Dissertation

Aspects of Nonequilibrium in Leptogenesis

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März 2016

Betreuer: Prof. Dr. Dietrich Bödeker

Gedruckt auf alterungsbeständigem Papier °° ISO 9706

Abstract

In this thesis, we study Leptogenesis in a regime where the heavy right-handed Majorana neutrinos are nonrelativistic when they decay and thereby produce a lepton - antilepton asymmetry. We motivate rate equations that are valid at leading order in the Yukawa couplings of the heavy neutrinos and to all orders in all the Standard Model interactions. We calculate all coefficients in these equations explicitly at leading order in all interactions and partially include next-to-leading order radiative corrections. Thereby, we additionally derive a relation between the production rate, the equilibration rate, and the self-energy of a general particle species. We also introduce systematic relativistic corrections. Finally, we solve the rate equations numerically and combine our solutions with experimental data on neutrino oscillations to obtain upper bounds on the light neutrino masses.

Contents

1 Introduction

One of the unsolved problems in modern cosmology is the asymmetry between baryonic and antibaryonic matter in the visible universe. The baryon to photon density ratio η_B has been measured by the WMAP satellite [1]

$$\eta_B^{\text{CMB}} = (6.19 \pm 0.14) \cdot 10^{-10}. \tag{1.1}$$

Since all our observations, here on earth as well as up to the scales of galaxy clusters, indicate that there is almost no antimatter present in the universe today [2, §6.2], this value can be understood as the asymmetry between baryons and antibaryons normalised to the photon density. The Standard Model of particle physics is not able to explain why this *Baryon Asymmetry of the Universe* exists [3]. It is also very unlikely that it is an initial condition of the universe because, if the Big Bang produced a baryon asymmetry, it would have been diluted away during inflation [4]. However, then there must be a mechanism that dynamically produced the asymmetry. In 1967, Andrei Sakharov pointed out that such a Baryogenesis mechanism needs to fulfill three necessary criteria [5]. They are known today as the Sakharov criteria:

- **B-violation**
  Obviously, if Baryon number B is always conserved, no asymmetry can be produced.

- **C- and CP-violation**
  The charge conjugation C flips the sign of all internal charges of a particle. CP is the composite transformation of charge conjugation and parity P, where parity flips the sign of all space coordinates. To see that both C and CP must be violated, let $X \to Y + B$ be a process which violates baryon number by $+1$. It may occur at the rate Γ. Then its charge conjugated process $\overline{X} \to \overline{Y} + \overline{B}$ which violates baryon number by -1 occurs at the rate $\overline{\Gamma}$. If now C was a symmetry, we would have $\Gamma = \overline{\Gamma}$ and no B-asymmetry would be produced. The same argument requires CP-violation. Consider the B-violating process $X \to q_L q_L$. If CP was a symmetry, $\Gamma(X \to q_L q_L) + \Gamma(X \to q_R q_R) = \Gamma\left(\overline{X} \to \overline{q}_L \overline{q}_L\right) + \Gamma\left(\overline{X} \to \overline{q}_R \overline{q}_R\right)$. The indices L and R mark left- and right-handed particles, respectively [6].

- **Departure from thermal equilibrium**
  The processes which produce the baryon asymmetry must occur out of thermal equilibrium because otherwise the expectation value of the baryon number $\langle B \rangle$ vanishes. To see this, recall the definition

$$\langle B \rangle = \frac{1}{Z} Tr\left(e^{-\beta H} B\right),$$

where H is the Hamiltonian and Z is the partition function. We assume that CPT is a symmetry and therefore $[CPT, H] = 0$. Then [7]

$$\begin{aligned}\langle B \rangle &= \frac{1}{Z} Tr\left((CPT)(CPT)^{-1} e^{-\beta H} B\right) \\ &= \frac{1}{Z} Tr\left((CPT)^{-1} e^{-\beta H} B (CPT)\right) \\ &= \frac{1}{Z} Tr\left(e^{-\beta H} (CPT)^{-1} B (CPT)\right) \\ &= -\frac{1}{Z} Tr\left(e^{-\beta H} B\right) \\ &= -\langle B \rangle.\end{aligned}$$

One could also state that in thermal equilibrium the decay rate is equal to the inverse decay rate. Consider again the process $X \to Y + B$ which changes baryon number bei $+1$. In thermal equilibrium, its rate is equal to that of the process $Y + B \to X$ which violates baryon number by -1 and again no net asymmetry can be produced [6].

Several Baryogenesis scenarios are under consideration. In this thesis, we will study *Baryogenesis via Leptogenesis*, a theory which was first proposed in 1986 by Fukugita and Yanagida [8]. The idea is that the decays of heavy right-handed Majorana neutrinos that are added to the Standard Model produce an asymmetry between leptons and antileptons. This asymmetry is partially converted into a baryon asymmetry via sphaleron processes.

This theses is organized as follows: We first discuss briefly the mechanisms that are required for Leptogenesis to work, i.e. we introduce neutrino masses via the type-I seesaw mechanism in Sec. 2 and derive a relation between lepton, or more precise $B - L$, number density and the baryon number density in Sec. 3. In Sec. 4 we present a description for Leptogenesis in the limit where the heavy Majorana neutrinos are nonrelativistic and establish the rate equations. We calculate all coefficients in these rate equations at leading order in all couplings and most of them up to next-to-leading order in the Standard Model couplings. In Sec. 5 we introduce systematic relativisitic corrections. In Sec. 6, we consider a more general setup to obtain a relation between production rates and equilibration rates. The numerical results are presented in Sec. 7. Finally, we summerize our results in Sec. 8.

Throughout this thesis, we use natural units, i.e. $c = \hbar = k_B = 1$.

2 Massive neutrinos

In the Standard Model, there are only left-handed, massless neutrinos [9, §1.2.2]. However, it could be that these neutrinos do have very small masses $\lesssim 2\text{eV}$ [10]. In fact, when neutrino oscillations were observed, this was considered a proof that at least two neutrinos cannot be massless because the mass-squared differences enter the oscillation rate [11]. From atmospheric and solar neutrino oscillations we know two different mass-squared differences, Δm^2_{atm} and Δm^2_{sol}, respectively. If not stated otherwise, we assume normal hierarchy, i.e. [12]

$$\Delta m^2_{\text{atm}} = m_3^2 - m_2^2 \approx 2.6 \cdot 10^{-3}\,(\text{eV})^2. \tag{2.1}$$

$$\Delta m^2_{\text{sol}} = m_2^2 - m_1^2 \approx 7.9 \cdot 10^{-5}\,(\text{eV})^2 \tag{2.2}$$

Today's experiments are consistent with neutrinos being Dirac or Majorana particles [9, §7.10.1]. Either way, to generate neutrino masses, we have to add right-handed neutrinos to the Standard Model. For illustration, let us consider the case of only one generation. For a right-handed Dirac neutrino ν_R, we introduce the gauge invariant Yukawa interaction term

$$\mathcal{L}_{\text{int}} = h\overline{\nu_R}\tilde{\varphi}^\dagger \ell + \text{h.c.}, \tag{2.3}$$

where ℓ is a left-handed lepton doublet, $\tilde{\varphi} = i\sigma^2\varphi^*$ is the conjugate Higgs doublet and h is the Yukawa coupling. After symmetry breaking this contains the Dirac mass term

$$\mathcal{L}_{\text{mass,D}} = -m_D\overline{\nu_R}\nu_L + \text{h.c.}, \tag{2.4}$$

where $m_D = hv$ and $v = 174$ GeV is the Higgs vacuum expectation value. The smallness of the neutrino masses could now only be explaned by very small Yukawa couplings h. On the other hand, if neutrinos were Majorana particles, it could be explaned via the seesaw mechanism [13]. Let us introduce an additional, Majorana massterm [14]

$$\mathcal{L}_{\text{mass,M}} = -\frac{m_M^L}{2}\left(\overline{\nu_L}\,(\nu_L)^C + \overline{(\nu_L)^C}\nu_L\right) - \frac{m_M^R}{2}\left(\overline{\nu_R}\,(\nu_R)^C + \overline{(\nu_R)^C}\nu_R\right), \tag{2.5}$$

where ν^C denotes a charge conjugated neutrino field. To ensure gauge invariance when we add this term and Eq. (2.3) to the Standard Model Lagrangian, $m_M^L = 0$ is required. Now, we can write $\mathcal{L}_{\text{mass}} = \mathcal{L}_{\text{mass,D}} + \mathcal{L}_{\text{mass,M}}$ as [14]

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\begin{pmatrix}\overline{\nu_L} & \overline{(\nu_R)^C}\end{pmatrix} \mathbf{M} \begin{pmatrix} (\nu_L)^C \\ \nu_R \end{pmatrix} + \text{h.c.} \tag{2.6}$$

with the neutrino mass matrix

$$\mathbf{M} = \begin{pmatrix} 0 & m_D \\ m_D & m_M^R \end{pmatrix} . \tag{2.7}$$

This matrix is diagonalizable: substract from the first line m_D/m_M^R times the second line (I), then add the new first line multiplied by m_M^R/m_D to the second line (II),

$$\begin{pmatrix} 0 & m_D \\ m_D & m_M^R \end{pmatrix} \overset{\text{(I)}}{\longrightarrow} \begin{pmatrix} -m_D^2/m_M^R & 0 \\ m_D & m_M^R \end{pmatrix} \overset{\text{(II)}}{\longrightarrow} \begin{pmatrix} -m_D^2/m_M^R & 0 \\ 0 & m_M^R \end{pmatrix} . \tag{2.8}$$

Finally, multiply the first line by -1. We obtain the eigenvalues

$$M = m_M^R \tag{2.9}$$

$$m = \frac{m_D^2}{m_M^R} = \frac{h^2 v^2}{M} . \tag{2.10}$$

The respective eigenfields are [14]

$$N \approx \left(\nu_R + (\nu_R)^C\right) + \frac{m_D}{M}\left(\nu_L + (\nu_L)^C\right) \tag{2.11}$$

$$\nu \approx \left(\nu_L - (\nu_L)^C\right) + \frac{m_D}{M}\left(\nu_R - (\nu_R)^C\right) . \tag{2.12}$$

For $M \gg m_D$, the field N contains almost only right-handed fields, while ν contains almost only left-handed fields [1]. If now the right-handed neutrino is very heavy, $M \gtrsim 10^8$ GeV, while at the same time the left-handed neutrino is very light as observed, $m \lesssim 2$ eV, the Yukawa coupling h could be of the same order as the Yukawa couplings of other Standard Model particles. In general, Eq. (2.10) states: the heavier the right-handed neutrino is, the lighter the left-handed one becomes. This is why it is called the *seesaw formula*.

In the case of three families, where m, M and h are matrices, Eq. (2.10) is generalized to [15]

$$m = v^2 h^T M^{-1} h, \tag{2.13}$$

and the effective light neutrino mass is defined as [16]

$$\tilde{m}_i \equiv \frac{\left(hh^\dagger\right)_{ii} v^2}{M_i} . \tag{2.14}$$

[1] Note, that $N = N^C$ and $\nu = \nu^C$.

To show that $\tilde{m}_1$ has an upper as well as a lower bound, we define a complex matrix R by [15, 17]

$$h = \frac{1}{v} D_{\sqrt{M}} R D_{\sqrt{m}} V^\dagger, \tag{2.15}$$

where $D_X = \mathrm{diag}\,(X_i)$ and V is the unitary PMNS-matrix that diagonalizes m in the sense that $D_m = V^T m V$ [2]. Then, working in a basis where $M = D_M$ is diagonal, which is always possible [15], we have

$$\begin{aligned} D_m &= V^T m_\nu V \\ &= v^2\, V^T h^T D_M^{-1} h\, V \\ &= v^2 \frac{1}{v^2} V^T V^* D_{\sqrt{m}} R^T D_{\sqrt{M}} D_{M^{-1}} D_{\sqrt{M}} R D_{\sqrt{m}} V^\dagger V \\ &= D_{\sqrt{m}} R^T R D_{\sqrt{m}}. \end{aligned} \tag{2.16}$$

Thus, R has to be orthogonal. It can be parametrized by three complex mixing angles φ_{12}, φ_{13}, φ_{23} and be written as a product of the rotations $R^{(ij)}(\varphi_{ij})$ in the three planes (ij) [17],

$$\begin{aligned} R &= D_{\pm 1}\, R^{(23)}(\varphi_{23})\, R^{(13)}(\varphi_{13})\, R^{(12)}(\varphi_{12}) \\ &= \begin{pmatrix} \pm c_{12}c_{13} & \mp c_{13}s_{12} & \pm s_{13} \\ \pm c_{23}s_{12} \pm c_{12}s_{13}s_{23} & \pm c_{12}c_{23} \mp s_{12}s_{13}s_{23} & \mp c_{13}s_{23} \\ \mp c_{12}c_{23}s_{13} \pm s_{12}s_{23} & \pm c_{23}s_{12}s_{13} \pm c_{12}s_{23} & \pm c_{13}c_{23} \end{pmatrix}. \end{aligned} \tag{2.17}$$

Here, c_{ij} and s_{ij} denote $\cos\varphi_{ij}$ and $\sin\varphi_{ij}$, respectively.
Using the matrix R, the effective light neutrino mass can be written as

$$\begin{aligned} \tilde{m}_1 &= \frac{v^2}{M_1} \cdot \frac{1}{v^2} \left(D_{\sqrt{M}} R D_{\sqrt{m}} V^\dagger V D_{\sqrt{m}} R^\dagger D_{\sqrt{M}} \right)_{11} \\ &= \sum_{i,j} \frac{1}{M_1} \left(D_{\sqrt{M}} R \right)_{1i} (D_m)_{ij} \left(R^\dagger D_{\sqrt{M}} \right)_{j1} \\ &= \sum_j \frac{1}{M_1} \left(D_{\sqrt{M}} R \right)_{1j} m_j \left(R^\dagger D_{\sqrt{M}} \right)_{j1} \\ &= \sum_j \frac{1}{M_1} M_1 m_j R_{1j} R^\dagger_{j1} \end{aligned}$$

[2] To see that such a matrix exists, consider the hermitian matrix $m^\dagger m$ with the real positive eigenvalues m_i^2. According to the spectral theorem, there is a unitary matrix V such that

$$D_{m^2} = V^\dagger m^\dagger m V = V^\dagger m^\dagger\, V^* V^T\, m V = \left(V^T m V \right)^\dagger \left(V^T m V \right).$$

Hence, $D_m = V^T m V$.

$$
\begin{aligned}
&= \sum_j m_j R_{1j} R_{1j}^* \\
&= \sum_j m_j |R_{1j}|^2. \qquad (2.18)
\end{aligned}
$$

Inserting the parametrization (2.17) gives,

$$\tilde{m}_1 = m_1 \ |\cos\varphi_{13}|^2 \, |\cos\varphi_{12}|^2 + m_2 \ |\cos\varphi_{13}|^2 \, |\sin\varphi_{12}|^2 + m_3 \ |\sin\varphi_{13}|^2 \, . \qquad (2.19)$$

Therefore, $\tilde{m}_1$ is independent of φ_{23}. Since $\Delta m^2_{\rm sol} \ll \Delta m^2_{\rm atm}$, we can assume $m_1 \simeq m_2 \ll m_3$ [3]. Then, defining $m_1' = m_1 \cosh\left(2\,{\rm Im}\varphi_{12}\right)$ and $\varphi_{13} = x + {\rm i}\, y$, and using that for complex arguments $z = x + {\rm i}\, y$,

$$
\begin{aligned}
|\cos z|^2 + |\sin z|^2 &= |\cos x \, \cosh y - {\rm i} \sin x \, \sinh y|^2 + |\sin x \, \cosh y + {\rm i} \cos x \, \sinh y|^2 \\
&= \cos^2 x \, \cosh^2 y + \sin^2 x \, \sinh^2 y + \sin^2 x \, \cosh^2 y + \cos^2 x \, \sinh^2 y \\
&= \cosh^2 y + \sinh^2 y \\
&= \cosh(2y), \qquad (2.20)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\tilde{m}_1 &\simeq m_1 \left(|\cos\varphi_{12}|^2 + |\sin\varphi_{12}|^2\right) |\cos\varphi_{13}|^2 + m_3 \ |\sin\varphi_{13}|^2 \\
&= m_1' \ |\cos x \cosh y - {\rm i} \sin x \sinh y|^2 + m_3 \ |\sin x \cosh y + {\rm i} \cos x \sinh y|^2 \\
&= m_1' \left(\cos^2 x \, \cosh^2 y + \sin^2 x \, \sinh^2 y\right) + m_3 \left(\sin^2 x \, \cosh^2 y + \cos^2 x \, \sinh^2 y\right) \\
&= \frac{m_1'}{2} \left(\cos(2x) + \cosh(2y)\right) - \frac{m_3}{2} \left(\cos(2x) - \cosh(2y)\right). \qquad (2.21)
\end{aligned}
$$

This yields

$$\cosh(2y) = \frac{2\tilde{m}_1 - (m_1' - m_3)\cos(2x)}{m_1' + m_3}. \qquad (2.22)$$

Using relations (2.20) and (2.22), we are able to derive an upper bound on $\tilde{m}_1$,

$$
\begin{aligned}
\tilde{m}_1 &< m_3 \left(|\cos\varphi_{13}|^2 + |\sin\varphi_{13}|^2\right) \\
&= m_3 \ \cosh(2y) \\
&= m_3 \ \left(\frac{2\tilde{m}_1}{m_1' + m_3} - \frac{m_1' - m_3}{m_1' + m_3} \ \cos(2x)\right) \\
&= m_3 \ \frac{m_1' - m_3}{m_1' + m_3} \ \cos(2x) \ \frac{1}{\frac{2m_3}{m_1'+m_3} - 1}
\end{aligned}
$$

[3] Recall, that we assume normal hierarchy.

$$
\begin{aligned}
&= m_3 \cos(2x + \pi) \\
&\leq m_3.
\end{aligned} \tag{2.23}
$$

On the other hand, it is [18]

$$
\tilde{m}_1 > m_1 \sum_i |R_{1i}|^2 \geq m_1 \left| \sum_i R_{1i}^2 \right| = m_1, \tag{2.24}
$$

where in the last step we used the orthogonality of R. Thus, $m_1 < \tilde{m}_1 < m_3$.

3 Baryon and lepton number violation in the Standard Model

Baryon and lepton number are violated in the Standard Model due to the triangle anomaly [19], which reads in the case of three generations [3]

$$\partial^\mu J^B_\mu = \partial^\mu J^L_\mu = \frac{3}{32\pi^2}\left(-g^2 W^I_{\mu\nu}\tilde{W}^{I\mu\nu} + g'^2 B_{\mu\nu}\tilde{B}^{\mu\nu}\right), \tag{3.1}$$

with the respective field strength tensors $W^I_{\mu\nu}$ and $B_{\mu\nu}$ of the $SU(2)$ and $U(1)$ gauge fields W^I_μ and B_μ, the $SU(2)$ and $U(1)$ gauge couplings g and g', and the B and L currents

$$J^B_\mu = \frac{1}{3}\sum_{\text{generations}}\left(\overline{q_L}\gamma_\mu q_L + \overline{u_R}\gamma_\mu u_R + \overline{d_R}\gamma_\mu d_R\right) \tag{3.2}$$

$$J^L_\mu = \sum_{\text{generations}}\left(\overline{\ell_L}\gamma_\mu \ell_L + \overline{e_R}\gamma_\mu e_R\right). \tag{3.3}$$

Because the electroweak theory is a nonabelian gauge theory, there are "infinitly many degenerate ground states" [3]. Imagine a pendulum. Classically, after a full 360° turn it is back in the same state as in the beginning. Here, however, the gauge configuration depends on the number of turns, the Chern-Simons number [3]

$$N_{CS} = \frac{g^3}{96\pi^2}\int d^3x \epsilon_{ijk}\epsilon^{IJK} W^{Ii} W^{Jj} W^{Kk}. \tag{3.4}$$

If one plots the potential against the Chern-Simons number, one sees a periodic structure with minima at integer N_{CS} (see Fig. 1). A jump from one minimum to another changes the Chern-Simons number by $\Delta N_{CS} = 1$ and, since [3]

$$\Delta B = \int_{t_i}^{t_f} dt \int d^3x\, \partial^\mu J^B_\mu = 3\,\Delta N_{CS}, \tag{3.5}$$

it changes the baryon number by $\Delta B = 3$. Because of (3.1) $\Delta L = 3$, too, and therefore, $B - L$ is conserved.

There are two possibilities to get from one groundstate to another, namely tunneling through the potential barrier or "jumping" over it. The tunneling, or instanton process, is highly suppressed, its rate is known to be of $\mathcal{O}(e^{-165})$ [3]. The "jump" over the barrier is called (electroweak) sphaleron process. Today, it is also suppressed, which is why in our observation of nature B and L are conserved, but in the early universe, when the temperature was larger than the sphaleron energy E_{Sp}, such processes occurred at a notable rate.

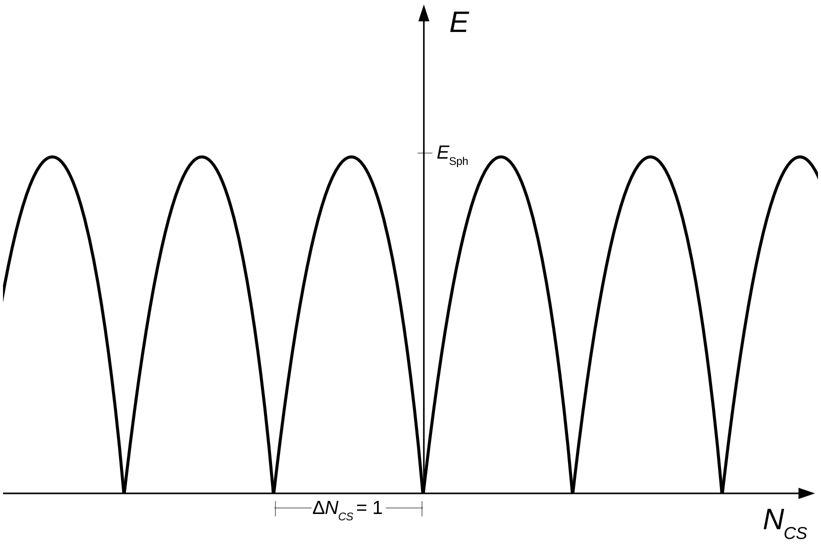

Figure 1: Sketch of the electroweak potential as a function of the Chern-Simons number for illustration. It has minima at integer N_{CS}. There are two ways to get from one minimum to another, tunneling through the potential barrier or "jumping" over it.

If there is a lepton asymmetry present in the universe, the sphalerons will contribute to the washout. Because they conserve $B - L$, the reduction of the L asymmetry will inevitably lead to a B asymmetry. In other words, the sphalerons provide a relation between the number densities of the baryons and $B - L$,

$$n_B = c_B n_{B-L} \tag{3.6}$$

with a coefficient c_B which we will now determine. Following Refs. [3] and [20], we assign chemical potentials μ to all particles, i.e. the left-handed quark doublets q_i, the right-handed quark singlets u_i and d_i, the left-handed lepton doublets ℓ_i, the right-handed lepton singlets e_i and the higgs φ. The electroweak sphalerons determine the relation

$$\sum_i 3\mu_{q_i} + \mu_{\ell_i} = 0, \tag{3.7}$$

and the requirement of a vanishing total isospin demands

$$\sum_i \mu_{q_i} + 2\mu_{u_i} - \mu_{d_i} - \mu_{\ell_i} - \mu_{e_i} + \frac{2}{3}\mu_\varphi = 0. \tag{3.8}$$

Furthermore, if all Yukawa interactions are in equilibrium, which is true for temperatures $T \ll 10^8$ GeV [4], we have the relations

$$\mu_{q_i} - \mu_\varphi - \mu_{d_i} = 0 \tag{3.9}$$
$$\mu_{q_i} + \mu_\varphi - \mu_{u_i} = 0 \tag{3.10}$$
$$\mu_{\ell_i} - \mu_\varphi - \mu_{e_i} = 0. \tag{3.11}$$

Now we have a set of eleven linearly independent [5] relations between the chemical potentials. Since we have sixteen chemical potentials entering these eleven equations, five of them will remain undetermined. We choose to express all chemical potentials in terms of μ_{ℓ_1}, μ_{ℓ_2}, μ_{ℓ_3}, μ_{q_1} and μ_{q_2} and obtain

$$\mu_{q_3} = -\frac{1}{3}\left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right) - \mu_{q_1} - \mu_{q_2} \tag{3.12}$$
$$\mu_{d_1} = -\frac{4}{21}\left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right) + \mu_{q_1} \tag{3.13}$$
$$\mu_{d_2} = -\frac{4}{21}\left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right) + \mu_{q_2} \tag{3.14}$$
$$\mu_{d_3} = -\frac{11}{21}\left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right) - \mu_{q_1} - \mu_{q_2} \tag{3.15}$$
$$\mu_{u_1} = \frac{4}{21}\left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right) + \mu_{q_1} \tag{3.16}$$
$$\mu_{u_2} = \frac{4}{21}\left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right) + \mu_{q_2} \tag{3.17}$$
$$\mu_{u_3} = -\frac{1}{7}\left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right) - \mu_{q_1} - \mu_{q_2} \tag{3.18}$$
$$\mu_{e_1} = \frac{1}{21}\left(17\mu_{\ell_1} - 4\mu_{\ell_2} - 4\mu_{\ell_3}\right) \tag{3.19}$$
$$\mu_{e_2} = \frac{1}{21}\left(-4\mu_{\ell_1} + 17\mu_{\ell_2} - 4\mu_{\ell_3}\right) \tag{3.20}$$
$$\mu_{e_3} = \frac{1}{21}\left(-4\mu_{\ell_1} - 4\mu_{\ell_2} + 17\mu_{\ell_3}\right) \tag{3.21}$$
$$\mu_\varphi = \frac{4}{21}\left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right). \tag{3.22}$$

The baryon- and lepton-number densities are

$$n_B = \frac{gT^2}{6} B \tag{3.23}$$

[4]For higher temperatures, $10^8\text{GeV} \lesssim T \lesssim 10^{12}\text{GeV}$, where the electroweak sphalerons are still in equilibrium, some Yukawa interactions are slow compared to the Hubble rate. Strictly speaking, in that regime this calculation does not hold. We ignore this complication nevertheless, since it is assumed to have only little effect [3].

[5]There is a twelfth relation from the QCD-sphalerons, $\sum_i 2\mu_{q_i} - \mu_{u_i} - \mu_{d_i} = 0$, but it is equal to the sum of Eqs. (3.9) and (3.10).

$$n_{L_i} = \frac{gT^2}{6} L_i, \tag{3.24}$$

and on the other hand

$$n_X - n_{\overline{X}} = \frac{gT^3}{6} \begin{cases} \frac{\mu_X}{T} + \mathcal{O}\left((\frac{\mu_X}{T})^3\right) & \text{(fermions)} \\ 2\frac{\mu_X}{T} + \mathcal{O}\left((\frac{\mu_X}{T})^3\right) & \text{(bosons).} \end{cases} \tag{3.25}$$

With the chemical potentials that we determined previously we now find

$$B = \sum_i (2\mu_{q_i} + \mu_{u_i} + \mu_{d_i}) = -\frac{4}{3} \left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right) \tag{3.26}$$

$$L = \sum_i (2\mu_{\ell_i} + \mu_{e_i}) = \frac{17}{7} \left(\mu_{\ell_1} + \mu_{\ell_2} + \mu_{\ell_3}\right), \tag{3.27}$$

and therefore,

$$c_B = \frac{n_B}{n_{B-L}} = \frac{B}{B-L} = \frac{28}{79}. \tag{3.28}$$

Note that the ratio n_B/n_γ is constant only after photon decoupling. To relate its value N_B at the time where the asymmetry was produced to its value η_B today, one has to take into account the dilution factor $n_\gamma(T_{\text{Leptogenesis}})/n_\gamma(T_{\text{Recombination}})$ due to "photon production from the onset of Leptogenesis till recombination" [21] [6]. One can estimate this factor as follows: Assume constant entropy density $s \propto g_{*S}(T)\, T^3 = \text{const.}$ [2, §3.4.]. Then

$$\frac{n_\gamma(T_{\text{Leptogenesis}})}{n_\gamma(T_{\text{Recombination}})} = \frac{g_{*S}(T_{\text{Recombination}})}{g_{*S}(T_{\text{Leptogenesis}})}. \tag{3.29}$$

Here, g_{*S} is the effective number of relativistic degrees of freedom, [2, §3.4.]

$$g_{*S}(T) = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T_\gamma}\right)^3, \tag{3.30}$$

where g_i is the number of degrees of freedom of the particle species i and T_i is its temperature, while T_γ denotes the temperature of the photon bath. At the onset of Leptogenesis all Standard Model particles and the lightest Majorana neutrino were relativistic and in thermal equilibrium with the photons, which leads to $g_{*S}(T_{\text{Leptogenesis}}) = 434/4$ [22]. After photon decoupling, only the photons and

[6] Actually, recombination does not play a role here. We could choose any point between photon decoupling and today instead, but the following considerations are easier at a time where all light neutrinos were relativistic.

the three light neutrinos were relativistic, and they had different temperatures. To calculate their temperature relation, we have to look at what happened between neutrino decoupling and photon decoupling [23, §2.4.4]. Neutrino decoupling occurred when the typical neutrino interaction rate became smaller than the Hubble rate. This happened at a temperature of about 1 MeV. At that time, the three neutrinos, electrons, positrons and photons contributed to the total entropy density [7], which was therefore $s_1 = 43\pi^2 T_1^3/90$. When the universe cooled down further, electrons and positrons annihilated via $e^+ + e^- \to \gamma + \gamma$ and transfered their energy to the photon bath (and not to the neutrinos), which is why afterwards $T_\gamma > T_\nu$. Now that electrons and positrons no longer contribute to the entropy density, it changes to $s_2 = 2\pi^2/45 \left(2T_\gamma^3 + 7/8 \sum_i 6T_{\nu_i}^3\right)$. The entropy density scales as a^{-3}, where a is the scaling factor of the universe. Thus, $s_1 a_1 = s_2 a_2$. Since the neutrinos are decoupled at both times, we have $T_1 a_1 = T_\nu a_2$. This leads to $T_\nu / T_\gamma = (4/11)^{1/3}$ [23, §2.4.4]. Therefore, $g_{*S}(T_{\text{Recombination}}) = 43/11$, and all in all [22]

$$\eta_B = \frac{n_\gamma(T_{\text{Leptogenesis}})}{n_\gamma(T_{\text{Recombination}})} = \frac{86}{2387} N_B. \tag{3.31}$$

[7]The contributions to the total entropy density are $\begin{cases} \frac{2\pi T^3}{45} & \text{for massless bosons} \\ \frac{7}{8}\frac{2\pi T^3}{45} & \text{for massless fermions,} \end{cases}$ while the contributions of massive particles are negligible [23, §2.4.4].

4 Leptogenesis in the nonrelativistic limit [8]

Let us extend the Standard Model by three heavy right-handed Majorana neutrinos with hierarchical masses, $M_1 \ll M_{2,3}$. Their only coupling to Standard Model particles may be via a Yukawa interaction

$$\mathcal{L}_{\text{int}} = \bar{N}_i h_{ij} \tilde{\varphi}^\dagger \ell_j + \text{h.c.}, \tag{4.1}$$

where N is the Majorana neutrino field, ℓ is a left-handed lepton doublet, $\tilde{\varphi} = i\sigma^2\varphi^*$ is the conjugate Higgs doublet and h is the Yukawa coupling matrix. We will assume that the Yukawa interaction of the heavy neutrinos is the only interaction that is of the same order as the expansion rate of the Universe. The Standard Model interactions are either much slower and we can neglect them, or they are much faster. Then, we call them *spectator processes* and we will revisit them later. This separation of time scales enables us to write the Hamiltonian as

$$H = H_0 + H_{\text{int}}, \tag{4.2}$$

where H_0 contains all Standard Model particles and their interactions as well as free heavy neutrinos and H_{int} is the corresponding Hamiltonian to $\mathcal{L}_{\text{int}}$ and contains only the interaction of the heavy neutrinos.

The heavy neutrinos can decay into a lepton and a Higgs boson. This decay violates lepton number [9]. If the couplings h carry CP-violating phases, the decay channel $N \leftrightarrow \ell\varphi$ can occur at a different rate than the channel $N \leftrightarrow \bar{\ell}\bar{\varphi}$. Finally, if the neutrinos are out of thermal equilibrium, decays and inverse decays can occur at different rates, so that all three Sakharov conditions are fulfilled and a lepton asymmetry can be produced.

We assume that the inverse decays of the lightest neutrino N_1 are effective enough so that any asymmetry that is produced by the heavier neutrinos $N_{2,3}$ is completely washed out. Then the final asymmetry is only produced by the N_1. Henceforth, we will call the lightest neutrino N and its mass $M_1 = M_N$.

If the temperature of the universe is so large that all lepton Yukawa interactions are slow compared to Hubble expansion, one cannot distinguish between e-, μ-, and τ-flavor. Therefore, one can introduce linear combinations ℓ_1, ℓ_2 and ℓ_3 of the three flavors in such a way, that N couples, for example, only to l_1 [12]. The result then is the same as if N would couple only to one specific flavor, like τ. For lower temperatures, where one or more lepton Yukawa couplings are fast, the asymmetry in the left-handed doublets is partially converted into an asymmetry in the right-handed singlets. Thus, the flavor whose Yukawa interaction is in equilibrium

[8] This section is mainly based on Ref. [24].

[9] Recall that the N's are Majorana particles and therefore do not carry lepton number.

can be distinguished from those whose is not, and one has to deal with different asymmetries for each flavor. Nevertheless, we will always assume that N couples to only one flavor. In this “single-flavor approximation“, the interaction of N is

$$\mathcal{L}_{\text{int}} = \overline{N} h \tilde{\varphi}^\dagger \ell + \text{h.c.} \tag{4.3}$$

In the nonrelativistic limit, we neglect the motion of the heavy right-handed neutrinos. Then the out-of-equilibrium state is fully specified by the number densities n_N and n_{B-L}. Their time evolution is a function of their respective deviation from thermal equilibrium and the temperature. Thus, we can write [10]

$$\left(\frac{d}{dt} + 3H\right) n_N = F_N\big((n_N - n_N^{\text{eq}}), n_{B-L}, T\big) \tag{4.4}$$

$$\left(\frac{d}{dt} + 3H\right) n_{B-L} = F_{B-L}\big((n_N - n_N^{\text{eq}}), n_{B-L}, T\big), \tag{4.5}$$

where the term proportional to the Hubble rate H accounts for the fact that the density decreases due to the expansion of the universe. We assume both $n_N - n_N^{\text{eq}}$ and n_{B-L} to be small, which is true in the so-called strong washout regime [20], where the tree-level neutrino decay rate is much larger than the Hubble rate at $T = M_N$ [11]. Then we can linearize the functions F so that

$$\left(\frac{d}{dt} + 3H\right) n_N = -\Gamma_N \left(n_N - n_N^{\text{eq}}\right) + \Gamma_{N,B-L}\, n_{B-L} \tag{4.6}$$

$$\left(\frac{d}{dt} + 3H\right) n_{B-L} = \Gamma_{B-L,N} \left(n_N - n_N^{\text{eq}}\right) - \Gamma_{B-L}\, n_{B-L}, \tag{4.7}$$

where the coefficients Γ depend only on the temperature. As we will see, the coefficient $\Gamma_{B-L,N}$ is small because it is suppressed by the CP-asymmetry, which is small itself. We expect that for $\Gamma_{N,B-L}$ as well and, since $n_{B-L} \ll n_N - n_N^{\text{eq}}$, we will neglect the term $\Gamma_{N,B-L}\, n_{B-L}$ in the following, so that Eq. (4.6) simplifies to

$$\left(\frac{d}{dt} + 3H\right) n_N = -\Gamma_N \left(n_N - n_N^{\text{eq}}\right). \tag{4.8}$$

These equations are valid at leading order in the Yukawa coupling of N and to all orders in the Standard Model couplings. The next step is now to calculate the coefficients Γ_N, $\Gamma_{B-L,N}$ and Γ_{B-L}.

[10] Note that we assume the equilibrium value of the asymmetry to be $n_{B-L}^{\text{eq}} = 0$.

[11] We will see later that this regime is favored by our current knowledge of neutrino masses.

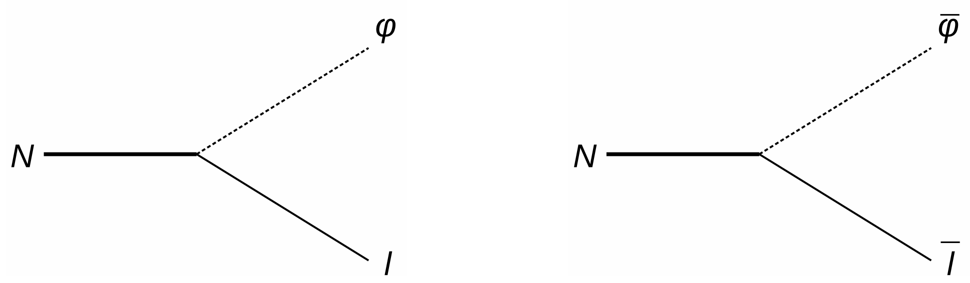

Figure 2: Tree-level Feynman graphs of the heavy neutrino decay.

4.1 Neutrino equilibration

From Eq. (4.8), the coefficient Γ_N can be interpreted as the equilibration rate of the heavy neutrinos. At leading order, it can be calculated from a Boltzmann equation, [2, §5.1]

$$\frac{df_N}{dt} - H|\mathbf{p}_N|^2 \frac{df_N}{E_N\, dE_N} = \frac{C[f_N]}{E_N}, \tag{4.9}$$

with the Hubble rate H and the collision term [12]

$$C[f_N] = -\int d\Pi_\ell d\Pi_\varphi (2\pi)^4 \delta^{(4)}(p_N - p_\ell - p_\varphi) \sum |\mathcal{M}|^2 \left(f_N - f_\ell^{\rm eq} f_\varphi^{\rm eq}\right), \tag{4.10}$$

where $\sum |\mathcal{M}|^2$ is the matrix element averaged over the spins of the *in*-states and summed over the spins of the *out*-states. It can be obtained from the leading order Feynman diagram of the neutrino decay, see Fig. 2. Using Feynman rules the amplitude is

$$\mathrm{i}\mathcal{M} = \mathrm{i}h_{11}\, u_s^{(\ell)}(p_\ell)\, u_r^{(N)}(p_N) \tag{4.11}$$

and therefore

$$\begin{aligned}
\sum |\mathcal{M}|^2 &= \frac{|h_{11}|^2}{2} \sum_{r,s} \bar{u}_r^{(N)} u_s^{(\ell)} \bar{u}_s^{(\ell)} u_r^{(N)} \\
&= |h_{11}|^2 \ \mathrm{Tr}\left((\not{p}_N + M_N)\, \not{p}_\ell \right) \\
&= |h_{11}|^2 \ (p_N^\mu p_\ell^\nu \, \mathrm{Tr}(\gamma_\mu \gamma_\nu) + M_N p_\ell^\mu \mathrm{Tr}(\gamma_\mu)) \\
&= |h_{11}|^2 \ (p_N^\mu p_\ell^\nu \, 2\eta_{\mu\nu} + M_N p_\ell^\mu \cdot 0) \\
&= 2\, |h_{11}|^2 \, p_N p_\ell.
\end{aligned} \tag{4.12}$$

[12] We neglect Bose enhancement and Pauli blocking terms.

The momenta of the leptons and Higgs bosons are saturated at $M_N/2$, so that we can treat them as ultrarelativistic and neglect their masses and also apply Boltzmann statistics for them. Then,

$$f_\ell^{\rm eq} f_\varphi^{\rm eq} = e^{E_\ell/T} e^{E_\varphi/T} = e^{E_N/T} = f_N^{\rm eq} \tag{4.13}$$

because of momentum conservation. Since the integration measure

$$d\Pi_i = \frac{d^3 p_i}{(2\pi)^3\, 2E_i} = \frac{1}{(2\pi)^3} d^4 p_i\, \delta_+(p_i^2 - m_i^2) \tag{4.14}$$

is Lorentz invariant, we can boost into the rest frame of the neutrino where $\sum |\mathcal{M}|^2 = 2\,|h_{11}|^2\, M_N\, |\mathbf{p}_\ell|$. Then the collision term becomes

$$\begin{aligned} C[f_N] &= -\frac{2(2\pi)^4\,|h_{11}|^2\, M_N}{(2\pi)^6} \int d^4 p_\ell\, d^4 p_\varphi \delta_+^{(4)}((p_\ell^0)^2 - |\mathbf{p}_\ell|^2) \delta_+^{(4)}(p_\varphi^2) \\ &\qquad \times \delta^{(4)}(p_N - p_\ell - p_\varphi)\, |\mathbf{p}_\ell|\, (f_N - f_N^{\rm eq}) \\ &= -\frac{|h_{11}|^2\, M_N}{2\pi^2} \int d^4 p_\ell\, \delta_+^{(4)}((p_\ell^0)^2 - |\mathbf{p}_\ell|^2) \delta^{(4)}(M_N^2 - 2M_N\,|p_\ell|)\, |\mathbf{p}_\ell|\, (f_N - f_N^{\rm eq}) \\ &= -\frac{|h_{11}|^2\, M_N}{2\pi^2}\, 4\pi \int dp_\ell^0\, d|\mathbf{p}_\ell|\, \delta_+^{(4)}((p_\ell^0)^2 - |\mathbf{p}_\ell|^2) \\ &\qquad \times \frac{1}{2M_N} \delta^{(4)}\left(\frac{M_N}{2} - |p_\ell|\right) |\mathbf{p}_\ell|^3\, (f_N - f_N^{\rm eq}) \\ &= -\frac{|h_{11}|^2\, M_N}{\pi} \int dp_\ell^0\, \frac{M_N^2}{8} \delta_+^{(4)}\left((p_\ell^0)^2 - \frac{M_N^2}{4}\right) (f_N - f_N^{\rm eq}) \\ &= -\frac{|h_{11}|^2\, M_N^2}{8\pi}\, (f_N - f_N^{\rm eq})\,. \end{aligned} \tag{4.15}$$

On the left-hand side of Eq. (4.9) we use that

$$\frac{dE}{d|\mathbf{p}|} = \frac{d}{d|\mathbf{p}|}\sqrt{|\mathbf{p}|^2 + m^2} = \frac{2|\mathbf{p}|}{2\sqrt{|\mathbf{p}|^2 + m^2}} \tag{4.16}$$

$$\Leftrightarrow E\, dE = |\mathbf{p}|\, d|\mathbf{p}|, \tag{4.17}$$

and the Boltzmann equation becomes

$$\frac{df_N}{dt} - H|\mathbf{p}_N| \frac{df_N}{d|\mathbf{p}_N|} = -\frac{M_N\, \Gamma_0}{E_N}\, (f_N - f_N^{\rm eq})\,, \tag{4.18}$$

where $\Gamma_0 = |h_{11}|^2\, M_N/8\pi$ is the tree level decay rate of the neutrinos. To get to an equation for the number density

$$n_N = (2s_N + 1) \int \frac{d^3\mathbf{p}}{(2\pi)^3} f_N, \tag{4.19}$$

where $s_N = 1/2$ is the spin of the neutrinos, we have to integrate Eq. (4.18) over the momentum. Using the fact that the neutrinos are nonrelativistic and approximating $1/E_N \approx 1/M_N$, we get after integrating

$$\left(\frac{d}{dt} + 3H\right) n_N = -\Gamma_0 \left(n_N - n_N^{\mathrm{eq}}\right). \tag{4.20}$$

Comparing this result to Eq. (4.8), we find $\Gamma_N = \Gamma_0$.

4.1.1 Radiative corrections

The production rate of the neutrinos has been calculated at $\mathcal{O}(g^2)$ in the nonrelativistic limit, [25, 26]

$$\left.\frac{df_N}{dt}\right|_{f_N=0} \equiv \Gamma^{\mathrm{pro}} = f_N^{\mathrm{eq}}\Gamma_0 \left\{1 - \frac{21}{2(4\pi)^2}\left|h_t\right|^2 + \frac{29}{8(4\pi)^2}\left(g_1^2 + 3g_2^2\right)\right\}. \tag{4.21}$$

Here, g_1 and g_2 are the $U(1)$ and $SU(2)$ gauge couplings, respectively, and h_t is the top-Yukawa coupling. We assume that we can use the radiative corrections of this rate for the equilibration rate so that

$$\Gamma_N = \Gamma_0 \left\{1 - \frac{21}{2(4\pi)^2}\left|h_t\right|^2 + \frac{29}{8(4\pi)^2}\left(g_1^2 + 3g_2^2\right)\right\}. \tag{4.22}$$

This implies that there is a simple relation between production and equilibration rates, $\Gamma^{\mathrm{pro}} = f^{\mathrm{eq}}\Gamma^{\mathrm{eq}}$. We will show in Sec. 6 that such a relation exists and the assumption is justified.

4.2 $B - L$ production

The production of the $B - L$ asymmetry is, at leading order, due to the decays of the heavy neutrinos N. Thus, it will be proportional to the decay rate Γ_0. It also has to be proportional to the size of the CP-violation in these decays to ensure the implementation of the second Sakharov condition. We respect that by multiplying a factor ϵ_1, so that the $B - L$ production rate reads

$$\Gamma_{B-L,N} = \epsilon_1 \Gamma_0, \tag{4.23}$$

where ϵ_1 is defined as

$$\epsilon_1 \equiv \frac{\Gamma\left(N \to \varphi l\right) - \Gamma\left(N \to \bar{\varphi}\bar{l}\right)}{\Gamma\left(N \to \varphi l\right) + \Gamma\left(N \to \bar{\varphi}\bar{l}\right)}. \tag{4.24}$$

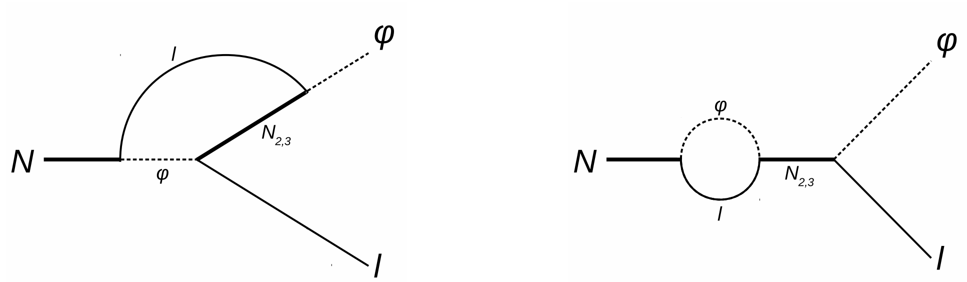

Figure 3: One-loop Feynman graphs of the heavy neutrino decay.

The tree-level decay rate Γ_0 is CP invariant, since it is proportional to $|h_{11}|$. Thus, one needs to take into account the interference term between the tree-level (cf. Fig. 2) and the one-loop diagrams (cf. Fig. 3) to find a nonvanishing ϵ_1. It has been calculated that [27]

$$\epsilon_1 = -\frac{1}{8\pi}\frac{1}{(hh^\dagger)_{11}}\sum_j \mathrm{Im}\left[\left(hh^\dagger\right)^2_{1j}\right]\left(\frac{2M_1M_j}{M_j^2-M_1^2}+\frac{M_j}{M_1}\ln\left[1+\frac{M_1^2}{M_j^2}\right]\right). \tag{4.25}$$

Using again the definition (2.15) and taking the limit $M_{2,3}/M_1 \to \infty$ leads to [27]

$$\epsilon_1 = -\frac{3}{16\pi}\frac{M_1}{v^2}\frac{\sum_i m_i^2\,\mathrm{Im}\left[R_{1i}^2\right]}{\sum_i m_i\left|R_{1i}\right|^2}. \tag{4.26}$$

Using the parametrization (2.17), the relation (2.22) and $M_1 = M_N$ one finds [27]

$$\begin{aligned}|\epsilon_1| &= \frac{3}{16\pi}\frac{M_1}{v^2}\frac{m_3^2-m_1^2}{\tilde{m}_1}\left|\mathrm{Im}\left[R_{11}^2\right]\right| \\ &\leq \frac{3}{16\pi}\frac{M_1(m_3-m_1)}{v^2}\begin{cases}1-\frac{m_1}{\tilde{m}_1} & m_1 \ll m_3 \\ \sqrt{1-\frac{m_1^2}{\tilde{m}_1^2}} & m_1 \simeq m_3.\end{cases}\end{aligned} \tag{4.27}$$

4.3 $B-L$ washout

The rate of the $B-L$-washout can be derived using the approach of Ref. [28]. In the nonrelativistic regime, at leading order in all couplings, it becomes

$$\begin{aligned}\Gamma_{B-L} &= \sum_i \frac{|h_{1i}|^2}{8\pi^3}M_N^3K_1\left(\frac{M_N}{T}\right)\Xi^{-1} \\ &= \frac{T^2}{\pi^2\Xi}\,\Gamma_0 z^2 K_1(z),\end{aligned} \tag{4.28}$$

where we introduced $z = M_N/T$ and $K_1(z)$ is a modified Bessel function of the second kind. The susceptibility Ξ can be calculated from the pressure P according to Ref. [28] via

$$\Xi_{ab} = \frac{\partial^2}{\partial\mu_a \partial\mu_b}\left(P(T,\mu)_{\big|\partial P/\partial\mu_{\bar{a}}=0}\right), \tag{4.29}$$

where $\bar{a}$ labels the charges that are conserved by both H_0 and $H_{\rm int}$, while a denotes the charges that are only conserved by H_0 but broken by $H_{\rm int}$. At leading order in the single-flavor case, this simplifies to [13]

$$\Xi = \frac{T^2}{12}\frac{\partial^2}{\partial\mu_a^2}\left(6\,\mathrm{Tr}(\mu_q^2) + 3\,\mathrm{Tr}(\mu_u^2) + 3\,\mathrm{Tr}(\mu_d^2) + 2\,\mathrm{Tr}(\mu_\ell^2) + \mathrm{Tr}(\mu_e^2) + 4\,\mathrm{Tr}(\mu_\varphi^2)\right)_{\big|\partial P/\partial\mu_{\bar{a}}=0}. \tag{4.30}$$

As one can see, T^2/Ξ is a dimensionless number.

Which charges are broken and which are conserved by $H_{\rm int}$ depends on the spectator processes that are present. In order to use Eq. (4.29) we have to reexpress the chemical potentials of the particle species in terms of those of the Q_a and $Q_{\bar{a}}$. If we assume that N only couples to ℓ_1, the only charge that is broken by $H_{\rm int}$ obviously is the lepton number carried by ℓ_1, $Q_a = \{L_{\ell_1}\}$. If we ignore spectator processes completely, no other charges are correlated with L_{ℓ_1} per definition, so that $Q_{\bar{a}} = \{\}$. Then the chemical potential of ℓ_1 is equal to that of Q_a, $\mu_{\ell_1} = \mu_a$, which we can plug into Eq. (4.29). This yields

$$\Xi = \frac{T^2}{12}\frac{\partial^2}{\partial\mu_a^2}\left(2\,\mu_a^2\right) = \frac{T^2}{3} \tag{4.31}$$

and therefore,

$$\Gamma_{B-L} = \frac{3}{\pi^2}\Gamma_0 z^2 K_1(z). \tag{4.32}$$

In the following, we will study the more realistic case including spectator processes and analyze in detail the different temperature regimes. We summerize the respective values for $T^2\Xi^{-1}$ in table 1.

$\mathbf{T \gg 10^{13}}$ GeV

In this regime, only the gauge couplings and the top Yukawa coupling h_t are much faster than the Yukawa coupling of N. Since we still cannot distinguish

[13] In general, Ξ is a matrix. Its dimension is equal to the number of charges that are broken by $H_{\rm int}$, which is, however, in the single-flavor case always one.

between e-, μ-, and τ-flavor, we can again choose $Q_a = \{L_{\ell_1}\}$. The conserved charges are the isospins and baryon numbers of the respective particles [28], $Q_{\bar{a}} = \{Y_{\ell_1,\varphi,q_3,u_3}, B_{q_3,u_3}\}$ [14]. The nonvanishing chemical potentials then are

$$\mu_{q_3} = \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{3} \tag{4.33}$$

$$\mu_{u_3} = \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{3} \tag{4.34}$$

$$\mu_{\ell_1} = \mu_a - \frac{\mu_{\bar{a}_1}}{2} \tag{4.35}$$

$$\mu_{\varphi} = \frac{\mu_{\bar{a}_1}}{2}. \tag{4.36}$$

The two conditions $\partial P/\partial\mu_{\bar{a}} = 0$ demand $\mu_{\bar{a}_1} = \mu_a/2$ and $\mu_{\bar{a}_1} = -\mu_a/2$. Now, we can easily obtain Ξ from Eq. (4.29),

$$\Xi = \frac{T^2}{12}\frac{\partial^2}{\partial\mu_a^2}\left\{\left[6\left(\frac{1}{6}+\frac{1}{3}\right)^2 + 3\left(\frac{2}{3}+\frac{1}{3}\right)^2 + 2\left(1-\frac{1}{2}\right)^2 + 1\right]\frac{\mu_a^2}{4}\right\} = \frac{T^2}{4}. \tag{4.37}$$

This agrees with the example presented in Ref. [28].

$\mathbf{T \gtrsim 10^{13}}$ GeV

If we switch on QCD-sphalerons in addition to h_t and the gauge couplings, $Y_{\ell_1,\varphi,q_3,u_3}$ and B_{q_3,u_3} are not conserved by H_0 any more. The QCD-sphalerons only conserve charges that contain an equal amount of left-handed and right-handed quarks. Therefore, they break the charges B_{q_1,q_2}, B_{u_1,u_2} and B_{d_1,d_2,d_3}, too. In the previous temperature regime, these charges were conserved, but we did not have to consider them there because they were not correlated with L_ℓ. Now, we can use them to build linear combinations that are conserved. One possible choice is

$$Q_{\bar{a}_1} = Y_{\ell_1,\varphi,q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3} \tag{4.38}$$

$$Q_{\bar{a}_2} = B_{q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3} \tag{4.39}$$

$$Q_{\bar{a}_3} = B_{q_1,q_2} + 2B_{u_1,u_2} \tag{4.40}$$

$$Q_{\bar{a}_4} = 3B_{u_1,u_2} - 2B_{d_1,d_2,d_3}. \tag{4.41}$$

Nothing has changed regarding H_{int}. It is still possible to choose $Q_a = L_{\ell_1}$. Then we proceed as before and reexpress the chemical potentials of the affected particles

$$\mu_{q_1} = \mu_{q_2} = \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{3} + \frac{\mu_{\bar{a}_3}}{3} \tag{4.42}$$

[14] We write $Q_a + Q_b + \cdots \equiv Q_{a,b,\ldots}$.

$$\mu_{q3} = \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{3} \tag{4.43}$$

$$\mu_{u_1} = \mu_{u_2} = \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{3} + \frac{2\,\mu_{\bar{a}_3}}{3} + \mu_{\bar{a}_4} \tag{4.44}$$

$$\mu_{u_3} = \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{3} \tag{4.45}$$

$$\mu_{d_1} = \mu_{d_2} = \mu_{d_3} = -\frac{\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{3} - \frac{2\,\mu_{\bar{a}_4}}{3} \tag{4.46}$$

$$\mu_{\ell_1} = \mu_a - \frac{\mu_{\bar{a}_1}}{2} \tag{4.47}$$

$$\mu_{\varphi} = \frac{\mu_{\bar{a}_1}}{2}. \tag{4.48}$$

From Eq. (4.29) we get $\Xi/T^2 = 23/90$.

$\mathbf{T = 10^{12} - 10^{13}}$ GeV

In this regime, the bottom-quark and τ-lepton Yukawa interactions enter equilibrium. For the single-flavor ansatz, the latter means in particular that the lepton number of the left-handed τ leptons L_{ℓ_3} is not conserved by H_0, because of the τ-Yukawa coupling, but the sum of the left- and right-handed leptons with τ flavor L_{ℓ_3,e_3} is. Thus, we have to distinguish between two cases: (i) No asymmetry in the τ flavor is produced, or, (ii) the asymmetry is solely produced in the τ flavor. In the case (i) we can again define a linear combination ℓ_1 of e and μ so that L_{ℓ_1} is the only charge that is broken by $H_{\rm int}$. The conseved charges can be chosen as

$$Q_{\bar{a}_1} = Y_{\ell_1,\ell_3,e_3,\varphi,q_3,u_3,d_3} \tag{4.49}$$

$$Q_{\bar{a}_2} = B_{q_3,u_3,d_3} \tag{4.50}$$

$$Q_{\bar{a}_3} = L_{\ell_3,e_3}. \tag{4.51}$$

The chemical potentials then are

$$\mu_{q3} = \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{3} \tag{4.52}$$

$$\mu_{u_3} = \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{3} \tag{4.53}$$

$$\mu_{d_3} = -\frac{\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{3} \tag{4.54}$$

$$\mu_{\ell_1} = \mu_a - \frac{\mu_{\bar{a}_1}}{2} \tag{4.55}$$

$$\mu_{\ell_3} = \mu_{\bar{a}_3} - \frac{\mu_{\bar{a}_1}}{2} \tag{4.56}$$

$$\mu_{e_3} = \mu_{\bar{a}_3} - \mu_{\bar{a}_1} \tag{4.57}$$

$$\mu_{\varphi} = \frac{\mu_{\bar{a}_1}}{2}, \tag{4.58}$$

and from Eq. (4.29) we get $\Xi/T^2 = 16/57$.

In the other case (ii), L_{ℓ_3,e_3} is broken by $H_{\rm int}$, while L_{ℓ_1} is now conserved. Therefore, we have to interchange the charges Q_a and $Q_{\bar{a}_1}$. The chemical potentials of the quarks and the Higgs do not change and

$$\mu_{\ell_3} = \mu_a - \frac{\mu_{\bar{a}_1}}{2} \quad (4.59)$$

$$\mu_{e_3} = \mu_a - \mu_{\bar{a}_1}. \quad (4.60)$$

Here, we find $\Xi/T^2 = 1/3$.

$\mathbf{T = 10^{11} - 10^{12}}$ GeV

The next processes to enter equilibrium are the electroweak sphalerons. Since they violate baryon and lepton number, L is not any more conserved by H_0, but $B-L$ is. Therefore, in the single-flavor scenario we can choose $B/3 - L_i$ as the conserved charge. As before, i can be (i) a linear combination ℓ_1 of e and μ flavor if no asymmetry is produced in the τ flavor, or (ii) it *is* the τ flavor and only the sum of the lepton number left- and-right-handed particles $L_{\ell_3 e_3}$ is conserved. So in case (i) we choose the broken charge to be $B/3 - L_{\ell_1}$ and the conserved charges

$$Q_{\bar{a}_1} = Y_{\ell_1,\ell_2,\ell_3,e_3,\varphi,q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3} \quad (4.61)$$

$$Q_{\bar{a}_2} = \frac{B_{q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3}}{3} - L_{\ell_2} \quad (4.62)$$

$$Q_{\bar{a}_3} = \frac{B_{q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3}}{3} - L_{\ell_3,e_3} \quad (4.63)$$

$$Q_{\bar{a}_4} = B_{u_1,u_2} - B_{d_1,d_2} \quad (4.64)$$

$$Q_{\bar{a}_5} = B_{q_2,u_2,d_2} - B_{q_3,u_3,d_3} \quad (4.65)$$

$$Q_{\bar{a}_6} = B_{q_1,u_1,d_1} - B_{q_2,u_2,d_2}. \quad (4.66)$$

Then,

$$\mu_{q_1} = \frac{\mu_a}{9} + \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_6}}{3} \quad (4.67)$$

$$\mu_{q_2} = \frac{\mu_a}{9} + \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_5}}{3} - \frac{\mu_{\bar{a}_6}}{3} \quad (4.68)$$

$$\mu_{q_3} = \frac{\mu_a}{9} + \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_5}}{3} \quad (4.69)$$

$$\mu_{u_1} = \frac{\mu_a}{9} + \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_4}}{3} + \frac{\mu_{\bar{a}_6}}{3} \quad (4.70)$$

$$\mu_{u_2} = \frac{\mu_a}{9} + \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_4}}{3} + \frac{\mu_{\bar{a}_5}}{3} - \frac{\mu_{\bar{a}_6}}{3} \quad (4.71)$$

$$\mu_{u_3} = \frac{\mu_a}{9} + \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_5}}{3} \quad (4.72)$$

$$\mu_{d_1} = \frac{\mu_a}{9} - \frac{\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_4}}{3} + \frac{\mu_{\bar{a}_6}}{3} \tag{4.73}$$

$$\mu_{d_2} = \frac{\mu_a}{9} - \frac{\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_4}}{3} + \frac{\mu_{\bar{a}_5}}{3} - \frac{\mu_{\bar{a}_6}}{3} \tag{4.74}$$

$$\mu_{d_3} = \frac{\mu_a}{9} - \frac{\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_5}}{3} \tag{4.75}$$

$$\mu_{\ell_1} = -\mu_a - \frac{\mu_{\bar{a}_1}}{2} \tag{4.76}$$

$$\mu_{\ell_2} = -\frac{\mu_{\bar{a}_1}}{2} - \mu_{\bar{a}_2} \tag{4.77}$$

$$\mu_{\ell_3} = -\frac{\mu_{\bar{a}_1}}{2} - \mu_{\bar{a}_3} \tag{4.78}$$

$$\mu_{e_3} = -\mu_{\bar{a}_1} - \mu_{\bar{a}_3} \tag{4.79}$$

$$\mu_{\varphi} = \frac{\mu_{\bar{a}_1}}{2}. \tag{4.80}$$

With Eq. (4.29) this leads to $\Xi/T^2 = 230/711$.

To obtain the result for case (ii), we simply interchange Q_a and $Q_{\bar{a}_3}$, while the other $Q_{\bar{a}}$ are left as they were. The expressions for the chemical potentials of the quarks and the Higgs boson are then not affected and

$$\mu_{\ell_1} = -\frac{\mu_{\bar{a}_1}}{2} - \mu_{\bar{a}_2} \tag{4.81}$$

$$\mu_{\ell_2} = -\frac{\mu_{\bar{a}_1}}{2} - \mu_{\bar{a}_3} \tag{4.82}$$

$$\mu_{\ell_3} = -\mu_a - \frac{\mu_{\bar{a}_1}}{2} \tag{4.83}$$

$$\mu_{e_3} = -\mu_a - \mu_{\bar{a}_1}. \tag{4.84}$$

The result is now $\Xi/T^2 = 115/318$.

$\mathbf{T = 10^8 - 10^{11}}$ GeV

In this regime, charm and strange quark Yukawa interactions are in equilibrium, too, as is the myon Yukawa interaction. The latter causes that we can now distinguish between all three lepton flavors. In the single-flavor approximation we, therefore, have to assume that the asymmetry is either produced in (i) e or in (ii) τ flavor [15]. The charges do slightly change compared to the previous paragraph. In case (i) we can again use $Q_a = B/3 - L_{\ell_1}$ as the broken charge and the conserved charges are

$$Q_{\bar{a}_1} = Y_{\ell_1,\ell_2,\ell_3,e_2,e_3,\varphi,q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3} \tag{4.85}$$

$$Q_{\bar{a}_2} = \frac{B_{q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3}}{3} - L_{\ell_2,e_2} \tag{4.86}$$

[15] Of course, one could assume that the asymmetry is produced in μ flavor, too. However, the result would be the same as for τ.

$$Q_{\bar{a}_3} = \frac{B_{q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3}}{3} - L_{\ell_3,e_3} \tag{4.87}$$

$$Q_{\bar{a}_4} = B_{u_1} - B_{d_1} \tag{4.88}$$

$$Q_{\bar{a}_5} = B_{q_2,u_2,d_2} - B_{q_3,u_3,d_3} \tag{4.89}$$

$$Q_{\bar{a}_6} = B_{q_1,u_1,d_1} - B_{q_2,u_2,d_2}. \tag{4.90}$$

Then,

$$\mu_{q_1} = \frac{\mu_a}{9} + \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_6}}{3} \tag{4.91}$$

$$\mu_{q_2} = \frac{\mu_a}{9} + \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_5}}{3} - \frac{\mu_{\bar{a}_6}}{3} \tag{4.92}$$

$$\mu_{q_3} = \frac{\mu_a}{9} + \frac{\mu_{\bar{a}_1}}{6} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_5}}{3} \tag{4.93}$$

$$\mu_{u_1} = \frac{\mu_a}{9} + \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_4}}{3} + \frac{\mu_{\bar{a}_6}}{3} \tag{4.94}$$

$$\mu_{u_2} = \frac{\mu_a}{9} + \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_5}}{3} - \frac{\mu_{\bar{a}_6}}{3} \tag{4.95}$$

$$\mu_{u_3} = \frac{\mu_a}{9} + \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_5}}{3} \tag{4.96}$$

$$\mu_{d_1} = \frac{\mu_a}{9} - \frac{\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_4}}{3} + \frac{\mu_{\bar{a}_6}}{3} \tag{4.97}$$

$$\mu_{d_2} = \frac{\mu_a}{9} - \frac{\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} + \frac{\mu_{\bar{a}_5}}{3} - \frac{\mu_{\bar{a}_6}}{3} \tag{4.98}$$

$$\mu_{d_3} = \frac{\mu_a}{9} - \frac{\mu_{\bar{a}_1}}{3} + \frac{\mu_{\bar{a}_2}}{9} + \frac{\mu_{\bar{a}_3}}{9} - \frac{\mu_{\bar{a}_5}}{3} \tag{4.99}$$

$$\mu_{\ell_1} = -\mu_a - \frac{\mu_{\bar{a}_1}}{2} \tag{4.100}$$

$$\mu_{\ell_2} = -\frac{\mu_{\bar{a}_1}}{2} - \mu_{\bar{a}_2} \tag{4.101}$$

$$\mu_{\ell_3} = -\frac{\mu_{\bar{a}_1}}{2} - \mu_{\bar{a}_3} \tag{4.102}$$

$$\mu_{e_2} = -\mu_{\bar{a}_1} - \mu_{\bar{a}_2} \tag{4.103}$$

$$\mu_{e_3} = -\mu_{\bar{a}_1} - \mu_{\bar{a}_3} \tag{4.104}$$

$$\mu_{\varphi} = \frac{\mu_{\bar{a}_1}}{2}. \tag{4.105}$$

From Eq. (4.29) we get $\Xi/T^2 = 358/1017$. In case (ii) we interchange the charges Q_a and $Q_{\bar{a}_3}$. This does not change the expressions for the Higgs and the quark chemical potentials. The chemical potentials of ℓ_1 and ℓ_3 interchange, but that does not affect P. However, the chemical potential of e_3 changes to $\mu_{e_3} = -\mu_a - \mu_{\bar{a}_1}$ which leads to $\Xi/T^2 = 179/422$.

$\mathbf{T \ll 10^8}$ GeV

The last regime that we want to consider here is that in which all the Standard Model Yukawa and gauge couplings as well as the strong and electroweak sphalerons are in equilibrium. The single-flavor approximation now always leads to the same result, independent of whether the asymmetry is produced in e, μ or τ flavor. We will now assume it is produced in the τ flavor. Then the broken charge to consider is $Q_a = B/3 - L_{\ell_3,e_3}$ and the conserved charges are

$$Q_{\bar{a}_1} = Y_{\ell_1,\ell_2,\ell_3,e_1,e_2,e_3,\varphi,q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3} \tag{4.106}$$

$$Q_{\bar{a}_2} = \frac{2\,B_{q_1,q_2,q_3,u_1,u_2,u_3,d_1,d_2,d_3}}{3} - L_{\ell_1,\ell_2,e_1,e_2} \tag{4.107}$$

$$\tag{4.108}$$

Then

$$\mu_{q_1} = \mu_{q_2} = \mu_{q_3} = \frac{\mu_a}{9} + \frac{\mu_{\bar{a}_1}}{6} + \frac{2\,\mu_{\bar{a}_2}}{9} \tag{4.109}$$

$$\mu_{u_1} = \mu_{u_2} = \mu_{u_3} = \frac{\mu_a}{9} + \frac{2\,\mu_{\bar{a}_1}}{3} + \frac{2\,\mu_{\bar{a}_2}}{9} \tag{4.110}$$

$$\mu_{d_1} = \mu_{d_2} = \mu_{d_3} = \frac{\mu_a}{9} - \frac{\mu_{\bar{a}_1}}{3} + \frac{2\,\mu_{\bar{a}_2}}{9} \tag{4.111}$$

$$\mu_{\ell_1} = \mu_{\ell_2} = -\frac{\mu_{\bar{a}_1}}{2} - \mu_{\bar{a}_2} \tag{4.112}$$

$$\mu_{\ell_3} = -\mu_a - \frac{\mu_{\bar{a}_1}}{2} \tag{4.113}$$

$$\mu_{e_1} = \mu_{e_2} = -\mu_{\bar{a}_1} - \mu_{\bar{a}_2} \tag{4.114}$$

$$\mu_{e_3} = -\mu_a - \mu_{\bar{a}_1} \tag{4.115}$$

$$\mu_\varphi = \frac{\mu_{\bar{a}_1}}{2}. \tag{4.116}$$

From Eq (4.29) we obtain the result $\Xi/T^2 = 237/514$.

Alternatively, one can derive the washout rate from the Boltzmann equation

$$\Gamma_{B-L} n_{B-L} = \int \prod_{a=N,\ell,\varphi} d\Pi_a \delta(p_\ell + p_\varphi - p_N)(f_\ell f_\varphi - f_{\bar{\ell}} f_{\bar{\varphi}}) \sum |\mathcal{M}|^2 \tag{4.117}$$

where $\sum |\mathcal{M}|^2$ is the spin-summed matrix element for the inverse decay of the heavy neutrino. It is, however, at leading order the same as the matrix element for the decays and therefore given by Eq. (4.12). Now, we expand the distribution functions f_x in the chemical potentials,

$$f_\ell f_\varphi - f_{\bar{\ell}} f_{\bar{\varphi}} \simeq \frac{2e^{-\beta E_N}}{T}(\mu_\ell + \mu_\varphi), \tag{4.118}$$

T (GeV)	spectators	Q_a	$T^2\Xi^{-1}$
$\gg 10^{13}$	h_t, gauge	L_{ℓ_1}	$4\left(1+\frac{1}{4}\frac{m_\varphi}{\pi T}\right)$
$\gtrsim 10^{13}$	+ QCD sphalerons	L_{ℓ_1}	$\frac{90}{23}\left(1+\frac{49}{230}\frac{m_\varphi}{\pi T}\right)$
10^{12} - 10^{13}	$+\ h_b, h_\tau$	$\begin{cases} L_{\ell_1} \\ L_{\ell_3,e_3} \end{cases}$	$\begin{cases} \frac{57}{16}\left(1+\frac{27}{304}\frac{m_\varphi}{\pi T}\right) \\ 3\left(1+\frac{3}{16}\frac{m_\varphi}{\pi T}\right) \end{cases}$
10^{11} - 10^{12}	+ EW sphalerons	$\begin{cases} B/3 - L_{\ell_1} \\ B/3 - L_{\ell_3,e_3} \end{cases}$	$\begin{cases} \frac{711}{230}\left(1+\frac{1681}{18170}\frac{m_\varphi}{\pi T}\right) \\ \frac{318}{115}\left(1+\frac{1176}{6095}\frac{m_\varphi}{\pi T}\right) \end{cases}$
10^{8} - 10^{11}	$+\ h_c, h_s, h_\mu$	$\begin{cases} B/3 - L_{\ell_1} \\ B/3 - L_{\ell_3,e_3} \end{cases}$	$\begin{cases} \frac{1017}{358}\left(1+\frac{1369}{40454}\frac{m_\varphi}{\pi T}\right) \\ \frac{422}{179}\left(1+\frac{3042}{37769}\frac{m_\varphi}{\pi T}\right) \end{cases}$
$\ll 10^{8}$	$+\ h_u,\ h_d,\ h_e$	$B/3 - L_{\ell_3,e_3}$	$\frac{517}{237}\left(1+\frac{864}{20303}\frac{m_\varphi}{\pi T}\right)$

Table 1: We present the values up to next-to-leading order for the number $T^2\Xi$ that enters the lepton number washout rate (4.28). In the second column we list the spectator processes that are present in the respective temperature regime. The third column indicates which charge is broken by heavy neutrino Yukawa interaction. In the single-flavor case that we consider here it is always only one charge broken, however, in some regimes we have to distinguish between the asymmetry produced in a flavor whose Yukawa interaction is active and one whose is not.

with $\beta = 1/T$. Then, we introduce coefficients c_ℓ and c_φ via [12]

$$n_\ell - n_{\bar{\ell}} = -c_\ell \, n_{B-L} \tag{4.119}$$
$$n_\varphi - n_{\bar{\varphi}} = -c_\varphi \, n_{B-L}. \tag{4.120}$$

With them, we can relate the chemical potentials of ℓ and φ to the asymmetry n_{B-L}. Here, it is important to note that one has to use quantum instead of Boltzmann statistics on the left-hand side of Eqs. (4.119) and (4.120), because the momenta of the leptons and Higgs bosons are saturated at $|\mathbf{p}| \sim T$. In Eq. (4.117) they are saturated at $|\mathbf{p}| \sim M_N/2$, which is why we could use Boltzmann statistics for all particles there. We expand the distribution functions in the chemical potentials and approximate $m_\ell = m_\varphi = 0$. Then,

$$f_x = \left(e^{\beta(E_x - \mu_x)} \pm 1\right)^{-1} \approx \frac{1}{e^{\beta E_x} \pm 1} + \frac{\beta \mu_x \, e^{\beta E_x}}{\left(e^{\beta E_x} \pm 1\right)^2}, \tag{4.121}$$

and therefore

$$\begin{aligned} n_x - n_{\bar{x}} &= 4\beta\mu_x \int \frac{d^3\mathbf{p}_x}{(2\pi)^3} \frac{e^{\beta E_x}}{\left(e^{\beta E_x} \pm 1\right)^2} \\ &\approx 4\beta\mu_x \int \frac{d^3\mathbf{p}_x}{(2\pi)^3} \frac{1}{e^{-\beta|\mathbf{p}_x|} + e^{\beta|\mathbf{p}_x|} \pm 2} \\ &= 2\beta\mu_x \frac{1}{\pi^2} \int d|\mathbf{p}_x| \frac{|\mathbf{p}_x|^2}{e^{-\beta|\mathbf{p}_x|} + e^{\beta|\mathbf{p}_x|} \pm 2} \\ &= \begin{cases} \frac{\mu_x T^2}{3}, & x = \ell \\ \frac{2\mu_x T^2}{3}, & x = \varphi \end{cases} . \end{aligned} \tag{4.122}$$

Plugging this into Eqs. (4.119) and (4.120) gives

$$\mu_\ell = \frac{3c_\ell}{T^2} n_{B-L} \tag{4.123}$$
$$\mu_\varphi = \frac{3c_\varphi}{2T^2} n_{B-L}. \tag{4.124}$$

Now, we can write Eq. (4.118) as

$$f_\ell f_\varphi - f_{\bar{\ell}} f_{\bar{\varphi}} \simeq \frac{6e^{-\beta E_N}}{T^3} \left(c_\ell + \frac{c_\varphi}{2}\right), \tag{4.125}$$

so that we obtain from Eq. (4.117)

$$\Gamma_{B-L} n_{B-L} = \frac{3}{(2\pi)^3 T^3} \left(c_\ell + \frac{c_\varphi}{2}\right) n_{B-L}$$

$$\cdot \int \frac{d^3\mathbf{p}_N}{E_N} \prod_{a=\ell,\varphi} d\Pi_a (2\pi)^4 \delta(p_\ell + p_\varphi - p_N) e^{-\beta E_N}. \tag{4.126}$$

The integrals over p_ℓ and p_φ have already been calculated in Eq. (4.15). We use that result and substitute $x = \sqrt{|\mathbf{p}_N|^2 + M_N^2}/T$,

$$\begin{aligned}
\Gamma_{B-L} &= \frac{3}{\pi^2 T^3} \left(c_\ell + \frac{c_\varphi}{2}\right) M_N \Gamma_0 \int_0^\infty d|\mathbf{p}_N|\, |\mathbf{p}_N|^2 \frac{e^{-x}}{xT} \\
&= \frac{3}{\pi^2 T} \left(c_\ell + \frac{c_\varphi}{2}\right) M_N \Gamma_0 \int_z^\infty dx\, \sqrt{x^2 - z^2} e^{-x} \\
&= \frac{3}{\pi^2} \left(c_\ell + \frac{c_\varphi}{2}\right) z^2 K_1(z) \Gamma_0.
\end{aligned} \tag{4.127}$$

Comparing this result to Eq. (4.28) one sees that the relation

$$T^2 \Xi^{-1} = \frac{3}{\pi^2} \left(c_\ell + \frac{c_\varphi}{2}\right) \tag{4.128}$$

must hold. Indeed, using for the coefficients c_ℓ and c_φ the values of Ref. [12] this is true for all temperature regimes.

If we use classical Maxwell-Boltzmann statistics for leptons and Higgs bosons in the calculation (4.122), Eq. (4.127) changes to [21]

$$\Gamma_{B-L}^{\text{cl.}} = \frac{1}{4} \left(c_\ell + c_\varphi\right) z^2 K_1(z) \Gamma_0. \tag{4.129}$$

The ratio of Eqs. (4.127) and (4.129) is $12/\pi^2$. This may look small at first glance, but we will show in Sec. 7.2 that it has a notable effect on the final lepton asymmetry.

4.3.1 Radiative corrections

The NLO corrections for the pressure are of order g [28]. From these terms, we directly obtain the leading corrections for the susceptibilities, which we denote as Ξ', with

$$\Xi'_{ab} = \frac{T^2}{12} \frac{\partial^2}{\partial\mu_a \partial\mu_b} \left(-4 \frac{3 m_\varphi}{2\pi T} \mu_\varphi^2\right)_{\big|\partial P/\partial\mu_{\bar{a}}=0} = -\frac{m_\varphi T}{\pi} \left(\frac{\partial^2 \mu_\varphi}{\partial\mu_a \partial\mu_b}\right)_{\big|\partial P/\partial\mu_{\bar{a}}=0}, \tag{4.130}$$

where m_φ is the thermal Higgs mass [29]

$$m_\varphi^2 = m_0^2 + \frac{T^2}{16} \left(g_1^2 + 3 g_2^2 + 4|h_t|^2 + 8\lambda\right). \tag{4.131}$$

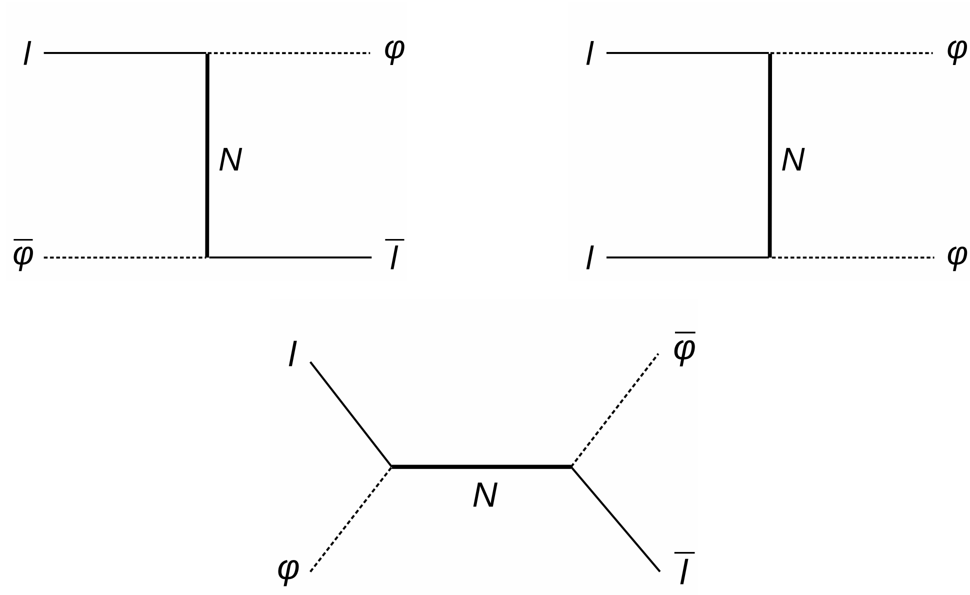

Figure 4: Feynman graph of a scattering process with heavy neutrino exchange, violating lepton number by $\Delta L = 2$.

Note that these terms only depend on the Higgs chemical potential. Since we already know from the previous calculations how μ_φ depends on the μ_a in the respective temperature regimes, we can easily get the correction terms. The results are included in table 1.

Ref. [28] also gives the $\mathcal{O}(g^2)$ corrections for the pressure, so it would be just as easy to derive the respective corrections for the susceptibilities. The reason why we abstain from this here is that at this order the corrections coming from the susceptibilities are not the only corrections that enter the washout rate at thus order. There are those coming from the spectral function, too. If one expands the spectral function in naive perturbation theory to order g^2, one finds that the diagrams are infrared divergent. These divergences can be cured by using massive Higgs propagators. Thus the NLO contribution to the spectral function is also due to Higgs mass resummation. However, one can estimate that the thermal Higgs mass does not lead to an order g correction, but to an order $g^2 \ln(g)$ correction which is even parametrically larger than the $\mathcal{O}(g^2)$ corrections [30].

4.3.2 $\Delta L = 2$ scatterings

There are also scattering processes like in Fig. 4 contributing to the washout rate. They are mediated by the exchange of heavy neutrinos and violate lepton number by $\Delta L = 2$. They are of $\mathcal{O}(h^4)$, but they are not exponentially suppressed in z.

Therefore, they may become important. It has been mentioned in Ref. [12], and we will confirm in Sec. 7, that they are negligible for $M_N \lesssim 10^{13}$ GeV. Up to next-to-leading order their rate can be written as [30]

$$\Gamma_{B-L}^{\Delta L=2} = c_{\Delta L=2}\, T^2 \Xi^{-1} \frac{\bar{m}^2 M_N^3}{v^4} z^{-3}, \tag{4.132}$$

where $\bar{m}^2 = m_1^2 + m_2^2 + m_3^2$ and $c_{\Delta L=2} \approx 0.01116$. To obtain this value, it is necessary to use quantum statistics for the leptons Higgs bosons and to take into account the Bose enhancement and Fermi blocking terms in the collision integral because the heavy neutrino in the propagator may be off-shell. Therefore, the momenta are not saturated at $|\mathbf{p}| = M_N/2$. In Refs. [21, 22] the leading-order $\Delta L = 2$-washout rate (4.132) is derived with Boltzmann statistic for all particles and neglecting the Bose-enhancement and Pauli-blocking terms. The result is

$$\Gamma_{B-L}^{\Delta L=2,\mathrm{cl.}} = \frac{3}{2\pi^5 n_\ell^{\mathrm{eq}}} \frac{\bar{m} T^6}{v^4}, \tag{4.133}$$

which corresponds to a value of $c_{\Delta L=2} \approx 0.00894$, if we use, like Refs. [21,22], $n_\ell^{\mathrm{eq}} = 3\zeta(3)/2\pi^2$ [16]. Thus, neglecting quantum effects underestimates the rate by 20%.

[16] It seems to be somehow inconsistent though to calculate n_ℓ^{eq} with Fermi-Dirac statistic, since it only appears in Eq. (4.133) due to the fact that the rate was obtained using Boltzmann statistic for the leptons.

5 Relativistic corrections [17]

The main idea of the nonrelativistic approximation was to set $1/E_N = 1/M_N$ in Eq. (4.18). Now, we will introduce relativistic corrections by taking higher orders of the Taylor expansion of $1/E_N$ into account,

$$\frac{1}{E_N} = \frac{1}{\sqrt{M_N^2 + \mathbf{p}_N^2}} = \frac{1}{M_N}\left(1 + \sum_{i=1}^{\infty}(-1)^i c_i \left(\frac{\mathbf{p}_N}{M_N}\right)^{2i}\right), \tag{5.1}$$

with real positive c_i. Plugging this into Eq. (4.18) and integrating over $\mathbf{p}_N$ gives us an equation of the form

$$\left(\frac{d}{dt} + 3H\right) n_N = -\Gamma_0 \left[(n_N - n_N^{\text{eq}}) + \sum_i (-1)^i (u_i - u_i^{\text{eq}})\right], \tag{5.2}$$

where

$$u_i^{(\text{eq})} = (2s_N + 1)\int \frac{d^3\mathbf{p}_N}{(2\pi)^3} c_i \left(\frac{\mathbf{p}_N}{M_N}\right)^{2i} f_N^{(\text{eq})}. \tag{5.3}$$

In the following, we will refer to the i-th term as $\mathcal{O}(\mathbf{v}^{2i})$-correction, where $\mathbf{v} \equiv \mathbf{p}_N/M_N$ is the velocity of the neutrinos. In particular, the $\mathcal{O}(\mathbf{v}^2)$-correction

$$u_1 = \frac{2}{M_N}\int \frac{d^3\mathbf{p}_N}{(2\pi)^3}\frac{\mathbf{p}_N^2}{2M_N} f_N \tag{5.4}$$

can be identified as the kinetic energy density of the neutrinos devided by their mass.

The quantities u_i are themselves governed by rate equations of the form (4.8). One gets them by multiplying Eq. (4.18) with p^{2i}. At leading order in all couplings, we find up to $\mathcal{O}(\mathbf{v}^{10})$

$$\left(\frac{d}{dt} + 5H\right) u_1 = -\Gamma_0 \left[(u_1 - u_1^{\text{eq}}) + \sum_{i=2}^{5}(-1)^i (u_i - u_i^{\text{eq}})\right] \tag{5.5}$$

$$\left(\frac{d}{dt} + 7H\right) u_2 = -\Gamma_0 \left[(u_2 - u_2^{\text{eq}}) - \sum_{i=3}^{5}(-1)^i (u_i - u_i^{\text{eq}})\right] \tag{5.6}$$

$$\left(\frac{d}{dt} + 9H\right) u_3 = -\Gamma_0 \left[(u_3 - u_3^{\text{eq}}) + \sum_{i=4}^{5}(-1)^i (u_i - u_i^{\text{eq}})\right] \tag{5.7}$$

[17] This section is mainly based on Ref. [24].

$$\left(\frac{d}{dt} + 11H\right) u_4 = -\Gamma_0 \left[(u_4 - u_4^{\rm eq}) + (u_5 - u_5^{\rm eq})\right] \tag{5.8}$$

$$\left(\frac{d}{dt} + 13H\right) u_5 = -\Gamma_0 \left(u_5 - u_5^{\rm eq}\right). \tag{5.9}$$

Relativistic corrections to the production rate of the neutrinos are given in Ref. [26],

$$\Gamma^{\rm pro} = f^{\rm eq}\Gamma_0 \frac{M_N}{E_N}\left\{a + \frac{p^2}{M_N^2} b + \mathcal{O}\left(\frac{p^4}{M_N^4}\right)\right\}, \tag{5.10}$$

with

$$a = 1 - \frac{\lambda}{z^2} - |h_t|^2 \left[\frac{21}{2(4\pi)^2} + \frac{7\pi^2}{60\, z^4}\right] + \left(g_1^2 + 3g_2^2\right)\left[\frac{29}{8(4\pi)^2} - \frac{\pi^2}{80\, z^4}\right] + \mathcal{O}\left(\frac{g^2}{z^6}, \frac{g^3}{z^2}\right)$$

$$b = -\left[|h_t|^2 \frac{7\pi^2}{45} + \left(g_1^2 + 3g_2^2\right)\frac{\pi^2}{60}\right]\frac{1}{z^4} + \mathcal{O}\left(\frac{g^2}{z^6}, \frac{g^3}{z^2}\right).$$

Here, λ is the Higgs self-coupling and $z = M_N/T$. Since (in thermal equilibrium) $\mathbf{p}^2/M_N \sim T$, we have $z^{-1} \sim \mathbf{v}^2$. Eq. (5.10), therefore, includes corrections up to $\mathcal{O}(g^2\mathbf{v}^{10})$. We define the $\mathcal{O}(g^2\mathbf{v}^{2i})$-coefficients a_i and b_i,

$$a_0 = 1 - \frac{21\,|h_t|^2}{2(4\pi)^2} + \frac{29\,(g_1^2 + 3g_2^2)}{8(4\pi)^2} \tag{5.11}$$

$$a_2 = -\frac{\lambda}{z^2} \tag{5.12}$$

$$a_4 = -\left[\frac{7\pi^2\,|h_t|^2}{60} - \frac{\pi^2\,(g_1^2 + 3g_2^2)}{80}\right]\frac{1}{z^4} \tag{5.13}$$

$$b_4 = -\left[\frac{7\pi^2\,|h_t|^2}{45} + \frac{\pi^2\,(g_1^2 + 3g_2^2)}{60}\right]\frac{1}{z^4}. \tag{5.14}$$

When putting everything together, we should not partially include terms of higher order than $g^2\mathbf{v}^{10}$ to be consistent. We get the following set of rate equations:

$$\left(\frac{d}{dt} + 13H\right) u_5 = -a_0\Gamma_0 \left(u_5 - u_5^{\rm eq}\right) \tag{5.15}$$

$$\left(\frac{d}{dt} + 11H\right) u_4 = -a_0\Gamma_0 \left(u_4 - u_4^{\rm eq}\right) - a_0\Gamma_0 \left(u_5 - u_5^{\rm eq}\right) \tag{5.16}$$

$$\left(\frac{d}{dt} + 9H\right) u_3 = -(a_0 + a_2)\Gamma_0 \left(u_3 - u_3^{\rm eq}\right) + a_0\Gamma_0 \left(u_4 - u_4^{\rm eq}\right)$$

$$- a_0\Gamma_0\,(u_5 - u_5^{\rm eq}) \tag{5.17}$$

$$\left(\frac{d}{dt} + 7H\right) u_2 = -(a_0 + a_2)\Gamma_0\,(u_2 - u_2^{\rm eq}) + (a_0 + a_2)\Gamma_0\,(u_3 - u_3^{\rm eq}) - a_0\Gamma_0\,(u_4 - u_4^{\rm eq}) + a_0\Gamma_0\,(u_5 - u_5^{\rm eq}) \tag{5.18}$$

$$\left(\frac{d}{dt} + 5H\right) u_1 = -(a_0 + a_2 + a_4)\Gamma_0\,(u_1 - u_1^{\rm eq}) + (a_0 + a_2)\Gamma_0\,(u_2 - u_2^{\rm eq}) - (a_0 + a_2)\Gamma_0\,(u_3 - u_3^{\rm eq}) + a_0\Gamma_0\,(u_4 - u_4^{\rm eq}) - a_0\Gamma_0\,(u_5 - u_5^{\rm eq}) \tag{5.19}$$

$$\left(\frac{d}{dt} + 3H\right) n_N = -(a_0 + a_2 + a_4)\Gamma_0\,(n_N - n_N^{\rm eq}) + (a_0 + a_2 + a_4 - 2b_4)\Gamma_0\,(u_1 - u_1^{\rm eq}) - (a_0 + a_2)\Gamma_0\,(u_2 - u_2^{\rm eq}) + (a_0 + a_2)\Gamma_0\,(u_3 - u_3^{\rm eq}) - a_0\Gamma_0\,(u_4 - u_4^{\rm eq}) + a_0\Gamma_0\,(u_5 - u_5^{\rm eq}) \tag{5.20}$$

$$\left(\frac{d}{dt} + 3H\right) n_{B-L} = \epsilon_1\Gamma_0 \left[(n_N - n_N^{\rm eq}) + \sum_{i=1}^{5}(-1)^i\,(u_i - u_i^{\rm eq})\right] - T^2\Xi^{-1}\left[\frac{\Gamma_0}{\pi^2} z^2 K_1(z) + 0.01116\,\frac{\bar{m}^2 M_N^3}{v^4} z^{-3}\right] n_{B-L}. \tag{5.21}$$

6 Equilibration rate of a general particle species [18]

To incorporate radiative corrections in the coefficient Γ_N, which we identified as the equilibration rate of the heavy neutrinos, we claimed that there is a relation between the production rate and the equilibration rate (see Sec. 4.1.1). We will now derive this relation. Let us therefore generalize the previous setup and consider some particle Φ that couples weakly to a hot plasma via the Lagrangian

$$\mathcal{L}_{\text{int}} = -\overline{J}\Phi - \overline{\Phi}J, \tag{6.1}$$

where J can be elementary or composite, but does not contain Φ. The equilibration of Φ shall be much slower than all other interactions within the plasma. Then we can write the Hamiltonian like in Eq. (4.2) where now H_0 contains all the interactions within plasma as well as free Φ fields and H_{int} is

$$H_{\text{int}} = \int d^3\mathbf{x}\, \left(\overline{J}\Phi + \overline{\Phi}J\right). \tag{6.2}$$

We will be concerend with the phase space density, or occupancy, $f_{\mathbf{p}}(t, \mathbf{x})$ of Φ. In thermal equilibrium, f would be given by the Fermi-Dirac or Bose-Einstein distribution. Here, we are interested in a state where f is out of thermal equilibrium. This state is specified by f itself and the temperature T [19]. Thus, the time evolution of f depends on these quantities, too. If f is close to equilibrium, $|f - f^{\text{eq}}| \ll 1$, we can linearize the time derivative of f in $\delta f \equiv f - f^{\text{eq}}$ [20],

$$\dot{f}_{\mathbf{p},\lambda} = -\sum_{\mathbf{p}',\lambda'} \widetilde{\Gamma}_{\mathbf{p}\mathbf{p}',\lambda\lambda'} \delta f_{\mathbf{p}',\lambda'}. \tag{6.3}$$

In general, there could be other quantities X_a that equilibrate slowly. One can always choose them such that their equilibrium value $X_a^{\text{eq}} = 0$. Then, there would also appear terms linear in the X_a on the right-hand side of Eq. (6.3) with different coefficients. For our purposes, it will be sufficient to determine the coefficient $\widetilde{\Gamma}_{\mathbf{p}\mathbf{p}',\lambda\lambda'}$, which is not affected by such other terms.

Using the Landau theory of quasistationary fluctuations [32, §118], we can compute $\widetilde{\Gamma}_{\mathbf{p}\mathbf{p}',\lambda\lambda'}$ via [28],

$$\widetilde{\Gamma}_{\mathbf{p}\mathbf{p}',\lambda\lambda'} = \frac{1}{2V} \sum_{\mathbf{p}'',\lambda''} \lim_{\omega\to 0} \frac{\rho_{\mathbf{p}\mathbf{p}'',\lambda\lambda''}(\omega)}{\omega} \left(\Xi^{-1}\right)_{\mathbf{p}''\mathbf{p}',\lambda''\lambda'}, \tag{6.4}$$

[18] This section is mainly based on Ref. [31].

[19] In general, it would also depend on chemical potentials, but we will assume for simplicity that none are present.

[20] We work in a finite volume. Therefore, the momenta $\mathbf{p}$ are discrete. For the infinite volume limit, take $V^{-1}\sum_{\mathbf{p}} \to (2\pi)^{-3}\int d^3\mathbf{p}$.

with the spectral function

$$\rho_{\mathbf{pp}',\lambda\lambda'}(\omega) = \int dt\, e^{\mathrm{i}\omega t} \left\langle \left[\dot{f}_{\mathbf{p},\lambda}(t), \dot{f}_{\mathbf{p}',\lambda'}(0) \right] \right\rangle \tag{6.5}$$

and the susceptibility matrix

$$\Xi_{\mathbf{pp}',\lambda\lambda'} \equiv \frac{1}{TV} \langle \delta f_{\mathbf{p},\lambda}\, \delta f_{\mathbf{p}',\lambda'} \rangle\,. \tag{6.6}$$

The thermal average $\langle \cdots \rangle \equiv \mathrm{tr}[\cdots \exp(-H_0/T)]/\mathrm{tr}\exp(-H_0/T)$ is performed with the Hamiltonian H_0, since we work at leading order in H_{int}. In order to calculate the right-hand sides of these equations we need to define an appropriate operator expression for the occupancy. Its explicit form will depend on whether Φ is a charged or an uncharged particle. Therefore, we will treat both cases separately.

6.1 Charged particles

The field operator Φ for a charged particle can be written in the interaction picture with respect to H_{int} as

$$\Phi_{\mathrm{I}}(x) = \sum_{\mathbf{p},\lambda} \frac{1}{\sqrt{2E_{\mathbf{p}}V}} \left[e^{-\mathrm{i}px}\, u_{\mathbf{p},\lambda} c_{\mathbf{p},\lambda} + e^{\mathrm{i}px}\, v_{\mathbf{p},\lambda} d^{\dagger}_{\mathbf{p},\lambda} \right]_{p^0=E_{\mathbf{p}}}, \tag{6.7}$$

where $c_{\mathbf{p},\lambda}$ and $d_{\mathbf{p},\lambda}$ are the annihilation operators of the particles and antiparticles, respectively. They are normalized such that

$$[c_{\mathbf{p},\lambda}, c^{\dagger}_{\mathbf{p}',\lambda'}]_{-\sigma} = \delta_{\mathbf{pp}'}\delta_{\lambda\lambda'}, \tag{6.8}$$

and similarly for $d_{\mathbf{p},\lambda}$, while all other (anti-)commutators vanish,

$$[c_{\mathbf{p},\lambda}, c_{\mathbf{p}',\lambda'}]_{-\sigma} = [c_{\mathbf{p},\lambda}, d_{\mathbf{p}',\lambda'}]_{-\sigma} = [c_{\mathbf{p},\lambda}, d^{\dagger}_{\mathbf{p}',\lambda'}]_{-\sigma} = 0. \tag{6.9}$$

Here, $\sigma = +1$ for bosons and $\sigma = -1$ for fermions. We then define the occupancy operator f_{I} in the interaction picture [21] as

$$(f_{\mathbf{p},\lambda})_{\mathrm{I}} \equiv c^{\dagger}_{\mathbf{p},\lambda} c_{\mathbf{p},\lambda}. \tag{6.10}$$

Now, we calculate Eq. (6.6). Using that $\langle f \rangle = f^{\mathrm{eq}}$, and applying Wick's theorem and Eq. (6.8),

$$\Xi_{\mathbf{pp}',\lambda\lambda'} = \frac{1}{TV} \left(\langle f_{\mathbf{p},\lambda} f_{\mathbf{p}',\lambda'} \rangle - \left\langle f_{\mathbf{p},\lambda} f^{\mathrm{eq}}_{\mathbf{p}',\lambda'} \right\rangle - \left\langle f^{\mathrm{eq}}_{\mathbf{p},\lambda} f_{\mathbf{p}',\lambda'} \right\rangle + \left\langle f^{\mathrm{eq}}_{\mathbf{p},\lambda} f^{\mathrm{eq}}_{\mathbf{p}',\lambda'} \right\rangle \right)$$

[21] The corresponding Heisenberg-picture operator is $f = e^{\mathrm{i}Ht} e^{-\mathrm{i}H_0 t} f_{\mathrm{I}} e^{\mathrm{i}H_0 t} e^{-\mathrm{i}Ht}$.

$$
\begin{aligned}
&= \frac{1}{TV}\left(\langle f_{\mathbf{p},\lambda} f_{\mathbf{p}',\lambda'}\rangle - \langle f_{\mathbf{p},\lambda}\rangle f^{\mathrm{eq}}_{\mathbf{p}',\lambda'} - f^{\mathrm{eq}}_{\mathbf{p},\lambda}\langle f_{\mathbf{p}',\lambda'}\rangle + f^{\mathrm{eq}}_{\mathbf{p},\lambda} f^{\mathrm{eq}}_{\mathbf{p}',\lambda'}\right)\\
&= \frac{1}{TV}\left(\left\langle c^\dagger_{\mathbf{p},\lambda} c_{\mathbf{p},\lambda} c^\dagger_{\mathbf{p}',\lambda'} c_{\mathbf{p}',\lambda'}\right\rangle - f^{\mathrm{eq}}_{\mathbf{p},\lambda} f^{\mathrm{eq}}_{\mathbf{p}',\lambda'}\right)\\
&= \frac{1}{TV}\left(\left\langle c^\dagger_{\mathbf{p},\lambda} c_{\mathbf{p}',\lambda'}\right\rangle \left\langle c_{\mathbf{p},\lambda} c^\dagger_{\mathbf{p}',\lambda'}\right\rangle - \langle c_{\mathbf{p},\lambda} c_{\mathbf{p}',\lambda'}\rangle \left\langle c^\dagger_{\mathbf{p},\lambda} c^\dagger_{\mathbf{p}',\lambda'}\right\rangle\right.\\
&\qquad \left. + \left\langle c^\dagger_{\mathbf{p},\lambda} c_{\mathbf{p},\lambda}\right\rangle \left\langle c^\dagger_{\mathbf{p}',\lambda'} c_{\mathbf{p}',\lambda'}\right\rangle - f^{\mathrm{eq}}_{\mathbf{p},\lambda} f^{\mathrm{eq}}_{\mathbf{p}',\lambda'}\right)\\
&= \frac{1}{TV}\left(\left\langle c^\dagger_{\mathbf{p},\lambda} c_{\mathbf{p}',\lambda'}\right\rangle \left\langle c_{\mathbf{p},\lambda} c^\dagger_{\mathbf{p}',\lambda'}\right\rangle\right)\\
&= \frac{1}{TV}\left(\left\langle c^\dagger_{\mathbf{p},\lambda} c_{\mathbf{p}',\lambda'}\right\rangle \left(\left[c^\dagger_{\mathbf{p}',\lambda'}, c_{\mathbf{p},\lambda}\right] + \sigma \left\langle c^\dagger_{\mathbf{p}',\lambda'} c_{\mathbf{p},\lambda}\right\rangle\right)\right)\\
&= \frac{\delta_{\lambda\lambda'}\delta_{\mathbf{p}\mathbf{p}'}}{TV} f^{\mathrm{eq}}_{\mathbf{p},\lambda}\Big(1 + \sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda}\Big). \qquad (6.11)
\end{aligned}
$$

Next, we will need the time derivative of the occupancy which is in the Heisenberg picture determined by the commutator with the Hamiltonian,

$$
\begin{aligned}
\dot f_{\mathbf{p},\lambda} &= \mathrm{i}[H, f_{\mathbf{p},\lambda}]\\
&= \mathrm{i}e^{\mathrm{i}Ht}e^{-\mathrm{i}H_0 t}[H_0 + H_{\mathrm{int}}, (f_{\mathbf{p},\lambda})_{\mathrm{I}}]e^{\mathrm{i}H_0 t}e^{-\mathrm{i}Ht}\\
&= \mathrm{i}e^{\mathrm{i}Ht}e^{-\mathrm{i}H_0 t}\left[\int d^3\mathbf{x}\left(\overline{J}_{\mathrm{I}}(x)\Phi_{\mathrm{I}}(x) + \overline{\Phi}_{\mathrm{I}}(x)J_{\mathrm{I}}(x)\right), c^\dagger_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda}\right]e^{\mathrm{i}H_0 t}e^{-\mathrm{i}Ht}\\
&= \int d^3\mathbf{x}\sum_{\mathbf{p}',\lambda'}\frac{\mathrm{i}}{\sqrt{2E_{\mathbf{p}'}V}}e^{\mathrm{i}Ht}e^{-\mathrm{i}H_0 t}\Big(\overline{J}_{\mathrm{I}}(x)e^{-\mathrm{i}p'x}\,u_{\mathbf{p}',\lambda'}[c_{\mathbf{p}',\lambda'}, c^\dagger_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda}]\\
&\qquad + \overline{u}_{\mathbf{p}',\lambda'}e^{\mathrm{i}p'x}\,J_{\mathrm{I}}(x)[c^\dagger_{\mathbf{p}',\lambda'}, c^\dagger_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda}]\Big)e^{\mathrm{i}H_0 t}e^{-\mathrm{i}Ht}\\
&= \frac{\mathrm{i}}{\sqrt{2E_{\mathbf{p}}V}}\int d^3\mathbf{x}\left(\overline{J}(x)e^{-\mathrm{i}px}u_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda} - \overline{u}_{\mathbf{p},\lambda}e^{\mathrm{i}px}J(x)c^\dagger_{\mathbf{p},\lambda}\right), \qquad (6.12)
\end{aligned}
$$

for a bosonic Φ. The result also holds for fermions, where the $c_{\mathbf{p},\lambda}$ fulfill the respective anticommutator relations because then

$$
\begin{aligned}
[c_{\mathbf{p},\lambda}, c^\dagger_{\mathbf{p}',\lambda'}c_{\mathbf{p}',\lambda'}] &= [c_{\mathbf{p},\lambda}, c^\dagger_{\mathbf{p}',\lambda'}]c_{\mathbf{p}',\lambda'} + c^\dagger_{\mathbf{p}',\lambda'}[c_{\mathbf{p},\lambda}, c_{\mathbf{p}',\lambda'}]\\
&= [c_{\mathbf{p},\lambda}, c^\dagger_{\mathbf{p}',\lambda'}]_+ c_{\mathbf{p}',\lambda'} - 2c^\dagger_{\mathbf{p}',\lambda'}c_{\mathbf{p},\lambda}c_{\mathbf{p}',\lambda'}\\
&\quad + c^\dagger_{\mathbf{p}',\lambda'}[c_{\mathbf{p},\lambda}, c_{\mathbf{p}',\lambda'}]_+ - 2c^\dagger_{\mathbf{p}',\lambda'}c_{\mathbf{p}',\lambda'}c_{\mathbf{p},\lambda}\\
&= [c_{\mathbf{p},\lambda}, c^\dagger_{\mathbf{p}',\lambda'}]_+ - 2c^\dagger_{\mathbf{p}',\lambda'}[c_{\mathbf{p},\lambda}, c_{\mathbf{p}',\lambda'}]_+\\
&= \delta_{\mathbf{p}\mathbf{p}'}\delta_{\lambda,\lambda'}c_{\mathbf{p}',\lambda'}. \qquad (6.13)
\end{aligned}
$$

With Eq. (6.12) and treating Φ as free fields, it follows that

$$
\left\langle\left[\dot f_{\mathbf{p},\lambda}(t), \dot f_{\mathbf{p}',\lambda'}(0)\right]\right\rangle
$$

$$
\begin{aligned}
=&\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}V}\int d^3\mathbf{x}d^3\mathbf{x}'\,\Big(e^{-\mathrm{i}E_{\mathbf{p}}t}e^{\mathrm{i}(\mathbf{p}\mathbf{x}-\mathbf{p}'\mathbf{x}')}\left\langle\Big[\overline{J}(x)u_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda},c^\dagger_{\mathbf{p}',\lambda'}\overline{u}_{\mathbf{p}',\lambda'}J(x')\Big]\right\rangle\\
&\qquad\qquad+e^{\mathrm{i}E_{\mathbf{p}}t}e^{-\mathrm{i}(\mathbf{p}\mathbf{x}-\mathbf{p}'\mathbf{x}')}\left\langle\Big[c^\dagger_{\mathbf{p},\lambda}\overline{u}_{\mathbf{p},\lambda}J(x),\overline{J}(x')u_{\mathbf{p}',\lambda'}c_{\mathbf{p}',\lambda'}\Big]\right\rangle\Big)\\
=&\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}V}\int d^3\mathbf{x}d^3\mathbf{x}'\,\Big(e^{-\mathrm{i}E_{\mathbf{p}}t}e^{\mathrm{i}(\mathbf{p}\mathbf{x}-\mathbf{p}'\mathbf{x}')}\Big(\Big\langle\overline{J}(x)u_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda}c^\dagger_{\mathbf{p}',\lambda'}\overline{u}_{\mathbf{p}',\lambda'}J(x')\Big\rangle\\
&\qquad\qquad-\Big\langle c^\dagger_{\mathbf{p}',\lambda'}\overline{u}_{\mathbf{p}',\lambda'}J(x')\overline{J}(x)u_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda}\Big\rangle\Big)\\
&\qquad\qquad+e^{\mathrm{i}E_{\mathbf{p}}t}e^{-\mathrm{i}(\mathbf{p}\mathbf{x}-\mathbf{p}'\mathbf{x}')}\Big(\Big\langle c^\dagger_{\mathbf{p},\lambda}\overline{u}_{\mathbf{p},\lambda}J(x)\overline{J}(x')u_{\mathbf{p}',\lambda'}c_{\mathbf{p}',\lambda'}\Big\rangle\\
&\qquad\qquad-\Big\langle\overline{J}(x')u_{\mathbf{p}',\lambda'}c_{\mathbf{p}',\lambda'}c^\dagger_{\mathbf{p},\lambda}\overline{u}_{\mathbf{p},\lambda}J(x)\Big\rangle\Big)\Big)\\
=&\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}V}\int d^3\mathbf{x}d^3\mathbf{x}'\,\Big(e^{-\mathrm{i}E_{\mathbf{p}}t}e^{\mathrm{i}(\mathbf{p}\mathbf{x}-\mathbf{p}'\mathbf{x}')}\Big(\Big\langle c_{\mathbf{p},\lambda}c^\dagger_{\mathbf{p}',\lambda'}\Big\rangle\left\langle\overline{J}(x)u_{\mathbf{p},\lambda}\overline{u}_{\mathbf{p}',\lambda'}J(x')\right\rangle\\
&\qquad\qquad-\Big\langle c^\dagger_{\mathbf{p}',\lambda'}c_{\mathbf{p},\lambda}\Big\rangle\left\langle\overline{u}_{\mathbf{p}',\lambda'}J(x')\overline{J}(x)u_{\mathbf{p},\lambda}\right\rangle\Big)\\
&\qquad\qquad+e^{\mathrm{i}E_{\mathbf{p}}t}e^{-\mathrm{i}(\mathbf{p}\mathbf{x}-\mathbf{p}'\mathbf{x}')}\Big(\Big\langle c^\dagger_{\mathbf{p},\lambda}c_{\mathbf{p}',\lambda'}\Big\rangle\left\langle\overline{u}_{\mathbf{p},\lambda}J(x)\overline{J}(x')u_{\mathbf{p}',\lambda'}\right\rangle\\
&\qquad\qquad-\Big\langle c_{\mathbf{p}',\lambda'}c^\dagger_{\mathbf{p},\lambda}\Big\rangle\left\langle\overline{J}(x')u_{\mathbf{p}',\lambda'}\overline{u}_{\mathbf{p},\lambda}J(x)\right\rangle\Big)\Big)\\
=&\frac{\delta_{\mathbf{p}\mathbf{p}'}\delta_{\lambda\lambda'}}{2E_{\mathbf{p}}V}\int d^3\mathbf{x}d^3\mathbf{x}'\,\Big(e^{-\mathrm{i}E_{\mathbf{p}}t}e^{\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{x}')}\Big((1+\sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda})\sigma\Delta^<_{\overline{u}J,\overline{J}u}(x'-x)\\
&\qquad\qquad-f^{\mathrm{eq}}_{\mathbf{p},\lambda}\Delta^>_{\overline{u}J,\overline{J}u}(x'-x)\Big)\\
&\qquad\qquad+e^{\mathrm{i}E_{\mathbf{p}}t}e^{-\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{x}')}\Big(f^{\mathrm{eq}}_{\mathbf{p},\lambda}\Delta^>_{\overline{u}J,\overline{J}u}(x-x')\\
&\qquad\qquad-(1+\sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda})\sigma\Delta^<_{\overline{u}J,\overline{J}u}(x-x')\Big)\Big)\\
=&\frac{\delta_{\mathbf{p}\mathbf{p}'}\delta_{\lambda\lambda'}}{2E_{\mathbf{p}}}\Big(e^{-\mathrm{i}E_{\mathbf{p}}t}\Big(\sigma(1+\sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda})\Delta^<_{\overline{u}J,\overline{J}u}(-t,\mathbf{p})-f^{\mathrm{eq}}_{\mathbf{p},\lambda}\Delta^>_{\overline{u}J,\overline{J}u}(-t,\mathbf{p})\Big)\\
&\qquad+e^{\mathrm{i}E_{\mathbf{p}}t}\Big(f^{\mathrm{eq}}_{\mathbf{p},\lambda}\Delta^>_{\overline{u}J,\overline{J}u}(t,\mathbf{p})-\sigma(1+\sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda})\Delta^<_{\overline{u}J,\overline{J}u}(t,\mathbf{p})\Big)\Big),
\end{aligned}
\tag{6.14}
$$

where we introduced the Wightman functions

$$
\Delta^>_{AB}(x)\equiv\langle A(x)B(0)\rangle \tag{6.15}
$$

$$
\Delta^<_{AB}(x)\equiv\sigma\langle B(0)A(x)\rangle. \tag{6.16}
$$

Therefore, the spectral function (6.5) is

$$
\begin{aligned}
\rho_{\mathbf{p}\mathbf{p}',\lambda\lambda'}(\omega)=\frac{\delta_{\mathbf{p}\mathbf{p}'}\delta_{\lambda\lambda'}}{2E_{\mathbf{p}}}&\Big(\sigma(1+\sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda})\Delta^<_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}-\omega,\mathbf{p})-f^{\mathrm{eq}}_{\mathbf{p},\lambda}\Delta^>_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}-\omega,\mathbf{p})\\
&+f^{\mathrm{eq}}_{\mathbf{p},\lambda}\Delta^>_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}+\omega,\mathbf{p})-\sigma(1+\sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda})\Delta^<_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}+\omega,\mathbf{p})\Big).
\end{aligned}
\tag{6.17}
$$

The Wightman functions can be written as

$$\Delta^{>}_{AB}(\omega) = (1 + \sigma f^{\mathrm{eq}}(\omega))\,\widetilde{\rho}_{AB}(\omega) \tag{6.18}$$
$$\Delta^{<}_{AB}(\omega) = \sigma f^{\mathrm{eq}}(\omega)\widetilde{\rho}_{AB}(\omega) \tag{6.19}$$

with the spectral function

$$\widetilde{\rho}_{AB}(p) = \int d^4x e^{\mathrm{i}px} \left\langle [A(x), B(0)]_{-\sigma} \right\rangle . \tag{6.20}$$

Then,

$$\begin{aligned}
\rho_{\mathbf{pp}',\lambda\lambda'}(\omega) = \frac{\delta_{\mathbf{pp}'}\delta_{\lambda\lambda'}}{2E_{\mathbf{p}}} \big(&\widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}} - \omega, \mathbf{p}) \big((1 + \sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda}) f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega) \\
&\qquad - f^{\mathrm{eq}}_{\mathbf{p},\lambda} (1 + \sigma f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega))\big) \\
&+ \widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}} + \omega, \mathbf{p}) \big(f^{\mathrm{eq}}_{\mathbf{p},\lambda} (1 + \sigma f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega)) \\
&\qquad - (1 + \sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda}) f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega)\big)\big) \\
= \frac{\delta_{\mathbf{pp}'}\delta_{\lambda\lambda'}}{2E_{\mathbf{p}}} &\widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}, \mathbf{p}) \big((1 + \sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda}) f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega) \\
&\qquad - f^{\mathrm{eq}}_{\mathbf{p},\lambda} (1 + \sigma f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega)) \\
&+ f^{\mathrm{eq}}_{\mathbf{p},\lambda} (1 + \sigma f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega)) \\
&\qquad - (1 + \sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda}) f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega)\big) + \mathcal{O}(\omega^2) \\
= \frac{\delta_{\mathbf{pp}'}\delta_{\lambda\lambda'}}{2E_{\mathbf{p}}} &\widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}, \mathbf{p}) \big(f^{\mathrm{eq}}_{\mathbf{p},\lambda} ((1 + \sigma f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega)) \\
&\qquad - (1 + \sigma f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega))) \\
&- (1 + \sigma f^{\mathrm{eq}}_{\mathbf{p},\lambda}) (f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega) \\
&\qquad - f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega))) + \mathcal{O}(\omega^2) \\
= -\frac{\delta_{\mathbf{pp}'}\delta_{\lambda\lambda'}}{2E_{\mathbf{p}}} &\widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}, \mathbf{p}) (f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega) - f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega)) + \mathcal{O}(\omega^2).
\end{aligned} \tag{6.21}$$

We expanded around small ω, since we are interested in the limit $\omega \to 0$. Taking this limit, we get the derivative of the distribution function,

$$\begin{aligned}
&\lim_{\omega\to 0} \frac{f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega) - f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega)}{\omega} \\
&= \lim_{\omega\to 0} \frac{f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} + \omega) - f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}})}{\omega} + \lim_{\omega\to 0} \frac{f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}} - \omega) - f^{\mathrm{eq}}_{\lambda}(E_{\mathbf{p}})}{-\omega} \\
&= 2 \cdot \frac{df^{\mathrm{eq}}_{\lambda}}{dE_{\mathbf{p}}}
\end{aligned}$$

$$= 2 \cdot \frac{d}{dE_{\mathbf{p}}} \left(e^{E_{\mathbf{p}}/T} - \sigma\right)^{-1}$$
$$= -\frac{2}{T} f_\lambda^{\text{eq}} \frac{e^{E_{\mathbf{p}}/T}}{e^{E_{\mathbf{p}}/T} - \sigma}$$
$$= -\frac{2}{T} f_\lambda^{\text{eq}} \frac{e^{E_{\mathbf{p}}/T} - \sigma + \sigma}{e^{E_{\mathbf{p}}/T} - \sigma}$$
$$= -\frac{2}{T} f_\lambda^{\text{eq}} \left(1 + \sigma f_\lambda^{\text{eq}}\right), \tag{6.22}$$

and find

$$\lim_{\omega \to 0} \frac{\rho_{\mathbf{pp}',\lambda\lambda'}(\omega)}{\omega} = \frac{1}{TE_{\mathbf{p}}} \delta_{\mathbf{pp}'} \delta_{\lambda\lambda'} f_{\mathbf{p},\lambda}^{\text{eq}} \left(1 + \sigma f_{\mathbf{p},\lambda}^{\text{eq}}\right) \widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}, \mathbf{p}). \tag{6.23}$$

Finally, with Eqs. (6.11) and (6.23) the coefficient $\widetilde{\Gamma}_{\mathbf{pp}',\lambda\lambda'}$ becomes

$$\widetilde{\Gamma}_{\mathbf{pp}',\lambda\lambda'} = \frac{1}{2V} \sum_{\mathbf{p}'',\lambda''} \frac{1}{TE_{\mathbf{p}}} \delta_{\mathbf{pp}''} \delta_{\lambda\lambda''} f_{\mathbf{p},\lambda}^{\text{eq}} \left[1 + \sigma f_{\mathbf{p},\lambda}^{\text{eq}}\right] \widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}, \mathbf{p})$$
$$\cdot \left[\frac{\delta_{\lambda''\lambda'} \delta_{\mathbf{p}''\mathbf{p}'}}{TV} f_{\mathbf{p}'',\lambda''}^{\text{eq}} \left[1 + \sigma f_{\mathbf{p}'',\lambda''}^{\text{eq}}\right]\right]^{-1}$$
$$= \frac{\delta_{\lambda\lambda'} \delta_{\mathbf{pp}'}}{2VTE_{\mathbf{p}}} f_{\mathbf{p},\lambda}^{\text{eq}} \left[1 + \sigma f_\sigma^{\text{eq}}(E_{\mathbf{p}})\right] \widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}, \mathbf{p}) \frac{TV}{f_{\mathbf{p},\lambda}^{\text{eq}} \left[1 + \sigma f_{\mathbf{p},\lambda}^{\text{eq}}\right]}$$
$$= \delta_{\lambda\lambda'} \delta_{\mathbf{pp}'} \frac{1}{2E_{\mathbf{p}}} \widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}, \mathbf{p})$$
$$\equiv \delta_{\lambda\lambda'} \delta_{\mathbf{pp}'} \Gamma_{\mathbf{p},\lambda}^{\text{eq}} \tag{6.24}$$

We call $\Gamma_{\mathbf{p},\lambda}^{\text{eq}}$ the equilibration rate of Φ.

A spectral function (6.20) can be obtained from the discontinuity of a Euclidean correlator $\Delta_{AB}(\mathrm{i}\omega_n, \mathbf{p})$,

$$\widetilde{\rho}_{AB}(p^0, \mathbf{p}) = \frac{1}{\mathrm{i}} \mathrm{Disc} \Delta_{AB}(p^0, \mathbf{p}), \tag{6.25}$$

with

$$\Delta_{AB}(\mathrm{i}\omega_n, \mathbf{p}) \equiv \int_0^\beta d\tau \int d^3x \, e^{\mathrm{i}(\omega_n \tau - \mathbf{p} \cdot \mathbf{x})} \left\langle A(-\mathrm{i}\tau, \mathbf{x}) B(0) \right\rangle. \tag{6.26}$$

Here, $\omega_n = n\pi T$ are the discrete Matsubara frequencies with even and odd integer n for bosons and fermions, respectively. In our case, where $A = \overline{u}_{\mathbf{p},\lambda} J$ and $B = A^\dagger$, we can write

$$\mathrm{Disc} \Delta_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}}, \mathbf{p}) = \overline{u}_{\mathbf{p},\lambda} \mathrm{Disc} \Sigma(E_{\mathbf{p}}, \mathbf{p}) u_{\mathbf{p},\lambda}, \tag{6.27}$$

with

$$\Sigma(\mathrm{i}\omega_n, \mathbf{p}) \equiv \int_0^\beta d\tau \int d^3x \, e^{\mathrm{i}(\omega_n\tau - \mathbf{p}\cdot\mathbf{x})} \left\langle J(-\mathrm{i}\tau, \mathbf{x})\overline{J}(0)\right\rangle . \tag{6.28}$$

We can identify $\Sigma(\mathrm{i}\omega_n, \mathbf{p})$ as the Euclidean Φ self-energy at leading order in H_{int} and conclude, that the equilibration rate of particles is proportional to the discontinuity of their self-energy,

$$\Gamma^{\mathrm{eq}}_{\mathbf{p},\lambda} = \frac{1}{2\mathrm{i}E_{\mathbf{p}}}\overline{u}_{\mathbf{p},\lambda}\mathrm{Disc}\Sigma(E_{\mathbf{p}}, \mathbf{p})u_{\mathbf{p},\lambda}. \tag{6.29}$$

Weldon [33] found the same relation for the equilibration of W-bosons. However, he used Boltzmann equations and his result is only valid at leading order in *all* interactions. We point out that the separation of time scales, which we considered here, cannot be applied in the case of W-boson production.

6.2 Uncharged particles

Let us now turn our attention to uncharged particles like, for example, Majorana fermions. These particles are their own antiparticles. Their field operator Φ has to be equal to its charge conjugate, $\Phi = \Phi^C \equiv S\left(\overline{\Phi}\right)^T$ and $\overline{\Phi} = \Phi^T S$, with an appropriate [22] matrix S. This implies $c = d$ and $v = u^C$ in Eq. (6.7). In this case, we can write the interaction as

$$\mathcal{L}_{\mathrm{int}} = \overline{\Phi}I = \overline{I}\Phi. \tag{6.30}$$

Now, we will determine $\widetilde{\Gamma}_{\mathbf{pp'},\lambda\lambda'}$ for an uncharged particle species. The susceptibility is the same as for charged particles, Eq. (6.11). For the time derivative of f we now obtain

$$\begin{aligned}
\dot{f}_{\mathbf{p},\lambda,\mathrm{uncharged}} &= \mathrm{i}[H, f_{\mathbf{p},\lambda}] \\
&= \mathrm{i}e^{\mathrm{i}Ht}e^{-\mathrm{i}H_0t}\left[\int d^3\mathbf{x}\left(\overline{I}_{\mathrm{I}}(x)\Phi_{\mathrm{I}}(x)\right), c^\dagger_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda}\right]e^{\mathrm{i}H_0t}e^{-\mathrm{i}Ht} \\
&= \int d^3\mathbf{x}\sum_{\mathbf{p}',\lambda'}\frac{\mathrm{i}}{\sqrt{2E_{\mathbf{p}'}V}}e^{\mathrm{i}Ht}e^{-\mathrm{i}H_0t}\Big(\overline{I}_{\mathrm{I}}(x)e^{-\mathrm{i}p'x}\,u_{\mathbf{p}',\lambda'}[c_{\mathbf{p}',\lambda'}, c^\dagger_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda}] \\
&\qquad + \overline{I}_{\mathrm{I}}(x)e^{\mathrm{i}p'x}\,v_{\mathbf{p}',\lambda'}[c^\dagger_{\mathbf{p}',\lambda'}, c^\dagger_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda}]\Big)e^{\mathrm{i}H_0t}e^{-\mathrm{i}Ht} \\
&= \frac{\mathrm{i}}{\sqrt{2E_{\mathbf{p}}V}}\int d^3\mathbf{x}\overline{I}(x)\left[e^{-\mathrm{i}px}u_{\mathbf{p},\lambda}c_{\mathbf{p},\lambda} - e^{\mathrm{i}px}v_{\mathbf{p},\lambda}c^\dagger_{\mathbf{p},\lambda}\right].
\end{aligned} \tag{6.31}$$

[22] It is $S^{-1} = S^T = -S$.

To obtain the equilibration rate, we follow exactly the steps (6.14)-(6.24), but replace $\overline{J}u$ by $\overline{I}u$ and $\overline{u}J$ by $\overline{I}v$ everywhere. Recalling that $v = u^C$ and $I = I^C$, we have

$$\begin{aligned}\overline{I}v &= -\left(\overline{I}v\right)^T \\ &= -v^T\left(\overline{I}\right)^T \\ &= v^T SI \\ &= (u^C)^T SI \\ &= \overline{u}S^T SI \\ &= \overline{u}I,\end{aligned} \tag{6.32}$$

and find

$$\Gamma^{\text{eq}}_{\mathbf{p},\lambda} = \frac{1}{2E_{\mathbf{p}}}\,\widetilde{\rho}_{\overline{u}I,\overline{I}u}(E_{\mathbf{p}},\mathbf{p}). \tag{6.33}$$

The self-energy of the uncharged particle Φ is simply obtained by replacing J by I in Eq. (6.28). Therefore, relation (6.29) between equilibration rate and self-energy holds for uncharged particles, too.

6.3 Relation to the production rate

Similar to the equilibration rate, one can define a production rate of Φ. It is the rate at which Φ particles are produced if none of them are present, so for $|f| \ll 1$

$$\dot{f}_{\mathbf{p},\lambda} = \Gamma^{\text{pro}}_{\mathbf{p},\lambda}. \tag{6.34}$$

We expanded around $f = 0$ and, therefore, neglected all terms at linear or higher orders in f. In general, there would, like in Eq. (6.3), appear terms linear in the X_a on the right-hand side of Eq. (6.34), because the interactions of Φ with the plasma will drive the plasma slightly away from equilibrium. We assume that this effect is sufficiently small so that we do not have to consider the evolutions of the X_a here. It has been shown [34, 35] that in this setup the production rate is related to the correlation funcions of J, too, [23]

$$\Gamma^{\text{pro}}_{\mathbf{p},\lambda} = \frac{1}{2E_{\mathbf{p}}} f^{\text{eq}}_{\sigma}(E_{\mathbf{p}})\widetilde{\rho}_{\overline{u}J,\overline{J}u}(E_{\mathbf{p}},\mathbf{p}). \tag{6.35}$$

If we compare this result to Eq. (6.24), we obtain the relation

$$\Gamma^{\text{pro}} = f^{\text{eq}}\Gamma^{\text{eq}}. \tag{6.36}$$

[23] In case of uncharged particles, J is replaced by I in Eq. (6.35).

This relation holds for charged as well as for uncharged particles at leading order in their coupling to the plasma and to all orders in all other interactions within the plasma. It can also be obtained from Refs. [36, 37], yet it is not stated there explicitly. Furthermore, Ref. [36] makes an ansatz for the nonequilibrium density matrix.

The heavy Majorana neutrino N can play the role of the particle Φ. Its interaction (4.3) is of the form (6.1) with $J = h\tilde{\varphi}^\dagger \ell$. We can convert it into the form (6.30). Since $N = N^C$, we have [24]

$$\begin{aligned}
\mathcal{L}_{\text{int}} &= -\overline{N}J - \overline{J}N \\
&= -\overline{N}J - \overline{J}N^C \\
&= -\overline{N}J - \overline{J}S\left(\overline{N}\right)^T \\
&= -\overline{N}J - \left(\overline{N}S^T\left(\overline{J}\right)^T\right)^T \\
&= -\overline{N}J + \overline{N}S^T\left(\overline{J}\right)^T \\
&= -\overline{N}J - \overline{N}S\left(\overline{J}\right)^T \\
&= -\overline{N}J - \overline{N}J^C \\
&= -\overline{N}I.
\end{aligned} \tag{6.37}$$

In the last step, we defined $I = J + J^C$. Therefore, the assumption in Sec. 4.1.1 was justified.

[24]Note, that N and J are fermionic fields, so that changing their order gives an additional factor -1.

7 Numerical results [25]

In order to numerically solve the rate equations (4.6) and (4.7), we eliminate the term proportional to the Hubble rate on the left-hand side by normalizing the number densities to T^3, i.e. we define new quantities $X = n/T^3$. Then

$$\begin{aligned}\frac{d}{dt}\left(XT^3\right) + 3Hn &= \frac{dX}{dt}T^3 + X\frac{d}{dt}T^3 + 3Hn \\ &= \frac{dX}{dt}T^3 + X\,3T^2\frac{dT}{dt} + 3Hn \\ &= \frac{dX}{dt}T^3 - 3HXT^3 + 3Hn \\ &= \frac{dX}{dt}T^3. \end{aligned} \tag{7.1}$$

We also substitute the time t by the parameter $z = M_N/T$,

$$\frac{dX}{dt} = \frac{dX}{dz}\frac{dz}{dt} = \left(-\frac{M_N}{T^2}\right)\frac{dX}{dz}\frac{dT}{dt} = \frac{dX}{dz}Hz, \tag{7.2}$$

so that the left-hand sides of Eq. (4.6) and (4.7) take the form

$$\left(\frac{d}{dt} + 3H\right)n = \frac{dX}{dz}HzT^3. \tag{7.3}$$

The Hubble rate in a radiation dominated universe is [21]

$$H \simeq 1.66 g_*^{1/2}\frac{T^2}{M_p} = 1.66 g_*^{1/2}\frac{M_N^2}{M_p z^2}. \tag{7.4}$$

Then,

$$\begin{aligned}\frac{\Gamma_0}{Hz} &= \frac{\Gamma_0}{1.66 g_*^{1/2}\frac{M_N^2}{M_p z^2}z} \\ &= \frac{\Gamma_0 M_p}{1.66 g_*^{1/2} M_N^2}\,z \\ &\equiv Kz. \end{aligned} \tag{7.5}$$

The parameter $K \equiv \Gamma_0/H(z=1)$ is called the washout strength. It is related to the effective light neutrino mass (2.14) by

$$K = \frac{\frac{|h_{11}|^2 M_N}{8\pi}}{1.66 g_*^{1/2}\frac{M_N^2}{M_p}}$$

[25] This section is mainly based on Ref. [24].

$$= \frac{M_p}{8\pi v^2 \cdot 1.66 g_*^{1/2}} \frac{|h_{11}|^2 v^2}{M_N}$$
$$\simeq 1.08 \cdot 10^3 \, \frac{\tilde{m}_1}{\text{eV}}. \tag{7.6}$$

In the last step the single-flavor approximation enters, i.e. we used $h_{12} = h_{13} = 0$. All in all, we can now rewrite the differential equations (4.6) and (4.7) in terms of the quantities $X(z)$

$$\frac{dX_N}{dz} = -zK \left(X_N - X_N^{\text{eq}}\right) \tag{7.7}$$
$$\frac{dX_{B-L}}{dz} = \epsilon_1 zK \left(X_N - X_N^{\text{eq}}\right) - \frac{T^2}{\pi^2 \Xi} z^3 K_1(z) K X_{B-L}. \tag{7.8}$$

Similarly, we normalize the relativistic corrections, too. There we write

$$X_{u_i} = \frac{M_N^{2i}}{T^{2i+3}}. \tag{7.9}$$

To determine the equilibrium densities, we have to calculate integrals of the form

$$\frac{2}{T^{2i+3}} \int \frac{d^3\mathbf{p}_N}{(2\pi)^3} \left|\mathbf{p}_N\right|^{2i} \exp\left(-\frac{\sqrt{\mathbf{p}_N^2 + M_N^2}}{T}\right), \tag{7.10}$$

where $i \in \mathbb{N}_0$. We work in polar coordinates where it takes the form

$$\frac{1}{\pi^2 T^{2i+3}} \int_0^\infty d\left|\mathbf{p}_N\right| \left|\mathbf{p}_N\right|^{2i+2} \exp\left(-\frac{\sqrt{\left|\mathbf{p}_N\right|^2 + M_N^2}}{T}\right). \tag{7.11}$$

Then we substitute $x = \sqrt{|\mathbf{p}_N| + M_N^2}/T$, so that $|\mathbf{p}_N| = T\sqrt{x^2 - z^2}$ and $d\,|\mathbf{p}_N| = T^2 x \,|\mathbf{p}_N|^{-1}\, dx$, which leads to

$$\frac{1}{\pi^2 T^{3+2i}} \int_{x(0)}^{x(\infty)} dx T^2 x T^{2i+1} (x^2 - z^2)^{i+1/2} e^{-x}$$
$$= \frac{1}{\pi^2} \int_z^\infty dx x (x^2 - z^2)^{i+1/2} e^{-x}$$
$$= \frac{2^{i+1}(i + 1/2)!}{(\pi)^{5/2}} z^{i+2} K_{i+2}(z), \tag{7.12}$$

where $K_{i+2}(z)$ is a modified Bessel function of the second kind. Using (7.12), we find

$$X_N^{\text{eq}} = \frac{1}{\pi^2} z^2 K_2(z) \tag{7.13}$$

$$X_{u_1}^{\rm eq} = \frac{3}{2\pi^2} z^3 K_3(z) \tag{7.14}$$

$$X_{u_1}^{\rm eq} = \frac{45}{8\pi^2} z^4 K_4(z) \tag{7.15}$$

$$X_{u_1}^{\rm eq} = \frac{525}{16\pi^2} z^5 K_5(z) \tag{7.16}$$

$$X_{u_1}^{\rm eq} = \frac{33075}{128\pi^2} z^6 K_6(z) \tag{7.17}$$

$$X_{u_1}^{\rm eq} = \frac{654885}{128\pi^2} z^7 K_7(z). \tag{7.18}$$

The $B-L$ - asymmetry can be written as the product of two quantities, namely the CP-asymmetry ϵ_1 in the heavy neutrino decays and the efficiency factor $\kappa(t)$, that is defined as

$$N_{B-L} = \frac{3}{4}\epsilon_1 \kappa_f \tag{7.19}$$

where $\kappa_f = \kappa(\infty)$ and $N_{B-L} = n_{B-L}/n_\gamma$ is the $B-L$-asymmetry density normalized to the photon density. Then, with Eqs. (3.28),(3.31) and $n_\gamma = 2\zeta(3)T^3/\pi^2$ we find a short expression for the asymmetry,

$$\eta_B \simeq 0.01 \epsilon_1 \kappa_f, \tag{7.20}$$

where, using Eq. (5.21), κ is the solution of the differential equation

$$\frac{d\kappa}{dz} = \frac{2\pi^2}{3\zeta(3)} zK \left[(X_N - X_N^{\rm eq}) + \sum_{i=1}^{5} (-1)^i \frac{M_N^2}{z^2} \left(X_{u_i} - X_{u_i}^{\rm eq} \right) \right] - T^2 \Xi^{-1} \left[\frac{1}{\pi^2} z^3 K_1(z)\, K + 0.0868 \left(\frac{\bar{m}}{\rm eV} \right)^2 \frac{M_N}{\rm GeV} z^{-2} \right] \kappa. \tag{7.21}$$

We will always start our calculations with $z = 1$, since the nonrelativistic approximation cannot be expected to work at earlier times, and we will always assume a vanishing $B-L$-asymmetry. For the heavy neutrino number density and the relativistic corrections, we will study different initial conditions.

Fig. 5 shows the time evolution of the efficiency factor κ for different washout strengthes K. Here we did not include relativistic or radiative corrections and no spectator processes. The efficiency factor reaches its final value at latest at $z = 20$. This result remains true also if we include the above mentioned corrections as long as $\Delta L = 2$ scatterings are neglected. Thus, it is sufficient to solve Eq. (7.21) numerically on the intervall $1 \leq z \leq 20$. However, if $\Delta L = 2$ scatterings are included, this is not sufficient any more. Therefore, let us use Eq. (7.21) to calculate $\kappa(20)$. Then, for $z \geq 20$, we use

$$\frac{d\kappa}{dz} = -0.0868\, T^2 \Xi^{-1} \left(\frac{\bar{m}}{\rm eV} \right)^2 \frac{M_N}{\rm GeV} z^{-2} \kappa, \tag{7.22}$$

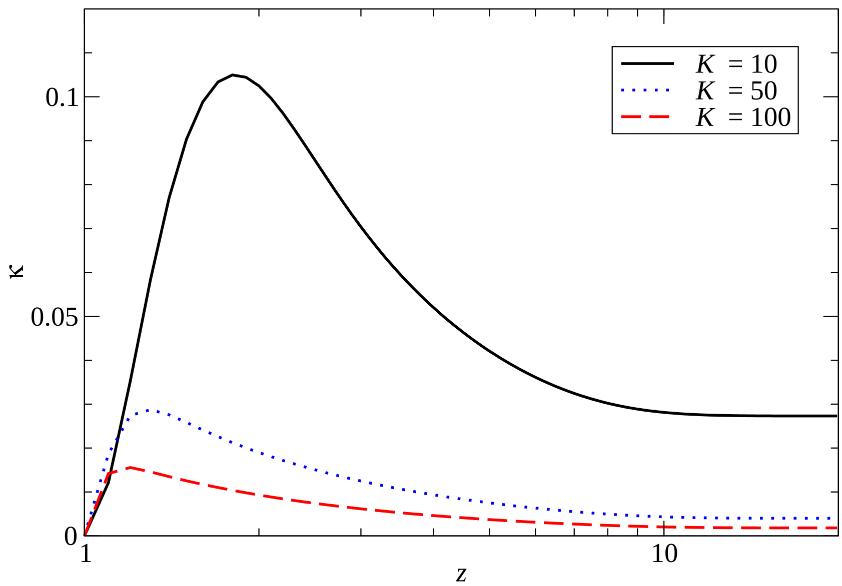

Figure 5: Time evolution of the efficiency factor κ for different washout strengthes K. Here, we did not include relativistic or radiative corrections and no spectator processes. The efficiency factor reaches its final value at latest at $z = 20$.

which we can solve analytically. The solution is

$$\kappa(z) = \kappa(\infty) \cdot \exp\left(0.0868\, T^2 \Xi^{-1} \left(\frac{\bar{m}}{\text{eV}}\right)^2 \frac{M_N}{\text{GeV}}\, z^{-1}\right). \tag{7.23}$$

The requirement of κ being continuous immediately gives

$$\kappa(\infty) = \kappa(20) \cdot C(M_N, \bar{m}), \tag{7.24}$$

$$\text{with} \quad C(M_N, \bar{m}) = \exp\left(-0.0868\, T^2 \Xi^{-1} \left(\frac{\bar{m}}{\text{eV}}\right)^2 \frac{M_N}{\text{GeV}}\, z^{-1}\right).$$

The results of a calculation of $C(M_N, \bar{m})$ for different parameters are given in table 2. There one can see nicely that the $\Delta L = 2$ scatterings dominate the washout for large $M_N \gtrsim 10^{14}$ GeV ($\kappa(\infty) \approx C$) and are negligible for $M_N \lesssim 10^{13}$ GeV ($\kappa(\infty) \approx \kappa(20)$).

7.1 Relativistic corrections

We calculate the efficiency factor κ including the $\mathcal{O}(\mathbf{v}^2)$ relativistic corrections and compare it to κ_{NR} which we obtain from the pure nonrelativistic approach. For

M_N/GeV	$C(M_N, \bar{m})$
10^{15}	$e^{-40} \approx 0$
10^{14}	e^{-4}
10^{13}	$e^{-0.4}$
10^{12}	$e^{-0.04} \approx 1$

Table 2: Values of C for $\bar{m} = 0.2$ eV and different masses M_N. $C = 1$ corresponds to the approximation where $\Delta L = 2$ scatterings are ignored. Thus, this approximation is sufficient for $M_N \lesssim 10^{13}$ GeV.

both κ and $\kappa_{\rm NR}$ we work at leading order in all couplings and use thermal initial conditions for n_N and u_1, respectively. In Fig. 6, we show the results in terms of the ratio $(\kappa - \kappa_{\rm NR})/\kappa$ at $z = 20$ as a function of the washout strength K for different temperature regimes. The plot shows that the relativistic corrections are smaller than 3% for $K \gtrsim 5$. In Fig. 7, we show higher order relativistic corrections in the regime $10^8 \lesssim T/\text{GeV} \lesssim 10^{11}$. We see that the nonrelativistic approximation breaks down for small K, because there the higher order relativistic corrections grow large. For larger K it does work very well. At $K = 10$, for example, the effect of the $\mathcal{O}(\mathbf{v}^2)$ corrections is 0.78%, while that of the $\mathcal{O}(\mathbf{v}^4)$ is 0.04% and that of the $\mathcal{O}(\mathbf{v}^6)$ even only 0.02%.

In Fig. 8, we compare the size of the relativistic corrections for different initial conditions. For thermal initial conditions, which we used before, the relativistic corrections remain very small even for small K. In the extreme case of zero initial values for n_N and u_1, however, the situation is different. For $K \gtrsim 5$ the $\mathcal{O}(\mathbf{v}^2)$ relativistic corrections remain small ($\lesssim 1.6\%$ for the parameters in Fig. 8), but for $K \lesssim 4$ the nonrelativistic approximation clearly breaks down [26]. These results are valid in the range $10^8 \lesssim T/\text{GeV} \lesssim 10^{11}$. For higher temperatures the effects are weaker.

That already the first relativistic corrections are so small, shows that the nonrelativistic approximation works very well. Beyond that, it indicates that also the approach of assuming kinetic equilibrium, which has been studied in detail by many papers [27], should work similarly well. In fact, the deviation from kinetic equilibrium corresponds to the difference $u_1 - u_1^{\rm eq}$, which we found to be small. Therefore it is interesting to compare our results to those obtained with the assumption of kinetic equilibrium. The main difference is an additional factor $K_1(z)/K_2(z)$ in the coefficients Γ_N and $\Gamma_{B-L,N}$ if one starts with the assumption of kinetic equilibrium. The washout rate of Ref. [21] also differs from ours,

[26] We point out that vanishing n_N and u_1 at $z = 1$ is somewhat unphysical because by that time for $K \gtrsim 1$ a substantial number of right handed neutrinos will have been thermally produced.

[27] See for example Ref. [21].

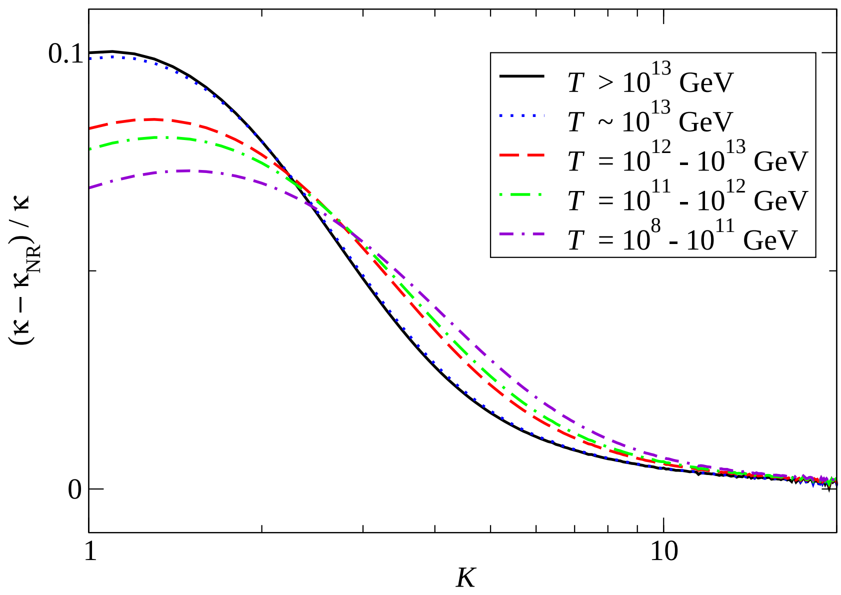

Figure 6: Relative size of the $\mathcal{O}(\mathbf{v}^2)$ relativistic corrections to the efficiency factor at $z = 20$. Here we used thermal initial conditions for n_N and u_1. The temperature regimes correspond to the different values of T^2/Ξ according to table 1. For $T \lesssim 10^{12}$ GeV we opted for the values corresponding to the right-handed neutrinos decaying into τ leptons.

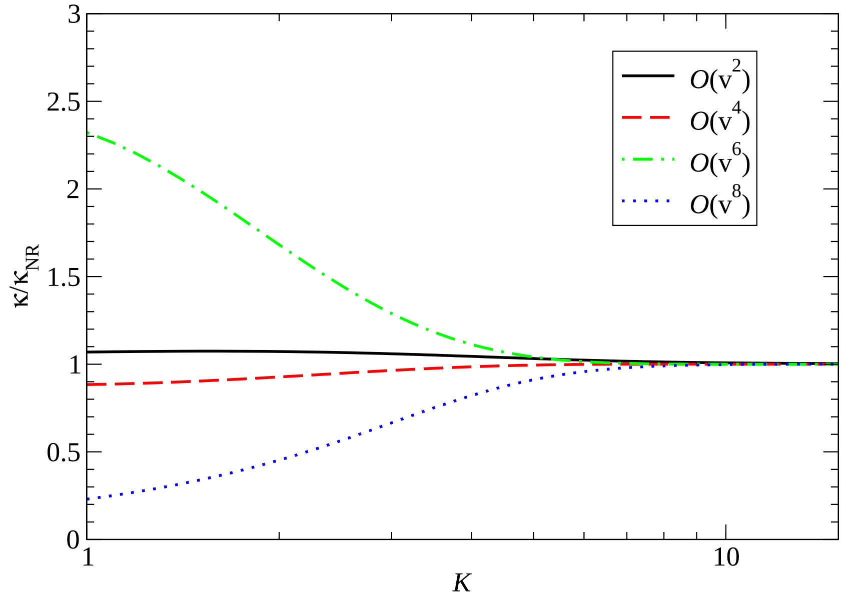

Figure 7: Relative size of higher order relativistic corrections in the regime $10^8 \lesssim T/\text{GeV} \lesssim 10^{11}$, assuming the right-handed neutrinos to decay into τ leptons. We used thermal initial conditions for n_N and all u_i.

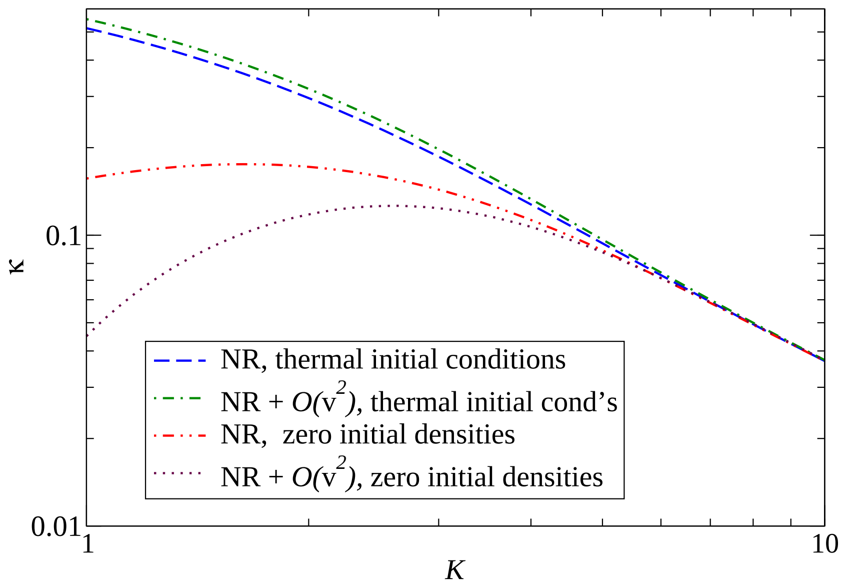

Figure 8: Efficiency factors at $z = 20$ for thermal and zero initial conditions for n_N and u_1. The curves have been obtained in the regime $10^8 \lesssim T/\text{GeV} \lesssim 10^{11}$, assuming the right-handed neutrinos to decay into τ leptons.

however, this discrepancy is not due to the kinetic equilibrium approach, but due to the fact that Ref. [21] used Boltzmann statistics for leptons and Higgs bosons to obtain a relation between the respective particle number densities and n_{B-L} (c.f. Eqs. (4.122)). We ignore this for a moment and only consider the changes due to the kinetic equilibrium approach. Then we find indeed, the difference between the nonrelativistic approach and that one assuming kinetic equilibrium is quite small, e.g. for $K = 8$ and $T = 10^{10}$ GeV they deviate by 0.87%. That is similar in size as the $\mathcal{O}(\mathbf{v}^2)$ corrections. If we add the $\mathcal{O}(\mathbf{v}^2)$ corrections in our calculation and compare that result to the kinetic equilibrium approach, the difference is even smaller, i.e. in the case mensioned above only 0.3%.

7.2 Statistics

Let us have a closer look at the discrepancy of the washout rate (4.127) which we obtained using Fermi-Dirac and Bose-Einstein statistics for leptons and Higgs bosons, respectively, in the calculation (4.122) and that of Ref. [21], which matches Eq. (4.129) and was obtained using classical Maxwell-Boltzmann statistics for leptons and Higgs bosons. A comparison shows a descrepancy of at least 20%, as we show in Fig. 9.

In principle, it is not necessary to make a nonrelativistic approximation or as-

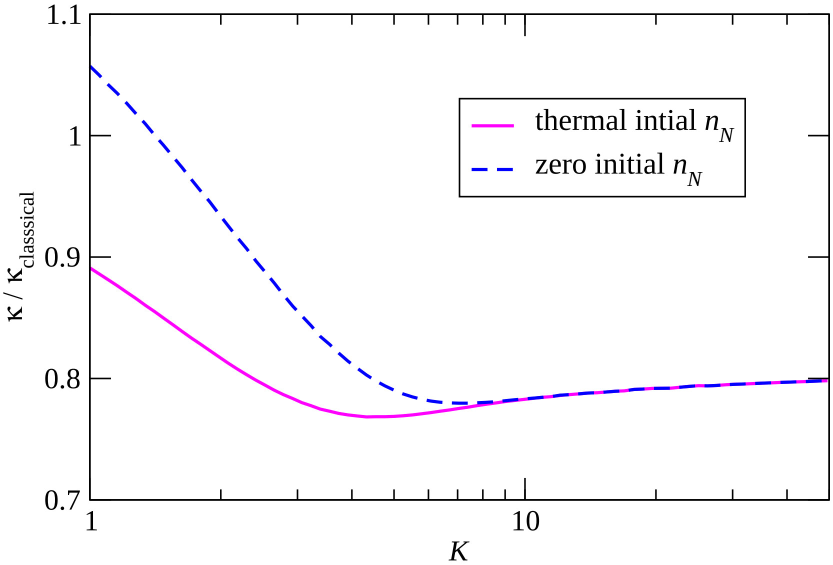

Figure 9: Ratio of efficiency factors computed with Bose-Einstein and Fermi-Dirac distributions in the washout term and with the classical Maxwell-Boltzmann statistics. Here we used the nonrelativistic approximation, evaluated at $z = 20$, without including any of the correction terms or spectator processes. Even at very large K the error caused by using classical statistics is of order 20%.

sume kinetic equilibrium, but one can also solve the momentum dependent Boltzmann equations. One would have to solve one equation for each momentum, but they are independent from one another, c.f. Eq. (6.24). This has been done in Refs. [38–40]. The results were compared to those using the kinetic equilibrium ansatz. Ref. [38] found a discrepancy of about 15% for $K > 5$, Ref. [39] found a discrepancy of 20% for $K > 1$. However, both groups used classical statistics with the kinetic equilibrium density and quantum statistics with the momentum dependent Boltzmann equations. Therefore, the discrepancies they found are mostly due to the statistics, but *not* due to taking into account the momentum distribution of the heavy neutrinos.

7.3 Radiative corrections

Now we want to study the size of the radiative corrections. We use thermal initial conditions for n_N and all u_i. As the renormalization scale for the Standard Model couplings we choose M_N.

First, we turn our attention to the corrections of $T^2\Xi$ in the coefficient Γ_{B-L}. We expect them to have the largest effect because they are of $\mathcal{O}(g)$. In Fig. 10 we show the results for the efficiency factors obtained from the purely nonrelativistic approximation where κ contains the $\mathcal{O}(g)$ corrections while $\kappa_{\rm LO}$ does not. The effect of these corrections is $\lesssim 3\%$ for high temperatures, $T > 10^{13}$ GeV, and much smaller for smaller temperatures.

Next we include the $\mathcal{O}(g^2)$ radiative corrections in the coefficients a_i and b_i (see Eqs. (5.11)-(5.14)). Since they are suppressed by $\mathcal{O}(\mathbf{v}^{2i})$, we have to consistently add the respective relativistic correction terms u_i in our calculation. For example, if we include the term $b_4 u_1$, then we should also include the term $a_0 u_5$ since both are of $\mathcal{O}(g^2\mathbf{v}^{10})$. In Fig. 11 we show results normalized to the efficiency factor without radiative corrections $\kappa_{\rm LO}$. In the strong washout regime, $K \gg 1$, the effects of these radiative corrections are smaller than the relativistic $\mathcal{O}(\mathbf{v}^2)$ corrections. For $K \lesssim 8$, the effect of the $\mathcal{O}(g^2\mathbf{v}^{10})$ contributions grows large. This supports the impression of Ref. [41] that the expansion of radiative corrections in powers of $\mathbf{v}^2$ does not converge in the regime where $\mathbf{v}$-dependent corrections are important. Nevertheless, the effect of the $\mathcal{O}(g^2)$ and the $\mathcal{O}(g^2\mathbf{v}^4)$ corrections does remain small for small K. The reason might be that the factor a_2 contains only the Higgs self-coupling which is small itself.

7.4 Neutrino mass bounds

The way to find an upper bound for the light neutrino mass scale is to determine the region in the parameter space where the maximal producable baryon asymmetry $\eta_B^{\rm max}$ is at least as large as the observed value (1.1). To find $\eta_B^{\rm max}$,

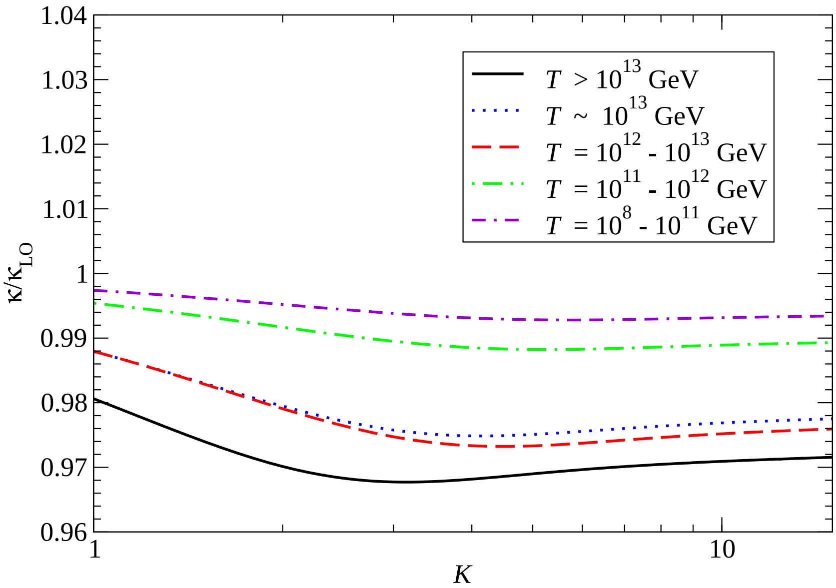

Figure 10: Relative size of the $\mathcal{O}(g)$-radiative corrections evaluated at $z = 20$. Here, we used the nonrelativistic approximation with thermal initial conditions for n_N. The temperature regimes correspond to the different values of T^2/Ξ according to table 1. For $T \lesssim 10^{12}$ GeV we used for the values corresponding to the right-handed neutrinos decaying into τ leptons.

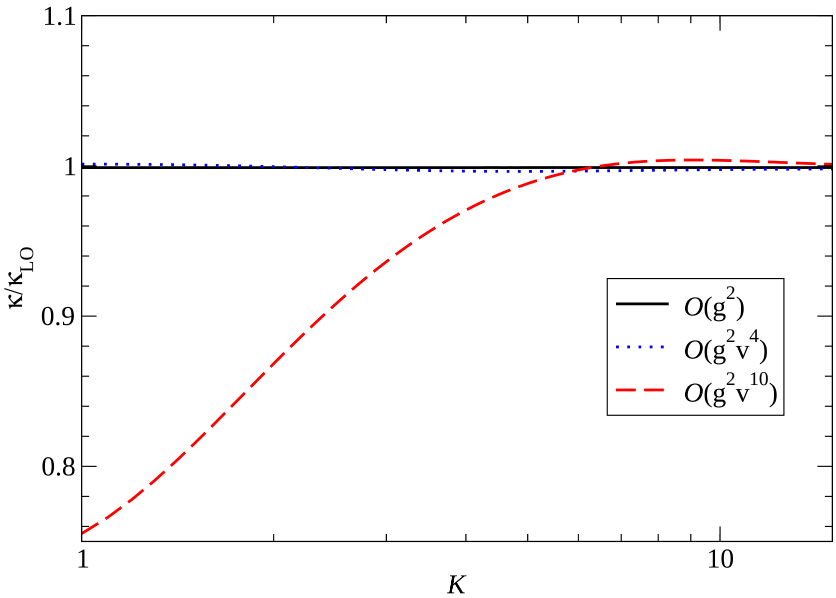

Figure 11: Relative size of the $\mathcal{O}(\mathbf{v}^2)$ corrections at $z = 20$. The different curves correspond to different orders included in the coefficients a_i and b_i. Both κ and κ_{LO} contain the respective relativistic corrections. The renormalization scale of the Standard Model coupling is chosen as the mass of the heavy neutrinos and we used thermal initial conditions for n_N and all u_i.

we estimate ϵ_1 in Eq. (7.20) by its upper bound (4.27). Then $\eta_B^{\max}$ is a function of M_N, $\tilde{m}_1$ and $\bar{m}^2 = \sum_i m_i^2$ [28]. While keeping $\bar{m}$ constant, we search for all pairs $(M_N, \tilde{m}_1)$ where the maximal producable baryon asymmetry is at least as large as the observed value. Repeating this procedure with larger and larger $\bar{m}$ will let the allowed region shrink more and more until no allowed $(M_N, \tilde{m}_1)$-pairs can be found [29]. The corresponding $\bar{m}$ can be considered an upper bound.

To find the correct bound the $\Delta L = 2$-scattering term in the washout rate $\Gamma_{B-L,N}$ is of vital importance. Since it is, unlike the inverse decays, not exponentially suppressed in z, it dominates the washout for large $M_N \gtrsim 10^{14}$ GeV and causes the efficiency factor to decrease faster than the CP-asymmetry increases with M_N.

Ref. [42] states the bound $\bar{m}^{\max} = 0.2$ eV. To obtain this bound they used the kinetic equilibrium approximation, assumed Boltzmann statistics for all particles, and did not include spectator processes. Furthermore, they used $\eta_B^{\text{CMB}} = 3.6 \cdot 10^{-10}$ and their bound on the CP-asymmetry $\epsilon_1^{\max}$ differs from Eq. (4.27). If we adapt their settings to our setup, but use the nonrelativistic approximation instead of assuming kinetic equilibrium, we find a similar bound, $\bar{m}^{\max} = 0.21$ eV. Changing only $\epsilon_1^{\max}$ back to Eq. (4.27) weakens the bound drastically to $\bar{m}^{\max} = 0.26$ eV. Using the value (1.1) strengthens the bound to $\bar{m}^{\max} = 0.23$ eV, which is in agreement with Ref. [27], who first noticed the mistake in the calculation of $\epsilon_1^{\max}$ in Ref. [42]. Including spectator processes strengthens the bound further to $\bar{m}^{\max} = 0.21$ eV. Switching to quantum statistics for leptons and Higgs bosons where necessary in the washout rate again slightly strengthens the bound about 5% to $\bar{m}^{\max} = 0.2$ eV. This is much less than the effect on κ_f, which we found to be at least 20%. Effects of the radiative and relativistic corrections of Sects. 4 and 5 are even much smaller than 5%.

For $\bar{m} \gtrsim 0.2$ eV only heavy neutrino masses around 10^{13} GeV are allowed. In this region, flavor effects are negligible. Therefore, the single-flavor approximation should give reliable results for an upper limit of the light masses. Spectator processes, however, are important in this region. It also implies that, if $\bar{m} \sim \bar{m}^{\max}$, the final $B - L$ asymmetry is produced first, and afterwards it is partially converted to a baryon asymmetry by the electroweak sphaleron processes. However, the conversion then starts at temperatures $T \lesssim 10^{12}$ GeV, where the relation (3.28) does not hold. It would be interesting to check if a modification of (3.28) would have a significant effect on the mass bound.

An overview of the current situation regarding neutrino masses is given in

[28] The measured mass squared differences Δm_{sol}^2 and Δm_{atm}^2 together with the assumption of normal (or inverse) hierarchy only leave one mass parameter in the light sector undetermined. We choose this parameter to be $\bar{m}$. Other common choices are m_1 or m_3.

[29] Note, that increasing $\bar{m}$ inevitably leads to quasidegenerate light neutrinos. Due to the fact that $m_1 \leq \tilde{m}_1 \leq m_3$, the allowed $\tilde{m}_1$ interval shrinks even without claiming successful Leptogenesis.

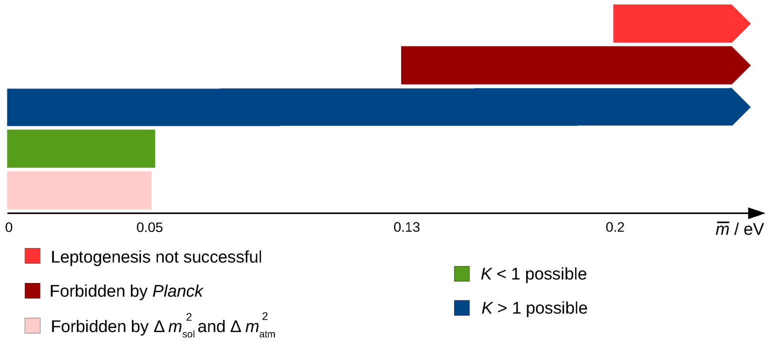

Figure 12: Overview of current information about allowed and excluded values of $\bar{m}$. The strongest upper bound comes from *Planck*'s observation of the cosmic microwave background. It is also interesting to note that the weak washout regime is almost completely ruled out by the neutrino oscillation data.

Fig. 12. It is interesting to mention that combining Eq. (2.24), (7.6) and (7.35) gives a lower bound on the washout strength K,

$$\begin{aligned} K \cdot \mathrm{eV} &\gtrsim 1.1 \cdot 10^3 m_1 \\ &= 1.1 \cdot 10^3 \sqrt{\bar{m}^2 - \Delta m^2_{\mathrm{atm}} - 2\Delta m^2_{\mathrm{sol}}}. \end{aligned} \tag{7.25}$$

Then on the other hand,

$$\bar{m} \lesssim \sqrt{\Delta m^2_{\mathrm{atm}} + 2\Delta m^2_{\mathrm{sol}} + 1.21 \cdot 10^{-6} K^2 (\mathrm{eV})^2}, \tag{7.26}$$

and $0 < K < 1$ implies

$$\sqrt{\Delta m^2_{\mathrm{atm}} + 2\Delta m^2_{\mathrm{sol}}} < \bar{m} < \sqrt{\Delta m^2_{\mathrm{atm}} + 2\Delta m^2_{\mathrm{sol}} + 1.21 \cdot 10^{-6} (\mathrm{eV})^2}. \tag{7.27}$$

With the values (2.1) and (2.2) from neutrino oscillation data, this is an extremely small interval, which means that the weak washout scenario $K \ll 1$ is almost ruled out and the strong washout regime, where the considerations of this thesis hold, is favored.

7.4.1 Neutrino mass bounds from experiments

One possibility to measure neutrinos is to look at beta decays. This is what, i.a., the KATRIN experiment does [43]. They analyze the Tritium decay channel

$$\mathrm{T} \to {}^3\mathrm{He} + e^- + \bar{\nu}_e. \tag{7.28}$$

In the rest frame of the Tritium atom T, energy conservation gives approximately (with $m_{\text{He}} \gg m_e$)

$$m_{\text{T}} = m_{\text{He}} + E_e + E_{\bar{\nu}_e}. \tag{7.29}$$

The masses of the Tritium atom and the 3-Helium atom are known and the energy of the electron is measured, so that

$$E_{\bar{\nu}_e} = m_T - m_{\text{He}} + E_e. \tag{7.30}$$

The smallest $E_{\bar{\nu}_e}$ then is an upper bound on the "antielectron neutrino mass". However, such a mass does not really exist, because the neutrino produced in this decay is in a flavor eigenstate and not in a mass eigenstate. Therefore, $m_{\bar{\nu}_e} = \sum_i V_{ei}\, m_i$ with the PNMS-matrix V. The coefficients V_{ei} then give the probability to measure the mass m_i. The upper bound on $m_{\bar{\nu}_e}$ has therefore to be transformed into a bound on the lightest neutrino mass, i.e. m_1 in the normal hierarchy case. However, the PMNS-matrix is not exactly known. Thus, by performing this transformation additional uncertainties arise [44].

There is also the possibility to obtain mass bounds from cosmology. The contribution of neutrinos to the energy density of the universe is [23, §2.4.4]

$$\Omega_\nu = \frac{\rho_\nu}{\rho_{\text{cr}}} = \sum_i \frac{\rho_{\nu_i}}{\rho_{\text{cr}}}, \tag{7.31}$$

with the energy density of each neutrino mass eigenstate

$$\rho_{\nu_i} = 2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{E_{\nu_i}}{e^{E_{\nu_i}/T} - 1} \tag{7.32}$$

and the critical energy density $\rho_{\text{cr}} = 3H_0^2/8\pi G$. Here, $G = 1/M_{\text{Pl}}^2$ is the gravitational constant and $H_0 = 2.133 \cdot 10^{-33}\, h$ eV is today's Hubble rate [2, §1.2]. Let us assume that all three neutrinos are nonrelativistic today. Then we can write the energy density as the product of the mass times the number density, $\rho_{\nu_i} = m_i\, n_{\nu_i}$. We can relate the neutrino number density to that of the photons. Due to the fact that neutrinos are fermions while the photons are bosons, the number density of the neutrinos will be smaller by a factor 3/4 [23, §2.4.4]. Since the number densities are proportional to the temperature cubed, we can write $n_{\nu_i} = 3/4\, (T_{\nu_i}/T_\gamma)^3\, n_\gamma$. The relation between the temperatures is $T_\nu/T_\gamma = (4/11)^{1/3}$ (c.f. Sec. 3). Putting all together finally gives [23, §2.4.4]

$$\Omega_\nu \approx \frac{\sum_i m_i}{94 h^2 \text{eV}}. \tag{7.33}$$

Cosmology therefore constraints the sum of the neutrino mass eigenvalues. The Planck Collaboration gives the bound $\sum_i m_i < 0.23$ eV [45]. To obtain this bound they combined the Planck temperature power spectrum with a WMAP polarization low-multipole likelihood, high-resolution CMB data and baryon acoustic oscillation surveys. In the normal hierarchy case, it is

$$\sum_i m_i = m_1 + \sqrt{m_1^2 + \Delta m_{\rm sol}^2} + \sqrt{m_1^2 + \Delta m_{\rm sol}^2 + \Delta m_{\rm atm}^2}, \tag{7.34}$$

which fixes a bound on m_1. On the other hand,

$$\begin{aligned} \bar{m} = \left(\sum_i m_i^2\right)^{1/2} &= \left(m_1^2 + m_1^2 + \Delta m_{\rm sol}^2 + m_1^2 + \Delta m_{\rm sol}^2 + \Delta m_{\rm atm}^2\right)^{1/2} \\ &= \left(3m_1^2 + 2\Delta m_{\rm sol}^2 + \Delta m_{\rm atm}^2\right)^{1/2}, \end{aligned} \tag{7.35}$$

so that all in all the Planck bound translates into $\bar{m} < 0.13$ eV.

8 Summary

We have obtained rate equations for single-flavor Leptogenesis in the nonrelativistic limit which are valid at leading order in the heavy neutrino Yukawa coupling and to all orders in the Standard Model couplings. Furthermore, we introduced a systematic expansion around the nonrelativistic limit and found that already the first relativistic corrections are quite small. This shows that the nonrelativistic approximation works very well.

The coefficients in the rate equations can explicitly be determined by perturbation theory. We calculated all coefficients at leading order in all couplings and some at next-to-leading order in the Standard Model couplings. In the case of the heavy sterile neutrino equilibration rate, we used the radiative corrections of the production rate. To show that this is possible, we derived a relation between the production and equilibration rate using the theory of quasistationary fluctuations. Thereby, we showed that the equilibration rate of a particle species is proportional to the discontinuity of its self energy. This relation holds if there is a separation of the time scales between the particle species whose equilibration is considered and all other present interactions. The same relation has been obtained by other groups under different assumptions.

From the solutions of the rate equations we obtained bounds on the light neutrino masses. Our bounds are in agreement with previously obtained values. The biggest improvements are achieved by using quantum statistics for leptons and Higgs bosons in the lepton number washout rate. However, the bounds obtained are not as strong as bounds obtained by experiments like the CMB analysis of the Planck satellite.

It would be interesting to further investigate the nonrelativistic approximation of Leptogenesis because it allows for including radiative corrections. The next steps would obviously be the calculation of the NLO correction of the asymmetry production rate as well as NLO contributions to spectral functions in the washout rate to accomplish a full description up to $\mathcal{O}(g^2)$ in the Standard Model couplings.

Acknowledgement

I like to thank my supervisor Dietrich Bödeker for his kind advise and support, Marc Sangel and Dennis Schröder for many interesting discussions, and Marc Sangel also for contributing the rate (4.132).
I am also grateful to *all* my collegues for a very friendly working atmosphere, to our secretaries Gudrun and Susi who always had solutions to all nonphysical questions and problems, and to my parents who not only supported me during the time I worked on this theses, but my whole life.
During my first year as a Ph.D. student, I was financially supported through the DFG funded Graduate School GRK 881.

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