

Dissertation

The Monoidal Structure on Strict Polynomial Functors

vorgelegt von
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1. Introduction

1.1. Motivation and Main Results

In 1901 Schur investigated representations of the complex general linear group $\mathrm{GL}_n(\mathbb{C})$ [Sch01]. As a tool he defined a new algebra, nowadays known as the *Schur algebra* and denoted by $S_{\mathbb{C}}(n, d)$. Its module category was shown to be equivalent to $M_{\mathbb{C}}(n, d)$, the category of *polynomial representations of fixed homogeneous degree d of $\mathrm{GL}_n(\mathbb{C})$* :

$$M_{\mathbb{C}}(n, d) \simeq S_{\mathbb{C}}(n, d) \mathbf{Mod}$$

Using this algebra, Schur obtained a connection between representations of $\mathrm{GL}_n(\mathbb{C})$ and those of the *symmetric group* \mathfrak{S}_d by defining a functor f , now called the *Schur functor*, between these representations. To be more precise, he showed that the polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ of fixed homogeneous degree d are equivalent to representations of the symmetric group \mathfrak{S}_d , whenever $n \geq d$. This correspondence is commonly known as *Schur–Weyl duality*.

Based on Schur’s ideas, Green developed a similar theory extending the ground field to an arbitrary infinite field k in 1981. In particular, he showed that the category of polynomial representations of the general linear group $\mathrm{GL}_n(k)$ of fixed degree d is equivalent to the category of modules over the Schur algebra $S_k(n, d)$ [Gre07]. Moreover, he also considered the Schur functor f , relating the module category of $S_k(n, d)$ to the one of the group algebra of the symmetric group:

$$M_k(n, d) \simeq S_k(n, d) \mathbf{Mod} \xrightarrow{f} k\mathfrak{S}_d \mathbf{Mod}$$

However, in this context the Schur functor does not generally induce an equivalence in positive characteristic, in contrast to Schur’s original setup.

Following the introduction of the Schur algebras, results about representations of the group algebra of the symmetric group have been used to infer properties of modules over the Schur algebra and thus about representations of the general linear group. Starting with Green’s monograph, the Schur algebra has become an object of interest in its own right and consequently results have been obtained independently of the group algebra of the symmetric group. Even more is true: findings for the Schur algebra have been used to obtain new results about symmetric group representations.

Schur algebras have been extensively investigated over the last years, among others by Donkin [Don86] [Don87] [Don94a] [Don94b]. He introduced *generalized Schur algebras* and showed the existence of Weyl filtrations for projective modules over Schur algebras. In particular, he showed that Schur algebras are *quasi-hereditary* and thus of finite global

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dimension. He also described explicitly the blocks of the Schur algebra. Moreover, in 1993 Erdmann determined those Schur algebras that are of *finite representation type* [Erd93].

In 1997 Friedlander and Suslin introduced the category of *strict polynomial functors*. Their definition is based on polynomial maps of finite dimensional vector spaces over an arbitrary field k . They showed in [FS97] that the category of strict polynomial functors of a fixed degree d , which we denote by $\text{Rep } \Gamma_k^d$, is equivalent to the category of modules over the Schur algebra $S_k(n, d)$ whenever $n \geq d$:

$$\text{Rep } \Gamma_k^d \xrightarrow{\sim} S_k(n, d) \text{ Mod} \quad (n \geq d)$$

We work more generally over an arbitrary commutative ring k and use a different description in terms of divided powers. This is convenient, because via Day convolution, the category of strict polynomial functors inherits a closed symmetric monoidal structure from the category of divided powers. This particular tensor product can be implicitly found in works by Chałupnik [Cha08] and Touzé [Tou13]; an explicit definition is given by Krause in [Kra13]. We will denote this tensor product by $- \otimes_{\Gamma_k^d} -$ and the corresponding internal hom by $\mathcal{H}om(-, -)$.

Using the equivalence proven by Friedlander and Suslin we obtain, via transport of structure, a tensor product for modules over the Schur algebra. Despite the fact that Schur algebras were invented more than a hundred years ago, this tensor product has only been discovered recently. Unfortunately, working with the tensor product is less profitable than one would hope for since its definition is not explicit: it is explicitly defined for representable functors only, i.e. for certain projective objects, and extended to arbitrary objects by taking colimits.

Apart from providing a tensor product for modules over the Schur algebra, the monoidal structure on strict polynomial functors is interesting on its own: Chałupnik in [Cha08] and Touzé in [Tou13] introduced a Koszul duality on the derived level of strict polynomial functors. This duality is given by taking the tensor product with exterior powers. Moreover, in [Tou13] Touzé established a connection between this Koszul duality in the category of strict polynomial functors and derived functors of non-additive functors, hence extending recent applications of the tensor product.

The main motivation of this thesis is to better understand the tensor product of strict polynomial functors and gain insights into related categorical structures.

A first step toward this goal is to strengthen the relation between strict polynomial functors and representations of the symmetric group, in particular to compare the monoidal structures on both sides. Since $k\mathfrak{S}_d$ is a group algebra, it has a Hopf algebra structure and thus the category of representations of the symmetric group possesses a closed symmetric monoidal structure. The tensor product of this monoidal structure is often called the *Kronecker product* and is denoted by $- \otimes_k -$.

In characteristic zero, the Kronecker product has been intensely studied over the last century. In this characteristic the group algebra of the symmetric group is semi-simple, thus understanding the Kronecker product reduces to the problem of how the Kronecker

product of two simple representations decomposes into a direct sum of simple representations. The multiplicities appearing in this decomposition are called *Kronecker coefficients*, for which only partial results are known: Murnaghan stated in [Mur38] a stability property of the Kronecker coefficients, i.e. for three partitions λ, μ, ν the Kronecker coefficient $g_{\lambda+n, \mu+n}^{\nu+n}$ is independent of n for large n . In [JK81] James and Kerber provided tables of Kronecker coefficients for symmetric groups of degree up to 8. In addition, Kronecker products for several special partitions, including hook partitions and 2-part partitions, have been computed, but a general description of Kronecker coefficients is still an open problem.

In the case of positive characteristic even less is known. Already the simplest non-trivial case, namely tensoring with the sign representation, is hard to compute. A combinatorial description, given by the *Mullineux map*, was conjectured by Mullineux in [Mul79] and proved by Ford and Kleshchev in [FK97], almost two decades later. Another known fact, proved by Bessenrodt and Kleshchev in [BK00], states that the Kronecker product of two simple representations of dimension greater than 1 is never indecomposable in odd characteristic.

Fortunately, there are also some positive results. For example, it is possible to describe the Kronecker product of two *permutation modules* explicitly, see Lemma 3.5. This description is even independent of the characteristic.

Using the aforementioned properties of the Kronecker product allows us to make progress in describing the tensor product for strict polynomial functors. This is an application from our first main result.

Theorem 3.23. *The Schur functor*

$$\mathcal{F}: \text{Rep } \Gamma_k^d \rightarrow k\mathfrak{S}_d \text{ Mod}$$

is a strong closed monoidal functor.

Extending our investigation of properties of the Schur functor \mathcal{F} , as next step we consider the fully faithful *left adjoint* \mathcal{G}_{\otimes} and *right adjoint* \mathcal{G}_{Hom} of \mathcal{F} . These adjoints have been studied in order to compare the cohomology of general linear groups to that of symmetric groups, see [DEN04], and to relate (dual) Specht filtrations of symmetric group modules to Weyl filtrations of modules over the general linear group in [HN04]. We focus on the relationship with the monoidal structure and show that the left adjoint of the Schur functor can be expressed in terms of the tensor product of strict polynomial functors. Denote by S^d the d -symmetric powers and let $X \in \text{Rep } \Gamma_k^d$.

Theorem 4.3. *There exists a natural isomorphism*

$$\mathcal{G}_{\otimes} \mathcal{F}(X) \cong S^d \otimes_{\Gamma_k^d} X.$$

Dually, we show that the right adjoint to the Schur functor can be expressed in terms of the internal hom of strict polynomial functors:

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Theorem 4.10. *There exists a natural isomorphism*

$$\mathcal{G}_{\text{Hom}}\mathcal{F}(X) \cong \mathcal{H}\text{om}(S^d, X).$$

Moreover we show that a *projection formula* holds:

Theorem 4.6. *For all $X \in \text{Rep } \Gamma_k^d$ and $N \in k\mathfrak{S}_d \text{Mod}$ there is an isomorphism*

$$\mathcal{G}_{\otimes}(\mathcal{F}(X) \otimes_k N) \cong X \otimes_{\Gamma_k^d} \mathcal{G}_{\otimes}(N).$$

The general results above allow us to draw conclusions about the tensor product in specific cases. We are now in a position to explicitly calculate the tensor product of (generalized) divided, symmetric and exterior powers, both among one another and between any two objects, see Corollary 5.6. For the tensor product of Weyl, respectively Schur filtered functors, we get partial results in special cases. In particular, we obtain in Proposition 5.10 the negative result that the subcategory of Weyl, respectively Schur filtered objects is not closed under the tensor product. By using Theorem 4.3, we give a necessary and sufficient condition whether the tensor product of two simple strict polynomial functors L_λ and L_μ is again simple:

Theorem 5.15. *Denote by Λ^d the d -th exterior powers and Q^d the truncated symmetric powers. Let k be a field of odd characteristic and $\lambda, \mu \in \Lambda_p^+(n, d)$. The tensor product $L_\lambda \otimes L_\mu$ is simple if and only if, up to interchanging λ and μ ,*

- $L_\lambda \cong \Lambda^d$ and all ν with $\text{Ext}^1(L_{m(\mu)}, L_\nu) \neq 0$ are p -restricted, or
- $L_\lambda \cong Q^d$ and all ν with $\text{Ext}^1(L_\mu, L_\nu) \neq 0$ are p -restricted.

In these cases $\Lambda^d \otimes L_\mu \cong L_{m(\mu)}$ and $Q^d \otimes L_\mu \cong L_\mu$.

In the case $n = d = p$ even a full characterization can be given [Theorem 5.18].

1.2. Outline

Following the outline, we fix some notation and recall widely known definitions and facts about monoidal and k -linear categories.

The second chapter serves as an introduction to *strict polynomial functors*. Most parts are collected from [Kra13] and [Kra14]. In Section 2.7.2 we introduce another dual for strict polynomial functors, the *monoidal dual*, and explicitly compute this dual for divided, symmetric and exterior powers.

In the third chapter we give a short introduction into *representations of the symmetric group \mathfrak{S}_d* . We recall the usual monoidal structure on the module category $k\mathfrak{S}_d \text{Mod}$ and definitions of important objects such as *permutation modules*, *Young modules*, *Specht modules* and *simple modules*. Furthermore, we investigate the *Schur functor \mathcal{F}* connecting the category of strict polynomial functors $\text{Rep } \Gamma_k^d$ to $k\mathfrak{S}_d \text{Mod}$. In particular, we show in Theorem 3.23 that \mathcal{F} preserves the closed monoidal structure. Closing the chapter, we describe the action of the Schur functor on duals in Corollary 3.24.

The fourth chapter deals with the *adjoints to the Schur functor* \mathcal{F} . We describe the fully faithful left and right adjoints of \mathcal{F} and show that they are inverses to \mathcal{F} when restricted to particular subcategories. We further prove in Theorems 4.3 and 4.10 that the composition of \mathcal{F} and its left, respectively right adjoint can be expressed in terms of the monoidal structure. As a direct consequence, we give a relation between the left and right adjoint in Proposition 4.14. Another conclusion from the previous findings is a *projection formula* [Theorem 4.6] for the Schur functor in the sense of [FHM03, (3.6)].

Utilizing results from the previous chapters, we are finally able to compute the *tensor product of various strict polynomial functors* in the fifth chapter. First of all we provide calculations for *divided, symmetric and exterior powers* which are summarized in Corollary 5.6.

Next, we focus on *Schur and Weyl functors*: although general calculations of the tensor product of two Schur, respectively Weyl functors have not been completed, we provide computations in special cases in Propositions 5.7 - 5.9 and give a negative answer to the question whether subcategories consisting of Schur, respectively Weyl filtered functors are closed under the tensor product in Proposition 5.10.

Finally, we study tensor products of *simple strict polynomial functors*. We develop a necessary and sufficient condition in terms of Ext-vanishing between certain simple functors for whether this tensor product is again simple [Theorem 5.15] and give a detailed analysis in the case $n = d = p$ [Theorem 5.18].

The sixth chapter is devoted to the *Schur algebra* and its connection to strict polynomial functors. In particular, we explain how the tensor product of $\mathbf{Rep} \Gamma_k^d$ is translated to $S_k(n, d) \text{Mod}$. This chapter contains fewer new results, but is rather an overview of the correspondences between several objects, morphisms and structures.

In the appendix we collect very explicit – sometimes combinatorial – calculations used to obtain the special relations between modules and morphisms of modules over the Schur algebra, group algebra of the symmetric group and strict polynomial functors. Moreover, we give a tabular overview of these correspondences.

1.3. Notations and Prerequisites

In the following we fix some notation used throughout the rest of this thesis and collect important prerequisites about monoidal categories.

1.3.1. Compositions, Partitions and Tableaux

Most of the notations are taken from [Ful97], [JK81] and [Mar93].

For positive integers n and d let

$$\Lambda(n, d) := \{\lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_i \in \mathbb{N}, \sum_i \lambda_i = d\}$$

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be the set of all *compositions of d into n parts*,

$$\Lambda(d) := \{\lambda \mid \lambda \in \Lambda(n, d) \text{ for some } n \in \mathbb{N}, \lambda_i > 0 \text{ for all } 1 \leq i \leq n\}$$

be the set of all *compositions of d* ,

$$\Lambda^+(n, d) := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n, d) \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}$$

the set of all *partitions of d into n parts*,

$$\Lambda^+(d) := \{\lambda \mid \lambda \in \Lambda^+(n, d) \text{ for some } n \in \mathbb{N}, \lambda_n > 0\}$$

the set of all *partitions of d* , and for $p > 0$

$$\Lambda_p^+(n, d) := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n, d) \mid \lambda_i - \lambda_{i+1} < p \text{ for } 1 \leq i \leq n-1, \lambda_n < p\}$$

the set of *p -restricted partitions of d into n parts*.

A partition $\lambda \in \Lambda^+(n, d)$ is called *p -regular* if every value occurs less than p times. A partition $\lambda \in \Lambda^+(n, d)$ is a *p -core* if it contains no p -rim hooks, see e.g. [Mat99, Section 3] or [JK81, Section 2.7] for detailed descriptions.

The *conjugate partition* λ' of $\lambda \in \Lambda^+(n, d)$ is given by $\lambda'_i := \#\{j \mid \lambda_j \geq i\}$. The set of d -tuples of positive integers smaller equal than n is denoted by

$$I(n, d) := \{\underline{i} = (i_1 \dots i_d) \mid 1 \leq i_l \leq n\}.$$

We say that $\underline{i} \in I(n, d)$ is represented by $\lambda \in \Lambda^+(n, d)$, and write $\underline{i} \in \lambda$, if \underline{i} has λ_l entries equal to l for $1 \leq l \leq n$. Two pairs of sequences $(\underline{j}, \underline{i})$ and $(\underline{j}', \underline{i}')$ in $I(n, d) \times I(n, d)$ are equivalent, denoted by $(\underline{j}, \underline{i}) \sim (\underline{j}', \underline{i}')$, if there exists a permutation σ of the entries such that $\underline{j}\sigma = \underline{j}'$ and $\underline{i}\sigma = \underline{i}'$.

Example 1.1. Let $n = 5$ and $d = 13$. Then $\lambda = (5, 3, 2, 2, 1) \in \Lambda^+(5, 13)$. The conjugate partition of λ is $\lambda' = (5, 4, 2, 1, 1)$. The sequence (2543121412131) belongs to λ . The partition λ is 5-restricted, but not 2-restricted since $\lambda_1 - \lambda_2 = 5 - 3 = 2 \not< 2$. It is 3-regular, but not 2-regular since the value 2 occurs twice.

Matrices. Let $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$. We define A_μ^λ to be the set of all $n \times m$ matrices $A = (a_{ij})$ with entries in \mathbb{N} such that $\lambda_i = \sum_j a_{ij}$ and $\mu_j = \sum_i a_{ij}$.

Symmetric group. Let S be any set and define \mathfrak{S}_S to be the group of permutations of elements in S , i.e. the group of bijections from S to itself. For a set S with $|S| = d$, we denote the *symmetric group on d elements* \mathfrak{S}_S by \mathfrak{S}_d . It depends only on the cardinality $|S|$ of the set S . For a composition $\lambda \in \Lambda(n, d)$ the *Young subgroup* is defined by

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\{1, \dots, \lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1, \dots\}} \times \dots \times \mathfrak{S}_{\{\dots, d-1, d\}}.$$

It is isomorphic to $\mathfrak{S}_{\lambda_1} \times \dots \times \mathfrak{S}_{\lambda_n}$ and we will identify both groups.

The *signature of a permutation* $\sigma \in \mathfrak{S}_d$ is denoted by $\text{sign}(\sigma)$.

Tableaux. Let $\lambda \in \Lambda^+(n, d)$. The *Young diagram* for λ is the subset

$$[\lambda] = \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq \lambda_i\} \subseteq \mathbb{Z}^2.$$

Example 1.2. A Young diagram can be visualized by drawing λ_1 boxes in a row, then λ_2 boxes in a row below and so on. For example let $\lambda = (5, 3, 2, 2, 1) \in \Lambda^+(5, 13)$, then we write

$$[\lambda] = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

The Young diagram $[\lambda']$ corresponding to the conjugate partition of λ is obtained from $[\lambda]$ by reflecting along the diagonal, i.e.

$$[\lambda'] = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

A λ -*tableau* is a map T^λ from $[\lambda]$ to a set. One can visualize T^λ as

$$T^\lambda = \begin{array}{|c|c|c|c|c|} \hline T^\lambda((1, 1)) & T^\lambda((1, 2)) & T^\lambda((1, 3)) & \dots & \dots \\ \hline T^\lambda((2, 1)) & T^\lambda((2, 2)) & \dots & & \\ \hline \vdots & \ddots & & & \\ \hline T^\lambda((n, 1)) & \dots & & & \\ \hline \end{array}$$

A *basic λ -tableau* is a bijective map from $[\lambda]$ to $\{1, \dots, d\}$, i.e. T^λ is basic if every integer from 1 to d occurs exactly once in T^λ .

Example 1.3. Let $\lambda = (5, 3, 2, 2, 1) \in \Lambda^+(5, 13)$, then one basic λ -tableau is

$$T^\lambda = \begin{array}{|c|c|c|c|c|} \hline 3 & 2 & 7 & 11 & 5 \\ \hline 4 & 13 & 10 & & \\ \hline 12 & 6 & & & \\ \hline 9 & 1 & & & \\ \hline 8 & & & & \\ \hline \end{array}$$

The group \mathfrak{S}_d acts on the set of basic λ -tableaux T^λ by interchanging the entries. The *row stabilizer* or *horizontal group* ([JK81]) $R(T^\lambda)$ of T^λ is the subgroup of \mathfrak{S}_d that preserves the entries in each row. The *column stabilizer* or *vertical group* ([JK81]) $C(T^\lambda)$ of T^λ is the subgroup of \mathfrak{S}_d that preserves the entries in each column.

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Example 1.4.

(i) Let

$$T^\lambda = \begin{array}{|c|c|c|c|c|} \hline 3 & 2 & 7 & 11 & 5 \\ \hline 4 & 13 & 10 & & \\ \hline 12 & 6 & & & \\ \hline 9 & 1 & & & \\ \hline 8 & & & & \\ \hline \end{array}$$

Then $R(T^\lambda) \cong \mathfrak{S}_{\{3,2,7,11,5\}} \times \mathfrak{S}_{\{4,13,10\}} \times \cdots \times \mathfrak{S}_{\{8\}}$ and $C(T^\lambda) \cong \mathfrak{S}_{\{3,4,12,9,8\}} \times \cdots \times \mathfrak{S}_{\{5\}}$.

(ii) Let T_R^λ be the tableau where the entries are given by $1, 2, \dots, d$ when read from left to right, from top to bottom, i.e.

$$T_R^\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & \dots & \lambda_1 \\ \hline \lambda_1 + 1 & \lambda_1 + 2 & \dots & & \\ \hline \vdots & & & & \\ \hline \dots & d & & & \\ \hline \end{array}$$

Then $R(T_R^\lambda) \cong \mathfrak{S}_\lambda$.

(iii) Let T_C^λ be the tableau where the entries are given by $1, 2, \dots, d$ when read from top to bottom, from left to right, i.e.

$$T_C^\lambda = \begin{array}{|c|c|c|} \hline 1 & \lambda'_1 + 1 & \dots \\ \hline 2 & \lambda'_1 + 2 & \dots \\ \hline \vdots & & \\ \hline \lambda'_1 & \dots & \\ \hline \end{array}$$

then $C(T_C^\lambda) \cong \mathfrak{S}_\lambda$.

1.3.2. Monoidal and k -Linear Categories

We briefly recall the definition of a monoidal category.

Definition 1.5. A *monoidal category* is a category \mathcal{M} together with

- an (*internal*) *tensor product* \otimes , i.e. a bifunctor $- \otimes -: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$,
- a *unit* object $\mathbf{1} \in \mathcal{M}$,
- a *left unitor* λ , i.e. a natural isomorphism with components $\lambda_X: \mathbf{1} \otimes X \rightarrow X$,
- a *right unitor* ϱ , i.e. a natural isomorphism with components $\varrho_X: X \otimes \mathbf{1} \rightarrow X$,
- an *associator* α , i.e. a natural isomorphism with components $\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$,

such that

- the pentagonal diagram

$$\begin{array}{ccc}
 & (W \otimes X) \otimes (Y \otimes Z) & \\
 \alpha_{W \otimes X, Y, Z} \nearrow & & \searrow \alpha_{W, X, Y \otimes Z} \\
 ((W \otimes X) \otimes Y) \otimes Z & & W \otimes (X \otimes (Y \otimes Z)) \\
 \alpha_{W, X, Y} \otimes \text{id}_Z \downarrow & & \uparrow \text{id}_W \otimes \alpha_{X, Y, Z} \\
 (W \otimes (X \otimes Y)) \otimes Z & \xrightarrow{\alpha_{W, X \otimes Y, Z}} & W \otimes ((X \otimes Y) \otimes Z)
 \end{array}$$

and

- the triangular diagram

$$\begin{array}{ccc}
 X \otimes (\mathbf{1} \otimes Y) & \xleftarrow{\alpha_{X, \mathbf{1}, Y}} & (X \otimes \mathbf{1}) \otimes Y \\
 \text{id}_X \otimes \lambda_Y \searrow & & \swarrow \varrho_X \otimes \text{id}_Y \\
 & X \otimes Y &
 \end{array}$$

commute for all $W, X, Y, Z \in \mathcal{M}$.

A *symmetric monoidal category* is a monoidal category \mathcal{M} that is equipped with a *braiding*, i.e. a natural isomorphism γ with components $\gamma_{X, Y}: X \otimes Y \rightarrow Y \otimes X$ such that for all X, Y, Z $\gamma_{Y, X} \circ \gamma_{X, Y} = \text{id}_{X \otimes Y}$ and the following diagrams commute:

$$\begin{array}{ccc}
 (X \otimes Y) \otimes Z & \xrightarrow{\gamma_{X, Y} \otimes \text{Id}_Z} & (Y \otimes X) \otimes Z \\
 \alpha_{X, Y, Z} \downarrow & & \downarrow \alpha_{Y, X, Z} \\
 X \otimes (Y \otimes Z) & & Y \otimes (X \otimes Z) \\
 \gamma_{X, Y} \otimes \text{id}_Z \downarrow & & \downarrow \text{Id}_Y \otimes \gamma_{X, Z} \\
 (Y \otimes Z) \otimes X & \xrightarrow{\alpha_{Y, Z, X}} & Y \otimes (Z \otimes X)
 \end{array}$$

$$\begin{array}{ccc}
 X \otimes \mathbf{1} & \xrightarrow{\gamma_{X, \mathbf{1}}} & \mathbf{1} \otimes X \\
 \varrho \searrow & & \swarrow \lambda \\
 & X &
 \end{array}$$

Later on we often omit the additional data, e.g. associator / unit / braiding, and talk only about a monoidal category \mathcal{M} whenever the parameters have been fixed before.

Definition 1.6. A *closed symmetric monoidal category* is a symmetric monoidal category in which for all $Y \in \mathcal{M}$ the functor $- \otimes Y: \mathcal{M} \rightarrow \mathcal{M}$ has a right adjoint. This adjoint is called the *internal hom* and is denoted by $\mathcal{H}om(Y, -)$.

Note that for all $Y \in \mathcal{M}$ the assignment $X \mapsto \mathcal{H}om(X, Y)$ defines a functor from $\mathcal{M}^{\text{op}} \rightarrow \mathcal{M}$. Thus, we actually obtain a bifunctor $\mathcal{H}om(-, -): \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{M}$. By definition we have for all $X, Y, Z \in \mathcal{M}$ natural isomorphisms

$$\text{Hom}_{\mathcal{M}}(X \otimes Y, Z) \cong \text{Hom}_{\mathcal{M}}(X, \mathcal{H}om(Y, Z)).$$

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Remark 1.7. It is sometimes convenient to specify in which category a specific (internal) tensor product, internal hom or the unit object lives. In this cases we write $- \otimes_{\mathcal{M}} -$, $\mathcal{H}om_{\mathcal{M}}(-, -)$ or $\mathbf{1}_{\mathcal{M}}$. We often omit the supplement “internal” when dealing with the internal tensor product and just write “tensor product”.

Example 1.8. Let k be a commutative ring. Then $\mathbf{Mod} k$, the category of all k -modules, is a closed symmetric monoidal category, where for $V, W, X \in \mathbf{Mod} k$

- the internal tensor product is just the usual tensor product over k :
 $V \otimes W = V \otimes_k W$,
- the unit object is $\mathbf{1} = k$, the regular representation,
- the left unitor $\lambda_V: \mathbf{1} \otimes V \rightarrow V$ is the usual isomorphism given by $r \otimes v \mapsto r \cdot v$,
- the right unitor $\rho_V: V \otimes \mathbf{1} \rightarrow V$ is the usual isomorphism given by $v \otimes r \mapsto v \cdot r$,
- the associator $\alpha_{V,W,X}: (V \otimes W) \otimes X \rightarrow V \otimes (W \otimes X)$ is given by the usual associativity isomorphism,
- the braiding $\gamma_{V,W}: V \otimes W \rightarrow W \otimes V$ is given by the usual commutativity isomorphism,
- the internal hom is given by $\mathcal{H}om(V, W) = \text{Hom}_k(V, W)$, the k -linear maps from V to W .

The additional conditions such as commutativity of certain diagrams are satisfied by the usual tensor product properties for modules. For example the adjointness property of the internal hom follows from the usual tensor-hom adjunction

$$\text{Hom}_k(V \otimes W, X) \cong \text{Hom}_k(V, \text{Hom}_k(W, X)).$$

Definition 1.9. A *lax monoidal functor* is a functor \mathcal{F} between monoidal categories \mathcal{M} and \mathcal{M}' together with a morphism $\varepsilon: \mathbf{1}_{\mathcal{M}'} \rightarrow \mathcal{F}(\mathbf{1}_{\mathcal{M}})$ and a natural transformation $\Phi_{X,Y}: \mathcal{F}X \otimes_{\mathcal{M}'} \mathcal{F}Y \rightarrow \mathcal{F}(X \otimes_{\mathcal{M}} Y)$ for all $X, Y \in \mathcal{M}$ such that the following three diagrams commute.

$$\begin{array}{ccc}
 (\mathcal{F}X \otimes \mathcal{F}Y) \otimes \mathcal{F}Z & \xrightarrow{\alpha'_{\mathcal{F}X, \mathcal{F}Y, \mathcal{F}Z}} & \mathcal{F}X \otimes (\mathcal{F}Y \otimes \mathcal{F}Z) \\
 \downarrow \Phi_{X,Y} \otimes \text{id}_{\mathcal{F}Z} & & \downarrow \text{id}_{\mathcal{F}X} \otimes \Phi_{Y,Z} \\
 \mathcal{F}(X \otimes Y) \otimes \mathcal{F}Z & & \mathcal{F}X \otimes \mathcal{F}(Y \otimes Z) \\
 \downarrow \Phi_{X \otimes Y, Z} & & \downarrow \Phi_{X, Y \otimes Z} \\
 \mathcal{F}((X \otimes Y) \otimes Z) & \xrightarrow{\mathcal{F}\alpha_{X,Y,Z}} & \mathcal{F}(X \otimes (Y \otimes Z))
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{F}X \otimes \mathbf{1}_{\mathcal{M}'} & \xrightarrow{\rho_{\mathcal{F}X}} & \mathcal{F}X \\
 \downarrow \text{id}_{\mathcal{F}X} \otimes \varepsilon & & \uparrow \mathcal{F}\rho_X \\
 \mathcal{F}X \otimes \mathcal{F}(\mathbf{1}_{\mathcal{M}}) & \xrightarrow{\Phi_{X, \mathbf{1}_{\mathcal{M}}}} & \mathcal{F}(X \otimes \mathbf{1}_{\mathcal{M}})
 \end{array}$$

$$\begin{array}{ccc}
 \mathbf{1}_{\mathcal{M}'} \otimes \mathcal{F}X & \xrightarrow{\lambda_X} & \mathcal{F}X \\
 \downarrow \varepsilon \otimes \text{id}_{\mathcal{F}X} & & \uparrow \mathcal{F}\lambda_X \\
 \mathcal{F}(\mathbf{1}_{\mathcal{M}}) \otimes \mathcal{F}X & \xrightarrow{\Phi_{\mathbf{1}_{\mathcal{M}}, \mathcal{F}X}} & \mathcal{F}(\mathbf{1}_{\mathcal{M}} \otimes X)
 \end{array}$$

A *strong monoidal functor* is a lax monoidal functor \mathcal{F} such that the maps ε and Φ are isomorphisms, i.e. $\mathcal{F}X \otimes \mathcal{F}Y \cong \mathcal{F}(X \otimes Y)$ and $\mathcal{F}\mathbf{1}_{\mathcal{M}} \cong \mathbf{1}_{\mathcal{M}'}$.

For the definition of strict polynomial functors it is important to know what a k -linear category is.

Definition 1.10. Let k be a commutative ring. A k -linear category is a category \mathcal{A} such that for all $X, Y, Z \in \mathcal{A}$ we have $\mathrm{Hom}_{\mathcal{A}}(X, Y) \in \mathbf{Mod} k$ and

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

is k -bilinear.

Let \mathcal{A} and \mathcal{B} be k -linear categories. A k -linear functor or k -linear representation of \mathcal{A} in \mathcal{B} is a functor \mathcal{F} such that for all objects $X, Y \in \mathcal{A}$ the map

$$\mathcal{F}: \mathrm{Hom}_{\mathcal{A}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a homomorphism of k -modules. The category of all k -linear functors from \mathcal{A} to \mathcal{B} is denoted by $\mathrm{Fun}_k(\mathcal{A}, \mathcal{B})$.

2. Strict Polynomial Functors

Strict polynomial functors were first defined by Friedlander and Suslin in [FS97], using polynomial maps of finite dimensional vector spaces over a field k . We work with a different, but equivalent, definition as in [Tou13] and [Kra13]. This definition uses the category of divided powers and has the advantage of transparently inducing a closed symmetric monoidal structure—the main object of this thesis. The monoidal structure has gained interest when a Koszul duality for strict polynomial functors had been established by Chałupnik in [Cha08] and Touzé in [Tou13], since this duality can be expressed in terms of the monoidal structure, see e.g. [Kra13, Section 3] for an elaboration and in particular for its connection to Ringel duality.

The aim of this chapter is to introduce the category of strict polynomial functors and in particular its monoidal structure. We collect known results about this tensor product as well as further structures such as the highest weight structure. We finally recall the definition of the Kuhn dual and then introduce a second dual, the monoidal dual. Since the latter is important for the explicit calculations in Chapter 5, we provide computations for this dual for divided, symmetric and exterior powers.

2.1. Prerequisites

Let k be a commutative ring and denote by \mathbf{P}_k the category of finitely generated projective k -modules and k -linear maps. Since k is commutative, this category is a k -linear category and equipped with a closed symmetric monoidal structure. The internal tensor product $V \otimes_{\mathbf{P}_k} W$ is given by the usual tensor product $V \otimes_k W$ over k , the internal hom is $\mathcal{H}om_{\mathbf{P}_k}(V, W) = \text{Hom}_k(V, W)$, i.e. all k -linear maps from V to W and the tensor unit is $\mathbf{1}_{\mathbf{P}_k} = k$, the regular representation. See Example 1.8 for more details.

We denote the usual dual in \mathbf{P}_k by $(-)^* = \text{Hom}_{\mathbf{P}_k}(-, k)$.

Divided, symmetric and exterior powers. For $V \in \mathbf{P}_k$ consider the d -fold tensor product $V^{\otimes d}$. The symmetric group on d variables, \mathfrak{S}_d , acts by place permutation on the right on it, i.e. for $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$ and $\sigma \in \mathfrak{S}_d$ define

$$v_1 \otimes \cdots \otimes v_d \cdot \sigma := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

We build new objects from $V \in \mathbf{P}_k$ as follows:

$\Gamma^d V = (V^{\otimes d})^{\mathfrak{S}_d} = \{v \in V^{\otimes d} \mid v\sigma = v \text{ for all } \sigma \in \mathfrak{S}_d\}$, the divided powers of degree d ,
 $S^d V = (V^{\otimes d})_{\mathfrak{S}_d} = V^{\otimes d} / \langle v \otimes w - w \otimes v \mid v, w \in V \rangle$, the symmetric powers of degree d ,
 $\Lambda^d V = V^{\otimes d} / \langle v \otimes v \mid v \in V \rangle$, the exterior powers of degree d .

2. Strict Polynomial Functors

Remark 2.1. We denote the inclusion map $\Gamma^d V \hookrightarrow V^{\otimes d}$ by $(\iota_\Gamma)_V$, the quotient map $V^{\otimes d} \twoheadrightarrow S^d V$ by $(\pi_S)_V$ and the quotient map $V^{\otimes d} \twoheadrightarrow \Lambda^d V$ by $(\pi_\Lambda)_V$. In addition, there is an isomorphism

$$\Lambda^d(V) \cong \left\{ \sum_{\sigma \in \mathfrak{S}_d} \text{sign}(\sigma) \cdot v\sigma \mid v \in V^{\otimes d} \right\}$$

and we denote the inclusion map $\Lambda^d V \hookrightarrow V^{\otimes d}$ by $(\iota_\Lambda)_V$.

Note that

- (i) $\Gamma^d V \times \Gamma^d W \subseteq \Gamma^d(V \times W)$ and $\Gamma^{d+e} V \subseteq \Gamma^d V \otimes \Gamma^e V$
- (ii) $\Gamma^d(V^*)^* \cong S^d(V)$
- (iii) $\Lambda^d(V^*)^* \cong \Lambda^d(V)$

For V projective, the modules $\Gamma^d V, S^d V, \Lambda^d V$ are still projective, see for example [Bou89, III.6.6]. Hence, they induce functors $\Gamma^d, S^d, \Lambda^d: \mathbf{P}_k \rightarrow \mathbf{P}_k$.

2.2. The Category of Divided Powers

We now define a new category which has the same objects as \mathbf{P}_k but different morphisms. This category inherits many properties from \mathbf{P}_k , in particular the closed symmetric monoidal structure.

Definition 2.2. The *category of divided powers* $\Gamma^d \mathbf{P}_k$ is the category with

- the same objects as $\Gamma^d \mathbf{P}_k$,
- morphisms given, for two objects V and W , by

$$\text{Hom}_{\Gamma^d \mathbf{P}_k}(V, W) := \Gamma^d \text{Hom}(V, W) = (\text{Hom}(V, W)^{\otimes d})^{\mathfrak{S}_d}.$$

Remark 2.3. We can identify $(\text{Hom}(V, W)^{\otimes d})^{\mathfrak{S}_d}$ with $\text{Hom}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$ where for $\sigma \in \mathfrak{S}_d, f \in \text{Hom}(V^{\otimes d}, W^{\otimes d})$ and $v_i \in V$ the action is given by

$$f\sigma(v_1 \otimes \cdots \otimes v_d) := f((v_1 \otimes \cdots \otimes v_d)\sigma^{-1})\sigma = f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)})\sigma.$$

In other words, the set of morphisms $\text{Hom}_{\Gamma^d \mathbf{P}_k}(V, W)$ is isomorphic to the set of \mathfrak{S}_d -equivariant morphisms from $V^{\otimes d}$ to $W^{\otimes d}$.

Monoidal structure. The closed symmetric monoidal structure on \mathbf{P}_k induces a closed symmetric monoidal structure on $\Gamma^d \mathbf{P}_k$. Namely, the (internal) tensor product $V \otimes_{\Gamma^d \mathbf{P}_k} W$ of two objects $V, W \in \Gamma^d \mathbf{P}_k$ is the same as the tensor product in \mathbf{P}_k . The tensor product $f \otimes_{\Gamma^d \mathbf{P}_k} f'$ of two morphisms $f \in \text{Hom}_{\Gamma^d \mathbf{P}_k}(V, W)$ and $f' \in \text{Hom}_{\Gamma^d \mathbf{P}_k}(V', W')$ is given as the image of the following composition of maps

$$\begin{aligned} \text{Hom}_{\Gamma^d \mathbf{P}_k}(V, W) \times \text{Hom}_{\Gamma^d \mathbf{P}_k}(V', W') &= \Gamma^d \text{Hom}(V, W) \times \Gamma^d \text{Hom}(V', W') \\ &\hookrightarrow \Gamma^d(\text{Hom}(V, W) \times \text{Hom}(V', W')) \\ &\rightarrow \Gamma^d(\text{Hom}(V, W) \otimes \text{Hom}(V', W')) \\ &\xrightarrow{\cong} \Gamma^d \text{Hom}(V \otimes V', W \otimes W') \\ &= \text{Hom}_{\Gamma^d \mathbf{P}_k}(V \otimes_{\Gamma^d \mathbf{P}_k} V', W \otimes_{\Gamma^d \mathbf{P}_k} W'). \end{aligned}$$

The internal hom on objects is again the same as the internal hom in \mathbf{P}_k , whereas the internal hom $\mathcal{H}om_{\Gamma^d \mathbf{P}_k}(f, f')$ of two morphisms $f \in \text{Hom}_{(\Gamma^d \mathbf{P}_k)^{\text{op}}}(V, W)$ and $f' \in \text{Hom}_{\Gamma^d \mathbf{P}_k}(V', W')$ is given as the image of the following composition of maps

$$\begin{aligned} \text{Hom}_{(\Gamma^d \mathbf{P}_k)^{\text{op}}}(V, W) \times \text{Hom}_{\Gamma^d \mathbf{P}_k}(V', W') &= \Gamma^d \text{Hom}(W, V) \times \Gamma^d \text{Hom}(V', W') \\ &\hookrightarrow \Gamma^d(\text{Hom}(W, V) \times \text{Hom}(V', W')) \\ &\rightarrow \Gamma^d(\text{Hom}(\text{Hom}(V, V'), \text{Hom}(W, W'))) \\ &= \text{Hom}_{\Gamma^d \mathbf{P}_k}(\text{Hom}(V, V'), \text{Hom}(W, W')). \end{aligned}$$

We have an isomorphism, natural in $U, V, W \in \Gamma^d \mathbf{P}_k$:

$$\text{Hom}_{\Gamma^d \mathbf{P}_k}(U \otimes_{\Gamma^d \mathbf{P}_k} V, W) \cong \text{Hom}_{\Gamma^d \mathbf{P}_k}(U, \mathcal{H}om_{\Gamma^d \mathbf{P}_k}(V, W))$$

2.3. The Category of Strict Polynomial Functors

We are now able to define the main object of this thesis, the category of strict polynomial functors. Originally, strict polynomial functors were defined using polynomial maps¹, see [FS97, Definition 2.1]. We use another approach, first introduced by [Kuh98], and follow mainly [Kra13]. This approach allows us, via Day convolution [Day71], to get a closed symmetric monoidal structure from the closed symmetric monoidal structure on the aforementioned category of divided powers.

Let $\mathbf{M}_k = \text{Mod } k$ denote the category of all k -modules.

Definition 2.4. The *category of strict polynomial functors* is

$$\text{Rep } \Gamma_k^d := \text{Fun}_k(\Gamma^d \mathbf{P}_k, \mathbf{M}_k),$$

the category of k -linear representations of $\Gamma^d \mathbf{P}_k$. The morphisms are given by natural transformations and are denoted by $\text{Hom}_{\Gamma_k^d}(X, Y)$ for two strict polynomial functors X and Y . The *degree* of $X \in \text{Rep } \Gamma_k^d$ is d .

Sometimes we need to restrict to the full subcategory of *finite representations*

$$\text{rep } \Gamma_k^d := \text{Fun}_k(\Gamma^d \mathbf{P}_k, \mathbf{P}_k),$$

consisting of all strict polynomial functors X such that $X(V)$ is finitely generated projective for all $V \in \Gamma^d \mathbf{P}_k$.

The category of strict polynomial functors is an abelian category, where (co)kernels and (co)products are computed pointwise over k .

Example 2.5. Let \otimes^d be the functor sending a module $V \in \Gamma^d \mathbf{P}_k$ to $V^{\otimes d} \in \mathbf{M}_k$ and a morphism $f \in \text{Hom}_{\Gamma^d \mathbf{P}_k}(V, W)$ to a morphism in $\text{Hom}_{\mathbf{M}_k}(V^{\otimes d}, W^{\otimes d})$ via the inclusion

$$\text{Hom}_{\Gamma^d \mathbf{P}_k}(V, W) = \Gamma^d \text{Hom}(V, W) \cong \text{Hom}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d} \subseteq \text{Hom}_{\mathbf{M}_k}(V^{\otimes d}, W^{\otimes d}).$$

In the same way we can define Γ^d, S^d respectively Λ^d on objects and morphisms of $\Gamma^d \mathbf{P}_k$ to obtain objects in $\text{Rep } \Gamma_k^d$, again denoted by Γ^d, S^d respectively Λ^d .

¹The name “strict polynomial functor” originated from this definition.

2. Strict Polynomial Functors

Embedding of divided powers. Embedding the category of divided powers into the category of strict polynomial functors serves as the main tool to transfer the symmetric monoidal structure to strict polynomial functors. This is done by using representable functors and the Yoneda lemma.

Definition 2.6. The strict polynomial functor *represented by the object* $V \in \Gamma^d \mathbf{P}_k$ is defined by

$$\Gamma^{d,V}(-) := \mathrm{Hom}_{\Gamma^d \mathbf{P}_k}(V, -) = \Gamma^d \mathrm{Hom}(V, -).$$

Let $X \in \mathrm{Rep} \Gamma_k^d$. Then by the Yoneda lemma we have

$$\mathrm{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X) = \mathrm{Hom}_{\Gamma_k^d}(\mathrm{Hom}_{\Gamma^d \mathbf{P}_k}(V, -), X) \cong X(V) \quad (2.1)$$

for every $V \in \Gamma^d \mathbf{P}_k$. Thus, there is an embedding

$$(\Gamma^d \mathbf{P}_k)^{\mathrm{op}} \hookrightarrow \mathrm{Rep} \Gamma_k^d$$

with image the full subcategory of $\mathrm{Rep} \Gamma_k^d$ consisting of representable functors. Furthermore, it follows from the Yoneda lemma that for all $V \in \Gamma^d \mathbf{P}_k$ the strict polynomial functor $\Gamma^{d,V}$ is a projective object in $\mathrm{Rep} \Gamma_k^d$.

Example 2.7. For $V = k$ one gets

$$\Gamma^{d,k}(-) = \mathrm{Hom}_{\Gamma^d \mathbf{P}_k}(k, -) = \Gamma^d \mathrm{Hom}(k, -) \cong \Gamma^d(-)$$

and thus $\Gamma^{d,k} \cong \Gamma^d$.

Colimits of representable functors. Taking colimits allows us to construct arbitrary strict polynomial functors from representable functors. The reason is an analogue of a free presentation of a module over a ring, see [ML98, III.7]. Let $X \in \mathrm{Rep} \Gamma_k^d$ and $V \in \Gamma^d \mathbf{P}_k$. By the Yoneda isomorphism, there is a correspondence

$$X(V) \ni v \longleftrightarrow F_v \in \mathrm{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X)$$

Let $\mathcal{C}_X = \{F_v \in \mathrm{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X) \mid V \in \Gamma^d \mathbf{P}_k, v \in X(V)\}$ be the category with

- objects the natural transformations F_v from representable functors $\Gamma^{d,V}$ to X , where V runs through all objects in $\Gamma^d \mathbf{P}_k$ and
- morphisms between F_v and F_w , with $v \in X(V)$, $w \in X(W)$, given by a natural transformation $\phi_{v,w} : \Gamma^{d,V} \rightarrow \Gamma^{d,W}$ such that $F_v = F_w \circ \phi_{v,w}$.

Define $\mathcal{F}_X : \mathcal{C}_X \rightarrow \mathrm{Rep} \Gamma_k^d$ to be the functor sending a natural transformation F_v to its domain, the representable functor $\Gamma^{d,V}$. Then $X = \mathrm{colim} \mathcal{F}_X$.

External tensor product. Until now we have considered only strict polynomial functors of one fixed degree. If we take strict polynomial functors (possibly of different degrees) we can form a new strict polynomial functor of as follows.

Definition 2.8. Let d and e be non-negative integers, $X \in \text{Rep } \Gamma_k^d$ and $Y \in \text{Rep } \Gamma_k^e$. The *external tensor product* of X and Y is a strict polynomial functor of degree $d + e$, denoted by $X \boxtimes Y$, and defined

- on an object $V \in \Gamma^{d+e}\mathbf{P}_k$ by $(X \boxtimes Y)(V) := X(V) \otimes Y(V)$, the usual tensor product in \mathbf{M}_k ,
- on a morphism $f \in \text{Hom}_{\Gamma^{d+e}\mathbf{P}_k}(V, W)$ by applying $X \otimes Y$ to the image of f under the following map

$$\begin{aligned} \text{Hom}_{\Gamma^{d+e}\mathbf{P}_k}(V, W) &= \Gamma^{d+e} \text{Hom}(V, W) \\ &\hookrightarrow \Gamma^d \text{Hom}(V, W) \otimes \Gamma^e \text{Hom}(V, W) = \text{Hom}_{\Gamma^d\mathbf{P}_k}(V, W) \otimes \text{Hom}_{\Gamma^e\mathbf{P}_k}(V, W). \end{aligned}$$

Generalized divided, symmetric and exterior powers. Of particular interest are external tensor products of divided, symmetric and exterior powers. Recall that for positive integers n and d we denote by $\Lambda(n, d)$ is the set of compositions of d into n parts, i.e. n -tuples of non-negative integers such that $\sum_i \lambda_i = d$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n, d)$ we can form representable functors $\Gamma^{\lambda_1, k} \in \text{Rep } \Gamma_k^{\lambda_1}, \dots, \Gamma^{\lambda_n, k} \in \text{Rep } \Gamma_k^{\lambda_n}$ and take their external tensor product to obtain a functor in $\text{Rep } \Gamma_k^d$

$$\Gamma^\lambda := \Gamma^{\lambda_1} \boxtimes \dots \boxtimes \Gamma^{\lambda_n}$$

and in the same way we define

$$\begin{aligned} S^\lambda &:= S^{\lambda_1} \boxtimes \dots \boxtimes S^{\lambda_n} \\ \Lambda^\lambda &:= \Lambda^{\lambda_1} \boxtimes \dots \boxtimes \Lambda^{\lambda_n}. \end{aligned}$$

We denote the external tensor product

$$\Gamma^{\lambda_1} \boxtimes \dots \boxtimes \Gamma^{\lambda_n} \xrightarrow{\iota_\Gamma \boxtimes \dots \boxtimes \iota_\Gamma} \overbrace{\Gamma^{\mathbf{1}, \dots, \mathbf{1}}}^{\lambda_1\text{-times}} \boxtimes \dots \boxtimes \overbrace{\Gamma^{\mathbf{1}, \dots, \mathbf{1}}}^{\lambda_n\text{-times}} = \overbrace{\Gamma^{\mathbf{1}, \dots, \mathbf{1}}}^{d\text{-times}}$$

of the inclusion maps ι_Γ (see Remark 2.1) again by ι_Γ and similarly for $\pi_S, \pi_\Lambda, \iota_\Lambda$.

Example 2.9. For the partition $\lambda = \omega = (1, 1, \dots, 1) \in \Lambda^+(d, d)$, the three objects defined above coincide, in fact

$$\Gamma^\omega \cong S^\omega \cong \Lambda^\omega \cong \otimes^d.$$

Remark 2.10. The external tensor product preserves projectivity and thus Γ^λ is a projective object in $\text{Rep } \Gamma_k^d$ for all $\lambda \in \Lambda(n, d)$. Moreover every projective object is a direct summand of a finite direct sum of objects of the form Γ^λ ([Kra13, Proposition 2.9]). In particular, there is a canonical decomposition of the strict polynomial functor represented by k^n

$$\Gamma^{d, k^n} = \bigoplus_{\lambda \in \Lambda(n, d)} \Gamma^\lambda. \quad (2.2)$$

Note that in general Γ^λ is not indecomposable, see Remark 3.21 for more details.

Definition 2.11. Let $\Gamma = \{\Gamma^\lambda\}_{\lambda \in \Lambda(d)}$ and $S = \{S^\lambda\}_{\lambda \in \Lambda(d)}$. Let as usual $\text{add } \Gamma$, respectively $\text{add } S$ denote the full subcategory of $\text{Rep } \Gamma_k^d$ whose objects are direct summands of finite direct sums of objects in Γ , respectively S .

2. Strict Polynomial Functors

Frobenius twist. If k is a field of characteristic p , we denote by $F: k \rightarrow k$ the Frobenius endomorphism, defined by $F(\alpha) = \alpha^p$ for all $\alpha \in k$. For $V \in \Gamma^d \mathbf{P}_k$ we denote by $V^{(1)}$ the module obtained by extending scalars via F , i.e. $V^{(1)} = k \otimes_F V$. For all $r \geq 1$ the r -th Frobenius twist $I^{(r)}$ is then defined inductively by

$$I^{(1)}(V) := V^{(1)} \quad \text{and} \quad I^{(r+1)}(V) := I^{(1)}(I^{(r)}(V)).$$

See [FFPS03, Pirashvili 1.2] for an explicit description of a basis. For an arbitrary strict polynomial functor X the twisted functor is defined by

$$X^{(r)} := X \circ I^{(r)}.$$

If X is of degree d , i.e. $X \in \text{Rep } \Gamma_k^d$, then $X^{(r)}$ is of degree dp^r , i.e. $X^{(r)} \in \text{Rep } \Gamma_k^{dp^r}$.

2.4. Morphisms Between Strict Polynomial Functors

For projectives Γ^λ and Γ^μ there exists an explicit description of the morphism set, see also [Kra14] and [Tot97]. For any partition $\nu \in \Lambda(m, e)$ there is an inclusion $\iota_\nu: \Gamma^e \hookrightarrow \Gamma^\nu$ given by

$$(\iota_\nu)_V: v \mapsto v$$

for $v \in \Gamma^e(V)$ and a product map $p_\nu: \Gamma^\nu \rightarrow \Gamma^e$ given by

$$(p_\nu)_V: v_1 \boxtimes \cdots \boxtimes v_m \mapsto \sum_{\sigma \in \mathfrak{S}_d / \mathfrak{S}_\nu} (v_1 \boxtimes \cdots \boxtimes v_m) \sigma$$

for $v_1 \boxtimes \cdots \boxtimes v_m \in \Gamma^{\nu_1}(V) \boxtimes \cdots \boxtimes \Gamma^{\nu_m}(V) = \Gamma^\nu(V)$. We now put these maps together.

Recall that A_μ^λ is the set of all $n \times m$ matrices $A = (a_{ij})$ with entries in \mathbb{N} such that $\lambda_i = \sum_j a_{ij}$ and $\mu_j = \sum_i a_{ij}$. We consider $a_{i-} := (a_{i1}, \dots, a_{im})$ as a partition of $\Lambda(m, \lambda_i)$ and in the same way $a_{-j} := (a_{1j}, \dots, a_{nj}) \in \Lambda(n, \mu_j)$.

Definition 2.12. Given a matrix $A = (a_{ij}) \in A_\mu^\lambda$ we define the corresponding *standard morphism* $\varphi_A \in \text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda)$ as the following composition:

$$\varphi_A: \Gamma^\mu = \prod_{j=1}^m \Gamma^{\mu_j} \xrightarrow{\prod_{j=1}^m (\iota_{a_{-j}})} \prod_{j=1}^m \left(\prod_{i=1}^n \Gamma^{a_{ij}} \right) \cong \prod_{i=1}^n \left(\prod_{j=1}^m \Gamma^{a_{ij}} \right) \xrightarrow{\prod_{i=1}^n (p_{a_{i-}})} \prod_{i=1}^n \Gamma^{\lambda_i} = \Gamma^\lambda$$

See Appendix A.1.4 for examples and a more explicit description. In particular, in the following we use an identification of $A \in A_\mu^\lambda$ with a pair of sequences $(\underline{j}, \underline{i})$ where $\underline{j} \in \mu$ and $\underline{i} \in \lambda$, see (A.1).

2.5. Monoidal Structure

Next let us explain how the symmetric monoidal structure on $\Gamma^d \mathbf{P}_k$ yields a symmetric monoidal structure on $\text{Rep } \Gamma_k^d$.

Definition 2.13. For representable functors $\Gamma^{d,V}$ and $\Gamma^{d,W}$ in $\text{Rep } \Gamma_k^d$ we define an internal tensor product by

$$\Gamma^{d,V} \otimes_{\Gamma_k^d} \Gamma^{d,W} := \Gamma^{d,V \otimes_{\Gamma_k^d} W}.$$

For arbitrary objects X and Y in $\text{Rep } \Gamma_k^d$ define

$$\begin{aligned} \Gamma^{d,V} \otimes_{\Gamma_k^d} X &:= \text{colim}(\Gamma^{d,V} \otimes_{\Gamma_k^d} \mathcal{F}_X), \\ X \otimes_{\Gamma_k^d} Y &:= \text{colim}(\mathcal{F}_X \otimes_{\Gamma_k^d} Y), \end{aligned}$$

where $\Gamma^{d,V} \otimes_{\Gamma_k^d} \mathcal{F}_X$, respectively $\mathcal{F}_X \otimes_{\Gamma_k^d} Y$ is the functor sending F_v to $\Gamma^{d,V} \otimes_{\Gamma_k^d} \mathcal{F}_X(F_v)$, respectively $\mathcal{F}_X(F_v) \otimes_{\Gamma_k^d} Y$ and \mathcal{F}_X is the functor sending a natural transformation F_v to $\Gamma^{d,V}$ for $v \in X(V)$ (cf. page 16).

Remark 2.14. The tensor unit is given by

$$\mathbf{1}_{\Gamma_k^d} := \Gamma^{d,k} \cong \Gamma^{(d)},$$

and also the associator α , the left unitor λ , the right unitor ϱ and the braiding γ are defined on representable functors using the corresponding morphisms in $\Gamma^d \mathcal{P}_k$ and then extended to arbitrary objects using colimits.

The closed monoidal structure on $\Gamma^d \mathcal{P}_k$ also yields an internal hom in $\text{Rep } \Gamma_k^d$.

Definition 2.15. For representable functors $\Gamma^{d,V}$ and $\Gamma^{d,W}$ in $\text{Rep } \Gamma_k^d$ we define

$$\mathcal{H}om_{\Gamma_k^d}(\Gamma^{d,V}, \Gamma^{d,W}) := \Gamma^{d, \mathcal{H}om_{\Gamma^d \mathcal{P}_k}(V, W)}.$$

For arbitrary objects X and Y in $\text{Rep } \Gamma_k^d$ define

$$\begin{aligned} \mathcal{H}om_{\Gamma_k^d}(\Gamma^{d,V}, X) &:= \text{colim}(\mathcal{H}om_{\Gamma_k^d}(\Gamma^{d,V}, \mathcal{F}_X)), \\ \mathcal{H}om_{\Gamma_k^d}(X, Y) &:= \text{lim}(\mathcal{H}om_{\Gamma_k^d}(\mathcal{F}_X, Y)). \end{aligned}$$

This is indeed an internal hom, i.e. we have a natural isomorphism for $X, Y, Z \in \text{Rep } \Gamma_k^d$, see [Kra13, Proposition 2.4],

$$\text{Hom}_{\Gamma_k^d}(X \otimes_{\Gamma_k^d} Y, Z) \cong \text{Hom}_{\Gamma_k^d}(X, \mathcal{H}om_{\Gamma_k^d}(Y, Z)). \quad (2.3)$$

We briefly collect some results concerning calculations of the internal tensor product.

Proposition 2.16. [Kra13, Proposition 3.4, Corollary 3.7]

- $\Lambda^d \otimes_{\Gamma_k^d} \Gamma^\lambda \cong \Lambda^\lambda$
- $\Lambda^d \otimes_{\Gamma_k^d} \Lambda^\lambda \cong S^\lambda$
- $S^d \otimes_{\Gamma_k^d} \Gamma^\lambda \cong \Lambda^d \otimes_{\Gamma_k^d} \Lambda^d \otimes_{\Gamma_k^d} \Gamma^\lambda \cong S^\lambda$

In Chapter 5 we calculate the (internal) tensor product of certain strict polynomial functors.

2.6. Highest Weight Structure

We assume in this section that k is a field. In this case, the category of strict polynomial functors is a highest weight category. This is explained in detail in [Kra14], in particular we refer to [Kra14, Section 6] for a definition of a highest weight category. We briefly collect the most important results and definitions, for example of (co)standard and simple objects.

2.6.1. Partial Order

The (co)standard and simple objects in $\text{Rep } \Gamma_k^d$ are each indexed by partitions of d , i.e. compositions λ of d such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots > 0$. Recall that the set of partitions of d is denoted by $\Lambda(d)^+$. This is a finite poset with ordering given by the *lexicographic order* which can be defined on $\Lambda(d)^+$ as follows:

$$\mu \leq \lambda \text{ if } \mu_1 < \lambda_1 \text{ or } \mu_i = \lambda_i \text{ for } 1 \leq i < r \text{ implies } \mu_r \leq \lambda_r.$$

2.6.2. Schur and Weyl Functors

The costandard objects $\nabla(\lambda)$ are given by the *Schur functors* and the standard objects $\Delta(\lambda)$ by their duals, the *Weyl functors*. For their definition we need the following maps (cf. [Kra14, Section 3]): let $\lambda \in \Lambda(d)^+$ and σ_λ be the permutation of \mathfrak{S}_d defined on $r = \lambda_1 + \dots + \lambda_{i-1} + j$ with $1 \leq j \leq \lambda_i$ by

$$\sigma_\lambda(r) = \sigma_\lambda(\lambda_1 + \dots + \lambda_{i-1} + j) = \lambda'_1 + \dots + \lambda'_{j-1} + i,$$

where λ' is the partition conjugate to λ . Note that every $r \in \{1, \dots, d\}$ can be written in a unique way as $r = \lambda_1 + \dots + \lambda_{i-1} + j$ for some i and j if $1 \leq j \leq \lambda_i$, thus $\sigma_\lambda(r)$ is well-defined.

Remark 2.17. For an explicit calculation of the permutation $\sigma_\lambda(r)$ we write down the numbers $1, 2, \dots$ into the Young diagram corresponding to λ first from left to right, from top to bottom. Afterwards we write down the numbers $1, 2, \dots$ into the same diagram but this time from top to bottom, left to right. Then the first number in each box is mapped under σ_λ to the second number in this box.

For $\lambda = (4, 3, 1) \in \Lambda(3, 8)$, for example, this looks as follows

1	2	3	4	
1	4	6	8	
5	6	7		
2	5	7		
8				
3				

That is, $\sigma_\lambda(1) = 1$, $\sigma_\lambda(2) = 4$, $\sigma_\lambda(3) = 6$ and so on $\dots \sigma_\lambda(8) = 3$.

Using the permutation $\sigma_\lambda \in \mathfrak{S}_d$, we can define a permutation morphism corresponding to λ .

Definition 2.18. Let $\lambda \in \Lambda^+(n, d)$ and $\omega = (1, 1, \dots, 1) \in \Lambda(d, d)$. The *permutation morphism* s_λ in $\text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\omega)$ is defined at $V \in \Gamma^d \mathbf{P}_k$ for $v_1 \otimes \dots \otimes v_d \in V^{\otimes d} = \Gamma^\omega(V)$ by

$$(s_\lambda)_V: \Gamma^\omega(V) \rightarrow \Gamma^\omega(V)$$

$$v_1 \otimes \dots \otimes v_d \mapsto v_{\sigma_\lambda(1)} \otimes \dots \otimes v_{\sigma_\lambda(d)}.$$

For an explicit description in terms of a matrix $A \in A_\mu^\lambda$ see (A.6).

Recall for $\lambda \in \Lambda(n, d)$ and its conjugate partition λ' the inclusion maps $\iota_\Gamma: \Gamma^\lambda \rightarrow \Gamma^\omega$ and $\iota_\Lambda: \Lambda^{\lambda'} \rightarrow \Gamma^\omega$ and the projection maps $\pi_S: \Gamma^\omega \rightarrow S^\lambda$ and $\pi_\Lambda: \Gamma^\omega \rightarrow \Lambda^{\lambda'}$. These maps can be composed to get

$$\varphi_{S_\lambda}: \Lambda^{\lambda'} \xrightarrow{\iota_\Lambda} \Gamma^\omega \xrightarrow{s_\lambda} \Gamma^\omega \xrightarrow{\pi_S} S^\lambda,$$

$$\varphi_{W_\lambda}: \Gamma^\lambda \xrightarrow{\iota_\Gamma} \Gamma^\omega \xrightarrow{s_{\lambda'}} \Gamma^\omega \xrightarrow{\pi_\Lambda} \Lambda^{\lambda'}.$$

Finally, we give the definition of Schur and Weyl functors.

Definition 2.19. The *Schur functor* S_λ is the image of the map $\varphi_{S_\lambda} = \pi_S \circ s_\lambda \circ \iota_\Lambda$. The *Weyl functor* W_λ is the image of the map $\varphi_{W_\lambda} = \pi_\Lambda \circ s_{\lambda'} \circ \iota_\Gamma$.

Remark 2.20. Unfortunately, the name ‘‘Schur functor’’ is commonly used for two very different kind of functors: the first use is dedicated to a series of functors, namely the above defined Schur functors S_λ , the duals of Weyl functors and indexed by partitions $\lambda \in \Lambda^+(d)$. Another use of this name is *the* Schur functor defined in Section 3.4 relating strict polynomial functors, respectively modules over the Schur algebra to modules over the group algebra of the symmetric group. It should be clear from the context, which Schur functor is meant.

Note that the Schur functor S_λ is a subfunctor of S^λ whereas the Weyl functor W_λ is a quotient functor of Γ^λ . To be more precise we have the following

Theorem 2.21 ([Kra14, Theorem 4.7 and Corollary 4.8]). *There are isomorphisms*

$$S_\lambda \cong \bigcap_{\mu \not\leq \lambda} \bigcap_{\varphi: S^\lambda \rightarrow S^\mu} \ker \varphi \quad \text{and} \quad W_\lambda \cong \Gamma^\lambda / \left(\sum_{\mu \not\leq \lambda} \sum_{\varphi: \Gamma^\mu \rightarrow \Gamma^\lambda} \text{im } \varphi \right).$$

Note that $W_\lambda(V^*) \cong (S_\lambda(V))^*$.

Definition 2.22. We denote by $\text{Filt}(\nabla)$ the subcategory of Schur filtered functors, i.e. the full subcategory consisting of all $X \in \text{Rep } \Gamma_k^d$ such that there exist a filtration

$$0 = X_0 \subseteq X_1 \subseteq \dots \subseteq X_s = X$$

with $X_i/X_{i-1} \cong S_\lambda$ for some $\lambda \in \Lambda(n, d)$ and for all $1 \leq i \leq s$. Similarly, we denote by $\text{Filt}(\Delta)$ the subcategory of Weyl filtered functors.

2.6.3. Simple Functors

Let $U(\lambda)$ be the maximal subfunctor of W_λ such that $\sum_{\varphi: \Gamma^\lambda \rightarrow U(\lambda)} \text{im } \varphi = 0$. Then $U(\lambda)$ is a maximal subfunctor of W_λ (see [Kra14, Proposition 4.9.(2)]).

Definition 2.23. The simple functor corresponding to the partition λ is defined by

$$L_\lambda := W_\lambda / U(\lambda).$$

The simple functor L_λ is isomorphic to the (simple) socle of S_λ (see [Kra14, Lemma 4.10]) and each composition factor L_μ of $U(\lambda)$ satisfies $\mu < \lambda$ (see [Kra14, Proposition 4.9.(3)]). Finally we cite the following

Theorem 2.24 ([Kra14, Theorem 6.1]). *The category $\text{Rep } \Gamma_k^d$ of strict polynomial functors is a highest weight category with respect to the set of partitions of weight d and the lexicographic order. The costandard objects $\nabla(\lambda)$ are given by Schur functors S_λ , the standard objects $\Delta(\lambda)$ are given by Weyl functors W_λ and the simple objects are $L_\lambda = W_\lambda / U(\lambda)$.*

2.7. Dualities

The category of strict polynomial functors admits two kinds of dual, one corresponding to the transpose duality for modules over the general linear group and the other one using the closed monoidal structure on $\text{Rep } \Gamma_k^d$. In this section we again work over an arbitrary commutative ring k .

2.7.1. The Kuhn Dual

Definition 2.25 ([Kuh94, 3.4]). For $X \in \text{Rep } \Gamma_k^d$ define its *Kuhn dual* X° by

$$\begin{aligned} (-)^\circ: (\text{Rep } \Gamma_k^d)^{\text{op}} &\rightarrow \text{Rep } \Gamma_k^d \\ X &\mapsto X^\circ \\ \text{with } X^\circ(V) &:= X(V^*)^* \text{ for } V \in \Gamma^d \mathbf{P}_k. \end{aligned}$$

Taking the Kuhn dual is a contravariant exact functor, sending projective objects to injective objects and vice versa.

Example 2.26.

- (i) Symmetric powers are duals of divided powers, i.e. $(\Gamma^d)^\circ = S^d$ (see Remark 2.1(ii)) and more generally $(\Gamma^\lambda)^\circ = S^\lambda$.
- (ii) Exterior powers are self-dual, i.e. $(\Lambda^\lambda)^\circ = \Lambda^\lambda$ (see Remark 2.1(iii)).
- (iii) Weyl functors are duals of Schur functors, i.e. $W_\lambda^\circ = S_\lambda$.
- (iv) Simple functors are self-dual, i.e. $L_\lambda^\circ \cong L_\lambda$.

For every $M, N \in \mathbf{M}_k$ we have $\text{Hom}(M, N^*) \cong \text{Hom}(N, M^*)$ and thus in particular

$$\begin{aligned} \text{Hom}(X(V), Y^\circ(V)) &= \text{Hom}(X(V), Y(V^*)^*) \cong \text{Hom}(Y(V^*), X(V)^*) \\ &\cong \text{Hom}(Y(V^*), X(V^{**})^*) = \text{Hom}(Y(V^*), X^\circ(V^*)). \end{aligned}$$

Thus, we get a natural isomorphism

$$\text{Hom}_{\Gamma^d \mathbf{P}_k}(X, Y^\circ) \cong \text{Hom}_{\Gamma^d \mathbf{P}_k}(Y, X^\circ). \quad (2.4)$$

For $Y = X^\circ$ we get $\text{Hom}_{\Gamma^d \mathbf{P}_k}(X^\circ, X^\circ) \cong \text{Hom}_{\Gamma^d \mathbf{P}_k}(X, X^{\circ\circ})$ and for X finite the evaluation map (the one corresponding to $\text{id}_{X^\circ} \in \text{Hom}_{\Gamma^d \mathbf{P}_k}(X^\circ, X^\circ)$) is an isomorphism $X \rightarrow X^{\circ\circ}$. Furthermore, the following results will be important.

Lemma 2.27. [Kra13, Lemma 2.7 and Lemma 2.8] For all $X, Y \in \text{Rep } \Gamma_k^d$ we have a natural isomorphism

$$\mathcal{H}om_{\Gamma^d \mathbf{P}_k}(X, Y^\circ) \cong \mathcal{H}om_{\Gamma^d \mathbf{P}_k}(Y, X^\circ).$$

If X is finitely presented we have natural isomorphisms

$$\begin{aligned} X \otimes_{\Gamma_k^d} Y^\circ &\cong \mathcal{H}om_{\Gamma_k^d}(X, Y)^\circ \\ (X \otimes_{\Gamma_k^d} Y)^\circ &\cong \mathcal{H}om_{\Gamma_k^d}(X, Y^\circ). \end{aligned}$$

2.7.2. The Monoidal Dual

Using the internal hom, one can define another dual in $\text{Rep } \Gamma_k^d$:

Definition 2.28. For $X \in \text{Rep } \Gamma_k^d$ define its *monoidal dual* X^\vee by

$$\begin{aligned} (-)^\vee: (\text{Rep } \Gamma_k^d)^{\text{op}} &\rightarrow \text{Rep } \Gamma_k^d \\ X &\mapsto X^\vee := \mathcal{H}om_{\Gamma_k^d}(X, \Gamma^d) \end{aligned}$$

By definition this functor is left exact, but in general not right exact. By Lemma 2.27, for X finitely presented, it holds that

$$X^\vee = \mathcal{H}om_{\Gamma_k^d}(X, \Gamma^d) \cong (X \otimes_{\Gamma_k^d} S^d)^\circ.$$

Divided powers. It follows immediately that $(\Gamma^d)^\vee \cong (\Gamma^d \otimes S^d)^\circ \cong \Gamma^d$. Moreover, by using Proposition 2.16, we get

$$(\Gamma^\lambda)^\vee = \mathcal{H}om(\Gamma^\lambda, \Gamma^d) \cong (\Gamma^\lambda \otimes S^d)^\circ \cong (S^\lambda)^\circ \cong \Gamma^\lambda.$$

Exterior and Symmetric Powers

Let us now calculate the monoidal duals of symmetric and exterior powers. We make use of the following fact.

Lemma 2.29. Let $\varphi_A: \Gamma^\mu \rightarrow \Gamma^\lambda$ be the morphism in $\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda)$ corresponding to the matrix $A \in A_\lambda^\mu$, then $\varphi_A^\vee = \varphi_{A^T}$, where A^T denotes the transposed matrix of A .

Proof. See Appendix A.1.4. □

2. Strict Polynomial Functors

Proof. Consider first the case $\lambda = (d)$. Take the following presentation

$$\bigoplus_{i=1}^{d-1} \Gamma^\omega \xrightarrow{[\rho_1 \cdots \rho_{d-1}]} \Gamma^\omega \xrightarrow{\pi_S} S^d \rightarrow 0$$

where ρ_i is defined as before. Applying the monoidal dual then yields an exact sequence

$$0 \rightarrow (S^d)^\vee \xrightarrow{(\pi_S)^\vee} (\Gamma^\omega)^\vee \xrightarrow{\sum \rho_i^\vee} \bigoplus (\Gamma^\omega)^\vee.$$

Since $A_{\text{id}}^T = A_{\text{id}}$ and $A_{\sigma_i}^T = A_{\sigma_i}$ we get $\rho_i^\vee = \rho_i$ and thus $(S^d)^\vee$ is the kernel of $\sum \rho_i$. Together with (2.6) this yields

$$(S^d)^\vee \cong \ker\left(\sum \rho_i\right) \cong \Gamma^d.$$

For arbitrary $\lambda \in \Lambda(n, d)$ we use Proposition 2.16, Lemma 2.27 and the isomorphism $\mathcal{H}om(X, Y \otimes \Gamma^\lambda) \cong \mathcal{H}om(X, Y) \otimes \Gamma^\lambda$ ([Kra13, Lemma 2.6]) to obtain

$$\begin{aligned} (S^\lambda)^\vee &= \mathcal{H}om(S^\lambda, \Gamma^d) \\ &\cong \mathcal{H}om(S^d, \Gamma^\lambda) \\ &\cong \mathcal{H}om(S^d, \Gamma^d \otimes \Gamma^\lambda) \\ &\cong \mathcal{H}om(S^d, \Gamma^d) \otimes \Gamma^\lambda \\ &\cong (S^d)^\vee \otimes \Gamma^\lambda \\ &\cong \Gamma^\lambda. \end{aligned}$$

□

As an immediate consequence we obtain the following result.

Corollary 2.32. $S^d \otimes S^d \cong S^d$.

Proof. From Lemma 2.27 we get $S^d \otimes S^d \cong \mathcal{H}om(S^d, \Gamma^d)^\circ = ((S^d)^\vee)^\circ$. We use Proposition 2.31 to obtain $((S^d)^\vee)^\circ \cong (\Gamma^d)^\circ \cong S^d$. □

3. Representations of the Symmetric Group

The study of representations has a long standing history. If k is a field of characteristic 0 or $p > d$, the group algebra $k\mathfrak{S}_d \text{Mod}$ is semi-simple and the simple modules can be parametrized by partitions and are well-known. These simple modules, called Specht modules, can be constructed integrally, but when reducing modulo the characteristic these modules are not simple in general if $p \leq d$. In particular, there are fewer simple modules in the non semi-simple case and an explicit full determination of the simple modules is still an open problem.

In this chapter we recall important definitions and concepts for modules over the group algebra of the symmetric group. Notably we describe the standard closed monoidal structure on $k\mathfrak{S}_d \text{Mod}$. We mostly use a characteristic-free approach and distinguish between cases only when necessary.

In the last section we introduce the Schur functor \mathcal{F} , relating strict polynomial functors to representations of the symmetric group. We show that \mathcal{F} induces an equivalence between certain subcategories and that it preserves the closed monoidal structure. This property is later used as an important tool to advance in the description of the tensor product on strict polynomial functors by exploiting known results of the Kronecker product.

Of course, it would be desirable to also obtain results on the Kronecker product, i.e. from calculations of the tensor product of strict polynomial functors. However, the understanding of the monoidal structure on strict polynomial functors is not yet sufficiently advanced.

Recall that \mathfrak{S}_d denotes the *symmetric group* of all permutations on d elements. We can form the group algebra

$$k\mathfrak{S}_d := \left\{ \sum_{\sigma \in \mathfrak{S}_d} k_\sigma \sigma \mid k_\sigma \in k \right\},$$

where the multiplication is induced by the group operation in \mathfrak{S}_d .

Definition 3.1. We define $k\mathfrak{S}_d \text{Mod}$ to be the category of all left $k\mathfrak{S}_d$ -modules. For $N, N' \in k\mathfrak{S}_d \text{Mod}$ we denote by $\text{Hom}_{k\mathfrak{S}_d}(N, N')$ the morphisms in $k\mathfrak{S}_d \text{Mod}$ i.e. $k\mathfrak{S}_d$ -module homomorphism from N to N' .

The full subcategory of all modules that are finitely generated projective over k is denoted by $k\mathfrak{S}_d \text{mod}$.

3. Representations of the Symmetric Group

If the characteristic of k does not divide the order of the group \mathfrak{S}_d , i.e. $\text{char } k \nmid d!$, then by Maschkes theorem $k\mathfrak{S}_d$ is semisimple. That means every module decomposes into simple modules. This is in particular the case when $k = \mathbb{C}$. If $\text{char } k$ divides $d!$, then $k\mathfrak{S}_d$ is not semisimple.

3.1. Monoidal Structure

Every group algebra carries automatically a *cocommutative Hopf algebra structure*. It is given by defining the comultiplication, counit, and antipode as the linear extension of the following maps defined on $\sigma \in \mathfrak{S}_d$ by

$$\begin{aligned}\Delta(\sigma) &:= \sigma \otimes \sigma \\ \epsilon(\sigma) &:= 1 \\ S(\sigma) &:= \sigma^{-1}.\end{aligned}$$

Every Hopf algebra equips its module category with a closed monoidal structure, see e.g. [Kas95, III.5]. The internal tensor product is given by taking the tensor product over k . The group algebra acts on it via composition with the comultiplication. In the case of the group algebra $k\mathfrak{S}_d$, this reads

$$\begin{aligned}k\mathfrak{S}_d \text{ Mod} \times k\mathfrak{S}_d \text{ Mod} &\rightarrow k\mathfrak{S}_d \text{ Mod} \\ (N, N') &\mapsto N \otimes_k N',\end{aligned}$$

$$\sigma \cdot (m \otimes m') = \Delta(\sigma)(\cdot m \otimes m') = \sigma \cdot m \otimes \sigma \cdot m',$$

the diagonal action. It is then linearly extended to all elements of $k\mathfrak{S}_d$.

The tensor unit is given by $\mathbf{1}_{k\mathfrak{S}_d} = k$, the trivial $k\mathfrak{S}_d$ -module. The associator α , the left unitor λ , the right unitor ϱ and the braiding γ are given by

$$\begin{aligned}\lambda_N &: k \otimes N \rightarrow N, & r \otimes m &\mapsto r \cdot m \\ \varrho_N &: N \otimes \mathbf{1} \rightarrow N, & r \otimes m &\mapsto m \cdot r \\ \alpha_{N,N',N''} &: (N \otimes N') \otimes N'' \rightarrow N \otimes (N' \otimes N''), & \text{the usual associativity map} \\ \gamma_{N,N'} &: N \otimes N' \rightarrow N' \otimes N, & m \otimes m' &\mapsto m' \otimes m.\end{aligned}$$

The antipode S of a Hopf algebra yields also an internal hom. In the case of $k\mathfrak{S}_d$ it is given by

$$\begin{aligned}(k\mathfrak{S}_d \text{ Mod})^{\text{op}} \times k\mathfrak{S}_d \text{ Mod} &\rightarrow k\mathfrak{S}_d \text{ Mod} \\ (N, N') &\mapsto \mathcal{H}om(N, N') = \text{Hom}_k(N, N'),\end{aligned}$$

where for $m \in N$ and $\sigma \in \mathfrak{S}_d$ the module action is $\sigma \cdot f(m) = \sigma \cdot f(S(\sigma) \cdot m) = \sigma \cdot f(\sigma^{-1} \cdot m)$.

Dual. This internal hom yields naturally a dual, namely

$$(-)^* := \mathcal{H}om(-, \mathbf{1}_{k\mathfrak{S}_d}) = \mathcal{H}om(-, k).$$

Let $N \in k\mathfrak{S}_d \text{ Mod}$ and $m \in N$. Since $k\mathfrak{S}_d$ acts trivially on $f(S(\sigma) \cdot m) \in k$, the module action is given by

$$\sigma \cdot f(m) = f(\sigma^{-1} \cdot m). \quad (3.1)$$

3.2. Permutation Modules

Let n be any positive integer. Recall that $I(n, d) := \{\underline{i} = (i_1 \dots i_d) \mid 1 \leq i_l \leq n\}$. Let E be an n -dimensional k -vector space with basis $\{e_1, \dots, e_n\}$. A basis for the d -fold tensor product $E^{\otimes d}$ can be indexed by the set $I(n, d)$. We write $e_{\underline{i}} = e_{i_1} \otimes \dots \otimes e_{i_d}$ for $\underline{i} = (i_1 \dots i_d) \in I(n, d)$.

Let the symmetric group act on the right by place permutation, i.e.

$$(v_1 \otimes \dots \otimes v_d)\sigma := v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)} \quad \text{for } \sigma \in \mathfrak{S}_d, v_1 \otimes \dots \otimes v_d \in E^{\otimes d}.$$

We can use this action to define an action on the left, namely we define for $\sigma \in \mathfrak{S}_d$ and $v_1 \otimes \dots \otimes v_d \in E^{\otimes d}$

$$\sigma(v_1 \otimes \dots \otimes v_d) := (v_1 \otimes \dots \otimes v_d)\sigma^{-1} = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(d)}. \quad (3.2)$$

By linear extension of this action, $E^{\otimes d}$ becomes a left $k\mathfrak{S}_d$ -module.

Definition 3.2. The *transitive permutation module* M^λ corresponding to the composition λ is the submodule of $E^{\otimes d}$ with k -basis $\{e_{\underline{i}} \mid \underline{i} \text{ belongs to } \lambda\}$.

Note that the set $\{e_{\underline{i}} \mid \underline{i} \text{ belongs to } \lambda\}$ is invariant under the action of \mathfrak{S}_d and thus M^λ is really a submodule of $E^{\otimes d}$. Each basis element $e_{\underline{i}}$ of $E^{\otimes d}$ belongs to exactly one M^λ and hence $E^{\otimes d}$ decomposes into a direct sum

$$E^{\otimes d} = \bigoplus_{\lambda \in \Lambda(n, d)} M^\lambda. \quad (3.3)$$

Note that $M^\lambda \cong M^\mu$ if and only if the compositions λ and μ yield the same partition after reordering. Thus, a complete set of isomorphism classes of permutation modules is indexed by all partitions.

Example 3.3. Let $\lambda = (d, 0, \dots, 0) \in \Lambda(n, d)$, the partition consisting only of one non-zero entry. Then $M^\lambda \cong k$, the trivial $k\mathfrak{S}_d$ -module.

Remark 3.4. Recall from Section 1.3.1 that \mathfrak{S}_λ denotes the Young subgroup $\mathfrak{S}_{\lambda_1} \times \dots \times \mathfrak{S}_{\lambda_n} \subseteq \mathfrak{S}_d$. For every coset $\bar{\sigma} \in \mathfrak{S}_d/\mathfrak{S}_\lambda$ we define $\bar{\sigma}e_{\underline{i}}$ to be $\sigma e_{\underline{i}}$ for some representative σ of $\bar{\sigma}$. If $\underline{i} \in \lambda$ this is independent of the choice of σ . We let \mathfrak{S}_d act on $\mathfrak{S}_d/\mathfrak{S}_\lambda$ in the usual way.

3. Representations of the Symmetric Group

Let $\underline{i}_\lambda = (1 \dots 1 \ 2 \dots 2 \dots n \dots n) \in \lambda$ be the weakly increasing sequence with λ_l entries equal to l . Then the set $\{e_{\underline{i}} \mid \underline{i} \in \lambda\}$ can be identified with the set $\{\bar{\sigma}e_{\underline{i}_\lambda} \mid \bar{\sigma} \in \mathfrak{S}_d/\mathfrak{S}_\lambda\}$ and this induces an isomorphism of \mathfrak{S}_d -modules

$$M^\lambda \cong k \langle \bar{\sigma} \mid \bar{\sigma} \in \mathfrak{S}_d/\mathfrak{S}_\lambda \rangle.$$

Furthermore we have an isomorphism of $k\mathfrak{S}_d$ -modules

$$k \langle \bar{\sigma} \mid \bar{\sigma} \in \mathfrak{S}_d/\mathfrak{S}_\lambda \rangle \cong k\mathfrak{S}_d \left(\sum_{\pi \in \mathfrak{S}_\lambda} \pi \right)$$

induced by the map sending $\bar{\sigma} \in \mathfrak{S}_d/\mathfrak{S}_\lambda$ to $\sigma \left(\sum_{\pi \in \mathfrak{S}_\lambda} \pi \right)$ where σ is a representative of $\bar{\sigma}$.

Thus, we get an isomorphism of $k\mathfrak{S}_d$ -modules

$$M^\lambda \cong k\mathfrak{S}_d \left(\sum_{\pi \in \mathfrak{S}_\lambda} \pi \right).$$

Permutation modules are self-dual. For every $\lambda \in \Lambda(n, d)$, there is a non-degenerate, \mathfrak{S}_d -invariant bilinear form

$$\begin{aligned} \beta: M^\lambda \times M^\lambda &\rightarrow k \\ (e_{\underline{i}}, e_{\underline{j}}) &\mapsto \delta_{\underline{i}\underline{j}}. \end{aligned}$$

Details can be found in [JK81, 7.1.6]; we here identify the basis element $e_{\underline{i}}$ for $\underline{i} = i_1 i_2 \dots i_d$ with the λ -tableau with j -th row consisting of all the integers k such that $i_k = j$. (i.e. the integers in the j -th row correspond to the positions in \underline{i} where a j is located). This bilinear form yields the following isomorphism:

$$\begin{aligned} M^\lambda &\rightarrow \mathcal{H}om_k(M^\lambda, k) = (M^\lambda)^* \\ e_{\underline{i}} &\mapsto \beta(e_{\underline{i}}, -) = e_{\underline{i}}^* = (e_{\underline{j}} \mapsto \delta_{\underline{i}\underline{j}}) \end{aligned}$$

Standard morphisms of permutation modules. Let $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$. and fix $\underline{i} = (11 \dots 2 \dots nn) \in \lambda$. For every $\underline{j} \in \mu$ we define a matrix A by $a_{ij} := \#\{l \mid i_l = i, j_l = j\}$. Let $A_{\underline{j}}$ be the composition $(a_{11}, a_{12}, \dots, a_{21}, a_{22}, \dots, a_{mn})$. We define $I_{\underline{j}}$ to be a complete set of representatives of $\mathfrak{S}_\lambda/\mathfrak{S}_{A_{\underline{j}}}$. As explained in Appendix A.1.1 a set of generators of $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$ is given by elements $\xi_{\underline{j}, \underline{i}}$, $\underline{j} \in \mu$, defined by

$$\begin{aligned} \xi_{\underline{j}, \underline{i}}: \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu) \\ e_{\underline{i}} \mapsto \sum_{\sigma \in I_{\underline{j}}} e_{\underline{j}} \sigma. \end{aligned}$$

Note that $\xi_{\underline{j}, \underline{i}} = \xi_{\underline{j}', \underline{i}}$ if and only if there exists a $\sigma \in \mathfrak{S}_d$ such that $\underline{i}\sigma = \underline{i}$ and $\underline{j}\sigma = \underline{j}'$. This set in turn can be identified with the set of matrices A_μ^λ . See Appendix A.1.1 for more details and explanations.

Tensor products of permutation modules. Recall that A_μ^λ is the set of all $n \times m$ matrices $A = (a_{ij})$ with entries in \mathbb{N} such that $\lambda_i = \sum_j a_{ij}$ and $\mu_j = \sum_i a_{ij}$.

For a field k of characteristic 0, James and Kerber showed in [JK81] how to decompose the tensor product of two permutation modules in terms of characters. The following is an analogue for an arbitrary commutative ring k :

Lemma 3.5. *Let $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$. The tensor product of the two permutation modules M^λ and M^μ can be decomposed into permutation modules as follows:*

$$M^\lambda \otimes_k M^\mu \cong \bigoplus_{A \in A_\mu^\lambda} M^A,$$

where A is regarded as the composition $(a_{11}, a_{12}, \dots, a_{21}, a_{22}, \dots, a_{mn})$.

Proof. The idea of the proof is taken from [JK81]. For $\underline{i} \in \lambda$ and $\underline{j} \in \mu$ denote by $\underline{i} + \underline{j}$ the sequence $(i_1 i_2 \dots i_n j_1 \dots j_m)$. A basis of $M^\lambda \otimes_k M^\mu$ is given by $\{e_{\underline{i} + \underline{j}} \mid \underline{i} \in \lambda, \underline{j} \in \mu\}$. We consider now the orbits of $M^\lambda \otimes_k M^\mu$ under the action of \mathfrak{S}_d . Two basis elements $e_{\underline{i} + \underline{j}}$ and $e_{\underline{i}' + \underline{j}'}$ belong to the same orbit if and only if $(\underline{j}, \underline{i}) \sim (\underline{j}', \underline{i}')$, that is there exists $\sigma \in \mathfrak{S}_d$ such that $\underline{j}\sigma = \underline{j}'$ and $\underline{i}\sigma = \underline{i}'$. Thus, we can decompose the $k\mathfrak{S}_d$ -module $M^\lambda \otimes_k M^\mu$ into a direct sum of submodules

$$M^\lambda \otimes_k M^\mu = \bigoplus (M^\lambda \otimes_k M^\mu)_{\overline{(\underline{j}, \underline{i})}},$$

where the sum is taken over a complete set of representatives $\overline{(\underline{j}, \underline{i})}$ with $\underline{j} \in \mu$ and $\underline{i} \in \lambda$ and the submodule $(M^\lambda \otimes_k M^\mu)_{\overline{(\underline{j}, \underline{i})}} \subseteq M^\lambda \otimes_k M^\mu$ is spanned by all $e_{\underline{i}' + \underline{j}'}$ such that $(\underline{j}', \underline{i}') \in \overline{(\underline{j}, \underline{i})}$. We identify the set $\{\overline{(\underline{j}, \underline{i})} \mid \underline{j} \in \mu, \underline{i} \in \lambda\}$ with the set $\{A = A_{\underline{j}, \underline{i}} \mid A \in A_\mu^\lambda\}$ via the correspondence defined in (A.1) and the module $(M^\lambda \otimes_k M^\mu)_{\overline{(\underline{j}, \underline{i})}}$ with $M^{A_{\underline{j}, \underline{i}}}$ via the isomorphism

$$e_{\underline{i}' + \underline{j}'} \mapsto e_{\underline{k}},$$

where $\underline{k} = (k_1 \dots k_d)$ is defined for all $1 \leq l \leq d$ by

$$k_l = \begin{cases} 1 & \text{if } i_l = 1, j_l = 1, \\ 2 & \text{if } i_l = 1, j_l = 2, \\ \vdots & \\ m & \text{if } i_l = 1, j_l = m, \\ m + 1 & \text{if } i_l = 2, j_l = 1, \\ \vdots & \\ m \cdot n & \text{if } i_l = n, j_l = m. \end{cases}$$

□

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Example 3.6. Let $\lambda = (3, 1) \in \Lambda(2, 4)$ and $\mu = (2, 1, 1) \in \Lambda(3, 4)$. For $\underline{i} = (1112) \in \lambda$ and $\underline{j} = (1312) \in \mu$, the orbit of $e_{\underline{i}+\underline{j}}$ and consists of all $e_{\underline{i}'+\underline{j}'}$ such that $(\underline{j}', \underline{i}') \sim (\underline{j}, \underline{i})$, i.e.

$$\begin{aligned} (\underline{j}', \underline{i}') \in \{ & ((1112), (1312)), \\ & ((1112), (3112)), \\ & ((1112), (1132)), \\ & \vdots \\ & ((1121), (1321)), \\ & \vdots \\ & ((1211), (1231)), \\ & \vdots \\ & ((2111), (2131)) \}. \end{aligned}$$

In particular, $(M^\lambda \otimes_k M^\mu)_{\overline{(\underline{j}, \underline{i})}}$ is spanned by those $e_{\underline{i}'+\underline{j}'}$. The corresponding matrix $A_{\underline{j}, \underline{i}}$ is given by $a_{ij} := \#\{l \mid i_l = i, j_l = j\}$ (see (A.1)) i.e.

$$A_{\underline{j}, \underline{i}} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and thus

$$(M^\lambda \otimes_k M^\mu)_{\overline{(\underline{j}, \underline{i})}} \cong M^{A_{\underline{j}, \underline{i}}} = M^{(2,0,1,0,1,0)} \cong M^{(2,1,1)}.$$

There are two more matrices belonging to A_μ^λ , namely:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

There corresponding orbits can be obtained by taking, for example, the elements

$$\underline{i}' = (1112) \text{ and } \underline{j}' = (1213) \text{ respectively } \underline{i}'' = (1112) \text{ and } \underline{j}'' = (1321),$$

which span the submodules $M^{(2,1,0,0,0,1)} \cong M^{(2,1,1)}$, respectively $M^{(1,1,1,1,0,0)} \cong M^{(1,1,1,1)}$. All in all we get

$$M^{(3,1)} \otimes M^{(2,1,1)} \cong M^{(2,1,1)} \oplus M^{(2,1,1)} \oplus M^{(1,1,1,1)}.$$

3.3. Cellular Structure

The group algebra of the symmetric group is a *cellular algebra*, in particular it possesses a special set of modules, the *cell modules*, given by the Specht modules. We briefly recall the definitions and refer to [Jam78], [Mat99] and [JK81] for more details on this subject.

Specht modules

Let $\lambda \in \Lambda^+(n, d)$ and T_R^λ be the λ -tableau with entries $1, 2, \dots, d$ when read from left to right, from top to bottom (cf. Example 1.4(ii)). Recall that $C(T_R^\lambda)$ is the column stabilizer, i.e. the subgroup of \mathfrak{S}_d fixing all columns of T_R^λ , and $R(T_R^\lambda)$ the row stabilizer of T_R^λ , i.e. the subgroup of \mathfrak{S}_d fixing all rows of T_R^λ .

Definition 3.7. The *Specht module* corresponding to $\lambda \in \Lambda^+(n, d)$ is

$$\mathrm{Sp}(\lambda) = k\mathfrak{S}_d \left(\sum_{\pi \in C(T_R^\lambda)} \mathrm{sign}(\pi)\pi \right) \left(\sum_{\pi \in R(T_R^\lambda)} \pi \right).$$

Note that, since $R(T_R^\lambda) \cong \mathfrak{S}_\lambda$ (see Example 1.4(ii)), by Remark 3.4 the permutation module M^λ is isomorphic to $k\mathfrak{S}_d(\sum_{\pi \in R(T_R^\lambda)} \pi)$. In particular, $\mathrm{Sp}(\lambda)$ is a submodule of M^λ .

Example 3.8. Let $\lambda = (d)$. Then $M^\lambda \cong \mathrm{Sp}(\lambda) \cong k$, the trivial representation.

Remark 3.9. We denote by $\mathrm{dSp}(\lambda)$ the *dual Specht module* $\mathrm{Sp}(\lambda)^*$. It is isomorphic to

$$k\mathfrak{S}_d \left(\sum_{\pi \in R(T_R^\lambda)} \pi \right) \left(\sum_{\pi \in C(T_R^\lambda)} \mathrm{sign}(\pi)\pi \right).$$

In the literature, the most common notation for (dual) Specht modules is S^λ and for its dual S_λ . To avoid confusion with our notation of the particular strict polynomial functors S^λ (generalized symmetric powers) and S_λ (Schur functors) we denote Specht modules by $\mathrm{Sp}(\lambda)$ and their duals by $\mathrm{dSp}(\lambda)$.

The connection between Specht and dual Specht modules is as follows:

Theorem 3.10 ([Jam78, Theorem 8.15]). *Over any field*

$$\mathrm{Sp}(\lambda) \otimes \mathrm{sgn}^d \cong \mathrm{dSp}(\lambda'),$$

where sgn^d denotes the alternating module and λ' is the conjugate partition of λ .

Simple modules

Assume that k is a field. If the characteristic of k is 0, Specht modules are already simple and they form a complete set of isomorphism classes of simple modules.

If k is a field of characteristic $p > 0$, Specht modules are not simple in general, but for λ p -regular, the Specht module $\mathrm{Sp}(\lambda)$ has a unique simple quotient ([JK81, Theorem 7.1.8, Theorem 7.1.14]). If λ is p -restricted, the dual Specht module $\mathrm{dSp}(\lambda)$ has a simple quotient ([Mar93, p. 97]).

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Definition 3.11. Let k be a field of characteristic $p > 0$. For λ p -regular, the *simple module corresponding to λ* is the simple quotient of $\mathrm{Sp}(\lambda)$ and is denoted by D^λ .

For λ p -restricted, the simple quotient of $\mathrm{dSp}(\lambda)$ is denoted by D_λ .

Remark 3.12. We briefly recall some properties of the simple $k\mathfrak{S}_d$ -modules.

- (i) The simple modules D^λ and D_λ are related as follows: for λ p -restricted, i.e. λ' p -regular, it holds that ([Mar93, Thm 4.2.9])

$$D^{\lambda'} \cong D_\lambda \otimes \mathrm{sgn}^d.$$

- (ii) The modules D^λ , where λ varies over all p -regular partitions of d , form a complete set of pairwise non-isomorphic simple $k\mathfrak{S}_d$ -modules ([JK81, Theorem 7.1.14]). The same holds for the modules D_λ , where λ varies over all p -restricted partitions of d .
- (iii) Since $D^\lambda \otimes \mathrm{sgn}^d$ is again simple, there must exist a partition $m_p(\lambda)$ such that $D^\lambda \otimes \mathrm{sgn}^d \cong D^{m_p(\lambda)}$. This defines a map $m_p: \Lambda^p(d) \rightarrow \Lambda^p(d)$. In [FK97] it has been shown that m_p equals the Mullineux map, defined by G. Mullineux in [Mul79].

Example 3.13. The only 1-dimensional $k\mathfrak{S}_d$ modules are given by

$$k \cong D^{(d)} \cong D_{m_p(1, \dots, 1)} \quad \text{and} \quad \mathrm{sgn}^d \cong D^{m_p(d)} \cong D_{(1, \dots, 1)}.$$

Young modules

In general, permutation modules are not indecomposable. For a decomposition

$$M^\lambda = \bigoplus_{i=1}^s Y_i$$

with Y_i indecomposable and $s \geq 1$, there is exactly one direct summand Y_i that contains the Specht module corresponding to λ (see [Erd94, 2.4]). This leads to the following definition:

Definition 3.14. The *Young module* Y^λ is the unique direct summand Y_i of M^λ such that $S^\lambda \subseteq Y_i$.

Remark 3.15. Every indecomposable direct summand of M^λ is isomorphic to Y^μ for some p -restricted partition μ . Thus, every M^λ decomposes into Young modules, i.e.

$$M^\lambda = \bigoplus_{\mu \in \Lambda^+(d)} K_{\mu\lambda} Y^\mu. \tag{3.4}$$

The coefficients $K_{\mu\lambda}$ depend on the characteristic of k and are called Kostka numbers (for k of characteristic 0) respectively p -Kostka numbers (for $\mathrm{char} k = p$). Their full determination is still an open problem, although some progress has been made in the last years (see e.g. (Klyachko Multiplicity Formula)[Kly84, Corollary 9.2], [Gil14], [Hen05], [FHK08]).

As in case of permutation modules, Young modules are self-dual, i.e. $(Y^\lambda)^* \cong Y^\lambda$. Young modules have Specht and dual Specht filtrations, to be more precise:

Theorem 3.16 ([Don87, 2.6]). *The Young module Y^λ has a Specht filtration, i.e. there exist a filtration*

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_s = Y^\lambda$$

for some $s \geq 0$ such that V_i/V_{i-1} is a Specht module for all $1 \leq i \leq s$. The multiplicity $[Y^\lambda : \text{Sp}(\mu)]$ of $\text{Sp}(\mu)$ in such a filtration is independent of the chosen filtration and is equal to $[\text{Sp}(\lambda) : D^\mu]$.

3.4. The Schur Functor

The connection between strict polynomial functors and representations over the symmetric group is given by the *Schur functor*. Originally it was defined by Issai Schur in his dissertation ([Sch01]) as a functor from the representations of the general linear group to the representations of the symmetric group in characteristic 0. Green extended this theory to infinite fields of arbitrary characteristic [Gre07]. He showed that the category of polynomial representations of the general linear group of fixed degree is equivalent to the category of modules over the Schur algebra. Since this, in turn, is equivalent to the category of strict polynomial functors (see Chapter 6) we deal here with the Schur functor from the category of strict polynomial functors to the category of representations over the symmetric group. Let $X \in \text{Rep } \Gamma_k^d$ be any strict polynomial functor. We obtain a functor

$$\text{Hom}_{\Gamma_k^d}(X, -): \text{Rep } \Gamma_k^d \rightarrow \text{Mod } \text{End}_{\Gamma_k^d}(X)$$

For $X = \Gamma^\omega$ with $\omega = (1, \dots, 1)$, the composition of d consisting of d times the value 1, we get $\text{Mod } \text{End}_{\Gamma_k^d}(X) = \text{Mod } \text{End}_{\Gamma_k^d}(\Gamma^\omega) \cong \text{Mod } k\mathfrak{S}_d^{\text{op}} \cong k\mathfrak{S}_d \text{Mod}$ (see (A.4)) and thus the following functor:

Definition 3.17. The *Schur functor* from the category of strict polynomial functors to the category of modules over the group algebra of the symmetric group is

$$\mathcal{F} = \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, -): \text{Rep } \Gamma_k^d \rightarrow k\mathfrak{S}_d \text{Mod}.$$

It is well-known that under the Schur functor Schur functors are mapped to Specht modules (see e.g. [Gre07, Theorem 6.3c]) and Weyl functors to dual Specht modules (see e.g. [Gre07, Theorem 6.3e]), i.e.

$$\mathcal{F}(S_\lambda) \cong \text{Sp}(\lambda) \quad \text{and} \quad \mathcal{F}(W_\lambda) \cong \text{dSp}(\lambda). \quad (3.5)$$

In the case that k is a field of characteristic $p > 0$, the simple functors indexed by p -restricted partitions are mapped to the simple modules, i.e.

$$\mathcal{F}(L_\lambda) \cong \begin{cases} D_\lambda & \text{if } \lambda \text{ is } p\text{-restricted,} \\ 0 & \text{else.} \end{cases}$$

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In characteristic 0 we have $L_\lambda \cong W_\lambda$ and $\mathcal{F}(L_\lambda) \cong D_\lambda \cong \mathrm{dSp}(\lambda)$ for all $\lambda \in \Lambda^+(n, d)$. Moreover, Schur proved that in this case the Schur functor is an equivalence of categories.

We now concentrate again on the general setting, i.e. k is an arbitrary commutative ring. In particular, the following results hold without any additional assumption on k :

Proposition 3.18. *The functors Γ^λ are mapped under \mathcal{F} to the corresponding permutation modules, i.e.*

$$\mathcal{F}(\Gamma^\lambda) = M^\lambda.$$

Proof. A k -basis of $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda)$ is indexed by matrices in the set A_ω^λ , see Section 2.4. Since $\omega = (1, \dots, 1)$ the set A_ω^λ is given by $\{A_{\underline{l}, \underline{i}} \mid \underline{i} \in \mathfrak{S}_\lambda / \mathfrak{S}_d\}$ for a fixed (arbitrary) $\underline{l} \in \lambda$. Let $A_{\underline{l}, \underline{i}} \in A_\omega^\lambda$ and $\varphi_{\underline{l}, \underline{i}} \in \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda)$ the corresponding morphism. From (A.5) we know that the $k\mathfrak{S}_d$ -module structure on $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda)$ is given by

$$\sigma \cdot \varphi_{\underline{l}, \underline{i}} = \varphi_{\underline{l}, \underline{i}\sigma^{-1}} \quad \text{for } \sigma \in \mathfrak{S}_d.$$

In particular, $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda)$ is cyclic as a $k\mathfrak{S}_d$ -module and for fixed \underline{l} , the map

$$\begin{aligned} M^\lambda &\rightarrow \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda) \\ e_{\underline{l}} &\mapsto \varphi_{\underline{l}, \underline{i}} \end{aligned} \tag{3.6}$$

is an isomorphism of $k\mathfrak{S}_d$ -modules. □

Corollary 3.19. *The representable functor $\Gamma^{d, k^n} = \bigoplus_{\lambda \in \Lambda(n, d)} \Gamma^\lambda$ is mapped under \mathcal{F} to $(k^n)^{\otimes d} = E^{\otimes d} \cong \bigoplus_{\lambda \in \Lambda(n, d)} M^\lambda$.*

An equivalence of categories. We assume for the remaining part of this chapter that $n \geq d$. Recall from Definition 2.11 that $\mathbf{add} \Gamma$ denotes the full subcategory of $\mathbf{Rep} \Gamma_k^d$ whose objects are direct summands of finite direct sums of Γ^λ for $\lambda \in \Lambda(n, d)$. Similarly, we define $\mathbf{add} M$ as the subcategory of $k\mathfrak{S}_d \mathbf{Mod}$ consisting of direct summands of finite direct sums of M^λ for $\lambda \in \Lambda(n, d)$. The following lemma has already been shown in [AR15, Lemma 4.3]:

Lemma 3.20. *The functor $\mathcal{F} = \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, -)$ restricts to an equivalence of categories between $\mathbf{add} \Gamma$ and $\mathbf{add} M$.*

Proof. Since $\Gamma^{d, k^n} = \bigoplus_{\lambda \in \Lambda(n, d)} \Gamma^\lambda$ we have $\mathbf{add} \Gamma = \mathbf{add} \Gamma^{d, k^n}$. Similarly one can see that $\mathbf{add} M = \mathbf{add} E^{\otimes d}$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Rep} \Gamma_k^d & \xrightarrow{\mathcal{F} = \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, -)} & k\mathfrak{S}_d \mathbf{Mod} \\ \uparrow & & \uparrow \\ \mathbf{add} \Gamma^{d, k^n} = \mathbf{add} \Gamma & \xrightarrow{\mathcal{F}|_{\mathbf{add} \Gamma}} & \mathbf{add} M = \mathbf{add} E^{\otimes d} \end{array}$$

where the object Γ^{d, k^n} is mapped under \mathcal{F} to $E^{\otimes d}$ by Proposition 3.18.

On morphisms, \mathcal{F} induces the following map:

$$\begin{aligned} \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda) &\rightarrow \mathrm{Hom}_{k\mathfrak{S}_d}(\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\mu), \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda)) \\ \varphi_{\underline{j}, \underline{i}} &\mapsto (\varphi_{\underline{j}, \underline{i}} \circ -): \varphi_{\underline{l}, \underline{j}} \mapsto \varphi_{\underline{j}, \underline{i}} \circ \varphi_{\underline{l}, \underline{j}} = \varphi_{\underline{l}, \underline{i}} \end{aligned}$$

Via the isomorphism (3.6) this translates to the map sending $e_{\underline{j}}$ to $e_{\underline{i}}$. Thus, the basis element $\varphi_{\underline{j}, \underline{i}}$ of $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda)$ is mapped to the basis element $\xi_{\underline{i}, \underline{j}}: e_{\underline{j}} \mapsto e_{\underline{i}}$ of $\mathrm{Hom}_{k\mathfrak{S}_d}(M^\mu, M^\lambda)$. In particular the map between the morphisms induced by \mathcal{F} is an isomorphism, i.e. \mathcal{F} is fully faithful. \square

Remark 3.21. The equivalence between $\mathrm{add} \Gamma$ and $\mathrm{add} M$ allows us to decompose every Γ^λ into indecomposable functors in the same way as the module M^λ in (3.4):

$$\Gamma^\lambda \cong \bigoplus_{\mu \in \Lambda^+(d)} K_{\mu\lambda} X^\mu,$$

where $K_{\mu\lambda}$ are the Kostka numbers and X^μ is the projective strict polynomial functor that is mapped under \mathcal{F} to Y^μ , the Young module.

Proposition 3.22. *For all $X \in \mathrm{add} \Gamma$ there exists an isomorphism*

$$\mathcal{F}(X^\circ) \cong \mathcal{F}(X)^*$$

which is natural in X .

Proof. By Lemma 3.20, there is an isomorphism

$$\begin{aligned} \mathrm{Hom}_{\Gamma_k^d}(X, (\Gamma^\omega)^\circ) &\cong \mathrm{Hom}_{\mathrm{End}(\Gamma^\omega)}(\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, X), \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, (\Gamma^\omega)^\circ)) \\ &= \mathrm{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X), \mathcal{F}((\Gamma^\omega)^\circ)). \end{aligned} \quad (3.7)$$

We can equip $\mathrm{Hom}_{\Gamma_k^d}(X, (\Gamma^\omega)^\circ)$ with a left $\mathrm{End}((\Gamma^\omega)^\circ)$ -module structure in the usual way, i.e. postcomposing $f \in \mathrm{Hom}_{\Gamma_k^d}(X, (\Gamma^\omega)^\circ)$ with $\varphi \in \mathrm{End}((\Gamma^\omega)^\circ)$. If we identify $k\mathfrak{S}_d$ and $\mathrm{End}((\Gamma^\omega)^\circ)$ (see (A.4)), this structure coincides with the left $k\mathfrak{S}_d$ -module structure on $\mathcal{F}(X^\circ) = \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, X^\circ) \cong \mathrm{Hom}_{\Gamma_k^d}(X, (\Gamma^\omega)^\circ)$.

Also $\mathrm{Hom}_{\mathrm{End}(\Gamma^\omega)}(\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, X), \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, (\Gamma^\omega)^\circ))$ is equipped with a left $\mathrm{End}((\Gamma^\omega)^\circ)$ -module structure, namely by postcomposing $\psi(g) \in \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, (\Gamma^\omega)^\circ)$ with $\varphi \in \mathrm{End}((\Gamma^\omega)^\circ)$. Thus, the isomorphism (3.7) is actually an isomorphism in $\mathrm{End}((\Gamma^\omega)^\circ) \mathrm{Mod} = k\mathfrak{S}_d \mathrm{Mod}$. Moreover, the left $\mathrm{End}((\Gamma^\omega)^\circ)$ -module structure on $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, (\Gamma^\omega)^\circ)$ coincides with the right $\mathrm{End}(\Gamma^\omega)$ -module structure on $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\omega)$ given by using the inverse. Thus, as bimodules $\mathcal{F}((\Gamma^\omega)^\circ) \cong M^\omega \cong k\mathfrak{S}_d$. It follows that

$$\mathrm{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X), \mathcal{F}((\Gamma^\omega)^\circ)) \cong \mathrm{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X), k\mathfrak{S}_d)$$

and since $k\mathfrak{S}_d$ is a symmetric algebra, we have

$$\mathrm{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X), \mathcal{F}((\Gamma^\omega)^\circ)) \cong \mathcal{H}om(\mathcal{F}(X), k)$$

and thus $\mathcal{F}(X^\circ) \cong \mathcal{H}om(\mathcal{F}(X), k) = \mathcal{F}(X)^*$. \square

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Closed monoidal structure. If we do not restrict to the subcategories $\mathbf{add} \Gamma$ and $\mathbf{add} M$ the functor $\mathcal{F} = \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, -)$ is not an equivalence in general. Nevertheless, we have the following result on the closed monoidal structure, see also [AR15, Theorem 4.4] for partial results.

Theorem 3.23. *The functor*

$$\mathcal{F} = \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, -): \mathbf{Rep} \Gamma_k^d \rightarrow k\mathfrak{S}_d \mathbf{Mod}$$

is a strong closed monoidal functor. In particular

$$\mathcal{F}(X \otimes_{\Gamma_k^d} Y) \cong \mathcal{F}(X) \otimes_k \mathcal{F}(Y) \quad (3.8)$$

$$\mathcal{F}(\mathcal{H}om(X, Y)) \cong \mathcal{H}om(\mathcal{F}(X), \mathcal{F}(Y)) \quad (3.9)$$

for all X and Y in $\mathbf{Rep} \Gamma_k^d$ and

$$\begin{aligned} \mathcal{F}(\mathbf{1}_{\Gamma_k^d}) &= \mathbf{1}_{k\mathfrak{S}_d}, \\ \text{i.e. } \mathcal{F}(\Gamma^d) &= k. \end{aligned}$$

Proof. First we show that the functor is strong monoidal. As observed in Section 2.3, every functor X in $\mathbf{Rep} \Gamma_k^d$ is a colimit of representable functors. Moreover, the functor $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, -)$ preserves colimits, since it has a right adjoint, see Chapter 4. Thus it is enough to show the isomorphism (3.8) for functors represented by free modules. Let $V = k^n$ and $W = k^m$ for some non-negative integers n and m . Using the definition of the internal tensor product and the canonical decomposition (2.2) we get

$$\Gamma^{d,k^n} \otimes_{\Gamma_k^d} \Gamma^{d,k^m} = \Gamma^{d,k^n \otimes k^m} \cong \Gamma^{d,k^{n \cdot m}} = \bigoplus_{\nu \in \Lambda(n \cdot m, d)} \Gamma^\nu.$$

Writing down the entries of $\nu \in \Lambda(n \cdot m, d)$ in an $n \times m$ matrix, we obtain a bijection between the set $\Lambda(n \cdot m, d)$ and the set of all $n \times m$ matrices with entries in \mathbb{N} such that the sum of all entries is d . Every such matrix $A = (a_{ij})$ defines a couple (λ, μ) with $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$ where λ_i is given by $\sum_j a_{ij}$ and μ_j is given by $\sum_i a_{ij}$, so that $A \in A_\mu^\lambda$. All in all we get a bijection of sets

$$\Lambda(n \cdot m, d) \longleftrightarrow \{A \in \mathbf{M}_{n \times m}(\mathbb{N}) \mid \sum_{st} a_{st} = d\} \longleftrightarrow \bigcup_{\substack{\lambda \in \Lambda(n, d) \\ \mu \in \Lambda(m, d)}} A_\mu^\lambda$$

and thus the following decomposition

$$\Gamma^{d,k^n} \otimes_{\Gamma_k^d} \Gamma^{d,k^m} \cong \bigoplus_{\nu \in \Lambda(n \cdot m, d)} \Gamma^\nu = \bigoplus_{\substack{\lambda \in \Lambda(n, d) \\ \mu \in \Lambda(m, d)}} \bigoplus_{A \in A_\mu^\lambda} \Gamma^A,$$

where the matrix $A = (a_{ij})$ is seen as the composition $(a_{11}, a_{12}, \dots, a_{21}, a_{22}, \dots, a_{mn})$. Using this decomposition we obtain isomorphisms

$$\begin{aligned}
 \Phi_{\Gamma^d, k^n, \Gamma^d, k^m} : \mathcal{F}(\Gamma^d, k^n) \otimes_k \mathcal{F}(\Gamma^d, k^m) &\xrightarrow{\cong} (k^n)^{\otimes d} \otimes_k (k^m)^{\otimes d} \\
 &\xrightarrow{\cong} \left(\bigoplus_{\lambda \in \Lambda(n, d)} M^\lambda \right) \otimes_k \left(\bigoplus_{\mu \in \Lambda(m, d)} M^\mu \right) \\
 &\xrightarrow{\cong} \bigoplus_{\substack{\lambda \in \Lambda(n, d) \\ \mu \in \Lambda(m, d)}} M^\lambda \otimes_k M^\mu \\
 &\xrightarrow{\cong} \bigoplus_{\substack{\lambda \in \Lambda(n, d) \\ \mu \in \Lambda(m, d)}} \bigoplus_{A \in A_\mu^\lambda} M^A \\
 &\xrightarrow{\cong} \mathcal{F} \left(\bigoplus_{\substack{\lambda \in \Lambda(n, d) \\ \mu \in \Lambda(m, d)}} \bigoplus_{A \in A_\mu^\lambda} \Gamma^A \right) \\
 &\xrightarrow{\cong} \mathcal{F}(\Gamma^d, k^n \otimes_{\Gamma_k^d} \Gamma^d, k^m).
 \end{aligned}$$

By Yoneda, we have a natural isomorphism

$$\begin{aligned}
 \text{Hom}_{\Gamma_k^d}(\Gamma^d, k^n, \Gamma^d, k^{n'}) &\cong \text{Hom}_{\Gamma^d \mathcal{P}_k}(k^{n'}, k^n) = \text{Hom}((k^{n'})^{\otimes d}, (k^n)^{\otimes d})^{\mathfrak{S}_d} \\
 &\cong \text{Hom}_{k\mathfrak{S}_d}((k^n)^{\otimes d}, (k^{n'})^{\otimes d})
 \end{aligned}$$

which coincides with the isomorphism $\text{Hom}_{\Gamma_k^d}(\Gamma^d, k^n, \Gamma^d, k^{n'}) \rightarrow \text{Hom}_{k\mathfrak{S}_d}((k^n)^{\otimes d}, (k^{n'})^{\otimes d})$ induced by \mathcal{F} . It follows that $\Phi_{\Gamma^d, k^n, \Gamma^d, k^m}$ is actually natural in Γ^d, k^n and Γ^d, k^m . We extend it to all strict polynomial functors by taking colimits to get a natural isomorphism

$$\Phi_{X, Y} : \mathcal{F}(X) \otimes_k \mathcal{F}(Y) \rightarrow \mathcal{F}(X \otimes_{\Gamma_k^d} Y).$$

For the tensor units we get the following isomorphism from Proposition 3.18:

$$\varepsilon : \mathbf{1}_{k\mathfrak{S}_d} = M^{(d)} \xrightarrow{\cong} \mathcal{F}(\Gamma^d) = \mathcal{F}(\mathbf{1}_{\Gamma_k^d})$$

It is straightforward to check that the diagrams

$$\begin{array}{ccc}
 (\mathcal{F}(\Gamma^d, k^n) \otimes \mathcal{F}(\Gamma^d, k^m)) \otimes \mathcal{F}(\Gamma^d, k^{m'}) & \xrightarrow{\alpha'_{\mathcal{F}(\Gamma^d, k^n), \mathcal{F}(\Gamma^d, k^m), \mathcal{F}(\Gamma^d, k^{m'})}} & \mathcal{F}(\Gamma^d, k^n) \otimes (\mathcal{F}(\Gamma^d, k^m) \otimes \mathcal{F}(\Gamma^d, k^{m'})) \\
 \downarrow \Phi_{\Gamma^d, k^n, \Gamma^d, k^m} \otimes \text{id}_{\mathcal{F}(\Gamma^d, k^{m'})} & & \downarrow \text{id}_{\mathcal{F}(\Gamma^d, k^n)} \otimes \Phi_{\Gamma^d, k^m, \Gamma^d, k^{m'}} \\
 \mathcal{F}(\Gamma^d, k^n \otimes \Gamma^d, k^m) \otimes \mathcal{F}(\Gamma^d, k^{m'}) & & \mathcal{F}(\Gamma^d, k^n) \otimes (\mathcal{F}(\Gamma^d, k^m \otimes \Gamma^d, k^{m'})) \\
 \downarrow \Phi_{\Gamma^d, k^n \otimes \Gamma^d, k^m, \Gamma^d, k^{m'}} & \xrightarrow{\mathcal{F}(\alpha_{\Gamma^d, k^n, \Gamma^d, k^m, \Gamma^d, k^{m'}})} & \downarrow \Phi_{\Gamma^d, k^n, \Gamma^d, k^m \otimes \Gamma^d, k^{m'}} \\
 \mathcal{F}((\Gamma^d, k^n \otimes \Gamma^d, k^m) \otimes \Gamma^d, k^{m'}) & \longrightarrow & \mathcal{F}(\Gamma^d, k^n \otimes (\Gamma^d, k^m \otimes \Gamma^d, k^{m'}))
 \end{array}$$

3. Representations of the Symmetric Group

$$\begin{array}{ccc}
\mathcal{F}(\Gamma^{d,k^n}) \otimes \mathbf{1}_{k\mathfrak{S}_d} & \xrightarrow{\varrho_{\mathcal{F}(\Gamma^{d,k^n})}} & \mathcal{F}(\Gamma^{d,k^n}) \\
\text{id}_{\mathcal{F}(\Gamma^{d,k^n})} \otimes \varepsilon \downarrow & & \uparrow \mathcal{F}(\varrho_{\Gamma^{d,k^n}}) \\
\mathcal{F}(\Gamma^{d,k^n}) \otimes \mathcal{F}(\mathbf{1}_{\mathcal{M}}) & \xrightarrow{\Phi_{\Gamma^{d,k^n}, \mathbf{1}_{\Gamma_k^d}}} & \mathcal{F}(\Gamma^{d,k^n} \otimes \mathbf{1}_{\Gamma_k^d})
\end{array}$$

$$\begin{array}{ccc}
\mathbf{1}_{k\mathfrak{S}_d} \otimes \mathcal{F}(\Gamma^{d,k^n}) & \xrightarrow{\lambda_{\Gamma^{d,k^n}}} & \mathcal{F}(\Gamma^{d,k^n}) \\
\varepsilon \otimes \text{id}_{\mathcal{F}(\Gamma^{d,k^n})} \downarrow & & \uparrow \mathcal{F}(\lambda_{\Gamma^{d,k^n}}) \\
\mathcal{F}(\mathbf{1}_{\Gamma_k^d}) \otimes \mathcal{F}(\Gamma^{d,k^n}) & \xrightarrow{\Phi_{\mathbf{1}_{\Gamma_k^d}, \mathcal{F}(\Gamma^{d,k^n})}} & \mathcal{F}(\mathbf{1}_{\Gamma_k^d} \otimes \Gamma^{d,k^n})
\end{array}$$

commute.

To show that \mathcal{F} is closed, we consider the following diagram of equivalences of categories

$$\begin{array}{ccc}
\text{add } \Gamma & \xrightarrow{\mathcal{F}} & \text{add } M \\
(-)^\circ \uparrow & & \downarrow (-)^* \\
(\text{add } S)^{\text{op}} & \xrightarrow{\mathcal{F}} & (\text{add } M)^{\text{op}}
\end{array}$$

By Proposition 3.22 this diagram commutes, i.e. $\mathcal{F}(X^\circ) \cong (\mathcal{F}(X))^*$ for all $X \in (\text{add } \Gamma)^{\text{op}}$. Since $\mathcal{H}om(\Gamma^\lambda, S^\mu)^\circ \cong \Gamma^\lambda \otimes_{\Gamma_k^d} \Gamma^\mu \in \text{add } \Gamma$ we get for $X^\circ = \mathcal{H}om(\Gamma^\lambda, S^\mu)$

$$\begin{aligned}
\mathcal{F}(\mathcal{H}om(\Gamma^\lambda, S^\mu)) &\cong (\mathcal{F}(\mathcal{H}om(\Gamma^\lambda, S^\mu)^\circ))^* \\
&\cong (\mathcal{F}(\Gamma^\lambda \otimes_{\Gamma_k^d} \Gamma^\mu))^* \\
&\cong (\mathcal{F}(\Gamma^\lambda) \otimes_{k\mathfrak{S}_d} \mathcal{F}(\Gamma^\mu))^* \\
&\cong \mathcal{H}om(\mathcal{F}(\Gamma^\lambda), \mathcal{F}(\Gamma^\mu)^*) \\
&\cong \mathcal{H}om(\mathcal{F}(\Gamma^\lambda), \mathcal{F}((\Gamma^\mu)^\circ)) \\
&\cong \mathcal{H}om(\mathcal{F}(\Gamma^\lambda), \mathcal{F}(S^\mu)).
\end{aligned}$$

For an arbitrary $Y \in \text{Rep } \Gamma_k^d$ we take an injective presentation

$$0 \rightarrow Y \rightarrow I^0(Y) \rightarrow I^1(Y).$$

Since $\mathcal{H}om(\Gamma^\lambda, -)$ is left exact, by first applying $\mathcal{F}\mathcal{H}om(\Gamma^\lambda, -)$ and second applying $\mathcal{H}om(\mathcal{F}(\Gamma^\lambda), \mathcal{F}(-))$ to the presentation, we get

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}\mathcal{H}om(\Gamma^\lambda, Y) & \longrightarrow & \mathcal{F}\mathcal{H}om(\Gamma^\lambda, I^0(Y)) & \longrightarrow & \mathcal{F}\mathcal{H}om(\Gamma^\lambda, I^0(Y)) \\
& & & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \mathcal{H}om(\mathcal{F}(\Gamma^\lambda), \mathcal{F}(Y)) & \longrightarrow & \mathcal{H}om(\mathcal{F}(\Gamma^\lambda), \mathcal{F}(I^0(Y))) & \longrightarrow & \mathcal{H}om(\mathcal{F}(\Gamma^\lambda), \mathcal{F}(I^0(Y)))
\end{array}$$

and thus $\mathcal{F}\mathcal{H}om(\Gamma^\lambda, Y) \cong \mathcal{H}om(\mathcal{F}(\Gamma^\lambda), \mathcal{F}(Y))$. The same arguments for arbitrary X and a projective resolution $P^1(X) \rightarrow P^0(X) \rightarrow X \rightarrow 0$ finally yields

$$\mathcal{F}(\mathcal{H}om(X, Y)) \cong \mathcal{H}om(\mathcal{F}(X), \mathcal{F}(Y)). \quad \square$$

The monoidal property of the Schur functor allows us to compare the Kuhn and the monoidal dual of strict polynomial functors with the dual for symmetric group representations:

Corollary 3.24. *For all $X \in \text{Rep } \Gamma_k^d$ we have*

$$\mathcal{F}(X^\circ) \cong \mathcal{F}(X)^* \cong \mathcal{F}(X^\vee).$$

Proof. By Lemma 2.27

$$X^\circ \cong (\Gamma^d \otimes X)^\circ \cong \mathcal{H}om(\Gamma^d, X^\circ) \cong \mathcal{H}om(X, S^d)$$

and thus, since $\mathcal{F}(X)^* = \mathcal{H}om(\mathcal{F}(X), k)$,

$$\begin{aligned} \mathcal{F}(X^\circ) &\cong \mathcal{F}(\mathcal{H}om(X, S^d)) \cong \mathcal{H}om(\mathcal{F}(X), \mathcal{F}(S^d)) \cong \mathcal{H}om(\mathcal{F}(X), k) \\ &\cong \mathcal{H}om(\mathcal{F}(X), \mathcal{F}(\Gamma^d)) \cong \mathcal{F}(\mathcal{H}om(X, \Gamma^d)) = \mathcal{F}(X^\vee). \end{aligned}$$

\square

4. The Adjoint of the Schur Functor

Having introduced the Schur functor \mathcal{F} in the preceding chapter, we now study its left and right adjoints. Our main concern will be the composition of \mathcal{F} with its adjoints. In particular, we present a connection between the adjoints of the Schur functor and the monoidal structure on strict polynomial functors, allowing us to deduce a projection formula for \mathcal{F} . Some results about the adjoints have been obtained in [DEN04] and [HN04]. For a generalization of \mathcal{F} and its adjoints, we refer to [Kuh02]. Note that most parts of this chapter have been published in [Rei16].

In the following we abbreviate $\text{End}_{\Gamma_k^d}(\Gamma^\omega)$ by $\text{End}(\Gamma^\omega)$ and $\text{End}_{\Gamma_k^d}(\Gamma^{d,V})$ by $\text{End}(\Gamma^{d,V})$.

4.1. The Left Adjoint of the Schur Functor

Let $N \in \text{Mod End}(\Gamma^\omega)$ and $X \in \text{Rep } \Gamma_k^d$. We consider $N \otimes_{\text{End}(\Gamma^\omega)} \Gamma^\omega$ as an object in $\text{Rep } \Gamma_k^d$ by sending a module V to $N \otimes_{\text{End}(\Gamma^\omega)} \Gamma^\omega(V)$. Note that by the usual tensor-hom adjunction we have the following isomorphism

$$\begin{aligned} \text{Hom}_{\Gamma_k^d}(N \otimes_{\text{End}(\Gamma^\omega)} \Gamma^\omega, X) &\cong \text{Hom}_{\text{End}(\Gamma^\omega)}(N, \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, X)) \\ &= \text{Hom}_{\text{End}(\Gamma^\omega)}(N, \mathcal{F}(X)). \end{aligned}$$

Thus, \mathcal{F} has a *left adjoint*, which we denote by \mathcal{G}_\otimes . It is given by

$$\begin{aligned} \mathcal{G}_\otimes: \text{Mod End}(\Gamma^\omega) &\rightarrow \text{Rep } \Gamma_k^d \\ N &\mapsto N \otimes_{\text{End}(\Gamma^\omega)} \Gamma^\omega. \end{aligned}$$

In terms of modules of the group algebra of the symmetric group this reads

$$\begin{aligned} \mathcal{G}_\otimes: k\mathfrak{S}_d \text{ Mod} &\rightarrow \text{Rep } \Gamma_k^d \\ N &\mapsto (-)^{\otimes d} \otimes_{k\mathfrak{S}_d} N. \end{aligned}$$

We denote the unit by $\eta_\otimes: \text{id}_{\text{End}(\Gamma^\omega)} \rightarrow \mathcal{F}\mathcal{G}_\otimes$ and the counit by $\varepsilon_\otimes: \mathcal{G}_\otimes\mathcal{F} \rightarrow \text{id}_{\text{Rep } \Gamma_k^d}$ and omit indices where possible. Since \mathcal{F} is exact and \mathcal{G}_\otimes is right exact, $\mathcal{F}\mathcal{G}_\otimes$ is right exact. Note that for the regular representation $\text{End}(\Gamma^\omega) \in \text{Mod End}(\Gamma^\omega)$ we get

$$\begin{aligned} \mathcal{F}\mathcal{G}_\otimes(\text{End}(\Gamma^\omega)) &= \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \text{End}(\Gamma^\omega) \otimes_{\text{End}(\Gamma^\omega)} \Gamma^\omega) \\ &\cong \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\omega) \\ &\cong \text{End}(\Gamma^\omega). \end{aligned}$$

4. The Adjoints of the Schur Functor

It follows that $\mathcal{F}\mathcal{G}_\otimes(N) \cong N$ for all $N \in \mathbf{Mod}\ \mathrm{End}(\Gamma^\omega)$ and hence the unit η_\otimes is an isomorphism and \mathcal{G}_\otimes is fully faithful.

We now focus on the composition $\mathcal{G}_\otimes\mathcal{F}$. Recall that $\mathbf{add}\ S$, respectively $\mathbf{add}\ M$ denotes the full subcategory of $\mathbf{Rep}\ \Gamma_k^d$, respectively $k\mathfrak{S}_d\mathbf{Mod}$ whose objects are direct summands of finite direct sums of S^λ , respectively of M^λ for $\lambda \in \Lambda(n, d)$.

Proposition 4.1. *There exists a natural isomorphism*

$$\mathcal{G}_\otimes\mathcal{F}(X) \cong X$$

for all $X \in \mathbf{add}\ S$.

Proof. Let $V \in \Gamma_k^d\mathbf{P}_k$ and $X \in \mathbf{add}\ S$. Using Lemma 2.27, the Yoneda isomorphism (2.1), and the equivalence of $\mathbf{add}\ S$ with $\mathbf{add}\ M$ we get the following sequence of isomorphisms,

$$\begin{aligned} (X)^\circ(V) &\cong \mathrm{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, (X)^\circ) \\ &\cong \mathrm{Hom}_{\Gamma_k^d}(X, (\Gamma^{d,V})^\circ) \\ &\cong \mathrm{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X), \mathcal{F}((\Gamma^{d,V})^\circ)) \\ &\cong \mathrm{Hom}_{\Gamma_k^d}(\mathcal{G}_\otimes\mathcal{F}(X), (\Gamma^{d,V})^\circ) \\ &\cong \mathrm{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, (\mathcal{G}_\otimes\mathcal{F}(X))^\circ) \\ &\cong (\mathcal{G}_\otimes\mathcal{F}(X))^\circ(V) \end{aligned}$$

and thus $\mathcal{G}_\otimes\mathcal{F}(X) \cong X$. □

Corollary 4.2. *The functor \mathcal{G}_\otimes restricted to $\mathbf{add}\ M$ is an inverse for $\mathcal{F}|_{\mathbf{add}\ S}$, i.e. we have the following equivalences of categories:*

$$\mathbf{add}\ S \begin{array}{c} \xleftarrow{\mathcal{G}_\otimes} \\ \xrightarrow{\mathcal{F}} \end{array} \mathbf{add}\ M \quad \square$$

Without restricting to the subcategory $\mathbf{add}\ S$, the composition $\mathcal{G}_\otimes\mathcal{F}$ is not isomorphic to the identity. Though, we can shown the following:

Theorem 4.3. *There exists a natural isomorphism*

$$\mathcal{G}_\otimes\mathcal{F}(X) \cong S^d \otimes_{\Gamma_k^d} X.$$

Proof. Recall the natural isomorphism $\Phi_{X,Y}: \mathcal{F}(X) \otimes_k \mathcal{F}(Y) \rightarrow \mathcal{F}(X \otimes_{\Gamma_k^d} Y)$ constructed in the proof of Theorem 3.23. Using this isomorphism and by the adjunction property we obtain the following isomorphism

$$\begin{aligned} \mathrm{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X) \otimes_k N, \mathcal{F}(X) \otimes_k N) &\cong \mathrm{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X) \otimes_k N, \mathcal{F}(X) \otimes_k \mathcal{F}\mathcal{G}_\otimes(N)) \\ &\cong \mathrm{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X) \otimes_k N, \mathcal{F}(X \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(N))) \\ &\cong \mathrm{Hom}_{\Gamma_k^d}(\mathcal{G}_\otimes(\mathcal{F}(X) \otimes_k N), X \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(N)), \end{aligned}$$

that maps $f \in \text{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(X) \otimes_k N, \mathcal{F}(X) \otimes_k N)$ to

$$(\varepsilon_\otimes)_{X \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(N)} \circ \mathcal{G}(\Phi_{X, \mathcal{G}_\otimes N} \circ (\text{id}_{\mathcal{F}(X)} \otimes_k (\eta_\otimes)N) \circ f).$$

In particular, the identity on $\mathcal{F}(X) \otimes_k N$ yields the map (omitting the indices)

$$\vartheta_{X, N} := (\varepsilon_\otimes) \circ \mathcal{G}_\otimes(\Phi \circ (\text{id} \otimes_k \eta_\otimes)): \mathcal{G}_\otimes(\mathcal{F}(X) \otimes_k N) \rightarrow X \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(N).$$

By setting $N := \mathbf{1}$, the trivial module, we obtain the following map

$$\vartheta_{X, \mathbf{1}}: \mathcal{G}_\otimes \mathcal{F}(X) \rightarrow X \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(\mathbf{1}).$$

We show that it is an isomorphism. Since $\mathcal{G}_\otimes \mathcal{F}(-)$ and $- \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(\mathbf{1})$ are right exact functors it is enough to show that $\vartheta_{X, \mathbf{1}}$ is an isomorphism for X projective. Thus, let $X = \Gamma^\lambda$. Since $\mathcal{F}(S^d) \cong \mathbf{1}$, we know by Proposition 4.1 that $\mathcal{G}_\otimes(\mathbf{1}) \cong S^d$. It follows from Proposition 2.16 that $\Gamma^\lambda \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(\mathbf{1}) \cong S^\lambda$. Thus, by Corollary 4.2

$$(\varepsilon_\otimes)_{\Gamma^\lambda \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(\mathbf{1})}: \mathcal{G}_\otimes \mathcal{F}(\Gamma^\lambda \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(\mathbf{1})) \rightarrow \Gamma^\lambda \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(\mathbf{1})$$

is an isomorphism. Both maps Φ and η_\otimes are isomorphisms and thus $\mathcal{G}_\otimes(\Phi \circ (\text{id} \otimes_k \eta_\otimes))$ is an isomorphism. It follows that $\vartheta_{\Gamma^\lambda, \mathbf{1}}: \mathcal{G}_\otimes \mathcal{F}(\Gamma^\lambda) \rightarrow \Gamma^\lambda \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(\mathbf{1})$ is an isomorphism. Identifying $\mathcal{G}_\otimes(\mathbf{1})$ with S^d we get the desired isomorphism for all $X \in \text{Rep } \Gamma_k^d$:

$$\vartheta_{X, \mathbf{1}}: \mathcal{G}_\otimes \mathcal{F}(X) \xrightarrow{\cong} X \otimes_{\Gamma_k^d} S^d \cong S^d \otimes_{\Gamma_k^d} X. \quad \square$$

Recall from Corollary 2.32 that $S^d \cong S^d \otimes S^d$. Thus, using the fact that \mathcal{F} preserves the monoidal structure, we get the following

Corollary 4.4. *The functor \mathcal{G}_\otimes is compatible with the tensor product, in the sense that*

$$\mathcal{G}_\otimes(N \otimes_k N') \cong \mathcal{G}_\otimes(N) \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(N'). \quad \square$$

Note, however, that the tensor unit $\mathbf{1}_{k\mathfrak{S}_d}$ is mapped under \mathcal{G}_\otimes to S^d which is not the tensor unit in $\text{Rep } \Gamma_k^d$. Using Lemma 2.27, we get the following description of the Schur functor composed with its left adjoint:

Corollary 4.5. *The endofunctor $\mathcal{G}_\otimes \mathcal{F}$ can be expressed by duals, namely*

$$\mathcal{G}_\otimes \mathcal{F}(X) \cong S^d \otimes_{\Gamma_k^d} X \cong \mathcal{H}om(X, \Gamma^d)^\circ = (X^\vee)^\circ. \quad \square$$

4.1.1. Projection Formula

As we have seen in the proof of Theorem 4.3, there is always a morphism

$$\vartheta_{X, N}: \mathcal{G}_\otimes(\mathcal{F}(X) \otimes_k N) \rightarrow X \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(N).$$

This is a very general fact, using only the properties of \mathcal{G}_\otimes being left adjoint to \mathcal{F} and \mathcal{F} being monoidal. This morphism $\vartheta_{X, N}$ does not need to be an isomorphism in general, but using Theorem 4.3 we can show that in our case¹:

¹I am very grateful to Paul Balmer who pointed me to this result.

4. The Adjoints of the Schur Functor

Theorem 4.6 (Projection formula). *For all $X \in \text{Rep } \Gamma_k^d$ and $N \in k\mathfrak{S}_d \text{ Mod}$ there is an isomorphism*

$$\mathcal{G}_\otimes(\mathcal{F}(X) \otimes_k N) \cong X \otimes_{\Gamma_k^d} \mathcal{G}_\otimes(N).$$

Proof. Recall that $N \cong \mathcal{F}\mathcal{G}_\otimes(N)$. Using Theorem 4.3 we get

$$\begin{aligned} \mathcal{G}_\otimes(\mathcal{F}(X) \otimes_k N) &\cong \mathcal{G}_\otimes(\mathcal{F}(X) \otimes_k \mathcal{F}\mathcal{G}_\otimes(N)) \\ &\cong \mathcal{G}_\otimes \mathcal{F}(X \otimes \mathcal{G}_\otimes(N)) \\ &\cong S^d \otimes X \otimes \mathcal{G}_\otimes(N) \\ &\cong X \otimes S^d \otimes \mathcal{G}_\otimes(N) \\ &\cong X \otimes \mathcal{G}_\otimes \mathcal{F}\mathcal{G}_\otimes(N) \\ &\cong X \otimes \mathcal{G}_\otimes(N). \end{aligned}$$

□

4.2. The Right Adjoint of the Schur Functor

This section provides analogous results to those in the preceding section, now for the right adjoint to the Schur functor. We show that also the right adjoint can be expressed in terms of the monoidal structure on strict polynomial functors.

Let $V \in \Gamma^d \mathcal{P}_k$, $X \in \text{Rep } \Gamma_k^d$ and $N \in \text{Mod End}(\Gamma^\omega)$. We consider $\text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X)$ as a right $\text{End}(\Gamma^{d,V})$ -module and $\text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^{d,V})$ as an $\text{End}(\Gamma^{d,V})$ - $\text{End}(\Gamma^\omega)$ -bimodule. By the usual tensor-hom adjunction we then get the following isomorphism

$$\begin{aligned} &\text{Hom}_{\text{End}(\Gamma^{d,V})}(\text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X), \text{Hom}_{\text{End}(\Gamma^\omega)}(\text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^{d,V}), N)) \\ &\cong \text{Hom}_{\text{End}(\Gamma^\omega)}(\text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X) \otimes_{\text{End}(\Gamma^{d,V})} \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^{d,V}), N). \end{aligned}$$

On the other hand, since $\text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, \Gamma^\omega)$ is finitely generated projective over $\text{End}(\Gamma^{d,V})$, we have

$$\begin{aligned} \text{Hom}_{\text{End}(\Gamma^{d,V})}(\Gamma^\omega(V), X(V)) &\cong \text{Hom}_{\text{End}(\Gamma^{d,V})}(\text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, \Gamma^\omega), \text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X)) \\ &\cong \text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X) \otimes_{\text{End}(\Gamma^{d,V})} \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^{d,V}) \end{aligned}$$

and thus

$$\begin{aligned} &\text{Hom}_{\text{End}(\Gamma^{d,V})}(\text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X), \text{Hom}_{\text{End}(\Gamma^\omega)}(\text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^{d,V}), N)) \\ &\cong \text{Hom}_{\text{End}(\Gamma^\omega)}(\text{Hom}_{\text{End}(\Gamma^{d,V})}(\Gamma^\omega(V), X(V)), N). \end{aligned}$$

Since $\text{Mod End}(\Gamma^{d,V}) \cong \text{Rep } \Gamma_k^d$ for $V := k^n$ if $n \geq d$ (see (6.3)) and $X \cong \text{Hom}_{\Gamma_k^d}(\Gamma^{d,-}, X)$ we finally obtain the following isomorphism

$$\begin{aligned} &\text{Hom}_{\Gamma_k^d}(X, \text{Hom}_{\text{End}(\Gamma^\omega)}(\text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^{d,-}), N)) \\ &\cong \text{Hom}_{\text{End}(\Gamma^\omega)}(\text{Hom}_{\Gamma_k^d}(\Gamma^\omega, X), N). \end{aligned}$$

Thus, $\mathcal{F} = \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, -)$ has a *right adjoint*, which we denote by \mathcal{G}_{Hom} . It is given by

$$\begin{aligned} \mathcal{G}_{\text{Hom}}: \text{Mod End}(\Gamma^\omega) &\rightarrow \text{Rep } \Gamma_k^d \\ N &\mapsto \text{Hom}_{\text{End}(\Gamma^\omega)}(\text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^{d,-}), N). \end{aligned}$$

In terms of modules of the group algebra of the symmetric group this reads

$$\begin{aligned} \mathcal{G}_{\text{Hom}}: k\mathfrak{S}_d \text{ Mod} &\rightarrow \text{Rep } \Gamma_k^d \\ N &\mapsto \text{Hom}_{k\mathfrak{S}_d}(\text{Hom}_{k\mathfrak{S}_d}((-)^{\otimes d}, k\mathfrak{S}_d), N). \end{aligned}$$

We denote by $\eta_{\text{Hom}}: \text{id}_{\text{End}(\Gamma^\omega)} \rightarrow \mathcal{G}_{\text{Hom}}\mathcal{F}$ the unit and by $\varepsilon_{\text{Hom}}: \mathcal{F}\mathcal{G}_{\text{Hom}} \rightarrow \text{id}_{\text{Rep } \Gamma_k^d}$ the counit. Note that

$$\begin{aligned} \mathcal{F}\mathcal{G}_{\text{Hom}}(M) &= \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \mathcal{G}_{\text{Hom}}(M)) \\ &\cong \text{Hom}_{\text{End}_{\Gamma_k^d}(\Gamma^\omega)}(\mathcal{F}(\Gamma^\omega), M) \\ &= \text{Hom}_{\text{End}_{\Gamma_k^d}(\Gamma^\omega)}(\text{End}_{\Gamma_k^d}(\Gamma^\omega), M) \\ &\cong M, \end{aligned}$$

i.e. the counit ε_{Hom} is an isomorphism and thus \mathcal{G}_{Hom} is fully faithful.

Again, we are interested in the endofunctor $\mathcal{G}_{\text{Hom}}\mathcal{F}$. Recall that $\mathbf{add } \Gamma$ denotes the full subcategory of $\text{Rep } \Gamma_k^d$ whose objects are direct summands of finite direct sums of Γ^λ .

Proposition 4.7. *There exists a natural isomorphism*

$$\mathcal{G}_{\text{Hom}}\mathcal{F}(X) \cong X$$

for all $X \in \mathbf{add } \Gamma$.

Proof. Let $V \in \Gamma^d \mathbf{P}_k$ and $X \in \mathbf{add } \Gamma$. Due to the the Yoneda isomorphism (2.1) and the equivalence of $\mathbf{add } \Gamma$ and $\mathbf{add } M$ we have the following sequence of isomorphisms

$$\begin{aligned} X(V) &\cong \text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X) \\ &\cong \text{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(\Gamma^{d,V}), \mathcal{F}(X)) \\ &\cong \text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, \mathcal{G}_{\text{Hom}}\mathcal{F}(X)) \\ &\cong \mathcal{G}_{\text{Hom}}\mathcal{F}(X)(V) \end{aligned}$$

and thus $\mathcal{G}_{\text{Hom}}\mathcal{F}(X) \cong X$. □

Corollary 4.8. *The functor \mathcal{G}_{Hom} restricted to $\mathbf{add } M$ is an inverse for $\mathcal{F}|_{\mathbf{add } \Gamma}$, i.e. we have the following equivalences of categories:*

$$\mathbf{add } \Gamma \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \xleftarrow{\mathcal{G}_{\text{Hom}}} \end{array} \mathbf{add } M \quad \square$$

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Remark 4.9. Suppose that k is a field of characteristic ≥ 5 . In [HN04, Theorem 3.8.1.] it is shown that on $\text{Filt}(\Delta)$, the full subcategory of Weyl filtered modules, \mathcal{G}_{Hom} is an inverse for \mathcal{F} . The subcategory $\text{Filt}(\Delta)$ contains the subcategory $\mathbf{add} \Gamma$, so in the case of a field of characteristic ≥ 5 , Corollary 4.8 follows from [HN04]. However, Corollary 4.8 is independent of any assumption on the commutative ring k .

Without restricting to the subcategory $\mathbf{add} \Gamma$, the composition $\mathcal{G}_{\text{Hom}}\mathcal{F}$ is not isomorphic to the identity. Though we have the following result, dual to Theorem 4.3:

Theorem 4.10. *There exists a natural isomorphism*

$$\mathcal{G}_{\text{Hom}}\mathcal{F}(X) \cong \mathcal{H}om(S^d, X).$$

Proof. Recall the natural isomorphism $\Psi_{X,Y}: \mathcal{H}om(\mathcal{F}(X), \mathcal{F}(Y)) \rightarrow \mathcal{F}(\mathcal{H}om(X, Y))$ constructed in the proof of Theorem 3.23. Using this isomorphism and the adjunction we get a sequence of isomorphisms

$$\begin{aligned} & \text{Hom}_{k\mathfrak{S}_d}(\mathcal{H}om(\mathcal{F}(X^\circ), N), \mathcal{H}om(\mathcal{F}(X^\circ), N)) \\ & \cong \text{Hom}_{k\mathfrak{S}_d}(\mathcal{H}om(\mathcal{F}(X^\circ), N), \mathcal{H}om(N^*, \mathcal{F}(X^\circ)^*)) \\ & \cong \text{Hom}_{k\mathfrak{S}_d}(\mathcal{H}om(\mathcal{F}(X^\circ), \mathcal{F}\mathcal{G}_{\text{Hom}}(N)), \mathcal{H}om(N^*, \mathcal{F}(X))) \\ & \cong \text{Hom}_{k\mathfrak{S}_d}(\mathcal{F}(\mathcal{H}om(X^\circ, \mathcal{G}_{\text{Hom}}(N))), \mathcal{H}om(N^*, \mathcal{F}(X))) \\ & \cong \text{Hom}_{\Gamma_k^d}(\mathcal{H}om(X^\circ, \mathcal{G}_{\text{Hom}}(N)), \mathcal{G}_{\text{Hom}}(\mathcal{H}om(N^*, \mathcal{F}(X)))) \\ & \cong \text{Hom}_{\Gamma_k^d}(\mathcal{H}om(\mathcal{G}_{\text{Hom}}(N)^\circ, X), \mathcal{G}_{\text{Hom}}(\mathcal{H}om(N^*, \mathcal{F}(X)))) \end{aligned}$$

Thus, the identity on $\mathcal{H}om(\mathcal{F}(X^\circ), N)$ yields a map

$$\kappa_{N,X}: \mathcal{H}om(\mathcal{G}_{\text{Hom}}(N)^\circ, X) \rightarrow \mathcal{G}_{\text{Hom}}(\mathcal{H}om(N^*, \mathcal{F}(X))).$$

By setting $N := \mathbf{1}$, the trivial module, we get a map

$$\kappa_{\mathbf{1},X}: \mathcal{H}om(\mathcal{G}_{\text{Hom}}(\mathbf{1})^\circ, X) \rightarrow \mathcal{G}_{\text{Hom}}(\mathcal{H}om(\mathbf{1}, \mathcal{F}(X))).$$

Similarly to the case of \mathcal{G}_\otimes this is an isomorphism. This time, we use the fact that since $\mathcal{H}om(\mathcal{G}_{\text{Hom}}(\mathbf{1}), -)$ and $\mathcal{G}_{\text{Hom}}(\mathcal{H}om(\mathbf{1}, \mathcal{F}(-)))$ are left exact functors it is enough to show that $\kappa_{\mathbf{1},X}$ is an isomorphism for $X = S^\lambda$ injective. But $\mathcal{F}(\Gamma^d) \cong \mathbf{1}$, thus we know by Proposition 4.7 that $\mathcal{G}_{\text{Hom}}(\mathbf{1}) \cong \Gamma^d$. It follows from Lemma 2.27 that

$$\mathcal{H}om(\mathcal{G}_{\text{Hom}}(\mathbf{1})^\circ, S^\lambda) \cong (S^d \otimes \Gamma^\lambda)^\circ \cong \Gamma^\lambda$$

and hence

$$(\eta_{\text{Hom}})_{\mathcal{H}om(\mathcal{G}_{\text{Hom}}(\mathbf{1})^\circ, S^\lambda)}: \mathcal{H}om(\mathcal{G}_{\text{Hom}}(\mathbf{1})^\circ, S^\lambda) \rightarrow \mathcal{G}_{\text{Hom}}\mathcal{F}(\mathcal{H}om(\mathcal{G}_{\text{Hom}}(\mathbf{1})^\circ, S^\lambda))$$

is an isomorphism by Corollary 4.8. Similarly to the case of $\vartheta_{I,X}$ in the proof of Theorem 4.3, $\kappa_{\mathbf{1},X}$ is the composition of this isomorphism and further isomorphisms, hence is itself an isomorphism. Identifying $\mathcal{G}_{\text{Hom}}(\mathbf{1})^\circ$ with S^d and $\mathcal{H}om(\mathbf{1}, \mathcal{F}(X))$ with $\mathcal{F}(X)$ we finally get for all $X \in \mathbf{Rep} \Gamma_k^d$

$$\kappa_{\mathbf{1},X}: \mathcal{H}om(S^d, X) \rightarrow \mathcal{G}_{\text{Hom}}\mathcal{F}(X). \quad \square$$

Corollary 4.11. *The functor \mathcal{G}_{Hom} preserves the internal hom up to duality, i.e.*

$$\mathcal{G}_{\text{Hom}}\mathcal{F}(\mathcal{H}om(X^\circ, Y)) \cong \mathcal{H}om(\mathcal{G}_{\text{Hom}}\mathcal{F}(X)^\circ, \mathcal{G}_{\text{Hom}}\mathcal{F}(Y))$$

and

$$\mathcal{G}_{\text{Hom}}\mathcal{H}om(N^*, N') \cong \mathcal{H}om(\mathcal{G}_{\text{Hom}}(N)^\circ, \mathcal{G}_{\text{Hom}}(N')). \quad \square$$

Remark 4.12. In general, \mathcal{G}_{Hom} does not preserve the internal tensor product, e.g. for sgn^d , the sign-representation in $k\mathfrak{S}_d \text{Mod}$, we get $\mathcal{G}_{\text{Hom}}(\text{sgn}^d) \cong \Lambda^d$ if 2 is invertible in k , but $\mathcal{G}_{\text{Hom}}(\mathbf{1}) \cong \Gamma^d$ and thus if $2 \in k^*$

$$\begin{aligned} \mathcal{G}_{\text{Hom}}(\text{sgn}^d \otimes_k \text{sgn}^d) &= \mathcal{G}_{\text{Hom}}(\mathbf{1}) \cong \Gamma^d \neq S^d \cong \Lambda^d \otimes_{\Gamma_k^d} \Lambda^d \\ &\cong \mathcal{G}_{\text{Hom}}(\text{sgn}^d) \otimes_{\Gamma_k^d} \mathcal{G}_{\text{Hom}}(\text{sgn}^d). \end{aligned}$$

Corollary 4.13. *Let $X \in \text{rep } \Gamma_k^d$, i.e. $X^{\circ\circ} \cong X$. The endofunctor $\mathcal{G}_{\text{Hom}}\mathcal{F}$ can be expressed by duals, namely*

$$\mathcal{G}_{\text{Hom}}\mathcal{F}(X) \cong \mathcal{H}om(S^d, X) \cong \mathcal{H}om(S^d, X^{\circ\circ}) \cong \mathcal{H}om(X^\circ, \Gamma^d) = (X^\circ)^\vee. \quad \square$$

4.3. Both Adjoints

The results in the previous two sections allow us to relate the left and the right adjoint.

Proposition 4.14. *The left and the right adjoints of the Schur functor are related by taking duals, namely*

$$(\mathcal{G}_{\otimes}\mathcal{F}(X))^\circ \cong \mathcal{G}_{\text{Hom}}\mathcal{F}(X^\circ).$$

In particular

$$\mathcal{G}_{\otimes}(N)^\circ \cong \mathcal{G}_{\text{Hom}}(N^*)$$

for all $X \in \text{Rep } \Gamma_k^d$ and $N \in k\mathfrak{S}_d \text{Mod}$.

Proof. Using Theorem 4.10 and Theorem 4.3 we get

$$(\mathcal{G}_{\otimes}\mathcal{F}(X))^\circ \cong (S^d \otimes X)^\circ \cong \mathcal{H}om(S^d, X^\circ) \cong \mathcal{G}_{\text{Hom}}\mathcal{F}(X^\circ).$$

By setting $N := \mathcal{F}(X)$ and using the fact that $\mathcal{F}(X^\circ) \cong \mathcal{F}(X)^*$ we get the second isomorphism. \square

Remark 4.15. In case of k being a field of characteristic p , this result has already been obtained in a more general setting by N. Kuhn in [Kuh02, Theorem 6.10, Lemma 6.11].

To summarise, we have the following commutative diagram

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$$\begin{array}{ccc}
 \text{rep } \Gamma_k^d & \begin{array}{c} \xleftarrow{\mathcal{G}_\otimes} \\ \xrightarrow{\mathcal{F}} \end{array} & k\mathfrak{S}_d \text{ mod} \\
 (-)^\circ \downarrow & & \downarrow (-)^* \\
 (\text{rep } \Gamma_k^d)^{\text{op}} & \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \xleftarrow{\mathcal{G}_{\text{Hom}}} \end{array} & (k\mathfrak{S}_d \text{ mod})^{\text{op}}
 \end{array}$$

where the vertical arrows are equivalences of categories. The horizontal arrows become equivalences when restricted to the following subcategories

$$\begin{array}{ccc}
 \text{add } S & \begin{array}{c} \xleftarrow{\mathcal{G}_\otimes} \\ \xrightarrow{\mathcal{F}} \end{array} & \text{add } M \\
 (-)^\circ \downarrow & & \downarrow (-)^* \\
 (\text{add } \Gamma)^{\text{op}} & \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \xleftarrow{\mathcal{G}_{\text{Hom}}} \end{array} & (\text{add } M)^{\text{op}}
 \end{array}$$

5. The Tensor Product on Strict Polynomial Functors

Following the discussion of the monoidal structure on strict polynomial functors in its generality, we now apply these findings to provide explicit calculations of the (internal) tensor product. Firstly, we compute the tensor product of (generalized) divided, symmetric and exterior powers. Secondly, we consider Schur and Weyl functors. Among other results it turns out that the category of Weyl, respectively Schur filtered functors is not closed under the tensor product. Finally, for k a field we focus on simple functors. In the case $\text{char } k = 0$ the Schur functor is an equivalence of categories and the internal tensor product of strict polynomial functors equals the Kronecker product of $k\mathfrak{S}_d$ -modules. Hence, we may concentrate on the case where $\text{char } k = p > 0$.

5.1. Divided, Symmetric and Exterior Powers

Some explicit calculations of tensor products of divided, symmetric and exterior powers are already known. Recall from Proposition 2.16 that:

- $\Lambda^d \otimes_{\Gamma_k^d} \Gamma^\lambda \cong \Lambda^\lambda$
- $\Lambda^d \otimes_{\Gamma_k^d} \Lambda^\lambda \cong S^\lambda$
- $S^d \otimes_{\Gamma_k^d} \Gamma^\lambda \cong S^\lambda$

We now aim to calculate the tensor product of any other two divided, symmetric and exterior power. We start with the tensor product of two divided powers and show that they behave in the same way as permutation modules, in particular the tensor product can be decomposed in divided powers again.

Recall that for $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$ the set A_μ^λ consists of all $n \times m$ matrices $A = (a_{ij})$ with entries in \mathbb{N} such that $\lambda_i = \sum_j a_{ij}$ and $\mu_j = \sum_i a_{ij}$. A matrix $A \in A_\mu^\lambda$ is considered as a composition by reading the entries from left to right and top to bottom.

Proposition 5.1. *For all $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$ we have*

$$\Gamma^\lambda \otimes \Gamma^\mu \cong \bigoplus_{A \in A_\mu^\lambda} \Gamma^A.$$

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Proof. Using the decomposition (2.2) applied to Γ^{d,k^n} we get

$$\begin{aligned} \bigoplus_{\nu \in \Lambda(n \cdot m, d)} \Gamma^\nu = \Gamma^{d, k^{n \cdot m}} &\cong \Gamma^{d, k^n} \otimes \Gamma^{d, k^m} = \left(\bigoplus_{\lambda \in \Lambda(n, d)} \Gamma^\lambda \right) \otimes \left(\bigoplus_{\mu \in \Lambda(m, d)} \Gamma^\mu \right) \\ &\cong \bigoplus_{\substack{\lambda \in \Lambda(n, d) \\ \mu \in \Lambda(m, d)}} \Gamma^\lambda \otimes \Gamma^\mu. \end{aligned}$$

Thus, $\Gamma^\lambda \otimes \Gamma^\mu$ belongs to $\mathbf{add} \Gamma$. Since by Lemma 3.20 the subcategory $\mathbf{add} \Gamma$ is equivalent to $\mathbf{add} M$, the decomposition of the tensor product of permutation modules given in Lemma 3.5 implies the claim. \square

Next we consider the dual objects, the symmetric powers. We obtain an analogous result, namely:

Proposition 5.2. *For all $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$*

$$S^\lambda \otimes S^\mu \cong \bigoplus_{A \in A_\mu^\lambda} S^A.$$

Proof. From Proposition 2.16 we know that $S^\lambda \cong S^d \otimes \Gamma^\lambda$ and thus

$$S^\lambda \otimes S^\mu \cong S^d \otimes \Gamma^\lambda \otimes S^d \otimes \Gamma^\mu \cong S^d \otimes S^d \otimes \Gamma^\lambda \otimes \Gamma^\mu.$$

By Corollary 2.32 $S^d \otimes S^d \cong S^d$ and by Proposition 5.1 we know $\Gamma^\lambda \otimes \Gamma^\mu \cong \bigoplus_{A \in A_\mu^\lambda} \Gamma^A$. Thus,

$$S^\lambda \otimes S^\mu \cong S^d \otimes \left(\bigoplus_{A \in A_\mu^\lambda} \Gamma^A \right) \cong \bigoplus_{A \in A_\mu^\lambda} (S^d \otimes \Gamma^A) \cong \bigoplus_{A \in A_\mu^\lambda} S^A. \quad \square$$

We are also able to calculate the tensor product of two exterior powers:

Proposition 5.3. *For all $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$*

$$\Lambda^\lambda \otimes \Lambda^\mu \cong \bigoplus_{A \in A_\mu^\lambda} S^A.$$

Proof. From Proposition 2.16 we know that $\Lambda^d \otimes \Lambda^d \cong S^d$ and $\Lambda^\lambda \cong \Lambda^d \otimes \Gamma^\lambda$ implying

$$\Lambda^\lambda \otimes \Lambda^\mu \cong \Lambda^d \otimes \Gamma^\lambda \otimes \Lambda^d \otimes \Gamma^\mu \cong S^d \otimes \left(\bigoplus_{A \in A_\mu^\lambda} \Gamma^A \right) \cong \bigoplus_{A \in A_\mu^\lambda} S^A. \quad \square$$

Now we consider the tensor product between divided powers and symmetric, respectively exterior powers:

Proposition 5.4. *For all $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$*

$$\Gamma^\lambda \otimes \Lambda^\mu \cong \bigoplus_{A \in A_\mu^\lambda} \Lambda^A \quad \text{and} \quad \Gamma^\lambda \otimes S^\mu \cong \bigoplus_{A \in A_\mu^\lambda} S^A.$$

Proof. This follows directly from Proposition 5.1 by using the isomorphism $\Lambda^\mu \cong \Lambda^d \otimes \Gamma^\mu$, respectively $S^\mu \cong S^d \otimes \Gamma^\mu$ (Proposition 2.16). \square

It remains to calculate the tensor product of exterior and symmetric powers. This time we have to take the ring k into account.

Proposition 5.5. *For all $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$ it holds*

$$\Lambda^\lambda \otimes S^\mu \cong \begin{cases} \bigoplus_{A \in A_\mu^\lambda} \Lambda^A & \text{if } 2 \text{ is invertible in } k, \\ \bigoplus_{A \in A_\mu^\lambda} S^A & \text{if } 2 \text{ is not invertible in } k. \end{cases}$$

Proof. We know from Proposition 2.16 that $S^\mu \cong S^d \otimes \Gamma^\mu$ and from Lemma 2.27 that $\Lambda^\lambda \otimes S^d \cong \mathcal{H}om(\Lambda^\lambda, \Gamma^d)^\circ = ((\Lambda^\lambda)^\vee)^\circ$. Thus

$$\Lambda^\lambda \otimes S^\mu \cong ((\Lambda^\lambda)^\vee)^\circ \otimes \Gamma^\mu.$$

We use Proposition 2.30 to obtain

$$((\Lambda^\lambda)^\vee)^\circ \cong \begin{cases} \Lambda^\lambda & \text{if } 2 \text{ is invertible in } k, \\ S^\lambda & \text{if } 2 \text{ is not invertible in } k. \end{cases}$$

The calculations in Proposition 5.4, respectively Proposition 5.2 finish the proof. \square

Let us summarize our computations:

Corollary 5.6. *Define*

$$\Gamma^{\lambda\mu} := \bigoplus_{A \in A_\mu^\lambda} \Gamma^A, \quad \Lambda^{\lambda\mu} := \bigoplus_{A \in A_\mu^\lambda} \Lambda^A, \quad S^{\lambda\mu} := \bigoplus_{A \in A_\mu^\lambda} S^A.$$

The internal tensor products of divided, symmetric and exterior powers are given by

\otimes	Γ^μ	S^μ	Λ^μ	\otimes	Γ^μ	S^μ	Λ^μ
Γ^λ	$\Gamma^{\lambda\mu}$	$S^{\lambda\mu}$	$\Lambda^{\lambda\mu}$	Γ^λ	$\Gamma^{\lambda\mu}$	$S^{\lambda\mu}$	$\Lambda^{\lambda\mu}$
S^λ	$S^{\lambda\mu}$	$S^{\lambda\mu}$	$\Lambda^{\lambda\mu}$	S^λ	$S^{\lambda\mu}$	$S^{\lambda\mu}$	$S^{\lambda\mu}$
Λ^λ	$\Lambda^{\lambda\mu}$	$\Lambda^{\lambda\mu}$	$S^{\lambda\mu}$	Λ^λ	$\Lambda^{\lambda\mu}$	$S^{\lambda\mu}$	$S^{\lambda\mu}$
	<i>if 2 is invertible in k</i>				<i>if 2 is not invertible in k</i>		

In particular, the tensor products of divided, symmetric and exterior powers behave like tensor products of permutation modules and their decompositions depend only on the set A_μ^λ . Thus, the computations reduce to tensor products of the form $X^d \otimes Y^d$ with $X, Y \in \{\Gamma, S, \Lambda\}$.

5.2. Schur and Weyl Functors

We already know that Weyl and Schur functors are connected by tensoring with the exterior powers:

Proposition 5.7. *[[Kra13, Proposition 4.16], [AB85]] For every partition λ , there is an isomorphism*

$$\Lambda^d \otimes W_\lambda \cong S_{\lambda'}.$$

It follows immediately that Weyl functors are obtained by taking the internal hom of exterior powers and Schur functors:

Corollary 5.8. *For every partition λ , there is an isomorphism*

$$\mathcal{H}om(\Lambda^d, S_\lambda) \cong W_{\lambda'}.$$

Proof. Since Λ^d is finitely presented, we can use Lemma 2.27 and obtain because of $W_\lambda^\circ \cong S_\lambda$

$$\mathcal{H}om(\Lambda^d, S_\lambda) \cong (\Lambda^d \otimes W_\lambda)^\circ \cong S_{\lambda'}^\circ \cong W_{\lambda'}. \quad \square$$

Relating Weyl and Schur functors in the other direction is not possible in general, but we can show:

Proposition 5.9. *For every partition λ there are isomorphisms*

$$\Lambda^d \otimes S_\lambda \cong \mathcal{G}_\otimes(\mathrm{dSp}(\lambda')) \quad \text{and} \quad \mathcal{H}om(\Lambda^d, W_\lambda) \cong \mathcal{G}_{\mathrm{Hom}}(\mathrm{Sp}(\lambda')).$$

Proof. By Proposition 5.7 we know

$$\Lambda^d \otimes S_\lambda \cong \Lambda^d \otimes \Lambda^d \otimes W_{\lambda'} \cong S^d \otimes W_{\lambda'}.$$

Using Theorem 4.3 one gets

$$S^d \otimes W_{\lambda'} \cong \mathcal{G}_\otimes \mathcal{F}(W_{\lambda'}) \cong \mathcal{G}_\otimes(\mathrm{dSp}(\lambda')),$$

where the last isomorphism is from (3.5). For W_λ we use Corollary 5.8 and Theorem 4.10 to obtain

$$\begin{aligned} \mathcal{H}om(\Lambda^d, W_\lambda) &\cong \mathcal{H}om(\Lambda^d, \mathcal{H}om(\Lambda^d, S_{\lambda'})) \cong \mathcal{H}om(S^d, S_{\lambda'}) \\ &\cong \mathcal{G}_{\mathrm{Hom}} \mathcal{F}(S_{\lambda'}) \cong \mathcal{G}_{\mathrm{Hom}}(\mathrm{Sp}(\lambda')). \end{aligned}$$

□

Recall from Definition 2.22 that $\mathrm{Filt}(\nabla)$, respectively $\mathrm{Filt}(\Delta)$ denotes the subcategory of all objects that are filtered by Schur, respectively Weyl functors. We now concentrate on the closedness of these subcategories under the tensor product.

Proposition 5.10. *In general, $\mathrm{Filt}(\Delta)$ and $\mathrm{Filt}(\nabla)$ are not closed under the tensor product.*

Proof. Recall that $\Lambda^d \in \text{Filt}(\Delta)$. Consider $\Lambda^d \otimes \Lambda^d \cong S^d$. This Schur functor is of course in $\text{Filt}(\nabla)$, but in general it is not in $\text{Filt}(\Delta)$: Let for example $n = p = d$. Then the composition series of $S^{(p)}$ is as follows:

$$S^{(p)} = \begin{matrix} L_{(p-1,1)} \\ L_{(p)} \end{matrix}$$

In particular, it is uniserial and since neither $L_{(p-1,1)}$ nor $L_{(p)}$ is a Weyl functor, $S^{(p)}$ cannot possess a Weyl filtration.

Now consider $S^\lambda, \Lambda^d \in \text{Filt}(\nabla)$. We know that

$$\mathcal{F}(S^\lambda \otimes \Lambda^d) \cong \mathcal{F}(S^\lambda) \otimes \mathcal{F}(\Lambda^d) \cong \text{Sp}(\lambda) \otimes \text{sgn}^d \cong \text{dSp}(\lambda').$$

It is known that in general the category of Specht filtered modules does not coincide with the category of dual Specht filtered modules, see [Hem07] for a discussion on modules that possess both filtrations. In particular, there exists dual Specht modules that do not have a Specht filtration and thus, in those cases, $S^\lambda \otimes \Lambda^d$ cannot have a Schur filtration. \square

Proposition 5.11. *Let k be a field of char $k \geq 5$. Then*

$$\text{Hom}(S^d, X) \cong X \text{ for } X \in \text{Filt}(\Delta) \quad \text{and} \quad S^d \otimes Y \cong Y \text{ for } Y \in \text{Filt}(\nabla).$$

Proof. We use [HN04, Theorem 3.8.1] showing that, for a field of char $k \geq 5$, G_{Hom} is an inverse of \mathcal{F} when restricted to the subcategory $\text{Filt}(\Delta)$, i.e. $\mathcal{G}_{\text{Hom}}\mathcal{F}(X) \cong X$ for all $X \in \text{Filt}(\Delta)$. It follows by Theorem 4.10 that for all $X \in \text{Filt}(\Delta)$

$$\text{Hom}(S^d, X) \cong \mathcal{G}_{\text{Hom}}\mathcal{F}(X) \cong X.$$

To compute $S^d \otimes Y$ for $Y \in \text{Filt}(\nabla)$ we use the fact that $Y \in \text{Filt}(\nabla)$ if and only if $Y^\circ \in \text{Filt}(\Delta)$. Thus, by the previous calculation we get $\text{Hom}(S^d, Y^\circ) \cong Y^\circ$. From Lemma 2.27, we also know that $\text{Hom}(S^d, Y^\circ) \cong (S^d \otimes Y)^\circ$ and thus $(S^d \otimes Y)^\circ \cong Y^\circ$. \square

5.3. Simple Functors

Throughout this section, let k be a field of characteristic p . In this case, the isomorphism classes of simple functors in $\text{Rep } \Gamma_k^d$ are indexed by partitions $\lambda \in \Lambda^+(n, d)$. Simple functors are self-dual, i.e. $L_\lambda^\circ \cong L_\lambda$, see e.g. [Kra14, Proposition 4.11].

In [Kuh02, Theorem 7.11] a generalized Steinberg Tensor Product Theorem has been proven: simple functors are given by the external tensor product of twisted simple functors. In our setting, this has been formulated also by Touz e:

Theorem 5.12 ([Tou15, Theorem 4.8]). *Let k be a field of characteristic p . Let $\lambda^0, \dots, \lambda^r$ be p -restricted partitions, and let $\lambda = \sum_{i=0}^r p^i \lambda^i$. There is an isomorphism:*

$$L_\lambda \cong L_{\lambda^0} \boxtimes L_{\lambda^1}^{(1)} \boxtimes \dots \boxtimes L_{\lambda^r}^{(r)},$$

where $L_{\lambda^i}^{(i)}$ denotes the i -th Frobenius twist of L_{λ^i} .

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Using this decomposition, Touzé showed that in order to calculate the internal tensor product of two simple functors, it is enough to consider p -restricted partitions. Namely one has for $\lambda = \sum_{i=0}^r p^i \lambda^i$ and $\mu = \sum_{i=0}^s p^i \mu^i$

$$L_\lambda \otimes L_\mu \cong \begin{cases} (L_{\lambda^0} \otimes L_{\mu^0}) \boxtimes (L_{\lambda^1} \otimes L_{\mu^1})^{(1)} \boxtimes \cdots \boxtimes (L_{\lambda^r} \otimes L_{\mu^r})^{(r)} & \text{if } r = s, |\lambda^i| = |\mu^i|, \\ 0 & \text{otherwise.} \end{cases}$$

(see [Tou15, Theorem 6.2]). We will see in Theorem 5.15 that the tensor product of two simple functors is almost never simple.

Mullineux map and truncated symmetric powers

Recall from Remark 3.12(iii) the Mullineux map $m: \Lambda_p^+(n, d) \rightarrow \Lambda_p^+(n, d)$ that relates simple $k\mathfrak{S}_d$ -modules, see e.g. [Mar93, Chapter 4.2] for an explicit definition. Denote by Q^d the truncated symmetric powers, i.e. the top of S^d . We have the following connection between tensor products of simple functors and the Schur functor and its left adjoint:

Lemma 5.13. *Let μ be a p -restricted partition, i.e. $\mu \in \Lambda_p^+(n, d)$. Then*

$$Q^d \otimes L_\mu \cong \mathcal{G}_\otimes \mathcal{F}(L_\mu) \quad \text{and} \quad \Lambda^d \otimes L_\mu \cong \mathcal{G}_\otimes \mathcal{F}(L_{m(\mu)}).$$

Proof. From [Tou15, Corollary 6.9, 6.10] we know that

$$Q^d \otimes L_\mu \cong Q^d \otimes \Lambda^d \otimes \Lambda^d \otimes L_\mu \cong \Lambda^d \otimes \Lambda^d \otimes L_\mu \cong S^d \otimes L_\mu.$$

By Theorem 4.3 this is the same as applying the Schur functor and its left adjoint to L_μ . By [Tou15, Corollary 6.9] we have $\Lambda^d \otimes L_\mu \cong Q^d \otimes L_{m(\mu)}$ and thus $\Lambda^d \otimes L_\mu \cong \mathcal{G}_\otimes \mathcal{F}(L_{m(\mu)})$ \square

The following lemma shows in which cases the left adjoint to the Schur functor sends simple modules to simple functors.¹

Lemma 5.14. *Let $\mu \in \Lambda_p^+(n, d)$. Then $\mathcal{G}_\otimes \mathcal{F}(L_\mu) \cong L_\mu$ if and only if all ν with $\text{Ext}^1(L_\mu, L_\nu) \neq 0$ are p -restricted.*

Proof. We use [DEN04, Section 3.2, Corollary] that $\mathcal{G}_\otimes \mathcal{F}(L_\mu)$ is the largest quotient of the projective cover P_μ of L_μ whose radical has only non p -restricted composition factors. This is simple if and only if the top of $\text{rad } P_\mu$ has only p -restricted composition factors. If we apply $\text{Hom}(-, L_\nu)$ to the exact sequence

$$0 \rightarrow \text{rad } P_\mu \rightarrow P_\mu \rightarrow L_\mu \rightarrow 0,$$

we get

$$\begin{aligned} 0 \rightarrow \text{Hom}(L_\mu, L_\nu) \rightarrow \text{Hom}(P_\mu, L_\nu) \rightarrow \text{Hom}(\text{rad } P_\mu, L_\nu) \\ \rightarrow \text{Ext}^1(L_\mu, L_\nu) \rightarrow \text{Ext}^1(P_\mu, L_\nu) = 0. \end{aligned}$$

¹I am very grateful to Karin Erdmann who pointed out the connection between the occurrence of composition factors in quotients of projective covers and Ext-vanishing of simple functors.

Since $\text{Hom}(L_\mu, L_\nu) \cong \text{Hom}(P_\mu, L_\nu)$, we obtain $\text{Hom}(\text{rad } P_\mu, L_\nu) \cong \text{Ext}^1(L_\mu, L_\nu)$. That means, L_ν is a composition factor of the top of $\text{rad } P_\mu$ if and only if $\text{Ext}^1(L_\mu, L_\nu) \neq 0$. So, we get

$$\mathcal{G}_\otimes \mathcal{F}(L_\mu) \text{ is simple} \iff \text{all } \nu \text{ with } \text{Ext}^1(L_\mu, L_\nu) \neq 0 \text{ are } p\text{-restricted.}$$

Since $\mathcal{F}\mathcal{G}_\otimes \mathcal{F}(L_\mu) \cong \mathcal{F}(L_\mu)$ we know that if $\mathcal{G}_\otimes \mathcal{F}(L_\mu)$ is simple it must be isomorphic to L_μ . \square

We get the following characterization of tensor products of simple functors corresponding to p -restricted partitions that are again simple.

Theorem 5.15. *Let k be a field of odd characteristic and $\lambda, \mu \in \Lambda_p^+(n, d)$. The tensor product $L_\lambda \otimes L_\mu$ is simple if and only if, up to interchanging λ and μ ,*

- $L_\lambda \cong \Lambda^d$ and all ν with $\text{Ext}^1(L_{m(\mu)}, L_\nu) \neq 0$ are p -restricted, or
- $L_\lambda \cong Q^d$ and all ν with $\text{Ext}^1(L_\mu, L_\nu) \neq 0$ are p -restricted.

In these cases $\Lambda^d \otimes L_\mu \cong L_{m(\mu)}$ and $Q^d \otimes L_\mu \cong L_\mu$.

Proof. First note that if $\dim \mathcal{F}(L_\lambda) \geq 2$ and $\dim \mathcal{F}(L_\mu) \geq 2$, then $L_\lambda \otimes L_\mu$ is not simple (see [Tou15, Corollary 6.6]). This follows from the fact that for simple $k\mathfrak{S}_d$ -modules of dimension ≥ 2 the Kronecker product is never simple [BK00]. There are only two $k\mathfrak{S}_d$ -modules with dimension 1: $L(\omega) = \text{sgn}^d$ with $\omega = (1, \dots, 1)$ and $M^{(d)} = k$. Now, $\mathcal{F}(L_\omega) = \text{sign}$ and $\mathcal{F}(Q^d) = k$. Thus, the only cases where the tensor product might be simple are $\Lambda^d \otimes L_\mu$ and $Q^d \otimes L_\mu$.

Consider first the case $Q^d \otimes L_\mu$. By Lemma 5.13, this is the same as $\mathcal{G}_\otimes \mathcal{F}(L_\mu)$ and by Lemma 5.14 it is simple if and only if the top of $\text{rad } P_\mu$ has only p -restricted composition factors.

For $\Lambda^d \otimes L_\mu$ use Lemma 5.13 and Lemma 5.14 to obtain that $\Lambda^d \otimes L_\mu \cong \mathcal{G}_\otimes \mathcal{F}(L_{m(\mu)})$ is simple if and only if all ν with $\text{Ext}^1(L_{m(\mu)}, L_\nu) \neq 0$ are p -restricted. \square

It is not known in general when $\text{Ext}^1(L_\mu, L_\nu) \neq 0$ for partitions $\mu, \nu \in \Lambda(n, d)$, so the question of when the internal tensor product of two simple functors is again simple is not yet answered completely. Also the computation of the Mullineux map m is not easy in general.

Corollary 5.16. *If μ is a p -core, then $Q^d \otimes L_\mu \cong L_\mu$ and $\Lambda^d \otimes L_{m(\mu)} \cong L_\mu$.*

Proof. If μ is a p -core, then it is the only simple in its block, i.e. $P_\mu = L_\mu$ and thus by [DEN04, 3.2 Corollary] $\Lambda^d \otimes L_{m(\mu)} \cong Q^d \otimes L_\mu \cong L_\mu$. \square

5.3.1. A Special Case

In the case $n = d$ we make use of the following result to obtain some partitions μ such that the tensor product $\Lambda^d \otimes L_\mu$, respectively $Q^d \otimes L_\mu$ is simple:

Proposition 5.17. [DEN04, 5.6 Proposition] *Let $n = d$ and $p > 2$. Assume μ is a p -restricted partition such that all λ with $m(\mu') \geq \lambda$ are also p -restricted. Then*

$$\mathcal{G}_\otimes \mathcal{F}(L_\mu) \cong L_\mu.$$

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In particular, every partition λ such that all smaller partitions are p -restricted, provides a partition $\mu = m(\lambda)'$ such that $\mathcal{G}_\otimes \circ \mathcal{F}(L_\mu) = L_\mu$. Starting with the partition $(1, \dots, 1)$ and going through the elements of $\Lambda_p^+(n, d)$ in lexicographic order, the first partitions of the sequence are all p -restricted. The smallest not p -restricted partition ν is $\nu := (p+1, 1, \dots, 1)$ if $d \geq p+1$ and $\nu := (d)$ if $d = p$. Thus, for every $\mu = (m(\lambda))'$ with $\lambda < \nu$ we get $Q^d \otimes L_\mu \cong \mathcal{G}_\otimes \mathcal{F}(L_\mu) \cong L_\mu$.

However, Proposition 5.17 only provides a sufficient condition, so we can not deal with the case of partitions λ such that $\lambda > \nu$.

The Case $n = p = d$

We can provide a full answer under the further restriction $n = p$.²

Theorem 5.18. *Let k be a field of characteristic $p > 2$ and $n = p = d$. The tensor product $L_\lambda \otimes L_\mu$ is simple if and only if, up to interchanging λ and μ ,*

- $\lambda = (1, \dots, 1)$ and $\mu \neq (3, 1, 1, \dots, 1)$, or
- $\lambda = (p-1, 1)$ and $\mu \neq (p-1, 1)$.

In these cases $L_{(1, \dots, 1)} \otimes L_\mu \cong L_{m(\mu)}$ and $L_{(p-1, 1)} \otimes L_\mu \cong L_\mu$.

Proof. It always holds that $\Lambda^d = L_{(1, \dots, 1)}$. If $n = d = p$, the truncated symmetric powers Q^d is the simple module indexed by the partition $(p-1, 1)$, i.e. $Q^d = L_{(p-1, 1)}$. Thus, by Theorem 5.15, all tensor products where $\lambda \neq (1, \dots, 1)$ and $\lambda \neq (p-1, 1)$ are not simple. It remains to check the cases where $\lambda = (1, \dots, 1)$ or $\lambda = (p-1, 1)$.

Now all partitions μ not of the form $(p-k, 1^k) = (p-k, 1, 1, \dots, 1)$ for $1 \leq k \leq p$ are p -cores, so in these cases by Corollary 5.16

$$L_{(p-1, 1)} \otimes L_\mu \cong L_\mu \quad \text{and} \quad L_{(1, \dots, 1)} \otimes L_\mu \cong L_{m(\mu)}.$$

Suppose now $\mu = (p-k, 1^k)$. There is only one partition which is not p -restricted, namely the partition (p) . We have $m((2, 1^{p-2})) = (p)$, thus all but the partition $\mu = (p-1, 1)$ fulfill the condition of Proposition 5.17 and we get

$$L_{(p-1, 1)} \otimes L_\mu \cong \mathcal{G}_\otimes \mathcal{F}(L_\mu) \cong L_\mu$$

for all $\mu = (p-k, 1^k)$ with $1 < k \leq p$. Since $m((p-1, 1)) = (3, 1^{p-3})$ we also get

$$L_{(1, \dots, 1)} \otimes L_\mu \cong \mathcal{G}_\otimes \mathcal{F}(L_{m(\mu)}) \cong L_{m(\mu)}$$

for all $\mu = (p-k, 1^k)$ with $1 \leq k < 3$ or $3 < k \leq p$.

The cases $\mu = (p-1, 1)$, respectively $\mu = (3, 1^{p-3})$ remain. We know that the composition series of $S^{(p)}$ is as follows:

$$S^{(p)} = \begin{array}{c} L_{(p-1, 1)} \\ L_{(p)} \end{array}$$

²I am very thankful to Karin Erdmann for her advice regarding the Mullineux map in this case as well as pointing out several composition series used in the proof of the following theorem.

That is, $S^{(p)}/\text{rad}(S^{(p)}) \cong L_{(p-1,1)}$ and hence there exists a surjection $P_{(p-1,1)} \twoheadrightarrow S^{(p)}$. But then $L_{(p)}$ is in the top of $\text{rad}(P_{(p-1,1)})$ and thus $\mathcal{G}_{\otimes} \mathcal{F}(L_{(p-1,1)}) \cong L_{(p-1,1)} \otimes L_{(p-1,1)}$ is not simple. Since $m((p-1,1)) = (3, 1^{p-3})$, the functor $L_{(1,\dots,1)} \otimes L_{(3,1^{p-3})} \cong L_{(p-1,1)} \otimes L_{(p-1,1)}$ is not simple. □

6. Modules over the Schur Algebra

Schur algebras, denoted by $S_k(n, d)$, were first defined by Issai Schur in [Sch01] and have sparked intense investigations ever since. Their relation to strict polynomial functors was described by Friedlander and Suslin in [FS97], which allows us to transfer the monoidal structure on $\mathbf{Rep} \Gamma_k^d$ to $S_k(n, d) \mathbf{Mod}$. Unfortunately, the description of the tensor product inside the latter category does not simplify and concrete calculations turn out to be still hard.

In this chapter, we present the correspondences between objects in $\mathbf{Rep} \Gamma_k^d$ and $S_k(n, d) \mathbf{Mod}$ and transfer results from strict polynomial functors over, where possible. In particular, we describe the image of projective strict polynomial functors under this equivalence in full detail. Our main reference for modules over the Schur algebra is [Gre07].

Definition 6.1. For positive integers n and d , let $k\mathfrak{S}_d$ act on the left on $(k^n)^{\otimes d}$ as defined in (3.2). The *Schur algebra* $S_k(n, d)$ is the endomorphism algebra of this representation:

$$S_k(n, d) := \mathrm{End}_{k\mathfrak{S}_d}((k^n)^{\otimes d})$$

Originally, the Schur algebra was defined as the dual space of a subalgebra of finitary functions from the general linear group to k . It has been shown that this is isomorphic to the above definition [Gre07, (2.6c)].

Basis and decomposition. One can decompose the $k\mathfrak{S}_d$ -module $(k^n)^{\otimes d}$ into permutation modules (see Section 3.3) and thus obtain the following decomposition

$$S_k(n, d) = \mathrm{End}_{k\mathfrak{S}_d}((k^n)^{\otimes d}) \cong \bigoplus_{\substack{\lambda \in \Lambda(n, d) \\ \mu \in \Lambda(n, d)}} \mathrm{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu). \quad (6.1)$$

A k -basis for $\mathrm{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$ can be indexed by the set $\xi_{\mu, \lambda} = \{\xi_{\underline{j}, \underline{i}} \mid \underline{j} \in \mu, \underline{i} \in \lambda\} / \sim$ where the equivalence relation \sim is given by $\xi_{\underline{j}, \underline{i}} \sim \xi_{\underline{j}', \underline{i}'}$ if there exist $\sigma \in \mathfrak{S}_d$ such that $\underline{j}\sigma = \underline{j}'$ and $\underline{i}\sigma = \underline{i}'$.

In particular $\xi_{\underline{i}, \underline{i}} \sim \xi_{\underline{i}', \underline{i}'}$ for \underline{i} and \underline{i}' represented by the same λ . This idempotent element $\xi_{\underline{i}, \underline{i}}$ corresponds to the identity on $\mathrm{End}_{k\mathfrak{S}_d}(M^\lambda)$ and we denote it by ξ_λ . From the decomposition above it follows that

$$\bigcup_{\lambda, \mu \in \Lambda(n, d)} \xi_{\mu, \lambda} = \left\{ \xi_{\underline{j}, \underline{i}} \mid \underline{j} \in \mu, \underline{i} \in \lambda \mid \mu, \lambda \in \Lambda(n, d) \right\} / \sim$$

forms a k -basis of $S_k(n, d)$. The multiplication is given by composition of morphisms (see Appendix A.2 for a combinatorial description).

Remark 6.2. Note that the set $\xi_{\mu,\lambda}$ can in turn be identified with the set $\{A \mid A \in A_\mu^\lambda\}$. See Appendix A.1.2 for more details.

The unit id in $S_k(n, d)$ is given by the identity $\text{id}_{(k^n)^{\otimes d}}$ of $\text{End}_{k^{\mathfrak{S}_d}}((k^n)^{\otimes d})$. It decomposes into identities $\text{id}_{M^\lambda} = \xi_\lambda$ of $\text{End}_{\mathfrak{S}_d}(M^\lambda)$ and thus we get an orthogonal decomposition

$$\text{id} = \sum_{\lambda \in \Lambda(n, d)} \xi_\lambda. \quad (6.2)$$

Note that the elements ξ_λ are not indecomposable in general, see below.

An equivalence of categories. Consider the functor

$$\text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -) : \text{Rep } \Gamma_k^d \rightarrow \text{Mod } \text{End}_{\Gamma_k^d}(\Gamma^{d, k^n}).$$

From the Yoneda isomorphism (2.1)) we get an isomorphism of k -algebras

$$\text{End}_{\Gamma_k^d}(\Gamma^{d, k^n}) \cong (\Gamma^{d, k^n}(k^n))^{\text{op}} \cong (\text{End}_{\Gamma^{d, \mathfrak{P}_k}}(k^n))^{\text{op}} \cong (\text{End}_{\mathfrak{S}_d}((k^n)^{\otimes d}))^{\text{op}} = S_k(n, d)^{\text{op}}$$

and it follows that $\text{Mod } \text{End}_{\Gamma_k^d}(\Gamma^{d, k^n}) \simeq S_k(n, d) \text{ Mod}$. Thus we obtain a functor from the category of strict polynomial functors to the category of modules over the Schur algebra. Even more, it has been shown to be an equivalence in certain cases:

Theorem 6.3 ([FS97, Theorem 3.2]).

$$\text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -) : \text{Rep } \Gamma_k^d \rightarrow S_k(n, d) \text{ Mod} \quad (6.3)$$

is an equivalence for $n \geq d$.

Remark 6.4. Since we have $\text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, X) \cong X(k^n)$ for all $X \in \text{Rep } \Gamma_k^d$ (see (2.1)), the functor $\text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -)$ is nothing other than evaluation at k^n . The action of $S_k(n, d)$ on $X(k^n)$ is given by acting by X applied to the corresponding element of $\text{End}_{\Gamma^{d, \mathfrak{P}_k}}(k^n)$.

From now on, we assume $n \geq d$. We describe the image of several objects under the functor $\text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -)$.

Since for every $m \in \mathbb{N}$ and every partition $\lambda \in \Lambda(m, d)$ there is a partition $\mu \in \Lambda(n, d)$ such that $\Gamma^\lambda \cong \Gamma^\mu$ we only consider partitions in $\Lambda(n, d)$.

6.1. Projective Objects

We can view $S_k(n, d)$ as a left module over itself, $S_k(n, d) = {}_{S_k(n, d)}S_k(n, d)$. The action of $\xi \in S_k(n, d)$ on $\xi' \in {}_{S_k(n, d)}S_k(n, d)$ is given by postcomposing with ξ , i.e. $\xi \cdot \xi' = \xi \circ \xi'$. We use the decomposition in (6.2) to decompose the projective module $S_k(n, d)$:

$$S_k(n, d) \cong \bigoplus_{\lambda \in \Lambda(n, d)} S_k(n, d)\xi_\lambda \cong \bigoplus_{\lambda \in \Lambda(n, d)} \text{End}_{k^{\mathfrak{S}_d}}((k^n)^{\otimes d})\xi_\lambda$$

Since ξ_λ is the identity on M^λ and the zero map on M^μ for $\mu \neq \lambda$ we get that $\text{End}_{\mathfrak{S}_d}((k^n)^{\otimes d})\xi_\lambda \cong \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})$. Thus

$$S_k(n, d) \cong \bigoplus_{\lambda \in \Lambda(n, d)} \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}). \quad (6.4)$$

Decomposition of projective objects. Since the idempotents ξ_λ are not indecomposable in general, the module $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})$ is not indecomposable in general. Using the decomposition of permutation modules into indecomposable summands (cf. (3.4)), one gets

$$\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}) \cong \bigoplus_{\mu \in \Lambda(d)} \text{Hom}_{k\mathfrak{S}_d}(Y^\mu, (k^n)^{\otimes d})^{\otimes K_{\mu\lambda}},$$

where $K_{\mu\lambda}$ are the (p -)Kostka numbers and Y^ν the Young modules.

Correspondence with divided powers. Denote by $\underline{i} = (1 \dots 1 2 \dots 2 \dots n \dots n) \in \lambda$ the weakly increasing sequence whose first λ_1 entries are 1, the next λ_2 entries are 2 etc. Since M^λ is generated by $e_{\underline{i}}$, a map in $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})$ is uniquely determined by the image of the element $e_{\underline{i}}$.

Lemma 6.5. *The map*

$$\begin{aligned} g: \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}) &\rightarrow \Gamma^\lambda(k^n) \\ (e_{\underline{i}} \mapsto v) &\mapsto v \end{aligned} \quad (6.5)$$

is an isomorphism of left $\text{End}_{k\mathfrak{S}_d}((k^n)^{\otimes d})$ -modules.

Proof. First of all, one has to check that the map g is well-defined. Since \mathfrak{S}_λ acts trivially on M^λ , it also acts trivially on v , thus $v \in \Gamma^\lambda(k^n)$. By the same arguments there is for every $v \in \Gamma^\lambda(k^n)$ a well-defined map in $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})$ that maps $e_{\underline{i}}$ to v . Thus, g is surjective. It is clearly injective, and since the action of $\text{End}_{\mathfrak{S}_d}((k^n)^{\otimes d})$ in both cases is given by applying an endomorphism to v , the map g is a morphism of $\text{End}_{\mathfrak{S}_d}((k^n)^{\otimes d})$ -modules. \square

Objects. Using this lemma, we get for the projective objects $\Gamma^\lambda \in \text{Rep } \Gamma_k^d$ the following correspondence:

$$\begin{aligned} \text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -): \text{Rep } \Gamma_k^d &\rightarrow \text{Mod } S_k(n, d) \\ \Gamma^\lambda &\mapsto \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}) \end{aligned}$$

Morphisms. Let $m_\xi = - \cdot \xi$ denote the right multiplication by ξ . From the isomorphism $(S_k(n, d))^{\text{op}} \xrightarrow{\cong} \text{End}_{S_k(n, d)}(S_k(n, d)S_k(n, d))$ given by $\xi \mapsto m_\xi$ and the decomposition (6.1) we get the following isomorphism:

$$\begin{aligned} \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu) &\xrightarrow{\cong} \text{Hom}_{S_k(n, d)}(\text{Hom}_{k\mathfrak{S}_d}(M^\mu, (k^n)^{\otimes d}), \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})) \\ \xi &\mapsto m_\xi = - \circ \xi \end{aligned}$$

6. Modules over the Schur Algebra

Thus the morphisms between projective objects are given by multiplication with elements of $S_k(n, d)$. By (6.5) $\text{Hom}_{S_k(n, d)}(\text{Hom}_{k\mathfrak{S}_d}(M^\mu, (k^n)^{\otimes d}), \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}))$ is isomorphic to $\text{Hom}_{S_k(n, d)}(\Gamma^\mu(k^n), \Gamma^\lambda(k^n))$, where the morphism $(- \circ \xi_{\underline{j}, \underline{i}})$ is mapped to $\varphi_A(k^n)$ for $A = A_{\underline{j}, \underline{i}}$ (see (A.1)). Thus, in conclusion, we get:

$$\begin{aligned} \text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -) &: \text{Rep } \Gamma_k^d \rightarrow \text{Mod } S_k(n, d) \\ \text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda) &\mapsto \text{Hom}_{S_k(n, d)}(\Gamma^\mu(k^n), \Gamma^\lambda(k^n)) \\ &\cong \text{Hom}_{S_k(n, d)}(\text{Hom}_{k\mathfrak{S}_d}(M^\mu, (k^n)^{\otimes d}), \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})) \\ &\cong \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu) \\ \varphi_{A_{\underline{j}, \underline{i}}} &\mapsto \varphi_{A_{\underline{j}, \underline{i}}}(k^n) = (- \circ \xi_{\underline{j}, \underline{i}}) \end{aligned}$$

In particular, since $\text{add } \Gamma \simeq \text{add } M$, there is an isomorphism

$$\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda) \cong \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu). \quad (6.6)$$

6.2. Monoidal Structure

Via the equivalence in Theorem 6.3, it is possible to transfer the monoidal structure on strict polynomial functors to the category of modules over the Schur algebra. We denote the internal tensor product by $- \otimes_S -$ and the internal hom by $\mathcal{H}om_S(-, -)$. Unfortunately, an explicit description is only available for the projective objects discussed before.

Proposition 6.6. *The monoidal structure on modules over the Schur algebra obtained from the one on strict polynomial functors is given by*

$$\begin{aligned} \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}) \otimes_S \text{Hom}_{k\mathfrak{S}_d}(M^\mu, (k^n)^{\otimes d}) &\cong \text{Hom}_{k\mathfrak{S}_d}(M^\lambda \otimes_k M^\mu, (k^n)^{\otimes d}) \\ \mathcal{H}om_S(\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}), \text{Hom}_{k\mathfrak{S}_d}(M^\mu, (k^n)^{\otimes d})) &\cong \text{Hom}_{k\mathfrak{S}_d}(\mathcal{H}om(M^\lambda, M^\mu), (k^n)^{\otimes d}) \\ \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})^\vee &\cong \text{Hom}_{k\mathfrak{S}_d}((M^\lambda)^*, (k^n)^{\otimes d}). \end{aligned}$$

Proof. From Proposition 5.1 we know that

$$\Gamma^\lambda \otimes \Gamma^\mu \cong \bigoplus_{A \in A_\mu^\lambda} \Gamma^A$$

and from Lemma 3.5 that

$$M^\lambda \otimes_k M^\mu \cong \bigoplus_{A \in A_\mu^\lambda} M^A.$$

Thus, by identifying $\Gamma^\lambda(k^n)$ with $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})$, we get

$$\begin{aligned} \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}) \otimes \text{Hom}_{k\mathfrak{S}_d}(M^\mu, (k^n)^{\otimes d}) &\cong \bigoplus_{A \in A_\mu^\lambda} \text{Hom}_{k\mathfrak{S}_d}(M^A, (k^n)^{\otimes d}) \\ &\cong \text{Hom}_{k\mathfrak{S}_d}\left(\bigoplus_{A \in A_\mu^\lambda} M^A, (k^n)^{\otimes d}\right) \\ &\cong \text{Hom}_{k\mathfrak{S}_d}(M^\lambda \otimes M^\mu, (k^n)^{\otimes d}). \end{aligned}$$

The computation of the internal hom follows by a similar argument and for the monoidal dual, we observe that the tensor unit is $\text{Hom}_{k\mathfrak{S}_d}(M^d, (k^n)^{\otimes d})$ and thus

$$\begin{aligned} \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})^\vee &= \mathcal{H}om_S(\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}), \text{Hom}_{k\mathfrak{S}_d}(M^d, (k^n)^{\otimes d})) \\ &\cong \text{Hom}_{k\mathfrak{S}_d}(\mathcal{H}om(M^\lambda, M^d), (k^n)^{\otimes d}) \\ &\cong \text{Hom}_{k\mathfrak{S}_d}((M^\lambda)^*, (k^n)^{\otimes d}). \end{aligned}$$

□

6.3. Highest Weight Structure

For all $n, d \geq 1$, the Schur algebra $S_k(n, d)$ is a quasi-hereditary algebra, i.e. its module category is a highest weight category. The (co)standard and simple modules are given by the images of the corresponding strict polynomial functors under $\text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -)$. They are indexed by the partially ordered set $\Lambda^+(n, d)$, see Section 2.6.1 for the order.

Weyl and Schur modules The costandard modules are given by *Schur modules* obtained by evaluating a Schur functor S_λ at k^n . The standard modules are given by *Weyl modules* obtained by evaluating a Weyl functor W_λ at k^n .

Simple modules In the case k is a field, the simple modules are given by quotients of Weyl modules or, equivalently, $L(\lambda) := L_\lambda(k^n)$. The set $\{L(\lambda) \mid \lambda \in \Lambda^+(n, d)\}$ forms a complete set of isomorphism classes of simple modules.

6.4. Duality

Analogously to the Kuhn dual for strict polynomial functors, there exist a dual for modules over the Schur algebra, sometimes called the *contravariant dual*. It is defined by (see [Gre07, (2.7c)])

$$V^\circ := V^* = \text{Hom}_k(V, k)$$

with action given for all $\xi \in S_k(n, d)$ by

$$(\xi \cdot f)(v) = f(J(\xi)v),$$

where $J: S_k(n, d) \rightarrow S_k(n, d)$ is the k -linear map defined on basis elements by $J(\xi_{i,j}) := \xi_{j,i}$.

6.5. The Schur Functor

Originally, the Schur functor was defined from the category of modules over the Schur algebra to those over the group algebra of the symmetric group. Namely, let $n \geq d$ and $\omega = (1, \dots, 1, 0, \dots, 0) \in \Lambda(n, d)$. Then $\xi_\omega S_k(n, d) \xi_\omega \cong k\mathfrak{S}_d$ and we have the following

Definition 6.7. With the assumption above the *Schur functor* f is defined as follows:

$$\begin{aligned} f: S_k(n, d) \text{ Mod} &\rightarrow \xi_\omega S_k(n, d) \xi_\omega \text{ Mod} \simeq k\mathfrak{S}_d \text{ Mod} \\ V &\mapsto \xi_\omega \cdot V. \end{aligned}$$

The relation to the Schur functor \mathcal{F} defined in Section 3.4 is given as follows:

Proposition 6.8. Let $n \geq d$. The Schur functor \mathcal{F} agrees with the composition of f and $\text{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, -)$, i.e. following diagram commutes:

$$\begin{array}{ccc} \text{Rep } \Gamma_k^d & \xrightarrow{\text{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, -)} & \text{Mod End}(\Gamma^{d,k^n}) \simeq S_k(n, d) \text{ Mod} \\ & \searrow \mathcal{F} & \downarrow f \\ & & \text{Mod End}(\Gamma^\omega) \simeq k\mathfrak{S}_d \text{ Mod} \end{array}$$

Proof. Let $\underline{i} \in \omega$. Under the isomorphism $S_k(n, d) \cong \text{End}(\Gamma^{d,k^n})^{\text{op}}$, the idempotent $\xi_\omega = \xi_{\underline{i}, \underline{i}}$ corresponds to the idempotent $\varphi_{\underline{i}, \underline{i}} \in \text{End}(\Gamma^\omega) \subseteq \text{End}(\Gamma^{d,k^n})$. Since $\varphi_{\underline{i}, \underline{i}}$ is the identity on Γ^ω , we have $\varphi_{\underline{i}, \underline{i}}(\Gamma^{d,k^n}) \cong \Gamma^\omega$ and in particular $\varphi_{\underline{i}, \underline{i}} \circ \text{End}(\Gamma^{d,k^n}) \circ \varphi_{\underline{i}, \underline{i}} \cong \text{End}(\Gamma^\omega)$. Thus $f = \xi_\omega \cdot -$ corresponds to the functor

$$- \cdot \varphi_{\underline{i}, \underline{i}}: \text{Mod End}(\Gamma^{d,k^n}) \rightarrow \text{Mod End}(\Gamma^\omega)$$

and $f \circ \text{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, -)$ corresponds to

$$(- \cdot \varphi_{\underline{i}, \underline{i}}) \circ \text{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, -) = \text{Hom}_{\Gamma_k^d}(\varphi_{\underline{i}, \underline{i}}(\Gamma^{d,k^n}), -) \cong \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, -).$$

□

In particular we can describe the Schur functor \mathcal{F} by

$$\begin{array}{ccccc} \text{Rep } \Gamma_k^d & \rightarrow & S_k(n, d) \text{ Mod} & \rightarrow & \xi_\omega S_k(n, d) \xi_\omega \text{ Mod} \\ X & \mapsto & X(k^n) & \mapsto & X(\xi_\omega) X(k^n) \end{array}$$

For $X = \Gamma^\lambda$ we have $X(\xi) = \Gamma^\lambda(\xi) = \xi$ and thus, this reads

$$\begin{array}{ccccc} \text{Rep } \Gamma_k^d & \rightarrow & S_k(n, d) \text{ Mod} & \rightarrow & \xi_\omega S_k(n, d) \xi_\omega \text{ Mod} \\ \Gamma^\lambda & \mapsto & \Gamma^\lambda(k^n) & \mapsto & \xi_\omega \cdot \Gamma^\lambda(k^n) \end{array}$$

7. Conclusion

Strengthening the relation between strict polynomial functors and representations of the symmetric group has served as a main tool to advance in the description of the tensor product of strict polynomial functors. In particular we have showed that the Schur functor is monoidal and provided a description of its adjoints in terms of the monoidal structure. This has enabled us to obtain explicit computations of the tensor product of strict polynomial functors from known calculations for symmetric group representations.

Partial results for the tensor product of divided, symmetric and exterior powers have been completed to a full description.

The tensor product of Schur and Weyl functors has turned out to be harder to describe. In particular, the tensor product of Schur, respectively Weyl filtered functors is not again Schur, respectively Weyl filtered in general. We have obtained partial results on the tensor product, but an exhaustive understanding could not be developed with these methods.

Lastly, we have characterized those tensor products of two simple functors that are again simple. The condition is based on Ext-vanishing between certain simple functors, thus it relies on homological properties. It would be interesting to extend this approach and find more relations between homological and representation theoretic characteristics. Since Ext-calculations between simple functors are not known in general, it would also be desirable to provide more explicit conditions on the tensor product of simple functors.

All computations of tensor products of strict polynomial functors can be transferred, via the Schur functor, to computations of the Kronecker product of symmetric group representations. The results obtained in this thesis do not lead to new results for the Kronecker product; some of the results on strict polynomial functors have been even derived from this structure. Since the Kronecker product is not yet fully understood, it would be interesting to obtain results on the tensor product of strict polynomial functors independently of symmetric group representations and in the best case use those results to advance in the understanding of the Kronecker product.

A. Appendix

In this appendix we present some very explicit and detailed calculations needed to show the exact correspondences between objects, respectively morphisms between strict polynomial functors, modules over the group algebra of the symmetric group and modules over the Schur algebra. At the end we provide a tabular overview over these correspondences.

A.1. Standard Morphisms

For this appendix we fix $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$. Recall from Section 1.3.1 the set

$$A_\mu^\lambda := \{A = (a_{ij})_{ij} \in \mathbf{M}_{n \times m}(\mathbb{N}) \mid \lambda_i = \sum_j a_{ij}, \mu_j = \sum_i a_{ij}\}.$$

That is, A_μ^λ consists of $n \times m$ matrices A with the following row and column sums:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \begin{array}{l} \sum a_{1j} = \lambda_1 \\ \sum a_{2j} = \lambda_2 \\ \vdots \\ \sum a_{nj} = \lambda_n \end{array}$$

$$\begin{array}{cccc} \sum a_{i1} & \sum a_{i2} & \dots & \sum a_{im} \\ \parallel & \parallel & & \parallel \\ \mu_1 & \mu_2 & & \mu_m \end{array}$$

We know that the set A_μ^λ indexes

- the standard morphisms for strict polynomial functors $\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda)$,
- the standard morphisms for modules over the symmetric group $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$,
- the basis elements of $S_k(n, d)$ contained in $\xi_{\mu, \lambda}$.

We explain this correspondence and how these standard morphisms are related to each other.

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First we need a bijection between the matrices $A = (a_{ij})_{ij} \in A_\mu^\lambda$ and pairs of sequences $(\underline{j}, \underline{i})$ with $\underline{j} \in \mu$ and $\underline{i} \in \lambda$. Formally, it is given by

$$(\underline{j}, \underline{i}) \mapsto a_{ij} := \#\{l \mid i_l = i, j_l = j\},$$

$$A = (a_{ij})_{ij} \mapsto \begin{cases} \underline{j} := \underbrace{1 \dots 1}_{a_{11}} \underbrace{2 \dots 2}_{a_{12}} \dots \underbrace{m \dots m}_{a_{1m}} \underbrace{1 \dots 1}_{a_{21}} \dots \underbrace{m \dots m}_{a_{nm}} \\ \underline{i} := \underbrace{1 \dots 1}_{\lambda_1} \underbrace{2 \dots 2}_{\lambda_2} \dots \underbrace{n \dots n}_{\lambda_n}. \end{cases} \quad (\text{A.1})$$

We denote the matrix A corresponding to $(\underline{j}, \underline{i})$ by $A_{\underline{j}, \underline{i}}$ and the corresponding morphism of strict polynomial functors $\varphi_{A_{\underline{j}, \underline{i}}}$ (see Section 2.4) by $\varphi_{\underline{j}, \underline{i}}$. Recall that $(\underline{j}, \underline{i}) \sim (\underline{j}', \underline{i}')$ if $\underline{j}\sigma = \underline{j}'$ and $\underline{i}\sigma = \underline{i}'$ for some $\sigma \in \mathfrak{S}_d$. Note that for such pairs we have $A_{\underline{j}, \underline{i}} = A_{\underline{j}', \underline{i}'}$.

To see how to deal with this correspondence in practice, we consider the following

Example A.1. Given a matrix $A \in A_\mu^\lambda$, we replace every integer a_{ij} in the matrix by a_{ij} many entries j , i.e. for example

$$A = \begin{bmatrix} 2 & 3 & 1 & 1 & 2 & \dots \\ 1 & 3 & 2 & \dots & & \\ 1 & 1 & 4 & \dots & & \\ \dots & & & & & \end{bmatrix} \rightsquigarrow \begin{bmatrix} 11 & 222 & 3 & 4 & 55 & \dots \\ 1 & 222 & 33 & \dots & & \\ 1 & 2 & 3333 & \dots & & \\ \dots & & & & & \end{bmatrix}$$

Now we obtain the pair $(\underline{j}, \underline{i})$ corresponding to the matrix A as follows:

- \underline{j} is the sequence reading the integers from left to right, from top to bottom, i.e. in the example above $\underline{j} = 112223455 \dots 122233 \dots 12333 \dots$ and
- \underline{i} is the sequence reading the integers from top to bottom, from left to right, i.e. in the example above $\underline{i} = 1111 \dots 222222 \dots 3333333 \dots 4 \dots 55 \dots$.

The other way around, for a given pair of sequences $(\underline{j}, \underline{i})$ we obtain the corresponding matrix $A_{\underline{j}, \underline{i}}$ as follows: first we take an equivalent pair (by permuting the sequences) such that \underline{i} is ordered as above, i.e. $\underline{i} = 11 \dots 2 \dots nn$. We write down the sequence \underline{j} under the sequence \underline{i} , i.e. for example

$$\begin{aligned} \underline{i} &= 11111 \dots 2222 \dots 33 \dots 44444 \\ \underline{j} &= 11567 \dots 1338 \dots 55 \dots 23337 \end{aligned}$$

Now a_{ij} is the number how often a pair $\begin{bmatrix} i \\ j \end{bmatrix}$ appears in the above, i.e. in this example

$$a_{11} = \# \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2, \quad a_{12} = \# \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0, \quad a_{43} = \# \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 3, \dots$$

and thus

$$A_{\underline{j}, \underline{i}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 2 & 0 & \dots & & & \\ 0 & 0 & 0 & 0 & 2 & \dots & & \\ 0 & 1 & 3 & 0 & \dots & & & \end{pmatrix}$$

After having established a correspondence between the sets A_μ^λ and $\{(j, \underline{i}) \mid j \in \mu, \underline{i} \in \lambda\}$, we are now going to explain their occurrence in our three different contexts:

A.1.1. Standard Morphisms of Modules over the Group Algebra of the Symmetric Group

Recall that a k -basis of $(k^n)^{\otimes d}$ is given by $\{e_{\underline{i}} \mid \underline{i} \in I(n, d)\}$ and one for M^λ is given by $\{e_{\underline{i}} \mid \underline{i} \in I(n, d), \underline{i} \in \lambda\}$, see Definition 3.2. Thus, a morphism $\varphi \in \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$ can be represented as a matrix B_φ where the columns, respectively rows are indexed by the elements $\{e_{\underline{i}} \mid \underline{i} \in \lambda\}$, respectively $\{e_{\underline{j}} \mid \underline{j} \in \mu\}$.

Since a morphism must be invariant under the action of \mathfrak{S}_d , the matrix B_φ has an entry x at position $(\underline{j}, \underline{i})$ if and only if it has the same entry x at every position $(j\pi, i\pi)$ for $\pi \in \mathfrak{S}_d$. Hence, a k -span of $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$ is given by matrices $B_{(\underline{j}, \underline{i})}$, for $(\underline{j}, \underline{i}) \in I(n, d) \times I(n, d)$, defined as follows, see also [Gre07, (2.6d)]:

$$(B_{(\underline{j}, \underline{i})})_{\underline{j}', \underline{i}'} = \begin{cases} 1 & \text{if } (\underline{j}', \underline{i}') \sim (\underline{j}, \underline{i}), \\ 0 & \text{else.} \end{cases}$$

Note that $B_{(\underline{j}, \underline{i})} = B_{(\underline{j}', \underline{i}')}$ if and only if $(\underline{j}', \underline{i}') \sim (\underline{j}, \underline{i})$. Moreover the matrix entries $(B_{(\underline{j}, \underline{i})})_{\underline{j}', \underline{i}'}$ and $(B_{(\underline{j}, \underline{i})})_{\underline{j}'\sigma, \underline{i}'\sigma}$ must be the same for all $\sigma \in \mathfrak{S}_d$.

In particular, for $\underline{i} = (11 \dots 2 \dots nn) \in \lambda$ the basis element $e_{\underline{i}}$ is mapped to a sum of elements of the form $\sum_{\sigma \in I_{\underline{j}}} e_{\underline{j}}\sigma$, where $I_{\underline{j}}$ is a full set of representatives of $\mathfrak{S}_\lambda / \mathfrak{S}_{A_{\underline{j}, \underline{i}}}$. Explicitly, if $\sigma \in \mathfrak{S}_\lambda$, we have $\underline{i}'\sigma = \underline{i}'$ and thus $(B_{(\underline{i}, \underline{j})})_{\underline{i}', \underline{j}'} = (B_{(\underline{i}, \underline{j})})_{\underline{i}'\sigma, \underline{j}'}$.

Now recall that every morphism in $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$ is uniquely determined by its image on one of the basis elements of M^λ , because every basis element generates the transitive permutation module M^λ . It follows that for $\underline{i} = (11 \dots 2 \dots nn) \in \lambda$, the morphisms $\xi_{\underline{j}, \underline{i}}$ given by the matrices $B_{(\underline{i}, \underline{j})}$ with $\underline{j} \in \mu$ are explicitly defined by

$$\begin{aligned} \xi_{\underline{j}, \underline{i}}: \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu) \\ e_{\underline{i}} \mapsto \sum_{\sigma \in I_{\underline{j}}} e_{\underline{j}}\sigma \end{aligned} \tag{A.2}$$

and span $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$. Note that $\xi_{\underline{j}, \underline{i}} \cong \xi_{\underline{j}', \underline{i}}$ if and only if $(\underline{j}, \underline{i}) \sim (\underline{j}', \underline{i})$ and thus the set $\{\xi_{\underline{j}, \underline{i}} \mid \underline{j} \in \mu, \underline{i} \in \lambda\} / \sim$ forms a k -basis of $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$. We identify the morphism $\xi_{\underline{j}, \underline{i}}$ with the matrix $A_{\underline{j}, \underline{i}}$.

A.1.2. Basis for the Schur Algebra

As seen in Chapter 6, the Schur algebra consists of homomorphisms between permutation modules. The basis elements $\xi_{\underline{j}, \underline{i}}$ of $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$ from the previous section thus form a basis of $S_k(n, d)$ when considering all $\lambda, \mu \in \Lambda(n, d)$. This basis is exactly the same basis as the one defined by Green, see [Gre07, Remark after (2.6d)]. We get a one-to-one correspondence between the subsets

$$\xi_{\mu, \lambda} = \{\xi_{\underline{j}, \underline{i}} \mid \underline{j} \in \mu, \underline{i} \in \lambda\}$$

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of the $S_k(n, d)$ -basis and the sets

$$A_\mu^\lambda = \{A = (a_{ij})_{ij} \mid \lambda_i = \sum_j a_{ij}, \mu_i = \sum_j a_{ij}\}.$$

A.1.3. Standard Morphisms vs. The Basis of Schur Algebras

We have seen that the matrices $A_{\underline{j}, \underline{i}}$ label a k -basis of the morphisms between modules over the Schur algebra and of the morphisms between certain strict polynomial functors. Recall that $\text{Rep } \Gamma_k^d$ and $S_k(n, d) \text{ Mod}$ are related by the functor $\text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -)$, see (6.3). We will show that the standard morphism in $\text{Rep } \Gamma_k^d$ labeled by a matrix A is mapped via this functor to the standard morphism of $S_k(n, d) \text{ Mod}$ labeled by the same matrix A .

Let $\varphi_A \in \text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda)$. Since the functor $\text{Hom}_{\Gamma_k^d}(\Gamma^{d, k^n}, -)$ sends an object $X \in \text{Rep } \Gamma_k^d$ to $X(k^n)$, the morphism φ_A is mapped to $\varphi_A(k^n) \in \text{Hom}_{S_k(n, d)}(\Gamma^\mu(k^n), \Gamma^\lambda(k^n))$. Recall from (6.5) that $\Gamma^\lambda(k^n)$ is isomorphic to $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})$ and thus we get

$$\text{Hom}_{S_k(n, d)}(\Gamma^\mu(k^n), \Gamma^\lambda(k^n)) \cong \text{Hom}_{S_k(n, d)}(\text{Hom}_{k\mathfrak{S}_d}(M^\mu, (k^n)^{\otimes d}), \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d})).$$

Under this isomorphism, the map

$$\begin{aligned} \varphi_A(V): \Gamma^\mu(k^n) &\rightarrow \Gamma^\lambda(k^n) \\ v &\mapsto \sum_{\sigma \in \mathfrak{S}_\lambda / \mathfrak{S}_A} v' \sigma \end{aligned}$$

is mapped to

$$\begin{aligned} \text{Hom}_{k\mathfrak{S}_d}(M^\mu, (k^n)^{\otimes d}) &\rightarrow \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, (k^n)^{\otimes d}) \\ (e_{\underline{j}} \mapsto v) &\mapsto (e_{\underline{i}} \mapsto \sum_{\sigma \in \mathfrak{S}_\lambda / \mathfrak{S}_A} v' \sigma) \end{aligned}$$

for $\underline{j} = (11 \dots 2 \dots nn) \in \mu$ and $\underline{i} = (11 \dots 2 \dots nn) \in \lambda$. Now let \underline{j}' such that $A = A_{\underline{j}', \underline{i}}$, then this map corresponds to the map

$$\begin{aligned} M^\lambda &\rightarrow M^\mu \\ e_{\underline{i}} &\mapsto \sum_{\sigma \in \mathfrak{S}_\lambda / \mathfrak{S}_A} e_{\underline{j}'} \sigma, \end{aligned}$$

which is $\xi_{\underline{j}', \underline{i}}$ and is identified with $A_{\underline{j}', \underline{i}}$.

A.1.4. Standard Morphisms of Strict Polynomial Functors

A k -basis of $\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda)$ can be identified with the set of matrices $A \in A_\mu^\lambda$. The definition of a morphism φ_A corresponding to some $A \in A_\mu^\lambda$ is given in Section 2.4, see also [Tot97, p. 8] and [Kra14, Lemma 4.3]. We here deal only with the special case where one obtains the group algebra of the symmetric group. We start this section by describing how to perform explicit calculations. Afterwards we show how to calculate the monoidal dual.

An explicit calculation. Recall that for a matrix $A \in A_\mu^\lambda$ and an object $V \in \Gamma^d \mathcal{P}_k$ the corresponding morphism $\varphi_A(V) \in \text{Hom}(\Gamma^\mu(V), \Gamma^\lambda(V))$ is given as the composition of three morphisms a, b, c defined by:

$$\varphi_A: \Gamma^\mu = \prod_{j=1}^m \Gamma^{\mu_j} \xrightarrow{a = \prod_{j=1}^m (\iota_{\alpha_j})} \prod_{j=1}^m \left(\prod_{i=1}^n \Gamma^{a_{ij}} \right) \xrightarrow{b} \prod_{i=1}^n \left(\prod_{j=1}^m \Gamma^{a_{ij}} \right) \xrightarrow{c = \prod_{i=1}^n (p_{\alpha_i})} \prod_{i=1}^n \Gamma^{\lambda_i} = \Gamma^\lambda$$

We now give an explicit description of this morphism $\varphi_A(V)$ on some element $v \in V^{\otimes d}$. To this end, we replace every integer in the matrix A by a box of this horizontal size. For example:

$$A = \begin{pmatrix} 2 & 3 & 1 & 1 & 2 & \dots \\ 1 & 3 & 2 & \dots & & \\ 1 & 1 & 4 & \dots & & \\ \dots & & & & & \end{pmatrix} \mapsto \left[\begin{array}{cccccc} \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \dots \\ \boxed{} & \boxed{} & \boxed{} & \dots & & \\ \boxed{} & \boxed{} & \boxed{} & \dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots \end{array} \right]$$

The first morphism a of φ_A corresponds to the following action: writing an element $v = v_1 \otimes v_2 \otimes \dots \otimes v_d \in V^{\otimes d}$ into the matrix from top to bottom, from left to right, i.e.

$$v_1 \otimes v_2 \otimes \dots \otimes v_d \rightsquigarrow \left[\begin{array}{ccc} \boxed{v_1 v_2} & \boxed{v_j v_{j+1} v_{j+2}} & \dots \\ \boxed{v_3} & \boxed{v_{j+3} v_{j+4} v_{j+5}} & \dots \\ \boxed{v_4} & \dots & \\ \dots & & \end{array} \right]$$

The second morphism b corresponds to reading out the entries of the matrix from left to right, from top to bottom, i.e. in the example above

$$(v_1 \otimes v_2) \otimes (v_j \otimes v_{j+1} \otimes v_{j+2}) \otimes \dots \otimes (v_3) \otimes (v_{j+3} \otimes v_{j+4} \otimes v_{j+5}) \otimes \dots \otimes \dots$$

We denote the element $b \circ a(v) \in V^{\otimes d}$ by v' . The last morphism c now permutes the factors of v' with permutations of $\mathfrak{S}_\lambda / \mathfrak{S}_A$, where A is considered as a partition of d . Thus we obtain

$$\varphi_A(V)(v) = \sum_{\sigma \in \mathfrak{S}_\lambda / \mathfrak{S}_A} v' \sigma.$$

Composition of morphisms. The composition $\varphi_{\underline{i}', \underline{k}} \circ \varphi_{\underline{j}, \underline{i}}$ of two morphisms $\varphi_{\underline{j}, \underline{i}} \in \text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda)$ and $\varphi_{\underline{i}', \underline{k}} \in \text{Hom}_{\Gamma_k^d}(\Gamma^\lambda, \Gamma^\nu)$ is given by multiplying the corresponding basis elements of the Schur algebra $\xi_{\underline{j}, \underline{i}} \cdot \xi_{\underline{i}', \underline{k}}$ (see Section A.2) and then taking the sum of the corresponding morphisms in $\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\nu)$.

Monoidal dual of a morphism. Let $\varphi_A \in \text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda)$. We are interested in the monoidal dual of this map, i.e. $(\varphi_A)^\vee \in \text{Hom}_{\Gamma_k^d}((\Gamma^\lambda)^\vee, (\Gamma^\mu)^\vee) \cong \text{Hom}_{\Gamma_k^d}(\Gamma^\lambda, \Gamma^\mu)$. In particular we want to know the matrix which represents it. For it, we fix $\underline{j} \in \mu$ and $\underline{i} \in \lambda$ and consider $\varphi_{\underline{j}, \underline{i}}$ at $V = k^n$ and use the isomorphism $\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda) \cong \text{Hom}_{k \mathfrak{S}_d}(M^\lambda, M^\mu)$,

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see (6.6). Recall from Section A.1.1 the corresponding element of $\text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$ is given by the matrix $B_{(\underline{j}, \underline{i})}$ where the columns, respectively rows are indexed by the elements $\{e_{\underline{i}'} \mid \underline{i}' \in \lambda\}$, respectively $\{e_{\underline{j}'} \mid \underline{j}' \in \mu\}$. Taking the dual of this map yields

$$(B_{(\underline{j}, \underline{i})})^* = (B_{(\underline{j}, \underline{i})})^T: (M^\mu)^* \rightarrow (M^\lambda)^*,$$

where $(B_{(\underline{j}, \underline{i})})^T$ is the matrix given with respect to the dual basis $\{e_{\underline{i}}^*\}$, respectively $\{e_{\underline{j}}^*\}$. But M^μ and M^λ are self-dual and we can identify the basis elements $e_{\underline{j}}^*$ and $e_{\underline{j}}$, respectively $e_{\underline{i}}^*$ and $e_{\underline{i}}$. It follows that

$$(B_{(\underline{j}, \underline{i})})_{\underline{j}', \underline{i}'}^* = \begin{cases} 1 & \text{if } (\underline{i}', \underline{j}') \in (\underline{j}, \underline{i}), \\ 0 & \text{else.} \end{cases}$$

Defining $(\underline{j}, \underline{i})^*$ to change the entries, i.e. $(\underline{j}, \underline{i})^* = (\underline{i}, \underline{j})$, the equation becomes

$$(B_{(\underline{j}, \underline{i})})_{\underline{j}', \underline{i}'}^* = \begin{cases} 1 & \text{if } (\underline{j}', \underline{i}') \in (\underline{j}, \underline{i})^*, \\ 0 & \text{else.} \end{cases}$$

Thus $(B_{(\underline{j}, \underline{i})})^*$ is the same as $B_{(\underline{j}, \underline{i})}^* = B_{(\underline{i}, \underline{j})}$ and hence via the isomorphism (6.6) we obtain that $(\varphi_{\underline{j}, \underline{i}})^\vee = \varphi_{\underline{i}, \underline{j}}$. But the matrix $A_{\underline{i}, \underline{j}}$, representing the morphism $\varphi_{\underline{i}, \underline{j}} = \varphi_{A_{\underline{i}, \underline{j}}}$ is exactly the transpose of the matrix $A_{\underline{j}, \underline{i}}$, representing the morphism $\varphi_{\underline{j}, \underline{i}}$. Hence, we obtain

$$(\varphi_A)^\vee = \varphi_{A^T}. \quad (\text{A.3})$$

The Symmetric Group. Let $\lambda = \mu = \omega = (1, \dots, 1)$. A k -basis of $\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda) = \text{End}_{\Gamma_k^d}(\Gamma^\omega)$ is given by matrices of the form

$$\begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & \dots & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & & & \dots & 0 & 1 & 0 \dots & 0 \\ 1 & 0 & \dots & & & & & & \\ & \ddots & & & & & & \ddots & \vdots \end{pmatrix} \begin{matrix} \sum = 1 \\ \sum = 1 \\ \sum = 1 \\ \sum = 1 \end{matrix}$$

$$\begin{matrix} \sum & \dots & \sum & \sum & \dots \\ \parallel & & \parallel & \parallel & \\ 1 & & 1 & 1 & \end{matrix}$$

i.e. matrices with a single entry 1 in each column and each row and the remaining entries being 0. Let $l = 12 \dots d \in \omega$. Then the identity matrix

$$\begin{pmatrix} 1 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & \\ \dots & 0 & 1 & 0 & \dots \\ & & & \ddots & 0 \\ & & & 0 & 1 \end{pmatrix}$$

identifies with $A_{l,l}$. It is obvious that each other matrix of the form above can be obtained by permuting the rows of the identity matrix.

$$\begin{aligned}
A_{\text{id}} = \begin{pmatrix} 1 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & \\ \dots & 0 & 1 & 0 & \dots \\ & & & \ddots & 0 \\ & & & 0 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{a_1} \\ \boxed{a_2} \\ \boxed{a_3} \\ \vdots \\ \boxed{a_d} \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} \boxed{a_{\sigma(1)}} \\ \boxed{a_{\sigma(2)}} \\ \boxed{a_{\sigma(3)}} \\ \vdots \\ \boxed{a_{\sigma(d)}} \end{pmatrix} \\
= \begin{pmatrix} 0 & \dots & & 1 & \dots \\ 0 & \dots & 1 & & \\ 0 & \dots & & & 1 & \dots \\ 1 & 0 & \dots & & & \\ \vdots & & & & & \end{pmatrix}
\end{aligned}$$

The matrix obtained from this permutation is exactly $A_{l\sigma,l}$. This can be seen as follows: we take an element $v_1 \otimes \dots \otimes v_d$ of $\Gamma^\omega(V) = V^{\otimes d}$ and write it into the matrix A as explained above, from bottom to top, from left to right, i.e.

$$v_1 \otimes v_2 \otimes \dots \otimes v_d \rightsquigarrow \begin{bmatrix} & & & \boxed{v_4} & & \\ & & \boxed{v_3} & & & \\ & & & & \boxed{v_i} & \\ \boxed{v_1} & \dots & & \dots & \dots & \dots \end{bmatrix}$$

Now reading out an element of $\Gamma^\omega(V) = V^{\otimes d}$ as above, i.e. from left to right, from top to bottom, yields $v_4 \otimes v_3 \otimes v_i \otimes v_1 \otimes \dots$ which is exactly $(v_1 \otimes \dots \otimes v_d)\sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(d)}$. Thus we get

$$v_1 \otimes \dots \otimes v_d \xrightarrow{\varphi_{l\sigma,l}} (v_1 \otimes \dots \otimes v_d)\sigma.$$

Composition of morphisms yields $\varphi_{l\sigma,l} \circ \varphi_{l\tau,l} = \varphi_{l\tau\sigma,l}$ and hence the assignment

$$\text{End}_{\Gamma_k^d}(\Gamma^\omega) \ni \varphi_{l\sigma,l} \leftrightarrow \sigma \in \mathfrak{S}_d$$

yields isomorphisms of algebras

$$\text{End}_{\Gamma_k^d}(\Gamma^\omega) = (k\mathfrak{S}_d)^{\text{op}} \quad \text{and} \quad \text{End}_{\Gamma_k^d}((\Gamma^\omega)^\circ) = k\mathfrak{S}_d. \quad (\text{A.4})$$

Module structure. The k -module $\text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda)$ is equipped with a right $\text{End}_{\Gamma_k^d}(\Gamma^\omega)$ -module structure via precomposition. It reads as follows: let $\varphi_{l,i} \in \text{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda)$ and $\varphi_{l\sigma,l} \in \text{End}_{\Gamma_k^d}(\Gamma^\omega)$. From Proposition A.4 we obtain

$$\varphi_{l,i} \cdot \varphi_{l\sigma,l} = \varphi_{l,i\sigma^{-1}}.$$

where the sum is taken over \mathfrak{S}_d -orbits $\underline{q}, \underline{p}$ in $I(n, d) \times I(n, d)$ and $Z_{\underline{q}, \underline{p}}$ depends on $\underline{i}, \underline{j}, \underline{k}, \underline{l}$. It is defined by

$$Z_{\underline{q}, \underline{p}} := \#\{\underline{s} \in I(n, d) \mid (\underline{l}, \underline{k}) \sim (\underline{q}, \underline{s}), (\underline{j}, \underline{i}) \sim (\underline{s}, \underline{p})\}.$$

Let $\underline{k} \in \varrho$ and $\underline{j} \in \mu$ for some $\varrho, \mu \in \Lambda(n, d)$. We have $\varrho \neq \mu$ if and only if $\underline{k} \not\sim \underline{j}$ and in this case by the rule above $\xi_{\underline{l}, \underline{k}} \cdot \xi_{\underline{j}, \underline{i}} = 0$. This is reflected by the fact that the composition of a morphism $\xi_{\underline{l}, \underline{k}} \in \text{Hom}_{k\mathfrak{S}_d}(M^\varrho, M^\nu)$ with a morphism $\xi_{\underline{j}, \underline{i}} \in \text{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\mu)$ is zero if $\varrho \neq \lambda$.

Thus, from now on we assume that $\underline{k} \sim \underline{j}$. Permuting the entries of \underline{l} or \underline{i} , we can even suppose that $\underline{k} = \underline{j}$. Moreover, we know that $Z_{\underline{q}, \underline{p}} = 0$ if $\underline{q} \not\sim \underline{l}$ or $\underline{p} \not\sim \underline{i}$. Hence, in (A.7) we only need to sum over those $(\underline{q}, \underline{p})$ with $\underline{q} = \underline{l}$ and \underline{p} a permutation of \underline{i} . Define $J_{\underline{l}, \underline{i}} \subset \mathfrak{S}_d$ to contain permutations such that $\{(\underline{l}, \underline{i}\pi) \mid \pi \in J_{\underline{l}, \underline{i}}\}$ represents different orbits in $I(n, d) \times I(n, d)$ and is maximal. Then, (A.7) reads

$$\xi_{\underline{l}, \underline{k}} \cdot \xi_{\underline{k}, \underline{i}} = \sum_{\pi \in J_{\underline{l}, \underline{i}}} Z_{\underline{l}, \underline{i}\pi} \xi_{\underline{l}, \underline{i}\pi} \quad (\text{A.8})$$

with $Z_{\underline{l}, \underline{i}\pi} = \#\{\underline{s} \in I(n, r) \mid (\underline{l}, \underline{k}) \sim (\underline{l}, \underline{s}), (\underline{k}, \underline{i}) \sim (\underline{s}, \underline{i}\pi)\}$.

We are interested in the computation of $\xi_{\underline{l}, \underline{k}} \cdot \xi_{\underline{k}, \underline{i}}$ in particular cases.

The case $\underline{l} = (12 \dots d)$. In this case, there exists only one permutation $\pi \in \mathfrak{S}_d$ such that $\underline{l}\pi = \underline{l}$, namely $\pi = \text{id}$. Hence, the multiplication rule reduces as follows:

Proposition A.3. *Let $\underline{l} = (12 \dots d)$. Then*

$$\xi_{\underline{l}, \underline{k}} \cdot \xi_{\underline{k}, \underline{i}} = \sum_{\sigma \in J} \xi_{\underline{l}, \underline{i}\sigma},$$

where J is a maximal subset of \mathfrak{S}_d such that $\underline{k}\sigma = \underline{k}$ and $\underline{i}\sigma$ are pairwise different for all $\sigma \in J$.

Proof. First, we show that for $\underline{l} = (12 \dots d)$ it holds $Z_{\underline{l}, \underline{i}\pi} \leq 1$ for all $\pi \in J_{\underline{l}, \underline{i}}$. By definition of $Z_{\underline{l}, \underline{i}\pi}$, we need to show that there exists at most one \underline{s} such that

$$(\underline{l}, \underline{k}) \sim (\underline{l}, \underline{s}) \quad \text{and} \quad (\underline{k}, \underline{i}) \sim (\underline{s}, \underline{i}\pi). \quad (\text{A.9})$$

But from the first condition it follows that $\underline{s} = \underline{k}$ since only $\underline{l}\text{id} = \underline{l}$.

Next, we want to determine those π such that $Z_{\underline{l}, \underline{i}\pi} \neq 0$, i.e. $Z_{\underline{l}, \underline{i}\pi} = 1$. From the second condition in (A.9) it follows that this holds if and only if

$$\underline{k}\sigma = \underline{s} = \underline{k} \quad \text{and} \quad \underline{i}\sigma = \underline{i}\pi \quad \text{for some } \sigma \in \mathfrak{S}_d. \quad (\text{A.10})$$

Since condition (A.10) states that $\underline{i}\sigma = \underline{i}\pi$, taking the sum over all $\xi_{\underline{l}, \underline{i}\pi}$ with $\pi \in \mathfrak{S}_d$ satisfying (A.10) is the same as taking the sum over all $\xi_{\underline{l}, \underline{i}\sigma}$ with $\sigma \in \mathfrak{S}_d$ such that $\underline{k}\sigma = \underline{k}$ and $\underline{i}\sigma$ are pairwise different. \square

A. Appendix

The case $\underline{l} = (12 \dots d)$ and $\underline{k} = \underline{l}\pi$. Here we assume in addition to the last case that \underline{k} is a permutation of $\underline{l} = (12 \dots d)$.

Proposition A.4. *Let $\underline{l} = (12 \dots d)$ and $\pi \in \mathfrak{S}_d$. Then*

$$\xi_{\underline{l}, \underline{l}\pi} \cdot \xi_{\underline{l}\pi, \underline{i}} = \xi_{\underline{l}, \underline{i}}.$$

Proof. Using Proposition A.3 we only need to determine the set J for the case $\underline{k} = \underline{l}\pi$, i.e. all $\sigma \in \mathfrak{S}_d$ such that $\underline{k}\sigma = \underline{k}$ and $\underline{i}\sigma$ are pairwise different. But $\underline{k}\sigma = \underline{l}\pi\sigma = \underline{l}\pi$ if and only if $\sigma = \text{id}$ and thus $J = \{\text{id}\}$. \square

A.3. Correspondence Between Objects and Morphisms

As explained in the previous sections, the objects and morphisms of strict polynomial functors, modules over the Schur algebra and symmetric group are closely related via the functors

$$\text{Hom}(\Gamma^{d, k^n}, -): \text{Rep } \Gamma_k^d \rightarrow S_k(n, d) \text{ Mod} \quad \text{and} \quad \mathcal{F}: \text{Rep } \Gamma_k^d \rightarrow k\mathfrak{S}_d \text{ Mod}.$$

Collecting various results of the appendix and other chapters, we provide an overview of the correspondence under these functors of certain objects defined in this thesis.

	$\xrightarrow{\text{Hom}(\Gamma^{d, k^n}, -)}$	$S_k(n, d) \text{ Mod}$	\xrightarrow{f}	$k\mathfrak{S}_d \text{ Mod}$
Γ^{d, k^n}	\mapsto	$S_k(n, d)$	\mapsto	$(k^n)^{\otimes d}$
Γ^λ	\mapsto	$S_k(n, d)\xi_\lambda$	\mapsto	M^λ
S^λ	\mapsto	\mapsto	\mapsto	M^λ
X	\mapsto	$X(k^n)$	\mapsto	
S_λ	\mapsto	$S_\lambda(k^n)$	\mapsto	$\text{Sp}(\lambda)$
W_λ	\mapsto	$W_\lambda(k^n)$	\mapsto	$\text{dSp}(\lambda)$
L_λ	\mapsto	\mapsto	\mapsto	$\begin{cases} D_\lambda & \text{if } \lambda \text{ is } p\text{-restricted} \\ 0 & \text{else.} \end{cases}$
$\varphi_{A_{j, i}}$	\mapsto	$\xi_{j, i}$	\mapsto	$\xi_{j, i}$

Glossary

A_μ^λ	matrices fulfilling certain row and column sum condition	6
$\text{add } \Gamma$	full subcategory of $\text{Rep } \Gamma_k^d$ of direct summands of finite direct sums of objects in $\Gamma = \{\Gamma^\lambda\}_{\lambda \in \Lambda(d)}$	17
$\text{add } M$	full subcategory of $k\mathfrak{S}_d \text{Mod}$ of direct summands of finite direct sums of M^λ for $\lambda \in \Lambda(n, d)$	36
$\text{add } S$	full subcategory of $\text{Rep } \Gamma_k^d$ of direct summands of finite direct sums of objects in $S = \{S^\lambda\}_{\lambda \in \Lambda(d)}$	17
$C(T^\lambda)$	column stabilizer of the tableau T^λ	7
$\Delta(\lambda)$	standard object of a highest weight category	20
$\text{dSp}(\lambda)$	dual Specht module corresponding to λ	33
D^λ	simple module corresponding to λ , simple quotient of $\text{Sp}(\lambda)$.	34
D_λ	simple quotient of $\text{dSp}(\lambda)$	34
f	Schur functor from $S_k(n, d) \text{Mod}$ to $k\mathfrak{S}_d \text{Mod}$	66
E	an n -dimensional k vector space	29
$e_{\underline{i}}$	$e_{i_1} \otimes \cdots \otimes e_{i_d}$ for $\underline{i} = (i_1 \dots i_d)$	29
\mathcal{F}	Schur functor from $\text{Rep } \Gamma_k^d$ to $k\mathfrak{S}_d \text{Mod}$	35
$\text{Filt}(\Delta)$	category of Weyl filtered functors	21
$\text{Filt}(\nabla)$	category of Schur filtered functors	21
$\text{Fun}_k(\mathcal{A}, \mathcal{B})$	category of all k -linear functors from \mathcal{A} to \mathcal{B}	11
Γ^d	divided powers of degree d	13
$\Gamma^{d, V}$	strict polynomial functor represented by V	16
$\Gamma^d \mathcal{P}_k$	category of divided powers	14
Γ^d	generalized divided powers	17
\mathcal{G}_\otimes	left adjoint to the Schur functor \mathcal{F}	43
\mathcal{G}_{Hom}	right adjoint to the Schur functor \mathcal{F}	47
$\underline{i}, \underline{j}, \underline{k}, \underline{l}$	elements of $I(n, d)$	
$I(n, d)$	set of d -tuples of positive integers smaller equal than n	6
$(\iota_\Gamma)_V$	inclusion map $\Gamma^d V \hookrightarrow V^{\otimes d}$	14
$(\iota_\Lambda)_V$	inclusion map $\Lambda^d V \hookrightarrow V^{\otimes d}$	14
$K_{\mu\lambda}$	(p) -Kostka numbers	34
k	commutative ring	
$k\mathfrak{S}_d$	group algebra of the symmetric group \mathfrak{S}_d	27
λ, μ, ν	elements of $\Lambda(n, d) / \Lambda^+(d) / \Lambda^+(n, d) / \Lambda_p^+(n, d)$	
$[\lambda]$	Young diagram	7
λ'	conjugate partition of λ	6
Λ^d	exterior powers of degree d	13

A. Appendix

Λ^λ	generalized exterior powers	17
$\Lambda(d)$	compositions of d	6
$\Lambda^+(d)$	partitions of d	6
$\Lambda(n, d)$	compositions of d into n parts	6
$\Lambda^+(n, d)$	partitions of d into n parts	6
$\Lambda_p^+(n, d)$	p -restricted partitions of d into n parts	6
L_λ	simple strict polynomial functor	22
$\mathbf{M}_k = \mathbf{Mod} k$	category of all k -modules	15
M^*	dual of a module M	29
M^λ	permutation module corresponding to λ	29
m_p	Mullineux map	34
$\nabla(\lambda)$	costandard object of a highest weight category	20
ω	partition $(1, 1, \dots, 1) \in \Lambda(d, d)$	
φ_A	standard morphism of strict polynomial functors corresponding to the matrix A	18
$\varphi_{\underline{j}, \underline{i}}$	standard morphism of strict polynomial functors corresponding to the pair $(\underline{j}, \underline{i})$	70
$(\pi_\Lambda)_V$	quotient map $V^{\otimes d} \twoheadrightarrow \Lambda^d V$	14
$(\pi_S)_V$	quotient map $V^{\otimes d} \twoheadrightarrow S^d V$	14
\mathbf{P}_k	category of finitely generated projective k -modules	13
P_μ	projective cover of L_μ	56
Q^d	truncated symmetric powers	56
$\mathbf{Rep} \Gamma_k^d$	category of strict polynomial functors	15
$\mathbf{rep} \Gamma_k^d$	category of finite strict polynomial functors	15
$R(T^\lambda)$	row stabilizer of the tableau T^λ	7
S^d	symmetric powers of degree d	13
S^λ	generalized symmetric powers	17
S_λ	Schur functor corresponding to λ	21
s_λ	permutation morphism in $\mathbf{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\omega)$	21
$S_k(n, d)$	Schur algebra	61
$\mathbf{Sp}(\lambda)$	Specht module corresponding to λ	33
\mathfrak{S}_d	symmetric group	6
\mathfrak{S}_λ	Young subgroup for λ	6
$\sigma_\lambda(r)$	sepcial element of \mathfrak{S}_d	20
$\text{sign}(\sigma)$	signature of σ	6
sgn^d	alternating module of $k\mathfrak{S}_d \mathbf{Mod}$	
T^λ	λ -tableau	7
T_R^λ	λ -tableau with entries $1, 2, \dots, d$ read from left to right, from top to bottom	8
T_C^λ	λ -tableau with entries $1, 2, \dots, d$ when read from top to bottom, from left to right	8
W_λ	Weyl functor corresponding to λ	21

A.3. Correspondence Between Objects and Morphisms

$\xi_{j,i}$	morphisms between permutation modules / bases elements of the Schur algebra.....	30 / 61
Y^λ	Young module corresponding to λ	34
id	unit object in a monoidal category	8
$- \otimes -$	(internal) tensor product of a monoidal category	8
$- \boxtimes -$	external tensor product of strict polynomial functors.....	17
$\mathcal{H}om(-, -)$	internal hom of a monoidal category	9
$\Phi_{X,Y}$	natural transformation from $\mathcal{F}X \otimes \mathcal{F}Y$ to $\mathcal{F}(X \otimes Y)$ for a monoidal functor \mathcal{F}	10
$X^{(r)}$	r -th Frobenius twist of a strict polynomial functor X	18
X°	Kuhn dual of a strict polynomial functor X	22
X^\vee	monoidal dual of a strict polynomial functor X	23

Bibliography

- [AB85] Kaan Akin and David A. Buchsbaum. Characteristic-Free Representation Theory of the General Linear Group. *Adv. in Math.*, 58(2):149–200, 1985.
- [AR15] Cosima Aquilino and Rebecca Reischuk. The monoidal structure on strict polynomial functors. *ArXiv e-prints*, 1503.05108, 2015.
- [BK00] Christine Bessenrodt and Alexander S. Kleshchev. On Tensor Products of Modular Representations of Symmetric Groups. *Bulletin of the London Mathematical Society*, 32:292–296, 2000.
- [Bou89] Nicolas Bourbaki. *Algebra I: Chapters 1-3*. Elements of Mathematics. Springer, 1989.
- [Cha08] Marcin Chałupnik. Koszul duality and extensions of exponential functors. *Advances in Mathematics*, 218(3):969–982, 2008.
- [Day71] Brian J. Day. Construction of biclosed categories. *Bulletin of the Australian Mathematical Society*, 5:139–140, 1971.
- [DEN04] Stephen R. Doty, Karin Erdmann, and Daniel K. Nakano. Extensions of modules over Schur algebras, symmetric groups and Hecke algebras. *Algebras and Representation Theory*, 7(1):67–100, 2004.
- [Don86] Stephen Donkin. On Schur Algebras and Related Algebras, I. *Journal of Algebra*, 104(2):310–328, 1986.
- [Don87] Stephen Donkin. On Schur Algebras and Related Algebras, II. *Journal of Algebra*, 111(2):354 – 364, 1987.
- [Don94a] Stephen Donkin. On Schur algebras and related algebras III: integral representations. *Mathematical Proceedings of the Cambridge Philosophical Society*, 116(1):37–55, 1994.
- [Don94b] Stephen Donkin. On Schur Algebras and Related Algebras. IV. The Blocks of the Schur Algebras. *Journal of Algebra*, 168(2):400–429, 1994.
- [Erd93] Karin Erdmann. Schur Algebras of Finite Type. *The Quarterly Journal of Mathematics. Oxford. Second Series*, 44(173):17–41, 1993.

BIBLIOGRAPHY

- [Erd94] Karin Erdmann. In Vlastimil Dlab and Leonard L. Scott, editors, *Finite Dimensional Algebras and Related Topics*, volume 424, chapter Symmetric Groups and Quasi-Hereditary Algebras, pages 123–161. Springer Netherlands, 1994.
- [FFPS03] Vincent Franjou, Eric M. Friedlander, Teimuraz Pirashvili, and Lionel Schwartz. *Rational representations, the Steenrod algebra and functor homology*, volume 16 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2003.
- [FHK08] Ming Fang, Anne Henke, and Steffen Koenig. Comparing GL_n -representations by characteristic-free isomorphisms between generalized Schur algebras. *Forum Mathematicum*, 20(1):45–79, 2008.
- [FHM03] Halvard Fausk, Po Hu, and J. Peter May. Isomorphisms between left and right adjoints. *Theory and Applications of Categories*, 11:No. 4, 107–131, 2003.
- [FK97] Ben Ford and Alexander Kleshchev. A proof of the Mullineux conjecture. *Mathematische Zeitschrift*, 226:267–308, 1997.
- [FS97] Eric M. Friedlander and Andrei Suslin. Cohomology of finite group schemes over a field. *Inventiones Mathematicae*, 127(2):209–270, 1997.
- [Ful97] William Fulton. *Young Tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, New York, 1997.
- [Gil14] Christopher C. Gill. Young module multiplicities, decomposition numbers and the indecomposable Young permutation modules. *Journal of Algebra and its Applications*, 13(5), 2014.
- [Gre07] James A. Green. *Polynomial representations of GL_n* , volume 830 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.
- [Hem07] David J. Hemmer. Symmetric group modules with Specht and dual Specht filtrations. *Communications in Algebra*, 35(11):3292–3306, 2007.
- [Hen05] Anne Henke. On p -kostka numbers and Young modules. *European Journal of Combinatorics*, 26(6):923 – 942, 2005.
- [HN04] David J. Hemmer and Daniel K. Nakano. Specht filtrations for Hecke algebras of type A. *Journal of the London Mathematical Society*, 69:623–638, 2004.
- [Jam78] Gordon James. *The Representation Theory of the Symmetric Groups*, volume 682 of *Lecture Notes in Mathematics*. Springer, 1978.
- [JK81] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981.

- [Kas95] Christian Kassel. *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics*. Springer, New York, 1995.
- [Kly84] Anton A. Klyachko. Direct summands of permutation modules. *Selecta Mathematica Sovietica*, 3(1):45–55, 1983/84.
- [Kra13] Henning Krause. Koszul, Ringel and Serre duality for strict polynomial functors. *Compositio Mathematica*, 149(6):996–1018, 2013.
- [Kra14] Henning Krause. The highest weight structure for strict polynomial functors. *ArXiv e-prints*, 1405.1691v1, 2014.
- [Kuh94] Nicholas J. Kuhn. Generic Representations of the Finite General Linear Groups and the Steenrod Algebra: I. *American Journal of Mathematics*, 116(2):327–360, 1994.
- [Kuh98] Nicholas J. Kuhn. Rational Cohomology and Cohomological Stability in Generic Representation Theory. *American Journal of Mathematics*, 120(6):1317–1341, 1998.
- [Kuh02] Nicholas J. Kuhn. A stratification of generic representation theory and generalized Schur algebras. *K-Theory*, 26(1):15–49, 2002.
- [Mar93] Stuart Martin. *Schur Algebras and Representation Theory*, volume 112 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.
- [Mat99] Andrew Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, volume 15 of *University Lecture Series*. American Mathematical Society, 1999.
- [ML98] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 1998.
- [Mul79] Glen Mullineux. Bijections of p -Regular Partitions and p -Modular Irreducibles of the Symmetric Groups. *Journal of the London Mathematical Society*, s2-20(1):60–66, 1979.
- [Mur38] Francis D. Murnaghan. The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group. *American Journal of Mathematics*, 60(3):761–784, 1938.
- [Rei16] Rebecca Reischuk. The Adjoints of the Schur Functor. *ArXiv e-prints*, 1601.03513, 2016.
- [Sch01] Issai Schur. *Ueber eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*. Dieterich in Göttingen, 1901.
- [Tot97] Burt Totaro. Projective resolutions of representations of $GL(n)$. *Journal für die reine und angewandte Mathematik*, 482(113):1–13, 1997.

BIBLIOGRAPHY

- [Tou13] Antoine Touzé. Ringel duality and derivatives of non-additive functors. *Journal of Pure and Applied Algebra*, 217(9):1642–1673, 2013.
- [Tou15] Antoine Touzé. Connectedness of cup products for polynomial representations of $GL(n)$ and applications. *ArXiv e-prints*, 1508.06049, 2015.