark 4.22, it follows that $\theta_{I,\mathbf{r}_I,\psi}$ is surjective and its fibres have cardinality

$$\prod_{k=1}^{\ell} q^{f_k(r_k-1)}.$$

This and Remark 4.22 readily imply the following

LEMMA 4.24. *Assume that $\mathfrak{g}$ has smooth and irreducible rank loci and let φ be a choice of rank-preserving lifts of level 2 and*

$$\psi = \xi_2^{-1} \circ \varphi \circ \xi_1.$$

Then, for all $I = \{i_1, \dots, i_\ell\} \subseteq [n-1]_0$, $|F_{I,\varphi}(\mathcal{R})| = |C_{I,\psi}(\mathfrak{g})|$.

Theorem 4.20 now follows from the definition of the Poincaré series (1.5).

4.3. Special linear Groups

In the rest of the chapter we apply the results in the previous sections to the linear algebraic groups $\mathrm{G} = \mathrm{SL}_h$ for $h \leq 5$. In order to do this, we need to make sure that the hypotheses of Theorem 4.20 are satisfied for almost all primes. We immediately see that $\mathfrak{g}$ admits a non-degenerate symmetric invariant bilinear form for almost all primes $\mathfrak{p}$. Indeed the Lie algebra $\mathfrak{sl}_h(\mathfrak{k})$ is semisimple. By Cartan's criterion this is equivalent to its Killing form being non-degenerate. The next section proves that $\mathfrak{g}$ has smooth rank loci. From now onwards, let $\mathfrak{p}$ be a non-zero prime ideal such that the normalized Killing form κ of $\mathfrak{sl}_h(\mathfrak{o})$ described in [3, Section 5] is non-degenerate.

4.3.1. Sheets of the special linear Lie algebra.

Let $\mathcal{L} = \mathfrak{sl}_h(\mathbb{C})$. For all $2k \leq d = n^2 - 1$ we define

$$\mathrm{X}^{2k}_{\mathcal{L}}(\mathbb{C}) = \{x \in \mathcal{L} \mid \dim_{\mathbb{C}} \mathrm{C}_{\mathcal{L}}(x) = 2k\}.$$

By [29, Section 1], these sets are algebraic varieties defined over $\mathbb{Z}$. Notice that, for $2k \leq d$, $\mathrm{X}^{2k}_{\mathcal{L}}(\mathfrak{o})$ are the $\mathfrak{o}$-rational points of $\mathrm{X}^{2k}_{\mathcal{L}}(\mathbb{C})$. By Lemma 4.11, it follows that $\mathfrak{sl}_h(\mathfrak{o})$ has smooth and irreducible rank loci if and only if $\mathrm{X}^{2k}_{\mathcal{L}}(\mathbb{C})$ is smooth, irreducible and has good reduction modulo $\mathfrak{p}$ for all k such that $\mathrm{X}^{2k}_{\mathcal{L}}(\mathbb{C}) \neq \emptyset$.

The irreducible components of $\mathrm{X}^{2k}_{\mathcal{L}}(\mathbb{C})$ $(2k \leq d)$ are called the *sheets* of $\mathcal{L}$. Every sheet of $\mathcal{L}$ corresponds in a 1-1 correspondence to a partition of h (see [29, Section 3.1]). Let $\mathbf{d} = [d_1, \dots, d_f]$ $(d_1 \geq \dots \geq d_f)$ be a partition of h, we denote with $\mathrm{S}_{\mathbf{d}}$ the sheet associated with $\mathbf{d}$. The dimension of an orbit $\mathcal{C} \subseteq \mathrm{S}_{\mathbf{d}}$ is given by equation (1) in [29, Section 3.1]:

$$\dim_{\mathbb{C}} \mathcal{C} = 2\,\mathrm{m}(\mathbf{d}), \text{ where } \mathrm{m}(\mathbf{d}) = (h^2 - \sum_{s \in \mathrm{D}(\mathbf{d})} s^2)/2.$$

and $\mathrm{D}(\mathbf{d}) = [s_i \mid i = 1, \dots, f]$ $(s_i = \#\{j \mid d_j \geq i\})$ is the dual partition of $\mathbf{d}$. For $\mathfrak{sl}_4(\mathbb{C})$ each partition of 4 gives a different orbit dimension. It follows that the varieties $\mathrm{X}^{2k}_{\mathcal{L}}(\mathbb{C})$, $2k = 6, 8, 10, 12$, are the sheets which are irreducible by definition (see Table 4.1). A similar computation reveals that the varieties of constant centralizer-dimension in $\mathfrak{sl}_5(\mathbb{C})$ coincide with its sheets too.

Let $e \in \mathbb{N}$ and K be an algebraically closed field of arbitrary characteristic. A result of Bongartz [6, Section, 3 Korollar 2] ensures that the sheets of $\mathfrak{sl}_e(K)$ are smooth.

TABLE 4.1. The sheets of $\mathfrak{sl}_4(\mathbb{C})$

Partition $\mathbf{d}$ of 4	Orbit dimension $2\,\mathrm{m}(\mathbf{d})$
$[1^4]$	0
$[2,1^2]$	6
$[2,2]$	8
$[3,1]$	10
$[4]$	12

This implies that as $\mathbb{Z}$-schemes they have good reduction for all primes $\mathfrak{p} \lhd \mathcal{O}$. In conclusion, $\mathfrak{sl}_h(\mathfrak{o})$ has smooth and irreducible rank loci, for $h \leq 5$. [29, Table 1] shows that $\mathfrak{sl}_6(\mathfrak{o})$ cannot have smooth and irreducible rank loci. It is, nontheless, interesting to notice that the sheets of $\mathfrak{sl}_h(\mathbb{C})$, while being smooth by [6, Section 3, Korollar 2], never intersect. Indeed, the intersection of two sheets of $\mathfrak{sl}_h(\mathbb{C})$ always contains a nilpotent orbit (cf. [7, Section 7.4]). However, it is a well known fact (see [32, Section 1.3]) that a nilpotent orbit of $\mathfrak{sl}_h(\mathfrak{o})$ cannot belong to two sheets. It follows that $\mathfrak{sl}_h(\mathfrak{o})$ has smooth rank loci.

4.3.2. Decreasing sequences of centralizers. The assumption $h \leq 5$ guarantees that the rank loci of $\mathfrak{g}$ are smooth and irreducible; hence, by Definition 4.16 and Remark 4.17, there exists ψ smooth choice of shadow preserving lifts of level 2. Let ψ be fixed for the rest of the section.

Recall that $\mathfrak{Sh}(\bar{\mathfrak{g}})$ is a transversal set for all isomorphism classes of group-shadows of level 1. In other words $\mathfrak{Sh}(\bar{\mathfrak{g}})$ is a transversal set for all isomorphism classes of group centralizers of elements in $\bar{\mathfrak{g}}$. Elements of this set are called *isomorphism types* of (group) centralizers. In order to preserve the 1-1 correspondence between Lie and group centralizers established in Lemma 3.3, we assume henceforth that $q > 2$.

DEFINITION 4.25. A *decreasing sequence of (group) centralizers over the finite field* is a set of of isomorphism types of group centralizers $\{\mathbf{S}_1, \dots, \mathbf{S}_\ell\}$ such that, for $0 < i < j \leq \ell$, $\dim_{\mathbb{F}_q} \mathbf{S}_i > \dim_{\mathbb{F}_q} \mathbf{S}_j$. We denote the set of all decreasing sequences of group centralizers with $\mathcal{Q}(\mathfrak{g})$. Let $I = \{i_1, \dots, i_\ell\}_< \subseteq [n-1]_0$ and let d_j be as in (4.3). We define

$$\mathcal{Q}_I(\mathfrak{g}) = \left\{\{\mathbf{S}_1, \dots, \mathbf{S}_\ell\} \in \mathcal{Q}(\mathfrak{g}) \mid \dim_{\mathbb{F}_q} \mathbf{S}_j = d_j \, \forall j \in \{1, \dots, \ell\}\right\}.$$

DEFINITION 4.26. Let $I = \{i_1, \dots, i_\ell\}_< \subseteq [n-1]_0$. Let $\mathcal{S} \in \mathcal{Q}_I(\mathfrak{g})$ and $C_{I,\psi}(\mathfrak{g})$ be as defined in Definition 4.18. We define

$$\mathcal{C}_{\mathcal{S},\psi}(\mathfrak{g}) = \left\{ (v_1, \dots, v_\ell) \in C_{I,\psi}(\mathfrak{g}) \mid \forall\, j = 1, \dots, \ell :\, \mathrm{C}_{\bar{G}}\left(\sum_{k \geq j} v_k\right) \cong \mathbf{S}_j \right\}.$$

REMARK 4.27. By Definitions 3.5 and 4.26

$$\sum_{\mathcal{S} \in \mathcal{Q}_I(\mathfrak{g})} |\mathcal{C}_{\mathcal{S},\psi}(\mathfrak{g})| = |C_{I,\psi}(\mathfrak{g})|.$$

Therefore, by Definition 4.19,

$$\mathrm{g}_{\mathfrak{g},I}(q) = \sum_{\mathcal{S} \in \mathcal{Q}_I(\mathfrak{g})} |\mathcal{C}_{\mathcal{S},\psi}(\mathfrak{g})| \cdot \prod_{i \in I} q^{-f_i}. \tag{4.14}$$

Definition 4.28. Let ψ be a choice of shadow-preserving lifts of level 2. Let $S \in \mathfrak{Sh}(\bar{G})$, we define

$$\delta(S) = \frac{1}{2}(d - d_S)$$
$$f_S = \dim \mathrm{X}_{\mathcal{L}}^{d_S}$$
$$\mathrm{g}_{G,\mathcal{S}}(q) = |\mathcal{C}_{\mathcal{S},\psi}(G)| \cdot \prod_{S \in \mathcal{S}} q^{-f_S}.$$

Notice that the $\delta(S)$ is an integer, indeed, by Lemma 4.11, $d - d_S$ is the number of invertible elementary divisors of an antisymmetric matrix and therefore even. Moreover, analogously to the observation before Definition 4.19, Theorem A and Theorem 2.12 imply that $|\mathcal{C}_{\mathcal{S},\psi}(G)|$ does not depend on the choice of shadow-preserving lifts. Hence $\mathrm{g}_{G,\mathcal{S}}(q)$ is well defined.

Let $I = \{i_1, \ldots, \imath_\ell\}_< \subseteq [n-1]_0$. If $\mathcal{S} = \{\mathbf{S}_1, \ldots, \mathbf{S}_\ell\} \in \mathcal{Q}_I(\mathfrak{g})$. Then, by (4.3) and (4.11), for all $1 \leq j \leq \ell$,

$$\delta(S_j) = d - d_j = n - i_j$$
$$f_{\mathbf{S}_j} = f_j.$$

By Theorem 4.20, (4.14), and Definition 4.28, it follows that

$$\mathcal{P}_{\mathfrak{g}}(s) = \sum_{\mathcal{S} \in \mathcal{Q}_{\bar{G}}(\mathfrak{g})} \mathrm{g}_{G,\mathcal{S}}(q) \cdot \prod_{S \in \mathcal{S}} \mathbf{gp}\left(q^{f_S - s \cdot \delta(S)}\right). \tag{4.15}$$

CHAPTER 5

The representation zeta function of $\mathrm{SL}_4^m(\mathfrak{o})$

This chapter contains the computation of the Poincaré series of $\mathfrak{sl}_4(\mathfrak{o})$. In order to apply (4.15), we first determine for which primes the normalized Killing form of $\mathfrak{sl}_4(\mathfrak{k})$ remains non-degenerate on $\mathfrak{sl}_4(\mathfrak{o})$. Indeed, (4.15) requires us to consider decreasing sequences of centralizers over the finite field: after having determined the non-regular centralizers that can occur (i.e. centralizers that do not have minimal dimension), we proceed to the computation of the coefficients $\mathrm{g}_{\mathrm{SL}_4(\mathfrak{o}),\mathcal{S}}(q)$ for all $\mathcal{S} \in \mathcal{Q}(\mathfrak{sl}_4(\mathfrak{o}))$. We do this by operating a case distinction according to the first group centralizer in the decreasing sequence. We then compute the contribution of all summands corresponding to these decreasing sequences. In Section 5.7, we compute the Poincaré series of $\mathfrak{sl}_4(\mathfrak{o})$ and consequently, applying Proposition 1.15, the representation zeta function of $\mathrm{SL}_4^m(\mathfrak{o})$ for permissible $m \in \mathbb{N}$. Let us adopt the notation conventions of Section 4.3 but applied to the specific case $\mathfrak{sl}_4(\mathfrak{o})$. In particular $d = 15$ and $n = 7$.

As observed just above Definition 4.19, the choice of shadow-preserving lifts can be arbitrary. Hence, let ψ be a choice of shadow-preserving lifts of level 2 coming from a smooth choice φ of rank-preserving lifts of level 2 as explained in Definition 4.16. Let $\mathcal{S} \in \mathcal{Q}(\mathfrak{sl}_4(\mathfrak{o}))$. As there is no risk of confusion we set

$$\mathrm{g}_{\mathcal{S}} = \mathrm{g}_{\mathrm{SL}_4(\mathfrak{o}),\mathcal{S}}(q)\,.$$

5.1. Non-degenerate Killing form

First of all we determine for which primes the normalized Killing form of $\mathfrak{sl}_4(\mathfrak{k})$ remains non-degenerate when restricted to $\mathfrak{sl}_4(\mathfrak{o})$.

LEMMA 5.1. *Assume that $2 \nmid q$. Then $\mathfrak{sl}_4(\mathfrak{o})$ admits a non-degenerate invariant symmetric bilinear form.*

PROOF. A choice for the non-degenerate bilinear form is the restriction κ of the normalized Killing form on $\mathfrak{sl}_4(\mathfrak{k})$. The latter is non-degenerate as the Lie algebra is semisimple. the assumption on q ensure that κ is also non-degenerate. Indeed, the normalized Killing form is $\kappa(X,Y) = \mathrm{tr}(XY)$. Let us fix a basis $\mathcal{B}$ for $\mathfrak{sl}_4(\mathfrak{o})$ comprising the elements

$$h_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$e_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_{13} = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad e_{24} = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad e_{14} = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

$$f_{21} = \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad f_{32} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \quad f_{43} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}$$

$$f_{31} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \quad f_{42} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{pmatrix}, \quad f_{41} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}$$

The matrix of κ with respect to $\mathcal{B}$ is the following 15×15 matrix:

$$\begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & & & & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \end{pmatrix}.$$

This has determinant 4, hence κ is non-degenerate if $2 \nmid q$ □

We assume henceforth that $2 \nmid q$.

5.2. Group centralizers in $\mathfrak{sl}_4(\mathfrak{o})$

We need to compute $|\mathcal{C}_{\mathcal{S},\psi}(\mathfrak{sl}_4(\mathfrak{o}))|$, for all decreasing sequences of group centralizers $\mathcal{S}$. The following definition is useful in order to take an inductive approach to the problem.

DEFINITION 5.2. Let $\mathcal{S} = \{\mathbf{S}, \mathbf{T}\} \in \mathcal{Q}(\mathfrak{sl}_4(\mathfrak{o}))$ and $x \in \mathfrak{sl}_4(\mathbb{F}_q)$ be such that

$$\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(x) \cong \mathbf{S}.$$

We define

$$\mathcal{C}_{\mathcal{S},\psi}(\mathrm{SL}_4(\mathfrak{o}))^x = \{(v_2, v_1) \in \mathcal{C}_{\mathcal{S},\psi}(\mathfrak{sl}_4(\mathfrak{o})) \mid v_2 = x\}.$$

The following lemma will help us in the computation of the cardinality of the sets introduced in the last definition.

LEMMA 5.3. *Let $\mathcal{S} = \{\mathbf{S}, \mathbf{T}\} \in \mathcal{Q}(\mathfrak{sl}_4(\mathfrak{o}))$ and $x \in \mathfrak{sl}_4(\mathbb{F}_q)$ be such that $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(x) \cong \mathbf{S}$. Let $I = \{\lfloor d_{\mathbf{T}}/2 \rfloor, \lfloor d_{\mathbf{S}}/2 \rfloor\}$, $\mathbf{r}_I = (1,1)$ and $\theta_{I,\mathbf{r}_I,\psi}$ be as in Definition 4.23. Then $|\mathcal{C}_{\mathcal{S},\psi}(\mathrm{SL}_4(\mathfrak{o}))^x|$ is equal to*

$$|\{w \in M^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathfrak{sl}_4(\mathfrak{o})) \mid \sigma(w) = x,\ \theta_{I,\mathbf{r}_I,\psi}(w) = (x,y) \text{ and } \mathrm{C}\,(x+y) \cong \mathbf{T}\}|.$$

PROOF. The result is a direct consequence of the definition of $\mathcal{C}_{\mathcal{S},\psi}(\mathrm{SL}_4(\mathfrak{o}))^x$ and of Proposition 4.14. □

Let us keep the notation of Lemma 5.3. This result allows us to use Proposition 3.7 in the computation of $\mathcal{C}_{\mathcal{S},\psi}(\mathrm{SL}_4(\mathfrak{o}))^x$. When $\mathcal{C}_{\mathcal{S},\psi}(\mathrm{SL}_4(\mathfrak{o}))^x$ does not depend on x but only on the isomorphism class of its group centralizer, by Lemma 5.3 and Definitions 5.2 and 4.26, it follows that, for an arbitrary decreasing sequence of group centralizers $\mathcal{S} = \{\mathbf{S}_1, \dots, \mathbf{S}_\ell\}$ (see Definition 4.25),

$$|\mathcal{C}_{\mathcal{S},\psi}(\mathrm{SL}_4(\mathfrak{o}))| = \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{S}_1) \cdot \prod_{i=1}^{\ell-1} |\mathcal{C}_{\{\mathbf{S}_i, \mathbf{S}_{i+1}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^{x_i}| \tag{5.1}$$

where $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(x_i) \cong \mathbf{S}$ for all $i = 1, \dots, \ell$. We shall therefore start by determining $\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{S})$ for all possible isomophism types of group centralizers over the finite field. Our strategy is to consider one sheet at each time. Indeed, let $c \in \{6, 8, 10, 12\}$ and let S_c be the sheet consisting of all orbits of dimension c, we have

$$\sum_{\substack{\mathbf{S} \in \mathfrak{Sh}(\mathfrak{sl}_4(\mathbb{F}_q)) \\ d_{\mathbf{S}} = d - c}} \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{S}) = |\mathrm{S}_c(\mathbb{F}_q)|.$$

Even more to the point, it is known that, over algebraically closed fields, each sheet contains an *affine cross-section*: a subset that transversally intersects each orbit exactly once and is isomorphic to an affine space. In practice, looking at elements in these affine spaces suffices to determine isomorphism types of group centralizers, hence the size of each orbit.

5.2.1. Affine cross-section. Let $h \in \mathbb{N}$. As first proved by D. Peterson [31, Chapter 3], every sheet S of $\mathfrak{sl}_h(\mathbb{C})$ contains an *affine cross-section* C: a subset of S that meets each orbit for the $\mathrm{SL}_h(\mathbb{C})$-action exactly once and is isomorphic to an affine space. An explicit construction is described also in [7, Section 1.4]. The orbits for the adjoint action of $\mathrm{SL}_h(\mathbb{C})$ coincide with the $\mathrm{GL}_h(\mathbb{C})$-orbits for the action by conjugation. It follows that the affine cross-section C also parameterizes the $\mathrm{GL}_h(\mathbb{C})$-orbits in $\mathfrak{sl}_h(\mathbb{C})$.

An equivalent construction to the one in [31, Chapter 3] but with base field $\overline{\mathbb{F}}_q$, an algebraic closure of $\mathbb{F}_q$, has been carried out in [6, Section 4]. However, the $\mathbb{F}_q$-rational points of an $\mathrm{SL}_h(\overline{\mathbb{F}}_q)$-orbit $\mathcal{C} \subseteq \mathfrak{sl}_h(\overline{\mathbb{F}}_q)$ might consist of a union of more than one $\mathrm{SL}_h(\mathbb{F}_q)$-orbit and the $\mathbb{F}_q$-rational points on C might no longer parameterize the $\mathrm{SL}_h(\mathbb{F}_q)$-orbits. In order to avoid this problem, we consider the $\mathrm{GL}_h(\mathbb{F}_q)$-action by conjugation on $\mathfrak{sl}_h(\mathbb{F}_q)$. Indeed, a consequence of Lang-Steinberg Theorem [28, Theorem 21.11] guarantees that the $\mathrm{GL}_h(\mathbb{F}_q)$-action on $\mathcal{C}(\mathbb{F}_q)$ remains transitive, while the following proposition ensures that we can replace the action of $\mathrm{SL}_h(\mathbb{F}_q)$ with the action of $\mathrm{GL}_h(\mathbb{F}_q)$.

LEMMA 5.4. *Let $a, b \in \mathfrak{sl}_h(\mathbb{F}_q)$ be $\mathrm{GL}_h(\mathbb{F}_q)$-conjugate. Let $g \in \mathrm{GL}_h(\mathbb{F}_q)$ be such that $\mathrm{C}_{\mathrm{GL}_h(\mathbb{F}_q)}(a)^g = \mathrm{C}_{\mathrm{GL}_h(\mathbb{F}_q)}(b)$. Then $\mathrm{C}_{\mathrm{SL}_h(\mathbb{F}_q)}(a)^g = \mathrm{C}_{\mathrm{SL}_h(\mathbb{F}_q)}(b)$.*

PROOF. We notice that $\mathrm{C}_{\mathrm{SL}_h(\mathbb{F}_q)}(a) \subseteq \mathrm{C}_{\mathrm{GL}_h(\mathbb{F}_q)}(a)$ and that conjugation by a fixed element in the group is an isomorphism that preserves determinant. □

It follows that each $\mathrm{GL}_h(\mathbb{F}_q)$-orbit is the union of $\mathrm{SL}_h(\mathbb{F}_q)$-orbits that have the same centralizer up to isomorphism. Before we are able to employ affine cross-sections defined over $\mathbb{F}_q$ in our computations, we need to make sure that the $\mathbb{F}_q$-rational points of an affine cross-section defined over $\overline{\mathbb{F}}_q$ still parameterize $\mathrm{GL}_h(\mathbb{F}_q)$-orbits in $\mathfrak{sl}_h(\mathbb{F}_q)$.

PROPOSITION 5.5. *A $\mathrm{GL}_h(\overline{\mathbb{F}}_q)$-orbit contains a $\mathbb{F}_q$-rational point if and only if its intersection with the* affine cross-section *contains a $\mathbb{F}_q$-rational point.*

PROOF. Let Frob be the Frobenius automorphism of $\overline{\mathbb{F}}_q$. We observe that an orbit containing an $\mathbb{F}_q$-rational point is Frob-stable while the affine cross-section is

Frob-stable because it is defined by equations with integer coefficients. It follows that their intersection, which consists of a single point, is Frob-stable and therefore $\mathbb{F}_q$-rational. □

5.3. Centralizers of dimension 3

An element with 3-dimensional centralizer is called regular and its centralizer is called regular too. Analogously, a 3-dimensional shadow is called regular. Table 5.1 records the isomorphism types of non-regular group centralizers in $\mathfrak{sl}_4(\mathbb{F}_q)$ paired with the numerical data for the geometric series in (4.15). In the rest of this chapter, we determine these isomorphism types and, by operating a case distinction according to the first group centralizer in the decreasing sequence of centralizers featuring in (4.15), we compute the Poincaré series of $\mathfrak{sl}_4(\mathfrak{o})$.

When the decreasing sequence of group centralizers begins with a regular centralizer it a singleton. Hence, for all isomorphism types $\mathbf{S}$ of 3-dimensional group centralizers,

$$|\mathcal{C}_{\{\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))| = \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{S}).$$

It follows that for our purposes we do not need to distinguish regular elements according to the isomorphism type of their centralizer. We therefore define

$$\begin{aligned} \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{R}) &= \sum_{\substack{\{\mathbf{S}\}\in\mathcal{Q}(\mathfrak{sl}_4(\mathfrak{o}))\\ d_{\mathbf{S}}=3}} \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{S}) \\ \mathrm{g}_{\{\mathbf{R}\}} &= \sum_{\substack{\{\mathbf{S}\}\in\mathcal{Q}(\mathfrak{sl}_4(\mathfrak{o}))\\ d_{\mathbf{S}}=3}} \mathrm{g}_{\{\mathbf{S}\}}\,. \end{aligned} \tag{5.2}$$

Let $\mathbf{S}$ be regular. By Lemma 4.11 and Definition 4.28, $f_{\mathbf{S}} = 15$ and $\delta(S) = 6$. In this notation the contribution to (4.15) given by summands corresponding to decreasing sequences of centralizers of the form $\{\mathbf{S}\}$ where $d_{\mathbf{S}} = 3$ is

$$\mathcal{P}_{\mathbf{R}}(s) = \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{R}) \cdot q^{-15}\frac{q^{15-6s}}{1-q^{15-6s}}. \tag{5.3}$$

TABLE 5.1. Non-regular centralizers with their structure

$\mathbf{S}$	$d_{\mathbf{S}}$	$\delta(\mathbf{S})$	f_S	$\mathbf{S}$ is isomorphic to	Reference
$\mathrm{SL}_4(\mathbb{F}_q)$	15			$\mathrm{SL}_4(\mathbb{F}_q)$	
$\mathbf{A}$	9	3	7	$\mathrm{GL}_3(\mathbb{F}_q)$	(5.70)
$\mathbf{B}$	9	3	7	$(\mathrm{Heis}(\mathbb{F}_q) \curlyvee \mathrm{Heis}(\mathbb{F}_q)) \rtimes \mathrm{GL}_2(\mathbb{F}_q)$	(5.80)
$\mathbf{C}$	7	4	9	$\mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{GL}_2(\mathbb{F}_q)$	(5.53)
$\mathbf{D}$	7	4	9	$C_{q+1} \times \mathrm{SL}_2(\mathbb{F}_{q^2})$	(5.64)
$\mathbf{E}$	7	4	9	$({\mathbb{F}_q}^+)^4 \rtimes \mathrm{SL}_2(\mathbb{F}_q)$	(5.32)
$\mathbf{F}$	5	5	12	$\mathrm{GL}_2(\mathbb{F}_q) \times {\mathbb{F}_q}^+$	(5.11)
$\mathbf{H}$	5	5	12	$(\mathrm{Heis}(\mathbb{F}_q) \times {\mathbb{F}_q}^+) \rtimes {\mathbb{F}_q}^\times$	(5.8)
$\mathbf{I}$	5	5	12	$\mathrm{SL}_2(\mathbb{F}_q) \times {\mathbb{F}_{q^2}}^\times$	(5.25)
$\mathbf{J}'$	5	5	12	$\mathrm{Heis}(\mathbb{F}_q) \rtimes ({\mathbb{F}_q}^\times \times {\mathbb{F}_q}^\times)$	(5.15)
$\mathbf{L}'$	5	5	12	$\mathrm{GL}_2(\mathbb{F}_q) \times {\mathbb{F}_q}^\times$	(5.21)

The sum in (5.3) will be determined in Section 5.6.4 by computing

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q),\mathbf{R}) = \sum_{\substack{\{\mathbf{S}\}\in\mathcal{Q}(\mathfrak{sl}_4(\mathfrak{o}))\\ d_{\mathbf{S}}=3}} \Lambda(\mathrm{SL}_4(\mathbb{F}_q),\mathbf{S}) = q^{15}-1-\sum_{\substack{\{\mathbf{S}\}\in\mathcal{Q}(\mathfrak{sl}_4(\mathfrak{o}))\\ d_{\mathbf{S}}\neq 3}} \Lambda(\mathrm{SL}_4(\mathbb{F}_q),\mathbf{S}).$$

5.4. Centralizers of dimension 5

We consider the affine cross-section on $\mathrm{S}_{[3,1]}$:

$$C_{[3,1]}(\alpha,\beta) = \begin{pmatrix} \alpha & 0 & 0 & 0\\ 0 & \alpha & 1 & 0\\ 0 & 0 & -\alpha & 1\\ 0 & 0 & \beta & -\alpha \end{pmatrix}.$$

for $\alpha,\beta\in\mathbb{F}_q$.

Definition 5.6. Every element $a\in\mathfrak{sl}_4(\mathbb{F}_q)$ admits a Jordan decomposition. When the semisimple part of a is diagonalizable over $\mathbb{F}_q$, we say that *a admits a Jordan normal form* or that its orbit *contains a Jordan normal form.*

We need a case distinction between orbits that contain a Jordan normal form and orbits that do not contain such a matrix: fix α,β and let $a=C_{[3,1]}(\alpha,\beta)$. A quick computation of the characteristic and minimal polynomial yields:

$$\chi_a(X) = (X-\alpha)^2(X^2+2\alpha X+\beta+\alpha^2) \tag{5.4}$$

$$m_a(X) = (X-\alpha)(X^2+2\alpha X+\beta+\alpha^2). \tag{5.5}$$

From this, we see that a admits a Jordan normal form if and only if $-\beta$ is a square, and it is diagonalizable if and only if $-\beta$ is a non-zero square and $4\alpha^2\neq-\beta$.

5.4.1. The nilpotent orbit.

We consider the case $\alpha=\beta=0$ first. For these values we obtain the unique nilpotent orbit with a 6-dimensional $\mathrm{GL}_4(\mathbb{F}_q)$-centralizer. The point on the affine cross-section is

$$a = C_{[3,1]}(0,0) = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix},$$

with centralizer

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) = \left\{ M = \begin{pmatrix} m_{11} & 0 & 0 & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ 0 & 0 & m_{22} & m_{23}\\ 0 & 0 & 0 & m_{22} \end{pmatrix} \;\middle|\; M\in\mathrm{GL}_4(\mathbb{F}_q) \right\}.$$

Let $\mathbf{H}=\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$, any other element in the same nilpotent $\mathrm{GL}_4(\mathbb{F}_q)$ orbit has $\mathrm{SL}_4(\mathbb{F}_q)$-centralizer isomorphic to $\mathbf{H}$. We choose $\mathbf{H}$ as isomorphism type for these elements. We pick the following basis $\mathcal{B}_{\mathfrak{h}}=\{b_0,\dots,b_5\}$ for $\mathfrak{h}=\mathrm{As}(\mathbf{H})$:

$$b_0 = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} b_1 = \begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} b_2 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$b_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} b_4 = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} b_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The non-zero commutator relations between the members of $\mathcal{B}$ are those implied by

$$[b_0, b_1] = -b_2,\ [b_0, b_4] = 4b_0,\ [b_1, b_4] = -4b_1. \tag{5.6}$$

By the 1-1 correspondence between Lie and group centralizers over $\mathbb{F}_q$(cf. Lemma 3.3), we deduce that the unipotent radical of $\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a)$ is $\mathrm{Heis}(\mathbb{F}_q) \times C_q$, where $\mathrm{Heis}(\mathbb{F}_q)$ is isomorphic to the Heisenberg group over $\mathbb{F}_q$. By [24, Theorem 7.1] we conclude

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) \cong ((\mathrm{Heis}(\mathbb{F}_q) \times {\mathbb{F}_q}^+) \rtimes {\mathbb{F}_q}^\times) \times {\mathbb{F}_q}^\times.$$

It follows that

$$|\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a)\,| = q^4(q-1)^2 \tag{5.7}$$

$$\mathbf{H} \cong (\mathrm{Heis}(\mathbb{F}_q) \times {\mathbb{F}_q}^+) \rtimes {\mathbb{F}_q}^\times. \tag{5.8}$$

Dividing the order of $\mathrm{GL}_4(\mathbb{F}_q)$ by the order of the centralizer in (5.7), we compute the cardinality of the $\mathrm{GL}_4(\mathbb{F}_q)$-orbit of a:

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{H}) = q^2 \cdot (q-1)^2 \cdot (q+1)^2 \cdot (q^2+1) \cdot (q^2+q+1). \tag{5.9}$$

The commutator relations in (5.6) allow us to compute the commutator matrix of $\mathfrak{h}$ with respect to $\mathcal{B}_{\mathfrak{h}}$:

$$\mathcal{R}_{\mathfrak{h}}(\mathbf{Y}) = \begin{pmatrix} 0 & -Y_2 & 0 & 0 & 4Y_0 \\ Y_2 & 0 & 0 & 0 & -4Y_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4Y_0 & 4Y_1 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{o}[Y_0, \dots, Y_4]$$

As a result, as we assumed $2 \nmid q$,

$$|\{\mathbf{x} = (x_1, \dots, x_4) \in \mathbb{F}_q^5 \mid \mathrm{rk}_{\mathbb{F}_q} \mathcal{R}_{\mathfrak{h}}(\mathbf{x}) = 2\}| = q^2 \cdot (q^3 - 1).$$

By Lemma 2.14 and Proposition 3.7, it follows that

$$\Lambda(\mathbf{H}, \mathbf{R}) = q^2 \cdot (q^3 - 1). \tag{5.10}$$

5.4.2. Orbits with 2 pairs of coincident non-zero eigenvalues. When $\beta = 0$ and $\alpha \neq 0$, the matrix on the cross-section is similar to

$$a = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}.$$

One computes that the centralizer of a is

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & 0 & m_{33} \end{pmatrix} \middle|\ M \in \mathrm{GL}_4(\mathbb{F}_q) \right\}.$$

Let $\mathbf{F} = \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$ (notice that this does not depend on the choice of α), it follows that

$$\mathbf{F} \cong \mathrm{GL}_2(\mathbb{F}_q) \times {\mathbb{F}_q}^+ \tag{5.11}$$

$$|\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}\,(a)\,| = q^2(q-1)^3(q+1). \tag{5.12}$$

By (5.12) we conclude that the cardinality of the $\mathrm{GL}_4(\mathfrak{o})$-orbit of a inside $\mathfrak{sl}_4(\mathbb{F}_q)$ is

$$q^4 \cdot (q-1) \cdot (q+1) \cdot (q^2+1) \cdot (q^2+q+1).$$

The number of such orbits is $q-1$, hence

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{F}) = q^4 \cdot (q-1)^2 \cdot (q+1) \cdot (q^2+1) \cdot (q^2+q+1). \tag{5.13}$$

By the structure of $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}\,(a)$ highlighted in (5.11) and Lemma 3.3, $\mathfrak{f} = \mathrm{As}(\mathbf{F}) \cong \mathfrak{gl}_2(\mathbb{F}_q) \oplus \mathbb{F}_q$. Choosing a basis of $\mathfrak{f}$ such that the first three elements are an $\mathfrak{sl}_2(\mathfrak{o})$-triple and the fourth and fifth are central, it follows from the well known shape of the commutator matrix of $\mathfrak{sl}_2(\mathfrak{o})$ that the rank-2 locus of the commutator matrix of $\mathfrak{f}$ is $\mathfrak{f} \smallsetminus \{0\}$. Therefore

$$\Lambda(\mathbf{F}, \mathbf{R}) = q^2 \cdot (q^3 - 1). \tag{5.14}$$

5.4.3. Orbits with 3 coincident eigenvalues. These are the orbits that we obtain when $-\beta$ is a non-zero square and $4\alpha^2 = -\beta$. In practice, this means that one of the zeroes of $(X^2 + 2\alpha X + \beta + \alpha^2)$ coincides with α, in other words, the orbit contains

$$a = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -3\alpha \end{pmatrix},$$

whose centralizer is

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}\,(a) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & 0 \\ 0 & m_{11} & 0 & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & 0 & 0 & m_{44} \end{pmatrix} \;\middle|\; M \in \mathrm{GL}_4(\mathbb{F}_q) \right\}.$$

Let

$$a_{\mathfrak{gl}_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{F}_q).$$

It can be easily computed that

$$\mathrm{C}_{\mathrm{GL}_3(\mathbb{F}_q)}\left(a_{\mathfrak{gl}_3}\right) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{11} & 0 \\ 0 & m_{32} & m_{33} \end{pmatrix} \;\middle|\; M \in \mathrm{GL}_3(\mathbb{F}_q) \right\}.$$

Moreover, it is known that $\mathrm{C}_{\mathrm{GL}_3(\mathbb{F}_q)}\left(a_{\mathfrak{gl}_3}\right) \cong \mathrm{Heis}(\mathbb{F}_q) \rtimes (\mathbb{F}_q{}^\times \times \mathbb{F}_q{}^\times)$ (see [4, Table 2.1]). We choose $\mathbf{J}' = \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}\,(a)$ as isomorphism type of centralizer for the $\mathrm{GL}_4(\mathbb{F}_q)$-conjugates of a. Imposing $\det(M) = 1$ to the elements of $\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}\,(a)$, we deduce that

$$\mathbf{J}' \cong \mathrm{Heis}(\mathbb{F}_q) \rtimes (\mathbb{F}_q{}^\times \times \mathbb{F}_q{}^\times). \tag{5.15}$$

We have

$$|\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}\,(a)\,| = q^3(q-1)^3. \tag{5.16}$$

It follows that these orbits consist of

$$q^3 \cdot (q-1) \cdot (q+1)^2 \cdot (q^2+1) \cdot (q^2+q+1)$$

points. The number of such orbits is given by

$$\frac{(q-1)}{2} \cdot 2 = (q-1).$$

Thus,

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{J}') = q^3 \cdot (q-1)^2 \cdot (q+1)^2 \cdot (q^2+1) \cdot (q^2+q+1). \tag{5.17}$$

Let $\mathfrak{j}' = \mathrm{As}(\mathbf{J}')$. We fix a basis $\mathcal{B}_{\mathfrak{j}'}$ for $\mathfrak{j}'$ comprising

$$e_1 = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The commutator matrix of $\mathfrak{j}'$ with respect to $\mathcal{B}_{\mathfrak{j}'}$ is

$$\mathcal{R}_{\mathfrak{j}'}(\mathbf{Y}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_1 & Y_2 \\ 0 & 0 & -Y_1 & 0 & -Y_3 \\ 0 & 0 & -Y_2 & Y_3 & 0 \end{pmatrix}. \tag{5.18}$$

Hence

$$|\{\mathbf{x} = (x_1, \ldots, x_4) \in \mathbb{F}_q^5 \mid \mathrm{rk}_{\mathbb{F}_q} \mathcal{R}_{\mathfrak{j}'}(\mathbf{x}) = 2\}| = q^2 \cdot (q^3 - 1).$$

By Lemma 2.14 and Proposition 3.7, it follows that

$$\Lambda(\mathbf{J}', \mathbf{R}) = q^2 \cdot (q^3 - 1). \tag{5.19}$$

5.4.4. Diagonalizable orbits. Among the orbits that contain the Jordan normal form of $a = C_{[3,1]}(\alpha, \beta)$, it remains to consider the cases in which $-\beta = \gamma^2$ with $\gamma \in \mathbb{F}_q$, and $4\alpha^2 \neq -\beta$. In this case, the minimal polynomial $m_a(X)$ in (5.5) splits in 3 distinct linear factors; this means that the orbit of the point on the cross-section contains a diagonal matrix

$$D(\alpha) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha+\gamma & 0 \\ 0 & 0 & 0 & -\alpha-\gamma \end{pmatrix}.$$

It follows that $\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) \cong \mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(D(\alpha)) \cong \mathrm{GL}_2(\mathbb{F}_q) \times {\mathbb{F}_q}^\times \times {\mathbb{F}_q}^\times$. We choose $\mathbf{L}' = \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(D(\alpha))$ as isomorphism type for the $\mathrm{GL}_4(\mathbb{F}_q)$-conjugates of a. Hence

$$|\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a)\,| = (q-1)^4(q+1)q \tag{5.20}$$

$$\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a) \cong \mathbf{L}' \cong \mathrm{GL}_2(\mathbb{F}_q) \times {\mathbb{F}_q}^\times. \tag{5.21}$$

Dividing the order of $\mathrm{GL}_4(\mathbb{F}_q)$ by the order of the centralizer in (5.20) we conclude that the orbit of a has cardinality

$$q^5 \cdot (q+1) \cdot (q^2+1) \cdot (q^2+q+1).$$

The number of elements on the cross-section $C_{[3,1]}(\alpha, \beta)$ whose orbit contains a diagonal matrix is

$$\frac{(q-1)}{2} \cdot (q-2).$$

By (5.21) and Lemma 3.3, $\mathfrak{l}' = \mathrm{As}(\mathbf{L}') \cong \mathfrak{gl}_2(\mathbb{F}_q) \oplus \mathbb{F}_q$. Choosing a basis of $\mathfrak{l}'$ that respects the decomposition above, it follows immediately that the rank-2 locus of the commutator matrix of $\mathfrak{l}'$ is $\mathfrak{l}' \smallsetminus \{0\}$. Hence

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{L}') = \frac{1}{2} \cdot q^5 \cdot (q-1) \cdot (q-2) \cdot (q+1) \cdot (q^2+1) \cdot (q^2+q+1) \tag{5.22}$$

$$\Lambda(\mathbf{L}', \mathbf{R}) = q^2 \cdot (q^3 - 1). \tag{5.23}$$

5.4.5. Orbits without Jordan normal form. We examine now the orbits that do not contain the Jordan normal form of the matrix $a = C_{[3,1]}(\alpha, \beta)$ on the cross-section. In other words $\chi_a(X)$ does not split in linear factors with coefficients in $\mathbb{F}_q$; this happens precisely when $-\beta \in \mathbb{F}_q$ is not a square. We can replace a with its Frobenius normal form

$$a = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -(\alpha^2+\beta) \\ 0 & 0 & 1 & -2\alpha \end{pmatrix}.$$

Now let $m = (m_{ij})_{i,j} \in \mathrm{Mat}_4(R)$. The Lie centralizer of a is the set of solutions to the linear system defined by $[a, m] = 0$. Since $-\beta$ is not a square we deduce that $m_{ij} = 0$ when $i \leq 2$, $j \geq 3$ and when $i \geq 3$, $j \leq 2$. Thus

$$\mathrm{C}_{\mathfrak{gl}_4(\mathbb{F}_q)}(a) = \left\{ \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & 2\alpha\, m_{43} + m_{44} & -(\alpha^2+\beta)\, m_{43} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{F}_q) \right\}.$$

Since $-\beta \in \mathbb{F}_q$ is not a square, the matrices

$$\begin{pmatrix} 2\alpha\, m_{43} + m_{44} & -(\alpha^2+\beta)\, m_{43} \\ m_{43} & m_{44} \end{pmatrix}$$

with $m_{43}, m_{44} \in \mathbb{F}_q$ form a Lie algebra isomorphic to $\mathbb{F}_{q^2}$, therefore $\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) \cong \mathrm{GL}_2(\mathbb{F}_q) \times \mathbb{F}_{q^2}{}^{\times}$. Let $\mathbf{I}$ be the isomorphism type of $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$. It follows

$$|\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a)\,| = |\,\mathrm{GL}_2(\mathbb{F}_q)\,| \cdot (q^2-1) = (q-1)^3(q+1)^2 q \tag{5.24}$$

$$\mathbf{I} \cong \mathrm{SL}_2(\mathbb{F}_q) \times \mathbb{F}_{q^2}{}^{\times}. \tag{5.25}$$

From (5.24) we conclude that the orbits without a Jordan normal form have cardinality

$$q^5 \cdot (q-1) \cdot (q^2+1) \cdot (q^2+q+1).$$

The number of these orbits is

$$q \cdot \frac{(q-1)}{2},$$

hence

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{I}) = \frac{1}{2} \cdot q^6 \cdot (q-1)^2 \cdot (q^2+1) \cdot (q^2+q+1). \tag{5.26}$$

By (5.25), the commutator matrix of $\mathfrak{i} = \mathrm{As}(\mathbf{I})$ with respect to a properly chosen basis looks like the commutator matrix of $\mathfrak{sl}_2(\mathbb{F}_q)$ with two more zero-columns and rows. Therefore the rank-2 locus has cardinality $q^5 - q^2$, and hence

$$\Lambda(\mathbf{I}, \mathbf{R}) = q^2 \cdot (q^3 - 1). \tag{5.27}$$

5.4.6. Computation of the contribution to (4.15). Now let $a \in \mathfrak{sl}_4(\mathbb{F}_q)$ such that $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a) \cong \mathbf{H}$. By Proposition 3.7, there are

$$q^{10} \cdot \Lambda(\mathbf{H}, \mathbf{R}) = q^{12}(q^3 - 1)$$

lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ that have regular shadow. Let $w \in \mathfrak{sl}_4(\mathfrak{o}_2)$ be such a lift and let $(a, y) = \theta_{I,\mathbf{r}_I,\psi}(w)$, for $I = \{1, 2\}$, $\mathbf{r}_I = (1, 1)$ and $\theta_{I,\mathbf{r}_I,\psi}$ as in Definition 4.23. Then, by Lemma 4.11 and Proposition 4.14, it follows that $\dim \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a + y) = 3$. Hence

$$|\mathcal{C}_{\{\mathbf{H},\mathbf{R}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^a| = q^{12}(q^3 - 1). \tag{5.28}$$

Since the choice of a is arbitrary, the last equality does not depend on a. The same as in (5.28) happens for all other 5-dimensional shadows suggesting the following notation: we define

$$\begin{aligned} \mathrm{g}_{\{\mathbf{U}\}} &= \sum_{d_{\mathbf{S}}=5} \mathrm{g}_{\{\mathbf{S}\}} \\ \mathrm{g}_{\{\mathbf{U},\mathbf{R}\}} &= \sum_{\substack{d_{\mathbf{S}}=5 \\ d_{\mathbf{T}}=3}} \mathrm{g}_{\{\mathbf{S},\mathbf{T}\}} . \end{aligned} \tag{5.29}$$

The values in (5.10), (5.14), (5.19), (5.23) and (5.27) are all equal. Therefore, combining (5.1) with the definition of the g's (Definition 4.28), we obtain

$$\begin{aligned} \mathrm{g}_{\{\mathbf{U}\}} &= q^{-12} \cdot \sum_{d_S=5} \Lambda(\mathrm{SL}_4(\mathbb{F}_q), S) \\ &= q^{-12}(q^{12} + q^{11} + 2q^{10} - q^9 - 2q^8 - 4q^7 - 2q^6 + 2q^4 + 2q^3 + q^2) \\ \mathrm{g}_{\{\mathbf{U},\mathbf{R}\}} &= q^{-5} \cdot q^2 \cdot (q^3 - 1) \cdot \mathrm{g}_{\{\mathbf{U}\}} \\ &= q^{-17}(q^{17} + q^{16} + 2q^{15} - 2q^{14} - 3q^{13} - 6q^{12} - q^{11} \\ &\quad + 2q^{10} + 6q^9 + 4q^8 + q^7 - 2q^6 - 2q^5 - q^4); \end{aligned} \tag{5.30}$$

Contribution to the Poincaré series. From (5.30) it follows immediately that the contribution of decreasing sequences beginning with a 5-dimensional shadow to the summation in (4.15) is

$$\begin{aligned} \mathcal{P}_{\mathbf{U}}(s) &= \mathrm{g}_{\{\mathbf{U}\}} \frac{q^{12-5s}}{(1 - q^{12-5s})} + \mathrm{g}_{\{\mathbf{U},\mathbf{R}\}} \frac{q^{12-5s}}{(1 - q^{12-5s})} \frac{q^{15-6s}}{(1 - q^{15-6s})} \\ &= \frac{\mathcal{F}_{\mathbf{U}}(q, q^{-s})}{\mathcal{G}_{\mathbf{U}}(q, q^{-s})}, \end{aligned} \tag{5.31}$$

where

$$\begin{aligned} \mathcal{F}_{\mathbf{U}}(q,t) = &- (q^{24} + q^{23} + 2\,q^{22} - q^{21} - 2\,q^{20} \\ &- 4\,q^{19} - 2\,q^{18} + 2\,q^{16} + 2\,q^{15} + q^{14})t^{11} \\ &+ (q^{12} + q^{11} + 2\,q^{10} - q^9 - 2\,q^8 - 4\,q^7 - 2\,q^6 + 2\,q^4 + 2\,q^3 + q^2)t^5 \\ \mathcal{G}_{\mathbf{U}}(q,t) = &\,(1 - q^{12}t^5)(1 - q^{15}t^6t). \end{aligned}$$

5.5. Centralizers of dimension 7

The affine cross-section in $\mathrm{S}_{[2,2]}$ is one-dimensional. The following is a parameterization of it in terms of $\alpha \in \mathbb{F}_q$:

$$C_{[2,2]}(\alpha) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \end{pmatrix}.$$

5.5.1. The nilpotent orbit. The nilpotent matrix on the affine cross-section is:

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The centralizer for this matrix has the following parameterization:

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{11} & 0 & m_{13} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & m_{31} & 0 & m_{33} \end{pmatrix} \middle| \, M \in \mathrm{GL}_4(\mathbb{F}_q) \right\}.$$

The unipotent radical $\mathrm{R}_u(\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a))$ is isomorphic to $({\mathbb{F}_q}^+)^4$; by [24, Theorem 7.1] we conclude that

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) \cong ({\mathbb{F}_q}^+)^4 \rtimes \mathrm{GL}_2(\mathbb{F}_q).$$

We choose $\mathbf{E} = \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$ as isomorphism type for the $\mathrm{GL}_4(\mathbb{F}_q)$-conjugate to a. Hence

$$\mathbf{E} \cong ({\mathbb{F}_q}^+)^4 \rtimes \mathrm{SL}_2(\mathbb{F}_q). \tag{5.32}$$

It follows

$$\begin{aligned} |\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a)\,| &= |\,\mathrm{GL}_2(\mathbb{F}_q)\,|q^4 = q^5(q-1)^2(q+1) \\ \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{E}) &= q \cdot (q+1) \cdot (q-1)^2 \cdot (q^2+1) \cdot (q^2+q+1). \end{aligned} \tag{5.33}$$

5.5.2. Elements without shadow-preserving lifts. The present section is not entirely needed in the computation of the Poincaré series of $\mathfrak{sl}_4(\mathfrak{o})$. It is however convenient to discuss here the reason why there are $r \in \mathbb{N}$ and elements $a \in \mathfrak{sl}_4(\mathfrak{o}_r)$ that do not admit any shadow-preserving lift to $\mathfrak{sl}_4(\mathfrak{o}_{r+1})$. Such elements may be found among lifts of elements with shadow $\mathbf{E}$.

5.5.2.1. *Lifts of elements with centralizer* $\mathbf{E}$. By Corollary 2.13 the possible shadows of a lift of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ correspond to possible stabilizers for the action of $\mathbf{E}$ on $\mathfrak{e}^\vee$, where $\mathfrak{e} = \mathrm{As}(\mathbf{E}) \cong \mathfrak{sl}_2(\mathbb{F}_q) \oplus (\mathbb{F}_q)^4$. We fix a basis $\mathcal{B} = \{e_0, \dots, e_7\}$ of $\mathfrak{e}$:

$$e_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} e_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Set $\mathbf{Y} = (Y_0, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$, the commutator matrix of $\mathfrak{e}$ relative to $\mathcal{B}$ is

$$\mathcal{R}_{\mathcal{B}}(\mathbf{Y}) = \begin{pmatrix} 0 & Y_2 & -2Y_0 & -2Y_4 & 0 & Y_3 & 0 \\ -Y_2 & 0 & 2Y_1 & 2Y_5 & -Y_3 & 0 & 0 \\ 2Y_0 & -2Y_1 & 0 & 0 & 2Y_4 & -2Y_5 & 0 \\ 2Y_4 & -2Y_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_3 & -2Y_4 & 0 & 0 & 0 & 0 \\ -Y_3 & 0 & 2Y_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.34}$$

Let $\mathcal{B}^\vee$ be the dual basis of $\mathcal{B}$ and $\mathbf{c}$ be the coordinates of $c \in \mathfrak{e}^\vee$ with respect to $\mathcal{B}^\vee$. By Lemma 2.14 , $\mathrm{As}(\mathrm{Stab}_{\mathbf{E}}(c)) = \ker \mathcal{R}_{\mathcal{B}}(\mathbf{c})$.

Let $\mathbf{c} = (c_0, \ldots, c_6)$, we observe that $\mathrm{rk}_{\mathbb{F}_q} \mathcal{R}_{\mathcal{B}}(\mathbf{c}) = 2$ if and only if at least one of the c_0, c_1, c_2 is non-zero and $c_3 = c_4 = c_5 = c_6 = 0$. Let c be of the aforementioned form, it follows from Lemma 1.13 that the unipotent radical $N = \mathrm{R}_u(\mathbf{E}) \cong ({\mathbb{F}_q}^+)^4$ acts trivially on c. The semisimple part is isomorphic to $\mathfrak{sl}_2(\mathbb{F}_q)$ and, once 0 is removed, it is the rank-2 locus of $\mathcal{R}_{\mathcal{B}}$. It follows that

$$\mathrm{Stab}_{\mathbf{E}}(c) \cong \mathrm{C}_{\mathrm{SL}_2(\mathbb{F}_q)}(c) \rtimes ({\mathbb{F}_q}^+)^4, \tag{5.35}$$

where the semidirect product is determined by the structure constants encoded in (5.34).

As a consequence we obtain different shadows according to whether c is semisimple diagonalizable, nilpotent or semisimple non-diagonalizable in $\mathfrak{sl}_2(\mathbb{F}_q)$, respectively

$$\mathbf{M} \cong {\mathbb{F}_q}^\times \rtimes ({\mathbb{F}_q}^+)^4 \tag{5.36}$$

$$\mathbf{N} \cong {\mathbb{F}_q}^+ \rtimes ({\mathbb{F}_q}^+)^4 \tag{5.37}$$

$$\mathbf{O} \cong C_{q+1} \rtimes ({\mathbb{F}_q}^+)^4. \tag{5.38}$$

Since there are $1/2 \cdot q \cdot (q^2 - 1)$ semisimple diagonalizable elements and $1/2 \cdot q \cdot (q-1)^2$ semisimple non-diagonalizable elements in $\mathfrak{sl}_2(\mathbb{F}_q)$, it follows that

$$\Lambda(\mathbf{E}, \mathbf{M}) = \frac{1}{2} \cdot q^2 \cdot (q^2 - 1) \tag{5.39}$$

$$\Lambda(\mathbf{E}, \mathbf{N}) = q \cdot (q^2 - 1) \tag{5.40}$$

$$\Lambda(\mathbf{E}, \mathbf{O}) = \frac{1}{2} \cdot q^2 \cdot (q - 1)^2. \tag{5.41}$$

All the other lifts of a that do not preserve $\mathbf{E}$ have a 3-dimensional shadow:

$$\Lambda(\mathbf{E}, \mathbf{R}) = q^4 \cdot (q^3 - 1). \tag{5.42}$$

5.5.2.2. *Lifts of elements with shadow* $\mathbf{M}$ *or* $\mathbf{N}$. Let a and $\mathcal{B}$ be as in Section 5.5.2.1. Let $x \in \mathfrak{sl}_4(\mathfrak{o}_2)$ be a lift of a with $\mathrm{Sh}_{\mathrm{SL}_4(\mathfrak{o}_2)}(x) = \mathbf{M}$ or $\mathrm{Sh}_{\mathrm{SL}_4(\mathfrak{o}_2)}(x) = \mathbf{N}$. In both cases $\mathfrak{s} = \mathrm{Sh}_{\mathfrak{sl}_4(\mathfrak{o}_2)}(x)$ has a basis $\mathcal{B}' = \{v, e_3, e_4, e_5, e_6\}$ with $v = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$ a non-zero $\mathbb{F}_q$-linear combination of the first three vectors of $\mathcal{B}$. The structure constants encoded in (5.34) allow us to compute

$$[v, e_3] = -2\alpha_0 e_4 + 2\alpha_1 e_5,\ [v, e_4] = -\alpha_1 e_3 + 2\alpha_2 e_4,\ [v, e_5] = \alpha_0 e_3 - 2\alpha_2 e_5.$$

Setting $\mathbf{X} = (X, Y_3, Y_4, Y_5, Y_6)$ it follows that $\mathcal{R}_{\mathcal{B}'}(\mathbf{X})$ is equal to

$$\begin{pmatrix} 0 & -2\alpha_0 Y_4 + 2\alpha_1 Y_5 & -\alpha_1 Y_3 + 2\alpha_2 Y_4 & \alpha_0 Y_3 - 2\alpha_2 Y_5 & 0 \\ 2\alpha_0 Y_4 - 2\alpha_1 Y_5 & 0 & 0 & 0 & 0 \\ \alpha_1 Y_3 - 2\alpha_2 Y_4 & 0 & 0 & 0 & 0 \\ -\alpha_0 Y_3 + 2\alpha_2 Y_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One checks that the non-zero entries above the diagonal of $\mathcal{R}_{\mathcal{B}'}(\mathbf{X})$ span a subspace of dimension 2 inside $\mathfrak{s}^\vee$. This implies that

$$\Lambda(\mathbf{M},\mathbf{R}) = q^3 \cdot (q^2-1) \tag{5.43}$$

$$\Lambda(\mathbf{N},\mathbf{R}) = q^3 \cdot (q^2-1) \tag{5.44}$$

$$\Lambda(\mathbf{O},\mathbf{R}) = q^3 \cdot (q^2-1). \tag{5.45}$$

We now prove the existence of an element that does not admit a shadow-preserving lift. Let

$$x = \begin{pmatrix} 0 & 1 & \pi & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_4(\mathfrak{o}_2)$$

one checks that $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(\sigma(x)) \cong \mathbf{E}$ and that $\mathrm{Sh}_{\mathrm{SL}_4(\mathfrak{o}_2)}(x) \cong \mathbf{N}$. The following is a lift of x

$$b = \begin{pmatrix} 0 & 1 & \pi & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_4(\mathfrak{o}_3)$$

that preserves the shadow of x. Then equations (5.43) to (5.45) together with Proposition 3.7 entail that there are exactly q^{13} shadow-preserving lifts of b to $\mathfrak{sl}_4(\mathfrak{o}_3)$. The sheet $\mathrm{S}_{[3,1]}$ is a smooth 12-dimensional scheme over $\mathbb{Z}$; thus, by the quantitative statement of Hensel's lemma (Proposition 1.18), it follows that $q^{13}-q^{12}$ of these lifts cannot have shadow-preserving lifts to $\mathfrak{sl}_4(\mathfrak{o}_4)$. The following matrix in $\mathfrak{sl}_4(\mathbb{Z}/27\mathbb{Z})$ is an example of a matrix that does not admit any shadow-preserving lift:

$$\begin{pmatrix} 9 & 10 & 21 & 0 \\ 0 & 18 & 9 & 21 \\ 0 & 9 & 0 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.46}$$

5.5.3. Decreasing sequences starting with E. Let $a \in \mathfrak{sl}_4(\mathbb{F}_q)$ such that $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a) \cong \mathbf{E}$. By Proposition 3.7 there are

$$q^8 \cdot (\Lambda(\mathbf{E},\mathbf{M}) + \Lambda(\mathbf{E},\mathbf{N}) + \Lambda(\mathbf{E},\mathbf{O})) = q^9 \cdot (q^3-1)$$

lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ that have 5-dimensional shadow. Let $w \in \mathfrak{sl}_4(\mathfrak{o}_2)$ be such a lift and let $(a, y) = \theta_{I,\mathbf{r}_I,\psi}(w)$, for $I = \{2,3\}$ and $\mathbf{r}_I = (1,1)$. Then, by Lemma 4.11 and Proposition 4.14, it follows that $\dim \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a+y) = 5$. Hence

$$\sum_{d_{\mathbf{S}}=5} \mathcal{C}_{\{\mathbf{E},\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^a = 2 \cdot q^{12} \cdot (q^3-1). \tag{5.47}$$

Since the choice of a is arbitrary, the last equality does not depend on a. The same happens for 3-dimensional centralizers, giving

$$\sum_{d_{\mathbf{S}}=3} \mathcal{C}_{\{\mathbf{E},\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^a = q \cdot (q-1)^2 \cdot (q^2+q+1)^2. \tag{5.48}$$

Now let $\mathbf{S},\mathbf{T}$ be isomorphism types of group centralizer and let $d_{\mathbf{S}} = 5$ and $d_{\mathbf{S}} = 3$. By the computations in Section 5.4 and by (5.1), in all the possible determinations of $\mathbf{S}$

$$\mathcal{C}_{\{\mathbf{E},\mathbf{S},\mathbf{T}\},\psi}(\mathrm{SL}_4(\mathfrak{o})) = \mathcal{C}_{\{\mathbf{E},\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o})) \cdot q^{-3} \cdot (q^3-1).$$

In (4.15), the geometric progressions depend only on the dimension of the centralizers involved in the decreasing sequence. As a consequence we can already collect all the relevant g's before we multiply by the geometric progressions, i.e. we can define

$$\begin{aligned} \mathrm{g}_{\{\mathbf{E},\mathbf{U}\}} &= \sum_{d_{\mathbf{S}}=5} \mathrm{g}_{\{\mathbf{E},\mathbf{S}\}} \\ \mathrm{g}_{\{\mathbf{E},\mathbf{R}\}} &= \sum_{d_{\mathbf{S}}=3} \mathrm{g}_{\{\mathbf{E},\mathbf{S}\}} \\ \mathrm{g}_{\{\mathbf{E},\mathbf{U},\mathbf{R}\}} &= \sum_{\substack{d_{\mathbf{S}}=5 \\ d_{\mathbf{T}}=3}} \mathrm{g}_{\{\mathbf{E},\mathbf{S},\mathbf{T}\}}\,. \end{aligned} \tag{5.49}$$

Thus we compute

$$\begin{aligned} \mathrm{g}_{\{\mathbf{E}\}} &= (q^8 - q^5 - q^4 + q) \cdot q^{-9} \\ \mathrm{g}_{\{\mathbf{E},\mathbf{U}\}} &= (q^{12} - 2q^9 - q^8 + q^6 + 2q^5 - q^2) \cdot q^{-13} \\ \mathrm{g}_{\{\mathbf{E},\mathbf{R}\}} &= (q^{15} - 2q^{12} - q^{11} + q^9 + 2q^8 - q^5) \cdot q^{-16} \\ \mathrm{g}_{\{\mathbf{E},\mathbf{U},\mathbf{R}\}} &= (q^{17} - 3q^{14} - q^{13} + 3q^{11} + 3q^{10} - q^8 - 3q^7 + q^4) \cdot q^{-18} \end{aligned} \tag{5.50}$$

Notice that this does use any peculiarity of the isomorphism type **E**. We shall therefore do the same also for the other isomorphism types of 7-dimensional group centralizers and even later when it is possible and it will simplify the computations and notation.

Contribution to the Poincaré series. The contribution to the Poincaré series in (4.15) given by summands corresponding to decreasing sequences beginning with **E** is

$$\begin{aligned} \mathcal{P}_{\mathbf{E}}(s) = \mathrm{g}_{\{\mathbf{E}\}} & \frac{q^{9-4s}}{1-q^{9-4s}} \\ &+ \mathrm{g}_{\{\mathbf{E},\mathbf{U}\}} \frac{q^{9-4s}}{1-q^{9-4s}} \frac{q^{13-5s}}{1-q^{13-5s}} \\ &+ \mathrm{g}_{\{\mathbf{E},\mathbf{R}\}} \frac{q^{9-4s}}{1-q^{9-4s}} \frac{q^{15-6s}}{1-q^{15-6s}} \\ &+ \mathrm{g}_{\{\mathbf{E},\mathbf{U},\mathbf{R}\}} \frac{q^{9-4s}}{1-q^{9-4s}} \frac{q^{13-5s}}{1-q^{13-5s}} \frac{q^{15-6s}}{1-q^{15-6s}} \\ &= \frac{\mathcal{F}_{\mathbf{E}}(q, q^{-s})}{\mathcal{G}_{\mathbf{E}}(q, q^{-s})}, \end{aligned} \tag{5.51}$$

where

$$\begin{aligned} \mathcal{F}_{\mathbf{E}}(q,t) &= \big(q^{29} - q^{26} - q^{25} + q^{22}\big)t^{15} - \big(q^{20} - q^{17} - q^{16} + q^{13}\big)t^{10} \\ &\quad - \big(q^{17} - q^{14} - q^{13} + q^{10}\big)t^9 + \big(q^8 - q^5 - q^4 + q\big)t^4 \\ \mathcal{G}_{\mathbf{E}}(q,t) &= \big(1 - q^9 t^4\big)\big(1 - q^{13} t^5\big)\big(1 - q^{15} t^6\big). \end{aligned}$$

5.5.4. Orbits with Jordan normal form. We now distinguish the remaining orbits according to whether or not the orbit contains a Jordan normal form of its point on the affine cross-section. Namely, the orbit contains a Jordan normal form of $C_{[2,2]}(\alpha)$ if and only if α is a square in $\mathbb{F}_q$.

We fix $\alpha \in \mathbb{F}_q$ such that $\alpha = \beta^2$ for $\beta \in \mathbb{F}_q$ and we consider $C_{[2,2]}(\alpha)$. Since the orbit of this matrix contains a diagonal matrix

$$D(\beta) = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix},$$

for the purposes of our computations we can replace a with $D(\beta)$. It follows that the centralizer of a is isomorphic to

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(D(\beta)) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix} \;\middle|\; M \in \mathrm{GL}_4(\mathbb{F}_q) \right\}.$$

Let $\mathbf{C} = \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(D(\beta))$. It follows that

$$|\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a)\,| = |\,\mathrm{GL}_2(\mathbb{F}_q)\,|^2 = q^2(q+1)^2(q-1)^4 \tag{5.52}$$

$$\mathbf{C} \cong \mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{GL}_2(\mathbb{F}_q)\,. \tag{5.53}$$

Therefore the cardinality of an orbit not containing a Jordan normal form for the matrix on the affine cross-section is

$$q^4 \cdot (q^2+1) \cdot (q^2+q+1).$$

Multiplying by the number of orbits containing a Jordan normal form of the element on the affine cross-section we obtain

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{C}) = \frac{1}{2} \cdot q^4 \cdot (q-1) \cdot (q^2+1) \cdot (q^2+q+1). \tag{5.54}$$

Finally by the structure of $\mathbf{C}$ given in (5.53) we discover that a can be lifted to elements with isomorphism type of shadow equal to $\mathbf{C}$, $\mathbf{L}'$, $\mathbf{F}$ or to elements with regular shadow. Therefore we have the following lifting behaviour

$$\Lambda(\mathbf{C}, \mathbf{L}') = 2 \cdot q^3 \cdot (q-1) \tag{5.55}$$

$$\Lambda(\mathbf{C}, \mathbf{F}) = 2 \cdot q \cdot (q^2-1) \tag{5.56}$$

$$\Lambda(\mathbf{C}, \mathbf{R}) = q \cdot (q-1)^2 \cdot (q^2+q+1)^2. \tag{5.57}$$

We can now obtain the quantities that we need in the computation of the Poincaré series. Let $a \in \mathfrak{sl}_4(\mathbb{F}_q)$ such that $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a) \cong \mathbf{C}$. By Proposition 3.7 there are

$$q^8 \cdot (\Lambda(\mathbf{C}, \mathbf{L}') + \Lambda(\mathbf{C}, \mathbf{F})) = 2 \cdot q^9 \cdot (q^3-1)$$

lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ that have 5-dimensional shadow. Let $w \in \mathfrak{sl}_4(\mathfrak{o}_2)$ be such a lift and $(a, y) = \theta_{I,\mathbf{r}_I,\psi}(w)$, for $I = \{2,3\}$ and $\mathbf{r}_I = (1,1)$. It follows that $\dim \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a+y) = 5$. Hence

$$\sum_{d_{\mathbf{S}}=5} \mathcal{C}_{\{\mathbf{C},\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^a = 2 \cdot q^9 \cdot (q^3-1). \tag{5.58}$$

Since the choice of a is arbitrary, the equality above does not depend on a. The same happens for 3-dimensional centralizers, giving

$$\sum_{d_{\mathbf{S}}=3} \mathcal{C}_{\{\mathbf{C},\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^a = q \cdot (q-1)^2 \cdot (q^2+q+1)^2. \tag{5.59}$$

Similar to what we have done at the end of Section 5.5.3 we define

$$\mathrm{g}_{\{\mathbf{C},\mathbf{U}\}} = \sum_{d_{\mathbf{S}}=5} \mathrm{g}_{\{\mathbf{C},\mathbf{S}\}}$$

$$\mathrm{g}_{\{\mathbf{C},\mathbf{R}\}} = \sum_{d_{\mathbf{S}}=3} \mathrm{g}_{\{\mathbf{C},\mathbf{S}\}}$$

$$\mathrm{g}_{\{\mathbf{C},\mathbf{U},\mathbf{R}\}} = \sum_{\substack{d_{\mathbf{S}}=5 \\ d_{\mathbf{T}}=3}} \mathrm{g}_{\{\mathbf{C},\mathbf{S},\mathbf{T}\}}$$

By (5.54), (5.58) and (5.59) and applying (5.1),

$$\begin{aligned}
\mathrm{g}_{\{\mathbf{C}\}} &= \frac{1}{2}(q^9+q^7-q^6-q^4)\cdot q^{-9}\\
\mathrm{g}_{\{\mathbf{C},\mathbf{U}\}} &= (q^{13}+q^{11}-2q^{10}-2q^8+q^7+q^5)\cdot q^{-13}\\
\mathrm{g}_{\{\mathbf{C},\mathbf{R}\}} &= \frac{1}{2}(q^{16}+q^{14}-3q^{13}-3q^{11}+3q^{10}+3q^8-q^7-q^5)\cdot q^{-16}\\
\mathrm{g}_{\{\mathbf{C},\mathbf{U},\mathbf{R}\}} &= (q^{18}+q^{16}-3q^{15}-3q^{13}+3q^{12}+3q^{10}-q^9-q^7)\cdot q^{-18}
\end{aligned} \tag{5.60}$$

Contribution to the Poincaré series. The contribution to the Poincaré series in (4.15) given by summands corresponding to decreasing sequences beginning with $\mathbf{C}$ is

$$\begin{aligned}
\mathcal{P}_{\mathbf{C}}(s) = {} & \mathrm{g}_{\{\mathbf{C}\}}\,\frac{q^{9-4s}}{1-q^{9-4s}}\\
& + \mathrm{g}_{\{\mathbf{C},\mathbf{U}\}}\,\frac{q^{9-4s}}{1-q^{9-4s}}\,\frac{q^{12-5s}}{1-q^{12-5s}}\\
& + \mathrm{g}_{\{\mathbf{C},\mathbf{R}\}}\,\frac{q^{9-4s}}{1-q^{9-4s}}\,\frac{q^{15-6s}}{1-q^{15-6s}}\\
& + \mathrm{g}_{\{\mathbf{C},\mathbf{U},\mathbf{R}\}}\,\frac{q^{9-4s}}{1-q^{9-4s}}\,\frac{q^{12-5s}}{1-q^{12-5s}}\,\frac{q^{15-6s}}{1-q^{15-6s}}\\
= {} & \frac{\mathcal{F}_{\mathbf{C}}(q,q^{-s})}{\mathcal{G}_{\mathbf{C}}(q,q^{-s})},
\end{aligned} \tag{5.61}$$

where

$$\begin{aligned}
\mathcal{F}_{\mathbf{C}}(q,t) = {} & \frac{1}{2}\,\Big(\big(q^{30}+q^{28}-q^{27}-q^{25}\big)t^{15}\\
& -\big(2\,q^{21}+2\,q^{19}-3\,q^{18}-3\,q^{16}+q^{15}+q^{13}\big)t^{10}\\
& +\big(q^{21}+q^{19}-3\,q^{18}-3\,q^{16}+2\,q^{15}+2\,q^{13}\big)t^{9}\\
& +\big(q^9+q^7-q^6-q^4\big)t^4\Big)\\
\mathcal{G}_{\mathbf{C}}(q,t) = {} & \big(1-q^9t^4\big)\big(1-q^{12}t^5\big)\big(1-q^{15}t^6\big).
\end{aligned}$$

5.5.5. Orbits without Jordan normal form. We complete our investigation by considering $C_{[2,2]}(\alpha)$ when α is not a square in $\mathbb{F}_q$.

Let us fix a non-square $\alpha\in\mathbb{F}_q$, let $a=C_{[2,2]}(\alpha)$. We compute

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14}\\ \alpha\, m_{12} & m_{11} & \alpha\, m_{14} & m_{13}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ \alpha\, m_{32} & m_{31} & \alpha\, m_{34} & m_{33}\end{pmatrix} \;\middle|\; M\in\mathrm{GL}_4(\mathbb{F}_q)\right\}. \tag{5.62}$$

We notice that the matrix M above consists of 4 block-elements in the subring of $\mathrm{Mat}_2(\mathbb{F}_q)$ given by

$$R = \left\{ \begin{pmatrix} x & y\\ \alpha\, y & x\end{pmatrix} \;\middle|\; x,y\in\mathbb{F}_q \right\}.$$

In fact $R\cong\mathbb{F}_q(\beta)\cong\mathbb{F}_{q^2}$ where $\beta^2=\alpha$. Now, let $N:\mathbb{F}_{q^2}\to\mathbb{F}_q$ be the norm function defined by $(x,y)\mapsto x^2-\alpha y^2$, $\det_{\mathbb{F}_{q^2}}$ and $\det_{\mathbb{F}_q}$ be the determinant function on $\mathrm{Mat}_2(\mathbb{F}_{q^2})$ and $\mathrm{Mat}_4(\mathbb{F}_q)$, respectively. For a matrix M as in (5.62), we have that

$$\det\nolimits_{\mathbb{F}_q}(M) = \mathrm{N}(\det\nolimits_{\mathbb{F}_{q^2}}(M)),$$

where N is the norm function on $\mathbb{F}_{q^2}$. Since the elements of norm 1 in $\mathbb{F}_{q^2}$ form a cyclic group of order $q+1$, while the fiber of $\det_{\mathbb{F}_{q^2}}$ over a non-zero point of $\mathbb{F}_{q^2}$

have cardinality $|\,\mathrm{SL}_2(\mathbb{F}_{q^2})\,|$. Let $\mathbf{D}$ be the isomorphism type of $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$. We conclude that

$$|\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a)\,| = |{\mathbb{F}_q}^{\times}|(q+1)|\,\mathrm{SL}_2(\mathbb{F}_{q^2})\,| = (q-1)(q+1)\big(q^4-1\big)q^2 \tag{5.63}$$

$$\mathbf{D} \cong C_{q+1} \times \mathrm{SL}_2(\mathbb{F}_{q^2})\,. \tag{5.64}$$

The order of the centralizer in (5.63) allows us to compute the cardinality of the orbit, which, multiplied by the number of non-squares in $\mathbb{F}_q$, is

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{D}) = \frac{1}{2} \cdot q^4 \cdot (q-1)^3 \cdot (q^2+q+1). \tag{5.65}$$

From (5.64) it follows that lifts of a can only preserve shadow or have regular shadow. We write

$$\Lambda(\mathbf{D}, \mathbf{R}) = \sum_{d_{\mathbf{S}}=3} \Lambda(\mathbf{D}, \mathbf{S}) = q \cdot (q^6-1). \tag{5.66}$$

We can now obtain the quantities that we need in the computation of the Poincaré series. Let $a \in \mathfrak{sl}_4(\mathbb{F}_q)$ such that $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a) \cong \mathbf{D}$. By Proposition 3.7 there are

$$q^8 \cdot \Lambda(\mathbf{D}, \mathbf{R}) = q^9 \cdot (q^6-1)$$

lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ that have 3-dimensional shadow. Let $w \in \mathfrak{sl}_4(\mathfrak{o}_2)$ be such a lift and $(a, y) = \theta_{I,\mathbf{r}_I,\varphi}(w)$, for $I = \{1, 3\}$ and $\mathbf{r}_I = (1, 1)$. Then $\dim \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a + y) = 3$, hence

$$\sum_{d_{\mathbf{S}}=3} \mathcal{C}_{\{\mathbf{D},\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^a = q^9 \cdot (q^6-1). \tag{5.67}$$

Again, analogously to the notation established in Section 5.5.3, we define

$$\mathrm{g}_{\{\mathbf{D},\mathbf{R}\}} = \sum_{d_{\mathbf{S}}=3} \mathrm{g}_{\{\mathbf{D},\mathbf{S}\}}$$

By (5.65) and (5.67)

$$\begin{aligned} \mathrm{g}_{\{\mathbf{D},\mathbf{R}\}} &= q^{-15} \cdot (q^{15} - q^9) \\ \mathrm{g}_{\{\mathbf{D}\}} &= q^{-9} \cdot \frac{1}{2} \cdot q^4 \cdot (q-1)^3 \cdot (q^2+q+1). \end{aligned} \tag{5.68}$$

Contribution to the Poincaré series. The contribution to (4.15) of summands corresponding to decreasing sequences beginning with $\mathbf{D}$ is

$$\begin{aligned} \mathcal{P}_{\mathbf{D}}(s) &= \mathrm{g}_{\{\mathbf{D}\}} \frac{q^{9-4s}}{(1-q^{9-4s})} \\ &\quad + \mathrm{g}_{\{\mathbf{D},\mathbf{R}\}} \frac{q^{9-4s}}{(1-q^{9-4s})} \frac{q^{15-6s}}{(1-q^{15-6s})} \\ &= \frac{\mathcal{F}_{\mathbf{D}}(q, q^{-s})}{\mathcal{G}_{\mathbf{D}}(q, q^{-s})}, \end{aligned} \tag{5.69}$$

where

$$\begin{aligned} \mathcal{F}_{\mathbf{D}}(q,t) &= -\frac{1}{2}\Big(\big(q^{18} - 2\,q^{17} + q^{16} - q^{15} + 2\,q^{14} - q^{13}\big)t^{10} \\ &\qquad -\big(q^9 - 2\,q^8 + q^7 - q^6 + 2\,q^5 - q^4\big)t^4\Big) \\ \mathcal{G}_{\mathbf{D}}(q,t) &= \big(1-q^9t^4\big)\big(1-q^{15}t^6\big). \end{aligned}$$

5.6. Centralizers of dimension 9

The affine cross-section in $\mathrm{S}_{[2,1^2]}$ is one-dimensional and, for $\alpha \in \mathbb{F}_q$, this is its parameterization in the affine space $\mathfrak{sl}_4(\mathbb{F}_q)$:

$$C_{[2,1^2]}(\alpha) = \begin{pmatrix} 3\alpha & 1 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}.$$

5.6.1. Semisimple orbits. Computing the characteristic and minimal polynomials of the matrices in $C_{[2,1^2]}$ one sees that non-nilpotent elements are semisimple and diagonalizable with three coincident eigenvalues. This implies that for $\alpha \neq 0$,

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}\left(C_{[2,1^2]}(\alpha)\right) \cong \mathrm{GL}_3(\mathbb{F}_q) \times \mathbb{F}_q^{\times}.$$

We choose $\mathbf{A} = \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}\left(C_{[2,1^2]}(\alpha)\right)$ as isomorphism type for the $\mathrm{GL}_4(\mathbb{F}_q)$-conjugates of $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}\left(C_{[2,1^2]}(\alpha)\right)$. It follows that

$$\mathbf{A} \cong \mathrm{GL}_3(\mathbb{F}_q)\,. \tag{5.70}$$

Considering that we have $(q-1)$ non-nilpotent orbits, there are

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{A}) = q^3 \cdot (q-1) \cdot (q+1) \cdot (q^2+1). \tag{5.71}$$

non-nilpotent elements on $\mathrm{S}_{[2,1^2]}$ in total.

We investigate now the lifting behaviour of elements with centralizer isomorphic to $\mathbf{A}$. By (3.3) and Lemma 3.13 this involves considering the action of $\mathbf{A}$ on $\mathfrak{a}^{\vee}$. This action is isomorphic to the $\mathrm{GL}_3(\mathbb{F}_q)$-conjugation on $\mathfrak{gl}_3(\mathbb{F}_q)$. The computations are analogous to the ones performed in Section 3.3, $\mathbf{L}'$ and $\mathbf{J}'$ in $\mathfrak{sl}_4(\mathfrak{o})$ correspond to the shadow-isomorphism types in $\mathfrak{sl}_3(\mathfrak{o})$ designated by $\mathbf{L}$ and $\mathbf{J}$, respectively. Notice that, by (5.18), we do not need to exclude the prime 3 here.

Let $a \in \mathfrak{sl}_4(\mathbb{F}_q)$ have group centralizer isomorphic to $\mathbf{A}$, there are

$$\Lambda(\mathbf{A}, \mathbf{R}) = q^2 \cdot (q-1) \cdot (q^6+q^5+q^4-q^2-2q-1) \tag{5.72}$$

lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ with regular shadow and

$$\begin{aligned} \Lambda(\mathbf{A}, \mathbf{L}') &= q \cdot (q^5 - q^2) \\ \Lambda(\mathbf{A}, \mathbf{J}') &= q \cdot (q^4+q^3-q-1) \end{aligned} \tag{5.73}$$

lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ with shadow $\mathbf{L}'$ and $\mathbf{J}'$ respectively.

We can now obtain the quantities that we need in the computation of the Poincaré series. By Proposition 3.7 there are

$$q^8 \cdot (\Lambda(\mathbf{A}, \mathbf{L}') + \Lambda(\mathbf{A}, \mathbf{J}')) = q \cdot (q^5+q^4+q^3-q^2-q-1)$$

lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ that have 5-dimensional shadow. Let $w \in \mathfrak{sl}_4(\mathfrak{o}_2)$ be such a lift and let $(a, y) = \theta_{I,\mathbf{r}_I,\psi}(w)$, for $I = \{2, 4\}$ and $\mathbf{r}_I = (1, 1)$. Then, by Lemma 4.11 and Proposition 4.14, $\dim \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a+y) = 5$. Hence

$$\sum_{d_{\mathbf{S}}=5} \mathcal{C}_{\{\mathbf{C},\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^a = q^7 \cdot (q^5+q^4+q^3-q^2-q-1). \tag{5.74}$$

Since the choice of a is arbitrary, the equality above does not depend on a. The same happens for 3-dimensional centralizers:

$$\sum_{d_{\mathbf{S}}=3} \mathcal{C}_{\{\mathbf{C},\mathbf{S}\},\psi}(\mathrm{SL}_4(\mathfrak{o}))^a = q^8 \cdot (q-1) \cdot (q^6+q^5+q^4-q^2-2q-1). \tag{5.75}$$

Similar to the notation conventions adopted at the end of Section 5.5.3, we define

$$\mathrm{g}_{\{\mathbf{A},\mathbf{U}\}} = \sum_{d_{\mathbf{S}}=5} \mathrm{g}_{\{\mathbf{A},\mathbf{S}\}}$$
$$\mathrm{g}_{\{\mathbf{A},\mathbf{R}\}} = \sum_{d_{\mathbf{S}}=3} \mathrm{g}_{\{\mathbf{A},\mathbf{S}\}}$$
$$\mathrm{g}_{\{\mathbf{A},\mathbf{U},\mathbf{R}\}} = \sum_{\substack{d_{\mathbf{S}}=5\\ d_{\mathbf{T}}=3}} \mathrm{g}_{\{\mathbf{A},\mathbf{S},\mathbf{T}\}}$$

By (5.70), (5.74) and (5.75), we compute

$$\begin{aligned}
\mathrm{g}_{\{\mathbf{A}\}} &= (q^4-1)\cdot q^{-4}\\
\mathrm{g}_{\{\mathbf{A},\mathbf{U}\}} &= (q^9+q^8+q^7-q^6-2q^5-2q^4-q^3+q^2+q+1)\cdot q^{-9}\\
\mathrm{g}_{\{\mathbf{A},\mathbf{R}\}} &= (q^{11}-q^8-2q^7-q^6+q^5+2q^4+q^3+q^2-q-1)\cdot q^{-11}\\
\mathrm{g}_{\{\mathbf{A},\mathbf{U},\mathbf{R}\}} &= (q^{12}+q^{11}+q^{10}-2q^9-3q^8-3q^7\\
&\quad +3q^5+3q^4+2q^3-q^2-q-1)\cdot q^{-12}.
\end{aligned} \tag{5.76}$$

Contribution to the Poincaré series. We compute part of the Poincaré series corresponding to decreasing sequences of centralizers beginning with **A**. The investigation of which decreasing sequences give non-zero coefficients is carried out in Section 5.6.1, the numerical data relative to the decreasing sequences under examination is summarized in (5.76). The contribution to the Poincaré series in (4.15) given by summands corresponding to decreasing sequences beginning with **A** is

$$\begin{aligned}
\mathcal{P}_{\mathbf{A}}(s) &= \mathrm{g}_{\{\mathbf{A}\}}\,\frac{q^{7-3s}}{1-q^{7-3s}}\\
&\quad + \mathrm{g}_{\{\mathbf{A},\mathbf{U}\}}\,\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{12-5s}}{1-q^{12-5s}}\\
&\quad + \mathrm{g}_{\{\mathbf{A},\mathbf{R}\}}\,\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{15-6s}}{1-q^{15-6s}}\\
&\quad + \mathrm{g}_{\{\mathbf{A},\mathbf{U},\mathbf{R}\}}\,\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{12-5s}}{1-q^{12-5s}}\,\frac{q^{15-6s}}{1-q^{15-6s}}\\
&= \frac{\mathcal{F}_{\mathbf{A}}(q,q^{-s})}{\mathcal{G}_{\mathbf{A}}(q,q^{-s})},
\end{aligned} \tag{5.77}$$

where

$$\begin{aligned}
\mathcal{F}_{\mathbf{A}}(q,t) &= \big(q^{26}-q^{22}\big)t^{14}\\
&\quad - \big(q^{19}+q^{18}+q^{17}-q^{16}-2\,q^{15}-q^{14}-q^{13}+q^{12}+q^{11}\big)t^9\\
&\quad + \big(q^{18}+q^{17}-q^{16}-q^{15}-2\,q^{14}-q^{13}+q^{12}+q^{11}+q^{10}\big)t^8\\
&\quad + \big(q^7-q^3\big)t^3\\
\mathcal{G}_{\mathbf{A}}(q,t) &= \big(1-q^7t^3\big)\big(1-q^{12}t^5\big)\big(1-q^7t^3\big)\big(1-q^{15}t^6\big).
\end{aligned}$$

5.6.2. The nilpotent orbit. The nilpotent matrix on the affine cross-section is

$$a = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The conjugation stabilizer for this matrix has the following parameterization:

$$\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{11} & 0 & 0 \\ 0 & m_{32} & m_{33} & m_{34} \\ 0 & m_{31} & m_{43} & m_{44} \end{pmatrix} \middle| \, M \in \mathrm{GL}_4(\mathbb{F}_q) \right\}.$$

Therefore

$$|\,\mathrm{C}_{\mathrm{GL}_4(\mathbb{F}_q)}(a)\,| = |\,\mathrm{GL}_2(\mathbb{F}_q)\,|(q-1)q^5 = q^6(q-1)^3(q+1).$$

It follows that the cardinality of the nilpotent orbit is:

$$(q-1)\cdot(q+1)\cdot(q^2+1)\cdot(q^2+q+1). \tag{5.78}$$

We choose $\mathbf{B} = \mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$ as isomorphism type for the $\mathrm{GL}_4(\mathbb{F}_q)$ conjugates of a. Let us investigate its structure. First of all we fix a basis for $\mathfrak{b} = \mathrm{C}_{\mathfrak{sl}_4(\mathbb{F}_q)}(a)$, say $\mathcal{B}_{\mathfrak{b}} = \{e_0, \ldots, e_8\}$ with

$$e_0 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, e_1 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, e_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{pmatrix}, e_4 = \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$e_5 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, e_6 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, e_7 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix},$$

$$e_8 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}.$$

Secondly we compute the commutator matrix of $\mathrm{C}_{\mathfrak{sl}_4(\mathbb{F}_q)}(a)$ relative to the basis $\mathcal{B}_{\mathfrak{b}}$:

$$\mathcal{R}_{\mathfrak{b}}(\mathbf{Y}) = \begin{pmatrix} 0 & 0 & Y_4 & 0 & 0 & 0 & Y_1 & Y_0 & -2Y_0 \\ 0 & 0 & 0 & Y_4 & 0 & Y_0 & 0 & -Y_1 & -2Y_1 \\ -Y_4 & 0 & 0 & 0 & 0 & -Y_3 & 0 & -Y_2 & 2Y_2 \\ 0 & -Y_4 & 0 & 0 & 0 & 0 & -Y_2 & Y_3 & 2Y_3 \\ 0&0&0&0&0&0&0&0&0 \\ 0 & -Y_0 & Y_3 & 0 & 0 & 0 & -Y_7 & 2Y_5 & 0 \\ -Y_1 & 0 & 0 & Y_2 & 0 & Y_7 & 0 & -2Y_6 & 0 \\ -Y_0 & Y_1 & Y_2 & -Y_3 & 0 & -2Y_5 & 2Y_6 & 0 & 0 \\ 2Y_0 & 2Y_1 & -2Y_2 & -2Y_3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.79}$$

Looking at the commutator relations in $\mathcal{R}_{\mathfrak{b}}$ we notice that the subgroup

$$H = \left\{ M = \begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ 0 & 1 & 0 & 0 \\ 0 & m_{32} & 1 & 0 \\ 0 & m_{31} & 0 & 1 \end{pmatrix} \middle| \; M \in \mathrm{GL}_4(\mathbb{F}_q) \right\} \leq \mathbf{B}$$

is isomorphic to the direct product $\mathrm{Heis}(\mathbb{F}_q) \curlyvee \mathrm{Heis}(\mathbb{F}_q)$ of two copies of the Heisenberg group $\mathrm{Heis}(\mathbb{F}_q)$ with amalgamation in the centre. Furthermore $\mathbf{B} = HS$ where

$$S = \left\{ M = \begin{pmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{11} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix} \middle| \; M \in \mathrm{SL}_4(\mathbb{F}_q) \right\} \cong \mathrm{GL}_2(\mathbb{F}_q)$$

and C commutes with both H and S. As a consequence

$$\mathbf{B} \cong (\mathrm{Heis}(\mathbb{F}_q) \curlyvee \mathrm{Heis}(\mathbb{F}_q)) \rtimes \mathrm{GL}_2(\mathbb{F}_q) \tag{5.80}$$

where the semidirect product is defined by the commutator relations in $\mathcal{R}_{\mathfrak{b}}$. We conclude that the centralizer of nilpotent elements on the sheet $\mathrm{S}_{[2,1^2]}$ is not isomorphic to $\mathbf{A}$. By (5.78), we write

$$\Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{B}) = (q-1) \cdot (q+1) \cdot (q^2+1) \cdot (q^2+q+1). \tag{5.81}$$

We therefore compute

$$\begin{aligned} \mathrm{g}_{\{\mathbf{B}\}} &= q^{-7} \cdot \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{B}) \\ &= (q^6+q^5+q^4-q^2-q-1) \cdot q^{-7}. \end{aligned} \tag{5.82}$$

5.6.2.1. *Lifts with* 7*-dimensional shadow.* As we did for the other isomorphism types, we proceed now to the investigation of the lifts of the element a. Let us identify $\mathrm{C}_{\mathfrak{sl}_4(\mathbb{F}_q)}(a)^\vee$ with $\mathbb{F}_q{}^9$. From (5.79) one deduces that $\mathcal{R}_{\mathfrak{b}}(c_0, \ldots, c_7, c_8)$ has rank 2 if and only if $c_0 = \cdots = c_4 = 0$ and at least one of c_5, c_6 and c_7 is non-zero. This means that the rank-2 variety is defined by the ideal $R_2 = (Y_0, \ldots, Y_4) \subseteq \mathbb{C}[Y_0, \ldots, Y_8]$ and that the rank-0 variety is defined by the ideal $(Y_0, \ldots, Y_7)$. By looking at the submatrix of $\mathcal{R}_{\mathfrak{b}}$ corresponding to the last 4 coordinates, we realize that the rank-2 variety of the matrix $\mathcal{R}_{\mathfrak{b}}$ is isomorphic to $\mathfrak{gl}_2(\mathbb{F}_q)$, and the $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$-action on it is isomorphic to the $\mathrm{GL}_2(\mathbb{F}_q)$-action on $\mathfrak{gl}_2(\mathbb{F}_q)$. So, analogously to what we did for elements with group centralizer isomorphic to $\mathbf{E}$, we may use Corollary 2.13 to deduce that the element a may lift to elements with 7-dimensional shadow of three distinct isomorphism types according to whether, in the correspondence of Theorem A, the lifting element corresponds to a semisimple diagonalizable, semisimple non-diagonalizable or nilpotent element of $\mathfrak{gl}_2(\mathbb{F}_q)$. We call these isomorphism types $\mathbf{Q}$, $\mathbf{V}$ and $\mathbf{W}$ respectively and we compute

$$\Lambda(\mathbf{B}, \mathbf{Q}) = \frac{1}{2} \cdot q^2 \cdot (q^2-1) \tag{5.83}$$

$$\Lambda(\mathbf{B}, \mathbf{V}) = \frac{1}{2} \cdot q^2 \cdot (q^2-1)^2 \tag{5.84}$$

$$\Lambda(\mathbf{B}, \mathbf{W}) = \frac{1}{2} \cdot q \cdot (q^2-1). \tag{5.85}$$

Remark 5.7. We do not determine explicitly the isomorphism type of $\mathbf{Q}$, $\mathbf{V}$ and $\mathbf{W}$ because this is not needed in our computation. However, using Corollary 2.13 and Lemma 2.14, a more detailed analysis of which isomorphism types occur among the lifts of a may be performed by looking at the kernels of the 9×9 commutator matrix when evaluated in the appropriate elements.

We derive now the coefficients in (4.15) corresponding to decreasing sequences beginning with **B**. Each orbit of lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ having a 7-dimensional centralizer contains an element of

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ \pi\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & \pi\alpha & 0 \end{pmatrix} \middle| \; \alpha \in \mathbb{F}_q \right\}, \tag{5.86}$$

and vice versa distinct elements of C are contained in a distinct orbits. Indeed this is just the correspondence of Theorem A made explicit. By Corollary 2.13, the isomorphism type of the orbit is determined by whether α is zero, a non-zero square or not a square in $\mathbb{F}_q$. For each $\alpha \in \mathbb{F}_q$ let y_α be the matrix such that

$$a + \pi y_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \pi\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & \pi\alpha & 0 \end{pmatrix}$$

The group centralizer isomorphism type of

$$a + y_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \end{pmatrix}$$

varies according to whether α is zero, a non-zero square or not a square in $\mathbb{F}_q$. All in all the situation is as described in Table 5.2. The same is valid for all the other elements of $\mathfrak{sl}_4(\mathbb{F}_q)$ that have centralizer isomorphic to **B**, for they are conjugate to a. We can therefore use (5.1) and compute

$$\begin{aligned} \mathrm{g}_{\{\mathbf{B},\mathbf{C}\}} =& \Lambda(\mathrm{SL}_4(\mathbb{F}_q), \mathbf{B}) \cdot q^6 \cdot \Lambda(\mathbf{B}, \mathbf{Q}) \cdot q^{-7} \cdot q^{-9} \\ & \frac{1}{2}(q^{10} + q^9 - q^7 - 2q^6 - q^5 + q^3 + q^2) \cdot q^{-10}. \end{aligned} \tag{5.87}$$

We adopt the convention of reuniting all the regular and subregular centralizer iso-types under **R** and **U** respectively, as done at the end of Section 5.5.3 we define

TABLE 5.2. Shadow and centralizer iso-types of elements of C.

Parameter α	Iso-type of shadow	Iso-type of $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a + y_\alpha)$
α a non-zero square in $\mathbb{F}_q$	**Q**	**C**
α not a square in $\mathbb{F}_q$	**V**	**D**
$\alpha = 0$	**W**	**E**

in order to simplify the computation. Analogously to (5.87) we compute

$$
\begin{aligned}
g_{\{\mathbf{B},\mathbf{C},\mathbf{U}\}} &= q^{-14}(q^{14}+q^{13}-2q^{11}-3q^{10}\\
&\quad -q^9+q^8+3q^7+2q^6-q^4-q^3)\cdot q^{-14}\\
g_{\{\mathbf{B},\mathbf{C},\mathbf{R}\}} &= \frac{1}{2}(q^{17}+q^{16}-3q^{14}-4q^{13}-q^{12}+3q^{11}+6q^{10}\\
&\quad +3q^9-q^8-4q^7-3q^6+q^4+q^3)\cdot q^{-17}\\
g_{\{\mathbf{B},\mathbf{C},\mathbf{U},\mathbf{R}\}} &= (q^{19}+q^{18}-3q^{16}-4q^{15}-q^{14}+3q^{13}+6q^{12}\\
&\quad +3q^{11}-q^{10}-4q^9-3q^8+q^6+q^5)\cdot q^{-19}\\
g_{\{\mathbf{B},\mathbf{D}\}} &= \frac{1}{2}(q^{10}-q^9-q^7+q^5+q^3-q^2)\cdot q^{-10}\\
g_{\{\mathbf{B},\mathbf{D},\mathbf{R}\}} &= \frac{1}{2}(q^{17}-q^{16}-q^{14}+q^{12}-q^{11}+2q^{10}\\
&\quad -q^9+q^8-q^6-q^4+q^3)\cdot q^{-17}\\
g_{\{\mathbf{B},\mathbf{E}\}} &= (q^9+q^8-q^6-2q^5-q^4+q^2+q)\cdot q^{-10}\\
g_{\{\mathbf{B},\mathbf{E},\mathbf{U}\}} &= (q^{13}+q^{12}-2q^{10}-3q^9-q^8+q^7\\
&\quad +3q^6+2q^5-q^3-q^2)\cdot q^{-14}\\
g_{\{\mathbf{B},\mathbf{E},\mathbf{R}\}} &= (q^{16}+q^{15}-2q^{13}-3q^{12}-q^{11}\\
&\quad +q^{10}+3q^9+2q^8-q^6-q^5)\cdot q^{-17}\\
g_{\{\mathbf{B},\mathbf{E},\mathbf{U},\mathbf{R}\}} &= (q^{18}+q^{17}-3q^{15}-4q^{14}-q^{13}+3q^{12}+6q^{11}\\
&\quad +3q^{10}-q^9-4q^8-3q^7+q^5+q^4)\cdot q^{-19}
\end{aligned}
\tag{5.88}
$$

5.6.3. Lifts with 5-dimensional shadow and regular lifts. In order to finish our investigation, we need to compute the cardinality of the rank-4 locus of $\mathcal{R}_\mathfrak{b}$. We can do it by looking at its equations. The following is a generating set for the radical of the ideal generated by the 6×6 Pfaffians of $\mathcal{R}_\mathfrak{b}$

$$
\begin{aligned}
&Y_0Y_3-Y_4Y_5\\
&Y_1Y_2-Y_4Y_6\\
&Y_0Y_2-Y_1Y_3-Y_4Y_7\\
&Y_2^2Y_5-Y_3^2Y_6-Y_2Y_3Y_7\\
&Y_1^2Y_5-Y_0^2Y_6+Y_0Y_1Y_7\\
&Y_1Y_3^2-Y_2Y_4Y_5+Y_3Y_4Y_7\\
&Y_1^2Y_3-Y_0Y_4Y_6+Y_1Y_4Y_7.
\end{aligned}
\tag{5.89}
$$

Let R_3 be the ideal of $\mathbb{C}[Y_0,\dots,Y_8]$ generated by the polynomials in (5.89) and let $\mathrm{V}^4_{\mathcal{R}_\mathfrak{b}}$ be the algebraic set defined by it. The rank-4 locus $\mathrm{L}^4_{\mathcal{R}_\mathfrak{b}}$ is the set where all the polynomials in R_3 but not all the polynomials in $R_2=(Y_0,\dots,Y_4)$ vanish. Now let $\mathbf{c}=(c_0,\dots,c_8)$ be a $\mathbb{F}_q$-rational point of $\mathrm{L}^4_{\mathcal{R}_\mathfrak{b}}$, i.e. $\mathbf{c}\in\mathrm{L}^4_{\mathcal{R}_\mathfrak{b}}(\mathbb{F}_q)$. We notice that by forcing $c_0,\dots,c_4$ to 0 we can project this point on $\mathrm{L}^2_{\mathcal{R}_\mathfrak{b}}(\mathbb{F}_q)$; this defines a function

$$
\begin{aligned}
\mathrm{proj}:\quad \mathrm{L}^4_{\mathcal{R}_\mathfrak{b}}(\mathbb{F}_q) &\longrightarrow \mathrm{V}^2_{\mathcal{R}_\mathfrak{b}}(\mathbb{F}_q)\\
(c_0,\dots,c_8) &\longmapsto (0,\dots 0,c_5,\dots,c_8).
\end{aligned}
$$

The rank-2 variety is stable under the action of $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$, so proj maps $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$-orbits to $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$-orbits and the cardinality of the fibres of proj is constant across $\mathrm{C}_{\mathrm{SL}_4(\mathbb{F}_q)}(a)$-orbits in the rank-2 variety. Let us identify $\mathrm{V}^2_{\mathcal{R}_\mathfrak{b}}(\mathbb{F}_q)$ with $\mathfrak{gl}_2(\mathbb{F}_q)$. In what follows we shall operate a case distinction according to the adjoint orbit in

$\mathfrak{gl}_2(\mathbb{F}_q)$. The elements in the centre of $\mathfrak{gl}_2(\mathbb{F}_q)$ are those for which $c_5 = \cdots = c_7 = 0$. Now we substitute the previous conditions in (5.89) and impose that at least one of the $c_0, \ldots, c_4$ is non-zero (we want to exclude points of the rank-2 locus inside the rank-4 variety). This gives that the fibre of proj above each one of these elements has cardinality

$$2q^3 - q - 1.$$

Now let us consider the elements in $\mathfrak{gl}_2(\mathbb{F}_q)$ that belong to a nilpotent orbit. These are the elements whose orbit contains an element defined by $c_5 = 1$, $c_6 = c_7 = 0$ and c_8 arbitrary. Substituting these relations into the (5.89) and imposing that at least one of the other variables is non-zero, we obtain that the fibre above a nilpotent point has cardinality

$$q^2 - 1.$$

The other orbits are parameterized by the following elements $c_5 = 1$, $c_7 = 0$ and $c_6 = \alpha \in \mathbb{F}_q^\times$. Again by substituting we see that there is no point in $\mathrm{L}^4_{\mathcal{R}_\mathfrak{b}}(\mathbb{F}_q)$ projecting down to a point in an orbit with α a non-square in $\mathbb{F}_q$. It remains to compute the cardinality of the fibre above points for which α is a non-zero square in $\mathbb{F}_q$ (semisimple diagonalizable points). Substituting this condition in (5.89) and imposing that the other variables are not all zero, gives that the cardinality of the fibre of proj above each of these points is

$$2 \cdot (q^2 - 1).$$

Considered that in $\mathfrak{gl}_2(\mathbb{F}_q)$ there are q central elements, $q \cdot (q^2-1)$ nilpotent elements and $q^2 \cdot (q^2 - 1)/2$ semisimple diagonalizable points, we obtain that

$$|\mathrm{L}^4_{\mathcal{R}_\mathfrak{b}}(\mathbb{F}_q)| = q \cdot (2q^3 - q - 1) + q \cdot (q^2 - 1)^2 + q^2 \cdot (q^2 - 1)^2 = q \cdot (q^5 + q^4 - 2q^2). \tag{5.90}$$

It follows that the number of lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ having 5-dimensional shadow is

$$\Lambda(\mathbf{B}, \mathbf{U}) = q \cdot (q^5 + q^4 - 2q^2), \tag{5.91}$$

while the number of lifts of a to $\mathfrak{sl}_4(\mathfrak{o}_2)$ having regular shadow is

$$\begin{aligned} \Lambda(\mathbf{B}, \mathbf{R}) = \sum_{d_\mathbf{S}=3} \Lambda(\mathbf{B}, \mathbf{S}) &= q^9 - 1 - \sum_{d_\mathbf{S} \leq 5} \Lambda(\mathbf{B}, \mathbf{S}) \\ &= q \cdot (q^8 - q^3 - q^2(q^3 + q^2 - 2)) = q^9 - q^6 - q^5 - q^4 + 2q^3. \end{aligned} \tag{5.92}$$

Analogously to the notation conventions adopted at the end of Section 5.5.3 we define , we can define

$$\begin{aligned} \mathrm{g}_{\{\mathbf{B},\mathbf{U}\}} &= \sum_{d_\mathbf{S}=5} \mathrm{g}_{\{\mathbf{B},\mathbf{S}\}} \\ \mathrm{g}_{\{\mathbf{B},\mathbf{R}\}} &= \sum_{d_\mathbf{S}=3} \mathrm{g}_{\{\mathbf{B},\mathbf{S}\}} \\ \mathrm{g}_{\{\mathbf{B},\mathbf{U},\mathbf{R}\}} &= \sum_{\substack{d_\mathbf{S}=5 \\ d_\mathbf{T}=3}} \mathrm{g}_{\{\mathbf{B},\mathbf{S},\mathbf{T}\}} \end{aligned}$$

the two equations (5.91) and (5.92) allow us to compute

$$\begin{aligned} \mathrm{g}_{\{\mathbf{B},\mathbf{U}\}} =& (q^{12} + 2q^{11} + 2q^{10} - q^9 - 3q^8 - 4q^7 \\ & - 2q^6 + q^5 + 2q^4 + 2q^3) \cdot q^{-13} \\ \mathrm{g}_{\{\mathbf{B},\mathbf{R}\}} =& (q^{15} + q^{14} + q^{13} - q^{12} - 3q^{11} - 4q^{10} \\ & - q^9 + 2q^8 + 4q^7 + 3q^6 - q^4 - 2q^3) \cdot q^{-16} \\ \mathrm{g}_{\{\mathbf{B},\mathbf{U},\mathbf{R}\}} =& (q^{17} + 2q^{16} + 2q^{15} - 2q^{14} - 5q^{13} - 6q^{12} - q^{11} + 4q^{10} + 6q^9 \\ & + 4q^8 - q^7 - 2q^6 - 2q^5) \cdot q^{-18}. \end{aligned} \tag{5.93}$$

Contribution to the Poincaré series. The investigation of which decreasing sequences beginning with $\mathbf{B}$ give non-zero coefficients in (4.15) is summarized in (5.82), (5.87), (5.88) and (5.93). The contribution to the Poincaré series given by summands corresponding to these coefficients is

$$\begin{aligned}
\mathcal{P}_{\mathbf{B}}(s) = {}& \mathrm{g}_{\{\mathbf{B}\}}\,\frac{q^{7-3s}}{1-q^{7-3s}}\\
&+\big(\mathrm{g}_{\{\mathbf{B},\mathbf{C}\}}+\mathrm{g}_{\{\mathbf{B},\mathbf{D}\}}+\mathrm{g}_{\{\mathbf{B},\mathbf{E}\}}\big)\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{9-4s}}{1-q^{9-4s}}\\
&+\big(\mathrm{g}_{\{\mathbf{B},\mathbf{C},\mathbf{U}\}}+\mathrm{g}_{\{\mathbf{B},\mathbf{E},\mathbf{U}\}}\big)\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{9-4s}}{1-q^{9-4s}}\,\frac{q^{12-5s}}{1-q^{12-5s}}\\
&+\big(\mathrm{g}_{\{\mathbf{B},\mathbf{C},\mathbf{R}\}}+\mathrm{g}_{\{\mathbf{B},\mathbf{D},\mathbf{R}\}}+\mathrm{g}_{\{\mathbf{B},\mathbf{E},\mathbf{R}\}}\big)\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{9-4s}}{1-q^{9-4s}}\,\frac{q^{15-6s}}{1-q^{15-6s}}\\
&+\big(\mathrm{g}_{\{\mathbf{B},\mathbf{C},\mathbf{U},\mathbf{R}\}}+\mathrm{g}_{\{\mathbf{B},\mathbf{E},\mathbf{U},\mathbf{R}\}}\big)\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{9-4s}}{1-q^{9-4s}}\,\frac{q^{12-5s}}{1-q^{12-5s}}\,\frac{q^{15-6s}}{1-q^{15-6s}}\\
&+\mathrm{g}_{\{\mathbf{B},\mathbf{U}\}}\,\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{12-5s}}{1-q^{12-5s}}\\
&+\mathrm{g}_{\{\mathbf{B},\mathbf{U},\mathbf{R}\}}\,\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{12-5s}}{1-q^{12-5s}}\,\frac{q^{15-6s}}{1-q^{15-6s}}\\
&+\mathrm{g}_{\{\mathbf{B},\mathbf{R}\}}\,\frac{q^{7-3s}}{1-q^{7-3s}}\,\frac{q^{15-6s}}{1-q^{15-6s}}.
\end{aligned}$$

Thus

$$\mathcal{P}_{\mathbf{B}}(s) = \frac{\mathcal{F}_{\mathbf{B}}(q,q^{-s})}{\mathcal{G}_{\mathbf{B}}(q,q^{-s})}, \tag{5.94}$$

where

$$\begin{aligned}
\mathcal{F}_{\mathbf{B}}(q,t) = {}& -\big(q^{34}+q^{33}+q^{32}-q^{30}-q^{29}-q^{28}\big)t^{18}\\
&+\big(q^{28}+q^{27}+q^{26}-q^{24}-q^{23}-q^{22}\big)t^{14}\\
&-\big(q^{28}+q^{27}-q^{26}-4\,q^{25}-4\,q^{24}-q^{23}+3\,q^{22}\\
&+5\,q^{21}+3\,q^{20}-2\,q^{18}-q^{17}\big)\,t^{13}\\
&+\big(q^{27}-q^{26}-2\,q^{25}-2\,q^{24}+4\,q^{22}+4\,q^{21}+3\,q^{20}\\
&-q^{19}-3\,q^{18}-2\,q^{17}-q^{16}\big)\,t^{12}\\
&-\big(q^{18}+2\,q^{17}+3\,q^{16}-2\,q^{14}-4\,q^{13}-3\,q^{12}+q^{10}+2\,q^{9}\big)t^{9}\\
&+\big(q^{17}+q^{16}-q^{15}-2\,q^{14}-3\,q^{13}-q^{12}+q^{11}+2\,q^{10}+2\,q^{9}\big)t^{8}\\
&+\big(q^{16}-2\,q^{13}-2\,q^{12}-q^{11}+2\,q^{9}+q^{8}+q^{7}\big)t^{7}\\
&+\big(q^{6}+q^{5}+q^{4}-q^{2}-q-1\big)t^{3}\\
\mathcal{G}_{\mathbf{B}}(q,t) = {}& \big(1-q^{7}t^{3}\big)\big(1-q^{9}t^{4}\big)\big(1-q^{12}t^{5}\big)\big(1-q^{15}t^{6}\big).
\end{aligned}$$

5.6.4. Number of elements with regular shadow. Now that we know the cardinalities of all other sheets we can compute the cardinality of the sheet containing all the regular elements:

$$\begin{aligned}
\Lambda(\mathrm{SL}_4(\mathbb{F}_q),\mathbf{R}) = {}& q^{15}-1-\sum_{d_{\mathbf{S}}\le 5}\Lambda(\mathrm{SL}_4(\mathbb{F}_q),\mathbf{S})\\
= {}& (q-1)\cdot(q+1)\cdot q^{3}\cdot(q^{10}+q^{8}-q^{7}-3q^{5}-q^{3}+2q^{2}+q+1).
\end{aligned}$$

Substituting in (5.3) we obtain

$$\mathcal{P}_{\mathbf{R}}(s) =(q^{15} - q^{12} - q^{11} - 2q^{10}+ 2q^8 + 2q^7 + 2q^6 - q^5 - q^4 - q^3) \cdot q^{-15} \frac{q^{15-6s}}{1 - q^{15-6s}}. \tag{5.95}$$

5.7. Poincaré series of $\mathfrak{sl}_4(\mathfrak{o})$

Adding the partial summands in (5.31), (5.51), (5.61), (5.69), (5.77), (5.94) and (5.95) we compute

$$\begin{aligned}\mathcal{P}_{\mathfrak{sl}_4(\mathfrak{o})}(s) &= 1 + \mathcal{P}_{\mathbf{R}}(s) + \mathcal{P}_{\mathbf{U}}(s) + \mathcal{P}_{\mathbf{C}}(s) + \mathcal{P}_{\mathbf{D}}(s) + \mathcal{P}_{\mathbf{E}}(s) + \mathcal{P}_{\mathbf{A}}(s) + \mathcal{P}_{\mathbf{B}}(s) \\ &= \frac{\mathcal{F}_{Poin}(q,t)}{\mathcal{G}_{Poin}(q,t)}\end{aligned} \tag{5.96}$$

where

$$\begin{aligned}\mathcal{F}_{Poin}(q,t) =& q^{28}t^{18} \\ & - \left(q^{28} + q^{27} + q^{26} + q^{25} - q^{24} - q^{23} - q^{22}\right)t^{15} \\ & + \left(q^{27} - 2\,q^{24} - q^{22} + q^{21}\right)t^{14} \\ & + \left(q^{26} + 2\,q^{25} + 2\,q^{24} - 2\,q^{22} - 4\,q^{21} - 2\,q^{20} - q^{19} + 2\,q^{18} + q^{17}\right)t^{13} \\ & - \left(q^{25} + q^{24} + q^{23} - 2\,q^{22} - 2\,q^{21} - 2\,q^{20} + 2\,q^{18} + q^{17} + q^{16}\right)t^{12} \\ & + \left(q^{21} + 2\,q^{19} + q^{17} - q^{16} - q^{15} - q^{14}\right)t^{11} \\ & + \left(q^{19} + q^{18} - 2\,q^{15} + q^{12}\right)t^{10} \\ & - \left(2\,q^{19} + q^{18} + q^{17} - q^{16} - 3\,q^{15} - 2\,q^{14} \right. \\ & \left. -3\,q^{13} - q^{12} + q^{11} + q^{10} + 2\,q^{9}\right) t^{9} \\ & + \left(q^{16} - 2\,q^{13} + q^{10} + q^{9}\right)t^{8} \\ & - \left(q^{14} + q^{13} + q^{12} - q^{11} - 2\,q^{9} - q^{7}\right)t^{7} \\ & - \left(q^{12} + q^{11} + 2\,q^{10} - 2\,q^{8} - 2\,q^{7} - 2\,q^{6} + q^{5} + q^{4} + q^{3}\right)t^{6} \\ & + \left(q^{11} + 2\,q^{10} - q^{9} - 2\,q^{8} - 4\,q^{7} - 2\,q^{6} + 2\,q^{4} + 2\,q^{3} + q^{2}\right)t^{5} \\ & + \left(q^{7} - q^{6} - 2\,q^{4} + q\right)t^{4} \\ & + \left(q^{6} + q^{5} + q^{4} - q^{3} - q^{2} - q - 1\right)t^{3} \\ & + 1 \\ \mathcal{G}_{Poin}(q,t) =& \left(1 - q^{7}t^{3}\right)\left(1 - q^{9}t^{4}\right)\left(1 - q^{12}t^{5}\right)\left(1 - q^{15}t^{6}\right).\end{aligned}$$

Operating the substitution in Proposition 1.15 we deduce Theorem B.

THEOREM B. *Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic* 0 *whose residue field has cardinality q and characteristic not equal to* 2. *Then, for all permissible m,*

$$\zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(s) = q^{15m} \frac{\mathcal{F}(q, q^{-s})}{\mathcal{G}(q, q^{-s})}$$

where

$$\begin{aligned}\mathcal{F}(q,t) =& qt^{18} \\ & - \left(q^{7} + q^{6} + q^{5} + q^{4} - q^{3} - q^{2} - q\right)t^{15} \\ & + \left(q^{8} - 2\,q^{5} - q^{3} + q^{2}\right)t^{14} \\ & + \left(q^{9} + 2\,q^{8} + 2\,q^{7} - 2\,q^{5} - 4\,q^{4} - 2\,q^{3} - q^{2} + 2\,q + 1\right)t^{13} \\ & - \left(q^{10} + q^{9} + q^{8} - 2\,q^{7} - 2\,q^{6} - 2\,q^{5} + 2\,q^{3} + q^{2} + q\right)t^{12}\end{aligned}$$

$$
\begin{aligned}
&+ \big(q^8 + 2\,q^6 + q^4 - q^3 - q^2 - q\big)t^{11}\\
&+ \big(q^8 + q^7 - 2\,q^4 + q\big)t^{10}\\
&- \big(2\,q^{10} + q^9 + q^8 - q^7 - 3\,q^6 - 2\,q^5 - 3\,q^4 - q^3 + q^2 + q + 2\big)t^9\\
&+ \big(q^9 - 2\,q^6 + q^3 + q^2\big)t^8\\
&- \big(q^9 + q^8 + q^7 - q^6 - 2\,q^4 - q^2\big)t^7\\
&- \big(q^9 + q^8 + 2\,q^7 - 2\,q^5 - 2\,q^4 - 2\,q^3 + q^2 + q + 1\big)t^6\\
&+ \big(q^{10} + 2\,q^9 - q^8 - 2\,q^7 - 4\,q^6 - 2\,q^5 + 2\,q^3 + 2\,q^2 + q\big)t^5\\
&+ \big(q^8 - q^7 - 2\,q^5 + q^2\big)t^4\\
&+ \big(q^9 + q^8 + q^7 - q^6 - q^5 - q^4 - q^3\big)t^3\\
&+ q^9\\
\mathcal{G}(q,t) =& q^9\big(1 - qt^3\big)\big(1 - qt^4\big)\big(1 - q^2t^5\big)\big(1 - q^3t^6\big).
\end{aligned}
$$

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