This thesis models a patent race with a differential game approach. Firms strategically invest in R&D and produce knowledge in order to innovate. Whether knowledge is substitutable to the existing knowledge pool affects firm's incentives to invest in R&D. After innovating successfully, the innovator patents on its techniques so that their opponents cannot infringe. However, a patent system indirectly ensures the monopoly power of the innovator in a product market, which reduces the social welfare. The endogenous benefit of a patent holder is evaluated and an optimal patent policy is suggested in order to balance out the loss of the social welfare and innovation incentives. Besides, the production process usually contains several techniques with different patents and this collection forms a patent portfolio. A model of a patent portfolio race explains why some innovation such as an open-source software without patenting can also be as successful as a private software product.

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R&D, patents and innovation: a differential game approach

Hsin-Mien Wang

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R&D, patents and innovation

A differential game approach

Invitation

to the Public Defense of the Doctoral Thesis

R&D, Patents and Innovation: a Differential Game Approach

by

Hsin-Mien Wang

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R&D, patents and innovation: a differential game approach

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aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
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Table of contents

Chapter 1

Introduction

In the digital age nowadays, a patent war among technology firms is nothing new. Firms invest in R&D, innovate and patent on their techniques so that their opponents cannot infringe. Better differentiated goods or higher efficient production by innovation drops opponents out of the market indirectly. In fact, patenting is a mechanism for encouraging innovation. However, it becomes a tool for firms to ensure their monopoly power in the market, which brings social welfare loss. The question is whether a patent really can encourage innovation. If so, what is the optimal patent policy that brings less social welfare loss?

Incentives of innovation in a patent race

Early literature discusses firms incentives of R&D investment of stochastic innovation against rivalry is Reinganum (1981, 1982). The follower firm is conservative, since R&D investment contributes to the chances of current innovation but future innovation. Strategically, the follower firm invests permissively since the leader has a relative great advantage of innovating successfully. This memoryless model is improved by Doraszelski (2003) by introducing knowledge accumulation. R&D effort contributes to current chances of innovation and accumulates as knowledge in the future. He argued the follower's incentives of R&D investment by a dynamic model of knowledge accumulation with differential game approach.

A differential game approach

Instead of growing model of R&D, this thesis uses a differential game approach to investigate firms interaction of R&D behavior in a patent race. Differential game approach gives a dynamic micro-view of strategic interaction among players. The best response with respect to opponents' best response gives time consistency. The Markov-perfect Nash equilibrium is subgame perfect so that firms do not deviate from their best choices. To characterize the equilibria, this thesis uses two methodologies: collocation method (Rui & Miranda, 1996) and auxiliary system

(Dockner & Wagener, 2013). Both methods can solve nonlinear system numerically; further, the later one can visualize the impact of different parameters given the state spaces.

Knowledge and spillovers

In general, knowledge is an intangible good that gives a special feature of a patent race. Since knowledge is not physical, its spillovers exist naturally. Further, whether knowledge is additive or multiplicative to the existing knowledge pool is another issue that affects firm's R&D investing behavior. In this case, does a patent still play a role of changing firm's behaviors of R&D investment?

Endogenous patent benefit and product market

A patent plays a role of innovation incentives by patenting benefits, which determines by its length and strength. A long patent life with strong protection preventing imitation gives a high benefit for a patent holder. This thesis estimates the endogenous benefit of a patent holder by process innovation in a product market, which allows discussing more details of patent length and strength than other exogenous patent race model.

Patent portfolio and n firms

Production usually contains several techniques with different patents and this collection owned by a single agent is a patent portfolio. In a patent war, firms holding more than one patent and compete with more than one competitor in the market. Even though the patent war is like a raging fire, there is still innovation developed by users volunteering without any patenting in the market. If the innovation can occur without patenting, what does the role of a patent play in this case? This thesis gives a model to explain why some innovation such as an open-source software without patenting can also be as successful as a private software product.

Thesis outline

Chapter 2 is a patent race with knowledge accumulation of the complementary property and spillovers, which focus on how the different features of knowledge affects firm R&D incentives. Chapter 3 is a patent race with knowledge accumulation and product market, which describes the optimal patent policy that balances the trade-off between monopoly welfare loss and innovation incentives. Chapter 4 is a patent portfolio race with knowledge accumulation, which gives more general outcomes for n firms with a collection of patents. All models follow differential game approach and Markov-perfect Nash equilibria is characterized numerically.

Chapter 2

A patent race with the complementary property and spillovers

2.1 Introduction

A patent race is a technological competition among firms. A firm making the first-hand innovation takes advantages. The early literature, Loury (1979), Lee and Wilde (1980) and Reinganum (1982), used the exponential distribution of success times to model patent race with knowledge stock. The property of the exponential distribution is memorylessness, which implies that firms' current R&D effort is not associated to knowledge stock that they have accumulated in the past. This is not realistic. To include history dependency into the model, multistage models are introduced by Fudenberg et al. (1983) and Harris and Vickers (1985). Even though there is history dependency from one stage to the next stage, memorylessness problem still exists within one stage. Further, R&D investment in multistage models is restricted to a couple of choices. The discussion to address patent policy and incentives for innovation is also limited in these multistage models.

Another way to deal with the problem of memorylessness is proposed by Doraszelski (2003). He extended Judd's (1985) concept that a hazard rate, which is the rate at which a firm makes the first discovery at a certain point of time in a patent race, depends on knowledge accumulation. He proposed that the knowledge stock can be carried to the next period due to learning but, at the same time, it depreciates over time due to forgetting. Since knowledge stock can be accumulated over time, the investment of previous periods helps the firm to innovate and win the race (Doraszelski, 2003); this solved the problem of memorylessness in the patent race model. The hazard rate model of knowledge accumulation explains the incentives of a follower to catch up to the leader. Because of the knowledge effect, a leader reduces its R&D effort by its large knowledge pool. The follower has chances to catch up to the leader by increasing its R&D effort even if it has small knowledge pool. However, this result only holds under the assumption of an additive hazard rate, which means knowledge and R&D efforts are

substitutable for innovation. This assumption allows firms to rely on their knowledge stock and innovate without making any current R&D effort. To lose this assumption, I propose a multiplicative hazard rate so that knowledge stock and R&D effort are complements. When R&D effort and knowledge stock are complementary for innovation, the follower with little knowledge stock initially loses its advantage in the race. The chances of innovation of the follower cannot be improved by making R&D effort without corresponding knowledge stock. Instead of a strong catching-up behavior of a follower, the leader has more incentives to invest and keep its dominance in the complementary case than the substitute case.

Further, Cohen and Levinthal (1989) maintained that R&D investment contributes to the chances of innovation not only by generating self knowledge but also assimilating spillovers from rival's knowledge. The ability to internalize spillovers is absorptive capacity (Cohen & Levinthal, 1989). I also include spillovers and absorptive capacity in the model of hazard rate, so that increasing R&D effort additionally contributes to innovation by absorbing opponent's knowledge stock. In fact, spillovers give more incentives to invest in R&D for the follower than the leader, which reinforces the catching-up behavior in the case of the substitute property. This contradicts with the argument of Halmenschlager (2004), which discussed spillovers and absorptive capacity given by three levels of R&D choices in the multistage model of Fudenberg et al. (1983). The reason is that limited R&D choices and restrictive R&D contribution of innovation within each stages underestimates the follower's incentives of R&D investment with costly spillovers. When R&D effort contributes to future chances of innovation and the strategy of R&D effort allows to locate at any level of a continuous interval rather than restrict to only three choices, spillovers give the follower incentives to innovate. This explains some empirical cases that increasing absorptive capacity is one of R&D strategies of innovation (Tilton, 1971; Allen, 1977; Mowery, 1983).

Even though spillovers increase firms' incentives to invest in R&D in the short run, R&D investment decreases in the long run. A strong patent protection that reduces the imitator's reward can encourage firms to invest in R&D in the long run. In contrast, in a race with weak patent protection, firms are not willing to invest because the imitation reward is high, which results in a slow process of innovation. As long as the innovator's reward is distinguished with the imitator's one, firms have incentives to invest and innovate eventually. Since investment is booming, time of innovation is shorter.

Overall, two types of characteristics of a patent race are introduced: the additive or multiplicative hazard rate and non-spillovers or spillovers, which generates four scenarios. The main research questions underlying the analysis will be the following:

- Does the follower invest more than the leader?

- Are firms aggressive to invest in R&D when their rival accumulates knowledge stock?

- Does patent protection encourage firms to innovate?

- How do firms interact during the race?

where different scenarios are compared. The next section will present models in different scenarios; section 3 is going to address the methodology of finding equilibria; in section 4, firms' strategies of R&D investment in different cases are discussed.

2.2 Model

Two firms ($i \in \{1,2\}$) compete to win a patent race, that is, firms compete to be an innovator. To innovate successfully, firms need to invest in R&D and accumulate knowledge stock. When firms invest in R&D, R&D effort u_i is produced at current time t, which will be accumulated as knowledge stock in the next period $t+1$. R&D effort and knowledge stock affect the distribution of success times of innovation. The way that they give impacts on innovation depends on the property between them.

Doraszelski (2003) assumed that R&D effort and knowledge stock are substitutive for innovation; in this paper, I consider them as complementary goods. This changes the form of the hazard rate h_i, which defines as

$$h_i(t) = \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(\tau_i \in [t, t+\Delta t] \big| \min[\tau_i, \tau_j] > t\right)}{\Delta t}$$

where τ_i is the innovation time of firm i. The hazard rate is the conditional probability of innovation given that neither of firms innovates. An additive form of the hazard rate implies that R&D effort and knowledge stock are substitutive for innovation.[1] I introduce a multiplicative form of the hazard rate, which considers the complementarity between R&D effort and knowledge stock.[2]

Another effect of making innovation is spillovers. Since spillovers change the way of knowledge accumulation, which changes the distribution of success times. Suppose knowledge stock is a private good. Doraszelski (2003) assumed that a firm's knowledge is accumulated by its R&D investment; this paper allows its knowledge to be additionally accumulated by spillovers from the opponent. Knowledge accumulation depends on R&D effort as well as spillovers.

By the property between R&D effort and knowledge stock for innovation and the way of knowledge accumulation, I introduce four scenarios of the race: the substitute property without spillovers, the complementary property without spillovers, the substitute property with spillovers, and the complementary property with spillovers. The model of scenario 1 follows Doraszelski (2003) and the other three scenarios are modeled as follows.

---

[1]See scenario 1 in the section 2.1.
[2]See scenario 2 in the section 2.2.

2.2.1   Scenario 1: substitute property without spillovers

According to Doraszelski (2003), the hazard rate of firm i making the first discovery follows the additive function:

$$h_i = \lambda u_i + \gamma z_i, \ \lambda \geq 0, \ \gamma \geq 0, \ i = 1,2$$

where the control variable u_i is current R&D effort of firm i at time t (a simplified notation of $u_i(t)$); the state variable z_i is firm i's knowledge stock at time t (a simplified notation of $z_i(t)$); λ is the effectiveness of current R&D effort; γ is the effectiveness of past R&D effort. By this additive characteristic, innovation without R&D effort is possible.

Further, he assumed there is no spillover when knowledge is accumulated and the state equation is

$$\dot{z}_i = u_i - \delta z_i, \ z_i(0) = z_i^0 \geq 0, \ \delta > 0, \ i = 1,2.$$

Knowledge stock is accumulated by a firm with its own R&D effort and, at the same time, depreciated at rate δ. The parameter δ is a depreciation rate of technology, which captures technological obsolescence (Doraszelski, 2003).

2.2.2   Scenario 2: complementary property without spillovers

In this scenario, all assumptions remain the same with scenario 1 except the form of the hazard rate. The state equation of accumulating knowledge stock without spillovers is

$$\dot{z}_i = u_i - \delta z_i, \ z_i(0) = z_i^0 \geq 0, \ \delta > 0, \ i = 1,2.$$

The hazard rate is changed from additive to multiplicative form. The distribution of success times with the complementary property is

$$h_i = u_i^\lambda z_i^\gamma, \ u_i \geq 0, \ z_i \geq 0, \ i = 1,2.$$

The hazard rate h_i of firm i depends on its current R&D effort and knowledge stock multiplicatively. The intuition of the multiplicative hazard rate is that current R&D effort and knowledge stock are complementary. Firms require both knowledge and R&D effort to innovate. The empirical study of Cohen and Levinthal (1989) observed that firms invest in R&D is to identify, assimilate and exploit knowledge pool. Making use of knowledge for innovation needs R&D investment. This complementary property of knowledge and R&D effort can be modeled by a multiplicative hazard rate, which is different from the case of the additive hazard rate in scenario 1. An additive hazard rate has the characteristic that current R&D effort and knowledge stock is substitutable, which implies that R&D effort is replaceable by knowledge stock for innovation. This assumption allows firms to innovate only with large knowledge stock rather than making R&D effort, which does not hold in scenario 2.

2.2.3   Scenario 3: substitute property with spillovers

In this scenario, all assumptions remain the same with scenario 1 except the state equation of accumulating knowledge stock. The hazard rate keeps an additive form as scenario 1:

$$h_i = \lambda u_i + \gamma z_i, \ \lambda \geq 0, \ \gamma \geq 0, \ i = 1, 2$$

which describes the substitute property between R&D effort and knowledge stock. Firms can accumulate knowledge stock by its current R&D effort and, additionally, spillovers from its opponent. The state equation of knowledge accumulation with spillovers is

$$\dot{z}_i = u_i - \delta z_i + \Phi(u_i) z_j, \ z_i(0) = z_i^0 \geq 0, \ \delta \geq 0, \ i, \ j = 1, 2, \ j \neq i.$$

How much spillovers can be used to build up its own knowledge base depends on its absorptive capacity $\Phi(u_i)$:

$$\Phi(u_i) = \beta u_i^\theta, \ 0 < \theta < 1, \ \beta > 0.$$

Absorptive capacity is the ability to obtain spillovers and this ability depends on R&D effort a firm invests in. Spillover parameter β controls how much percentage of current knowledge stock leaks out from its rival; θ is a parameter of absorptive capacity, which controls the marginal effect of current R&D effort that satisfies $\Phi_{u_i}(u_i) > 0$ and $\Phi_{u_i u_i}(u_i) < 0$. The marginal decreasing R&D effort implies that internalizing knowledge spillovers is diminishing with R&D effort, which follows the assumption of absorptive capacity by Cohen and Levinthal (1989).

2.2.4   Scenario 4: complementary property with spillovers

In the last scenario, the complementary property makes the firm with a small amount of knowledge stock hard to innovate. The distribution of success times with the complementary property follows

$$h_i = u_i^\lambda z_i^\gamma, \ \lambda \geq 0, \ \gamma \geq 0, \ i = 1, 2.$$

The way to increase chances of innovation is to expand knowledge stock by absorbing knowledge spillovers. The state equation of knowledge accumulation with spillovers is

$$\dot{z}_i = u_i - \delta z_i + \Phi(u_i) z_j, \ z_i(0) = z_i^0 \geq 0, \ \delta \geq 0, \ i, \ j = 1, 2, \ j \neq i.$$

Doraszelski (2003) stated that the follower firm has incentives to catch up to the leader in some empirical studies and the patent race model with history dependency gives evidences for this observation. By investing in R&D, making current R&D effort and accumulating knowledge, the follower firm has chances to innovate. In fact, the empirical observation also shows that one of the main reasons firms invest in R&D is to be able to facilitate other's

technology (Tilton, 1971; Allen, 1977; Mowery, 1983). Further, internalizing knowledge needs effort (Cohen & Levinthal, 1989) and making a successful innovation requires both knowledge stock and R&D effort. Therefore, I argued that the chances of a follower catching up to a leader is little since the follower has little knowledge stock in the beginning of the race. The main incentive of a follower investing in R&D is to make use of knowledge spillovers, which increases its chances to innovate.

2.2.5   Hamilton-Jacobi-Bellman (HJB) equation

Firms' optimal strategy of the game is the amount of R&D effort that can maximize their expected discounted innovation profits net of investment costs subject to their state of knowledge accumulation. There are control variables u_i, and state variables z_i, $i \in \{1,2\}$. Firms do not know when they will innovate and face a stochastic transition between non-innovating and innovating. The objective function of firm i over an infinite horizon is the present expected payoff J^i:

$$J^i = \mathbb{E} \left[\int_0^{\min[\tau_i, \tau_j]} e^{-rt}[-C(u_i)]dt + e^{-r\min[\tau_i, \tau_j]} P_i \right]$$

subject to the state equations of knowledge accumulation of two firms where $P_i = \overline{P}$ if firm i wins the race (i.e. $\tau_i < \tau_j$) and $P_i = \underline{P}$ if firm i loses the race (i.e. $\tau_i > \tau_j$); $r > 0$ is the discount rate; $C(u_1) = \frac{1}{2}u_1^2 > 0$ is the R&D investment function.

This game is a type of a piecewise deterministic game by Dockner et al. (2000). The race is deterministic like a general optimal control problem with an infinite horizon when no firm innovates; however, there is a stochastic transition as soon as one firm innovates. Firms face a stochastic transition between modes: innovating successfully, rival innovating successfully, and no one innovating successfully. Denote three modes:

$$mode = \begin{cases} m_1 & \text{if firm 1 wins the patent} \\ m_2 & \text{if firm 1 loses} \\ m_3 & \text{if none of the firms have obtained the patent yet.} \end{cases}$$

As soon as one firm wins the race, the game is over. The firm who loses the race does not allow to continue on, so modes cannot be switched between winning (m_1) and losing (m_2).

Then, each firm has two transition rates by switching between different modes: from pending (m_3) to winning (m_1) as well as from pending (m_3) to losing (m_2). The transition function of modes for firm 1 is

$$q_{m_a m_b} = \begin{cases} h_1 & (a,b) = (3,1) \\ h_2 & (a,b) = (3,2) \end{cases}$$

The transition rate from pending (m_3) to winning (m_1) is the hazard rate of innovation of firm 1. Since there are only two firms in the game, the transition rate of firm 1 from pending (m_3) to losing (m_2) is the hazard rate of firm 2 winning the race,

Thus, the value of the game of firm 1 for three modes is

$$
\begin{cases}
\overline{P} & mode = m_1 \\
\underline{P} & mode = m_2 \\
V^1(z_1, z_2, m_3) & mode = m_3.
\end{cases}
$$

The value of the game for firm 1 is $\overline{P}$ in mode m_1 and $\underline{P}$ in mode m_2. In mode m_3, while the race is pending, the value function of firm 1 depends on two state variables of knowledge stock: $V^1(z_1, z_2, m_3)$.[3]

The aim is to find a Markov-perfect equilibrium, which gives a firm's best strategy of R&D investment with respect to its rival's best strategy. Given the best strategy u_j of firm j, the value function $V^i(z_1, z_2, m_3)$ solves the Hamilton-Jacobi-Bellman (HJB) equation:

$$
rV^i(z_1, z_2, m_3) = \max_{u_i \geq 0} \Bigg[h_i(\overline{P} - V^i(z_1, z_2, m_3)) + h_j(\underline{P} - V^i(z_1, z_2, m_3))
$$
$$
- C(u_i) + V_1^i(z_1, z_2, m_3)\dot{z}_1 + V_2^i(z_1, z_2, m_3)\dot{z}_2 \Bigg]
$$

where $V_i^i(z_1, z_2, m_3) = \frac{\partial}{\partial z_i} V^i(z_1, z_2, m_3)$. If the transversality condition holds for $V^i(z_1, z_2)$, that is,

$$
\limsup_{t \to \infty} e^{-rt} V^i(z_1(t), z_2(t)) \leq 0,
$$

the strategic function $u_1^*(z_1, z_2)$ maximizing the right hand side of the HJB equation is a Markov-perfect equilibrium.

2.2.6 Symmetric Nash equilibria in feedback strategies

The strategic function of firm 1 in different scenarios can be obtained as follows.

- scenario 1

$$
u_1^*(z_1, z_2) = \lambda (\overline{P} - V^1(z_1, z_2)) + V_1^1(z_1, z_2)
$$

  To get more intuition, I simplify the effectiveness of current R&D effort $\lambda = 1$. The optimal R&D effort is $\overline{P} - V^1(z_1, z_2) + V_1^1(z_1, z_2)$. In the scenario of the substitute property without spillovers, firms' optimal strategies of R&D effort consists of two terms: the net payoff of winning the race and the marginal value of the game. The optimal

---

[3] I assume that firm 1 and firm 2 are homogeneous. The transition rate from m_3 to m_1 is h_2 and the value of the game for firm 2 is $\overline{P}$ in mode m_1. The transition rate from m_3 to m_2 is h_1 and the value of the game is $\underline{P}$ in mode m_2. In mode m_3, while the race is pending, the value function of firm 2 depends on two state variables of knowledge stock: $V^2(z_2, z_1, m_3)$

R&D effort is determined by the effective capital gain of winning, $\overline{P} - V^1(z_1, z_2)$, and the marginal capital gain of its knowledge stock, $V_1^1(z_1, z_2)$. There is no direct effect of knowledge stock.

- scenario 2

$$u_1^*(z_1, z_2) = \lambda z_1^\gamma \left(\overline{P} - V^1(z_1, z_2)\right) \left(u_1^*(z_1, z_2)\right)^{\lambda - 1} + V_1^1(z_1, z_2)$$

Assume $\lambda = \gamma = 1$. Then, the optimal R&D effort is $z_1\left(\overline{P} - V^1(z_1, z_2)\right) + V_1^1(z_1, z_2)$. By the complementary property without spillovers, firm 1's knowledge stock z_1 directly affects its optimal R&D effort u_1, which is different from scenario 1. This attributes to complements of R&D effort and knowledge stock.

- Scenario 3

$$u_1^*(z_1, z_2) = \lambda \left(\overline{P} - V^1(z_1, z_2)\right) + V_1^1(z_1, z_2) \left(1 + \frac{\theta z_2}{u_1^*(z_1, z_2)} \Phi(u_1^*(z_1, z_2))\right)$$

Assume $\theta = \lambda = 1$. Then, $\overline{P} - V^1(z_1, z_2) + V_1^1(z_1, z_2)(1 + \beta z_2)$. Optimal R&D effort depends on not only the net payoffs of winning the race but also a marginal value with a markup of spillovers. Compared with scenario 1, the influence of the marginal value is bigger due to the markup $\beta z_2 > 0$, and the magnitude of the markup depends on the opponent's knowledge stock z_2 and the spillovers parameter β. Under the assumption of the absorptive capacity $\Phi = \beta u_i$, the benefit of the marginal value of a firm increases if the marginal absorptive capacity Φ_{u_i} increases. In other words, if firms have the absorptive capacity to assimilate knowledge spillovers, the benefit of the additional value by additional knowledge stock increases so that they have incentives to invest more in R&D under spillovers.

- Scenario 4

$$\begin{aligned} u_1^*(z_1, z_2) = &\lambda z_1^\gamma \left(\overline{P} - V^1(z_1, z_2)\right) \left(u_1^*(z_1, z_2)\right)^{\lambda - 1} \\ &+ V_1^1(z_1, z_2) \left(1 + \frac{\theta z_2}{u_1^*(z_1, z_2)} \Phi(u_1^*(z_1, z_2))\right) \end{aligned}$$

Assume $\theta = \lambda = \gamma = 1$ and the optimal R&D effort simplify as $z_1\left(\overline{P} - V^1(z_1, z_2)\right) + V_1^1(z_1, z_2)(1 + \beta z_2)$. Clearly, the effect of complementary property contributes to the first term, $z_1(\overline{P} - V^1)$, and the effect of spillovers contributes to the second term, $V_1^1(1 + \beta z_2)$. There exists the direct effects of both firm 1 knowledge stock z_1 and its opponents knowledge stock z_2 because of the complementary property and spillovers respectively. When the spillover parameter β increases, the optimal R&D effort increases. The optimal R&D effort increases more if the opponent has a larger knowledge pool. In the case of a

follower with a smaller knowledge pool, the optimal R&D effort depends more highly on spillovers. This explains the reason that firms mainly invest in R&D to assimilate spillovers in many empirical studies (Tilton, 1971; Henderson & Cockburn, 1996). If the leader has a large knowledge stock with spillovers, the follower's best strategy is to invest its absorptive capacity.

Assume firm 1 and firm 2 are homogeneous. The optimal R&D effort of firm 2 in different scenarios are

- Scenario 1

$$u_2^*(z_1, z_2) = \lambda \left(\overline{P} - V^2(z_1, z_2) \right) + V_2^2(z_1, z_2)$$

- Scenario 2

$$u_2^*(z_1, z_2) = \lambda z_2^\gamma \left(\overline{P} - V^2(z_1, z_2) \right) \left(u_2^*(z_1, z_2) \right)^{\lambda - 1} + V_2^2(z_1, z_2)$$

- Scenario 3

$$u_2^*(z_1, z_2) = \lambda \left(\overline{P} - V^2(z_1, z_2) \right) + V_2^2(z_1, z_2) \left(1 + \frac{\theta z_1}{u_2^*(z_1, z_2)} \Phi(u_2^*(z_1, z_2)) \right)$$

- Scenario 4

$$\begin{aligned} u_2^*(z_1, z_2) = & \lambda z_2^\gamma \left(\overline{P} - V^2(z_1, z_2) \right) \left(u_2^*(z_1, z_2) \right)^{\lambda - 1} \\ & + V_2^2(z_1, z_2) \left(1 + \frac{\theta z_1}{u_2^*(z_1, z_2)} \Phi(u_2^*(z_1, z_2)) \right) \end{aligned}$$

The HJB equation of firm 1 with feedback strategies can be written as

$$\begin{aligned} rV^1(z_1, z_2) = & \left[\overline{P} - V^1(z_1, z_2) \right] h_1(u_1^*(z_1, z_2), z_1) + \left[\underline{P} - V^1(z_1, z_2) \right] h_2(u_2^*(z_1, z_2), z_2) \\ & + V_1^1(z_1, z_2) \dot{z}_1(u_1^*(z_1, z_2), z_1, z_2) + V_2^1(z_1, z_2) \dot{z}_2(u_2^*(z_1, z_2), z_1, z_2) \\ & - C(u_1^*(z_1, z_2)). \end{aligned}$$

2.3   Computation

To characterize the Markov-perfect equilibrium, the collocation method by Rui and Miranda (1996) is used. First, assume the state space in a two-dimensional interval $(z_1, z_2) \in [a_1, b_1] \times [a_2, b_2]$ where $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$. The value function is approximated as a Chebychev polynomial of degree n:

$$\hat{V}(z_1, z_2) \approx \mathbf{c} \left[\mathbf{T}_{i-1} \otimes \mathbf{T}_{j-1} \right]$$

$$\mathbf{T}_{i-1} = \begin{pmatrix} T_0(-1 + \frac{2(z_1 - b_1)}{a_1 - b_1}) \\ T_1(-1 + \frac{2(z_1 - b_1)}{a_1 - b_1}) \\ \vdots \\ T_{n-1}(-1 + \frac{2(z_1 - b_1)}{a_1 - b_1}) \end{pmatrix}$$

$$\mathbf{T}_{j-1} = \begin{pmatrix} T_0(-1 + \frac{2(z_2 - b_2)}{a_2 - b_2}) \\ T_1(-1 + \frac{2(z_2 - b_2)}{a_2 - b_2}) \\ \vdots \\ T_{n-1}(-1 + \frac{2(z_2 - b_2)}{a_2 - b_2}) \end{pmatrix}$$

$$T_k(x) = \cos[k \cos^{-1}(x)]$$

I suppose there exists a coefficient matrix $\mathbf{c}_{1 \times ij}$ of a Chebychev polynomial function such that the HJB equation holds on a set of Chebychev nodes

$$\hat{z}_{1i} = \frac{a_1 + b_1}{2} + \frac{b_1 - a_1}{2}(\cos \frac{n - i + 0.5}{n}\pi), i = 1, \cdots, n$$

$$\hat{z}_{2j} = \frac{a_2 + b_2}{2} + \frac{b_2 - a_2}{2}(\cos \frac{n - j + 0.5}{n}\pi), j = 1, \cdots, n$$

To find the coefficient matrix $\mathbf{c}_{1 \times ij}$, the iteration $\mathbf{c}^\tau$ for $\tau = 0, 1, \cdots, \kappa$ is constructed with the HJB equation:

$$[r + \mathbf{h_1} + \mathbf{h_2}]V^\tau(z_1, z_2) - \dot{\mathbf{z}}_1 V_1^\tau(z_1, z_2) - \dot{\mathbf{z}}_2 V_2^\tau(z_1, z_2) = \overline{P}\mathbf{h_1} + \underline{P}\mathbf{h_2} - \mathbf{C}$$

where $\mathbf{h_1}$, $\mathbf{h_2}$, $\dot{\mathbf{z}}_1$, $\dot{\mathbf{z}}_2$ and $\mathbf{C}$ are functions:

$$\mathbf{h_1} = h_1\left(z_1, u^*(z_1, z_2, V^{\tau-1}(z_1, z_2))\right)$$

$$\mathbf{h_2} = h_2\left(z_2, u^*(z_2, z_1, V^{\tau-1}(z_2, z_1))\right)$$

$$\dot{\mathbf{z}}_1 = \dot{z}_1\left(z_1, z_2, u^*(z_1, z_2, V^{\tau-1}(z_1, z_2))\right)$$

$$\dot{\mathbf{z}}_2 = \dot{z}_2\left(z_1, z_2, u^*(z_2, z_1, V^{\tau-1}(z_2, z_1))\right)$$

$$\mathbf{C} = C\left(u^*(z_1, z_2, V^{\tau-1}(z_1, z_2))\right)$$

and

$$V^\tau(z_1, z_2) = \mathbf{c}^\tau[\mathbf{T}_{i-1} \otimes \mathbf{T}_{j-1}]$$

$$V_1^\tau(z_1, z_2) = \mathbf{c}^\tau[\frac{\partial}{\partial z_1}\mathbf{T}_{i-1} \otimes \mathbf{T}_{j-1}]$$

$$V_2^\tau(z_1, z_2) = \mathbf{c}^\tau[\mathbf{T}_{i-1} \otimes \frac{\partial}{\partial z_2}\mathbf{T}_{j-1}]$$

The iteration starts with an initial guess of the value function $V^0(\hat{z}_{1i}, \hat{z}_{2j}) = h_1\overline{P} + h_2\underline{P} - c(u_1)$ where $u_1 = \delta\hat{z}_{1i}$ and $u_2 = \delta\hat{z}_{2j}$ for all Chebychev nodes i, j= 0,1,...,n. Then, $\mathbf{c}^0 = V^0(\hat{z}_{1i}, \hat{z}_{2j})[\mathbf{T}_{i-1} \otimes \mathbf{T}_{j-1}]^{-1}$. The iteration stops when the coefficient matrix $\mathbf{c}^\kappa$ that satisfies the following for all Chebychev nodes i, j= 0,1,...,n has been found:

$$\frac{|\varepsilon_{ij}|}{V^\kappa(\hat{z}_{1i}, \hat{z}_{2j})} < 0.01$$

where

$$\begin{aligned}
\varepsilon_{ij} = &\left(\overline{P} - V^\kappa(\hat{z}_{1i}, \hat{z}_{2j})\right) h_1\left(\hat{z}_{1i}, u^*(\hat{z}_{1i}, \hat{z}_{2j}, V^\kappa(\hat{z}_{1i}, \hat{z}_{2j}))\right) \\
&+ \left(\underline{P} - V^\kappa(\hat{z}_{1i}, \hat{z}_{2j})\right) h_2\left(\hat{z}_{2j}, u^*(\hat{z}_{2j}, \hat{z}_{1i}, V^\kappa(\hat{z}_{2j}, \hat{z}_{1i}))\right) \\
&+ V_1^\kappa(\hat{z}_{1i}, \hat{z}_{2j}) \left[\dot{z}_1\left(\hat{z}_{1i}, \hat{z}_{2j}, u^*(\hat{z}_{1i}, \hat{z}_{2j}, V^\kappa(\hat{z}_{1i}, \hat{z}_{2j}))\right)\right] \\
&+ V_2^\kappa(\hat{z}_{1i}, \hat{z}_{2j}) \left[\dot{z}_2\left(\hat{z}_{1i}, \hat{z}_{2j}, u^*(\hat{z}_{2j}, \hat{z}_{1i}, V^\kappa(\hat{z}_{2j}, \hat{z}_{1i}))\right)\right] \\
&- C\left(u^*(\hat{z}_{1i}, \hat{z}_{2j}, V^\kappa(\hat{z}_{1i}, \hat{z}_{2j}))\right).
\end{aligned}$$

Deviation ε_{ij} is divided by the value function in order to be scale-invariant. When it is small enough, the deviation from the HJB equation is small at nodes. The simulation is running with 100 Chebychev nodes, that is, 10 nodes for each state dimension $n = 10$.

2.3.1 Parameter

Martin (2002) distinguished input and output spillovers. Input spillovers are spillovers that flow between firms during accumulating knowledge stock in the R&D competition, which means that R&D investment takes the firm as well as its rivals closer to success by spillovers. Input spillovers are controlled by spillover parameter β in this model. The spillover parameter determines the percentage of knowledge stock leaking out among firms, and absorptive capacity parameter θ controls the learning ability of absorbing rival's knowledge stock. Here I set $\beta = \theta = 1$ for the benchmark of scenarios with spillovers.

Further, output spillovers are spillovers from the benefits of the new discovery by the successful firm (Martin, 2002). The successful firm, who wins the R&D competition, take advantages of process or/and product innovation due to its new discovery. Its rival needs to buy a license or patents of the new discovery in order to imitate. Higher output spillovers imply that the successful firm benefits less from the patent of its discovery. The output spillovers depend on the degrees of patent protection and can be described as a ratio between $\overline{P}$ and $\underline{P}$. If $\underline{P} = 0$, there is no benefits from imitation, which means patent protection is perfect. If $\overline{P} = \underline{P}$, the benefit from both innovation and imitation are the same, which means there is no patent protection. The benchmark of patent protection is $\underline{P}/\overline{P} = 0.2$ according to the investigation of Henderson and Cockburn (1994, 1996) where the market share of chemical patents was 82% in 1990.

(i) Scenario 1

(ii) Scenario 2

(iii) Scenario 3
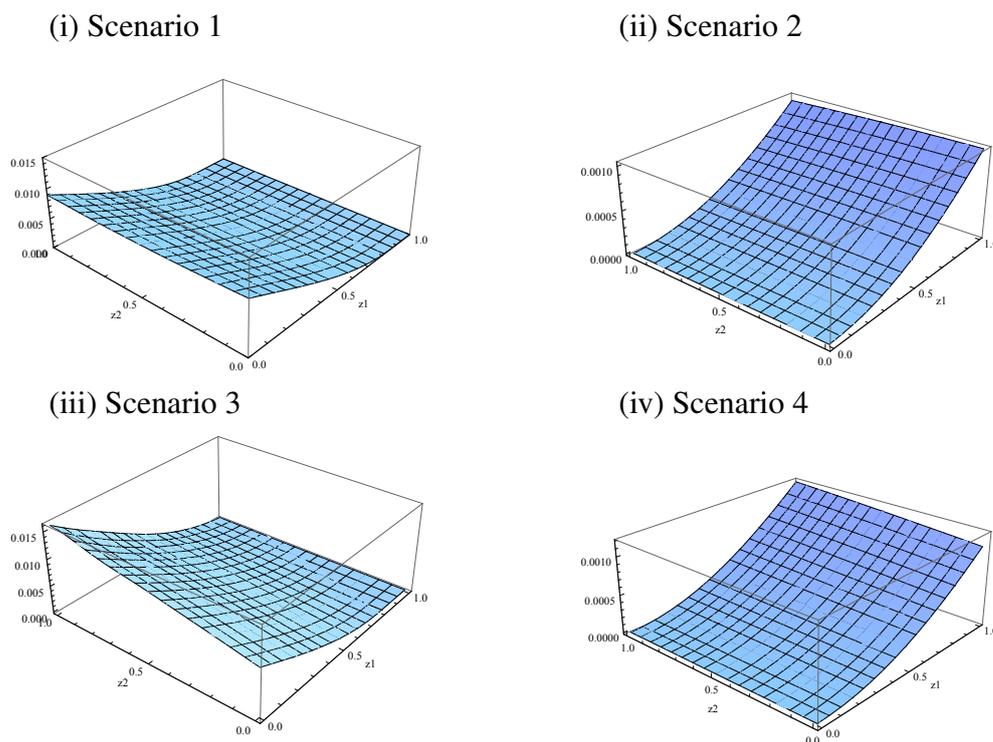
(iv) Scenario 4

Fig. 2.1 R&D investment of firm 1. *Figure (i) is $C(u^*(z_1, z_2))$ in scenario 1 with the additive hazard rate but spillovers; figure (ii) is $C(u^*(z_1, z_2))$ in scenario 2 with the multiplicative hazard rate but spillovers; figure (iii) is $C(u^*(z_1, z_2))$ in scenario 3 with the additive hazard rate and spillovers; figure (iv) is $C(u^*(z_1, z_2))$ in scenario 4 with the multiplicative hazard rate and spillovers. Parameters are $\lambda = \gamma = \theta = 1$, $r = 0.105$, $\delta = 0.2$, $\overline{P} = 0.0435$, $\underline{P} = 0.0087$. The parameter $\beta = 1$ for the scenario of spillovers.*

Besides, the rest of the parameter setting follows Dorazelski (2000) that used several empirical studies of the pharmaceutical chemistry for the parameter setting. Parameter $\delta = 0.2$ is by Henderson & Cockburn (1994, 1996) who found that knowledge stock depreciate 20% over time. The discounted rate is set as 10.5%, i.e., r=0.105 according to Grabowski & Vernon's (1994). The innovating reward is $\overline{P} = 0.0435$, where is calculated given a three-year expected duration of the race in the case of the linear hazard rate by Doraszelski (2003).

2.4 Equilibria and strategies

The equilibrium feedback function of R&D investment is given in figure 2.1 under different scenarios. In the scenarios of the substitute property, firm 1 decreases R&D investment when it has larger knowledge stock; instead, in the scenario of the complementary property, firm 1 increases R&D investment when it has larger knowledge stock. Substitution of R&D effort and knowledge stock reduces firms' incentives to invest with large knowledge stock, since

the chance of innovation depends on both factors additively. In this case, the firm with little knowledge stock also has chances to innovate if it makes high enough R&D effort. This effort is also accumulated as knowledge stock in the next period of time, which contributes to the chances of innovation in the future. Thus, firms invest more when it has little knowledge stock and invest less when it has large knowledge stock. However, in the scenarios of the complementary property, the chance of innovation is little if firms have not enough amount of knowledge stock. Innovation requires both R&D effort and knowledge stock relatively. The firm with large knowledge stock has high chances of innovation as long as it makes enough R&D effort. That is the reason that firms invest more when it has larger knowledge stock.

In the scenarios of spillovers, a slightly increasing of firm 1's R&D investment occurs when its opponent has large knowledge stock. In this case, R&D investment also contributes to absorptive capacity, which additionally increases knowledge accumulation. Firms invest in R&D to accumulate knowledge stock and, at the same time, absorb the opponent's knowledge spillovers. If the opponent's has large knowledge stock, knowledge spillovers increase. The chances of innovation increase if firms can absorb spillovers into its knowledge accumulation by investing in the absorptive capacity. Hence, the incentive to invest in R&D increases when knowledge spillovers are large. Compared to the scenarios without spillovers, a positive relation between R&D investment and opponent's knowledge stock is more significant.

2.4.1   Pure knowledge effect

A patent race with knowledge accumulation can capture the catching-up behavior of the follower firm and the reason is the pure knowledge effect (Doraszelski, 2003). The pure knowledge effect indicates that firms decrease their incentives to invest in R&D when they have large knowledge stock. The chances of innovation by investing in additional R&D effort reduce when knowledge stock becomes large. This effect explains the incentive of the follower firms to catch up to the leader, since decreasing leader's R&D investment increases the follower's chances to innovate relatively. The definition of the pure knowledge effect is given:

$$\frac{\partial}{\partial z_1}(C(u^*(z_1, z_2))) < 0.$$

Since the cost function $C(u_i)$ is monotonic, $\frac{\partial}{\partial z_1}u^*(z_1, z_2) < 0$ holds if the pure knowledge effect is satisfied. In the case of the additive hazard rate, there exists the pure knowledge effect if $V_1(z_1, z_2) > V_{11}(z_1, z_2)$.[4]

In scenario 2, $\frac{\partial}{\partial z_1}u^*(z_1, z_2) = \overline{P} - V(z_1, z_2) - z_1V_1(z_1, z_2) + V_{11}(z_1, z_2))$. Since $\overline{P} - V(z_1, z_2) > 0$, which is a positive net return of innovation, the pure knowledge effect depends on $-z_1V_1(z_1, z_2) + V_{11}(z_1, z_2)$. Compared with scenario 1, the pure knowledge effect in the case of the multiplica-

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[4]In scenario 1 where $\lambda = \gamma = 1$, the optimal R&D effort is $u^*(z_1, z_2) = \overline{P} - V(z_1, z_2) + V_1(z_1, z_2)$. Then, $\frac{\partial}{\partial z_1}u^*(z_1, z_2) = -V_1(z_1, z_2) + V_{11}(z_1, z_2)$.

(i) Scenario 1 & 3   (ii) Scenario 2 & 4
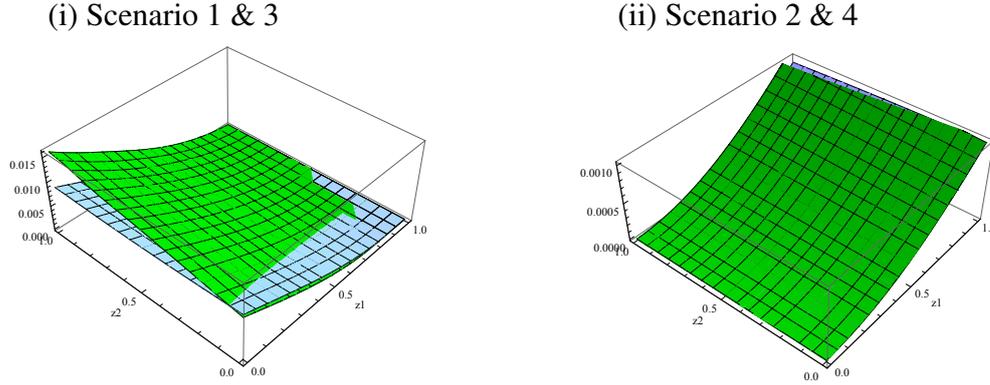
Fig. 2.2 R&D investment of firm 1 under comparison of spillovers and non-spillovers. *Figure (i) is $C(u^*(z_1, z_2))$ under the additive hazard rate; figure (ii) is $C(u^*(z_1, z_2))$ under the multiplicative hazard rate. The green surface is the R&D investment with spillovers and the blue one is without spillovers. Parameters are $\lambda = \gamma = \theta = 1$, $r = 0.105$, $\delta = 0.2$, $\overline{P} = 0.0435$, $\underline{P} = 0.0087$. The parameter $\beta = 1$ for the scenario of spillovers.*

tive hazard rate is harder to reach since $z_1 V_1(z_1, z_2) < V_1(z_1, z_2)$ such that $0 < z_1 < 1$. In fact, figure 2.1 (ii) and (iv) shows no pure knowledge effect in the case of the multiplicative hazard rate. The reason that the pure knowledge effect does not occur in this case is that R&D effort and knowledge stock are complementary. Even though a firm has large knowledge stock, it still needs corresponding amount of R&D effort to innovate, which has no reason to decrease R&D investment as knowledge stock increases.

To see the impact of spillovers on the pure knowledge effect, I check the scenario 3: $\frac{\partial}{\partial z_1} u^*(z_1, z_2) = -V_1(z_1, z_2) + (1 + \beta z_2)V_{11}(z_1, z_2)$. Compared with scenario 1, the additional term of spillovers for the pure knowledge effect is $\beta z_2 V_{11}(z_1, z_2)$. If $V_{11}(z_1, z_2) > 0$, $\frac{\partial}{\partial z_1} u^*(z_1, z_2)$ is larger in the case of spillovers than the case without it. This means that the pure knowledge effect is weakened by spillovers, which also shows in figure 2.1 (iii). The pure knowledge effect in figure 2.1 (iii) is not as strong as it in figure 2.1 (i).

2.4.2 Strategic R&D investment of the leader and the follower

By the pure knowledge effect, the firm with larger knowledge stock invests less than the firm with less knowledge stock. Strategically, the follower firm has an incentive to invest more because it knows that reducing R&D investment of the leader increases its chances to win the race relatively. To exam this in different scenarios, assume firm 1 is a follower, which implies it has less knowledge stock than firm 2, $z_1 < z_2$. The follower investing more in R&D than the leader can be expressed as

$$C(u^*(z_1, z_2)) > C(u^*(z_2, z_1)), \ z_1 < z_2$$

which is called action reaction (Doraszelski, 2003). On the other hand, if firm 1 is the leader and it invests more than the follower to keep its leader dominance, which is called increasing dominance:

$$C(u^*(z_1, z_2)) > C(u^*(z_2, z_1)), \ z_1 > z_2$$

Compared R&D investment of two regions in figure 2.1 (i) and (iii): region 1 where $z_1 < z_2$ and region 2 where $z_1 > z_2$, investment in region 1 is higher than it in region 2 in the case of the additive hazard rate. This shows that firm 1 invests more as a follower than it as a leader if knowledge stock and R&D effort are substitutive. Because of the pure knowledge effect in the scenario of the substitute property, a follower firm invests more when it considers a reducing R&D investment of the leader.

Nevertheless, the case of the multiplicative hazard rate does not have the pure knowledge effect and the R&D investment in region 1 is lower than it in region 2 in figure 2.1 (ii) and (iv). As a follower, it knows the leader will not reduce R&D investment; instead, the leader will invest in the amount of R&D corresponding to knowledge stock due to the complementary property. The follower firm loses its incentives to catch up to the leader, and the leader will keep its dominance by investing in R&D.

In scenario 3, even though spillovers weaken the pure knowledge effect, it still exists with the substitute property. The follower still invests more than the leader under spillovers as it in scenario 1. The property of substitutes between knowledge stock and R&D effort is the main force of action reaction between the leader and the follower. The effect of spillovers is only to increase both leader and follower's R&D investment since $\frac{\partial}{\partial \beta} C(u^*(z_1, z_2)) = u^*(z_1, z_2) z_2 V_1(z_1, z_2) > 0$ if $V_1(z_1, z_2) > 0$. This means that spillovers have a positive effect of R&D investment if firm 1's value of the race is positively correlated to its knowledge stock. Figure 2.2 gives evidence that most R&D investment is slightly greater with spillovers than without spillovers.

2.4.3   Aggressive and submissive responses

Aggressive and submissive responses are defined as follows. A firm increases R&D investment as its rival's knowledge stock increases, which is an aggressive response:

$$\frac{\partial}{\partial z_2} (C(u^*(z_1, z_2))) > 0.$$

A firm decides to invest less as its rival's knowledge stock increases, which is a submissive response:

$$\frac{\partial}{\partial z_2} (C(u^*(z_1, z_2))) < 0.$$

In scenario 1, if firm 1's marginal value with respect to its opponent knowledge stock is relatively small or negative $V_2(z_1, z_2) < V_{12}(z_1, z_2)$, it invests aggressively $\frac{\partial}{\partial z_2} u^*(z_1, z_2) = V_{12}(z_1, z_2) - V_2(z_1, z_2) > 0$. However, in scenario 2, this aggressive response can be weak. If
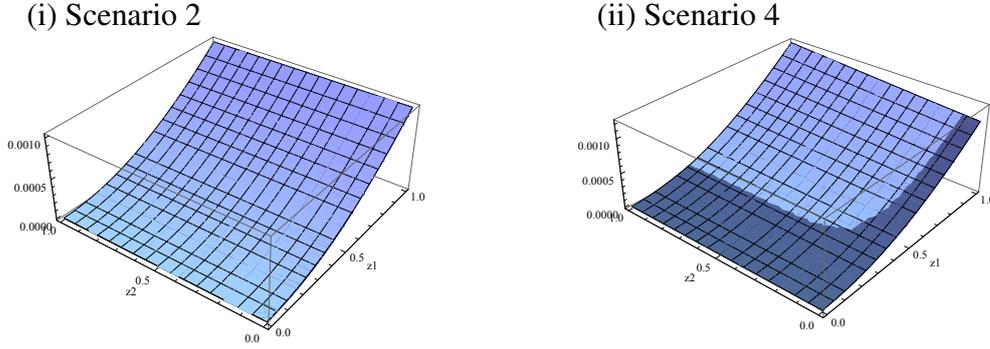
Fig. 2.3 R&D investment of firm 1 under comparison of different degrees of patent protection in the scenario of the complementary property. *Figure (i) is $C(u^*(z_1, z_2))$ without spillovers; figure (ii) is $C(u^*(z_1, z_2))$ with spillovers. The dark blue surface is R&D investment without patent protection $\frac{P}{\overline{P}} = 1$ and the light blue one is with perfect patent protection $\frac{P}{\overline{P}} = 0$. Parameters are $\lambda = \gamma = \theta = 1$, $r = 0.105$, $\delta = 0.2$, $\overline{P} = 0.0435$. The parameter $\beta = 1$ for the scenario of spillovers.*

$V_2(z_1, z_2) < 0$, $\frac{\partial}{\partial z_2} u^*(z_1, z_2) = V_{12}(z_1, z_2) - z_1 V_2(z_1, z_2) < V_{12}(z_1, z_2) - V_2(z_1, z_2)$. This implies firm 1 under the complementary property is not aggressive as it under the substitute property when its value of the race is negatively correlated to the opponent's knowledge stock.

If there are spillovers, aggressive behaviors are more intense. The response in scenario 3 follows $\frac{\partial}{\partial z_2} u^*(z_1, z_2) = V_{12}(z_1, z_2) - V_2(z_1, z_2) + \beta(z_2 V_{12}(z_1, z_2) + V_1(z_1, z_2))$ and the additional term is $\beta(z_2 V_{12}(z_1, z_2) + V_1(z_1, z_2))$ due to spillovers. If $V_1(z_1, z_2) > 0$ and $V_{12}(z_1, z_2) > 0$, then $\beta(z_2 V_{12}(z_1, z_2) + V_1(z_1, z_2)) > 0$. This explains that a firm chooses to invest aggressively in order to benefit from spillovers if its additional knowledge stock can increase its value.

2.4.4   Degree of patent protection

Here I define the degree of patent protection by the ratio $\frac{P}{\overline{P}}$. A lower ratio means stronger patent protection, that is, the reward of losing the race (imitator's reward) is relatively less than the reward of winning the race (innovator's reward). Then, perfect patent protection means zero imitator's reward $\frac{P}{\overline{P}} = 0$, and no patent protection means the same innovator and imitator's rewards $\frac{P}{\overline{P}} = 1$. Perfect patent protection brings a positive externality on R&D investment since the losing firm gets 0 imitator's reward. Firms are likely to invest in R&D, to win the race, and to get the innovator's reward. In contrast, firms lose their incentives to invest and innovate if a losing firm can get the same reward as a winning firm. A negative externality of R&D investment occurs when there is no patent protection.

However, in the case of the multiplicative hazard rate, the positive externality under perfect patent protection is not significant, because R&D investment is not sufficient to bring a firm to win the innovator's reward. The complementary property requires both R&D effort and
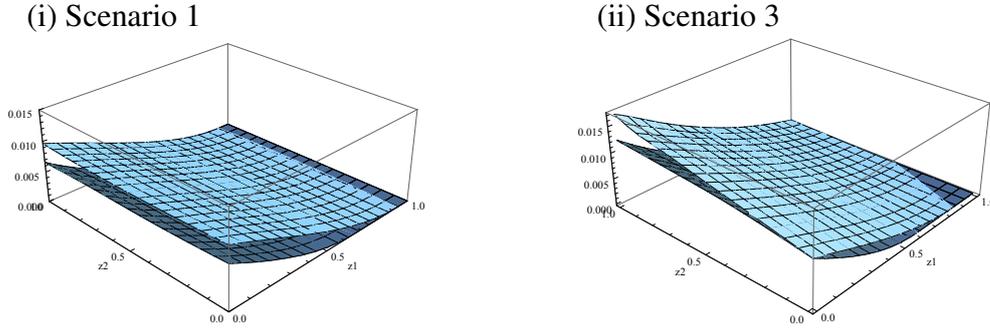
Fig. 2.4 R&D investment of firm 1 under comparison of different degrees of patent protection in the scenario of the substitute property. *Figure (i) is $C(u^*(z_1, z_2))$ without spillovers; figure (ii) is $C(u^*(z_1, z_2))$ with spillovers. The dark blue surface is R&D investment without patent protection $\frac{P}{\overline{P}} = 1$ and the light blue one is with perfect patent protection $\frac{P}{\overline{P}} = 0$. Parameters are $\lambda = \gamma = \theta = 1$, $r = 0.105$, $\delta = 0.2$, $\overline{P} = 0.0435$. The parameter $\beta = 1$ for the scenario of spillovers.*

knowledge stock for innovation. In this situation, patent protection does not bring significant externality to improve R&D investment. In fact, a firm's marginal value function with respect to its rival's knowledge stock does not change significantly and the difference of the strategic investment function between perfect patent protection and no patent protection is very little (see figure 2.3).

Consider a case of the additive hazard rate with spillovers. Spillovers give a positive externality of R&D investment especially for the follower. Since the leader is forced to share its knowledge pool through spillovers, the chances of the follower to catch up to the leader increase relatively. Further, spillovers reinforce the positive externality of R&D investment under patent protection. When the imitator reward is little and the chances of catching up to the leader increases, the follower best strategy is to invest aggressively. Figure 2.4 shows the action reaction is more significant under spillovers and patent protection.

2.4.5   Expected innovation time

To give a close look of the race and understand the impact on innovation time by different scenarios, I first introduce expected innovation time. According to Polasky et al. (2011), the probability of innovation before time t can be calculated by the hazard rate $h(t)$:

$$\mathbb{P}(\tau < t) = 1 - \mathrm{e}^{-\int_0^t h(s)ds}$$
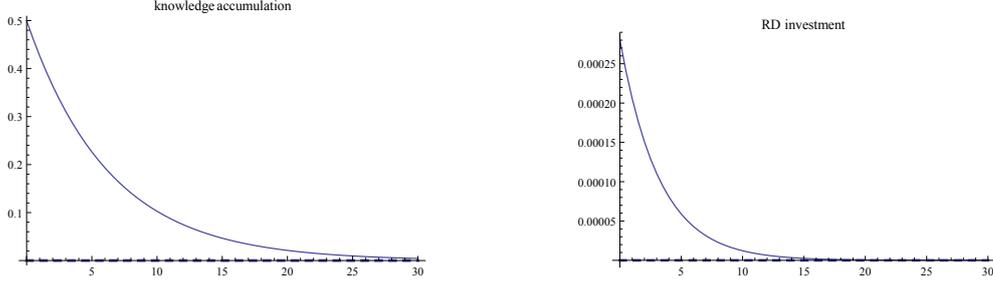
Fig. 2.5 Time path of knowledge accumulation and R&D investment in the scenario of the complementary property without spillovers. *Solid curves represent a leading firm; dashed curves represent a follower firm. Parameters are* $\lambda = \gamma = \theta = 1$, $r = 0.105$, $\delta = 0.2$, $\overline{P} = 0.0435$, $\underline{P} = 0.0087$, $z_1(0) = 0.5$, $z_2(0) = 0$.
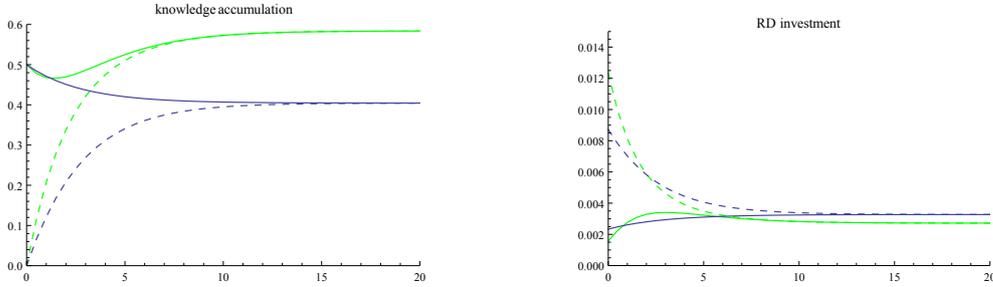
Fig. 2.6 Time path of knowledge accumulation, R&D investment and expected innovation time in the scenario of the substitute property. *Blue curves are the case without spillovers; green curves are the case with spillovers. Solid curves represent a leading firm; dashed curves represent a follower firm. Parameters are* $\lambda = \gamma = \theta = 1$, $r = 0.105$, $\delta = 0.2$, $\overline{P} = 0.0435$, $\underline{P} = 0.0087$, $z_1(0) = 0.5$, $z_2(0) = 0$. *The parameter* $\beta = 1$ *for the case with spillovers.*

The expected innovation time τ that evaluates the expected occurrence of innovation from time t_i to t_{i+1}, which follows

$$
\begin{aligned}
\tau &\approx \sum t_i[\mathbb{P}(\text{innovation before } t_{i+1}) - \mathbb{P}(\text{innovation before } t_i)] \\
&\approx \sum t_i[1 - e^{-\int_0^{t_{i+1}} h(s)ds} - (1 - e^{-\int_0^{t_i} h(s)ds})] \\
&\approx \int_0^\infty t h(t) e^{-\int_0^t h(s)ds} dt.
\end{aligned}
$$

The expected innovation time is calculated numerically under firms' optimal strategies $u^*(z_1(t), z_2(t))$ given initial knowledge stock $z_i(0)$. I obtain the steady state of knowledge stock by numerically solving two state dynamics: $\dot{z}_i = 0, i \in \{1, 2\}$.

I find that, in the case of multiplicative hazard rate, the leader and follower's R&D investment is close to 0 and also knowledge stock approaches to 0 at the steady state (see figure 2.5). The reason is that firms play an one-shot game under complementary property. The
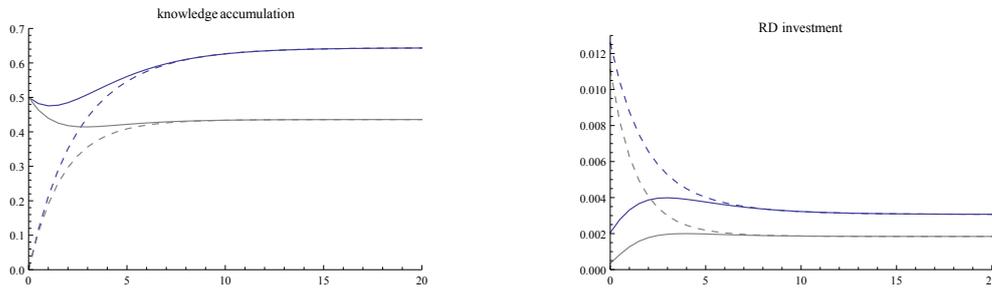
Fig. 2.7 Time path of knowledge accumulation, R&D investment and expected innovation time with different degrees of patent protection. *Blue curves are the case with patent protection $\frac{P}{\overline{P}} = 0$; gray curves are the case without patent protection $\frac{P}{\overline{P}} = 1$. Solid curves represent a leading firm; dashed curves represent a follower firm. Parameters are $\lambda = \gamma = \theta = \beta = 1$, $r = 0.105$, $\delta = 0.2$, $\overline{P} = 0.0435$, $z_1(0) = 0.5$, $z_2(0) = 0$.*

complements of knowledge stock and R&D effort make the follower hard to catch up to the leader, so the follower is not willing to increase investment from the beginning to the end of the race. The leader invests at the beginning of the race and decreases its investment along with the decreasing knowledge stock in the long run. Since knowledge stock depreciated overtime, the biggest chance of innovation is at the beginning of the race. If firms cannot innovate at the beginning, the chances of innovation become inferior over time. Thus, the steady state of R&D investment and knowledge stock are almost 0 and the chances of innovation is little in the long run. The probability of innovation before an infinitive time is only 3.84%. If there are spillovers in the case of the complementary property, the probability of innovation before an infinitive time only increases 0.001%.

Spillovers become a more significant factor that affects expected innovation time in the scenarios of the substitute property. One unit of R&D effort contributes to chances of innovation and knowledge accumulation that additionally increases the chances of innovation. If there are spillovers, one unit of R&D effort also contributes to the absorptive capacity that additionally increases the chances of innovation by absorbing spillovers into knowledge accumulation, which brings a positive effect for the follower and the leader loses its advantage relatively. Figure 2.6 shows that the leading firm with positive initial knowledge stock loses its incentives to invest in R&D under spillovers. The more knowledge the leader accumulates, the more knowledge spills out to its opponent. In contrast, the follower firm enters the race with 0 stock, and invests more with spillovers than without spillovers in the beginning of the race. Apparently, spillovers are attractive to the follower and not appealing to the leader at the start point of the race. The follower invests a lot and then sharply decreases its investment during the race until a lower level of investment is reached; the leader invests little initially, then sharply increases and then decreases investing. This overshooting investment during the beginning period attributes to spillovers. In the long run, knowledge accumulation reaches a higher level and R&D investment reaches a lower level under spillovers, which brings a smaller expected

innovation time.[5] Namely, a shorter innovation time attributes to knowledge accumulation of spillovers.

Since spillovers result in a lower expected innovation time in the long run, does patent protection change the effect of spillovers? Compare two cases in the scenario with spillovers: perfect patent protection and no patent protection. Both leading and follower firms invest more with patent protection at the start point of the race and in the long run as well (see figure 2.7). Since R&D investment and knowledge accumulation eventually reach a higher level with patent protection, the expected innovation time is less.[6] The reason is that perfect patent protection ensures 0 reward for a losing firm so that firms are motivated to invest and innovate.

2.5   Conclusions

I use the patent race model of knowledge accumulation, which considers history dependency of R&D investment on innovation. Firms make R&D effort by investing and this effort is accumulated as knowledge stock in the next period of time. The hazard rate of innovation depends on firm's knowledge stock that has been accumulated previously and R&D effort that is made in the current time. The follower has incentives to invest more than the leader if R&D effort and knowledge stock are substitutive in innovation. Because of the pure knowledge effect where the leader reduces its investment along with the large amount of knowledge stock, the follower strategically increases its investment, which increases its chances of innovation. However, the pure knowledge effect does not exist when R&D effort and knowledge stock are complementary. The follower does not have corresponding knowledge stock for innovation and the chance of innovation of the leader is relatively high. Thus, the follower loses its incentive to invest, and the leader invests more to keep its dominance in the patent race. The different types of knowledge and R&D effort and the relation between them causes different technology strategies of the leader and the follower. This result corresponds to the empirical study by Kylaheiko et al. (2011). In the high-tech industries, the follower firms are likely to focus R&D on substitutable knowledge assets of the leading firm. For example, in 1989, Qualcomm did not invest in R&D that is complementary with knowledge assets of 2G mobile phone, which has been developed by the leading firm such as Nokia or Mototrola. It turned to make R&D effort on Code Division Multiple Access (CDMA) of 3G mobile phone, which has substitutive property with original knowledge, and eventually innovates successfully.

Except for the different strategies of R&D investment corresponding to different types of knowledge and R&D effort, another strategic consideration of R&D investment refers to knowledge spillovers. Several empirical studies of high-tech industries such as semiconductors

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[5]The expected innovation time is 1.8648 in the scenario 1 and 1.787 in the scenario 3. The calculation follows the same parameter setting in figure 2.6.

[6]The expected innovation time is 1.7306 with perfect patent protection and 2.02 without patent protection. The calculation follows the same parameter settings in figure 2.7.

and pharmaceuticals show that firms' incentives to invest in R&D is to learn from opponents (Tilton, 1971; Henderson & Cockburn, 1996). To support this, the absorptive capacity is included in knowledge accumulation in the model so that firms' R&D investment additionally contributes to absorbing spillovers, which indirectly increases the chances of innovation. This factor is insignificant if R&D effort is complementary with knowledge stock. Nevertheless, in the substitute property, spillovers give a positive effect on R&D investment of the follower in the beginning of the race. Even though the leader loses its incentive to invest initially, it overinvests in the later on periods due to spillovers. In fact, both firms invest less with spillovers than without them in the long run, but knowledge stock reaches a higher level eventually, which decreases the expected time of innovation. To increase the long-term level of R&D investment, a perfect patent protection can encourage firms to invest. Since spillovers exist and patent protection ensures the innovator's reward, the high level of both R&D investment and knowledge stock leads to a shorter innovation time.

Overall, whether the follower invests more than the leader is associated with the pure knowledge effect, which is decided by the characteristics of R&D effort and knowledge stock. The pure knowledge effect does not appear under the complementary property, which results in increasing dominance. In addition, spillovers encourage firms to invest aggressively in R&D against rival's knowledge stock. At the same time, there should be a corresponding patent to protect the innovator's reward so that firms will actively invest in R&D rather than passively using spillovers for innovation. Patent protection that ensures the innovator's reward is an effective policy tool to increase the level of R&D investment in the long run and shorten the innovation time.

Chapter 3

A patent race with knowledge accumulation and product market

3.1 Introduction

In the technology intensive industry, firms invest in R&D for innovation. After innovating, an innovator applies for patents to protect its innovation. Since opponents cannot infringe patents, they are not allowed to imitate the innovator's technique to produce goods in the market, which usually drives opponents out of the market. The innovator becomes a monopolist, which reduces the social welfare. In fact, patents are a mechanism to protect the innovator, which increases firms' incentive to innovate; however, patents also indirectly ensure the innovator's market power, which increases the social welfare loss. This paper discusses the social welfare of patent policy by building a model connecting two different research lines: (i) a patent race and (ii) process innovation.

Firms face a patent race to be the first innovator. This research line discusses innovation incentives in a patent race (Reinganum, 1981; Fudenberg et al.,1983; Harris and Vickers, 1987). Firms compete to be the first innovator that can benefit from an exogenous patent reward. They make current R&D effort by investing in R&D for innovation in one period of time, which, nevertheless, does not contribute to the next period of innovation. Later on, Doraszelski (2003) introduced knowledge accumulation so that the current R&D efforts can be accumulated as knowledge stock that can also contribute to chances of innovation in the following periods. These models of the patent race are too simplified on the patent policy, which only features on exogenous patent benefits rather than patent length and strength. This paper includes two factors, patent length and patent strength, in the model, and introduces ex-post patents to evaluate the value of patents.

It has been discussed for a long time that which patent policy can optimize the trade-off between the social welfare loss and the innovation incentives. Gilbert and Shapiro (1990) proposed a general static model of patent benefits and concluded that the optimal length of

patents is infinite by maximizing the social welfare. However, their model did not capture competition among firms. Kamien and Schwartz (1974) argued that longer patents result in a more intensive rivalry, which affects firms' expected benefits of patents. DeBrock (1985) included the feature of rivalry by modeling ex-post patents with two stages, which captures firms' expected patent benefits better. In this ex-post patent system, firms invest in R&D optimally according to their expected payoff of innovation after patenting. Kitch (1977) also showed empirically that firms develops innovation after evaluating their expected payoff of innovation. This paper uses a differential game approach to capture better feature of ex-post patents in continuous time.

After a patent race, firms evaluate their payoff of innovation and decide the optimal strategies of R&D investment. They develop process innovation by investing in R&D to reduces marginal cost of production. Cellini and Lambertini (2009) discussed process innovation of rivalry, which follows D'Aspremont and Jacquemin (1988) with a differential game approach and calculated the social welfare of innovation in a dynamic system. They concluded that R&D cartel preforms better on social welfare than independent ventures since R&D cooperation generates lower production cost by more efficient R&D than non-cooperation. This result corresponds to Kamien et al. (1992) static R&D model, which shows that research joint ventures (RJV) cartel performs better on welfare than RJV competition by yielding higher profits and a lower price in the market. Instead of comparing cartels and competition, another dynamic model of process innovation by Hinloopen et al. (2013) focused more on firms' incentive to innovate in a view of the choke price. In this line of research, patent policy is not captured. This paper include the patent policy in the process innovation by connecting with a patent race model. Except for forming cartels to boost innovation, Judd et al. (2012) maintained that a policy maker can use a patent system as a tool to increases incentives of innovation. This paper compares social welfare between cartels and patenting to see which tool generates better welfare in a dynamic framework.

In fact, firms face both research and market competition when they want to innovate. They first invest on knowledge R&D to innovate in a patent race and then invest on process R&D to reduce production cost in a market competition. However, there is not many research combining a patent race and a market competition. Steinmetz (2010) connected a patent race and the market competition in a two-stage model that follows a patent race model of Doraszelski (2003) in the first stage and Bertrand competition in the second stage. Since Cournot competition is more efficient than Bertrand competition in a dynamic model of R&D under the conditions of productive R&D, highly substitutable goods and R&D spillovers (Breton et al., 2004), this paper considers a Cournot model that follows the dynamic process innovation as Cellini and Lambertini (2009) for the later stage.

The objective of the research is to know the optimal innovation strategy of firms under different patent policies and the optimal patent policy with the maximal social welfare. The model is a three-stage dynamic model with a differential game approach. In the first stage, two
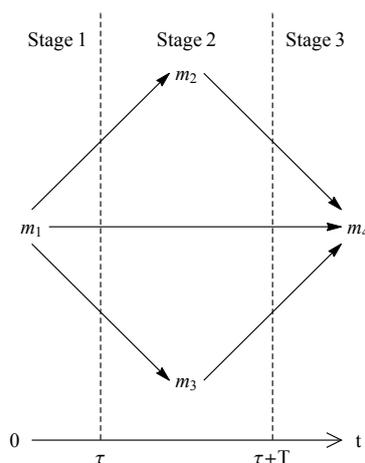
Fig. 3.1 Transition among different phases in the three-stage game overtime

firms invest in R&D, accumulate knowledge and compete in a patent race. In the second stage, the successfully innovating firm monopolizes the market until the patent expires. After the patent expires the market becomes duopoly, which is in the final stage. Monopoly does not always occur if the strength of a patent system is extremely weak. The three-stage game of two players will be solved by backward induction numerically.

3.2   Model

Assume two firms compete in a patent race and then compete in a product market. In a patent race, both firms invest in R&D effort that is accumulated as knowledge stock over time in order to innovate. After the race is over, the innovator can enter the product market and reduce its marginal costs by process innovation. Further, the innovator earns a patent, which drives the imitator out of the market. The length of the patent determines how long the innovator can monopolize the market; after the patent expires, the imitator enters the market. The imitator can imitate innovation without additional cost, which allows it to produce efficiently, that is, the innovator and the imitator compete with the same initial marginal cost in a duopoly market. This can be modeled as a three-stage differential game: (1) the first stage where both firms do not innovate yet, that is, both firms are in a patent race, (2) the second stage where one firm innovates and monopolizes the product market due to patent protection, and (3) the final stage where the imitator enters and the product market becomes duopoly after the patent expires. The

status of the firm follows four modes in different stages:

$$
mode = \begin{cases}
m_1 & \text{both firms compete in a patent race} \\
m_2 & \text{firm 1 wins the patent race and monopolize the product market} \\
& \text{by process innovation} \\
m_3 & \text{firm 1 loses the patent race and cannot enter the product market} \\
m_4 & \text{both firms compete in a duopoly market.}
\end{cases}
$$

Figure 3.1 shows the transition between different phases in the three-stage game. From stage 1 to stage 2, if the firm innovates and obtains the patent at time *tau* in a patent race, the firm status goes from mode 1 to mode 2; otherwise, the status goes from mode 1 to mode 3. After the patent expires at time T, the firm's status ends up at mode 4 in stage 3. It is also possible that the innovator does not have chances to monopolize the market due to innovation diffusion. If the strength of patents is weak, the imitator can easily enter the product market right after the patent race. In this case, the game goes from stage 1 (m_1) to stage 3 (m_4) directly.

Since I use the backward induction to solve the three-stage differential game, I will introduce the model of the third stage first, then second stage and the first stage in the last.

3.2.1 Third stage: process innovation in duopoly (m_4)

R&D effort of process innovation plays a role of reducing marginal cost in order to produce homogeneous goods. Assume that firms compete in Cournot competition. The reason to model process innovation by Cournot competition is that Cournot competition is more efficient with productive R&D and highly substitutable goods (Breton et al., 2004). The value function of firm 1 at m_4 is

$$
V_{m_4}^1(c_1, c_2) = \max_{q_1, k_1} \int_{\tau+T}^{\infty} e^{-\rho t} \left[(1 - q_1 - q_2 - c_1)q_1 - \frac{1}{2}k_1^2 \right] dt
$$

subject to

$$
\dot{c}_1 = (-k_1 + \delta_c)c_1
$$
$$
\dot{c}_2 = (-k_2 + \delta_c)c_2
$$

where $q_i = q_i(t)$, $k_i = k_i(t)$ and $c_i = c_i(t)$ are the quantity, R&D effort and marginal cost of firm i respectively; δ_c is a constant depreciation rate measuring decreasing productivity; ρ is the discounted rate; T is the length of the patent; τ is the time at which the patent race is over. The market becomes duopoly after the patent race is over and the patent expires, that is, the market competition begins from time $\tau + T$. The firm's marginal cost is reduced by R&D investment overtime, and the firm produces its optimal quantity to maximize its present value of production profits net off production cost and R&D cost over time.

This differential game of duopoly with process innovation has a normalized demand function and no spillovers, which is a simplified model of Cellini and Lambertini (2009). Instead of looking for open-loop strategies as Cellini and Lambertini (2009), feedback strategies are more appropriate since they are subgame perfect where firms are time-consistent. The Hamilton-Jacobi-Bellman (HJB) equation of firm 1 is

$$\rho V_{m_4}^1 = \max_{q_1,k_1}(1 - q_1 - q_2 - c_1)q_1 - \frac{1}{2}k_1^2 + \dot{c}_1 \frac{\partial V_{m_4}^1}{\partial c_1} + \dot{c}_2 \frac{\partial V_{m_4}^1}{\partial c_2}.$$

where $V_{m_4}^i(c_1, c_2)$ satisfies the transversality condition, that is,

$$\limsup_{t \to \infty} e^{-\rho t} V_{m_4}^i(c_1(t), c_2(t)) \leq 0, \ i = 1, 2.$$

Thus, the feedback strategies of symmetric firms are

$$q_{im_4}^\star = \frac{1 - 2c_i + c_j}{3}, \ i = 1, 2$$

$$k_{im_4}^\star = -c_i \frac{\partial V_{m_4}^i}{\partial c_i}, \ i = 1, 2$$

where $1 > c_i > 0$, $q_{im_4}^\star > 0$ and $k_{im_4}^\star > 0$. This implies $\frac{\partial V_{m_4}^i}{\partial c_i} < 0$. The firm's optimal R&D effort is positive if and only if the value of the race decreases as the marginal cost increases. Further, firms produce with positive profits in the duopoly market, that is, $(1 - q_i - q_j - c_i)q_i - \frac{1}{2}k_i^2 > 0$ if

$$c_i < \frac{1 + c_j}{4 + 3\sqrt{2}\frac{\partial V_{m_4}^i}{\partial c_i}}.$$

Then, duopoly holds when $\frac{-2\sqrt{2}}{3} < \frac{\partial V_{m_4}^i}{\partial c_i} < 0$ and $0 < c_i < \frac{1}{2}$.[1]

The feedback strategies of the quantity are the same in the dynamic model and the static model of Cournot competition. Firm's optimal quantity of production increases as its marginal cost decreases. If its opponent production is more marginally costly, the firm strategically produces more because of reducing production of its opponent. The feedback strategies of R&D investment positively depends on the marginal cost and the marginal value with respect to marginal cost. If a firm has less production efficiency than another firm, its optimal R&D investment is higher.

---

[1]Since firms are symmetric, I restrict to the state space where firms are allowed to perform symmetrically. Both firms have positive production in the state space $\{c_1, c_2\} \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$.

3.2.2 Second stage: process innovation in monopoly (m_2) or out of the market (m_3)

In the second stage, the patent race is over at time τ and the innovator holds a patent during the time period of a valid patent, T. If firm 1 innovates successfully (m_2), it obtains a patent and monopolizes the product market. Its objective is

$$V^1_{m_2}(c_1,t;\tau) = \max_{q_1,k_1} \int_t^{\tau+T} e^{-\rho(\tilde{t}-\tau)} \left[(1-q_1-c_1)q_1 - \frac{1}{2}k_1^2\right] d\tilde{t} + e^{-\rho(\tau+T-t)}V^1_{m_4}(c_1,c_2), \ t \geq \tau$$

subject to

$$\dot{c}_1 = (-k_1+\delta_c)c_1.$$

The patent is expired at time $\tau+T$ and the imitator, firm 2, enters the market. Assume that there is no cost of imitation so that firm 2 enters the market with the same marginal cost as firm 1. At the transition time $\tau+T$, the terminal value of the race at m_2 and the initial value of the race at m_4 are the same given that the imitator's initial cost at m_4 is the same with the terminal cost of innovator at m_2:

$$V^1_{m_2}(c_1,\tau+T) = V^1_{m_4}(c_1,c_2^{ini}), \ c_2^{ini} = c_1.$$

The HJB equation at m_2 is

$$\rho V^1_{m_2} - \frac{\partial V^1_{m_2}}{\partial t} = \max_{q_1,k_1}[(1-q_1-c_1)q_1 - \frac{1}{2}k_1^2 + \dot{c}_1\frac{\partial V^1_{m_2}}{\partial c_1}]$$

such that

$$V^1_{m_2}(c_1,\tau+T) = V^1_{m_4}(c_1,c_2^{ini}), \ c_2^{ini} = c_1.$$

The feedback strategies are

$$q^\star_{1m_2} = \frac{1-c_1}{2}, \ k^\star_{1m_2} = -c_1\frac{\partial V^1_{m_2}}{\partial c_1}.$$

where $q^\star_{1m_2} > 0$ and $k^\star_{1m_2} > 0$. The optimal R&D effort depends on the state of marginal cost and its marginal value with respect to the marginal cost, which is the same with the duopoly case. The marginal value of the race must decrease with the increasing marginal cost so that the monopoly market with process innovation can sustain. If the marginal cost brings more marginal value of monopoly than duopoly, optimal R&D effort in monopoly is larger than it in duopoly. Further, the firm only produces under a positive profit, that is, $(1-q_1-c_1)q_1 - \frac{1}{2}k_1^2 > 0$ if

$$c_i < \frac{1}{1+\sqrt{2}\frac{\partial V^1_{m_2}}{\partial c_1}}.$$

Then, monopoly only holds when $\frac{-1}{\sqrt{2}} < \frac{\partial V_{m_2}^1}{\partial c_1} < 0$.

In contrast, if the innovator and the patent holder is firm 2, firm 1 cannot infringe the patent, which means that firm 1 cannot offer the same good and drops out of the market (m_3). According to firm 2's state, $\dot{c}_2 = (-k_2 + \delta_c)c_2$, and the patent length, firm 1 faces its value function $V_{m_3}^1(c_2, t)$ where $V_{m_3}^1(c_2(t), T) = V_{m_4}^1(c_1(0), c_2(t))$. The terminal value of m_2 and m_4 should be the same at the transition time $\tau + T$. Assume no imitation cost so that firm 1 enters the market with the same marginal cost as the innovator, $c_1(0) = c_2(T)$, after the patent is expired at time $\tau + T$. The HJB equation at m_3 is

$$\rho V_{m_3}^1 - \frac{\partial V_{m_3}^1}{\partial t} = \dot{c}_2 \frac{\partial V_{m_3}^1}{\partial c_2}$$

such that

$$V_{m_3}^1(c_2, \tau + T) = V_{m_4}^1(c_1^{ini}, c_2), \ c_1^{ini} = c_2$$

There is no feedback strategies since firm 1 is out of the market.

3.2.3   First stage: a patent race (m_1)

In a patent race, R&D effort plays a role of innovating. The function of R&D in this stage is different from it in the second and third stages. R&D effort of first stage contributes to make innovation occur; R&D effort of second and third stages contributes to production efficiency after innovation. In this stage, firms invest in R&D to accumulate knowledge, which increases chances of innovation. The race is over as long as one firm innovates. Firms face a stochastic process of innovation and the transition rate of changing the status of a firm, which is the switching rate among different mode, is

$$\zeta_{m_a m_b}^1 = \begin{cases} h_1 \kappa & (a,b) = (1,2) \\ h_2 \kappa & (a,b) = (1,3) \\ h_1(1-\kappa) + h_2(1-\kappa) & (a,b) = (1,4) \end{cases}$$

where $h_i = h_i(t)$ is the distribution of successful innovation of firm i, which is a hazard rate that a firm innovates successfully in a small time interval at time t; κ is the strength of a patent. The chances of switching the firm's status from an innovator to a monopolist depends on the hazard rate h_i and the strength of a patent κ. If a firm innovates successfully with a rate h_1, the probability of a patent holder being a monopolist depends on the patent strength $\kappa \in [0,1]$. It is also possible that the market becomes duopoly where the imitator enters with a probability $1 - \kappa$. A higher κ implies a stronger patent, which indicates that the innovator has higher chances to monopolize the market. The transition rate $\zeta_{m_1 m_2}^1$ of firm 1 from innovating in a patent race to being a monopolist is $h_1 \kappa$; the transition rate $\zeta_{m_1 m_3}^1$ of firm 1 losing the patent race is the transition rate of firm 2 innovating in the patent race and being a monopolist, $h_2 \kappa$.

The probability of firm i innovating and a patent failing to protect innovation is $h_i(1-\kappa)$, so the transition rate $\zeta^1_{m_1 m_4}$ is $\sum^2_{i=1}(1-\kappa)h_i$, which indicates the probability of either one firm innovating and facing duopoly from a patent race phase to the market phase.

The hazard rate of successful innovation depends on firm's R&D effort $k_i(t)$ and its knowledge stock $z_i(t)$. The form of the hazard rate determines whether the follower firms have incentives to innovate (see chapter 2). To capture whether the patent policy affects incentives of the follower firm , I consider an additive hazard rate, $h_i = k_i + z_i$, where the follower firm has the incentive to catch up to the leader. The additive hazard rate captures the feature of substitution between R&D effort and knowledge so that the follower with less knowledge can increases chances of innovation by investing in sufficient R&D.

This stochastic mode-switching model of the patent race is a piecewise deterministic game where the game is deterministic during the race and is stochastic when the status of firms changes in the end of the race (Dockner et al., 2000). The HJB equation of the piecewise deterministic game of the patent race of firm 1 is

$$\rho V^1_{m_1} = \max_{k_1} \left[h_1 \kappa \left(V^1_{m_2} - V^1_{m_1} \right) + h_2 \kappa \left(V^1_{m_3} - V^1_{m_1} \right) + (h_1 + h_2)(1-\kappa)\left(V^1_{m_4} - V^1_{m_1} \right) - k^2_1 \right.$$
$$\left. + \dot{z}_1 \frac{\partial V^1_{m_1}}{\partial z_1} + \dot{z}_2 \frac{\partial V^1_{m_1}}{\partial z_2} \right]$$

where $\dot{z}_i = k_i - \delta_z z_i$ for $i = 1,2$; δ_z is a depreciation rate of knowledge stock, which implies the rate of technology obsolescence; $V^1_{m_i}$ is the value function of firm 1 at different modes. The net expected payoff of firm 1 switching to a monopolist at m_1 is the switching rate $h_1 \kappa$ multiplying the net payoff of being a monopolist $V^1_{m_2} - V^1_{m_1}$; the same argument goes for m_3 and m_4. If $V^i_{m_1}(z_1,z_2)$ satisfies the transversality condition

$$\limsup_{t \to \infty} e^{-\rho t} V^i_{m_1}(z_1(t), z_2(t)) \le 0, \ i = 1, 2,$$

the feedback strategy of symmetric firms is

$$k^\star_{im_1} = \frac{1}{2} \left[\kappa V^i_{m_2} + (1-\kappa)V^i_{m_4} - V^i_{m_1} + \frac{\partial V^i_{m_1}}{\partial z_i} \right], \ i = 1, 2$$

where $1 > z_i > 0$ and $k^\star_{im_1} > 0$. The optimal R&D of a patent race depends on the patent strength, the value of different market structures and the marginal value of knowledge dynamics. If the strength of patent is high, the proportion of the value of a monopolist is high. This implies that a firm has more incentives to invest in R&D if patents can efficiently protect it to obtain benefits as a monopolist.

Compared with a patent race with exogenous patent rewards, the patent reward of the model endogenously depends on the net total expected payoff in the latter stage. The strength of

patents and the value of different market structure determines the benefits of the patent. Firms'
optimal R&D effort of a patent race depends on the amount of benefits that an innovator with a
patent can generate in different market structures. In fact, the benefit structure of innovation
affects whether patent protection supports innovation (see chapter 4). It is important to use
a model that can capture the benefit structure of innovation better. Kitch (1977) argued that
patents come at the beginning of innovation process and its benefit is generated afterward
in most empirical cases. This paper uses three-stage model with differential game approach,
which allows to estimate ex-post benefits of innovation.

3.3   Computation

At Markov perfect Nash equilibria, HJB equations should hold for the feedback strategies in
the state space. Since the analytical solution cannot be derived from a non-linear system of
HJB equations, I use a numerical method to approximate the value function. Rui and Miranda
(1996) found that the collocation method of Chebychev nodes and a Chebychev polynomial
can estimate an unknown non-linear curve better than the other methods. The following will
introduce how to use the collocation method to approach the value function in the state space.

In the two state space $X_i = \{x_{i1}, x_{i2}, ..., x_{in}\} \in [a_i, b_i]$ for $i = 1, 2$, the Chebychev polynomial
of degree n with coefficient matrix $\mathbf{C}_{1 \times nn}$ is

$$\hat{V}(X_1, X_2) \approx \mathbf{C}\left[\mathbf{T}_{X_1} \otimes \mathbf{T}_{X_2}\right]$$

$$\mathbf{T}_{X_i} = \begin{pmatrix} T_0(-1 + \frac{2(x_{i1} - b_i)}{a_i - b_i}) \\ T_1(-1 + \frac{2(x_{i2} - b_i)}{a_i - b_i}) \\ \vdots \\ T_{n-1}(-1 + \frac{2(x_{in} - b_i)}{a_i - b_i}) \end{pmatrix}$$

where $T_y(w) = \cos[y \cos^{-1}(w)]$. A good approximated value function should hold for the HJB
equation at all Chebychev nodes in the state space. Chebychev nodes $\hat{X}_i = \{\hat{x}_{i1}, \hat{x}_{i2}, ..., \hat{x}_{in}\}$ for
$i = 1, 2$ are

$$\hat{x}_{ik} = \frac{a_i + b_i}{2} + \frac{b_i - a_i}{2}(\cos \frac{n - k + 0.5}{n}\pi), k = 1, \cdots, n, \ i = 1, 2.$$

Assume the initial guess of the approximated value function and calculate its value at Chebychev
nodes, $\hat{V}_{m_i}^{i,ini}(\hat{X}_1, \hat{X}_2)$. Obtain $\mathbf{C}_{m_i}^{i,ini}$ such that $\hat{V}_{m_i}^{i,ini}(\hat{X}_1, \hat{X}_2) = \mathbf{C}_{m_i}^{i,ini}\left[\mathbf{T}_{\hat{X}_1} \otimes \mathbf{T}_{\hat{X}_2}\right]$ at nodes. Then,
the estimated value function of the initial guess follows $\hat{V}_{m_i}^{i,ini}(X_1, X_2) = \mathbf{C}_{m_i}^{i,ini}[\mathbf{T}_{X_1} \otimes \mathbf{T}_{X_2}]$ by
which the control variables of the initial guess can also be estimated. The HJB equation
holds for the control variables of the initial guess, state variables, and a new value function
$\hat{V}_{m_i}^{i,1}(\hat{X}_1, \hat{X}_2) = \mathbf{C}_{m_i}^{i,1}\left[\mathbf{T}_{\hat{X}_1} \otimes \mathbf{T}_{\hat{X}_2}\right]$ at all nodes so that the new coefficient matrix $\mathbf{C}_{m_i}^{i,1}$ can be

calculated. Then, the new estimated value function of this iteration is generated $\hat{V}_{m_i}^{i,1}(X_1,X_2) = \mathbf{C}_{m_i}^{i,1}[\mathbf{T}_{X_1} \otimes \mathbf{T}_{X_2}]$. The same argument goes on for the next iteration until the estimated value function makes HJB equations scale-invariantly deviate from an error ε at all nodes.

By backward induction, I look for the approximated value function of two state variables at m_4 first, then m_2, m_3 and finally m_1. The HJB equations with feedback strategies in the state space at different modes are

$$\rho V_{m_2}^1(c_1,t) - \frac{\partial V_{m_2}^1(c_1,t)}{\partial t} = \left(\frac{1-c_1}{2}\right)^2 + c_1 \frac{\partial V_{m_2}^1(c_1,t)}{\partial c_1}\left(\frac{c_1}{2}\frac{\partial V_{m_2}^1(c_1,t)}{\partial c_1} + \delta_c\right)$$
$$, \{c_1,t\} \in [0,0.5] \times [0,30];$$
$$\rho V_{m_3}^1(c_2,t) - \frac{\partial V_{m_3}^1(c_2,t)}{\partial t} = c_2 \frac{\partial V_{m_3}^1(c_2,t)}{\partial c_2}\left(c_2 \frac{\partial V_{m_3}^2(c_2,t)}{\partial c_2} + \delta_c\right)$$
$$, \{c_2,t\} \in [0,0.5] \times [0,30]$$

$$\rho V_{m_1}^1(z_1,z_2) = A\kappa V_{m_2}^1(c_1,T) + B\kappa V_{m_3}^1(c_2,T) + (A+B)(1-\kappa)V_{m_4}^1(c_1,c_2)$$
$$+ [A - (1+\delta_z)z_1]\frac{\partial V_{m_1}^1(z_1,z_2)}{\partial z_1} + [B - (1+\delta_z)z_2]\frac{\partial V_{m_1}^1(z_1,z_2)}{\partial z_2}$$
$$- (A+B)V_{m_1}^1(z_1,z_2) - \frac{1}{2}(A - z_1)^2$$
$$, \{z_1,z_2\} \in [0,1] \times [0,1];$$
$$\rho V_{m_4}^1(c_1,c_2) = \left(\frac{1-2c_1+c_2}{3}\right)^2 + \frac{1}{2}\left(c_1 \frac{\partial V_{m_4}^1(c_1,c_2)}{\partial c_1}\right)^2 + c_2^2 \frac{\partial V_{m_4}^1(c_1,c_2)}{\partial c_2}\frac{\partial V_{m_4}^2(c_1,c_2)}{\partial c_2}$$
$$+ \delta_c\left(c_1 \frac{\partial V_{m_4}^1(c_1,c_2)}{\partial c_1} + c_2 \frac{\partial V_{m_4}^1(c_1,c_2)}{\partial c_2}\right)$$
$$, \{c_1,c_2\} \in [0,0.5] \times [0,0.5];$$

where $A = \kappa V_{m_2}^1(c_1,T) + (1-\kappa)V_{m_4}^1(c_1,c_2) - V_{m_1}^1(z_1,z_2,c_1,c_2) + \frac{\partial V_{m_1}^1(z_1,z_2,c_1,c_2)}{\partial z_1} + z_1$ and $B = \kappa V_{m_2}^2(c_2,T) + (1-\kappa)V_{m_4}^2(c_1,c_2) - V_{m_1}^2(z_1,z_2,c_1,c_2) + \frac{\partial V_{m_1}^2(z_1,z_2,c_1,c_2)}{\partial z_2} + z_2$. Since two firms are symmetric, $\frac{\partial V_{m_3}^2(c_2,t)}{\partial c_2} = \frac{\partial V_{m_2}^1(c_1,t)}{\partial c_1}$ at m_3, which has been calculated in m_2. The HJB equation at m_3 is a simple partial differential equation of two variables given terminal conditions, which can be solved directly numerically by the Mathematica program. The collocation method is used on solving HJB equations of m_1, m_2 and m_4. Initial guess of the estimated value function at those modes are

$$\hat{V}_{m_1}^{i,ini}(z_1,z_2) = h_i\kappa\left(V_{m_2}^i - V_{m_1}^i\right) + h_2\kappa\left(V_{m_3}^i - V_{m_1}^i\right) + (h_i+h_j)(1-\kappa)\left(V_{m_4}^i - V_{m_1}^i\right) - \frac{1}{2}k_i^2$$
$$, k_i = \delta_z z_i;$$
$$\hat{V}_{m_2}^{i,ini}(c_1,t) = \frac{1}{4r}\left((1-c_1^2) - \delta_c^2\right)\left(1 - e^{-\rho(T-t)}\right) + e^{-\rho(T-t)}V_{m_4}^i(c_1,c_1(T));$$
$$\hat{V}_{m_4}^{i,ini}(c_1,c_2) = (1 - q_1 - q_2 - c_1)q_1 - \frac{1}{2}k_1^2, \ q_i = \frac{1-2c_1}{3}, \ k_1 = \delta_c c_1.$$

(i)   (ii)
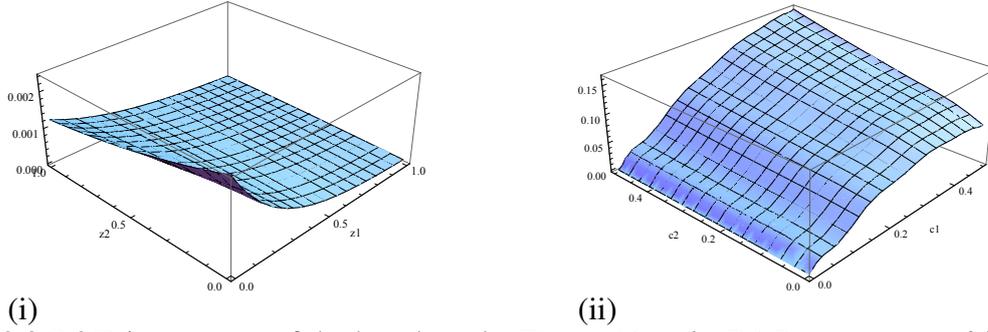
Fig. 3.2 R&D investment of the benchmark. *Figure (i) is the R&D investment of firm 1 at m_1; figure (ii) is R&D investment of firm 1 at m_4. The parameters are $\rho = 0.105$, $\delta_c = \delta_z = 0.2$, $\kappa = 0.5$. The patent length is $T = 10$.*

For the numerical estimation, the degree of Chebychev polynomial of the collocation method is $n = 10$ and its deviation error is $\varepsilon = 0.0001$.

3.4   Equilibria and strategies

The benchmark model is assumed the technology obsolescence at a rate $\delta_z = 0.2$. As soon as one firm becomes innovator in the patent race, it starts to produce by process innovation with an initial marginal cost $c_{im_2}(0) = 0.5$ where productive efficiency is depreciated at a rate $\delta_c = 0.2$. A patent with neutral strength $\kappa = 0.5$ protects the innovator from imitation by 50% chances, which is valid for 10 periods $T = 10$. Otherwise, the imitator enters the product market with the same process innovation $c_{im_4}(0) = c_{jm_4}(0) = 0.5, i \neq j$. The race runs under a discounted rate $\rho = 0.105$. The optimal strategies of R&D investment of the benchmark shows in figure 3.2.

The follower firm, which has less knowledge stock, has incentives to invest more in R&D than the leader, which has more knowledge stock. In figure 3.2(i), $\frac{1}{2}\left(k^\star_{1m_1}(z_1, z_2)\right)^2 > \frac{1}{2}\left(k^\star_{1m_1}(z_2, z_1)\right)^2$ when $z_1 < z_2$. This catching-up behavior is a result of knowledge effect (Doraszelski, 2003). The feature of reducing R&D in terms of increasing knowledge contributes to a less R&D investment of the leader, which increases incentives of the follower strategically. This effect can also be observed in figure 3.2(i) that R&D investment decreases with increasing knowledge accumulation. The substitution effect of R&D effort and knowledge for innovation is the main reason of the negative marginal R&D investment (see chapter 2). R&D effort can compensate insufficient knowledge stock for innovation, which gives follower's incentive to innovate.

The firm with less production efficiency has incentives to invest more in R&D than the firm with better efficiency in a product market. In figure 3.2(ii), $\frac{1}{2}\left(k^\star_{1m_4}(c_1, c_2)\right)^2 < \frac{1}{2}\left(k^\star_{1m_4}(c_2, c_1)\right)^2$ when $c_1 < c_2$. To be competitive in the market, firms try to be efficient on its production by process innovation. Better production efficiency induces firms to produce more,
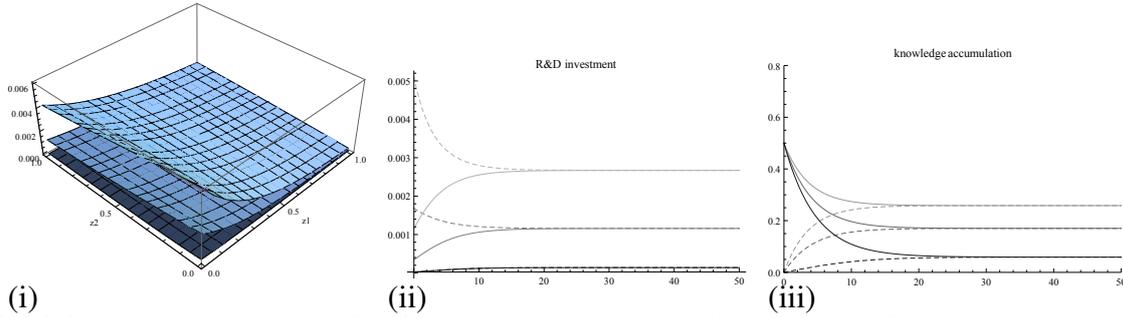
Fig. 3.3 Patent strength of R&D investment and knowledge accumulation. *Figure (i) is R&D investment of firm 1 with respect to knowledge state under different strength of patents; figure (ii) is R&D investment over time; figure (iii) is knowledge accumulation over time. Curves and the surface of light gray is for $\kappa = 1$; gray curves and the gray surface is for $\kappa = 0.5$; black curves and the black surface is for $\kappa = 0$. The dashed line represents the follower firm with initial knowledge $z_i(0) = 0$, and the solid line represents the leader with initial knowledge $z_j(0) = 0.5$. Other parameters follow benchmark: $\rho = 0.105$, $\delta_c = \delta_z = 0.2$.*

since the benefit of a lower production cost induces firms to produce more. In contrast, if the firm with low production efficiency, it strategically reduces its production due to increasing production of its opponent. To sustain its production in the market, the firm must invest in R&D for a more efficient production. That is the reason that the less efficient firm invests more in R&D. Figure 3.2(ii) shows that R&D investment increases when the marginal cost increases.

3.4.1 Patent strength

A stronger patent implies that the innovator has a higher chance to monopolize the market in the period of a valid patent, that is, κ approaches 1. A weaker patent indicates that the innovator has a higher chance to compete in a duopoly market after innovation, that is, κ approaches to 0. Since a strong patent protects the innovator from the imitator, the benefit of the innovator is more significant than the case with a weak patent. This gives firms more incentives to invest in R&D of a patent race with a strong patent. Figure 3.3(i) shows that firms' R&D investment is larger in the case of $\kappa = 1$ than the other cases.

Besides, assume that a follower firm has less knowledge stock than a leader firm in the beginning of the race. When they compete to innovate successfully in a patent race, a strong patent encourages the follower firm to invest more R&D than the leader. In figure 3.3(ii), the follower invests more in R&D than the leader at the beginning of the race under a strong patent. When the patent is stronger, the difference of R&D investment between the leader and the follower is more significant. If there is no patent to protect innovation, that is $\kappa = 0$, the catching-up behavior of the follower's R&D investment becomes insignificant. In the long run, a strong patent encourages frims to invest in R&D and reach a high level of knowledge accumulation.
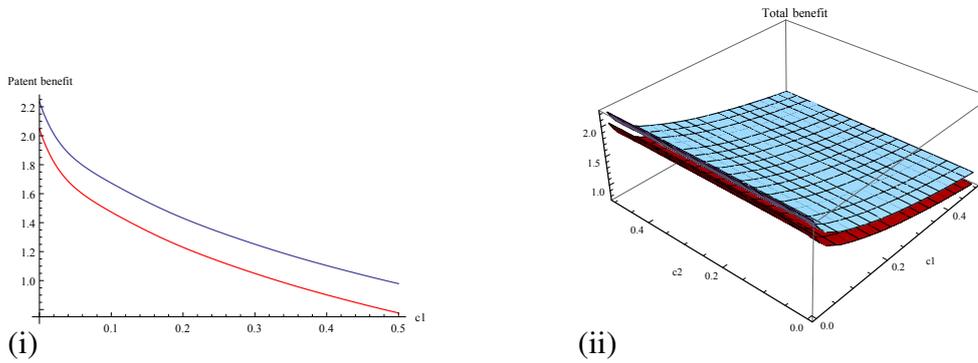
Fig. 3.4 Patent benefit structures of firm 1. *Figure (i) is innovation benefit of firm 1 with respect to its marginal cost; figure (ii) is total benefits with respect to two firms' cost state. The red curve and surface corresponds to 10 periods of patent length; The blue curve and surface corresponds to 20 periods patent length. Other parameters follow the benchmark: $\rho = 0.105$, $\delta_c = \delta_z = 0.2$, $\kappa = 0.5$.*

3.4.2 Benefit structure of patents

The benefit structure of patents affects firms' incentives to innovate under different degree of patent protection. If an innovator can acquire a certain share of the total profits, strong patent protection encourages firms to invest in R&D and innovate. In contrast, innovation is more likely to obtain with weak patent protection, if the profit of the innovator is independent of the degree of patent protection (see chapter 4).

To see which benefit structure of the endogenous benefit model belongs to, first define the benefit of a patent holder of firm 1 is $V^1_{(m_2)}(c_1, 0)$. Figure 3.4(i) shows that more efficient production and longer patent length generate more patent benefits. If firm 1 does not innovate, it gets an imitator's reward $V^1_{(m_3)}(c_2, 0)$. The total benefit of firm 1 is $V^1_{(m_2)}(c_1, 0) + V^1_{(m_3)}(c_2, 0)$. In figure 3.4(ii), the total benefit is high if the firm production is more efficient then its opponent. If the patent length is longer, the total benefits is higher. When c_1 decreases, both patent benefit and total benefit increases. This implies that an innovator is more likely to acquire a fixed share of total benefits and a longer patent brings more benefits.

3.4.3 Patent length, innovation time and social welfare

The optimal patent length has been discussed in several contributions. On the one hand, since the patent length implies the periods of monopoly of an innovator, an optimal patent length is associated with the minimal social welfare loss. On the other hand, benefits of a patent holder as a monopoly gives firms incentives to invest in R&D in the first stage of a patent race. A longer patent brings more benefits to the innovator, which induces innovation happens earlier. According to Polasky et al. (2011), the probability of innovation before time t can be calculated

(i)   (ii)
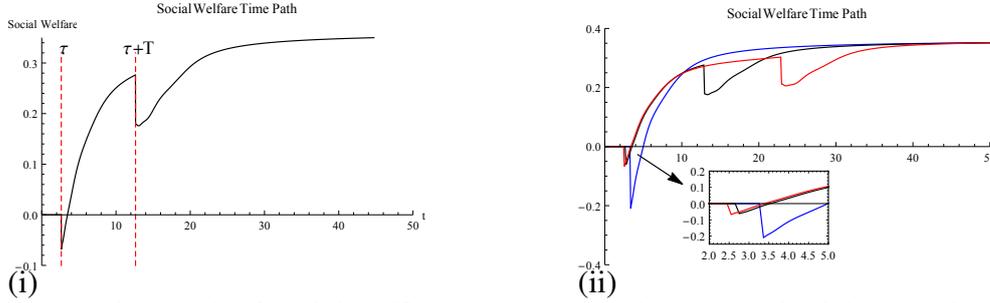
Fig. 3.5 The time path of social welfare. *Figure (i) is the time path of the social welfare under 10 periods of a patent length, $T = 10$ and the expected innovation time is at time $\tau = 2.63$ in this case. Figure (ii) is the comparison of social welfare under different patent length. The red curve follows $T = 20$; the black curve follows $T = 10$; the blue curve follows $T = 0$. The parameters are $\rho = 0.105$, $\delta_c = \delta_z = 0.2$ and $\kappa = 0.5$.*

by the hazard rate $h(t)$:

$$\mathbb{P}(\tau < t) = 1 - e^{-\int_0^t h_i(s)ds}$$

The expected innovation time that happens innovation from time t_i to t_{i+1} is

$$\mathbb{E}(t) = \int_0^\infty t d\mathbb{P}(\tau < t) \approx \int_0^\infty t h_i(t) e^{-\int_0^t h_i(s)ds} dt.$$

Figure 3.5(i) shows the expected innovation time in the case of 10 periods patent length, $\tau = 2.63$. Before innovation, firms invest in R&D in a patent race and the social welfare over time is $\sum_{i=1,2} -k_i(t)$. After innovation, the social welfare over time SW is the sum of consumer surplus over time CS and producer surplus over time PS. In the period of a valid patent T, the time path of social welfare of monopoly M is

$$SW_M(t) = CS_M(t) + PS_M(t),$$

$$CS_M(t) = \frac{\left(q^\star_{1m_2}(t)\right)^2}{2}, \quad PS_M(t) = \left(1 - q^\star_{1m_2}(t) - c_1(t)\right) q^\star_{1m_2}(t) - \frac{1}{2}\left(k^\star_{1m_2}(t)\right)^2.$$

In the beginning of the valid patent, the social welfare is negative because the R&D cost and producing cost surpass the revenue. After some time periods the producing cost decreases due to the process innovation and the social welfare becomes positive. After the patent expires, the opponent enters the market and the social welfare of duopoly over time D is

$$SW_D(t) = CS_D(t) + PS_D(t),$$

$$CS_D(t) = \frac{\left(q^\star_{1m_4}(t) + q^\star_{2m_4}(t)\right)^2}{2},$$
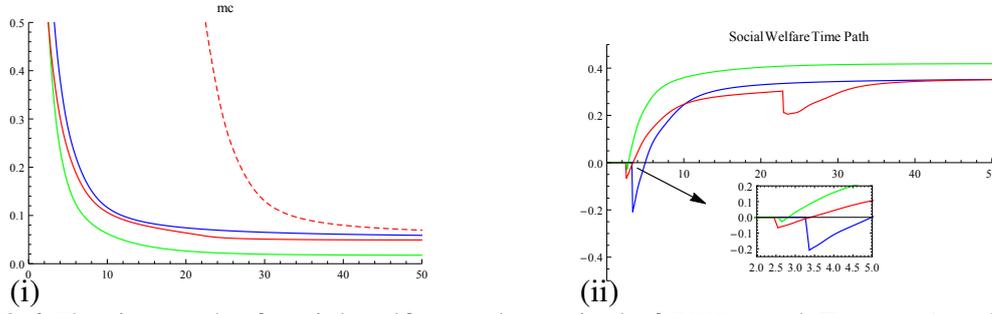
Fig. 3.6 The time path of social welfare and marginal of RJV cartel.*Figure (i) is the time path of marginal cost; figure (ii) is the time path of social welfare. The red curve represents the case under 20 periods of patent protection $T = 20$ (dashed line represents the opponent); the green curve represents RJV cartel; the blue curve represents duopoly competition. The parameters are $\rho = 0.105$, $\delta_c = \delta_z = 0.2$ and $\kappa = 0.5$.*

$$PS_D(t) = \left(1 - q^{\star}_{1m_4}(t) - q^{\star}_{2m_4}(t) - c_1(t)\right) q^{\star}_{1m_4}(t) - \frac{1}{2}\left(k^{\star}_{1m_4}(t)\right)^2$$
$$+ \left(1 - q^{\star}_{1m_4}(t) - q^{\star}_{2m_4}(t) - c_2(t)\right) q^{\star}_{2m_4}(t) - \frac{1}{2}\left(k^{\star}_{2m_4}(t)\right)^2.$$

The innovator lost its benefits as a monopoly after the patent expires, which is the main reason that social welfare drops at the time of patent expiration.

A long period of patent brings monopoly power for the innovator, which reduces the consumer surplus by a high market price. At the same time, a patent also increases firm's incentives to innovate, since the gain of monopoly is protected. Figure 3.5(ii) shows that innovation occurs early if the patent lasts longer. In the time periods from 3 to 8, social welfare of the case without patenting suffers from a slow process innovation due to the low R&D incentives of no monopoly benefits. This leads to a low social welfare level in the case without patenting. Nevertheless, after time 8, the effect of low incentives of process innovation becomes less and the effect of market competition increases. The consumer surplus of duopoly surpasses the consumer surplus of monopoly. The social welfare in the case without patenting is higher than it with patenting from time 8. If the periods of patenting is longer, the difference of social welfare between monopoly and duopoly is accumulatively more. In fact, there exists a critical time that balance out the trade-off between loss of innovation incentives and the loss of monopoly of social welfare, which implies the optimal patent length is finite. In contrast, Gilbert and Shapiro (1990) give a general model of patent policy and concluded that the optimal patent length is infinitive. After considering strategic interaction among firms, intertemporal choices and R&D incentives, the optimal length of patents should be finite.

3.4.4   Research joint venture (RJV) cartel

Research joint venture and R&D cartel gives more efficient innovation by eliminating dupli-
cation R&D effort. Kamien et al. (1992) discuss social welfare under four different cases
of research joint venture(RJV) and cartels: (i) R&D competition, (ii) R&D cartel, (iii) RJV
competition, and (iv) RJV cartel in a static model. Cellini and Lambertini (2009) introduce
a dynamics model and state that R&D cartel can improve social welfare. Assume that firms
form an RJV cartel, that is, firms share R&D effort and innovation completely, $k_1 = k_2 = k$
and $c_1 = c_2 = c$. Since both firms R&D effort contributes to process innovation fully, the
dynamics of marginal cost of the third stage follows $\dot{c} = (-2k + \delta_c)c$, which follows a special
case of fully technological spillovers in Cellini and Lambertini (2009). After a patent race at
m_1 the innovator and its opponent join RJV cartel at m_4. The Hamilton-Jacobi-Bellman (HJB)
equation of firm 1 under RJV cartels at m_4 becomes

$$\rho V_{m_4}^1 = \max_{q_1,k}(1 - q_1 - q_2 - c)q_1 - \frac{1}{2}k^2 + \dot{c}\frac{\partial V_{m_4}^1}{\partial c}$$

where $V_{m_4}^i(c_1, c_2)$ satisfies the transversality condition, that is,

$$\limsup_{t \to \infty} e^{-\rho t} V_{m_4}^i(c(t)) \leq 0, \ i = 1, 2.$$

Thus, the feedback strategies of symmetric firms are

$$q_{im_4}^\star = \frac{1-c}{3}, \ i = 1, 2$$

$$k_{m_4}^\star = -c\frac{\partial V_{m_4}^i}{\partial c}, \ i = 1, 2$$

where $1 > c > 0$, $q_{im_4}^\star > 0$ and $k_{m_4}^\star > 0$.

The marginal cost and social welfare of RJV cartels over time show in figure 3.6. RJV cartels
generates most social welfare over the time path by most efficient R&D, which corresponds to
the result of Kamien et al. (1992) and Cellini and Lambertini (2009). Further, RJV cartels gives
more incentives to innovate earlier than the case of completed duopoly due to efficient process
innovation. However, a long enough length of patents can intriguer the innovation occurs even
more earlier than the case of RJV cartels.

3.5   Conclusions

The research combines a patent race and process innovation by a three-stage model with
a differential game approach. Firms invest in R&D in ex-ante and ex-post of patents. In
the first stage, firms invest and accumulate knowledge competitively, which increases their

chances of being the first innovator. In the second stage, the innovator increases its production efficiency by ex-post R&D investment and becomes a monopolist in a valid period due to patent protection. In the last stage, the patent is expired and the imitator enters the market with the same production efficiency as the innovator. Firms compete in the duopoly market by their own process innovation.

This model characterizes the dynamics of ex-ante and ex-post R&D investment, which can capture endogenous benefits of prospect patents. According to different benefit structures, the patent has either a negative or positive impact on innovation (see chapter 4). Endogenous benefits can describe the strategic interaction between an innovator and a free-rider. When the free-rider problem is significant, the benefit is less due to the reducing innovator's incentives, which discourages innovation.

Except for the benefit of patents, the model also allows analyzing the impact of patent strength and patent length. Strong patent ensures the patent holder's benefit by monopolizing the market, which encourages innovation. Even though innovation brings efficient production and increases producer surplus, monopoly of an innovator scarifies consumer surplus at the same time. The optimal length of patent, which balances the trade-off between efficient production and loss of consumer surplus, is finite, since a long period patent brings more welfare loss than production efficiency. Research joint venture

Chapter 4

A patent portfolio race with knowledge accumulation

Coauthored with Florian Wagener

4.1 Introduction

A patent race or R&D race is a game in which firms compete with each other to make a certain innovation first. The winner of the race usually obtains the advantage of getting a patent on the innovation. Early models of patent races were given by Loury (1979), Lee and Wilde (1980), Dasgupta and Stiglitz (1980a,b), Reinganum (1981, 1982) and Kamien and Schwartz (1982). These models do not feature history dependence: the probability of successful innovation depends only on current R&D effort, but not on past R&D experience. Doraszelski (2003) introduced an R&D race model with knowledge accumulation, where R&D efforts in each period of time contribute to building a knowledge stock and affect the hazard rate of successful innovation. This rate is assumed to be linear in R&D effort and in the knowledge stock, which implies that it is possible to innovate successfully without actually exerting R&D efforts. This is not realistic: Cohen and Levinthal (1990) argued that firms need to make an effort to internalize knowledge. The model in this paper uses a hazard rate that depends on R&D efforts and knowledge stock multiplicatively. In this way, successful innovation is only possible if there is a positive knowledge stock as well as positive R&D efforts of the innovating firm.

Patents generate revenues because of their monopolizing power; moreover, R&D intensive firms usually hold many patents. It is however not the case that for each patent a specific knowledge stock is build, only to be discarded when the patent is awarded; rather a single R&D knowledge stock is used to obtain many patents on successive related innovations. In this article, we study therefore a patent portfolio race, where firms use a single knowledge stock to innovate multiple times in order to build up their patent portfolio. This is different from the direct contribution of investment to the stock of patents in Denicolo and Zanchettin (2012). In

their model, investing in R&D increases the stock of patents directly; this stock accumulates over time to form a patent portfolio. A shortcoming of their model is that investing in R&D leads with certainty to innovation and the acquisition of a patent. Sieberta and Graevenitz (2010) improved on this by adding a hazard rate model for successful innovation, introduced by Loury (1979) and Reinganum (1981, 1982), to a patent portfolio race model. Firms choose the level of the hazard rate to decide whether to innovate and whether to write a cross-license contract on their patent portfolios. Our model is close to their approach, but instead of using a memoryless model, we use a hazard rate that depends on the stock of past investments into R&D like in Doraszelski (2003).

Since we consider knowledge accumulation in a patent portfolio race, the nature of knowledge is an important issue. A common assumption is that each firm has a private knowledge stock (e.g. Doraszelski 2003). We argue however that the knowledge stock may be assumed to be fully public. Nelson (1959) and Arrow (1962) already posited that firms can obtain knowledge by various ways, which is not necessarily by paying, and the knowledge is not reduced if other firms obtain it. We moreover follow Romer (1986) by assuming that knowledge, at least for low knowledge stock levels, exhibits increasing returns to scale: If little is known, knowledge consists of isolated, almost unconnected results. Each new results suggests connections between what is already known, giving rise to 'synergy' network effects. Only if the knowledge stock is mature, marginal returns to developing additional knowledge diminish.

Moreover, we assume that for the most part the R&D efforts of the firm are directed to internalising public knowledge. Although this can be refined, we are already able to capture much relevant behaviour with this approach. Breton et al. (2006) developed a dynamic R&D model with public knowledge in a duopoly market. In their model, firms invest in a laboratory in order to assimilate public knowledge to innovate successfully. Firms face binary R&D decision, either investing 0 or 1, and their decisions over time affect their probability of successful innovation. If only one firm innovates, it establishes a monopoly. If two firms innovate, they form a duopoly, which is less profitable than a monopoly, but the firms have lower production costs as there are innovation spillovers from the other firm. Breton et al. find that firms invest in R&D only if the level of public knowledge is sufficiently high. Moreover, they find that for intermediate knowledge levels, the Pareto optimal outcome can be improved by policy intervention.

As an increase in the stock of public knowledge facilitates private enterprises, policy makers have an incentive to make big investments into public knowledge, in the hope that this will entail the formation of high-tech clusters. Breshnahan and Gambardella (2004) write: "The positive feedback elements of a successful cluster also make it difficult to learn anything from clusters that don't take off; an implication of 'nothing succeeds like success' is that 'nothing fails like failure'." It is the inherent inobservability of a knowledge stock that has failed to materialize that constitutes the added value of a model-based approach. In the present article, our objective is to investigate the effects of institutional parameters, which presumably can be

set or at least influenced by the policy maker, like the degree of patent protection, and external parameters, which are beyond the policy maker's control, like the size of the resulting market, on the establishment of a mature knowledge stock and a flourishing patent portfolio race.

A typical result is shown in figure 4.1, where the effects of the degree of patent protection — an institutional parameter — and firm patience — an external parameter — are shown. We find three different regimes: if firms are sufficiently forward-looking, R&D competition will lead to the establishment of a mature knowledge stock that is completely market-driven ("market-driven innovation"): there is nothing here for a social planner to do. On the other hand, if the planning horizon of the firms is too short, they will phase out R&D investments, whatever the initial knowledge stock; the probability that they make any innovation is very small, and it will drop to zero over time ("innovation unlikely"). Again, a social planner has little to do here. More interesting to her is the third region, that separates the first two. There, market competition can sustain a mature knowledge stock, provided the initial knowledge stock is larger than some critical level ("innovation after initial subsidy"). It is here that the planner has to decide whether to provide the means to move the initial knowledge stock above the critical level, in order to get industrial activity going that probably generates all kinds of external benefits.

Our model does not say anything about this decision. It does two other things: it identifies the region where the action of the social planner can make a difference, and it shows how changing institutional parameters in the model affects the location of this critical region. For instance, still referring to figure 4.1, installing stronger patent protection will increase both the "market-driven innovation" region as well as the "innovation after initial subsidy" region. However, the figure also indicates that the effect of increasing patent protection is small, as the boundary curves between the regions are almost flat. In the analysis, we focus on three parameters, one institutional — patent protection — and two external — innovation benefit and patience. We find that an initial subsidy helps establishing a self-sustaining innovation regime only if total innovation benefit is not too small or if firms are sufficiently patient. The impact of the degree of patent protection is, by comparison, relatively small. Moreover, our results give the cross-effects of changing two parameters simultanously in detail.

A methodological contribution of this paper is to use the auxiliary system approach of Dockner and Wagener (2014) to explore the characteristics of the Markov perfect Nash equilibria of the patent portfolio race. This allows us to deal with the arising non-differentiable value functions. In particular, we develop a bifurcation theory for Markov perfect Nash equilibria of differential games of a single state variable; we believe that we are the first to do so. Figure 4.1 is a bifurcation diagram, visualising how the qualitative characteristics of a dynamic Markov perfect Nash equilibrium depends on different parameters.

Section 2 models a patent portfolio race between two firms and introduces the auxiliary system. Section 3 discusses the mathematics of the bifurcation theory involved. In section 4 we
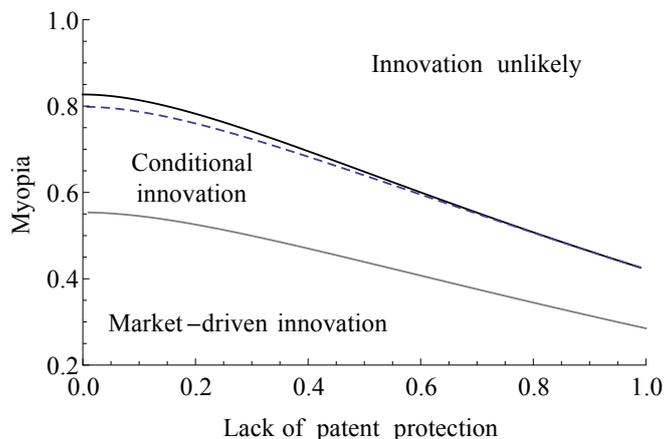
Fig. 4.1 The effect of patent protection and time preference on the resulting innovation regime.

investigate the dependence of the the expected innovation time on different parameters. We extend the two firm game to n firms in section 5. Section 6 concludes.

4.2   Markov perfect Nash equilibria of a patent portfolio race

4.2.1   Model

We model a patent portfolio race between two firms as a differential game, with a public knowledge stock as the single state variable. All R&D efforts from different firms will contribute to the public knowledge stock. To innovate, a firm needs to invest not only in R&D but also into labour: the knowledge stock is interpreted as generic knowledge, which needs to be assimilated by labour for innovation (Kylaheiko et al., 2011). The ability to exploit knowledge is called absorptive capacity by Cohen and Levinthal (1990); it is higher if a firm invests more in labour. In the model, firms can invest strategically in both labour and R&D: a firm with large knowledge stock and labour force has a higher probability to innovate successfully.

We think of the knowledge stock as a large collection of small fundamental ideas, indexed by a continuous variable $x \in [0, Z]$, where Z is the size of the knowledge stock. Total investment $I = I_1 + I_2$ in R&D increases the probability of a new fundamental idea being found, and hence increases z. On the other hand, ideas are forgotten or being made obsolete at a constant rate δ; after such an event, the ideas are relabeled. The index $x \in [0, Z]$ consequently indicates the vintage of a fundamental idea: $x = Z$ labels the oldest idea, while $x = 0$ corresponds to the newest. This model results in the knowledge evolution equation

$$\frac{\mathrm{d}Z}{\mathrm{d}\tau} = I_1 + I_2 - \delta Z, \qquad Z(0) = Z_0. \tag{4.1}$$

We turn to innovations: these can occur when two or more fundamental ideas are combined. However, not all combinations of ideas result in innovations: knowledge absorption by a firm consists, in our model, by going through the range of possible combinations of fundamental ideas, looking for those combinations that yield marketable innovations.

For simplicity, we restrict to the situation where a marketable innovation combines only two fundamental ideas. We then obtain a knowledge network, where the fundamental ideas are the nodes and the marketable innovations form the links. If two ideas are new, they are likely to generate an innovation. To model this, we assume that the density $\mathrm{d}\mu$ of links is proportional to

$$\mathrm{d}\mu(x_1 \text{ and } x_2 \text{ generate innovation}) \propto f(x_1 + x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2, \tag{4.2}$$

where f is a positive decreasing function.

If an innovation is made at time $\tau = T_1$, the innovator will benefit from its patent; the imitator benefits from innovation spillovers. Since firms can innovate multiple times, the race continues: the second innovation is made at $\tau = T_2$, etc.

Innovations occur at random times. Denote by H_i the hazard rate of innovation by firm i; then

$$H_i(\tau) = \lim_{\Delta\tau \to 0} \frac{\mathbb{P}(\text{firm } i \text{ innovates in } [\tau, \tau + \Delta\tau) \mid \text{race not over at time } \tau)}{\Delta\tau}.$$

That is, the probability that firm i innovates in the interval $(\tau, \tau + \Delta\tau)$ is approximately $H_i(\tau)\Delta\tau$. The rate $H_i(\tau) = H_i(Z(\tau))$ at which innovations are found is assumed to be proportional to

$$H_i(Z) \propto \int_{\{\text{all pairs of nodes}\}} \mathrm{d}\mu = \int_{0 \le x_2 \le x_1 \le Z} f(x_1 + x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2. \tag{4.3}$$

In particular, H_i is a positive increasing function satisfying $0 \le H_i(Z) \le cZ^2$ for some $c > 0$. We shall moreover assume that the innovation rate is bounded by a constant λ as a function of Z.

For instance, in the illustrations below, we assume that the hazard rate of firm i depends on the effectiveness of current effort λ, firm i's integrative labour L_i, public knowledge stock Z and a threshold parameter θ in the following way

$$HI_i(L_i, Z) = \lambda L_i H_i(Z) = \lambda L_i \frac{Z^2}{Z^2 + \theta^2}. \tag{4.4}$$

We can interpret HI_i as the production function of innovations with labour and knowledge as inputs; this production function has, for small knowledge stocks, increasing returns to scale, which are caused by knowledge network effects. Increasing returns in knowledge production have been proposed very early on (Arrow, 1962; Romer 1986). We note however that our results do not depend on the particular parametrisation; they can even be obtained for knowledge production functions without increasing returns to scale.

The firm exerts integrative labour effort $L_i = L_i(\tau) \geq 0$ and R&D effort $I_i = I_i(\tau) \geq 0$, incurring running effort costs

$$\frac{c_L}{1 + 1/\eta} L_i^{1+1/\eta} + \frac{c_I}{1 + 1/\xi} I_i^{1+1/\xi};$$

here $c_L, c_I, \eta, \xi > 0$. Firm i innovates with probability $H_i(\tau)\Delta\tau$; if it innovates, it reaps innovation benefits $\overline{P}$; the other firm innovates with probability $H_{-i}(\tau)\Delta\tau$, and in that case firm i reaps imitation profits $\underline{P}$, which are less than the innovation benefits.

Given the competing firm's effort schedules $L_{-i}(Z)$ and $I_{-i}(Z)$, firm i tries to maximise

$$J_i(L_i, I_i) = \int_0^\infty e^{-\rho\tau} \Big(HI_i(L_i, Z)\overline{P} + HI_{-i}(L_{-i}(Z), Z)\underline{P}$$

$$-\frac{c_L}{1 + 1/\eta} L_i^{1+1/\eta} - \frac{c_I}{1 + 1/\xi} I_i^{1+1/\xi} \Big)\, d\tau,$$

subject to (4.1) and the restrictions $L_i \geq 0$, $I_i \geq 0$. Let

$$V_i(Z) = \sup J_i(L_i, I_i),$$

denote where the supremum is taken over all bounded measurable functions $L_i(Z)$ and $I_i(Z)$ satisfying the restrictions.

Then $V_i(Z)e^{-\rho\tau}$ is the value of firm i at time τ and knowledge stock level Z. Over the time interval $[\tau, \tau + \Delta\tau)$, the value evolves as follows. At the end of the interval, the knowledge stock Z has evolved to $Z + \Delta Z$.

Choosing integrative labour effort and knowledge investment levels optimally and discounting future profits with rate ρ yields

$$V_i(Z)e^{-\rho\tau} = \max_{L_i, I_i} \Big[-\Big(\frac{c_L}{1 + 1/\eta} L_i^{1+1/\eta} + \frac{c_I}{1 + 1/\xi} I_i^{1+1/\xi} \Big) e^{-\rho\tau}\Delta\tau$$

$$+ HI_i(L_i, Z)\overline{P}e^{-\rho\tau}\Delta\tau + HI_{-i}(L_{-i}, Z)\underline{P}e^{-\rho\tau}\Delta\tau$$

$$+ V_i(Z + \Delta Z)e^{-\rho(\tau+\Delta\tau)} + o(\Delta\tau) \Big].$$

Dividing by $e^{-\rho\tau}\Delta\tau$, rewriting and taking the limit $\Delta\tau \to 0$ then yields the Hamilton-Jacobi-Bellman (HJB) equation for firm i:

$$\rho V_i(Z) - \max_{L_i, I_i} \Big[-\Big(\frac{c_L}{1 + 1/\eta} L_i^{1+1/\eta} + \frac{c_I}{1 + 1/\xi} I_i^{1+1/\xi} \Big) \tag{4.5}$$

$$+ HI_i(L_i, Z)\overline{P} + HI_{-i}(L_{-i}, Z)\underline{P} + V_i'(Z)\frac{dZ}{d\tau} \Big] = 0.$$

Note that this is equivalent to the HJB equation of a firm with deterministic revenues

$$HI_1(L_1, Z)\overline{P} + HI_2(L_2, Z)\underline{P}.$$

The benefit of imitation $\underline{P}$ derives from innovation spillovers that contribute less to a patent portfolio. We introduce the parameter κ as the ratio of innovation and imitation benefits

$$\kappa = \underline{P}/\overline{P}.$$

This measures the lack of patent protection. There is full freedom to imitate, and hence complete lack of patent protection, if innovation benefit and imitation benefit are equal, i.e. if $\kappa = 1$; there is no freedom to imitate and perfect patent protection if the imitator does not benefit from the innovation, i.e. if $\kappa = 0$.

If Π denotes the total benefit

$$\Pi = \overline{P} + \underline{P};$$

then the innovation benefit can be expressed in terms of total benefit and equals

$$\overline{P} = \frac{\Pi}{1 + \kappa}.$$

That is, if there is no patent protection ($\kappa = 1$), innovation benefits are just one half of total benefits, while if there is full protection ($\kappa = 0$), innovation benefits equal total benefits.

4.2.2 Natural units

The dynamics simplify if we introduce 'natural' units. This allows us to concentrate on the underlying relations between the variables of the model, rather than be distracted by complicated looking factors, which are only related to the measurement scales. It is straightforward to return to the actual units used.

For instance, the inverse $t_0 = \delta^{-1}$ of the obsolescence/forgetting rate δ determines a natural time scale of the knowledge evolution process. Likewise, if the innovation rate H_i is specified according to (4.4), the threshold parameter $Z_0 = \theta$ determines a natural scale Z_0 for the knowledge stock. Hence, by introducing a 'natural' time t by $\tau = t_0 t = t/\delta$, a 'natural' stock z by $Z = Z_0 z$, and a natural investment rate u_i by $I_i = I_0 u_i$, the knowledge evolution equation takes the form

$$\dot{z} = \frac{I_0}{\delta Z_0}(u_1 + u_2) - z; \tag{4.6}$$

here, as in the following, the superimposed dot denotes derivation with respect to t. This equation simplifies to

$$\dot{z} = u_1 + u_2 - z.$$

if the scale I_0 of the investment rate is chosen in terms of the natural scales for knowledge and time, that is, as $I_0 = Z_0/t_0 = \delta Z_0$.

We introduce natural scales for value V_i, integrative labour effort L_i, and benefits $\overline{P}$ and $\underline{P}$ by setting

$$V_i(Z) = V_0 v_i(z), \quad L_i = L_0 \ell_i, \quad \overline{P} = P_0 \pi, \quad \underline{P} = P_0 \kappa \pi.$$

We define V_0, L_0 and P_0 in terms of the following equalities:

$$V_0 = \frac{c_I t_0}{\alpha} I_0^{1+1/\xi} = c_L t_0 L^{1+1/\eta} = \lambda P_0 L_0;$$

the constant α will be determined later in a convenient manner. Moreover, we introduce

$$r = \rho t_0 \quad \text{and} \quad h_i(z) = H_i(Z).$$

The HJB equation for firm i then takes the form

$$rv_i(z) - \max_{\ell_i, u_i} \left[\pi \ell_i h_i(z) + \kappa \pi \ell_{-i} h_{-i}(z) - \frac{\ell_i^{1+1/\eta}}{1+1/\eta} - \alpha \frac{u_i^{1+1/\xi}}{1+1/\xi} + v_i'(z)(u_i + u_{-i} - z) \right] = 0. \tag{4.7}$$

4.2.3 Determining Markov perfect Nash equilibria

Equation (4.7) is the full HJB equation of the value function of firm i at points where v_i is differentiable. By choosing labour and investment schedules, the two firms play a differential game. We focus on the situation that firms use feedback strategies, determining their effort levels as functions of the knowledge stock. The corresponding equilibrium concept is that of Markov perfect Nash equilibria (MPNE).

It can be shown that the value v_i of each firm is increasing in the knowledge stock, which is also intuitively clear. The optimal R&D and labour efforts are then obtained by maximising the right hand side of (4.7), yielding

$$\ell_i^*(z) = (\pi h(z))^\eta \quad \text{and} \quad u_i^*(z) = \left(\frac{v_i'(z)}{\alpha} \right)^\xi, \quad \text{for} \quad i = 1, 2.$$

We concentrate on the symmetric situation, where $h_1(z) = h_2(z) = h(z)$ and $v_1(z) = v_2(z) = v(z)$ for all z. The HJB equations for a symmetric MPNE can be written as

$$rv(z) - G(z, v'(z)) = 0, \tag{4.8}$$

where

$$G(z, p) \stackrel{\text{def}}{=} \left(\frac{1}{1+\eta} + \kappa \right) (\pi h(z))^{1+\eta} + \alpha^{-\xi} \frac{2+\xi}{1+\xi} p^{1+\xi} - pz. \tag{4.9}$$

We now make the choice $\alpha = (2+\xi)^{1/\xi}$, to cancel out the factor $(2+\xi)$. Moreover, we introduce the 'effective innovation rate' $\Phi(z)$, as well as the 'effective innovation revenue' μ, which is net of integration costs, by

$$\mu = \left(\frac{1}{1+\eta} + \kappa\right)\pi^{1+\eta}, \quad \text{and} \quad \Phi(z) = h(z)^{1+\eta}.$$

The 'game Hamiltonian' G takes then the form

$$G(z,p) = \mu\Phi(z) + \frac{p^{1+\xi}}{1+\xi} - pz.$$

It is well-known that HJB equations like (4.8) need not have a solution $v(z)$ that is continuously differentiable and satisfies (4.8) pointwise. For this class of equations, Crandall and Lions (1983) have developed a weaker solution concept, that of 'viscosity solution', whose definition is given in the appendix. In economics, these solutions have been known for a long time under the name 'Skiba solutions' (see Skiba, 1978; cf. also Sethi, 1977, Dechert and Nishimura, 1983). They are characterised by points of nondifferentiability in the value and jump discontinuities in the actions; the mathematical definition of viscosity solution basically describes which kind of jumps are allowed.

In single-player problems, these jump points carry important information, as they bound 'policy regimes': for two initial states on either side of a jump point, the optimal policies are qualitatively different. In a multi-player problem, as considered in the present article, the situation is more involved; but in certain cases, jump points may have the same 'dynamical separation' property; examples of this are given below.

The existence of a value is settled by the following theorem.

Theorem 1. *The game HJB equation* (4.8) *has a unique uniformly continuous and bounded viscosity solution* $v(x)$, *which is positive, non-decreasing and Lipschitz continuous.*

To characterize the corresponding MPNE strategies, we use the auxiliary system approach of Dockner and Wagener (2014), which enables us to analyze the MPNE geometrically. A significant advantage of this approach is that it allows us to discuss MPNE in non-differentiable and non-continuous strategies. This approach starts with the classical remark that the graph $p = p(z)$ of the function $p(z) = v'(z)$ is traced out by the solution curves $(z(s), p(s))$ of the 'auxiliary' system of differential equations, which take for $p \geq 0$ the form

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\partial G(z,p)}{\partial p} = p^\xi - z, \tag{4.10}$$

$$\frac{\mathrm{d}p}{\mathrm{d}s} = rp - \frac{\partial G(z,p)}{\partial z} = (1+r)p - \mu\Phi'(z). \tag{4.11}$$
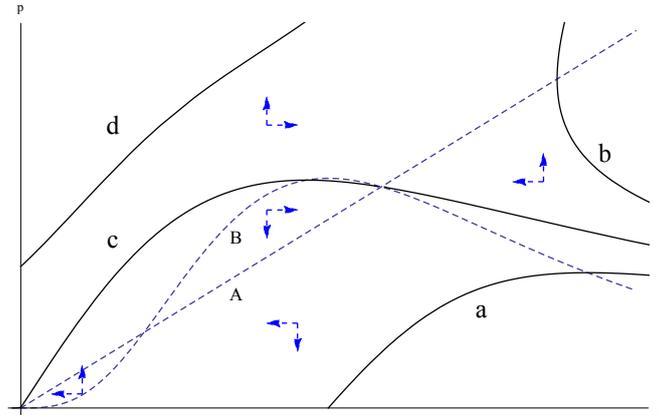
Fig. 4.2 Trajectories (solid curves *a*, *b*, *c*, and *d*) and isoclines $\frac{\mathrm{d}z}{\mathrm{d}s} = 0$ (dashed curve A), $\frac{\mathrm{d}p}{\mathrm{d}s} = 0$ (dashed curve B) of the auxiliary system given $\xi = \eta = 1$.

However, unlike the optimal control case, the curve parameter *s* does not have an interpretation as time (see Dockner and Wagener, 2014, for details). The system has to be complemented by the dynamic equation

$$\dot{z} = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{2}{2+\xi} p^{\xi} - z, \quad z(0) = z_0. \tag{4.12}$$

Figure 4.2 gives a phase diagram of the auxiliary system given $\xi = \eta = 1$. The dashed curves *A* and *B* denote respectively the isoclines $\mathrm{d}z/\mathrm{d}s = 0$ and $\mathrm{d}p/\mathrm{d}s = 0$; the solid curves *a*, *b*, *c* and *d* denote trajectories of the auxiliary system. However, not all of these correspond to a MPNE. Firstly, a MPNE gives rise to a function $p = p(z)$ that is defined on all of state space (Rowat, 2007): this requirement disqualifies for instance the curve *b* immediately.

But this requirement does not eliminate sufficiently many candidates. So let $p = p(z)$ be the graph of a function that is traced out by trajectories of the auxiliary system and that is defined for all $z \geq 0$. The corresponding knowledge dynamics can then be determined by substituting $p = p(z)$ in (4.12), yielding the state evolution $z^* = z^*(t)$.

As the value function is Lipschitz continuous, its derivative $p(z)$ has to be bounded, which rules out the trajectory *d*; moreover, since *v* is non-decreasing, necessarily $p(z) \geq 0$, and trajectories cannot cross into the region $p < 0$: this rules out *a*.

On the other hand, trajectory *c* corresponds to a MPNE. In view of theorem 1, it is sufficient to check that it gives rise to a continuous and bounded value function that is defined over all knowledge states.

4.3   Parameter dependence of innovation regimes

In this section, we discuss the dynamics of the knowledge stock in the patent portfolio race, given that the players choose MPNE strategies. These dynamics can be classified in a small

(a) Candidate curves   (b) Jump point
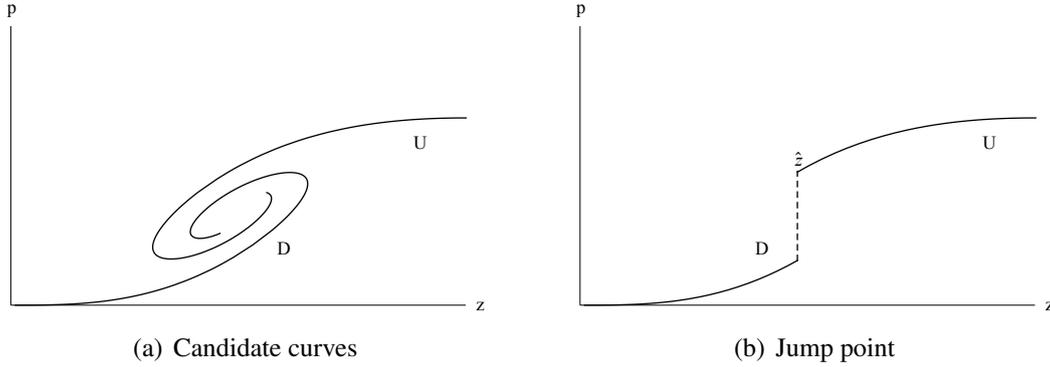
Fig. 4.3 Discontinuous MPNE. *Curves U and D are both trajectories of the auxiliary system. At the jump point, the jump condition $G(z, p_L) = G(z, p_U)$ is satisfied.*

number of qualitatively different types, which we call innovation regimes; a system can only change its innovation regime by going through a bifurcation. The bifurcation theory relevant to model is developed in appendix A.2. Here we analyse the implications of the resulting diagrams, which depict the parameter dependence of the innovation regimes. Specifically, we make the specific choices $\eta = \xi = 1$; we take moreover $h(z) = z^2/(1+z^2)$, so that the hazard rate is marginally increasing for small values of the knowledge stock and marginally decreasing if z is large. We note that this convex-concave behaviour of the hazard rate does not drive our results; in fact, the result presented in this section can also be obtained for a fully concave hazard rate.

The four qualitatively different types of Markov perfect Nash equilibria are illustrated in figure 4.4, panels (a)-(d). They correspond to three main types of innovation regimes, one regime having two subtypes. The main types are, respectively, a market-driven innovation regime, a conditional innovation regime and a regime where it is unlikely that many innovations will be made.

4.3.1   The innovation regimes

The first regime is characterised by market-driven innovation. Figure 4.4(a) shows the non-cooperative equilibrium effort levels $u_{\text{Nash}}(z)$ for this regime relative to the effort level $u_{ss}(z)$ that would be necessary to keep the knowledge stock constant at the state z. It follows that if $u_{\text{Nash}}(z) > u_{ss}(z)$, the knowledge level will increase, while if $u_{\text{Nash}}(z) < u_{ss}(z)$, it will decrease.

The market-driven innovation level is characterised by the fact that there is a single globally attracting positive steady state knowledge stock. Innovation is entirely driven by the competition amongst the two firms. We shall see that market-driven innovation occurs for high innovation benefits or low rates of time preference.

Figures 4.4(b) and 4.4(c) illustrate two situations that are mathematically different, but dynamically very similar: both are instances of a situation where market driven innovation

(a) Market-driven innovation

(b) Conditional innovation with slow separation

(c) Conditional innovation with fast separation

(d) Unlikely innovation
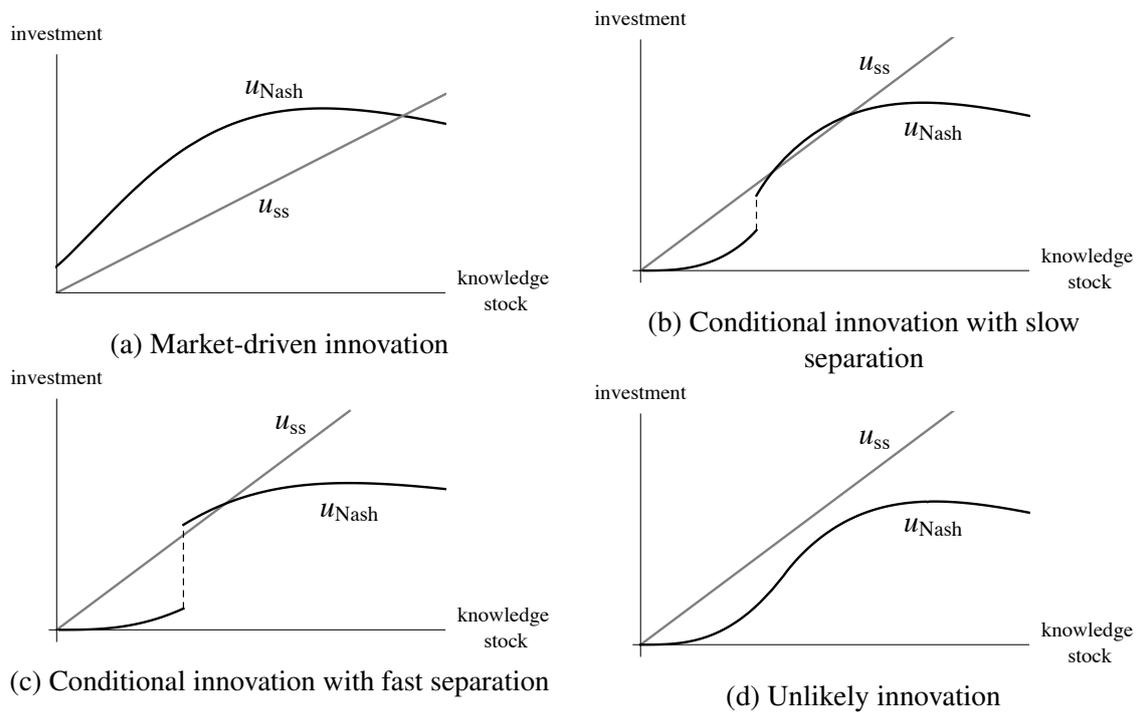
Fig. 4.4 Innovation regimes. *The figures in the left colum show the four generic types of Markov perfect Nash equilibrium investment strategies u_{Nash}, as well as the investment level u_{ss} necessary to maintain a steady state knowledge stock: if$u_{\text{Nash}} > u_{ss}$, the stock increases, if $u_{\text{Nash}} < u_{ss}$, it decreases.*

(a) Constant time preference

(b) Constant total profit

(c) Constant patent protection
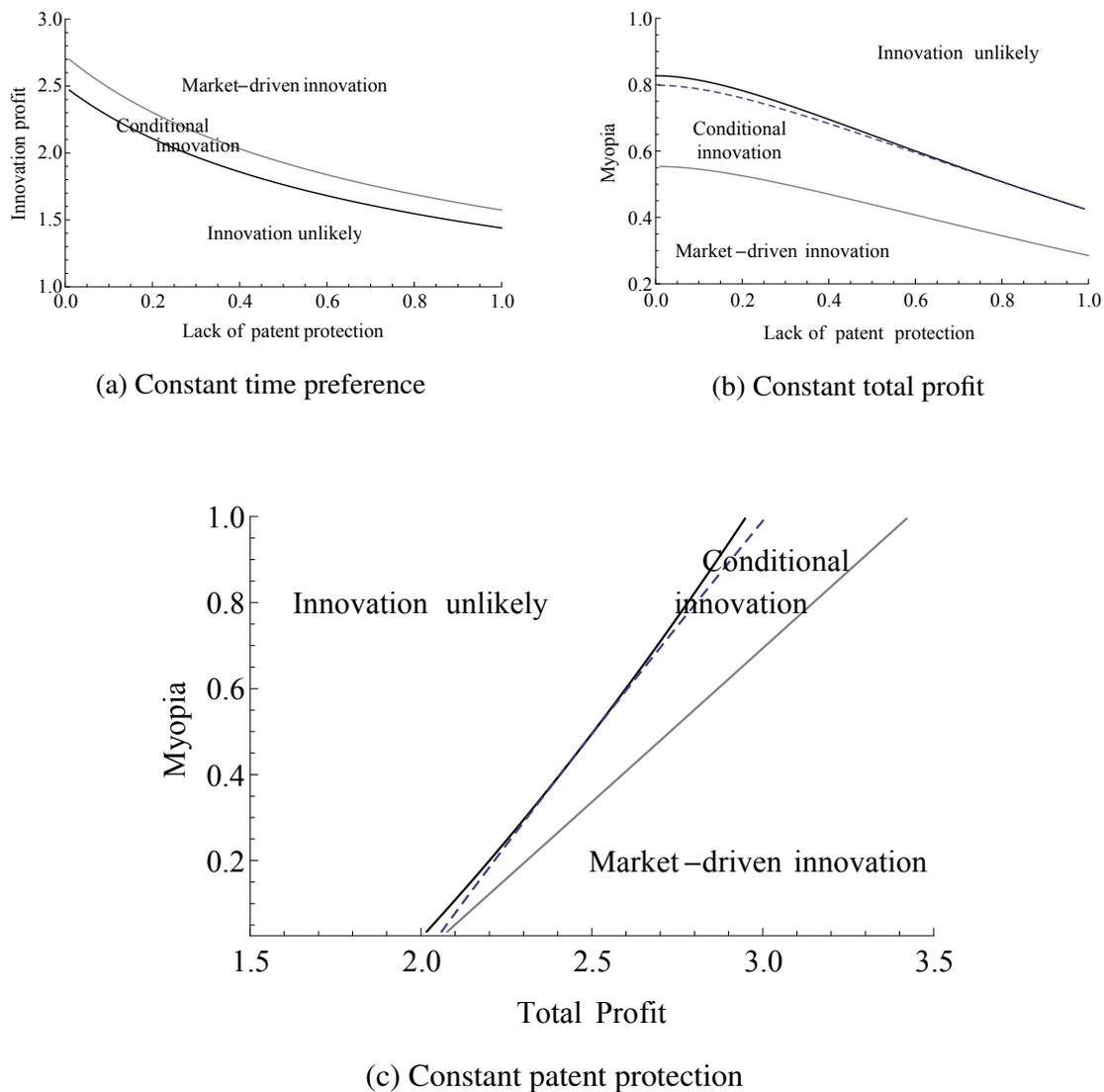
Fig. 4.5 Parameter regimes. *The figures show the dependence of the innovation regimes on innovation profit π versus lack of patent protection κ; time preference rate r versus lack of patent protection κ; and time preference rate r versus total profit $(1 + \kappa)\pi$. Parameters that are not varied are kept at the values $r = 0.525$, $\kappa = 0.2$, $\Pi = 2.765$.*

is possible, conditional on the initial knowledge stock being sufficiently high. For instance, in figure 4.4(b), there is a critical knowledge stock level at $\hat{z} \approx 0.43$. If the initial stock is less than this critical value, the effect of the strategic interaction between the firms is letting public knowledge peter out in the long run. The difference between the two situations is the nature of the critical level: if it is a repeller, the stock level will linger for a while at this level, before going either on an upward or a downward trajectory; if it is an indifference point, the knowledge stock will move away from it much more quickly.

In the regime where there is conditional innovation only if the initial knowledge stock is sufficiently high, the intervention of a policy maker might propel the initial stock past the critical value; competition takes over from there. Note that the policy maker's decision problem is not treated in this paper; it will of course depend on the public benefits of having an active knowledge industry in this field.

Finally, the third regime is characterised by a knowledge stock that eventually decays to zero with respect to any initial level or any amount of subsidy, cf. figure 4.4(d). As long as knowledge is at a positive level, there is some probability for an invention, but this decays quickly, and even the probability of a single invention is smaller than 1. This 'unlikely innovation' regime typically obtains if the innovation benefit is too low or the depreciation rate is too high.

4.3.2 Results

In the analysis of the dependence of these three regimes on parameters, we focus on the lack patent protection κ, the innovation benefit π or the related quantity total benefit Π, and the rate of time preference r. The resulting classification diagrams are shown as 4.5(a)–4.5(c).

The first of these, figure 4.5(a), shows the impact of varying the degree of patent protection and the innovation profit, while keeping the time preference rate constant. We conclude from it that higher innovation profits simulate market-driven innovation, which is as expected, but also that strong patent protection has an adverse effect on innovation, which is at first glance surprising. It is explained by the fact that total profit

$$\Pi = \pi + \kappa\pi \tag{4.13}$$

does actually depend on the degree of patent protection: decreasing the amount of protection, that is, increasing κ, increases the total profit. In this situation patent protection is restrictive, as it stops the imitating firm from achieving a profit it would otherwise have made. An example of this would be a software patent which would stop the second firm from writing an application that does not compete with the application developed by the first firm.

A different situation is obtained if π and κ are constrained by the requirement that total profit is constant, that is, by equation (4.13). In figure 4.6 a level curve of Π is shown as
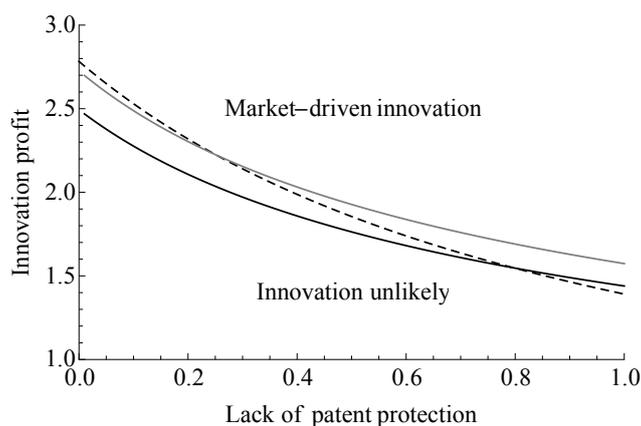
Fig. 4.6 Effect of patent protection if total profit is constant. In that situation, innovation profits depend on the degree of patent protection as indicated by the broken line.

a broken line. Along this curve, the effect of patent protection is reversed: strong patent protection fosters market-driven innovation, while weak patent protection stifles it.

In figure 4.5(c), the degree of patent protection is held constant, and total profit and the time preference rate are varied against each other. Again it is immediately apparant that an increase in the level of total profit is advantageous for market-driven innovation, while myopia of the firms is disadvantageous.

More specifically, there appears to be a critical value of total profit, below which the industry is always in the 'unlikely innovation' regime. The conditional innovation regime is wedge-shaped: this means that for small values of the total profit — for instance a highly specialised market — conditional innovation is possible if the competing firms can afford to have a long time horizon. This description might fit specialised industries cooperating closely with a well-informed banking sector that allows them to take a long term view. Also note that for small values of the time preference rate, there is virtually no situation where a social planner might want to intervene.

This is different for the situation where the time preference is large, which would be typical for fast-moving industries, where knowledge is quickly obsolete. Firstly, the profits have to be much larger for market-driven innovation to be sustainable, and secondly, the region where actions by a social planner may influence the outcome is much larger. This would lend support for governments pursuing an active campaign of supporting knowledge industries.

Finally, figure 4.5(b) considers the situation that total profit is constant, and that the degree of patent protection and the time preference rate are varied. Here we see the same kind of result as in figure 4.6, that stronger patent protection supports market-driven innovation. The effect seems however to be relatively weak.

4.4   Patent portfolio race for n firms

In this section, we generalise the model to a patent portfolio race for n firms that either innovate or imitate. As before, an innovation is made by a single firm, which is rewarded by the innovation benefit π; the non-innovating firms all obtain the same imitation benefit $\kappa\pi$.

We want to study the resulting innovation regime as a function of the number of competing firms. From our discussion in section 4.3.2 it follows that we have to distinguish between at least three situations: total benefits are constant, that is

$$\pi + (n-1)\kappa\pi = const;$$

the imitation benefit $\kappa\pi$ is constant and distributed evenly over the imitating firms, each of which receives $\kappa\pi/(n-1)$; and the situation that each imitating firm obtains a benefit $\kappa\pi$, independent on the number of firms.

We shall therefore begin with the situation that no a priori relation is assumed between innovation benefits π and imitation benefits $\kappa\pi$. Working again in natural variables, the value function of firm i is then as follows:

$$v_i = \sup \int_0^\infty e^{-rt} \left(\pi \ell_i h_i(z) + \sum_{j \neq i} \kappa\pi \ell_j h_j(z) - \frac{\ell_i^{1+1/\eta}}{1+1/\eta} - \alpha \frac{u_i^{1+1/\xi}}{1+1/\xi} \right) \, dt,$$

subject to the requirements that $\ell_i \geq 0$, $u_i \geq 0$ and that the knowledge stock z evolves according to

$$\dot{z} = \sum_{i=1}^n u_i - z.$$

The Hamilton-Jacobi-Bellman equation for the case of n firms is

$$rv_i(z) - \max_{\ell_i, u_i} \left[\left(\pi \ell_i h_i(z) + \kappa\pi \sum_{j \neq i} \ell_j h_j(z) \right) - \frac{\ell_i^{1+1/\eta}}{1+1/\eta} - \alpha \frac{u_i^{1+1/\xi}}{1+1/\xi} + v_i'(z) \left(\sum_{j=1}^n u_j - z \right) \right] = 0.$$

Optimal R&D effort and labour levels are

$$\ell_i = (\pi h_i(z))^\eta \quad \text{and} \quad u_i = \left(\frac{v_i'(z)}{\alpha} \right)^\xi, \quad \text{for} \quad i = 1, \ldots, n.$$

Again, we assume that $h_i(z) = h(z)$ for all i and consider only symmetric MPNE, where $v_i(z) = v(z)$ for all i. Set as before $p(z) = v'(z)$; choose $\alpha = (n + (n-1)\xi)^{1/\xi}$, and introduce the effective innovation benefits and the effective innovation rate as

$$\mu = \left(\frac{1}{1+\eta} + \kappa(n-1) \right) \pi^{1+\eta} \quad \text{and} \quad \Phi(z) = h(z)^{1+\eta}. \tag{4.14}$$

The HJB equation then takes the same form as above

$$rv(z) - G(z,p) = 0,$$

with

$$G(z,p) = \mu\Phi(z) + \frac{p^{1+\xi}}{1+\xi} - pz.$$

The auxiliary system together with the equation for the knowledge dynamics takes the form

$$\frac{\mathrm{d}z}{\mathrm{d}s} = p^{\xi} - z, \tag{4.15}$$

$$\frac{\mathrm{d}p}{\mathrm{d}s} = (1+r)p - \mu\Phi'(z), \tag{4.16}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{1 + (1 - 1/n)\xi}p^{\xi} - z. \tag{4.17}$$

We prove the following fundamental result about this auxiliary system in appendix A.3.

Theorem 2. *Assume that the function* Φ *is two times continuously differentiable, that* $\Phi(0) = 0$, $\Phi'(z) > 0$ *for all* $z > 0$ *and* $\Phi(z) \to 1$ *as* $z \to \infty$; *assume moreover that* $\Phi'(z)$ *has a finite number of critical points and that*

$$\frac{\Phi'(z)}{z^{1/\xi}} \to 0 \quad as \quad z \downarrow 0.$$

Then there are values $\mu_* = \mu_*(\xi, r)$ *and* $\mu^* = \mu^*(\xi, r)$ *of* μ, *such that for all n, the following holds. If* $0 < \mu < \mu_*$, *the symmetric Markov perfect Nash equilibrium is in the 'unlikely innovation' regime; if* $\mu > \mu^*$, *it is in the 'market driven innovation' regime.*

As in the situation with two firms, there is a significant difference between the situations that either total profit or the innovation benefit is independent of the degree of patent protection.

4.4.1   Constant total profit

The first situation we consider is that innovator profits are a fixed 'cake' of size Π, of which the innovator obtains a share π and each imitator $\kappa\pi$; moreover, the degree κ of patent protection is independent of the number of firms in the market.

This is the original situation for which patents have been developed: to give firms an incentive to innovate, the value that is added by the innovation should go to the innovator. It is expected that a working patent protection policy of this kind will then encourage many firms to work in a knowledge industry.

We model this by postulating that the innovating firm and the $n-1$ imitating firms share a fixed total profit Π according to the equality

$$\Pi = \pi + (n-1)\kappa\pi.$$

Consequently, the innovator profit depends on the number of imitators and the patent protection policy as

$$\pi = \frac{\Pi}{1 + (n-1)\kappa}.$$

Substituting this into the effective innovation benefits μ yields

$$\mu = \frac{\frac{1}{1+\eta} + (n-1)\kappa}{(1 + (n-1)\kappa)^{1+\eta}} \Pi^{1+\eta}.$$

For instance, under full patent protection $\kappa = 0$, effective innovation benefits are independent of the number of firms

$$\mu = \frac{\Pi^{1+\eta}}{1+\eta},$$

while if patent protection is absent ($\kappa = 1$), we have

$$\mu = \frac{1 + (n-1)(1+\eta)}{n^{1+\eta}} \frac{\Pi^{1+\eta}}{1+\eta} \sim \frac{\Pi^{1+\eta}}{n^{\eta}}, \quad \text{as} \quad n \to \infty.$$

More generally

$$\mu \sim \frac{\Pi^{1+\eta}}{\kappa^{\eta} n^{\eta}} \quad \text{as} \quad n \to \infty. \tag{4.18}$$

By theorem 2, a self-sustaining positive knowledge stock exists if $\mu > \mu^*$. The above results show that if patent protection is perfect ($\kappa = 0$), a knowledge industry can support any number of firms if total profit is sufficiently large. Of course, this is a formulation of the rationale for the patent system. If however patent protection is imperfect, there is a maximum number of firms that can be active in the knowledge industry, before the strategic incentive to invest in the publicly available fundamental knowledge disappears. This number is obtained by solving $\mu = \mu^*$ for n, where we take for μ the asymptotic approximation from equation (4.18), leading to

$$n_{\max} \sim \frac{\Pi^{1+1/\eta}}{\kappa(\mu^*)^{1/\eta}}. \tag{4.19}$$

We summarise this discussion in the following result.

Theorem 3. *Consider the situation that the total profit generated by a single innovation is independent of the number of competitors and the degree of patent protection. Then there is a number n_{max}, approximately given by (4.19), such that for any number $n \leq n_{max}$ of firms, a positive knowledge stock can be supported.*

Under full patent protection ($\kappa = 0$), the number of firms that can be supported is unbounded, if total profit is sufficiently large; if total profit is too small, a positive knowledge stock can never be supported.

4.4.2   Constant benefits from innovation

There is a second situation, where the innovation benefits are constant, but where imitation may create additional value. Consider for instance the situation of software patents or software licences. These days, application development builds on a huge stack of pre-existing software, some covered by commercial licences, some by open source licences. A new application of a different kind that is developed using the same low-level building blocks will not affect the market share, and the benefits, of that application for which these building blocks were developed originally.

To model this situation, we assume that innovation profit π as well as imitation profit $\kappa\pi$ are both independent of the number n of firms in the market. The effective innovation benefit μ as given in (4.14) is then linearly increasing in the number of firms if $\kappa > 0$; note moreover that if $\kappa = 0$, we are back in the situation of the previous section.

In the following theorem, the notation $[x]$ is used for the largest integer number that is smaller than or equal to x.

Theorem 4. *Consider the situation that the innovation benefit π is independent on the number n of imitators or on the imitation benefits $\kappa\pi$. Assume $\kappa > 0$.*

Then there is a minimal number n_{min} of firms that can support a positive knowledge stock. We have

$$1 + \left[\frac{1}{\kappa}\left(\frac{\mu_*}{\pi^{1+\eta}} - \frac{1}{1+\eta}\right)\right] \leq n_{min} \leq 1 + \left[\frac{1}{\kappa}\left(\frac{\mu^*}{\pi^{1+\eta}} - \frac{1}{1+\eta}\right)\right].$$

Note that $n_{\min}$ decreases with innovation profits π: for small innovation profits, a large number of firms are needed to sustain a positive knowledge stock. Also, $n_{\min}$ decreases with κ: if the patent regime is weak, or absent as in the case $\kappa = 1$, it is easier to sustain a stock of public knowledge. This reflects for instance the pioneering 'software sharing culture' described by Stallman (1999), and which is at the basis of the 'open software' culture (Lerner & Tirole, 2002).

4.5   Conclusions

We have modeled a patent portfolio race with public knowledge accumulation where symmetric firms compete with each other to innovate and build up their patent portfolio. If a firm does not innovate, it obtains rewards from innovation spillovers from its rival; the size of these rewards is determined by the degree of patent protection. We follow Arrow and Romer by assuming that, at least for small knowledge levels, knowledge production has increasing returns to scale. Firms invest strategically in the public knowledge stock as well as in the labour necessary to exploit the public stock.

In this game, we have found three different types of Markov perfect Nash equilibria, corresponding to three innovation regimes: market-driven innovation, where the non-cooperative

competition between the firms leads always to the creation of a positive steady state knowledge stock; conditional innovation, where the competing firms can sustain a positive knowledge stock indefinitely, but they need help creating a sufficiently large initial stock; and unlikely innovation, where the equilibrium outcome is to let the stock disappear.

We have investigated the dependence of the resulting innovation regime on a number of parameters: innovation profit, total profit, degree of patent protection and the time preference rate. We found that the benefit structure has significant effects on the outcome. If innovations only help to acquire a certain share of the total profits, then market-driven innovation fares better with a strong patent protection scheme. We showed that in this situation the maximal number of competing firms that can sustain a long-term positive knowledge stock increases much more rapidly with the total profit level under strong patent protection than under weak patent protection. This finding reinforces the traditional support of a patent protection scheme.

If however the profits of the innovator are independent of the degree of patent protection, market-driven innovation is more likely to obtain if patent protection is weaker. In fact, there is a critical degree of patent protection, below which the market can sustain an infinite number of firms. While it is true that both their costs and their benefits are small, this is not unrealistic. Lerner and Tirole (2001) remark, in the context of open source projects, that part of the reason why people contribute is that the cost of contributing is not that high.

We also investigate the impact of the average planning horizon of the competing firms, which is often dictated by their financing. We find that market-driven innovation can already occur if total profits are relatively small, if the planning horizon is sufficiently large; an example would be a highly specialised small firm that has a long-term relation with a bank or which is fully self-financed.

Chapter 5

Summary

Firms accumulate their knowledge stock by investing in R&D, which increases their chances of innovation. They compete with each other to become the first innovator and get the innovation benefits. This game is a patent race. All three chapters are based on the model of a patent race and the objective is to find the Markov perfect Nash equilibrium that firms' optimal strategies of R&D investment with respect to opponents' optimal strategies.

Chapter 2 features on the property between R&D effort and knowledge. The property of knowledge plays a more important role of R&D investment especially for the follower firms. If knowledge and R&D effort are complementary, the follower with less knowledge than the leader is relatively inferior in a patent race. Its incentive to innovate becomes very low. In this case, the chances of innovation become little in the long run with any patent policy. In contrast, if R&D and knowledge additively contribute to chances of innovation, high R&D effort compensates the followers' insufficient knowledge pool. Because of the pure knowledge effect the leader firm reduces its R&D investment by its large knowledge pool, which increases the follower chances of innovation relatively. Strategically, the follower's incentives to innovate increases.

In the case of additive property, a patent can encourage innovation. However, a patent holder monopolized the market indirectly. Opponents cannot infringe the patent, naturally become less competitive, and finally leave the market. This monopoly of innovator with a patent harms social welfare. Chapter 3 connects a patent race model with products market in order to discuss the impact of the patent policy on social welfare. Longer patents intriguer higher innovation payoff, which leads to earlier innovation, but gives longer harm of monopoly on welfare. Instead of patenting, research joint venture (RJV) cartels allows firms to cooperate on R&D but compete in duopoly. RJV cartels bring more efficient process innovation on production, which results in an early innovation and even increase welfare in a market competition under efficient R&D.

In reality, firms face more than one competitor and more than one patent in a race. In Chapter 4, firms allow to innovate several times and compete with a collection of patents,

which is a patent portfolio race. In this patent portfolio race model, the benefit structure of innovation has significant effect of R&D incentives in terms of the patent policy. If a patent portfolio can ensure the innovator to acquire certain share of innovation profits, R&D incentives increase with strong protection. If patents cannot ensure the share of benefits, firms would rather innovate without patent protection. This describes the case of successful innovation of open-source software that has been developed without patent protection.

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Appendix A

A.1 Existence and uniqueness

We need the notion of sub- and superdifferentials of a function at a point (cf. Bardi and Capuzzo-Dolcetta, 2008, chapter I, lemma 1.7). Let $z \in \mathbb{R}$, and let $v(z)$ be a continuous function. Then $p \in \mathbb{R}$ is an element of the *superdifferential $D^+v(\bar{z})$* at $\bar{z}$, if there exists a continuously differentiable function $\varphi(z)$ such that $\varphi'(\bar{z}) = p$ and $v(z) - \varphi(z)$ takes a (local) maximum at $\bar{z}$; $p \in \mathbb{R}$ is in the *subdifferential $D^-v(\bar{z})$*, if there $\varphi'(\bar{z}) = p$ and $v(z) - \varphi(z)$ takes a (local) minimum at $\bar{z}$.

These notions are generalisations of the notion of sub- and supergradient, usually introduced in the context of convex functions. The definition of supergradient is entirely analogous to that of superdifferential, except that the 'test function' φ is restricted to the class of linearly affine functions. For instance, the supergradient of the function $v(z) = z^2$ is empty for all z, while $D^-v(z) = D^+v(z) = \{2z\}$ for all z.

A function $v(z)$ is a *viscosity subsolution* of an equation

$$F(z, v(z), v'(z)) = 0, \tag{A.1}$$

if $F(z, v(z), p) \leq 0$ for all z and all $p \in D^+v(z)$; it is a *viscosity supersolution* if $F(z, v(z), p) \geq 0$ for all z and all $p \in D^-v(z)$. If it is both a sub- and a supersolution, $v(z)$ is called a viscosity solution.

It should be noted that if $v(z)$ is a subsolution of equation (A.1), then it is a supersolution of the equation $-F(z, v(z), v'(z)) = 0$.

Theorem 5. *Let $\mu, \xi, r > 0$ be given constants. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable non-decreasing function, such that $\Phi(0) = 0$ and $\lim_{z \to \infty} \Phi(z) = 1$, and let*

$$G(z, p) = \mu \Phi(z) + \frac{|p|^{1+\xi}}{1+\xi} - pz. \tag{A.2}$$

In the class of bounded continuous functions, there is a unique viscosity solution $v(z)$ of the HJB equation

$$rv(z) - G(z, v'(z)) = 0. \tag{A.3}$$

This function is moreover Lipschitz continuous and non-decreasing as a function of z.

Proof. Let $M > 0$ be a constant. Let $w_M(z_0)$ value function of the optimal control problem that tries to minimise the functional

$$J(z, u) = \int_0^\infty e^{-rt} \left(\frac{|u|^{1+1/\xi}}{1 + 1/\xi} - \mu \Phi(z) \right) \mathrm{d}t$$

over the set $\mathscr{U}_M = \{u : [0, \infty) \to [-M, M] : u \text{ measurable}\}$, subject to the restriction

$$\dot{z} = u - z, \qquad z(0) = z_0.$$

This auxiliary problem is introduced to show the existence and uniqueness of the solution of the game HJB equation, as well as to show that the solution is non-decreasing. We shall apply results from Bardi and Capuzzo-Dolcetta (2008). In the remainder of this section, references to chapters or propositions shall all be taken to refer to that book. For instance, a reference like "proposition III.2.1" means "proposition 2.1 of chapter III in Capuzzo-Dolcetta (2008)"; "equation III.(2.9)" will refer to equation (2.9) in chapter III.

We shall show that for M sufficiently large, $v(z) = -w_M(z)$ does not depend on M and is a viscosity solution of (A.3). To show this, let $U = [-M, M]$ and $f(z, u) = u - z$ if $z \geq 0$ and $f(z, u) = 0$ if $z < 0$. Then f satisfies assumptions A_0, A_1, and A_3 of chapter III in BC. Set $\ell(z, u) = |u|^{1+1/\xi}/(1 + 1/\xi) - \mu \Phi(z)$. Since Φ is bounded and $|u| \leq M$, also assumption A_4 from the same chapter is satisfied.

Proposition III.2.1 implies that the value function w_M of this minimisation problem exists, is continuous and even uniformly Hölder regular. By choosing $u(t) = 0$ for all t, it is clear that $v(z) \leq J(z_0, u) \leq 0$. As moreover $\Phi(z) \leq 1$ for all z, we obtain the bound $-\mu/r \leq v(z) \leq 0$ for v.

The Hamilton function of the minimisation problem, as defined in equation III.(2.9), is

$$\mathscr{H}_M(z, p) = \sup_{|u| \leq M} \left[-pf(z, u) - \ell(z, u) \right] = \mu \Phi(z) + \frac{|p|^{1+\xi}}{1 + \xi} + pz,$$

where the last equality holds only for $|p| \leq M^{1/\xi}$. Proposition III.2.8 then implies that the value function w_M is a viscosity solution of the equation

$$rw(z) + \mathscr{H}_M(z, w'(z)) = 0. \tag{A.4}$$

As $\mathscr{H}$ is coercive, that is $\mathscr{H}(x,p) \to \infty$ as $|p| \to \infty$, and w_M is bounded, continuous, and a viscosity solution of (A.4), it follows from proposition II.4.1 that w_M is Lipschitz continuous, with Lipschitz constant C. It follows moreover from the proof that C is independent of M, if M is sufficiently large, say, larger than $M_1 > 0$. Equations III.(4.4) and III.(4.6) then imply that the super- and subdifferentials of w_M are contained in the interval $[-C, C]$. If M is taken larger than both M_1 and C^ξ, then the restriction $|p| \leq M^{1/\xi}$ is never binding, and $w(z) = w_M(z)$ is a viscosity solution of the equation

$$rw(z) + \mathscr{H}(z, w'(z)) = 0,$$

where

$$\mathscr{H}(z,p) = \sup_u \left[-pf(z,u) - \ell(z,u) \right]. = \mu\Phi(z) + \frac{|p|^{1+\xi}}{1+\xi} + pz,$$

From this, we infer that $v(z) = -w(z)$ is a viscosity solution of

$$rv(z) - \mathscr{H}(z, -v'(z)) = rv(z) - G(z, v(z)) = 0.$$

This shows existence of a Lipschitz continuous viscosity solution.

Uniqueness in the class of bounded continuous function follows directly from theorem III.2.12.

To show non-decreasingness of v, fix z_{10} and $\varepsilon > 0$, and find $u(t)$ such that if $z_1(t)$ satisfies

$$\dot{z}_1 = u - z_1, \quad z_1(0) = z_{10},$$

then

$$J(z_1, u) \leq w(z_{10}) + \varepsilon.$$

Take $z_{20} > z_{10}$, and let $z_2(t)$ be the solution of

$$\dot{z}_2 = u - z_2, \quad z_2(0) = z_{20}.$$

Then $y(t) = z_2(t) - z_1(t)$ satisfies

$$\dot{y} = -y, \quad y(0) = z_{20} - z_{10} > 0,$$

implying that $z_1(t) < z_2(t)$ for all t. As Φ is non-decreasing, it follows that

$$w(z_{10}) = J(z_1, u) - \varepsilon \geq J(z_2, u) - \varepsilon \geq w(z_{20}) - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we conclude that $w(z_{10}) \geq w(z_{20})$ whenever $z_{10} < z_{20}$. This implies that w is non-increasing and that $v(z) = -w(z)$ is non-decreasing. $\square$

A.2 Bifurcation theory of differential games

A MPNE strategy is given by a function $p = p(z) = v'(z)$, which is such that v satisfies the HJB equation (4.8).

A dynamic equilibrium is obtained if the graph $p = p(z)$ of the MPNE strategy intersects the dynamic isocline $\dot{z} = 0$. Figure A.1 illustrates a typical situation. The graph of the discontinuous MPNE intersects the dynamic isocline at SS_H, SS_R and SS_L (see figure A.1(a)), which are therefore steady states of the knowledge dynamics, given in figure A.1(b). As $\dot{z} > 0$ in the area above the dynamic isocline, the knowledge stock is increasing there; below the isocline, it is decreasing. We conclude that SS_L and SS_H are attractors, while SS_R is a repeller.

For different parameter values, we may have a different number of attractors, repellers and jump points. These numbers change at bifurcations.

One of the main determinants of a bifurcation is its codimension: this is the minimal number of parameters that needs to be varied in order to encounter the bifurcation 'in a stable way'. 'Stable' means in this context that the bifurcation also occurs in a system that differs only slightly from the original one. Bifurcations of codimension one are encountered most often, as they occur stably even in one-parameter families. We shall encounter also bifurcations of codimension two, but not of any higher codimension.

A different way of thinking about codimensions is the following: the codimension of a bifurcation is the difference between the dimension of the parameter space and the dimension of the set of parameter values associated to the bifurcation. For instance, in a one-parameter model, the parameter set of a codimension one bifurcation will have dimension zero, that is, it will consist of isolated points. Likewise, in a two-parameter model, codimension one bifurcation values will form one-dimensional sets, i.e. curves, and codimension two bifurcations occur at isolated points. Note that all this dimension counting proceeds from the assumption of the system being generic, and the bifurcation sets being in 'general position'.

More precisely, we shall discuss four different types of codimension 1 bifurcations of MPNE: saddle node (SN), indifference attractor (IA), game indifference attractor (GIA) and game indifference repeller (GIR) bifurcations. In addition to these codimension one bifurcations, we discuss a codimension two bifurcation that occurs in the model, a game indifference saddle node (GISN) bifurcation. The bifurcation diagram shows which parameter values result in different kinds of dynamics. We focus on the parameters κ, π and r in order to study how patent protection, innovation benefits and far-sightedness help to accumulate public knowledge in the long run.

A.2.1 Saddle-node bifurcation

A dynamical system experiences a saddle-node bifurcation if, on varying a parameter, two steady states coalesce and disappear. For a general differential game, we define the bifurcation analogously: a Markov perfect Nash equilibrium that depends continuously on the game's

(a) MPNE and dynamic isocline

(b) Knowledge dynamics
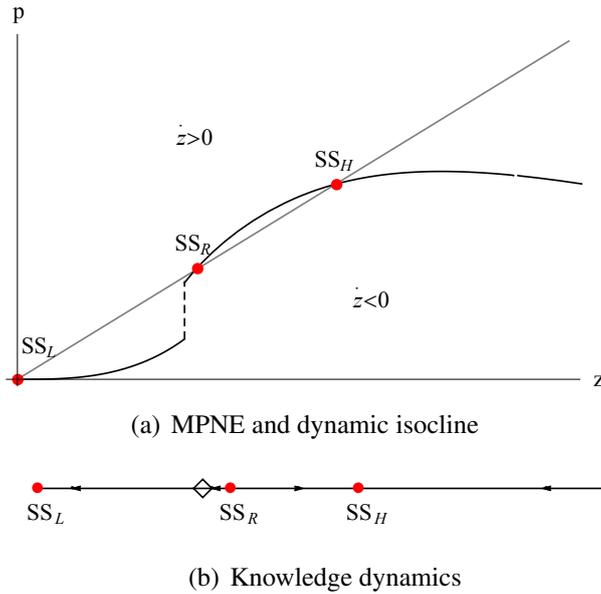
Fig. A.1 State dynamics. *Panel A.1(a): steady states correspond to intersections of the MPNE and the dynamics isocline; panel A.1(b): the resulting state dynamics. Black dots denote steady states, the diamond denotes a jump point .*

parameters experiences a saddle-node bifurcation if the dynamics that result from both players choosing their equilibrium feedback strategy goes through a saddle-node bifurcation.

Figure A.2 illustrates the bifurcation scenario. In A.2(d), there are a repeller and an attractor right of the jump state, which is marked with a diamond. At bifurcation — figure A.2(e) — they coalesce and disappear thereafter (figure A.2(f)), leaving only the 'no knowledge' steady state $z = 0$, which is then a global attractor. From figure A.2(b), we learn that this coalescing happens at the parameter value for which the graph of the MPNE strategy is tangent to the dynamic isocline.

More exactly, the saddle-node bifurcation is defined by the requirement that this tangency occurs and is nondegenerate. We have the following necessary conditions, formulated in terms of the parametrisation $(z(s), p(s))$ of the solution curve of the auxiliary system:

$$\begin{cases} \text{Dynamic equilibrium: } p_U(s_U) = \frac{3}{2} z_U(s_U) \\ \text{Tangency: } p'_U(s_U) = \frac{3}{2} z'_U(s_U). \end{cases}$$

Since the auxiliary system is highly non-linear, it is unlikely that an analytical solution exists. Hence, we have to resort to numerical methods to find the MPNE trajectories as well as the bifurcating parameter values.

In the present model, the saddle-node bifurcation separates the regime with a positive knowledge stock that is sustained by the competing R&D efforts of the two firms and that generates a stream of inventions (figures A.2(a) and A.2(d)) and the regime where no positive
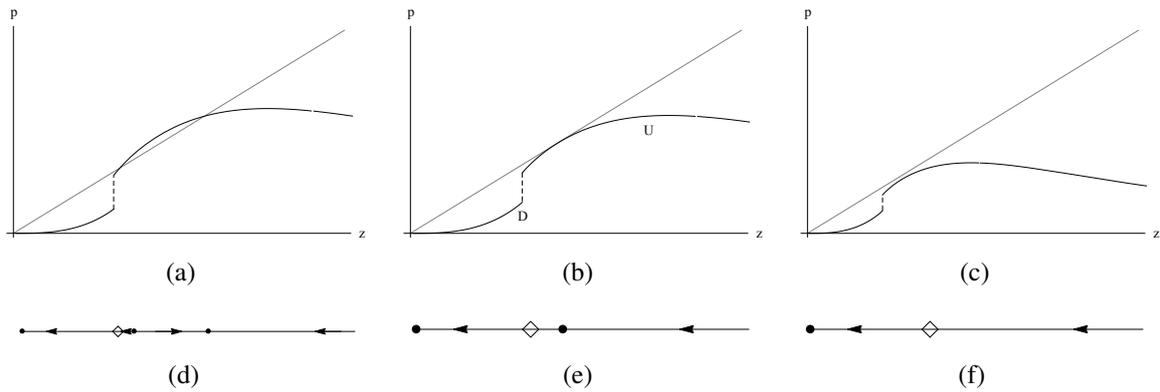
Fig. A.2 Saddle-node bifurcation scenario. *Panels A.2(a),A.2(b), and A.2(c) show the MPNE and the dynamic isocline; A.2(d), A.2(e), and A.2(f) give the corresponding state dynamics. The sequence illustrates a saddle-node bifurcation scenario, with the bifurcation occurring at A.2(b) and A.2(e).*

stock of knowledge can be supported in this way, and where by consequence even the probability of making a first innovation is quite low. Note however that the first regime can only be entered if the initial knowledge stock is already sufficiently high, or if it is pushed to the self-sustainable regime by a sufficiently large initial knowledge subsidy of a policy maker. The lower boundary of the sustainable regime is marked by an indifference point, which features in the next bifurcation.

## A.2.2   Indifference-attractor bifurcation
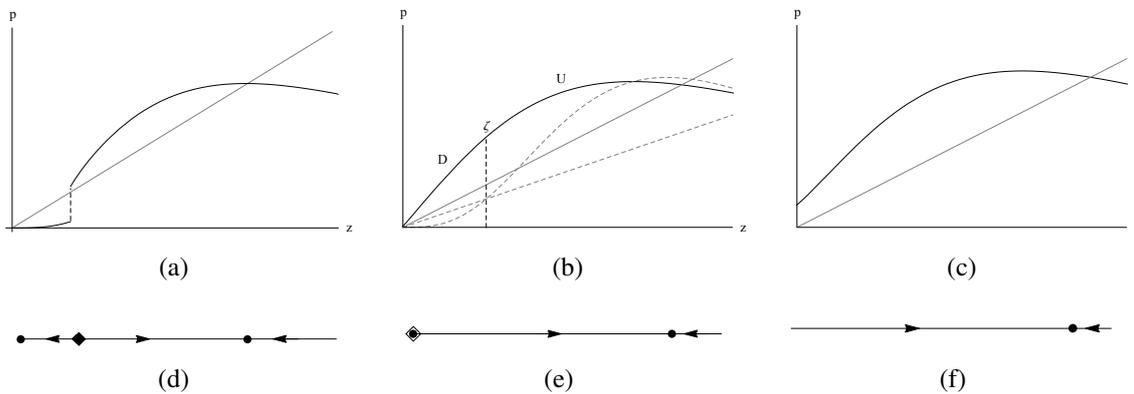
Fig. A.3 Indifference-attractor bifurcation scenario. *Panels as in figure A.2. The sequence illustrates an indifference-attractor bifurcation at the origin. The gray dashed curves are the isoclines of the auxiliary system.*

Before we can introduce the indifference-attractor bifurcation, we first need to discuss indifference points. In economic dynamic optimisation models, these are often associated with the name of Skiba (1978) and Dechert and Nishimura (1983), although they have been described earlier. These are points at which the optimal policy function, expressed as a function of the state variable, has a jump discontinuity, and where the agent is consequently indifferent between two or more different policies. Note however that in the single-agent situation, the optimal policy will always lead away from the jump point: as a function of time, the action of the agent is continuous.

The situation is slightly different in dynamic games. There Markov perfect Nash equilibrium strategies also can have jump discontinuities. But, when compared to the single player situation, there is now the additional possibility that the dynamics steers the state through a discontinuity, and that the actions of the agents, viewed as functions of time, can have simultaneous jumps (Dockner and Wagener, 2014).

To be consistent with the usage in dynamic optimisation, we shall call an indifference point of a Markov perfect Nash equilibrium strategy a point at which the strategy has a jump discontinuity, but which is such that no orbit of the resulting dynamics passes through this point. If the latter condition is not satisfied, we shall call such a point merely a jump point.

An indifference-attractor bifurcation is for the rest much like a saddle-node bifurcation, only that here an attractor and an indifference point coalesce and disappear. In figure A.3, an attractor at the origin and an indifference point in A.3(d), which marked with a black diamond, meet at the origin in A.3(e) and have disappeared in A.3(f).

The geometric characterisation is however not as straightforward as in the situation of the saddle-node. At bifurcation, there is a heteroclinic connection[1] between the unstable manifold D of the origin and stable manifold U of the steady state of the auxiliary system with the largest z coordinate (cf. figure 4.2).

To find a heteroclinic connection numerically, first a vertical line $z = \zeta$ has to be chosen inbetween the two saddles of the auxiliary system. At the connection, the following conditions are satisfied:

$$\begin{cases} p_D(s_D) = p_U(s_U) \\ z_D(s_D) = z_U(s_U) = \zeta \end{cases}$$

where $(z_U(s_U), p_U(s_U))$ and $(z_D(s_D), p_D(s_D))$ parameterize respectively the branches U and D in A.3(b); ζ is a constant. Note that in practice, the equations on the second line are used to determine s_D and s_U; once these are obtained, the difference $\Delta = p_D(s_D) - p_U(s_U)$ can be computed. A parameter value where this difference vanishes is then a heteroclinic bifurcation value.

In the present model, we have found the indifference-attractor bifurcation to occur only at the origin. It separates the regime where there is a self-sustainable knowledge stock that can be

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[1]A heteroclinic connection is formed when the unstable and stable manifold coincide; see Wagener (2003) for an exposition in the setting of dynamic optimisation models.

reached if the initial knowledge stock is sufficiently high from a regime with a self-sustainable knowledge stock that can be reached from every initial point. In the latter regime, the R&D competition between the firms will always lead to a stream of innovations, and no initial subsidy is needed to manipulate the initial stock of knowledge.

## A.2.3 Game indifference-attractor and game indifference-repeller bifurcation
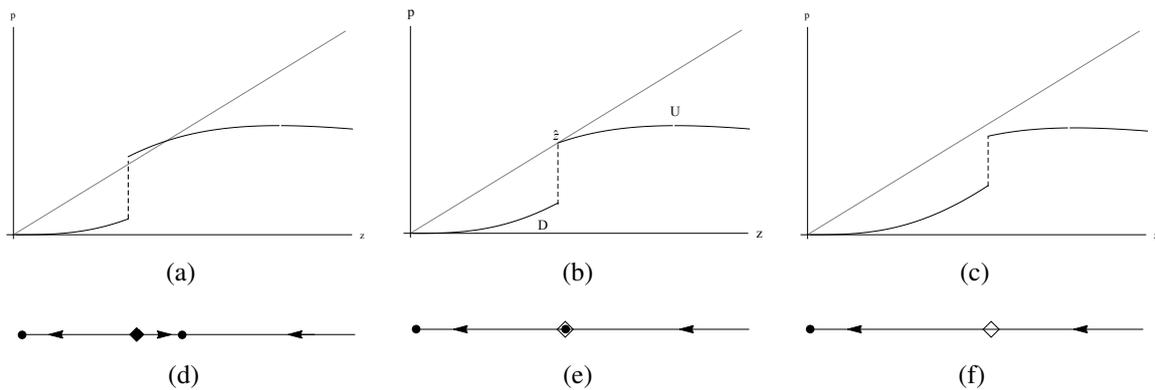
Fig. A.4 Game indifference-attractor bifurcation scenario. *Panels as in figure A.2. White diamonds are jump points; the black diamond is an indifference point.*
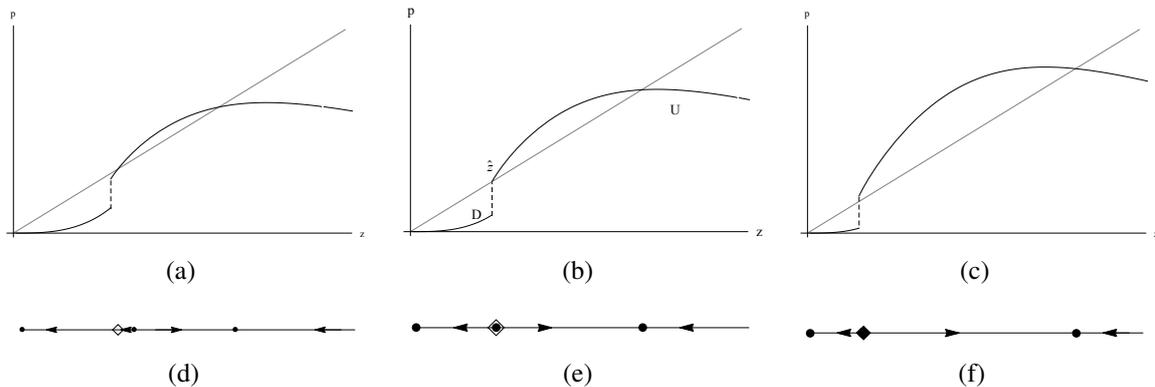

Fig. A.5 Game indifference-repeller bifurcation scenario. *Panels as in figure A.2.*

At a game indifference-attractor bifurcation an attractor and an indifference point coalesce; after the bifurcation a jump point remains. It is illustrated in figure A.4. Dynamically, it is rather similar to the saddle-node bifurcation, the indifference point taking the part of the repeller.

The game indifference-repeller bifurcation is similar to the game indifference-attractor bifurcation, except that now a repeller and a jump point coalesce, and after the bifurcation an

indifference point remains. It is illustrated in figure A.5. The 'game' in both names indicates that these bifurcations have no counterpart in the bifurcation theory of dynamic optimisation problems. This is of course due to the appearance of the jump point.

We observe in figures A.4(b) and A.5(b) that both bifurcation are similarly characterised: a jump point coincides with an intersection of the stable manifold U of the rightmost saddle of the auxiliary vector field and the dynamic isocline. The kind of intersection determines the kind of bifurcation. The observation leads to the following conditions:

$$\begin{cases} \text{Dynamic equilibrium: } p_U(s_U) = \frac{3}{2}z_U(s_U) \\ \text{Jump: } \hat{z} = z_U(s_U) = z_D(s_D) \quad \text{and} \quad G(\hat{z}, p_U(s_U)) = G(\hat{z}, p_D(s_D)). \end{cases}$$

where as before $(z_U(s_U), p_U(s_U))$ and $(z_D(s_D), p_D(s_D))$ parameterize the branches U and D in both figure A.4 and figure A.5. The conditions are used in the following order: from the dynamic equilibrium condition, the value of s_U is determined. Using the first of the jump conditions then yields the values of both $\hat{z}$ and s_D, and lets us compute the difference $\Delta = G(\hat{z}, p_U(s_U)) - G(\hat{z}, p_D(s_D))$. Parameter values for which this quantity vanishes are at bifurcation.

Even though the conditions of the two bifurcation are very similar, their effects are widely different. The game indifference attractor bifurcation is similar to the saddle node bifurcation in that it marks the boundary between the 'no innovation' and the 'conditional innovation' regimes. The game indifference repeller bifurcation is less interesting dynamically, as it seperates the situation where the state move slowly away from the repeller to the situation that the state moves quickly away from the indifference point. Both situations, though mathematically distinct, are in the 'conditional innovation' regime.

A.2.4   Game indifference-saddle-node bifurcation

The bifurcations discussed so far, saddle-node, indifference-attractor and the game indifference-attractor and indifference-repeller bifurcation, are all codimension one bifurcations that are encountered in the analysis of the patent portfolio race. In the two-parameter bifurcation diagrams which we consider below, these bifurcations are therefore represented by one-dimensional curves.

We have encountered a single codimension two bifurcation in the analysis, which occurs as terminal point of several of the codimension one bifurcation curves. This bifurcation we call, in analogy to a similar bifurcation in optimal control, game indifference-saddle-node bifurcation (cf. Kiseleva and Wagener, 2010, 2015).

The game indifference-attractor and the game indifference-repeller bifurcation are both characterised, by the coincidence of a transversal intersection of the dynamic isocline and the stable manifold U with a jump point. The game indifference-saddle-node bifurcation is similarly characterised by the coincidence of a tangency of U with a jump point. This is a
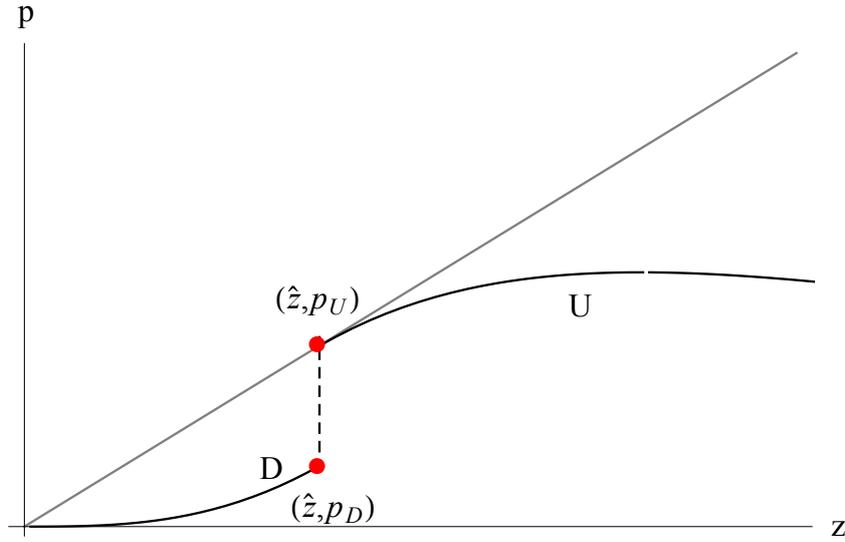
Fig. A.6 A game indifference-saddle-node bifurcation point. *At this bifurcation, two simpler bifurcation conditions are satisfied simultaneously: the stable manifold of the largest saddle of the auxiliary system is tangent to the dynamic isocline (saddle-node condition), and a jump point coincides with a dynamic steady state (indifference-attractor/indifference-repeller condition). In the case of constant patent protection, the GISN is at $\Pi = 2.45$ and $r = 0.445$; in the case of constant total profit, the GISN is at $\kappa = 0.94$ and $r = 0.447$. Parameters that are not varied are kept at the values $r = 0.525$, $\kappa = 0.2$, $\Pi = 2.765$.*

codimension 2 bifurcation determined by the conditions

$$
\begin{cases}
\text{Dynamic equilibrium:} & p_U(s_U) = \frac{3}{2} z_U(s_U) \\
\text{Tangency:} & p'_U(s_U) = \frac{3}{2} z'_U(s_U) \\
\text{Jump:} & \hat{z} = z_U(s_U) = z_D(s_D) \quad \text{and} \quad G(\hat{z}, p_U(s_U)) = G(\hat{z}, p_D(s_D)).
\end{cases}
$$

As before $(z_U(s_U), p_U(s_U))$ and $(z_D(s_D), p_D(s_D))$ parameterize the branches U and D in figure A.6. Note that these are four equations, containing two free parameters s_U and s_D. Hence at least two parameters are necessary to satisfy the equations generically, making this a codimension 2 bifurcation. Of course, a more extensive analysis will also have to check nondegeneracy conditions.

The GISN bifurcation point has no immediate economic significance, as it does not divide separate parameter regimes. But as it is, in a two-parameter bifurcation diagram, the endpoint of several codimension one bifurcation curves, it organises these curves. Such bifurcation points are sometimes called organising centres.
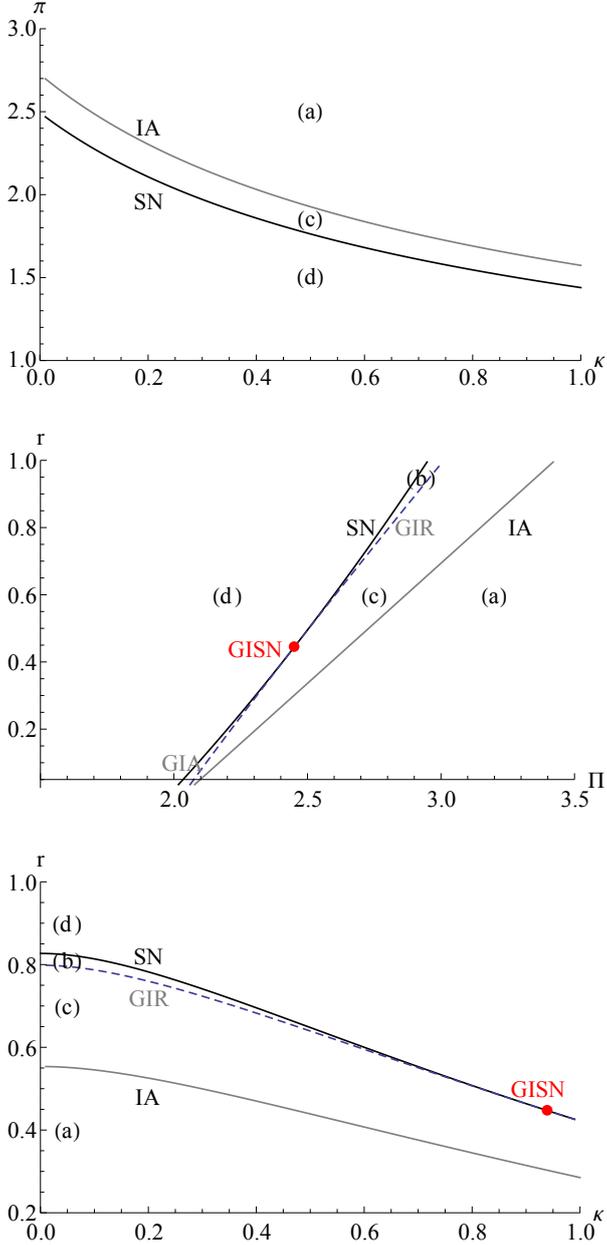
Fig. A.7 Bifurcation diagrams. *The dynamics of the parameters region (a), (b), (c) and (d) corresponds to figure 4.4. Parameters that are not varied are kept at the values r = 0.525, κ = 0.2, Π = 2.765.*

A.2.5 The bifurcation diagrams

Figure A.7 shows the resulting bifurcation diagrams, in the (κ, π), (Π, r) and (κ, r) parameter spaces respectively. An interpreted version of these diagrams is given in figure 4.4.

A.3 Proof of theorem 2

We repeat the statement of theorem 2.

Theorem. *Assume that the function Φ is two times continuously differentiable, that $\Phi(0) = 0$, $\Phi'(z) > 0$ for all $z > 0$ and $\Phi(z) \to 1$ as $z \to \infty$; assume moreover that $\Phi'(z)$ has a finite number of critical points and that*

$$\frac{\Phi'(z)}{z^{1/\xi}} \to 0 \quad as \quad z \downarrow 0.$$

Then there are values $\mu_* = \mu_*(\xi, r)$ and $\mu^* = \mu^*(\xi, r)$ of μ, such that for all n, the following holds. If $0 < \mu < \mu_*$, the symmetric Markov perfect Nash equilibrium is in the 'unlikely innovation' regime; if $\mu > \mu^*$, it is either the 'conditional innovation' or the 'market driven innovation' regime.

We construct the value function by analysing the auxiliary system

$$\frac{\mathrm{d}z}{\mathrm{d}s} = A_1(z, p) = p^\xi - z, \quad \frac{\mathrm{d}p}{\mathrm{d}s} = A_2(z, p) = (1+r)p - \mu\Phi'(z) \tag{A.5}$$

for $z \geq 0$. We call $A = A(z, p) = (A_1, A_2)$ the auxiliary vector field.

Proof, first part: small μ. We show that for $\mu > 0$ sufficiently small, the point $(z, p) = (0, 0)$ is the unique steady state. Then we show that there is a unique trajectory in the region $0 < p < z^{1/\xi}$ that converges to $(0, 0)$; this trajectory parametrises the graph of a continuously differentiable bounded function $p = p(z)$, which is such that $v(z) = \int_0^z p(x)\,\mathrm{d}x$ satisfies the game HJB equation.

Any steady state (z, p) has to satisfy the equations

$$p^\xi - z = 0, \quad (1+r)p - \mu\Phi'(z) = 0.$$

Eliminating p yields

$$\frac{1+r}{\mu} = \frac{\Phi'(z)}{z^{1/\xi}}. \tag{A.6}$$

If the right hand side is defined for $z = 0$ to equal 0, then, by hypothesis, it is a continuous function of z. As $\Phi'(z) \to 0$ for $z \to \infty$, it has to take a maximum M at a point $z = \bar{z}$. Set $\mu_* = (1+r)/M$. If $0 < \mu < \mu_*$, equation (A.6) has no solution, and

$$z^{1/\xi} > \frac{\mu}{1+r}\Phi'(z)$$

for all $z > 0$.

Consider the family of trajectories $\gamma_y(s) = (z_y(s), p_y(s))$ of (A.5) that satisfy $(z_y(0), p_y(0)) = (\bar{z}, y)$. As γ_0 intersects the horizontal axis transversally at $(\bar{z}, 0)$, for $y > 0$ sufficiently small γ_y also intersects the horizontal axis, at $(\bar{z}_y, 0)$. Since $\dot{p}/\dot{z} > 0$ for $z > 0$ and $p = 0$, z_y decreases as y increases.

Note that γ_y cannot enter the closed region

$$R_1 = \{(z,p) : z \geq 0, A_1(z,p) \geq 0\} = \{(z,p) : 0 \leq z \leq p^\xi\}$$

before intersecting the horizontal axis: as A points into this region on every boundary point of R_1, the region is forward invariant, and no trajectory can leave it again. In particular, no trajectory that has entered the region can intersect the horizontal axis afterwards.

Let y^* be the supremum of all y such that $\bar{z}_y > 0$. Then $\gamma_{y^*}(s) \to (0,0)$ as $s \to \infty$. Construct a function q as follows:

$$q(z_{y^*}(s)) = p_{y^*}(s) \tag{A.7}$$

for all s such that $0 < z_{y^*}(s) < \bar{z}$. This function is well-defined since the distance from $\gamma_y(s)$ to R_1 is always positive, and hence z_y is strictly decreasing. Moreover, after differentiating its defining relation with respect to s, it is seen to satisfy

$$A_2(z, q(z)) = q'(z)A_1(z, q(z)); \tag{A.8}$$

since A_1 is bounded away from zero on every compact interval $[\varepsilon, \bar{z}]$, it follows that on $(0, \bar{z}]$, the function q is continuously differentiable. Moroever, since $(z, q(z)) \notin R_1$, it follows that $q(z) \leq z^{1/\xi}$. Finally, equation (A.8) is equivalent to

$$\frac{\partial G}{\partial p}(z, q(z))q'(z) + \frac{\partial G}{\partial z}(z, q(z)) - rq(z) = 0.$$

Integration yields

$$G(z, q(z)) - rv(z) = 0, \tag{A.9}$$

where $v(z) = \int_0^z q(x)\,dx$.

We extend $q(z)$ by using (A.7) for all negative values of s. The region

$$R_2 = \{(z,p) : z \geq \bar{z}, 0 \leq p \leq M\}$$

is backward invariant and contained in the complement of R_1: hence γ_{y^*} cannot leave R_2 for negative s, and A_1 is bounded away from 0 on R_2. Again this implies that z_{y^*} is strictly decreasing for all $s < 0$ as well as $z_{y^*}(s) \to \infty$ as $s \to -\infty$.

Equation (A.8) holds also for the extended function; we conclude that $v(z) = \int_0^z q(x)\,dx$ is a continuously differentiable solution of the game HJB equation (A.9).

Finally, we note that for $(z, p) \notin R_1$, we have that $p^{\xi} < z$, and consequently that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{1 + (1 - 1/n)\xi} p^{\xi} - z < -\frac{(1 - 1/n)\xi}{1 + (1 - 1/n)\xi} z < 0$$

for $z > 0$. Hence, for $0 < \mu < \mu_*$, in the symmetric Markov perfect Nash equilibrium, the knowledge stock always decays towards 0. $\square$
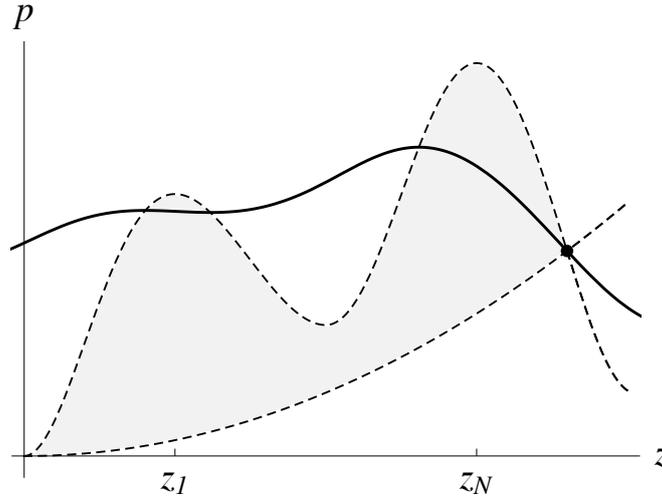

Fig. A.8 Illustration of the second part of the proof of theorem 2. The dashed curves are the isoclines $A_1 = 0$ and $A_2 = 0$; the solid curve is the stable manifold of the equilibrium e_μ of A, as well as the graph of $p(z) = v'(z)$. The region Q^- bounded by the isoclines is shaded; in this region, $\mathrm{d}p/\mathrm{d}z < 0$.

Proof, second part: large μ. We shall show that for μ sufficiently large, the stable manifold of the rightmost equilibrium of the auxiliary vector field is the graph of a function $q(z)$, such that $v(z) = \int_0^z q(x)\,\mathrm{d}x$ is a continuously differentiable solution to the game HJB equation.

By hypothesis, the number of critical points of $\Phi'(z)$ is finite. Let $z_1 < \ldots < z_N$ denote the positive critical points of Φ'. As $\Phi'(z) > 0$ and $\Phi'(z) \to 0$ as $z \downarrow 0$, it follows that $\Phi''(z) > 0$ for all $0 < z < z_1$; likewise, as $\Phi'(z) \to 0$ as $z \to \infty$, it follows that $\Phi''(z) < 0$ for all $z > z_N$.

We have seen above that an equilibrium of the auxiliary system has to satisfy (A.6), which can be rewritten as

$$F(z, \mu) = (1 + r)z^{1/\xi} - \mu\Phi'(z) = 0.$$

Clearly $F(z_N, \mu) < 0$ if $\mu > \mu_1 = (1 + r)z_N^{1/\xi}/\Phi'(z_N)$; moreover $F(z, \mu) \to \infty$ as $z \to \infty$, and $\partial F/\partial z > 0$ for all $z > z_N$. We conclude that for each $\mu > \mu_1$, there is a unique solution z_μ of $F(z_\mu, \mu) = 0$ in the interval $[z_N, \infty)$. Set $p_\mu = z_\mu^{1/\xi}$; then $e_\mu = (z_\mu, p_\mu)$ is an equilibrium of the auxiliary vector field.

Let Q^+ be the region to the right of the line $z = z_\mu$ that is bounded by the isoclinics $\{A_1(z,p) = 0\}$ and $\{A_2(z,p) = 0\}$; that is

$$Q^+ = \left\{ (z,p) : z > z_\mu, \, z^{1/\xi} < p < \frac{\mu}{1+r}\Phi'(z) \right\}.$$

Note that the auxiliary vector field A is outward pointing on all points of the boundary of Q^+; hence it contains a part $\gamma^+(s) = (z^+(s), p^+(s))$ of the stable manifold of e_μ, which is such that $\gamma^+(s) \to e_\mu$ as $s \to \infty$. But as Q^+ is backward invariant, necessarily $\gamma^+(s) \in Q^+$ for all s. By definition $A_1 \neq 0$ in Q^+ and hence z^+ is strictly decreasing in s. As before, $\gamma^+(s)$ defines a continuously differentiable function q on $z > z_\mu$ by setting

$$q(z^+(s)) = p^+(s).$$

Note that by differentiating this relation, we obtain

$$\frac{dq}{dz}(z^+(s)) = \frac{(p^+)'(s)}{(z^+)'(s)} = \frac{A_2}{A_1} < 0$$

for all s; hence $q(z)$ is stricly decreasing for $z > z_\mu$.

We proceed to show that q can be extended to a function defined for all $z \geq 0$. For this purpose, let

$$Q^- = \left\{ (z,p) : 0 < z < z_\mu, \, z^{1/\xi} < p < \frac{\mu}{1+r}\Phi'(z) \right\}.$$

Locally around e_μ, the auxiliary vector field is outward pointing from Q^-; hence it contains a part $\gamma^-(s) = (z^-(s), p^-(s))$ of the stable manifold of e_μ.

Let $p_i = (\mu/(1+r))\Phi'(z_i)$ for all critical points z_i, and introduce

$$p_{\min} = \min\{p_\mu, p_1, \ldots, p_N\}, \quad p_{\max} = \max\{p_1, \ldots, p_N\}.$$

Since $(\mu/(1+r))\Phi'(z_N) > p_\mu$ and $\Phi'(0) = 0$, there are solutions of $(1+r)p_\mu - \mu\Phi'(z) = 0$ in the interval $0 < z < z_N$. As $\Phi'(z)$ has finitely many critical points, it is piecewise monotonic, and there are only finitely many solutions to this equation. Let $z_{\min}$ denote the smallest of these.

The region

$$S_1 = \{(z,p) : z_{\min} < z < z_\mu, \, p_{\min} < p < p_{\max}\}$$

has then the property that A is inward pointing only on the segment $\Sigma = \{(z_{\min}, p) : p_{\min} < p < p_{\max}\}$ of the boundary, and outward pointing on the rest of the boundary. Moreover, $A_1 < 0$ for all points of S_1.

We claim that the trajectory $\gamma^-(s)$ has to intersect Σ for some s. To see this, choose s such that $\gamma^-(0) \in Q^-$ is so close to e_μ, that $z^-(s)$ is strictly increasing for all s. Note that there is then a constant $c < 0$ such that $A_1(z,p) \leq c$ for all points in $S_1 \cap \{z \leq z^-(0)\}$. As $\gamma^-(s)$ cannot

intersect other parts of the boundary of S_1 except Σ, there is some $\bar{s} < 0$ such that $z^-(\bar{s}) = z_{\min}$ and z^+ is strictly increasing for all $s > \bar{s}$.

The final part of the proof is to show that for μ sufficiently large, the trajectory $\gamma^-(s)$ intersects the positive vertical axis for some $\hat{s} < \bar{s}$, while satisfying $(z^+)'(s) > 0$ for all $\hat{s} < s \leq \bar{s}$.

For this we introduce the region

$$S_2 = \{(z, p) : 0 \leq z \leq z_{\min}, \ p^\xi - z \geq 1\}.$$

Note that $A_1(z, p) \geq 1$ on S_2.

We claim that $p_{\min} = p_{\min}(\mu) \to \infty$ as $\mu \to \infty$. It is clear from their definition that $p_i \to \infty$ as $\mu \to \infty$. Moreover, for every fixed $z > z_N$, we have that $F(z, \mu) < 0$ for $\mu > (1+r)z^{1/\xi}/\Phi'(z)$; it follows that $z_\mu > z$ for those values of μ. Hence $z_\mu \to \infty$ and $p_\mu = z_\mu^{1/\xi} \to \infty$ as $\mu \to \infty$. This proves the claim.

Consequently, for μ sufficiently large, we have $(z_{\min}, p_{\min}) \in S_2$ and consequently also $\gamma^-(\bar{s}) \in S_2$, as $p^-(\bar{s}) \geq p_{\min}$.

We shall extend the trajectory γ^{-1} for smaller values of s, until it hits the boundary of S_2. While in S_2, the trajectory traces out the graph of a function q, defined in the usual way, that satisfies

$$q'(z) = \frac{A_2(z, q(z))}{A_1(z, q(z))} \leq A_2(z, q(z)) \leq (1+r)q(z), \quad q(z_{\min}) = p^-(\bar{s}).$$

the first inequality holds as $A_1 \geq 1$ on S_2, and the second inequality follows from the fact that $\Phi'(z) \geq 0$.

By Gronwall's inequality, it follows that for $z_0 < z$

$$q(z) \leq q(z_0)e^{(1+r)(z-z_0)},$$

and consequently, replacing z by $z_{\min}$ and z_0 by z, that

$$q(z) \geq p^-(\bar{s})e^{(1+r)(z-z_{\min})} \geq p_{\min}e^{(1+r)(z-z_{\min})}.$$

It follows that

$$q(z)^\xi - z = p_{\min}^\xi e^{-\xi(1+r)z_{\min}}(1 + \xi(1+r)z) - z > 0$$

for all $0 \leq z \leq z_{\min}$, if $p_{\min}$ is larger than some constant $C > 0$. Let $\mu_* > \mu_1$ be such that this condition is satisfied for all $\mu > \mu_*$. Then the graph of q is in S_2 for all $0 \leq z \leq z_{\min}$, and $\gamma^-(s)$ leaves S_2 through the positive vertical axis at some $s = \hat{s} < \bar{s}$.

We conclude that for $\mu \geq \mu^*$, by setting

$$q(z^-(s)) = p^-(s)$$

for $s \geq \hat{s}$, the function q is defined for all $0 \leq z < z_\mu$; moreover, it is a continuously differentiable function.

We conclude that we have found a continuously differentiable function $q(z)$, defined for all $z \geq 0$, such that

$$q'(z)A_1(z, q(z)) = A_2(z, q(z));$$

as in the first part of the proof, this implies that $v(z) = \int_0^z q(x)\, dx$ is a continuously differentiable solution of the game HJB equation (A.9). $\qquad\square$