Center for
Mathematical Economics
Working Papers

557

May 2016

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ISSN: 0931-6558

Hypothesis Testing Equilibrium in Signaling Games

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Version: 23. May. 2016

Abstract

In this paper, we propose a definition of Hypothesis Testing Equilibrium (HTE) for general signaling games with non-Bayesian players nested by an updating rule according to Hypothesis Testing model characterized by Ortoleva (2012). An HTE may be different from a sequential Nash equilibrium because of the dynamic inconsistency. However, when player 2 only takes zero-probability message as an unexpected news, an HTE is a refinement of sequential Nash equilibrium and it survives Intuitive Criterion, but not vice versa. We provide existence theorem covering a broad class of signaling games often studied in economics, and the *constrained HTE* is unique in such signaling games. *Keywords*: Signaling Games, Hypothesis Testing Equilibrium, Equilibrium Refinement.

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This research is granted by European Erasmus Mundus Joint Doctoral Program. I would like to thank Prof. Frank Riedel, Prof. Pietro Ortoleva, Prof. Christoph Kuzmics, and Prof. Raphaël Giraud, for their insightful comments and precious advice.

1 Introduction

Ortoleva (2012) models an agent who does not update according to Bayes' Rule, but would "rationally" choose a new prior among a set of priors when her original prior assigned a small probability on an realized event. He provides axiomatic foundations for his model in the form of a Hypothesis Testing representation theorem for suitably defined preferences. Both the testing threshold and the set of priors are subjective, therefore, the agent who follows this updating rule is aware of and can anticipate her updating behavior when formulating plans.

In details, we consider the preferences of an agent over acts $\mathcal{F}$ which are functions from state space Ω to a set of consequence X. If the preference relation is Dynamic Coherence together with other standard postulates, then the agent's behavior can be represented by a Hypothesis Testing model (u, ρ, ϵ). According to this representation, the agent has a utility function u over consequences; a prior over priors ρ; and a threshold $\epsilon \in [0, 1)$. She then acts as follows: Before any information comes, she has a set of priors $\pi \in \Pi$ with probability assessment ρ over Π. She chooses π_Ω with highest probability $\rho(\pi_\Omega)$ among all $\pi \in supp(\rho)$ as her original prior. Then she forms her preference as the standard expected utility maximizer. As new information (an event) A is revealed, the agent evaluates the probability of occurrence of the event as $\pi_\Omega(A)$. She keeps her original prior π_Ω and proceeds Bayesian update π_Ω using A if the event A is not unexpected, i.e., $\pi_\Omega(A) > \epsilon$. However, if $\pi_\Omega(A) \leq \epsilon$, she doubts her original prior π_Ω and looks for a new prior π^* among $supp(\rho)$ such that π^* is the most likely one conditional on event A, that is, $\pi^* = \underset{\pi \in supp(\rho)}{\operatorname{argmax}} \mathbb{P}(\pi|A)$, where

$$\begin{aligned} \mathbb{P}(\pi|A) &= \frac{\mathbb{P}(A|\pi)\mathbb{P}(\pi)}{\int_{\pi' \in supp(\rho)} \mathbb{P}(A|\pi')\mathbb{P}(\pi')d\pi'} \\ &= \frac{\pi(A)\rho(\pi)}{\int_{\pi' \in supp(\rho)} \pi'(A)\rho(\pi')d\pi'}. \end{aligned} \tag{1}$$

Using this π^*, she proceeds Bayesian update and forms her preference by maximizing expected utility.

Ortoleva (2012) applied his model in the "Beer-Quiche" game and defined a Hypothesis Testing Equilibrium (HTE) when $\epsilon = 0$ for this specific game. In this game, there exists unique HTE which coincides with the selection of the Intuitive Criterion of Cho and Kreps (1987). This paper develops the idea of nesting the updating model in general signaling games with finite states and proposes a general concept of Hypothesis Testing Equilibrium. For the general definition of HTE, we allow the testing threshold $\epsilon \geq 0$. If player 2 has a testing threshold $\epsilon > 0$, then she changes her beliefs when a small (but non-zero) probability event happens. This dynamic inconsistency leads to a result that an HTE may deviate from sequential Nash equilibrium. However, we show that when $\epsilon = 0$, an HTE is a refinement of sequential Nash equilibria. In this case, player 2 only considers the zero-probability event as an unexpected event. In order to compare with other refinement criteria, we mainly focus on the properties of this class of HTE. We have three main findings: (a). As a method of refinement, an HTE survives Intuitive Criterion, but not vice verse; (b). A general HTE exists in a broad class of signaling games studied in economics which satisfy the Single Crossing Property together with other standard assumptions. (c). We proposed a concept of *constrained HTE* in which the set of alternative beliefs of player 2 is restricted to be around her original belief. We show that the *constrained HTE* is unique under the previous assumptions. As an example, we present this result in Milgrom-Roberts' limit pricing model and we get a unique HTE for each interesting case.

This paper focuses on the signaling games, a class of games where an informed player (player 1) conveys private information to an uninformed player (player 2) through messages, and player 2 tries to make inferences about hidden information and takes an action which can influence both players' payoffs. There is an enormous literature that analyzes and utilizes signaling games in applications of a wide range of economic problems as reviewed in Riley (2001) and Sobel (2007), see Spence's model in labor market (Spence, 1974), Milgrom-Roberts' model of limit pricing (Milgrom and Roberts, 1982), bargaining models (Fudenberg and Tirole, 1983), and models in finance (Leland and Pyle, 1977), for example. Typically, signaling games give rise to many sequential Nash equilibria because under the assumption of Bayesian updating rule, at equilibrium, there is no other restrictions on the message m that is sent with zero probability by player 1 except that player 2's responses to m can be rationalized by *some* belief of player 2. Therefore, the natural idea to refine the sequential Nash equilibria is to impose additional restrictions on the out-of-equilibrium beliefs as we can see in the literature reviewed in Govindan and Wilson (2008, 2009), Hillas and Kohlberg (2002), Kohlberg (1990), and van Damme (2002).

One branch of the refinement criteria, which has been widely applied in signaling games in order to reduce the set of sequential equilibria, is motivated by the concept of strategic stability for finite games addressed by Kohlberg and Mertens (1986). The Intuitive Criterion, D1, D2 Criteria (Cho and Kreps, 1987), Divinity (Banks and Sobel, 1987), for example, are all weaker versions of strategic stability that are defined more easily for signaling games. These refinements interpret the meaning of the out-of-equilibrium messages depending on the current equilibrium, which means that, at a reasonable equilibrium, sending an out-of-equilibrium message is costly and unattractive to player 1. There is also a branch of refinements trying to define a new concept of equilibrium, for example, perfect sequential equilibria proposed by Grossman and Perry (1986), different versions of Perfect Bayesian Equilibrium (PBE) discussed by Fudenburg and Tirole (1991), and forward induction equilibrium defined by Govindan and Wilsons (2009) and modified by Man (2012), Consistent Forward Induction Equilibrium Path proposed by Umbauer (1991), etc. There is also a few refinements that are base on the idea of hypothesis test, see Mailath, Okuno-Fujiwara, and Postlewaite (1993) and Ortoleva (2012), for example. There is no consensus in the literature that one refinement is better than the other. One refinement can be favorable in some settings and unfavorable in other settings.

All these refinements mentioned above are dealing with signaling games with Bayesian players. However, the behavior of deviation from Bayesian updating has been observed by psychologists[1] and these experiments motivated a growing interest in properties of non-Bayesian updating, see, for example, the model of temptation and self-control proposed by Gul and Pesendorfer (2001, 2004), characterized axiomatically by Epstein (2006), and extended by Epstein et al., (2008, 2010), models of learning in social networks by Golub and Jackson (2010) and Jadbabaie et al., (2012), arguments of rational beliefs by Gilboa et. al, (2008, 2009, 2012) and Teng (2014), and hypothesis testing model of Ortoleva (2012), etc. For signaling games with non-Bayesian players, this paper proposes a concept of Hypothesis Testing equilibrium, and provides a refinement based on the idea of non-Bayesian reactions to small probability messages according to Hypothesis Testing model. The non-Bayesian updating rule is nested in the signaling games as follows: Before player 1 moves, player 2 has a prior over a (finite) set of strategies that player 1 may use and she chooses the most likely strategy that player 1 would use. After she observes a message sent by player 1, she evaluates the probability of the message using her original belief induced by the strategy of player 1. She keeps her original belief and uses it to proceed Bayesian update if the probability of the massage she observed is greater than her testing threshold. However, if the probability is less than or equals to the threshold, she will discard her original belief (she thinks that player 1 may use a different strategy from her original conjecture), then she looks for a new belief which can be induced by another "rational" strategy of player 1 such that it is the most likely one conditional on the observed message. If there exist such beliefs to support the occurrence of the messages (for both on-the-equilibrium path and off-the-equilibrium path massages), then the sequentially rational strategies profile form an HTE. The difficulty is that how player 2 selects the set of strategies of player 1 (which can induce a set of

[1]For example Tversky and Kahneman (1974), Camerer (1995), Rabin (1998, 2002), and Mullainathan (2000).

beliefs of player 2) and how to assign probability distribution over these possible strategies. Here we allow all the beliefs which can be "rationalized" by at least one strategy of player 2 under consideration, which is a weak restriction on the beliefs available to player 2.

This paper is organized as follows. In the next section, we briefly recall the basic concepts and definitions from Ortoleva (2012) on the updating rule of Hypothesis Testing model and the framework of general signaling games. Section 3 is the heart of the paper, where we define the general Hypothesis Testing Equilibrium (HTE), discuss the main properties of HTE and prove the existence and uniqueness theorem. Section 4 compares the refinements of HTE and Intuitive Criterion. Section 5 analyzes the HTE of Milgrom-Roberts' limit pricing model in a finite framework and section 6 provides the conclusion and remarks.

2 Formulations and Preliminaries

2.1 The updating rule of Hypothesis Testing Model

Firstly we recall the basic concepts, definitions and main results of Hypothesis Testing Model in general decision theory. Adopting the notations in Ortoleva (2012), consider such a probability space $(\Omega, \Sigma, \Delta(\Omega))$, where Ω is finite (nonempty) state space, Σ is set of all subsets of Ω, and $\Delta(\Omega)$ is the set of all probability measures (beliefs) on Ω. Write $\Delta(\Delta(\Omega))$ as the set of all beliefs over beliefs. Let

$$BU(\pi, A)(B) = \frac{\pi(A \cap B)}{\pi(A)}$$

denote the Bayesian update of $\pi \in \Delta(\Omega)$ using $A \in \Sigma$ such that $\pi(A) > 0$. As we discussed in the introduction, equation (1) gives the Bayesian update of the second order prior $\rho \in \Delta(\Delta(\Omega))$ using $A \in \Sigma$ such that $\pi(A) > 0$ for some $\pi \in \text{supp}(\rho)$. We denote it as:

$$BU(\rho, A)(\pi) := \frac{\pi(A)\rho(\pi)}{\int\limits_{\Delta(\Omega)} \pi'(A)\rho(\pi')d\pi'}.$$

Let's consider the preferences of an agent over acts $\mathcal{F}$ which are functions from state space Ω to a set of consequence X. For example, X could be a set of possible prizes depending on the realizations of the state.

Definition 2.1. (Ortoleva, 2012) A class of preferences relations $\{\succeq_A\}_{A\in\Sigma}$ admits a Hypothesis Testing Representation if there exists a nonconstant affine function $u : X \to \mathbb{R}$, a prior over priors $\rho \in \Delta(\Delta(\Omega))$ with finite support, and $\epsilon \in [0, 1)$ such that, for any $A \in \Sigma$, there exist $\pi_A \in \Delta(\Omega)$ such that:

(i) for any $f, g \in \mathcal{F}$

$$f \succeq_A g \Leftrightarrow \sum_{\omega\in\Omega} \pi_A(\omega)u(f(\omega)) \geq \sum_{\omega\in\Omega} \pi_A(\omega)u(g(\omega))$$

(ii) $\{\pi_\Omega\} = \underset{\pi\in\Delta(\Omega)}{\text{argmax}}\, \rho(\pi)$

(iii)

$$\pi_A = \begin{cases} BU(\pi_\Omega, A) & \pi_\Omega(A) > \epsilon \\ BU(\pi^*_A, A) & otherwise, \end{cases}$$

where $\{\pi^*_A\} = \underset{\pi\in\Delta(\Omega)}{\text{argmax}}\, BU(\rho, A)(\pi)$.

Under this definition, if a decision maker's preference is represented by the updating rule according to Hypothesis Testing Model, then she proceeds the update in the following procedure:

Step 0. The agent is uncertain about some important state of the nature. Instead of a single subjective probability distribution over the alternative possibilities, she has a set of probability distributions (priors) Π and a probability distribution (second order prior) ρ on Π, and $\text{supp}(\rho) \neq \emptyset$. The agent has a subjective threshold ϵ for Hypothesis Test.

Step 1. Before any new information is revealed, the agent chooses a prior $\pi_\Omega \in \text{supp}(\rho)$ which is the most likely prior according to her belief ρ. In this hypothesis test, π_Ω serves as a *null hypothesis* and all the other priors $\pi \in \text{supp}(\rho)$ as *alternative hypothesis*.

Step 2. As new information (an event) A is revealed, the agent evaluates the probability of the occurrence of A as $\pi_\Omega(A)$. The null hypothesis will not be rejected if $\pi_\Omega(A) > \epsilon$, and the agent can proceed Bayes' rule to the prior π_Ω. However, the null hypothesis will be rejected if $\pi_\Omega(A) \leq \epsilon$. The agent doubts her original prior π_Ω because an unexpected event occurred. The agent will choose an alternative prior $\pi^* \in \text{supp}(\rho)$ which is the most likely one conditional on the event A. Then she proceeds Bayes' rule to the prior π^*.

This paper aims to nest the non-Bayesian updating rule according to the Hypothesis Testing model into signaling games, therefore, we now briefly introduce the general framework of signaling games with Bayesian players first.

2.2 Signaling Games

Nature selects the type of player 1 according to some probability distribution μ over a finite set T with $\text{supp}(\mu) \neq \emptyset$ (For simplicity, we take $T = \text{supp}(\mu)$). Player 1 is informed of his type $t \in T$ but player 2 is not. After player 1 learnt his type, he chooses to send a message m from a finite set M. Observing the message m, player 2 updates his beliefs on the type of player 1 and makes a response r in a finite action set R. The game ends with this response and payoffs are made to the two players. The payoff to player i, $i = 1, 2$, is given by a function $u_i : T \times M \times R \to \mathbb{R}$. The distribution μ and the description of the game are common knowledge.

2.2.1 Sequencial Nash Equilibrium

A behavioral strategy of player 1 is a function $\sigma : T \to \Delta(M)$ such that $\sum_{m \in M} \sigma(m;t) = 1$ for all $t \in T$. The type t of player 1 chooses to send message m with probability $\sigma(m;t)$ for all $t \in T$. A behavioral strategy of player 2 is a function $\tau : M \to \Delta(R)$ such that $\sum_{r \in R} \tau(r;m) = 1$ for all $m \in M$. Player 2 takes respond r following the message m with probability $\tau(r;m)$. We adopt the notations in Cho and Kreps (1978), write $\mathrm{BR}(m,\mu)$ for the set of best responses of player 2 observing m if she has posterior belief $\mu(\cdot|m)$.

$$\mathrm{BR}(m,\mu) = \operatorname*{argmax}_{r \in R} \sum_{t \in T} u_2(t,m,r)\mu(t|m).$$

If $T' \subseteq T$, let $\mathrm{BR}(T',m)$ denote the set of best responses of player 2 to posteriors concentrated on the set T'. That is,

$$\mathrm{BR}(T',m) = \bigcup_{\{\mu:\mu(T'|m)=1\}} \mathrm{BR}(m,\mu).$$

Let $\mathrm{BR}(T',m,\mu)$ be the set of best responses by player 2 after observing m if she has posterior belief $\mu(\cdot|m)$ concentrated on the subset T', and MBR denote the set of mixed best responses by player 2. Since we concentrate on the finite sets of T, M, and R, the sequential Nash equilibrium can be defined straightforward.

Proposition 2.1. *A profile of players' behavioral strategies (σ^*, τ^*) forms a sequential Nash equilibrium (SNE) in a finite signaling game if it satisfies the following conditions:*

(*i*) *Given player 2's strategy τ^*, each type t evaluates the expected utility from sending message m as $\sum_{r \in R} u_1(t,m,r)\tau^*(r;m)$ and $\sigma^*(\cdot;t)$ puts weight on m only if it is among the maximizing m's in this expected utility.*

(*ii*) *Given player 1's strategy σ^*, for all m that are sent by some type t with positive probability $\mu(t|m) > 0$, every response $r \in R$ such that $\tau^*(r;m) > 0$ must be a best response to m given beliefs $\mu(t|m)$, that is,*

$$\tau^*(\cdot;m) \in MBR(m, \mu(\cdot|m)), \tag{2}$$

where $\mu(t|m) = \frac{\sigma^*(m;t)\mu(t)}{\sum_{t'\in T}\sigma^*(m;t')\mu(t')}$.

(*iii*) *For every message* m *that is sent with zero probability by player 1 (for all* m *such that* $\sum_t \sigma^*(m;t)\mu(t) = 0$*), there must be some probability distribution* $\mu(\cdot|m)$ *over types* T *such that* (2) *holds.*

In an SNE, given the strategy of player 1, player 2 proceeds three steps: she computes the probability of an observed message m as $\mathbb{P}(m) = \sum_{t\in T}\sigma^*(m;t)\mu(t)$. If $\mathbb{P}(m) > 0$, that is, there exists some $t \in T$, such that $\sigma^*(m;t) > 0$, then she uses Bayes' rule to compute the posterior assessment $\mu(\cdot|m)$, and then she chooses her best response to m compatible with her belief $\mu(\cdot|m)$. If $\mathbb{P}(m) = 0$, then there only needs to exist some belief $\mu(\cdot|m)$ such that her responses to the out-of-equilibrium message is rational.

What happens if player 2 is a non-Bayesian player? In the next section, we aim to define an alternative equilibrium in such signaling games where player 2 uses non-Bayesian update rule according to Hypothesis Testing model.

3 Hypothesis Testing Equilibrium

3.1 Definition of HTE in general signaling games

In a signaling game, player 2 does not observe player 1's strategies but the messages sent by player 1, therefore, it's helpful to imagine that player 2 has a conjecture, $\hat{\sigma}(\cdot;t), \forall t \in T$, about player 1's behavior. She attempts to make inference using her conjecture. Similarly, player 1 also has a conjecture, $\hat{\tau}$, about player 2's behavior and, at equilibrium, the conjectures $(\hat{\sigma}, \hat{\tau})$ coincide with the strategies (σ^*, τ^*) that players actually use. Each one conjecture $\hat{\sigma}$ of player 2 about player 1's behavior induces a prior (belief) π on the state space $\Omega = T \times M$. For every realization (t, m), $\pi(t, m)$ is the probability of type t sending message m which is

$$\pi(t, m) = \hat{\sigma}(m;t)\mu(t),$$

and satisfies

$$\sum_{t\in T}\sum_{m\in M}\pi(t,m)=\sum_{t}\mu(t)=1.$$

If player 2 is non-Bayesian player and her preference is represented by the Hypothesis Testing model, then player 2 has a set of "rational" conjectures (priors) and she has a probability distribution ρ on the set of conjectures (priors). Firstly, we address some proper requirements for the second order prior ρ.

Definition 3.1. Consider a finite signaling game $\Gamma(\mu)$ where player 2's preference is represented by a Hypothesis Testing model (ρ, ϵ). ρ is consistent if it satisfies the following requirements:

(i). $\forall \pi \in \text{supp}(\rho)$, π is compatible with the initial information of the game, that is $\sum_{m\in M}\pi(t,m)=\mu(t)$.

(ii). $\forall \pi \in \text{supp}(\rho)$, π can be rationalized by at least one possible strategy of player 2. that is, there exists some strategy $\tau : M \to \Delta(R)$ of player 2 such that $\pi(t,m)=0, \forall t\in T, \forall m \in M$, if the type-message pair (t,m) is not a best response to τ.

As addressed in Ortoleva (2012), the requirement for rationality is a weak condition in the sense that player 2 can take any conjecture under consideration as long as it is compatible with player 1's best responding to some possible strategy τ of player 2. Now we are ready to define Hypothesis Testing Equilibrium in signaling games.

Definition 3.2. In a finite signaling game $\Gamma(\mu)$, a profile of behavioral strategies (σ^*, τ^*) is a Hypothesis Testing equilibrium (HTE) based on a Hypothesis Testing model (ρ, ϵ) if

(i). ρ is consistent.

(ii). The support of ρ contains π_Ω induced by σ^* such that

$$\pi_\Omega = \underset{\pi\in \text{supp}(\rho)}{\text{argmax}}\, \rho(\pi).$$

Let

$$M^E = \{m \in M : \sum_{t \in T} \pi_\Omega(m|t)\mu(t) > \epsilon\},$$

then for any $m \in M \backslash M^E$, there exists some $\pi_m \in \text{supp}(\rho)$, such that

$$\pi_m = \underset{\pi \in \text{supp}(\rho)}{\text{argmax}}\, BU(\rho, m)(\pi).$$

(iii).For all $t \in T$, $\sigma^*(m;t) > 0$ implies m maximizes the expected utility of player 1, that is $\sum_{r \in R} u_1(t,m,r)\tau^*(r;m)$. $\forall m \in M$,

$$\tau^*(\cdot;m) \in \text{MBR}(m, \mu(\cdot|m)),$$

where

$$\mu(t|m) = \begin{cases} \pi_\Omega(t|m) = \frac{\sigma^*(m;t)\mu(t)}{\sum_{t'} \sigma^*(m;t')\mu(t')} & \text{if} \quad m \in M^E \\ \pi_m(t|m) = \frac{\pi_m(m|t)\mu(t)}{\sum_{t'} \pi_m(m|t')\mu(t')}, & \text{otherwise.} \end{cases}$$

The idea behind this definition is similar as Nash equilibrium except that we allow non-Bayesian reactions for out-of-equilibrium messages when $\epsilon > 0$. Therefore, the dynamic consistency is violated but only up to ϵ. However, the dynamic consistency holds when $\epsilon = 0$ and the update rule is also well defined after zero-probability messages. According to the definition, for a given ϵ, in order to prove a profile of strategies is an HTE based on (ρ, ϵ), we just need to find a proper ρ to support the equilibrium.

Example 3.1. As an illustration, we apply this definition to the simple game depicted in Figure 1. In this game, it's very easy to check that there is one separating sequential Nash equilibrium: the t_1 type of player 1 chooses message m_1 and type t_2 chooses message m_2; player 2, regardless of which message is observed, chooses r_1. If player 2 has a threshold $\epsilon = 5\%$, then this equilibrium is an HTE supported by a Hypothesis Testing model (ρ, ϵ), where supp(ρ) only contains one element π induced by the strategy of player 1. That is π satisfies the following conditions:

$$\pi(t_1, m_1) + \pi(t_1, m_2) = 0.05, \quad \pi(t_2, m_1) + \pi(t_2, m_2) = 0.95;$$
$$\pi(m_1; t_1) = 1, \quad \pi(m_2; t_2) = 1.$$

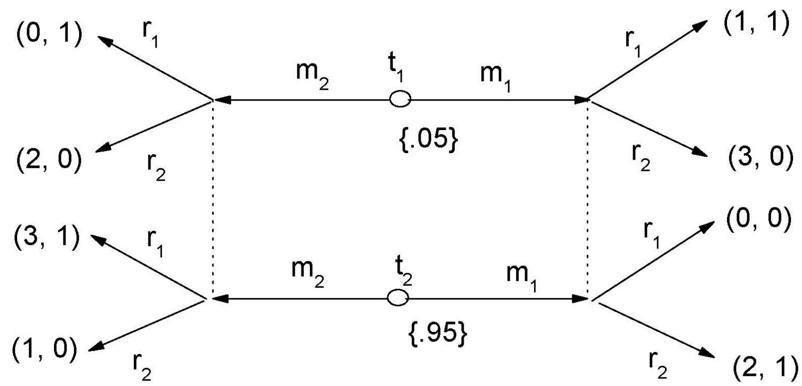

Figure 1

This prior (belief) π can be rationalized by a strategy of player 2 which is choosing r_1 regardless of which message is observed.

In this example, there also exists an HTE which is not a sequential equilibrium:

$$\sigma^*(m_1; t_1) = 1, \quad \sigma^*(m_2; t_2) = 1;$$
$$\tau^*(r_2; m_1) = 1, \quad \tau^*(r_1; m_2) = 1.$$

The support of ρ contains two elements π and $\hat{\pi}$ such that: $0 < \rho(\hat{\pi}) < \rho(\pi) < 0.9524$, and

$$\pi(t_1) = 0.05, \quad \pi(t_2) = 0.95, \quad \pi(m_1|t_1) = 1, \quad \pi(m_2|t_2) = 1;$$
$$\hat{\pi}(t_1) = 0.05, \quad \hat{\pi}(t_2) = 0.95, \quad \hat{\pi}(m_1|t_1) = 1, \quad \hat{\pi}(m_1|t_2) = 1.$$

π can be rationalized by the strategy τ^*, and $\hat{\pi}$ can be rationalized by choosing r_2 regardless of the message observed. Now let's check that (σ^*, τ^*) is an HTE supported by (ρ, ϵ). Given τ^*,

$$u_1(t_1, m_1, \tau^*) = 3 > u_1(t_1, m_2, \tau^*) = 0;$$
$$u_1(t_2, m_2, \tau^*) = 3 > u_1(t_2, m_1, \tau^*) = 2,$$

therefore, σ^* maximizes player 1's expected payoff for both types. Given σ^*, player 2 starts with belief π since $\rho(\pi) > \rho(\hat{\pi})$ and keeps π if she observes

m_2 since

$$\pi(m_2) = \pi(m_2|t_1)\pi(t_1) + \pi(m_2|t_2)\pi(t_2) = 0.95 > \epsilon.$$

With the posterior belief $\pi(\cdot|m_2)$, observing m_2, player 2's best response is r_1. She switches to belief $\hat{\pi}$ if she observes m_1 since

$$\pi(m_1) = \pi(m_1|t_1)\pi(t_1) + \pi(m_1|t_2)\pi(t_2) = 0.05 \leq \epsilon,$$

and

$$BU(\rho, m_1)(\hat{\pi}) > BU(\rho, m_1)(\pi).$$

Therefore, observing m_1, player 2 has the posterior assessments of $\hat{\pi}(t_1|m_1) = 0.05$ and $\hat{\pi}(t_2|m_1) = 0.95$. With this posterior belief, player 2 computes her expected payoffs as:

$$u_2(r_1; m_1, \sigma^*) = \hat{\pi}(t_1|m_1) \times 1 + \hat{\pi}(t_2|m_1) \times 0 = 0.05;$$
$$u_2(r_2; m_1, \sigma^*) = \hat{\pi}(t_1|m_1) \times 0 + \hat{\pi}(t_2|m_1) \times 1 = 0.95,$$

which implies that r_2 is the best response to m_1. Therefore, (σ^*, τ^*) is an HTE but it's not a Nash Equilibrium, since r_2 is not a best response of player 2 to the message m_1 if she only has one belief π.

3.2 Properties of Hypothesis Testing Equilibrium

From the previous example, we can immediately get the following property:

Proposition 3.1. *In a finite signaling game $\Gamma(\mu)$, if a profile of behavioral strategies (σ, τ) is an HTE supported by a Hypothesis Testing model $(\rho, 0)$, then it's also an HTE supported by a Hypothesis Testing model $(\rho_\epsilon, \epsilon)$, for all $\epsilon > 0$.*

Proof. Let M_0^E and M_ϵ^E denote the sets of messages which are sent with zero probability by player 1 and less than or equal to ϵ, respectively. Then $M_\epsilon^E \subseteq M_0^E$. We can simply take

$$\rho_\epsilon = \rho = \{\pi_\Omega, \pi_m, m \in M \backslash M^E\}.$$

Player 2 starts with $\pi_\Omega = \text{argmax}_{\pi \in \text{supp}(\rho)} \rho(\pi)$, he keeps π_Ω and proceeds Bayesian update if she observed $m \in M_\epsilon^E$. If $m \in M_0^E \backslash M_\epsilon^E$, there exists $\pi_m = \pi_\Omega$ such that σ and τ are sequentially rational. If $m \notin M_0^E$, there exists π'_m which is identical with π_m such that σ and τ are sequentially rational. □

As we can see in the previous example, if $\epsilon > 0$, then any message sent with probability less than or equal to ϵ is an off-the-equilibrium message. Because of the dynamic inconsistency, an HTE may deviate from a sequential Nash equilibrium. However, when $\epsilon = 0$, only the messages sent with zero-probability are off-the-equilibrium path, therefore, it is not surprising that there is a close relationship between this special class of HTE and sequential Nash equilibrium.

Proposition 3.2. *In a finite signaling game $\Gamma(\mu)$, an HTE supported by a Hypothesis Testing model $(\rho, 0)$ is a refinement of SNE.*

This requires no proof, it's just a matter of definitions of HTE and SNE. If a profile of strategies (σ^*, τ^*) is an HTE supported by $(\rho, 0)$, then for any message sent by player 1 such that $\sigma^*(m; t) > 0$ for some $t \in T$, player 2's posterior belief derived by Bayesian update using σ^*. And for any message sent with zero probability, that is, $\sigma^*(m; t) = 0$ for all $t \in T$, there exists some belief on the side of player 2 to rationalize her behavior. In addition, (σ^*, τ^*) are sequentially rational. Therefore, (σ^*, τ^*) is an SNE.

On the other side, according to definition 3.2, we require that any belief in the support of ρ must can be "rationalized" by at least one strategy of player 2, which means, in addition to the requirement of equation (2), we impose a further restriction on the off-the-equilibrium beliefs of player 2, therefore, it's not surprise that an SNE may not be an HTE supported by some $(\rho, 0)$ as the "Quiche-Beer" game in Ortoleva (2012). Here we also give a simple example[2] to show this property.

[2]This example is from Cho and Kreps (1987)

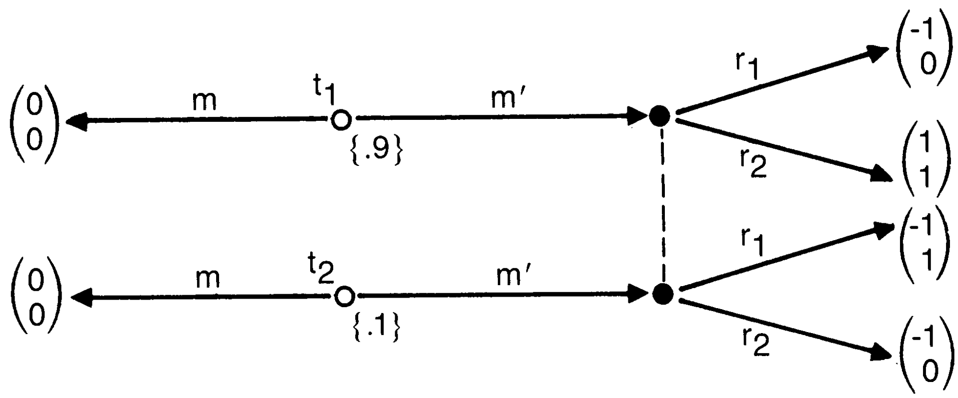

Figure 2

As the game depicted in Figure. 2, $(t_1, m), (t_2, m)$ is an SNE supported by a belief of player 2 such that $\mu(t_1|m) = 0.9$ and $\mu(t_2|m') \geq 0.5$. But this is not an HTE supported by $(\rho, 0)$ because there does not exist a ρ such that supp(ρ) contains a "rationalized" belief $\pi_{m'}$ for the out-of-equilibrium message m'. By contradiction, assume that there exists $\pi_{m'}$ such that $\pi_{m'}(t_2|m') \geq 0.5$ and $\pi_{m'}$ can be rationalized by some strategy of player 2. If $\pi_{m'}(t_2|m') \geq 0.5$, then $\pi_{m'}(t_2, m') > 0$ which implies for any strategy τ rationalized $\pi_{m'}$, m' is a best response of t_2 given τ. But for type t_2, m' is strictly dominated by m, which implies that there doesn't exist such strategy τ such that m' is a best response of t_2. This is contrary to the condition (ii) in the consistency definition 3.2.

3.3 Existence of HTE

3.3.1 Definition, notations, and assumptions

Proposition 3.1 tells us that, in a signaling game, HTE supported by (ρ, ϵ) exists if HTE supported by $(\rho, 0)$ exists. And an HTE supported by $(\rho, 0)$ is a refinement of SNE, which interests us to compare our definition of HTE with other criteria. Therefore, in the analysis to follow, we just need to restrict our attention on this class of HTE imposing $\epsilon = 0$. Since the mixed strategies are not needed for the existence, we only consider the Pure Sequential Nash Equilibrium (PSNE). A pure strategy of player 1 is a mapping $s_1 : T \to M$, and a pure strategy of player 2 is a response function $s_2; M \to R$.

Definition 3.3. In a finite signaling game $\Gamma(\mu)$, a profile of strategies (s_1^*, s_2^*) forms a PSNE if there exists $\beta_m \in \Delta(T)$, $\forall\ m \notin M^E$, such that:

(i). Given s_2^*,

$$u_1(t, s_1^*(t), s_2^*(s_1^*(t))) \geq u_1(t, m, s_2^*(m)), \quad \forall m \in M, \quad \forall t \in T,$$

(ii).Given s_1^*, for any $m \in M$, $s_2^*(m) \in \text{BR}(m, \mu_2(\cdot|m))$, where

$$\mu_2(t|m) = \begin{cases} \beta(t|m) & \text{if} \quad m \in M^E \\ \beta_m(t|m) & \text{otherwise,} \end{cases}$$

and

$$\beta(t|m) = \begin{cases} \frac{\mu(t)}{\sum_{t' \in T_{m,s_1^*}} \mu(t')} & \text{if} \quad t \in T_{m,s_1^*} \\ 0, & \text{otherwise,} \end{cases}$$

where $T_{m,s_1^*} = \{t \in T : s_1^*(t) = m\}$.

Before we go to the prove of existence, we provide some notations that we may use in the statements.

T_{m,s_1}: the subset of types of player 1 who send message m under strategy s_1, that is,

$$T_{m,s_1} = \{t \in T : s_1(t) = m\}.$$

$M^E_{(s_1,s_2)}$: the set of on-the-equilibrium messages if $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$, that is,

$$M^E_{(s_1,s_2)} = \{m \in M : \exists t \in T, s.t. s_1(t) = m\}.$$

$u_1(t; s_1, s_2)$: the payoff of type t under strategy (s_1, s_2), that is,

$$u_1(t; s_1, s_2) = u_1(t, s_1(t), s_2(s_1(t))).$$

$\beta_{T_m} \in \Delta(T)$: the probability assessment concentrating on types $t \in T_m$, that is,

$$\beta_{T_m}(t) = \begin{cases} \frac{\mu(t)}{\sum_{t' \in T_m} \mu(t')} & \text{if} \quad t \in T_m \\ 0 & \text{otherwise.} \end{cases}$$

$\beta_t \in \Delta(T)$: the probability assessment concentrating on type t, that is,

$$\beta_t(t') = \begin{cases} 1 & \text{if} \quad t' = t \\ 0 & \text{otherwise} \end{cases}$$

We cannot be confident that there exists an HTE for a game that is randomly selected from the space of signaling games with finite states, we prove the existence theorem in the class of signaling games that satisfy the following assumptions.

Assumption 1. *T, M, and R are finite. The type of player 1 has a probability distribution, $\mu \in \Delta(T)$, with full support. Further, $u_i(t, s_1, s_2)$, $i \in \{1, 2\}$, exists and is finite for all $t \in T$ and all nondecreasing functions $s_1 : T \to M$ and $s_2 : M \to R$.*

Assumption 2. *For any $t \in T$, for any fixed $r \in R$, if we connect the points $\{u_1(t, m, r) : m \in M\}$ in order by a smooth line, then $u_1(t)$ is strictly concave in m, and u_2 is strictly concave in r.*

Assumption 3. *First order stochastic dominance: $\forall t \in T$, $\forall m \in M$, $\forall \beta, \beta' \in \Delta(T)$, whenever β stochastically dominants β', that is*

$$\sum_{t' \leq t} \beta'(t') \geq \sum_{t' \leq t} \beta(t'), \quad \forall t \in T,$$

and strictly inequality holds for some $t \in T$, then

$$u_1(t, m, BR(m, \beta)) > u_1(t, m, BR(m, \beta')).$$

Assumption 4. *Single Crossing Property:*
(i). *For all $m > m'$, and all $t' > t$,$\forall r, r' \in R$,*

$$u_1(t, m, r) \geq (>) u_1(t, m', r'), \quad \textit{implies}$$
$$u_1(t', m, r) \geq (>) u_1(t', m', r').$$

(ii). *For all $\hat{r} > r$, and all $\hat{m} > m$, $\forall\ t \in T$,*

$$u_2(t, m, \hat{r}) \geq (>) u_2(t, m, r), \quad \textit{implies}$$
$$u_2(t, \hat{m}, \hat{r}) \geq (>) u_2(t, \hat{m}, r).$$

Assumption 1 is primarily a technical assumption to fit our definition of HTE. Assumption 2 insures that only pure strategies are under consideration for both players. Assumption 3 says that all types of player 1 prefer the best response of player 2 when player 2 believes that player 1 is more likely to be a higher type. The forth assumption is the Milgrom-Shannon single crossing property (SCP) for both players, which is a widely used assumption in the signaling games to model many economic problems. It says that if type t prefers a higher message-response pair (m, r) to a lower message-response pair (m', r'), then any higher type $t' > t$ also prefers the higher message-response pair (m, r). This captures the idea that higher messages are easier to send by a higher type. The utility of player 2 also satisfies SCP: if $\hat{r}$ is a better response to a message m sent by type t than r, then it is also a better response to a higher message $\hat{m}$ sent by t than r.

Before we proceed the existence theorem, let us review the concept of lexicographically dominance introduced by Mailath. et, al., (1993).

Definition 3.4. In a signaling game $\Gamma(\mu)$, a strategy profile $(s_1^*, s_2^*) \in \text{PSNE}(\Gamma(\mu))$ lexicographically dominates (l-dominates) another strategy profile $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$ if there exists $j \in T$, such that

$$u_1(t; s_1^*, s_2^*) > u_1(t; s_1, s_2) \quad if \quad t = j$$
$$u_1(t; s_1^*, s_2^*) \geq u_1(t; s_1, s_2) \quad if \quad t \geq j + 1.$$

A strategy profile $(s_1^*, s_2^*) \in \text{PSNE}(\Gamma(\mu))$ is a *lexicographically maximum sequential equilibrium* (LMSE) if there doesn't exist an $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$ l-dominates (s_1^*, s_2^*).

If we restrict player 1's types in a subset of T, we can define a truncated game from G. Formally, for any $j \in T$, let

$$T^j = \{1, ..., j\}, \quad \mu^j(t) = \beta_{T^j}.$$

A truncated game G^j is defined by substituting T^j for T, and the T^j-conditional prior μ^j for the prior μ in original game. Then we can get the following properties:

Proposition 3.3. *Assume* $(s_1, s_2) \in PSNE(\Gamma(\mu))$, $\forall\ j \in T$, *if* $s_1(t) \neq s_1(j)$, $\forall t > j$, *then* $(s_1^j, s_2^j) \in PSNE(\Gamma^j(\mu^j))$, *where* $s_1^j(t) = s_1(t)$, $\forall\ t \leq j$, *and* $s_2^j(m) = s_2(m)\ \forall\ m \in M$.

The following lemma derived in Mailath et al (1993) is very important in our proof. The reader is urged to read their paper to obtain a detailed analysis of this result.

Proposition 3.4. *Mailath et al (1993): Under A1-A4, suppose* $(s_1, s_2) \in PSNE(\Gamma(\mu))$, $(\hat{s}_1, \hat{s}_2) \in PSNE(\Gamma^j(\mu^j))$, *for some* $j \in T$. *Suppose further that*

$$u_1(j; \hat{s}_1, \hat{s}_2) > u_1(j; s_1, s_2),$$

then there exists $(s_1^*, s_2^*) \in \Gamma(\mu)$, *such that:*

$$\begin{aligned} u_1(t; s_1^*, s_2^*) &\geq u_1(t; \hat{s}_1, \hat{s}_2) \quad \textit{for all} \quad t \leq j \quad \textit{and} \\ u_1(t; s_1^*, s_2^*) &\geq u_1(t; s_1, s_2) \quad \textit{for all} \quad t > j. \end{aligned}$$

That is, (s_1^*, s_2^*) *l-dominates* (s_1, s_2).

3.3.2 Existence of HTE supported by a Hypothesis Testing model $(\rho, 0)$

Theorem 3.5. *Under* $A1 - A4$, *an LMSE is an HTE.*

In order to prove the theorem, we need the following critical results:

Lemma 3.6. *(Athey, 2001): Under A1 and A4, there exists a PSNE in* $\Gamma(\mu)$.

Therefore, an *LMSE* exists. More over, both players play nondecreasing strategies:

Lemma 3.7. *Under A1 and A4,* $\forall\ (s_1^*, s_2^*) \in PSNE(\Gamma(\mu))$, $s_1^*(t) \leq s_1^*(t')$ *if* $t < t'$.

Proof. At equilibrium (s_1^*, s_2^*) , $\forall t, t' \in T$,

$$u_1(t; s_1^*, s_2^*) \geq u_1(t, s_1^*(t'), s_2^*(s_1^*(t'))).$$

Suppose $s_1^*(t) > s_1^*(t')$, and $t' > t$, by assumption of SCP,

$$u_1(t'; s_1^*, s_2^*) > u_1(t'; s_1^*(t'), s_2^*(s_1^*(t'))) = u_1^*(t'; s_1^*, s_2^*),$$

which upsets the equilibrium. □

Lemma 3.8. *Under A1 and A4,* $\forall\ (s_1^*, s_2^*) \in PSNE(\Gamma(\mu))$, $s_2^*(m) \leq s_2^*(m')$ *if* $m < m'$.

Proof. At equilibrium (s_1^*, s_2^*) , $\forall m' > m \in M$, since $s_2^*(m)$ is a best response to message m for any t,

$$u_2(t', m, s_2^*(m)) \geq u_2(t', m, s_2^*(m')).$$

Suppose $s_2^*(m) > s_2^*(m')$, from assumption A4

$$u_2(t', m', s_2^*(m)) \geq u_2(t', m', s_2^*(m')),$$

which is contrary to the fact that $s_2^*(m')$ is a best response to m'. □

Lemma 3.9. *For an* $(s_1^*, s_2^*) \in PSNE(\Gamma(\mu))$, *let*

$$T(r) = \{t \in T : u_1(t, m, r) > u_1(t; s_1^*, s_2^*)\}, \tag{3}$$

then under A1 and A4, $T(r)$ *is convex.*

Proof. For all t', $t'' \in T(r)$, $t' < t''$, suppose $\exists\ t \in [t', t'']$, such that

$$u_1(t, m, r) \leq u_1(t; s_1^*, s_2^*).$$

If $m > s_2^*(t)$, since $t' < t$, by A4, we obtain

$$u_1(t', m, r) \leq u_1(t', s_1^*(t), s_2^*(s_1^*(t))).$$

And at equilibrium,

$$u_1(t', s_1^*(t), s_2^*(s_1^*(t))) \leq u_1(t'; s_1^*, s_2^*),$$

therefore,

$$u_1(t', m, r) \leq u_1(t'; s_1^*, s_2^*),$$

which is contrary to the assumption that $t' \in T(r)$. We can analogously prove the other case where $m < s_2^*(t)$ to get a contradiction with $t'' \in T(r)$. Therefore, $\forall\ t \in [t', t'']$, $t \in T(r)$, which implies that $T(r)$ is convex. □

Given a response r of player 2 to message m, $T(r)$ is the set of types who are willing to deviate from the equilibrium strategy.

Lemma 3.10. *If $r < r'$, then $T(r) \subseteq T(r')$.*

Proof. $\forall\ t \in T(r)$,

$$u_1(t, m, r) < u_1(t, m, r')$$
$$u_1(t, m, r) > u_1(t; s_1^*, s_2^*).$$

The first inequality holds because of A3. Therefore

$$u_1(t; s_1^*, s_2^*) < u_1(t, m, r'),$$

which implies that $t \in T(r')$. □

This lemma implies that a higher response to message m induces more types of player 1 to deviate from the equilibrium strategy. Now let us prove Theorem 3.5:

Proof. Let (s_1, s_2) be an *LMSE* in $\Gamma(\mu)$. In order to prove (s_1, s_2) is an HTE, we just need to prove that for any out-of-equilibrium message m, there exists a belief $\beta(\cdot|m) \in \Delta(T)$, such that $s_2(m) = \mathrm{BR}(m, \beta(\cdot|m))$, can be rationalized by some strategy $\tilde{s}_{2,m} : M \to R$. Then we can construct the hypothesis testing model $(\rho, 0)$ in which the support of ρ contains priors derived from beliefs of on-the-quilibrium messages and out-of-equilibrium messages. Suppose m is an out-of-equilibrium message. Let

$$R_m = \{r \in R : \exists t \in T, \quad s.\ t. \quad u_1(t, m, r) > u_1(t; s_1, s_2)\}. \tag{4}$$

Case (i). $R_m = \emptyset$. In this case, no type would deviate to m from his equilibrium strategy given any response of player 2, that is, $\beta(t|m) = 0$, $\forall$ $t \in T$. Such β can be rationalized by s_2.
Case (ii). $R_m \neq \emptyset$. Due to lemma 3.8, let us consider $r_m = \min R_m$, then for all $r > r_m$, $\exists$ $t_0 \in T$,

$$u_1(t_0; s_1, s_2) < u_1(t_0, m, r),$$

which implies that any $\beta(\cdot|m) \in \Delta(T)$ supporting the equilibrium (s_1, s_2) must satisfy

$$\mathrm{BR}(m, \beta(\cdot|m)) < r_m.$$

According to lemma 3.9, we can denote $T(r_m) = [i, j]$, where

$$i = \min\{t \in T : u_1(t, m, r_m) > u_1(t; s_1, s_2)\}$$
$$j = \max\{t \in T : u_1(t, m, r_m) > u_1(t; s_1, s_2)\}.$$

We denote $m_j = s_1(j)$ and $k = \max\{t \in T : s_1(t) = s_1(j)\}$. If $m > m_j$, then $k = j$ by assumption A4. If $m < m_j$, then $u(t, m_j, r_m) > u(t, m, r_m)$, for all $t \in [j, k]$ because of the concavity assumption A2. Now let's consider the k-truncated game $\Gamma^k(\mu^k)$. We claim that there doesn't exist a profile of strategies $(\hat{s}_1^k, \hat{s}_2^k) \in \mathrm{PSNE}(\Gamma^k(\mu^k))$ such that $\hat{s}_1^k(t) = m$ and $\hat{s}_2^k(m) = r_m$ for any $t \in [i, j]$. Suppose, contrary to the assertion, there exists such equilibrium $(\hat{s}_1^k, \hat{s}_2^k)$, and at equilibrium, $\exists j_0 \in [i, j]$,

$$\hat{s}_1^k(j_0) = m, \quad \text{and}$$
$$\hat{s}_2^k(m) = r_m.$$

We denote

$$h = \max\{t \in [i,k], \hat{s}_1^k(t) = \hat{s}_1^k(j_0)\},$$

$h < j$ because either $j = k$ or $\hat{s}_1^k(t) > m$. $\forall t \in [j,k]$. By Prop. 3.3, $(\hat{s}_1^k, \hat{s}_2^k)$ is a PSNE in h-truncated game $\Gamma^h(\mu^h)$ by just simply dropping the strategies of types higher than h, and

$$u_1(h; \hat{s}_1^k, \hat{s}_2^k) > u_1(h; s_1, s_2),$$

therefore, Prop. 3.4 implies that there exists (s_1^*, s_2^*) l-dominates (s_1, s_2), which is contrary to the assumption that (s_1, s_2) is *LMSE*. This analysis means that for any $(\hat{s}_1^k, \hat{s}_2^k) \in \text{PSNE}(\Gamma^k(\mu^k))$, $\text{BR}(m, \beta_{[i,j]}) < r_m$. Especially, (s_1, s_2) is a PSNE of $\Gamma^k(\mu^k)$ by simply deleting the strategies of the types higher than k, therefore, $s_2(m) < r_m$. To sum up the argument above, for any out-of-equilibrium message m, for any belief $\beta(\cdot|m) \in \Delta(T)$, such that $s_2(m) = \text{BR}(m, \beta(\cdot|m))$, can be rationalized by the strategy $\tilde{s}_2$ of player 2:

$$\begin{aligned}
&\tilde{s}_2(m) = r_m, \\
&\tilde{s}_2(m') = s_2(m'), \quad \forall m' \neq m.
\end{aligned}$$

For all $t \in T(r_m)$,

$$\begin{aligned}
u_1(t, m, \tilde{s}_2(m)) &= u_1(t, m, r_m) \\
&> u_1(t; s_1, s_2) \\
&\geq u_1(t, m', s_2(m')) \quad \forall m' \in M \\
&= u_1(t, m', \tilde{s}_2(m')) \quad \forall m' \in M,
\end{aligned}$$

and for all $t \notin T(r_m)$, $\exists\ s_1(t) \neq m$, such that $u_1(t, m, \tilde{s}_2(m)) \leq u_1(t; s_1, s_2)$. Therefore,

$$\begin{aligned}
&\beta(m|t) = 1, \quad \forall t \in T(r_m), \\
&\beta(m|t) = 0, \quad \forall t \notin T(r_m).
\end{aligned}$$

We can construct a Hypothesis Testing model $(\rho, 0)$ in which

$$\text{supp}(\rho) = \{\pi_\Omega, \{\pi_m : \forall m \notin M^E_{(s_1,s_2)}\}\},$$

where

$$\pi_\Omega(\cdot|m) = \beta_{T_{m,s_1}}, \quad \forall m \in M^E_{(s_1,s_2)}$$
$$\pi_m(t|m) = \beta(t|m) = \beta_{T(r_m)}, \quad \forall t \in T, \quad \forall m \notin M^E_{(s_1,s_2)},$$

with

$\rho(\pi_m) = 0, \quad \text{if} \quad m \notin M^E_{(s_1,s_2)} \quad \text{and} \quad R_m = \emptyset.$

$0 < \rho(\pi_m) = \rho(\pi'_m) < \rho(\pi_\Omega) \quad \text{if} \quad m, m' \notin M^E_{(s_1,s_2)}, \quad R_m \neq \emptyset \quad \text{and} \quad R'_m \neq \emptyset.$

By construction, (s_1, s_2) is an HTE supported by $(\rho, 0)$. □

3.4 Uniqueness of *Constrained HTE*

As we mentioned before, the condition (ii) for the requirements of consistency of ρ is a weak condition in the sense that player 2 is allowed to take any strategy as long as player 1 is rational to this strategy, which enlarges the set of alternative beliefs of player 2. However, it is a natural idea if we restrict the strategies of player 2 such that her alternative beliefs are around her original belief.

Definition 3.5. An HTE $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$ is called *constrained HTE* if for any $m \notin M^E_{(s_1,s_2)}$, and $R_m \neq \emptyset$, for any posterior belief $\beta(\cdot|m) \in \Delta(T)$ such that $s_2(m) = \text{BR}(m, \beta(\cdot|m))$, then $\beta(\cdot|m)$ can be rationalized by

$$\begin{aligned} \tilde{s}_2(m) &= r_m \\ \tilde{s}_2(m') &= s_2(m') \quad \forall m' \neq m, \end{aligned} \tag{5}$$

where $r_m = \min R_m$ defined in Equ. (4).

Remark 1. In a *constrained HTE*, any belief of an out-of-equilibrium message supporting the equilibrium can be rationalized by a strategy of player 2 which is not far from her equilibrium strategy. This idea is quite intuitive, when player 2 observes a message deviated from her original belief, she looks for the most likely types who have potential incentive to send this message and forms her new belief through only perturbing her original belief on this message.

Remark 2. We notice that $\tilde{s}_2$ may not be a best response to $\tilde{s}_1$, only $\tilde{s}_1$ is required to be a best response to $\tilde{s}_2$. If $(\tilde{s}_1, \tilde{s}_2)$ forms an equilibrium, then at this equilibrium, $u_1(t, m, r_m) \leq u_1(t; s_1, s_2), \forall\, t \in T$. This case coincides with the *Consistent Forward Induction Equilibrium Path* proposed by Umbhauer (1991).

Remark 3. Go through the proof of the existence theorem, we can see that an *LMSE* is a *constrained HTE*.

Proposition 3.11. *Under A1-A4, LMSE is unique.*

Proof. Suppose both (s_1^*, s_2^*) and (s_1, s_2) are *LMSE*, since (s_1^*, s_2^*) is not *l*-dominates (s_1, s_2), for any $t_0 \in T$, such that:

$$u_1(t_0; s_1^*, s_2^*) > u_1(t_0; s_1, s_2),$$

$\exists t_1 > t_0,\quad$ such that

$$u_1(t_1; s_1^*, s_2^*) < u_1(t_1; s_1, s_2).$$

We have the same expression for (s_1, s_2), and T is finite, therefore, (s_1^*, s_2^*) and (s_1, s_2) are identical. □

Clearly, all other strategies profile $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$ must be *l*-dominated by the unique *LMSE*. Let us denote the unique *LMSE* as (s_1^{LM}, s_2^{LM}),

Theorem 3.12. *Under A1-A4, if the unique LMSE is complete separating, and M is rich enough, then the outcome of constrained HTE supported by $(\rho, 0)$ is unique.*

Lemma 3.13. *Under A1-A4, assume $(s_1, s_2) \in PSNE(\Gamma(\mu))$ is a completely separating equilibrium, let $j = \min\{t \in T : u_1(t; s_1^{LM}, s_2^{LM}) > u_1(t; s_1, s_2)\}$. If $\forall\, t \in T$, $s_1(t) \neq s_1^{LM}(j)$, then (s_1, s_2) is not a constrained HTE.*

Proof. If $j = \min\{t \in T : u_1(t; s_1^{LM}, s_2^{LM}) > u_1(t; s_1, s_2)\}$, We denote:

$$s_1^{LM}(j) = m, \quad s_2^{LM}(m) = r_j^*,$$
$$s_1(j) = m_j, \quad s_2(m_j) = r_j;$$

We assume that $\beta(\cdot|m) \in \Delta(T)$ is a posterior belief supporting the equilibrium (s_1, s_2), then $R(m) \neq \emptyset$ since $u_1(j, m, r_j^*) > u_1(j; s_1, s_2)$. Let $r_m = \min R_m$, then $r_j^* \geq r_m$. Further, we can show that $r_m = r_j^*$. This is true because of the following argument: By assumption, we have can obtain that:

$$\begin{aligned}
&u_1(j, m, r_j^*) > u_1(j, m_j, r_j) \quad \text{and} \\
&u_1(j, m_j, r_j) > u_1(j, m, s_2(m)), \\
\Rightarrow &u_1(j, m, r_j^*) > u_1(j, m, s_2(m)) \\
\Rightarrow &s_2(m) < r_j^* \\
\Rightarrow &\text{BR}(m, \beta(\cdot|m)) < \text{BR}(m, \beta_j),
\end{aligned}$$

since (s_1^{LM}, s_2^{LM}) is a completely separating equilibrium. Therefore,

$$\sum_{t \in T} \beta(t|m) < \mu(j), \quad \text{and} \quad \exists t_0 < j, \quad \beta(t_0|m) > 0.$$

Let

$$T(r_m) = \{t \in T : u_1(t, m, r_m) > u_1(t, s_1, s_2)\}.$$

By contradiction, suppose (s_1, s_2) is a *constrained HTE*, then $\beta(\cdot|m)$ can be rationalized by $\tilde{s}_2$ given in Equ. (5). And $\beta(t_0|m) > 0$ implies that m is a best response of t_0, that is, $u_1(t_0, m, \tilde{s}_2) > u_1(t_0, m', \tilde{s}_2(m'))$, $\forall\ m' \in M$. Therefore, $t_0 \in T(r_m)$. However,

$$u_1(t; s_1^{LM}, s_2^{LM}) \leq u_1(t; s_1, s_2) \quad \forall t < j,$$

which together with

$$u_1(t; s_1^{LM}, s_2^{LM}) \geq u_1(t; m, r_j^*),$$

implies that

$$u_1(t, m, r_j^*) < u_1(t; s_1, s_2) \quad \forall t < j.$$

Therefore, $\forall\ r < r_j^*$,

$$u_1(t_0, m, r) \leq u_1(t_0; m, r_j^*) < u_1(t_0; s_1, s_2).$$

Therefore, $r_m \geq r_j^*$, which induces $r_m = r_j^*$. However, with this strategy $\tilde{s}_2$,

$$u_1(j, m, \tilde{s}_2) = u_1(j, m, r_j^*) > u_1(j; s_1, s_2) \geq u_1(j, m', \tilde{s}_2(m')), \quad \forall m' \in M$$

which implies that $j \in T_{r_m}$. Therefore, $\beta(j|m) \geq \mu(j)$. We have a contradiction with $\sum_{t \in T} \beta(t|m) < \mu(j)$. □

Now let us prove Theorem 3.12.

Proof. We just need to show that any $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$ l-dominated by (s_1^{LM}, s_2^{LM}) is not a *constrained HTE.* For any $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$, let

$$j = \min\{t \in T : u_1(t; s_1^{LM}, s_2^{LM}) > u_1(t; s_1, s_2)\},$$

and $s_1^{LM}(j) = m_j$. If m_j is an out-of-equilibrium message under (s_1, s_2), by lemma 3.14, we can get the conclusion immediately. However, if $m_j \in M^E_{(s_1,s_2)}$, according to our assumption, M is rich enough, we can select an m slightly greater than m_j but less than $s_1^{LM}(j+1)$ such that $m \notin M^E_{(s_1,s_2)}$ and form a new PSNE (s_1^*, s_2^*) by just perturbing (s_1^{LM}, s_2^{LM}) at j. Then (s_1^*, s_2^*) still l-dominates (s_1, s_2). This can be done because (s_1^{LM}, s_2^{LM}) is completely separating equilibrium. Again, lemma 3.13 implies that (s_1, s_2) is not a *constrained HTE.* □

Corollary. *Under A1- A4, if there only exist completely separating equilibria, then a "Pareto-dominant equilibrium" is the unique constrained HTE.*

Proof. A "Riley equilibrium" maximizes the payoffs of all the types in the completely separating equilibria set, which means that it is the unique *LMSE*, therefore, it is the unique *constrained HTE.* □

This proposition ensures that our HTE concept can catch the well-known "Pareto-dominant separating equilibrium" or "Riley outcome" that is often selected in applications.

4 Comparison with Intuitive Criterion

Intuitive Criterion: (Cho and Kreps, 1987) Fixed a sequential equilibrium outcome and let $u_1^*(t)$ be the payoff of a type t of player 1 in this equilibrium. For each out of equilibrium message m, form the set

$$S(m) = \{t \in T : \quad u_1^*(t) > \max_{r \in \mathrm{BR}(T(m),m)} u_1(t,m,r)\}, \tag{6}$$

If, for some out of equilibrium message m there exists a type $t' \in T$ such that

$$u_1^*(t') < \min_{r \in \mathrm{BR}(T \backslash S(m),m)} u_1(t',m,r), \tag{7}$$

then the equilibrium outcome fails the Intuitive Criterion.

Proposition 4.1. *In a finite signaling game $\Gamma(\mu)$, if $(\sigma_1, \sigma_2) \in SNE(\Gamma(\mu))$ fails the Intuitive Criterion, it is also not an HTE supported by a Hypothesis Testing model $(\rho, 0)$.*

Proof. If $(\sigma_1, \sigma_2) \in \mathrm{SNE}(\Gamma(\mu))$ fails the Intuitive Criterion, then for some $m \notin M^E_{(\sigma_1,\sigma_2)}$, $\exists\ t' \in T$, such that condition (7) holds. We prove that for this m, any belief $\pi_m \in \Delta(T \times M)$, such that

$$u_1(t; s_1, s_2) \geq u_1(t, m, \mathrm{BR}(m, \pi_m(\cdot|m))), \quad \forall t \in T, \tag{8}$$

there doesn't exist a strategy of player 2 can rationalize $\pi_m(\cdot|m)$. By contradiction, suppose there exists $\tilde{s}_2 : M \to R$ rationalizes $\pi_m(\cdot|m)$, then for any $m' \in M$, there exists a posterior distribution $\mu_2(\cdot|m') \in \Delta(T)$, such that $\tilde{s}_2 = \mathrm{BR}(t, m', \mu_2(\cdot|m'))$. By the definition of $S(m)$, m would never be a best response of type $t \in S(m)$ to $\tilde{s}_2$, therefore,

$$\pi_m(t,m) = 0, \quad \forall t \in S(m),$$

which implies that $\mathrm{BR}(\pi_m, T\backslash S, m) = \mathrm{BR}(\pi_m, T, m)$. Since

$$\begin{aligned} u_1^*(t') &< \min_{r \in \mathrm{BR}(T(m)\backslash S(m),m)} u_1(t',m,r) \\ &\leq u_1(t',m,r), \quad r \in \mathrm{BR}(\pi_m, T(m)\backslash S(m), m) \\ &\leq u_1(t',m,r), \quad r \in \mathrm{BR}(\pi_m, T(m), m), \end{aligned}$$

type t' could achieve a payoff strictly higher than his expected payoff at this equilibrium by sending the message m. This is contrary to condition (8).

$\square$

Example 4.1. We don't need to restrict in pure strategies to get the proposition above. The simple game depicted in Figure 3 provides an example to show this property. There are two PSNE in this game. At the first one, the

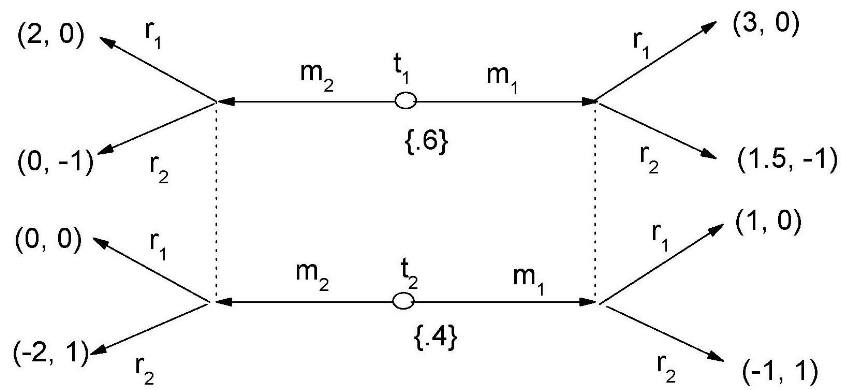

Figure 3

strategies of player 1 and player 2 are given by:

$$\begin{aligned} &\sigma^*(m_1; t_1) = 1, \quad \sigma^*(m_1; t_2) = 1; \\ &\tau^*(r_1; m_1) = 1, \end{aligned} \tag{9}$$

This is supported by any belief of player 2 on out-of-equilibrium message m_2. We can easily check that this equilibrium can pass Intuitive Criterion and it is also an HTE. The second equilibrium is:

$$\begin{aligned} &\sigma^*(m_2; t_1) = 1, \quad \sigma^*(m_2; t_2) = 1; \\ &\tau^*(r_1; m_2) = 1. \end{aligned} \tag{10}$$

This is supported by a belief such that $\mu_2(t_1|m_1) < 0.5$. Firstly, we check that this PSNE can survive Intuitive Criterion. For the out-of-equilibrium message m_1, we form the set

$$S(m_1) = \{t \in T : u_1^*(t) > \max_{r \in \mathrm{BR}(T, m_1)} u_1(t, m_1, r)\}.$$

Since

$$u_1^*(t_1) = 2 < \max_{r \in \mathrm{BR}(T, m_1)} u_1(t_1, m_1, r) = u_1(t_1, m_1, r_1) = 3,$$

and

$$u_1^*(t_2) = 0 < \max_{r \in \mathrm{BR}(T, m_1)} u_1(t_2, m_1, r) = u_1(t_1, m_1, r_1) = 1,$$

we have $S(m_1) = \emptyset$. Therefore, there doesn't exists some type t' such that

$$u_1^*(t') < \min_{r \in \mathrm{BR}(T \setminus S(m_1), m_1)} u_1(t', m_1, r).$$

Secondly, we can show that this PSNE is not an HTE. By contradiction, if it's an HTE, then for the out-of-equilibrium message m_1, there must exist a belief π' such that $\pi'(t_1|m_1) < 0.5$, which implies that $\pi'(t_2, m_1) > 0$. Then according to the condition (ii) of the consistency definition 3.2 , any strategy of player 2 that rationalizes π' must be such that m_1 is a best response for type t_2. If we write such strategy of player 2 as $\phi(r_1; m_1) = x \in [0, 1]$ and $\phi(r_1; m_2) = y \in [0, 1]$, then x and y must satisfy the following condition:

$$\begin{aligned} & u_1(t_2, m_1, \phi) \geq u_1(t_2, m_2, \phi) \\ \Rightarrow\ & 1 \cdot x + (-1) \cdot (1 - x) \geq 0 \cdot y + (-2) \cdot (1 - y) \\ \Rightarrow\ & 2x - 2y + 1 \geq 0. \end{aligned}$$

With such strategy, the payoffs of type t_1 are:

$$\begin{aligned} u_1(t_1, m_1, \phi) &= 3 \cdot x + 1.5 \cdot (1 - x) = 2.5x + 1.5, \\ u_1(t_1, m_2, \phi) &= 2 \cdot y + 0 \cdot (1 - y) = 2y. \end{aligned}$$

Since

$$\begin{cases} 2x - 2y + 1 \geq 0 \\ x \in [0, 1], y \in [0, 1] \end{cases} \Rightarrow 2.5x + 1.5 > 2y,$$

which means that m_1 is the unique best response for type t_1. That is $\pi'(m_1|t_1) = 1$. Therefore

$$\pi'(t_1|m_1) \geq \pi'(m_1|t_1)\pi'(t_1) = 0.6,$$

which is contrary to $\pi'(t_1|m_1) < 0.5$.

5 HTE of the Milgrom-Roberts Model

In this section we present an example of simplified version of the Milgrom and Roberts' (1982) limit pricing entry model to illustrate how to find an HTE. In this example, there are two periods and two firms. Firm 1, the incumbent, has private information concerning his production costs. Firm 1 chooses a first-period quantity Q_1. In the second period, firm 2, the entrant, observes the quantity and decides to enter the market or not. It pays a fixed cost $K > 0$ if it enters. Then the private information is revealed and last stage of the game played, either by both firms competing in a Cournot duopoly or by only the incumbent acting as a monopolist[3]. Suppose there are two possible costs for firm 1, (c_L, c_H). The common prior $\mu(c_1 = c_L) = 1 - \mu(c_1 = c_H) = x$. Firm 2's cost $c_2 > 0$ is common knowledge.

We assume that the inverse market demand is given by $P(Q) = a - bQ, a, b > 0$. Let $\Pi_1^t(Q_1)$ denote type t of firm 1's monopoly profit in the first period if its production quantity is Q_1, that is

$$\Pi_1^t(Q_1) = (a - bQ_1 - c_t)Q_1 \quad t \in \{L, H\},$$

then it is easily to see that $\Pi_1^t(Q_1)$ is strictly concave in Q_1. Let Q_M^L and Q_M^H denote firm 1's monopoly quantities that maximize its short-run profit when its cost is low and high. With our linear demand function, we can explicitly get

$$Q_M^L = \frac{a - c_L}{2b} \quad \text{and} \quad Q_M^H = \frac{a - c_H}{2b}.$$

[3]J. Tirole (1988) analyzed the similar model that the two firms play Bertrand Competition game if entry occurs

Let Π_M^L and Π_M^H denote the monopoly profits respectively, that is

$$\Pi_M^L = \Pi_1^L(Q_M^L) = \frac{(a-c_L)^2}{4b} \quad \text{and} \quad \Pi_M^H = \Pi_1^H(Q_M^H) = \frac{(a-c_H)^2}{4b}.$$

Since we assume (as Milgrom and Roberts, 1982) that firm 2 learns firm 1's cost immediately if firm 2 decides to enter the market, we can explicitly compute the two firm's Cournot duopoly profits with complete information.

$$\Pi_{1C}^t = \frac{(a+c_t-2c_2)^2}{9b}, \quad \text{and} \quad \Pi_{2C}^t = \frac{(a+c_2-2c_t)^2}{9b} - K,$$

where $t \in \{L, H\}$. We assume that the discount factors $\delta_1 = \delta_2 = \delta$. To make things interesting, assume that $\Pi_{2C}^H > 0 > \Pi_{2C}^L$, which implies that, under complete information, firm 2 enters the market if and only if firm 1 is high cost type. And $E_x(\Pi_{2C}) < 0$ which implies that firm 2 won't enter the market if it only has the prior information.

In order to fit our HTE concept, we assume that firm 1 only has four choices of Q_1, that is $Q_1 \in \{Q_M^L, Q_M^H, Q_1^L, Q_1^H\}$, where $Q_1^t > Q_M^t$, $\forall\ t \in \{L, H\}$ and satisfy the following conditions:

$$\Pi_1^L(Q_1^t) + \delta\Pi_M^t > \Pi_M^t + \delta\Pi_{1C}^t. \quad \forall t \in \{L, H\}$$

This condition implies that the low-cost (high-cost) type of firm 1 prefers to deter the entrant by choosing a higher quantity Q_1^L (Q_1^H) if the monopoly quantity Q_M^L (Q_M^H) induces entry. Let $\tilde{Q}_1^t$ for $t \in \{L, H\}$ denote the quantity at which level the type t of firm 1 is indifferent in deterring and not deterring the entry. That is

$$\Pi_1^t(\tilde{Q}_1^t) + \delta\Pi_M^t = \Pi_M^t + \delta\Pi_{1C}^t, \quad \text{for} \quad t \in \{L, H\}.$$

Due to the strictly concavity of $\Pi_1^t(Q_1^t)$, we have $\tilde{Q}_1^t < Q_1^t$. Now there are only three interesting cases left to us to analyze the equilibria:

$$(i). \quad \tilde{Q}_1^L > Q_1^L > Q_M^L > \tilde{Q}_1^H > Q_1^H > Q_M^H,$$
$$(ii). \quad \tilde{Q}_1^L > Q_1^L > \tilde{Q}_1^H > Q_1^H > Q_M^L > Q_M^H,$$
$$(iii). \quad \tilde{Q}_1^L > Q_1^L > \tilde{Q}_1^H > Q_M^L > Q_1^H > Q_M^H.$$

In this example, we only consider the pure strategies of the firms, let

$$s : \{L, H\} \to \{Q_M^L, Q_M^H, Q_1^L, Q_1^H\}$$

and

$$t : \{Q_M^L, Q_M^H, Q_1^L, Q_1^H\} \to \{0, 1\}$$

be the strategies of firm 1 and firm 2 respectively. We can check that, in this game, the assumptions A1-A4 are satisfied, therefore, an HTE exists.

Proposition 5.1. *In case (i), there exist two separating PSNE:*

$$\begin{aligned} &s(L) = Q_M^L, \quad \textit{and} \quad s(H) = Q_M^H \\ &t(Q_1) = \begin{cases} 0 & \textit{if} \quad Q_1 \geq Q_M^L \\ 1 & \textit{otherwise} \end{cases} \end{aligned} \qquad (PSNE \quad 5.1.1)$$

and

$$\begin{aligned} &s(L) = Q_1^L, \quad \textit{and} \quad s(H) = Q_M^H \\ &t(Q_1) = \begin{cases} 0 & \textit{if} \quad Q_1 \geq Q_1^L \\ 1 & \textit{otherwise} \end{cases} \end{aligned} \qquad (PSNE \quad 5.1.2)$$

Proposition 5.2. *Under condition in case (i), both of the equilibria can pass the Intuitive Criteria, but only the efficient separating equilibrium (PSNE 5.1.1) is an HTE.*

The proof is in appendix A.

Proposition 5.3. *In case (ii), there exist two pooling PSNE and one separating PSNE:*

$$\begin{aligned} &s(L) = s(H) = Q_M^L \\ &t(Q_1) = \begin{cases} 0 & \textit{if} \quad Q_1 \geq Q_M^L \\ 1 & \textit{otherwise,} \end{cases} \end{aligned} \qquad (PSNE \quad 5.3.1)$$

$$s(L) = s(H) = Q_1^H$$
$$t(Q_1) = \begin{cases} 0 & if \quad Q_1 \geq Q_1^H \\ 1 & otherwise, \end{cases} \qquad (PSNE \quad 5.3.2)$$

and

$$s(L) = Q_1^L, \quad and \quad s(H) = Q_M^H$$
$$t(Q_1) = \begin{cases} 0 & if \quad Q_1 \geq Q_1^L \\ 1 & otherwise \end{cases} \qquad (PSNE \quad 5.3.3)$$

Proposition 5.4. *Under condition in case (ii), all these three PSNE can pass the Intuitive Criterion, but only the efficient equilibrium (PSNE 5.3.1) is an HTE.*

Proof is given in Appendix B.

Proposition 5.5. *In case (iii), there exist one separating PSNE and one pooling PSNE:*

$$s(L) = s(H) = Q_M^L$$
$$t(Q_1) = \begin{cases} 0 & if \quad Q_1 \geq Q_M^L \\ 1 & otherwise \end{cases} \qquad (PSNE \quad 5.5.1)$$

and

$$s(L) = Q_1^L, \quad and \quad s(H) = Q_M^H$$
$$t(Q_1) = \begin{cases} 0 & if \quad Q_1 \geq Q_1^L \\ 1 & otherwise \end{cases} \qquad (PSNE \quad 5.5.2)$$

Proposition 5.6. *Under conditions in case (iii), both of the PSNE can pass the Intuitive Criterion, but only the efficient pooling equilibrium (PSNE 5.5.1) is an HTE.*

We can use almost the same argument as the proof of prop. 5.4 to prove this proposition. So we omit the proof here.

To summarize this analysis of the limit pricing entry deterrence model: In each case, there exists a unique HTE. At an HTE, the hight-cost type firm 1 either chooses its monopoly quantity and allows entry or makes pooling with the low-cost type and deters the entry depending on the cost of pooling. The low-cost type choose its monopoly quantity and entry doesn't occur. By contrast, in each case, there exist multi equilibria which can survive Intuitive Criterion. At separating equilibrium (PSNE 5.1.2; PSNE 5.3.3; PSNE 5.5.2), even though it is costly for the high-cost type to pool with the low-cost type, the low-cost type firm 1 still needs to sacrifice its short-run profit and chooses a higher quantity to distinguish himself from the high-cost type. We argue that this type of equilibrium can't be an HTE because there doesn't exist a rational belief system to support the equilibrium when an out-of-equilibrium message Q_M^L is observed.

6 Conclusions

In this paper, we propose a general definition of Hypothesis Testing Equilibrium (HTE) in a framework of general signaling games with Non-Bayesian players. We focus on the analysis of a special class of HTE with threshold $\epsilon = 0$ as a method of equilibrium refinement which can survive the Intuitive Criterion. For a broad class of signaling games, a Lexicographically Maximum sequential equilibrium is an HTE and the *constrained HTE* is unique. In the example of limit pricing enter deterrence game, we show that there exists unique HTE in each interesting case. However, there are aspects of HTE we have not considered. In particular, the existence and uniqueness of general HTE in more general signaling games, which we can't get a concrete conclusion. Natural extensions of HTE is to apply the concept in signaling games in which the state space is infinite or for general dynamic games. Without dynamic consistency, there will be some difficulties for these extensions. Nevertheless, a further study of the general HTE in dynamic games is encouraged.

APPENDIX

A. proof of proposition 5.2

Proof. To prove

$$s(L) = Q_M^L, \quad \text{and} \quad s(H) = Q_M^H$$
$$t(Q_1) = \begin{cases} 0 & \text{if} \quad Q_1 \geq Q_M^L \\ 1 & \text{otherwise} \end{cases}$$

is an HTE, we construct a Hypothesis Testing Model $(\rho, 0)$ to support this equilibrium. The support of ρ contains three elements: π_0, π_1, and π_2 such that $\rho(\pi_0 > \rho(\pi_1) = \rho(\pi_2) > 0$. And they satisfy the following conditions:
(i). $\pi_i(L) = 1 - \pi_i(H) = x$, $\forall\ i \in \{0, 1, 2\}$;
(ii). $\pi_0(Q_M^L|L) = 1$, $\pi_0(Q_M^H|H) = 1$;
(ii). $\pi_1(Q_1^L|L) = 1$, $\pi_1(Q_M^H|H) = 1$;
(ii). $\pi_2(Q_M^L|L) = 1$, $\pi_2(Q_1^H|H) = 1$.
We can check that π_0 can be rationalized by the strategy

$$t(Q_1) = \begin{cases} 0 & \text{if} \quad Q_1 \geq Q_M^L \\ 1 & \text{otherwise;} \end{cases}$$

π_1 can be rationalized by the strategy

$$t(Q_1) = \begin{cases} 0 & \text{if} \quad Q_1 \geq Q_1^L \\ 1 & \text{otherwise;} \end{cases}$$

and π_2 can be rationalized by the strategy

$$t(Q_1) = \begin{cases} 0 & \text{if} \quad Q_1 \geq Q_1^H \\ 1 & \text{otherwise.} \end{cases}$$

Firm 2 starts with π_0 and keeps it for Bayesian updating if it observes Q_M^L or Q_M^H.If firm 2 observes Q_1^L, it switches to π_1 and proceeds Bayesian Updating because

$$\pi_1(Q_1^L)\rho(\pi_1) = \pi_1(L)\rho(\pi_1) > 0$$
$$\pi_0(Q_1^L)\rho(\pi_0) = \pi_2(Q_1^L)\rho(\pi_2) = 0$$

Analogously, if firm 2 observes Q_1^H, it switches to π_2. Clearly, given the strategy s of firm 1, t is a best response of firm 2. Given t, under the condition in case (i), we have:

$$u_1(L, Q_M^L, t(Q_M^L) = 0) > u_1(L, Q_1^L, t(Q_1^L) = 0),$$
$$u_1(L, Q_M^L, t(Q_M^L) = 0) > u_1(L, Q_1^H, t(Q_1^H) = 1),$$
$$u_1(L, Q_M^L, t(Q_M^L) = 0) > u_1(L, Q_M^H, t(Q_M^H) = 1),$$

and

$$u_1(H, Q_M^H, t(Q_M^H) = 1) > u_1(H, Q_1^L, t(Q_1^L) = 0),$$
$$u_1(H, Q_M^H, t(Q_M^H) = 1) > u_1(H, Q_1^H, t(Q_1^H) = 1),$$
$$u_1(L, Q_M^H, t(Q_M^H) = 1) > u_1(H, Q_M^H, t(Q_M^H) = 1),$$

Therefore, (s, t) is an HTE.

Now let's prove that

$$s(L) = Q_1^L, \quad \text{and} \quad s(H) = Q_M^H$$

$$t(Q_1) = \begin{cases} 0 & \text{if} \quad Q_1 \geq Q_1^L \\ 1 & \text{otherwise} \end{cases}$$

is not an HTE. We can check that when the out-of-equilibrium message Q_M^L is observed, for any $\mu_2(\cdot|Q_M^L)$, such that

$$\mu_2(L|Q_M^L)\Pi_{2C}^L + (1 - \mu_2(L|Q_M^L))\Pi_{2C}^H > 0,$$

there doesn't exist a strategy of player 2 by which $\mu_2(\cdot|Q_M^L)$ can be rationalized. Since $\Pi_{2C}^L < 0 < \Pi_{2C}^H$, then $\mu_2(H|Q_M^L) = 1 - \mu_2(L|Q_M^L)$ must be big enough, which implies $\mu_2(Q_M^L|H) > 0$. By definition 3.1 (iii), any strategy of firm 2 that rationalizes μ_2 must be such that Q_M^L is a best response for type H. However, under condition in case (i), $Q_M^L > \tilde{Q}_1^H$, Q_M^L would never be a best response for type H because Q_M^L is strictly dominated by $\tilde{Q}_1^H$.

Now let's check that both of the equilibria can pass the Intuitive Criterion. In PSNE 5.3.1, $S(Q_1^L) = \{H\}$, and $S(Q_1^H) = \{L\}$, however, there doesn't exist t' such that condition (6) holds for Q_1^L and Q_1^H.

□

B. proof of proposition 5.4

Proof. (PSNE 5.3.1) is an HTE supported Hypothesis Testing model $(\rho, 0)$ where the support of ρ contains four elements: π_0, π_1, π_2 and π_3 such that $\rho(\pi_\Omega) > \rho(\pi_1) = \rho(\pi_2) = \rho(\pi_3) > 0$. And they satisfy the following conditions:
(i). $\pi = 1 - \pi = x$, for all $\pi \in supp(\rho)$;
(ii). $\pi_0(Q_M^L|L) = 1$, $\pi_0(Q_M^L|H) = 1$;
(iii). $\pi_1(Q_M^L|L) = 1$, $\pi_1(Q_M^H|H) = 1$;
(iv). $\pi_2(Q_1^H|L) = 1$, $\pi_2(Q_1^H|H) = 1$;
(v). $\pi_3(Q_1^L|L) = 1$, $\pi_2(Q_M^H|H) = 1$.

(PSNE 5.3.2) is not an HTE. When Q_M^L is observed, we show there doesn't exist a strategy of player 2 to rationalize any belief μ_2 satisfying the following conditions:

$$\mu_2(L) = 1 - \mu_2(H) = x, \quad \text{and}$$
$$\mu_2(L|Q_M^L)\Pi_{2C}^L + \mu_2(H|Q_M^L)\Pi_{2C}^H > 0.$$

By assumption:

$$\Pi_{2C}^L < 0 < \Pi_{2C}^H \quad \text{and} \quad x\Pi_{2C}^L + (1-x)\Pi_{2C}^H < 0,$$

we can obtain that $\mu_2(H|Q_M^L) > x$, therefore, $\mu_2(H, Q_M^L) > 0$, which implies that any strategy of firm 2 rationalizes μ_2 must be such that Q_M^L is a best response for type H. Since we restrict pure strategy, the strategy $t : M \to R$

must satisfy $t(Q_M^L) = 0$. But with this strategy, Q_M^L is the unique best response for type L, that is $\mu_2(Q_M^L|L) = 1$. Then

$$\begin{aligned}\mu_2(L|Q_M^L) &= \frac{\mu_2(Q_M^L|L)\mu_2(L)}{\mu_2(Q_M^L|L)\mu_2(L) + \mu_2(Q_M^L|H)\mu_2(H)} \\ &= \frac{x}{x + \mu_2(Q_M^L|H)(1-x)} \\ &\geq x,\end{aligned}$$

which is contradiction with $\mu_2(L|Q_M^L) < x$.

We can use the similar argument to prove that PSNE (5.3.3) is not an HTE. There doesn't exist rational prior to support the out-of-equilibrium message Q_M^L. We also can show that this equilibrium can pass the Intuitive Criterion because any out-of-equilibrium message $m \in \{Q_M^L, Q_1^H\}$, we have $S(m) = \emptyset$. □

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