Center for
Mathematical Economics
Working Papers

563

July 2016

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Tiziano De Angelis, Giorgio Ferrari and John Moriarty

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

Nash equilibria of threshold type for two-player nonzero-sum games of stopping *

Tiziano De Angelis[†] Giorgio Ferrari[‡] John Moriarty[§]

July 19, 2016

Abstract. This paper analyses two-player nonzero-sum games of optimal stopping on a class of regular diffusions with singular boundary behaviour (in the sense of Itô and McKean (1974) [19], p. 108). We prove that Nash equilibria are realised by stopping the diffusion at the first exit time from suitable intervals whose boundaries solve a system of algebraic equations. Under mild additional assumptions we also prove uniqueness of the equilibrium.

Keywords: nonzero-sum Dynkin games; Nash equilibrium; smooth-fit principle; regular diffusions; free boundary problems.

MSC2010 subject classification: 91A05, 91A15, 60G40, 60J60, 35R35.

1 Introduction

Given a real-valued Markov process $(X_t)_{t\geq 0}$ with $X_0 = x$, a two-player *Dynkin game* [13] is defined by the following cost functionals:

$$\mathcal{J}_i(\tau_1, \tau_2; x) := \mathsf{E}_x\Big[e^{-r\tau_i}G_i(X_{\tau_i})\mathbb{1}_{\{\tau_i<\tau_j\}} + e^{-r\tau_j}L_i(X_{\tau_j})\mathbb{1}_{\{\tau_i\geq\tau_j\}}\Big], \quad i = 1, 2, \; j \neq i. \tag{1.1}$$

Here player i chooses a stopping time τ_i and the game is terminated at time $\tau_1 \wedge \tau_2$, with the cost to player i equal to either $G_i(X_{\tau_i})$ or $L_i(X_{\tau_i})$ (continuously discounted at the rate $r > 0$), depending on who stops first. Player i aims to minimise the cost functional $\mathcal{J}_i(\tau_1, \tau_2; x)$, and central questions are the existence and uniqueness of Nash equilibria. Despite the fundamental relevance of nonzero-sum Dynkin games, especially in modern economic theory, they have received relatively little rigorous mathematical treatment and this has largely focused on the existence of Nash equilibria rather than the structure of the corresponding stopping times.

With applications in mind we go beyond the question of existence of Nash equilibria in (1.1) for a wide class of stochastic processes X found in optimisation problems. We identify sufficient conditions on the problem data to yield the structure of the corresponding optimal stopping times and exactly characterise the optimal stopping boundaries. Our method is to adapt the geometric approach to optimal stopping originally introduced in [14] in a natural way to the pair of cost functionals (1.1), and we also provide easily verified sufficient conditions for uniqueness of the equilibrium.

*The first and the third authors were supported by EPSRC grant EP/K00557X/1; financial support by the German Research Foundation (DFG) via grant Ri–1128–4–2 is gratefully acknowledged by the second author.

[†]School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom; tiziano.deangelis@manchester.ac.uk

[‡]Center for Mathematical Economics, Bielefeld University, Universitätsstrasse 25, D-33615 Bielefeld, Germany; giorgio.ferrari@uni-bielefeld.de

[§]School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom; john.moriarty@manchester.ac.uk

1.1 Background

Both in discrete and in continuous time, a large mathematical literature exists for the *zero-sum* case of (1.1) when $G_1 = -G_2$ and $L_1 = -L_2$ (see, for example, [3], [4], [5], [9], [15], [25], [34], [38], among others). It has found applications in mathematical finance especially in connection to pricing of new American contingent claims, such as the Israeli options introduced by Kifer [23] (see also [24]) or convertible bonds (see, e.g., [37] and references therein).

In the nonzero-sum setting, sufficient conditions have been provided in discrete time for the existence of a Nash equilibrium by employing martingale methods and a fixed point theorem for monotone mappings in [30]. In [28] the existence of an equilibrium is proved in the case of a game with monotone rewards by making use of Tarski's fixed point theorem, while equilibrium values in a two-player nonzero-sum game have been constructed by backward induction in [32]. The results of [28] are generalised in [33] by relaxing the monotonicity requirement on the rewards. More recently it has been proved that two-player nonzero-sum games in discrete time admit an ε-equilibrium in randomised stopping times [36].

In continuous time, both analytic and probabilistic methods have been employed to establish the existence of Nash equilibria in nonzero-sum Dynkin games. When the Markov process $(X_t)_{t\geq 0}$ is the solution to a stochastic differential equation (SDE) and the time horizon is finite, the seminal paper [6] reduces the problem of finding a Nash equilibirium to one of finding a sufficiently regular solution to a quasi-variational inequality and the corresponding stopping times are characterised in terms of hitting times to moving boundaries. The latter paper makes intensive use of partial differential equation (PDE) methods and requires smooth bounded coefficients for the SDE. Quasi-variational inequalities were also used in [31] from a probabilistic point of view, to study the existence of an equilibrium in games of symmetric Markov processes.

Among those papers employing fully probabilistic methods, the existence of Nash equilibrium is shown in [16] using the general theory of stochastic processes. The existence of a quasi-Markov Nash equilibrium (i.e. equilibria for a.e. initial condition of the underlying process) was shown by Cattiaux and Lepeltier [8] using the potential theory of Ray-Markov processes. The existence of Nash equilibria in a non-Markovian setting was proved more recently in [17] (generalisations to the N-player case can be found in [18]) and a BSDE approach to nonzero-sum games of control and stopping is presented in [22]. In [25] it is proven that every two-player nonzero-sum game in continuous time admits an ε-equilibrium over randomised stopping times, thus generalising the result in [36] (see also [26]). Finally, a notion of subgame-perfect equilibrium in possibly mixed strategies is introduced in [35].

1.2 Main results

In the present paper X is a weak solution of a stochastic differential equation with drift μ, volatility σ and state space an interval $\mathcal{I} \subset \mathbb{R}$ whose upper endpoint is *natural*, whereas the lower one is either *natural, exit or entrance* (see for instance Ch. 2, pp. 18–20, of [7]). Examples of such diffusions include Brownian motion, geometric Brownian motion, the Ornstein-Uhlenbeck process and Bessel processes (depending on their parameters), plus related processes such as the CIR (Cox-Ingersoll-Ross) and CEV (constant elasticity of variance) process.

We denote by $\mathbb{L}_X$ the infinitesimal generator of X and work with the following natural conditions on the problem data, which rule out equilibria which are either trivial or overly complex. For $i = 1, 2$ we assume:

(a) The functions L_i, G_i are continuous with $L_i < G_i$, so that waiting is incentivised relative to stopping for each player for each given value of the process X. Since we take a positive discount rate r in (1.1) we also assume that G_i is strictly negative for at least one value of X, so that stopping in finite time is incentivised relative to waiting indefinitely,

(b) for any value of the process X at least one player has a running benefit from waiting, in the sense that the cost functions G_i are sufficiently smooth (twice differentiable) and the sets $\{x : (\mathbb{L}_X - r)G_1(x) > 0\}$ and $\{x : (\mathbb{L}_X - r)G_2(x) > 0\}$ are disjoint,

(c) the equation $(\mathbb{L}_X - r)G_i(x) = 0$ has a single root so that, for each player, the running benefit from waiting changes sign at most once with respect to the value of the process X.

Beginning with natural boundaries (thus addressing for example Brownian motion, geometric Brownian motion and the Ornstein-Uhlenbeck process), we establish that conditions (a)–(c) (plus standard integrability assumptions) are sufficient for the existence of a Nash equilibrium, which we characterise via a system of algebraic equations. The corresponding optimal stopping policies are then both of threshold type and the smooth fit principle holds between the cost functional $\mathcal{J}_i(\tau_1, \tau_2; x)$ and the payoff function G_i. While the payoff functions L_i are not in general required to be smooth, we provide easy to check sufficient conditions on L_i, $i = 1, 2$ for uniqueness of the Nash equilibrium. In order to address examples of Bessel, CIR and CEV processes we extend these results to allow the lower endpoint to be either an exit or entrance boundary. Finally we indicate in Appendix A.2 the extension to a state dependent discount factor; note that other combinations of natural, exit and entrance boundaries may in principle also be addressed via our approach (indeed this is immediate by symmetry when the lower boundary is natural).

Our work complements recent closely related work [1, 2][1] using probabilistic methods: we establish results for a wide variety of stochastic processes used in applications under a natural condition on the payoff functions G_i (see Assumption 3.3) while in [1, 2] both endpoints of $\mathcal{I}$ are absorbing for the process X and a number of classes of payoff functions are explored. At the level of methodology we adapt to Dynkin games a constructive solution method for optimal stopping problems which is also originally due to Dynkin [14], Chapter 3, and proceeds by studying the geometry of the problem. The potential value in having this alternative, constructive approach is discussed.

The rest of the paper is organised as follows. In Section 2.1 we introduce the dynamics and recall some properties of linear diffusions, and the nonzero-sum game of optimal stopping is described in Section 2.2. The construction and characterisation of Nash equilibria is provided in Section 3 for the different combinations of diffusion boundary behaviours. In Appendix we provide the generalisation of our results to state dependent discount factors and we also give details on the solution of two optimal stopping problems needed in Section 3.

2 Setting

2.1 The underlying diffusion

Let $(\Omega, \mathcal{F}, \mathsf{P})$ denote a complete filtered probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ under usual hypotheses, $W = (W_t)_{t\geq 0}$ a one dimensional standard Brownian motion adapted to $\mathbb{F}$ and $X = (X_t)_{t\geq 0}$ a continuous adapted process with values in an open interval $\mathcal{I} \subseteq \mathbb{R}$. The triple $(\Omega, \mathcal{F}, \mathsf{P}), \mathbb{F}, (X, W)$ is a weak solution (if it exists) of the stochastic differential equation (SDE)

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x \in \mathcal{I}, \tag{2.1}$$

for some Borel functions $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ to be specified (see Assumption 2.1 below). To account for the dependence of X on its initial position, from now on we shall write X^x where appropriate and P_x to refer to the probability measure such that $\mathsf{P}_x(\cdot) = \mathsf{P}(\cdot|X_0 = x)$, $x \in \mathcal{I}$. Throughout the paper we will equivalently use the notations $\mathsf{E}[f(X^x_t)]$ and $\mathsf{E}_x[f(X_t)]$, $f : \mathbb{R} \to \mathbb{R}$ Borel-measurable and integrable, to refer to expectations under the measure P_x.

[1] These manuscripts became available during the final drafting of the present paper.

We denote by $\overline{\mathcal{I}}$ the closure of $\mathcal{I}$ and we assume that $\mathcal{I} = (\underline{x}, \overline{x}) \subseteq \mathbb{R}$ with $\underline{x}$ and $\overline{x}$ (not necessarily finite) boundary points for X. We assume that the upper boundary point $\overline{x}$ is natural, whereas the lower one $\underline{x}$ is either natural, exit or entrance (see for instance Ch. 2, pp. 18–20, of [7] for a characterisation of the boundary behaviour of diffusions). For the coefficients of the SDE (2.1) we make the following assumption.

Assumption 2.1. *The functions μ and σ are continuous in $\mathcal{I}$ with $\sigma^2 > 0$ in $\mathcal{I}$. Moreover*

$$\int_{y-\varepsilon_o}^{y+\varepsilon_o} \frac{1+|\mu(\xi)|}{|\sigma(\xi)|^2}\, d\xi < +\infty, \quad \textit{for some } \varepsilon_o > 0 \textit{ and every } y \in \mathcal{I}.$$

Assumption 2.1 guarantees that (2.1) has a weak solution that is unique in the sense of probability law (up to a possible explosion time, cf. [21], Ch. 5.5).

We will now recall some basic properties of diffusions. We refer the reader to Ch. 2 of [7] for a detailed exposition. Under Assumption 2.1, the diffusion process X is regular in $\mathcal{I}$; that is, if $\tau(z) := \inf\{t \geq 0 : X_t = z\}$ one has $\mathsf{P}_x(\tau(z) < \infty) > 0$ for every x and z in $\mathcal{I}$ so that the state space cannot be decomposed into smaller sets from which X cannot exit. The continuity of μ and σ imply that the scale function has density

$$S'(x) := \exp\Big(- \int_{x_o}^{x} \frac{2\mu(\xi)}{\sigma^2(\xi)} d\xi \Big), \quad x \in \mathcal{I},$$

for some reference point $x_o \in \mathcal{I}$, and the speed measure has density

$$m'(x) := \frac{2}{\sigma^2(x) S'(x)}, \quad x \in \mathcal{I}.$$

Denoting by

$$\big(\mathbb{L}_X u\big)(x) := \frac{1}{2}\sigma^2(x) u''(x) + \mu(x) u'(x), \qquad x \in \mathcal{I}$$

the infinitesimal generator of X, under Assumption 2.1 there always exist two linearly independent, strictly positive solutions of the ordinary differential equation $\mathbb{L}_X u = ru$, $r > 0$, satisfying a set of boundary conditions based on the boundary behaviour of X (see, e.g., pp. 18–19 of [7]). These functions span the set of solutions of $\mathbb{L}_X u = ru$, $r > 0$, and are uniquely defined up to multiplication if one of them is required to be strictly increasing and the other one to be strictly decreasing. We denote the strictly increasing solution ψ_r and the strictly decreasing one ϕ_r. For $x, y \in \mathcal{I}$ and $\tau(y) := \inf\{t \geq 0 : X_t = y\}$ one has

$$\mathsf{E}_x\big[e^{-r\tau(y)}\big] = \begin{cases} \frac{\psi_r(x)}{\psi_r(y)}, & x < y, \\ \frac{\phi_r(x)}{\phi_r(y)}, & x > y. \end{cases} \tag{2.2}$$

Also, it is well known that the Wronskian

$$W := \frac{\psi_r'(x)\phi_r(x) - \phi_r'(x)\psi_r(x)}{S'(x)}, \quad x \in \mathcal{I}, \tag{2.3}$$

is a positive constant and we introduce the Green function

$$r(x,y) := W^{-1} \cdot \begin{cases} \psi_r(x)\phi_r(y), & x \leq y, \\ \phi_r(x)\psi_r(y), & x \geq y. \end{cases}$$

For $\sigma_{\mathcal{I}} := \inf\{t \geq 0 \,:\, X_t \notin \mathcal{I}\}$ one has

$$\mathsf{E}\Big[\int_0^{\sigma_{\mathcal{I}}} e^{-rt} f(X_t^x) dt\Big] = \int_{\mathcal{I}} f(y) r(x,y) m'(y) dy, \quad x \in \mathcal{I}, \tag{2.4}$$

for any continuous real function f such that the integrals are well defined. Moreover the following useful equations hold for any $\underline{x} < a < b < \overline{x}$ (cf. par. 10, Ch. 2 of [7]):

$$\frac{\psi_r'(b)}{S'(b)} - \frac{\psi_r'(a)}{S'(a)} = r\int_a^b \psi_r(y)m'(y)dy, \qquad \frac{\phi_r'(b)}{S'(b)} - \frac{\phi_r'(a)}{S'(a)} = r\int_a^b \phi_r(y)m'(y)dy. \tag{2.5}$$

2.2 The nonzero-sum Dynkin game

In the setting of Section 2.1, consider now the following *two-player nonzero-sum game of optimal stopping.* Denoting

$$\mathcal{T} := \{\tau : \tau \text{ is an } \mathbb{F}\text{-stopping time and } \tau \le \sigma_{\mathcal{I}}\},$$

player i aims to *minimise* the cost functional $\mathcal{J}_1(\tau_1, \tau_2; x)$ in (1.1) by optimally choosing their $\mathbb{F}$-stopping time τ_i.

Definition 2.2. *For $x \in \mathcal{I}$ we say that a couple $(\tau_1^*, \tau_2^*) \in \mathcal{T} \times \mathcal{T}$ is a Nash equilibrium for the two-player nonzero-sum game of optimal stopping if and only if*

$$\begin{cases} \mathcal{J}_1(\tau_1^*, \tau_2^*; x) \le \mathcal{J}_1(\tau_1, \tau_2^*; x), & \forall\, \tau_1 \in \mathcal{T}, \\ \mathcal{J}_2(\tau_1^*, \tau_2^*; x) \le \mathcal{J}_2(\tau_1^*, \tau_2; x), & \forall\, \tau_2 \in \mathcal{T}. \end{cases} \tag{2.6}$$

We also say that $v_i(x) := \mathcal{J}_i(\tau_1^*, \tau_2^*; x)$ is the value of the game for the i-th player relative to the equilibrium.

In the rest of the paper we will be concerned with equilibria attained by hitting times of X to suitable sets. We introduce the following classes of functions:

Definition 2.3. *Let $\mathcal{A}$ be the class of real valued functions $H \in C^2(\mathcal{I})$ such that*

$$\limsup_{x\to\underline{x}} \Big|\frac{H}{\phi_r}\Big|(x) = 0, \;\; \limsup_{x\to\overline{x}} \Big|\frac{H}{\psi_r}\Big|(x) = 0 \tag{2.7}$$

$$\textit{and} \quad \mathsf{E}\bigg[\int_0^{\sigma_{\mathcal{I}}} e^{-rt}\big|h(X_t^x)\big|dt\bigg] < \infty \tag{2.8}$$

for all $x \in \mathcal{I}$ and with $h(x) := (\mathbb{L}_X H - rH)(x)$. We denote by $\mathcal{A}_1$ (respectively $\mathcal{A}_2$) the set of all $H \in \mathcal{A}$ such that h is strictly positive (resp. negative) on $(\underline{x}, x_h)$ and strictly negative (resp. positive) on $(x_h, \overline{x})$ for some $x_h \in \mathcal{I}$ with $\liminf_{x\to\underline{x}} h(x) > 0$ (resp. < 0) and $\limsup_{x\to\overline{x}} h(x) < 0$ (resp. > 0).

Remark 2.4. *The cost functions G_i of* (1.1) *will be drawn from these classes. For natural and entrance boundaries we have $\phi_r(x) \uparrow +\infty$ as $x \downarrow \underline{x}$ and $\psi_r(x) \uparrow +\infty$ as $x \uparrow \overline{x}$ in* (2.7)*, so the classes are not overly restrictive for applications. It is possible to relax* (2.7) *to some extent in our approach, and this is done in Section 3.3 in the case of exit boundaries. We also note that all the results in this paper remain true if in the definition of $\mathcal{A}$ the regularity of H is weakened by requiring $H \in W^{2,\infty}_{loc}(\mathcal{I})$.*

Before closing this section we provide some explicit formulae for functions in $\mathcal{A}$ which will prove useful in the next sections. If $H \in \mathcal{A}$ (see Definition 2.3) we can apply Itô's formula (and a standard localization argument) to obtain

$$H(x) = -\mathsf{E}\bigg[\int_0^{\sigma_{\mathcal{I}}} e^{-rt} h(X_t^x)dt\bigg], \quad x \in \mathcal{I}. \tag{2.9}$$

Then applying representation (2.4) we get the equivalent expression

$$H(x) = -W^{-1}\left[\phi_r(x)\int_{\underline{x}}^{x}\psi_r(y)h(y)m'(y)dy + \psi_r(x)\int_{x}^{\overline{x}}\phi_r(y)h(y)m'(y)dy\right] \tag{2.10}$$

and straightforward calculations also give

$$\left(\frac{H}{\phi_r}\right)'(x) = -\frac{F_r'(x)}{W}\int_{x}^{\overline{x}}\phi_r(y)h(y)m'(y)dy. \tag{2.11}$$

3 Construction of Nash equilibria

In this section we prove our main results, i.e. existence and then sufficient additional conditions for uniqueness of a Nash equilibrium of threshold type for the nonzero-sum Dynkin game of Section 2.2. Moreover we provide an explicit characterisation of the optimal thresholds in terms of a system of two algebraic equations in two unknowns. We begin in Section 3.1 under the assumption that the endpoints $\underline{x} < \overline{x}$ of $\mathcal{I}$ are natural for X, considering an entrance lower boundary $\underline{x}$ in Section 3.2 and an exit lower boundary in Section 3.3.

3.1 The case of natural boundaries

The results in this section are proved under the assumption that the endpoints of $\mathcal{I}$ are both natural for X. For the sake of completeness, recall that a boundary point ξ is *natural* for a diffusion process if it is both non-entrance and non-exit. That is, ξ cannot be a starting point for the process and it cannot be reached in finite time (cf. for instance [7], Ch. 2, p. 15). For $\underline{x}$ and $\overline{x}$ natural boundary points one also has (see par. 10, Sec. 2 of [7])

$$\lim_{x\downarrow\underline{x}}\psi_r(x) = 0, \quad \lim_{x\downarrow\underline{x}}\phi_r(x) = \infty, \quad \lim_{x\uparrow\overline{x}}\psi_r(x) = \infty, \quad \lim_{x\uparrow\overline{x}}\phi_r(x) = 0, \tag{3.1}$$

$$\lim_{x\downarrow\underline{x}}\frac{\psi_r'(x)}{S'(x)} = 0, \quad \lim_{x\downarrow\underline{x}}\frac{\phi_r'(x)}{S'(x)} = -\infty, \quad \lim_{x\uparrow\overline{x}}\frac{\psi_r'(x)}{S'(x)} = \infty, \quad \lim_{x\uparrow\overline{x}}\frac{\phi_r'(x)}{S'(x)} = 0. \tag{3.2}$$

As in [11], eq. (4.6), we define the strictly increasing function

$$F_r(x) := \frac{\psi_r(x)}{\phi_r(x)}, \qquad x \in \mathcal{I}, \tag{3.3}$$

together with its inverse $F_r^{-1}(y)$, $y > 0$, and for any continuous real function H we set

$$\hat{H}(y) := \begin{cases} \left(\frac{H}{\phi_r}\right)\circ F_r^{-1}(y), & y > 0, \\ 0, & y = 0. \end{cases} \tag{3.4}$$

We now relate (3.4) to the classes $\mathcal{A}_1$ and $\mathcal{A}_2$ of Section 2.2.

Lemma 3.1. *Let $H \in \mathcal{A}_1$ (respectively $\mathcal{A}_2$) and set $\hat{y} := F_r(x_h)$, $h(x) := (\mathbb{L}_X - r)h(x)$ and $\hat{H}$ as in* (3.4)*. Then $\hat{H}$:*

i) is convex (resp. concave) on $[0, \hat{y})$ and concave (resp. convex) on $(\hat{y}, \infty)$,

ii) satisfies $\hat{H}(0+) = 0$ and $\hat{H}'(0+) = -\infty$ (resp. $+\infty$);

iii) has a unique global minimum (resp. maximum) at some $\overline{y} \in [0, \hat{y})$ and $\lim_{y\to\infty}\hat{H}(y) = +\infty$ (resp. $-\infty$); hence it is monotonic increasing (resp. decreasing) on $(\hat{y}, +\infty)$.

Proof. Assume first that $H \in \mathcal{A}_1$. *i)* From Section 6, p. 192 in [11] we know that $\hat{H}$ is strictly convex if and only if[2] $h > 0$, whereas $\hat{H}$ is strictly concave if and only if $h < 0$. Therefore, we have that $\hat{H}$ is convex on $[F_r(\underline{x}), \hat{y})$, and concave on $(\hat{y}, F_r(\overline{x}))$. Note that $F_r(\underline{x}) = 0$ and $F_r(\overline{x}) = +\infty$ due to (3.1) and the limit at zero of $\hat{H}$ is verified from the definition of $\mathcal{A}$. If we now show that

$$\text{(a)}\ \liminf_{y\uparrow\infty} \hat{H}(y) = +\infty, \qquad \text{and} \qquad \text{(b)}\ \lim_{y\downarrow 0} \hat{H}'(y) = -\infty, \tag{3.5}$$

we can then conclude parts *ii)* and *iii)*.

First we prove (a) above. By the definition of $\mathcal{A}_1$, for each given $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that $h(z) \le -\varepsilon_\delta$ for any $z \in [x_h + \delta, \overline{x})$. Moreover, for any $x \in [x_h + \delta, \overline{x})$, (2.10) implies

$$\begin{aligned} H(x) &= -W^{-1}\Big[\phi_r(x)\int_{\underline{x}}^{x_h+\delta} \psi_r(z)h(z)m'(z)dz + \phi_r(x)\int_{x_h+\delta}^{x} \psi_r(z)h(z)m'(z)dz \\ &\qquad + \psi_r(x)\int_x^{\overline{x}} \phi_r(z)h(z)m'(z)dz\Big] \qquad (3.6)\\ &\ge -W^{-1}\Big[\phi_r(x)C_\delta - \phi_r(x)\varepsilon_\delta\int_{x_h+\delta}^{x} \psi_r(z)m'(z)dz - \varepsilon_\delta\psi_r(x)\int_x^{\overline{x}} \phi_r(z)m'(z)dz\Big] \end{aligned}$$

with $C_\delta := \int_{\underline{x}}^{x_h+\delta} \psi_r(z)h(z)m'(z)dz$. Note that the last two terms above may be rewritten using (2.5) as

$$\int_{x_h+\delta}^{x} \psi_r(z)m'(z)dz = \frac{1}{r}\left[\frac{\psi_r'(x)}{S'(x)} - \frac{\psi_r'(x_h+\delta)}{S'(x_h+\delta)}\right]$$

and

$$\int_x^{\overline{x}} \phi_r(z)m'(z)dz = -\frac{1}{r}\frac{\phi_r'(x)}{S'(x)}.$$

Hence rearranging terms in (3.6) and recalling the expression for the Wronskian W in (2.3) we get

$$H(x) \ge -W^{-1}\left[C_\delta + \frac{\varepsilon_\delta}{r}\frac{\psi_r'(x_h+\delta)}{S'(x_h+\delta)}\right]\phi_r(x) + \frac{\varepsilon_\delta}{r}, \tag{3.7}$$

which implies

$$\frac{H(x)}{\phi_r(x)} \ge -W^{-1}\left[C_\delta + \frac{\varepsilon_\delta}{r}\frac{\psi_r'(x_h+\delta)}{S'(x_h+\delta)}\right] + \frac{\varepsilon_\delta}{r\phi_r(x)}. \tag{3.8}$$

Using (3.1) we obtain

$$\liminf_{x\uparrow\overline{x}} \frac{H(x)}{\phi_r(x)} = +\infty,$$

and since $\lim_{y\to\infty} F_r^{-1}(y) = \overline{x}$, we also get

$$\liminf_{y\uparrow\infty} \hat{H}(y) = \liminf_{y\uparrow\infty} \left(\frac{H}{\phi_r}\right)\circ F_r^{-1}(y) = +\infty$$

and (a) is proved.

To prove (b) let $y = F_r(x) < F_r(x_h - \delta)$ for some $\delta > 0$, recall (2.11) and note that since $H \in \mathcal{A}_1$ we have

$$\begin{aligned} \hat{H}'(y) &= -\frac{1}{W}\Big[\int_{x_h-\delta}^{\overline{x}} \phi_r(z)h(z)m'(z)dz + \int_x^{x_h-\delta} \phi_r(z)h(z)m'(z)dz\Big] \qquad (3.9)\\ &\le -\frac{1}{W}\Big[C_\delta + \varepsilon_\delta\int_x^{x_h-\delta} \phi_r(z)m'(z)dz\Big] \end{aligned}$$

[2]Although this is a well known result we thought that an unfamiliar reader may benefit from a proof of this fact, which we account for in Appendix A.1.

for some $\varepsilon_\delta > 0$ and with $C_\delta := \int_{x_h-\delta}^{\overline{x}} \phi_r(z)h(z)m'(z)dz$. Now using (2.5) we also obtain

$$\hat{H}'(y) \le -W^{-1}C_\delta - \frac{\varepsilon_\delta}{W\,r}\Big(\frac{\phi_r'(x_h-\delta)}{S'(x_h-\delta)} - \frac{\phi_r'(x)}{S'(x)}\Big) \tag{3.10}$$

and then, letting $y \to 0$ (equivalently $x \to \underline{x}$) and using (3.2), we conclude $\hat{H}'(0+) = -\infty$. The case $H \in \mathcal{A}_2$ follows by exactly symmetric arguments. □

For $i = 1, 2$ we set

$$\hat{G}_i(y) := \Big(\frac{G_i}{\phi_r}\Big) \circ F_r^{-1}(y), \qquad \hat{L}_i(y) := \Big(\frac{L_i}{\phi_r}\Big) \circ F_r^{-1}(y), \qquad y > 0. \tag{3.11}$$

Definition 3.2. *For $i = 1, 2$, if $G_i \in \mathcal{A}_i$ then we define*

1. *$g_i(x) := (\mathbb{L}_X - r)G_i(x)$, $x \in \mathcal{I}$;*
2. *$\hat{x}_i$ the unique point at which the sign of $g_i(x)$ changes and we set $\hat{y}_i := F_r(\hat{x}_i)$;*
3. *$\overline{y}_i$ is the unique stationary point of $\hat{G}_i$;*

and if $L_i \in \mathcal{A}_i$ then we define

1. *$\ell_i(x) := (\mathbb{L}_X - r)L_i(x)$, $x \in \mathcal{I}$;*
2. *$\check{x}_i$ the unique point at which the sign of $\ell_i(x)$ changes and we set $\check{y}_i := F_r(\check{x}_i)$;*
3. *$\widetilde{y}_i$ is the unique stationary point of $\hat{L}_i$.*

We now give the key assumption in this paper, which will subsequently play the role of a sufficient condition for the existence of Nash equilibria of threshold type.

Assumption 3.3. *For $i = 1, 2$ we have L_i, $G_i \in C(\mathcal{I};\mathbb{R})$ with*

$$L_i < G_i \ \text{on } \mathcal{I}, \qquad \text{and} \qquad \inf_{x\in\mathcal{I}} G_i(x) < 0.$$

Further we require $G_1 \in \mathcal{A}_1$, $G_2 \in \mathcal{A}_2$ and $\hat{x}_1 < \hat{x}_2$ and we assume that

$$\limsup_{x\to\underline{x}} \Big|\frac{L_i}{\phi_r}\Big|(x) < +\infty \quad \text{and} \quad \limsup_{x\to\overline{x}} \Big|\frac{L_i}{\psi_r}\Big|(x) < +\infty \qquad i = 1, 2. \tag{3.12}$$

We are now ready to prove our main results. For u, $v \ge 0$ let us introduce the functions

$$\mathcal{L}_1(u, v) := \hat{G}_1(u) - \hat{L}_1(v) - \hat{G}_1'(u)(u - v), \tag{3.13}$$

$$\mathcal{L}_2(u, v) := \hat{G}_2(u) - \hat{L}_2(v) - \hat{G}_2'(u)(u - v), \tag{3.14}$$

then the following theorem holds.

Theorem 3.4. *Under Assumption 3.3 there exists a Nash equilibrium of the form*

$$\tau_1^* := \inf\{t \ge 0 : X_t^x \le x_1^*\}, \quad \tau_2^* := \inf\{t \ge 0 : X_t^x \ge x_2^*\}, \tag{3.15}$$

with $x_1^* := F_r^{-1}(y_1^*) \in (\underline{x}, \hat{x}_1)$ and $x_2^* := F_r^{-1}(y_2^*) \in (\hat{x}_2, \overline{x})$ and where the couple $(y_1^*, y_2^*) \in (0, \hat{y}_1) \times (\hat{y}_2, +\infty)$ solves the system

$$\begin{cases} \mathcal{L}_1(y_1, y_2) = 0 \\ \mathcal{L}_2(y_2, y_1) = 0. \end{cases} \tag{3.16}$$

Proof. We proceed by constructing each player's best reply to the other player's stopping rule. Denote

$$\tau_1(z) := \inf\{t \geq 0 \,:\, X^x_t \leq z\} \quad \text{and} \quad \tau_2(z) := \inf\{t \geq 0 \,:\, X^x_t \geq z\}, \tag{3.17}$$

for any $x, z \in \mathcal{I}$. By doing so we want to associate to P_1 hitting times of half-intervals of the form $(\underline{x}, z]$ and to P_2 hitting times of half-intervals of the form $[z, \overline{x})$. These are natural choices, since it is easy to see that the i-th player will never stop in the set where g_i is strictly negative (cf. (2.9)). Indeed let us assume for example that $x > \hat{x}_1$ and let P_1 pick the suboptimal stopping time $\tau_1(\hat{x}_1)$. Then, for any finite $s > 0$ and any stopping time σ chosen by P_2, we denote $\theta := \sigma \wedge \tau_1(\hat{x}_1) \wedge s$ and obtain (see (1.1))

$$\mathcal{J}_1(\tau_1(\hat{x}_1) \wedge s, \sigma) \leq \mathsf{E}_x\Big[e^{-r\theta} G_1(X_\theta)\Big] = G_1(x) + \mathsf{E}_x\left[\int_0^\theta e^{-ru} g_1(X_u) du\right] < G_1(x)$$

by using that $L_1 < G_1$ on $\mathcal{I}$ and Dynkin's formula. Hence stopping at once costs to P_1 more than continuing at least until $\tau_1(\hat{x}_1) \wedge s$ regardless of P_2's stopping time σ.

1. Let us assume that P_1 picks $z \in (\underline{x}, \hat{x}_1)$ and decides to stop at $\tau_1(z)$. Then P_2 is faced with an optimal stopping problem of the form

$$\inf_{\tau \in \mathcal{T}} \mathsf{E}_x\Big[e^{-r\tau} G_2(X_\tau) \mathbb{1}_{\{\tau < \tau_1(z)\}} + L_2(z) e^{-r\tau_1(z)} \mathbb{1}_{\{\tau \geq \tau_1(z)\}}\Big], \quad x \geq z. \tag{3.18}$$

This is a canonical problem whose solution is provided for completeness in Appendix A.3.1. It is shown there that an optimal stopping time for P_2 is $\tau_2(x_2)$ with $x_2 = x_2(z) := F_r^{-1}(y_2(\zeta))$ where $\zeta := F_r^{-1}(z)$ and $y_2(\zeta)$ a solution (if it exists and it is unique) in $(\hat{y}_2, \infty)$ of the equation

$$\mathcal{L}_2(\,\cdot\,, \zeta) = 0. \tag{3.19}$$

We now show existence and uniqueness of a solution in $(\hat{y}_2, \infty)$ to (3.19). Here $\zeta \in [0, \hat{y}_1)$ is given and fixed, and note that by concavity of $\hat{G}_2$ on $(0, \hat{y}_2)$ one has

$$\hat{G}_2'(\hat{y}_2)(\hat{y}_2 - \zeta) < \int_\zeta^{\hat{y}_2} \hat{G}_2'(s) ds = \hat{G}_2(\hat{y}_2) - \hat{G}_2(\zeta),$$

and therefore (cf. (3.14))

$$\mathcal{L}_2(\hat{y}_2, \zeta) > \hat{G}_2(\hat{y}_2) - \hat{L}_2(\zeta) - \hat{G}_2(\hat{y}_2) + \hat{G}_2(\zeta) > 0,$$

because $\hat{G}_2 > \hat{L}_2$. Moreover, $\frac{\partial}{\partial u}\mathcal{L}_2(u, \zeta) = -\hat{G}_2''(u)(u - \zeta) < 0$ for any $u \in (\hat{y}_2, \infty)$. For existence of a unique $y_2(\zeta) \in (\hat{y}_2, \infty)$ solving (3.19) it is sufficient to show $\mathcal{L}_2(u, \zeta) \to -\infty$ as $u \uparrow +\infty$. To this end, first note that

$$\sup_{\zeta \in (0, \hat{y}_1)} \big|\hat{L}_2(\zeta)\big| \leq C \tag{3.20}$$

for some finite constant $C > 0$, thanks to continuity of L_2 on $\mathcal{I}$ and (3.12). On the other hand, for $u_0 > 0$ sufficiently large we must have $\hat{G}_2'(u) \leq 0$ for $u \geq u_0$ (see *iii*) of Lemma 3.1) and therefore

$$\lim_{u \to \infty} \big[\hat{G}_2(u) - \hat{G}_2'(u)(u - \zeta)\big] \leq \lim_{u \to \infty} \big[\hat{G}_2(u) - \hat{G}_2'(u) u\big]. \tag{3.21}$$

If the latter limit equals $-\infty$ then also $\lim_{u \to \infty} \mathcal{L}_2(u, \zeta) = -\infty$ due to (3.20).

Note that direct computation and (2.11) (with $h = g_2$) give

$$\hat{G}_2'(u)u = \frac{u}{F_r'\big(F_r^{-1}(u)\big)}\Big(\frac{G_2}{\phi_r}\Big)'\big(F_r^{-1}(u)\big) = -\frac{u}{W}\int_{F_r^{-1}(u)}^{\overline{x}}\phi_r(t)g_2(t)m'(t)dt.$$

Setting $u = F_r(s)$ for notational convenience and recalling the first equation in (3.11) and (2.10) (with $h = g_2$) we also get

$$\hat{G}_2(u) - \hat{G}_2'(u)u = -W^{-1}\int_{\underline{x}}^{s}\psi_r(t)g_2(t)m'(t)dt. \tag{3.22}$$

Pick an arbitrary $\delta > 0$ and recall the existence of $\varepsilon_\delta > 0$ such that $g_2(y) \geq \varepsilon_\delta$ for $y \in [\hat{x}_2+\delta, \overline{x})$, which follows from the definition of $\mathcal{A}_2$. Since we are looking at the limit as $u \to \infty$ (i.e. $s \to \overline{x}$) with no loss of generality we assume $s > \hat{x}_2 + \delta$ and by using (2.5) we obtain

$$\begin{aligned}\hat{G}_2(u) - \hat{G}_2'(u)u &= -\frac{1}{W}\Big[\int_{\underline{x}}^{\hat{x}_2+\delta}\psi_r(t)g_2(t)m'(t)dt + \int_{\hat{x}_2+\delta}^{s}\psi_r(t)g_2(t)m'(t)dt\Big]\\ &\leq -\frac{\varepsilon_\delta}{Wr}\Big(\frac{\psi_r'(s)}{S'(s)} - \frac{\psi_r'(\hat{x}_2+\delta)}{S'(\hat{x}_2+\delta)}\Big) - \frac{C_\delta}{W}\end{aligned} \tag{3.23}$$

where $C_\delta := \int_{\underline{x}}^{\hat{x}_2+\delta}\psi_r(t)g_2(t)m'(t)dt$. In the limit as $u \to \infty$ one has $s = F_r^{-1}(u) \to \overline{x}$ and by (3.2) we conclude that $\lim_{u\to\infty}\big[\hat{G}_2(u) - \hat{G}_2'(u)u\big] = -\infty$ and hence $\lim_{u\to\infty}\mathcal{L}_2(u,\zeta) = -\infty$.

Equation (3.19) is the geometric version of the so called smooth-fit equation. From the arbitrariness of $z \in (\underline{x}, \hat{x}_1)$ and a simple application of the implicit function theorem we obtain that the map $z \mapsto x_2(z)$ is continuous on $(\underline{x}, \hat{x}_1)$ (or equivalently $y_2(\,\cdot\,) \in C([0, \hat{y}_1))$) (see, e.g., Th. 10.2.1 at p. 270 of [12]).

2. In a completely symmetric way we now look at the case where P_2 picks $z \in (\hat{x}_2, \overline{x})$ and decides to stop at $\tau_2(z)$. Then P_1 is faced with an optimal stopping problem of the form

$$\inf_{\tau\in\mathcal{T}}\mathsf{E}_x\Big[e^{-r\tau}G_1(X_\tau)\mathbb{1}_{\{\tau<\tau_2(z)\}} + L_1(z)e^{-r\tau_2(z)}\mathbb{1}_{\{\tau\geq\tau_2(z)\}}\Big], \quad x \leq z \tag{3.24}$$

whose standard method of solution is illustrated in Appendix A.4. An optimal stopping time for P_1 is $\tau_1(x_1)$ with $x_1 = x_1(z) := F_r^{-1}(y_1(\zeta))$ where $\zeta := F_r^{-1}(z)$ and $y_1(\zeta)$ is the unique solution in $(0, \hat{y}_1)$ of the equation

$$\mathcal{L}_1(\,\cdot\,, \zeta) = 0, \tag{3.25}$$

and with $\mathcal{L}_1$ as in (3.13). A proof of the existence of such a solution can be obtained following arguments similar to those employed for equation (3.19) above. Notice however that here we need extra care when using this argument as in fact (3.25) only holds if $y_1(\zeta) > 0$. However this is guaranteed in this setting by observing that $\hat{G}_1'(0+) = -\infty$ (see Lemma 3.1).

Again, the map $z \mapsto x_1(z)$ is continuous on $[\hat{x}_2, \overline{x})$ (or equivalently $y_1(\,\cdot\,) \in C([\hat{y}_2, +\infty))$) by the implicit function theorem and arbitrariness of z.

3. The optimal stopping times in parts 1 and 2 above determine each player's best reply to the other player's stopping rule, provided that the latter is of threshold type. To prove that a Nash equilibrium exists in this class of stopping times we now need to prove that it is possible to find $\underline{x} < x_1^* < x_2^* < \overline{x}$ such that both the following hold:

i) $\tau_1(x_1^*)$ is optimal for P_1, given that P_2 stops on hitting $[x_2^*, \overline{x})$ (i.e. at $\tau_2^*(x_2^*)$),

ii) $\tau_2(x_2^*)$ is optimal for P_2, given that P_1 stops on hitting $(\underline{x}, x_1^*]$ (i.e. at $\tau_1^*(x_1^*)$).

This is equivalent to finding an intersection point (x_1^*, x_2^*) for the curves $z \mapsto x_1(z)$ and $z \mapsto x_2(z)$, which is in turn equivalent to establishing the fixed points $x_1(x_2(x_1^*)) = x_1^*$ and $x_2(x_1(x_2^*)) = x_2^*$. For this it is convenient to use the transformed variables y_1 and y_2 along with $\mathcal{L}_1$ and $\mathcal{L}_2$.

For each $\zeta \in (\hat{y}_2, +\infty)$, part 2 above defines a unique $y_1(\zeta) \in (0, \hat{y}_1)$. We now seek $y_2^* \in (\hat{y}_2, +\infty)$ such that $\mathcal{L}_2(y_2^*, y_1(y_2^*)) = 0$ by analysing the map $\zeta \mapsto \mathcal{L}_2(\zeta, y_1(\zeta))$ for $\zeta \in (\hat{y}_2, +\infty)$. First we prove that $\mathcal{L}_2(\hat{y}_2, y_1(\hat{y}_2)) > 0$. For this it suffices to note that $\hat{G}_2$ is strictly concave on $(0, \hat{y}_2)$ (by Lemma 3.1) and hence

$$\hat{G}_2'(\hat{y}_2)(\hat{y}_2 - y_1(\hat{y}_2)) < \int_{y_1(\hat{y}_2)}^{\hat{y}_2} \hat{G}_2'(s)ds \tag{3.26}$$

since $\hat{y}_2 > \hat{y}_1 > y_1(\hat{y}_2)$. The above implies (see (3.14))

$$\mathcal{L}_2(\hat{y}_2, y_1(\hat{y}_2)) > \hat{G}_2(\hat{y}_2) - \int_{y_1(\hat{y}_2)}^{\hat{y}_2} \hat{G}_2'(s)ds - \hat{L}_2(y_1(\hat{y}_2)) = \hat{G}_2(y_1(\hat{y}_2)) - \hat{L}_2(y_1(\hat{y}_2)) > 0. \tag{3.27}$$

By continuity of G_2, L_2 and y_1 we get $\mathcal{L}_2(\,\cdot\,, y_1(\,\cdot\,)) \in C([\hat{y}_2, +\infty))$ and, if now we prove that $\mathcal{L}_2(\zeta, y_1(\zeta)) \to -\infty$ as $\zeta \uparrow +\infty$, then we get existence of at least one root $y_2^* \in (\hat{y}_2, +\infty)$. To this end, first note that $\sup_{\zeta \in (\hat{y}_2, \infty)} \big|\hat{L}_2(y_1(\zeta))\big| \le \sup_{\xi \in (0, \hat{y}_1)} \big|\hat{L}_2(\xi)\big| \le C$, for some finite constant $C > 0$, thanks to continuity of L_2 on $\mathcal{I}$ and (3.12). Then, we can employ arguments analogous to those used in part 1 above to obtain (3.23) with u replaced by ζ and s replaced by $z := F_r^{-1}(\zeta)$ (notice in particular that by arguing as in (3.21) we remove the dependence on $y_1(\zeta)$). Hence there exists at least one $y_2^* \in (\hat{y}_2, +\infty)$ such that $\mathcal{L}_2(y_2^*, y_1(y_2^*)) = 0$ and the couple y_2^* and $y_1^* := y_1(y_2^*)$ is a solution of (3.16). Finally the stopping times (3.15) produce a Nash equilibrium with $x_1^* := F_r^{-1}(y_1^*)$ and $x_2^* := F_r^{-1}(y_2^*)$. □

The above theorem states that a Nash equilibrium exists, together with a couple (y_1^*, y_2^*) solving (3.16) in a suitable domain. Now we show that any couple (y_1, y_2) solving (3.16) in $(0, \hat{y}_1) \times (\hat{y}_2, +\infty)$ may be used to construct stopping times that provide a Nash equilibrium. Before providing the result it is convenient to recall that $\hat{G}_i(y) = (G_i/\phi_r)(x)$ for any $y = F_r(x)$ and $i = 1, 2$.

Proposition 3.5. *Let Assumption 3.3 hold and let $(y_1, y_2) \in (0, \hat{y}_1) \times (\hat{y}_2, +\infty)$ be a solution of (3.16). Then for $x \in \mathcal{I}$, $x_i := F_r^{-1}(y_i)$, $i = 1, 2$ the couple*

$$\hat{\tau}_1 := \inf\{t \ge 0 \,:\, X_t^x \le x_1\}, \quad \hat{\tau}_2 := \inf\{t \ge 0 \,:\, X_t^x \ge x_2\} \tag{3.28}$$

is a Nash equilibrium. Moreover the functions

$$v_1(x) := \begin{cases} G_1(x), & x \le x_1, \\ m_1 \psi_r(x) + q_1 \phi_r(x), & x_1 < x < x_2, \\ L_1(x), & x \ge x_2, \end{cases} \tag{3.29}$$

and

$$v_2(x) := \begin{cases} L_2(x), & x \le x_1, \\ m_2 \psi_r(x) + q_2 \phi_r(x), & x_1 < x < x_2, \\ G_2(x), & x \ge x_2, \end{cases} \tag{3.30}$$

with

$$m_1 := \frac{(G_1/\phi_r)(x_1) - (L_1/\phi_r)(x_2)}{F_r(x_1) - F_r(x_2)}, \qquad q_1 := \frac{L_1}{\phi_r}(x_2) - m_1 F_r(x_2), \tag{3.31}$$

$$m_2 := \frac{(G_2/\phi_r)(x_2) - (L_2/\phi_r)(x_1)}{F_r(x_2) - F_r(x_1)}, \qquad q_2 := \frac{L_2}{\phi_r}(x_1) - m_2 F_r(x_1), \tag{3.32}$$

are the value functions of the two players, i.e. $v_i(x) = \mathcal{J}_i(\tau_1, \tau_2; x)$, $i = 1, 2$. *Also* $v_1 \in C(\mathcal{I})$ *with* $v_1 \in W^{2,\infty}_{loc}(\underline{x}, x_2)$ *and* $v_2 \in C(\mathcal{I})$ *with* $v_2 \in W^{2,\infty}_{loc}(x_1, \overline{x})$.

Proof. 1. Let us start by proving the required regularity of v_i. Consider first v_1 and observe that $v_1 \in C(\mathcal{I})$ and lies in $C^1(\underline{x}, x_2)$ if and only if analogous properties hold for v_1/ϕ_r. Then using that $(y_1, y_2) = (F_r(x_1), F_r(x_2))$ solves (3.16) and changing variables to $y = F_r(x)$ for computational convenience, it is matter of simple algebra to check the claims. We can proceed in a similar way for v_2.

2. By continuity one has $v_1 = G_1$ in $(\underline{x}, x_1]$, $v_1 = L_1$ in $[x_2, \overline{x})$ and $v_2 = G_2$ in $[x_2, \overline{x})$, $v_2 = L_2$ in $(\underline{x}, x_1]$. The C^1 regularity (smooth fit) also implies $v_1'(x_1+) = G_1'(x_1)$ and $v_2'(x_2-) = G_2'(x_2)$. Moreover we claim that the following variational inequalities hold:

$$(\mathbb{L}_X - r)v_i(x) = 0, \qquad x_1 < x < x_2,\ i = 1, 2 \tag{3.33}$$

$$(\mathbb{L}_X - r)v_1(x) > 0, \qquad \underline{x} < x < x_1 \tag{3.34}$$

$$(\mathbb{L}_X - r)v_2(x) > 0, \qquad x_2 < x < \overline{x} \tag{3.35}$$

$$v_i \le G_i, \qquad x \in \mathcal{I},\ i = 1, 2. \tag{3.36}$$

The first three expressions follow directly from the definition of v_i and by the fact that $x_1 < \hat{x}_1$ and $x_2 > \hat{x}_2$. For the final inequality (the obstacle conditions) again we resort to the change of variable inspired by the geometric approach. Notice indeed that $v_i \le G_i$ if and only if $(v_i/\phi_r) \le (G_i/\phi_r)$. Set $x = F_r^{-1}(y)$ and note that $\hat{v}_1(y) := (v_i/\phi_r) \circ F_r^{-1}(y)$ is a *straight line* in (y_1, y_2) and it is tangent to $\hat{G}_1$ in y_1 by smooth fit. Moreover $\hat{G}_1$ is convex in $(0, \hat{y}_1)$, it reaches its unique global minimum therein and it is concave in $(\hat{y}_1, +\infty)$; then given that $\hat{v}_1(y_2) = \hat{L}_1(y_2) < \hat{G}_1(y_2)$ we must also have $\hat{v}_1 \le \hat{G}_1$ on (y_1, y_2). Hence we get $v_1 \le G_1$ in $(\underline{x}, x_2)$ and $v_1 = L_1 < G_1$ in $[x_2, \overline{x})$. Symmetric arguments hold for v_2.

3. Given the regularity of v_i, $i = 1, 2$ we can now apply Dynkin's formula and use the above variational characterisation to conclude the proof. Let σ be an arbitrary stopping time and $\hat{\tau}_2$ as in (3.28), then using standard localisation arguments we get

$$\begin{aligned} v_1(x) =& \mathsf{E}_x\Big[e^{-r(\sigma\wedge\hat{\tau}_2)}v_1(X_{\sigma\wedge\hat{\tau}_2}) - \int_0^{\sigma\wedge\hat{\tau}_2} e^{-rt}(\mathbb{L}_X - r)v_1(X_t)dt\Big] \\ \le& \mathsf{E}_x\Big[e^{-r\sigma}G_1(X^x_\sigma)\mathbb{1}_{\{\sigma<\hat{\tau}_2\}} + e^{-r\hat{\tau}_2}L_1(X_{\hat{\tau}_2})\mathbb{1}_{\{\sigma\ge\hat{\tau}_2\}}\Big] = \mathcal{J}_1(\sigma, \hat{\tau}_2; x) \end{aligned} \tag{3.37}$$

and analogously $v_2(x) \le \mathcal{J}_2(\hat{\tau}_1, \sigma'; x)$ for another arbitrary σ' and $\hat{\tau}_1$ as in (3.28). If we now pick $\sigma = \hat{\tau}_1$ in (3.37) we obtain $v_1(x) = \mathcal{J}_1(\hat{\tau}_1, \hat{\tau}_2; x)$ whereas putting $\sigma' = \hat{\tau}_2$ we also get $v_2(x) = \mathcal{J}_2(\hat{\tau}_1, \hat{\tau}_2; x)$, and hence the Nash equilibrium condition. □

The next characterisation of the value functions v_1 and v_2 associated to a Nash equilibrium is probabilistic and follows from arguments similar to those in the proof of Proposition 3.5.

Corollary 3.6. *Let* $(\hat{\tau}_1, \hat{\tau}_2)$ *be as in Proposition 3.5 and* v_i, $i = 1, 2$ *the related values for the two players. For* $i, j = 1, 2$ *and* $i \ne j$ *set*

$$Y^i_t := e^{-rt}v_i(X_t), \quad G^i_t := e^{-rt}G_i(X_t), \quad L^i_t := e^{-rt}L_i(X_t), \quad t \ge 0 \tag{3.38}$$

then $(Y^i_{t\wedge\hat{\tau}_j})_{t\ge0}$ *is a continuous sub-martingale,* $(Y^i_{t\wedge\hat{\tau}_i\wedge\hat{\tau}_j})_{t\ge0}$ *is a continuous martingale,* $Y^i_t \le G^i_t$ *for all* $t \ge 0$ *and* $Y^i_{\hat{\tau}_i\wedge\hat{\tau}_j} = G^i_{\hat{\tau}_i}\mathbb{1}_{\{\hat{\tau}_i<\hat{\tau}_j\}} + L^i_{\hat{\tau}_j}\mathbb{1}_{\{\hat{\tau}_i\ge\hat{\tau}_j\}}$.

From Proposition 3.5 our nonzero-sum game may have multiple Nash equilibria, and so we now provide sufficient conditions under which the equilibrium of Theorem 3.4 is unique. First it is convenient to consider the auxiliary problem

$$\inf_{\tau\in\mathcal{T}} \mathsf{E}\big[e^{-r\tau}G_2(X^x_\tau)\big], \qquad x\in\mathcal{I}, \tag{3.39}$$

which corresponds to the optimal stopping problem for P_2 if P_1 decides not to stop at all. From standard theory and relying upon the geometry of $\hat{G}_2$ it is not hard to see that an optimal stopping time for (3.39) is

$$\tau_2^\infty := \inf\{t\ge 0\,:\, X^x_t \ge x_2^\infty\} \tag{3.40}$$

for some $x_2^\infty > \hat{x}_2$. In particular $y_2^\infty := F_r(x_2^\infty)$ can be determined as the unique $y>\hat{y}_2$ solving $\hat{G}_2'(y)y-\hat{G}_2(y)=0$. The latter is the tangency condition for a straight line passing through the origin and tangent to $\hat{G}_2$ at a point in $(\hat{y}_2,+\infty)$. If a solution to that equation exists then the convexity of $\hat{G}_2$ in $(\hat{y}_2,+\infty)$ implies that it must be unique. For existence it is sufficient to observe that

$$\hat{G}_2'(\hat{y}_2)\hat{y}_2 < \int_0^{\hat{y}_2}\hat{G}_2'(s)ds = \hat{G}_2(\hat{y}_2)$$

since $\hat{G}_2$ is strictly concave in $(0,\hat{y}_2)$. Recalling (3.23) we get $\lim_{y\to\infty}[\hat{G}_2'(y)y-\hat{G}_2(y)]=+\infty$ and therefore there exists a unique $y_2^\infty\in(\hat{y}_2,+\infty)$ corresponding to (3.40).

Now we also consider the auxiliary problem

$$\inf_{\tau\in\mathcal{T}} \mathsf{E}_x\Big[e^{-r\tau}G_1(X_\tau)\mathbb{1}_{\{\tau<\tau_2^\infty\}}+e^{-r\tau_2^\infty}L_1(x_2^\infty)\mathbb{1}_{\{\tau\ge\tau_2^\infty\}}\Big], \qquad x\in\mathcal{I}, \tag{3.41}$$

which corresponds to the optimal stopping problem P_1 is faced with when P_2 stops at τ_2^∞. Again an optimal stopping time for this problem is of the form $\tau_1^\infty := \inf\{t\ge 0\;\, X_t\le x_1^\infty\}$ with $y_1^\infty := F_r(x_1^\infty)\in(0,\hat{y}_1)$ obtained as the unique solution to $\mathcal{L}_1(\,\cdot\,,y_2^\infty)=0$ (see (3.13)). Since $\hat{G}_1'(0+)=-\infty$ there exists a unique such y_1^∞ by simple geometric considerations (see part 2 in the proof of Theorem 3.4).

We recall the definitions of $\widetilde{y}_i$, $i=1,2$, from Definition 3.2 and we are now ready to state our uniqueness result.

Theorem 3.7. *Let Assumption 3.3 hold and let $L_i\in\mathcal{A}_i$, $i=1,2$. Let us also assume that $\widetilde{y}_2>\hat{y}_1$ and that $\hat{G}_1'(y_1^\infty)<\hat{L}_1'(y_2^\infty)$. Then there exists a unique couple (x_1^*,x_2^*) with $\underline{x}<x_1^*<\hat{x}_1$ and $\hat{x}_2<x_2^*<\overline{x}$, such that (τ_1^*,τ_2^*) as in* (3.15) *constitutes a Nash equilibrium for the game. The couple (y_1^*,y_2^*) with $y_i^*:=F_r(x_i^*)$, $i=1,2$ is then the unique solution of the system* (3.16).

Proof. The main idea of the proof is to show that the maps $\zeta\mapsto y_i(\zeta)$ found in the proof of Theorem 3.4 are monotonic. We adopt the notation of the latter theorem and observe immediately that under the additional regularity assumptions on L_i we obtain $y_i(\,\cdot\,)\in C^1(\mathcal{O}_i)\cap C(\overline{\mathcal{O}}_i)$ with $i=1,2$ and $\mathcal{O}_1:=(\hat{y}_2,+\infty)$[3], $\mathcal{O}_2:=(0,\hat{y}_1)$. In fact denoting by $\partial_k\mathcal{L}_i$ the partial derivative of $\mathcal{L}_i$ with respect to the k-th variable $k=1,2$, the implicit function theorem gives

$$y_i'(\zeta) = -\frac{\partial_2\mathcal{L}_i}{\partial_1\mathcal{L}_i}(y_i(\zeta),\zeta) = \frac{\hat{G}_i'(y_i(\zeta))-\hat{L}_i'(\zeta)}{\hat{G}_i''(y_i(\zeta))(y_i(\zeta)-\zeta)}, \quad \zeta\in\mathcal{O}_i,\ i=1,2. \tag{3.42}$$

Since $L_2\in\mathcal{A}_2$, Lemma 3.1 implies that $\hat{G}_2''(y_2(\zeta))(y_2(\zeta)-\zeta)>0$ for $\zeta\in(0,\hat{y}_1)$ because $y_2(\zeta)>\hat{y}_2>\zeta$ and $\hat{G}_2$ is convex on $(\hat{y}_2,+\infty)$. By assumption $\widetilde{y}_2>\hat{y}_1$ and then $\hat{L}_2'(\zeta)>0$

[3] Here we take $\overline{\mathcal{O}}_1 := [\hat{y}_2,+\infty)$.

for $\zeta \in (0, \hat{y}_1)$; since $\hat{G}_2'(y_2(\zeta)) < 0$ as well then $y_2'(\zeta) < 0$ for $\zeta \in (0, \hat{y}_1)$. It follows that $y_2(\,\cdot\,)$ decreases monotonically on $(0, \hat{y}_1)$ and from (3.39) we find that its maximum value is $y_2(0) = y_2^\infty$. Therefore it is now sufficient to analyse $\zeta \mapsto y_1(\zeta)$ on the interval $\zeta \in (\hat{y}_2, y_2^\infty)$.

First we observe that since $y_1(\zeta) \in (0, \hat{y}_1)$ and $\zeta \in (\hat{y}_2, +\infty)$, item *i)* of Lemma 3.1 implies $\hat{G}_1''(y_1(\zeta))(y_1(\zeta) - \zeta) < 0$. The sign of $y_1'(\cdot)$ on $(\hat{y}_2, y_2^\infty)$ is then determined by the sign the numerator in (3.42). Note that τ_1^∞ provides the best reply of P_1 when P_2 stops at τ_2^∞ (see (3.41)) and therefore $y_1^\infty = y_1(y_2^\infty)$ determines the optimal boundary in that case. Since $\hat{G}_1'(y_1(y_2^\infty)) = \hat{G}_1'(y_1^\infty)$ and we are assuming $\hat{G}_1'(y_1^\infty) < \hat{L}_1'(y_2^\infty)$, then

$$y_1'(y_2^\infty) > 0. \tag{3.43}$$

Let us now study the sign of $\zeta \mapsto \hat{G}_1'(y_1(\zeta)) - \hat{L}_1'(\zeta)$ in $(\hat{y}_2, y_2^\infty)$. Assume that there exists $\zeta_1^o \in (\hat{y}_2, y_2^\infty)$ such that $\hat{G}_1'(y_1(\zeta_1^o)) - \hat{L}_1'(\zeta_1^o) = 0$, then by definition of $y_1(\cdot)$ this means that there is a straight line which is at the same time tangent to $\hat{L}_1$, at ζ_1^o, and to $\hat{G}_1$, at $y_1(\zeta_1^o)$. Since $\hat{L}_1$ is convex for $y < \check{y}_1$ and $\hat{L}_1 < \hat{G}_1$ it is easy to see that we must have $\zeta_1^o > \check{y}_1$. Now we claim that if such ζ_1^o exists, then it must be unique and

$$\hat{G}_1'(y_1(\zeta)) < \hat{L}_1'(\zeta) \quad \text{and} \quad y_1'(\zeta) > 0 \quad \text{for } \zeta \in (\hat{y}_2, \zeta_1^o) \tag{3.44}$$

$$\hat{G}_1'(y_1(\zeta)) > \hat{L}_1'(\zeta) \quad \text{and} \quad y_1'(\zeta) < 0 \quad \text{for } \zeta \in (\zeta_1^o, y_2^\infty). \tag{3.45}$$

The latter contradicts (3.43) so if we prove (3.44) and (3.45) it will follow that $y_1'(\cdot)$ is positive on $(\hat{y}_2, y_2^\infty)$.

To verify the claims we observe that

$$\frac{d}{d\zeta}\Big(\hat{G}_1'(y_1(\zeta)) - \hat{L}_1'(\zeta)\Big)\Big|_{\zeta=\zeta_1^0} = \Big(\hat{G}_1''(y_1(\zeta))y_1'(\zeta) - \hat{L}_1''(\zeta)\Big)\Big|_{\zeta=\zeta_1^0} = -\hat{L}_1''(\zeta_1^0) > 0, \tag{3.46}$$

where we have used that $y_1'(\zeta_1^0) = 0$ for the second equality, and the fact that $\zeta_1^o > \check{y}_1$ for the last inequality. Hence the function $\zeta \mapsto \hat{G}_1'(y_1(\zeta)) - \hat{L}_1'(\zeta)$ may only take zero values with strictly positive derivative and (3.44), (3.45) hold. We therefore conclude that $y_1(\,\cdot\,)$ is monotonic increasing in $(\hat{y}_2, y_2^\infty)$.

The map $(\xi, \zeta) \mapsto (y_1(\xi), y_2(\zeta))$ is a mapping $(\hat{y}_2, y_2^\infty) \times (0, \hat{y}_1) \mapsto (0, \hat{y}_1) \times (\hat{y}_2, y_2^\infty)$ and we know from Theorem 3.4 that the curves $y_1(\,\cdot\,)$ and $y_2(\,\cdot\,)$ must meet at least once in the rectangle $(0, \hat{y}_1) \times (\hat{y}_2, y_2^\infty)$. The monotonicity of both y_1 and y_2 obtained here implies that these can only meet once. Hence there exists a unique couple (y_1^*, y_2^*) solving $y_1^* = y_1(y_2(y_1^*))$ and $y_2^* = y_2(y_1(y_2^*))$ or equivalently (3.16). □

Remark 3.8. *The value of geometric analyses of optimal stopping problems, such as that involved in this section, has been demonstrated in a number of recent papers. While beyond the scope of the present analysis in which smooth fit plays a central role, it also applies in principle to non-smooth payoff functions (see, for example, [11], [29]).*

3.2 The case of an entrance boundary

In this section we assume that $\underline{x}$ is an entrance boundary and $\overline{x}$ is a natural boundary for X. This setting includes for example CIR and Bessel processes for suitable choices of their parameters (see for instance [20]). We will see that the methodology developed above may be extended to this setting with minimal additional requirements. For the fundamental solutions ϕ_r and ψ_r we have that (3.1), (3.2), (2.5) continue to hold if we replace

$$\lim_{x \downarrow \underline{x}} \psi_r(x) = 0 \quad \text{by} \quad \lim_{x \downarrow \underline{x}} \psi_r(x) > 0, \tag{3.47}$$

$$\lim_{x \downarrow \underline{x}} \frac{\phi_r'(x)}{S'(x)} = 0 \quad \text{by} \quad \lim_{x \downarrow \underline{x}} \frac{\phi_r'(x)}{S'(x)} > -\infty. \tag{3.48}$$

In this setting Lemma 3.1 holds in a slightly different form and some of the arguments require trivial changes. In particular functions in $\mathcal{A}_i$, $i=1,2$ may now have finite derivative at zero with either positive or negative sign and we have

Lemma 3.9. *Let $H \in C^2(\mathcal{I})$ with $h(x) := (\mathbb{L}_X - r)h(x)$ and $\hat{H}$ as in (3.4).*

i) If $H \in \mathcal{A}_1$ then for $\hat{y} := F_r(x_h)$, one has $\hat{H}$ convex on $[0,\hat{y})$, concave on $(\hat{y},\infty)$ with $\hat{H}(0+) = 0$, also $\lim_{y\to\infty}\hat{H}(y) = +\infty$ and H is monotonic increasing on $(\hat{y},+\infty)$; if in addition $\inf_{x\in\mathcal{I}} H(x) < 0$ then $\hat{H}'(0+) \in [-\infty, 0)$ and $\hat{H}$ has a unique global minimum at some $\overline{y} \in [0,\hat{y})$.

ii) If $H \in \mathcal{A}_2$ then, for $\hat{y} := F_r(x_h)$, one has $\hat{H}$ concave on $[0,\hat{y})$, convex on $(\hat{y},\infty)$ with $\hat{H}(0+) = 0$; moreover if $\hat{H}'(0+) > 0$ then $\hat{H}$ has a unique global maximum at some $\overline{y} \in [0,\hat{y})$ and $\lim_{y\to\infty}\hat{H}(y) = -\infty$; finally $\hat{H}$ is monotonic decreasing on $(\hat{y},+\infty)$.

Note that the proof of the asymptotic behaviour as $y \to +\infty$ follows exactly the same arguments as in Lemma 3.1 since the upper endpoint of $\mathcal{I}$ is again natural.

The main difference here compared to the previous section is that for $G_1 \in \mathcal{A}_1$ we could have $\hat{G}_1'(0+)$ negative but finite (recall also that $\inf_{x\in\mathcal{I}} G_1(x) < 0$ from Assumption 3.3). Because of this we can now find Nash equilibria which did not arise when both boundaries were natural:

Proposition 3.10. *Let Assumption 3.3 hold and let $\tau_2^\infty := \inf\{t \geq 0 : X_t \geq x_2^\infty\}$ be optimal for (3.39). Then $(+\infty, \tau_2^\infty)$ is a Nash equilibrium if and only if*

$$(0>)\,\hat{G}_1'(0+) \geq \frac{\hat{L}_1(y_2^\infty)}{y_2^\infty} \tag{3.49}$$

with $y_2^\infty = F_r(x_2^\infty)$.

Proof. 1. First we show sufficiency. Suppose (3.49) holds and let P_2 choose the stopping time τ_2^∞ which is optimal in problem (3.39), so that P_1 is faced with solving (3.41). Due to condition (3.49), the largest convex minorant W_1 of $\hat{G}_1$ on $[0, y_2^\infty]$ such that $W_1(y_2^\infty) = \hat{L}_1(y_2^\infty)$ is given by the straight line starting from the origin and passing through $(y_2^\infty, \hat{L}_1(y_2^\infty))$. Therefore due to strict convexity of $\hat{G}_1$ at zero, P_1's best reply to τ_2^∞ is the stopping time $\tau_1(\underline{x}) = \inf\{t \geq 0 : X_t = \underline{x}\} = +\infty$ a.s. (since the entrance boundary $\underline{x}$ is unattainable in finite time). Since τ_2^∞ is also P_2's best reply to $\tau_1(\underline{x})$ we have a Nash equilibrium.

2. We show necessity by contradiction. Suppose that $(+\infty, \tau_2^\infty)$ is a Nash equilibrium and that (3.49) does not hold. The latter implies that an optimal stopping time for (3.41) may be constructed (as in Appendix A.4) and is given by $\tau_1^\infty = \inf\{t \geq 0 : X_t^x \leq x_1^\infty\}$. Here $x_i^\infty = F_r^{-1}(y_i^\infty)$, $i=1,2$ and $y_i^\infty \in (0,\hat{y}_1)$ solves $\mathcal{L}_1(y_1^\infty, y_2^\infty) = 0$.

Therefore $\tau_1(\underline{x}) = +\infty$ a.s. and τ_1^∞ are both optimal for the optimal stopping problem (3.41) (although τ_1^∞ does not necessarily lead to a Nash equilibrium), i.e.

$$\begin{aligned} &\mathsf{E}_x\Big[e^{-r\tau_2^\infty} L_1(X_{\tau_2^\infty})\Big] \\ &= \mathsf{E}_x\Big[e^{-r\tau_1^\infty} G_1(X_{\tau_1^\infty})\mathbb{1}_{\{\tau_1^\infty < \tau_2^\infty\}} + e^{-r\tau_2^\infty} L_1(X_{\tau_2^\infty})\mathbb{1}_{\{\tau_1^\infty \geq \tau_2^\infty\}}\Big]. \end{aligned} \tag{3.50}$$

We now use well known analytic representation formulae (see for instance eqs. (4.3)–(4.6) in [11]) to express both sides above as

$$\hat{L}_1(y_2^\infty)\frac{y}{y_2^\infty} = \hat{G}_1(y_1^\infty)\left(\frac{y_2^\infty - y}{y_2^\infty - y_1^\infty}\right) + \hat{L}_1(y_2^\infty)\left(\frac{y - y_1^\infty}{y_2^\infty - y_1^\infty}\right), \qquad y \in (y_1^\infty, y_2^\infty) \tag{3.51}$$

where we have set $y = F_r^{-1}(x)$ everywhere. Simple algebra shows that (3.51) may be reduced to the equivalent equation

$$\frac{\hat{L}_1(y_2^\infty)}{y_2^\infty} = \frac{\hat{G}_1(y_1^\infty)}{y_1^\infty}. \tag{3.52}$$

Using the fact that $\mathcal{L}_1(y_1^\infty, y_2^\infty) = 0$ in (3.52) one easily gets $\hat{G}_1(y_1^\infty) = \hat{G}_1'(y_1^\infty)y_1^\infty$ which is in contradiction with the strict convexity of $\hat{G}_1$ on $(0, \hat{y}_1)$ and the proof is completed. □

If $\lim_{y\to\infty} \hat{L}_1(y) > -\infty$ then $\widehat{\mathcal{B}} := \{y > 0 : \hat{G}'(0+)y = \hat{L}_1(y)\} \neq \emptyset$ as the straight line $r(y) := \hat{G}'(0+)y$ will certainly cross $\hat{L}_1$ at least once. In a similar way we also introduce the subset $\mathcal{B} := \{y > \hat{y}_2 : \hat{G}'(0+)y = \hat{L}_1(y)\} \subset \widehat{\mathcal{B}}$ and notice that for any $y \in \mathcal{B}$ the function $\mathcal{L}_1(\,\cdot\,, y)$ cannot have a root in $(0, \hat{y}_1)$. Hence the tangency condition used in the proof of Theorem 3.4 to determine $y_1(\,\cdot\,)$ no longer applies at points of $\mathcal{B}$. This makes the proof of the existence of an equilibrium more technical in general and there may be cases where no equilibrium can be found. However Proposition 3.5 continues to hold, and for specific choices of L_i and G_i it may be possible to check the existence of equilibria by studying (3.16) either analytically or numerically.

In the next proposition we make an assumption which is sufficient to rule out such technicalities and allows us to extend results of the previous section to the present setting.

Proposition 3.11. *Let*

$$\hat{G}_1'(0+) < \frac{\hat{L}_1(y_2^\infty)}{y_2^\infty} \tag{3.53}$$

and assume $\lim_{y\to\infty} \hat{L}_1(y) > -\infty$ *and* $y_T \le \hat{y}_2$*, where*

$$y_T := \sup\{y > 0 \,:\, \hat{G}'(0+)y = \hat{L}_1(y)\}. \tag{3.54}$$

then Theorem 3.4 continues to hold when $\underline{x}$ *is an entrance boundary.*

Proof. Since $\hat{G}_1'(0+)y$ does not cross $\hat{L}_1$ for $y > y_T$, it must be that $\hat{L}_1(y) - \hat{G}_1'(0+)y$ is either strictly positive or strictly negative for $y > y_T$ but the latter is impossible otherwise (3.53) would be violated. Then we must have $\hat{L}_1(y) - \hat{G}_1'(0+)y > 0$ for $y > \hat{y}_2$ and hence for $\zeta > \hat{y}_2$, $\mathcal{L}_1(\,\cdot\,, \zeta)$ has a unique root $y_1(\zeta) \in (0, \hat{y}_1)$ by strict convexity of $\hat{G}_1$ and a simple geometric argument. In fact one can always construct a unique straight line passing through $(\zeta, \hat{L}_1(\zeta))$ and tangent to $\hat{G}_1$ at a point of $(0, \hat{y}_1)$.

Now steps 1 and 3 of the proof of Theorem 3.4 can be repeated in the same way and the results follow. □

Remark 3.12. *1. It is important to note that for the existence of an equilibrium we have never examined whether or not* $\hat{L}_2$ *and* $\hat{G}_2$ *have maxima, nor the particular values of* $\hat{G}_2'$ *and* $\hat{L}_2'$ *at zero (see the proof of Theorem 3.4). Instead we use these details to establish uniqueness of the equilibrium. In fact in the setting of the above proposition and under the additional assumption that* $\hat{L}_2'(0+) > 0$ *we find that Theorem 3.7 holds and we obtain uniqueness of the Nash equilibrium.*

2. Even though $\phi_r(x) \uparrow +\infty$ *as* $x \downarrow \underline{x}$*, when* $\underline{x}$ *is an entrance boundary condition* (2.7) *may become more restrictive. For instance for a Bessel process with index* $\nu = 1/2$ *(i.e. dimension* $\delta = 3$*) one has* $\phi_r(x) \sim 1/x$ *as* $x \to 0$ *(see [7] Appendix 2, pp. 638 and 654). In this case, in order to address asymptotically linear costs we may relax* (2.7) *for* G_1 *by requiring*

$$\lim_{x\downarrow\underline{x}} \frac{G}{\phi_r}(x) = A_{G_1} \in (-\infty, +\infty)$$

then all the above arguments can be easily adapted to recover the existence and then uniqueness of a Nash equilibrium. We omit further details here because in the next section we provide a complete analysis of a similar situation in the case when $\underline{x}$ is an exit boundary and (2.7) *becomes a serious restriction.*

3.3 The case of an exit boundary

Here we extend the analysis carried out in the previous two sections by addressing the case of a diffusion with a lower exit boundary $\underline{x}$ and an upper natural boundary $\overline{x}$. We sketch the proof, drawing out key differences with the previous arguments. Equations (3.1), (3.2) and (2.5) continue to hold if we replace

$$\lim_{x\downarrow\underline{x}}\phi_r(x)=+\infty \quad\text{by}\quad \lim_{x\downarrow\underline{x}}\phi_r(x)<+\infty, \tag{3.55}$$

$$\lim_{x\downarrow\underline{x}}\frac{\psi_r'(x)}{S'(x)}=0 \quad\text{by}\quad \lim_{x\downarrow\underline{x}}\frac{\psi_r'(x)}{S'(x)}>0. \tag{3.56}$$

We see that $\phi_r(\underline{x}+)$ is now finite so that imposing (2.7) on the cost functions requires them to vanish at $\underline{x}$. Hence from now on we shall change the definition of the set $\mathcal{A}$ by replacing the condition (2.7) by

$$\lim_{x\downarrow\underline{x}}\frac{H}{\phi_r}(x)=A_H \tag{3.57}$$

for some $A_H\in\mathbb{R}$ depending on H. For any $H\in\mathcal{A}$ Dynkin's formula, standard localisation and (2.4) give

$$H(x)=A_H\phi_r(x)-W^{-1}\Big[\phi_r(x)\int_{\underline{x}}^{x}\psi_r(y)h(y)m'(y)dy+\psi_r(x)\int_{x}^{\overline{x}}\phi_r(y)h(y)m'(y)dy\Big] \tag{3.58}$$

and for $(H/\phi_r)'(x)$ we have expressions analogous to (2.11). We remark that the limit as $x\to\underline{x}$ of (ϕ_r'/S') is again equal to $-\infty$ as in the natural boundary case. Hence one can prove as in (3.9)–(3.10) that

$$H\in\mathcal{A}_1\Rightarrow \hat{H}'(0+)=-\infty \quad\text{and}\quad H\in\mathcal{A}_2\Rightarrow \hat{H}'(0+)=+\infty. \tag{3.59}$$

Thanks to the latter observation one has that, under the new definition of $\mathcal{A}$, Lemma 3.1 holds for $\hat{G}_i$ and $\hat{L}_i$, $i=1,2$ in the same form with only the exception of the lower boundary conditions: now indeed we have $\hat{G}_i(0+)=A_{G_i}$ and $\hat{L}_i(0+)=A_{L_i}$. As one may expect the sign of A_{G_1} plays a crucial role in determining the existence of Nash equilibria and we study the two possible cases below.

Proposition 3.13. *If $A_{G_1}\le 0$ then Theorem 3.4 holds when $\underline{x}$ is an exit boundary.*

Proof. Condition (3.59) implies that the construction of an equilibrium follows as in the proof of Theorem 3.4 up to trivial adjustments. □

We now consider $A_{G_1}>0$. In this case there exists a unique straight line passing through the origin and tangent to $\hat{G}_1$ at a point $(y_S,\hat{G}_1(y_S))$ where $y_S>0$ is the unique solution of

$$\hat{G}_1(y_S)=y_S\hat{G}_1'(y_S). \tag{3.60}$$

Repeating arguments as in the proof of Proposition 3.10, up to straightforward modifications, we obtain a similar result:

Proposition 3.14. *Let $A_{G_1} > 0$ and let Assumption 3.3 hold with (3.57) in place of (2.7). Let also $\tau_2^\infty := \inf\{t \geq 0 \,:\, X_t \geq x_2^\infty\}$ be optimal for (3.39). Then $(+\infty, \tau_2^\infty)$ is a Nash equilibrium if and only if*

$$\hat{G}_1'(y_S) > \frac{\hat{L}_1(y_2^\infty)}{y_2^\infty} \tag{3.61}$$

with $y_2^\infty = F_r(x_2^\infty)$.

We now introduce $\hat{y}_T := \sup\{y \geq y_S,\ \hat{G}_1'(y_S)y = \hat{L}_1(y)\}$ which will play a similar role as y_T of the previous section. Before stating the next result we recall that P_2 will never stop optimally in $[0, \hat{x}_2)$ (see the discussion in the first paragraph of the proof of Theorem 3.4).

Proposition 3.15. *Assume that $A_{G_1} > 0$ and that*

$$\hat{G}_1'(y_S) \leq \frac{\hat{L}_1(y_2^\infty)}{y_2^\infty} \tag{3.62}$$

holds. Assume also $\hat{y}_T < \hat{y}_2$ and $\lim_{y\to\infty} \hat{L}_1(y) > -\infty$. Set $x_S := F_r^{-1}(y_S)$ and $\sigma_S := \inf\{t \geq 0 \,:\, X_t^x \geq x_S\} \wedge \sigma_{\mathcal{I}}$, then under Assumption 3.3 (with (3.57) in place of (2.7)) one has

a) if $x \in (\underline{x}, x_S]$ then the couple (σ_S, τ) is a Nash equilibrium for any $\tau = \inf\{t \geq 0 \,:\, X_t^x \geq z\}$, $z > \hat{x}_2$ (i.e. for any τ optimally chosen by P_2);

b) if $x > x_S$ then Theorem 3.4 continues to hold when $\underline{x}$ is an exit boundary.

Proof. The proof is a simple repetition of arguments employed several times above in this paper and therefore we will skip it here. The only difference here is that one should notice that for any $y_0 > \hat{y}_2$ the convex minorant W_1 of $\hat{G}_1$ passing through $(y_0, \hat{L}_1(y_0))$ has two straight portions: $i)$ the usual one connecting $\hat{L}_1(y_0)$ to $\hat{G}_1$ via the smooth-fit equation $\mathcal{L}_1(y_1(y_0), y_0) = 0$ and $ii)$ the straight line $r_S(y) := \hat{G}_1'(y_S)\, y$ for $y \in [0, y_S]$.

Hence $a)$ follows by observing that P_1 will always stop prior to P_2 (with probability one) at $\sigma_{\mathcal{I}} \wedge \sigma_S$ and $b)$ is obtained as in the proof of Theorem 3.4. □

As for the case of natural boundaries, once again we obtain uniqueness of the equilibrium under the additional assumptions of Theorem 3.7 both in the setting of the proposition above and for $A_{G_1} \leq 0$. We remark that for $A_{G_1} \leq 0$ Proposition 3.5 holds in the same form whereas for $A_{G_1} > 0$ it holds in a slightly more complex form. We provide a full statement for completeness but skip the proof as it is the same as the original one up to minor adjustments.

Proposition 3.16. *Let all the assumptions of Proposition 3.15 hold. Let $(y_1, y_2) \in (0, \hat{y}_1) \times (\hat{y}_2, +\infty)$ be a solution of (3.16) and for $x_i := F_r^{-1}(y_i)$, $i = 1, 2$ set*

$$\hat{\tau}_1 := \inf\{t \geq 0 \,:\, X_t^x \leq x_1\}, \quad \hat{\tau}_2 := \inf\{t \geq 0 \,:\, X_t^x \geq x_2\}. \tag{3.63}$$

Then for $x \in [x_S, x_2)$ the couple $(\hat{\tau}_1, \hat{\tau}_2)$ is a Nash equilibrium whereas for $x \in (\underline{x}, x_S)$ the couple (σ_S, τ) (with τ as in the statement of Prop. 3.15) is a Nash equilibrium.

Moreover the functions

$$v_1(x) := \begin{cases} p_1\psi_r(x), & \underline{x} < x < x_S, \\ G_1(x), & x_S \leq x \leq x_1, \\ m_1\psi_r(x) + q_1\phi_r(x) & x_1 < x < x_2, \\ L_1(x), & x \geq x_2, \end{cases} \tag{3.64}$$

and

$$v_2(x) := \begin{cases} p_2\psi_r(x), & \underline{x} < x < x_S, \\ L_2(x), & x_S \le x \le x_1, \\ m_2\psi_r(x) + q_2\phi_r(x), & x_1 < x < x_2 \\ G_2(x), & x \ge x_2, \end{cases} \tag{3.65}$$

with m_i, q_i, $i = 1,2$ as in Proposition 3.5, $p_1 := G_1(x_S)/\psi_r(x_S)$ and $p_2 := L_2(x_S)/\psi_r(x_S)$ are the value functions of the two players, i.e. $v_i(x) = \mathcal{J}_i(\tau_1, \tau_2; x)$, $i = 1,2$. Also $v_1 \in C(\mathcal{I})$ with $v_1 \in W^{2,\infty}_{loc}(\underline{x}, x_2)$ and $v_2 \in C(\mathcal{I})$ with $v_2 \in W^{2,\infty}_{loc}(x_1, \overline{x})$.

A Appendix

A.1 Convexity of $\hat{H}$

We show here that $\hat{H}$ of (3.4) is convex at $y > 0$ if and only if $(\mathbb{L}_X - r)H(x) > 0$ at $x = F_r^{-1}(y)$. We simply work out explicitly calculations indicated by [11, Sec. 6]. For $y = F_r(x)$ it is obvious that $\hat{H}'(y) = g(x)$ with $g(x) := (H/\phi_r)'(x)/F_r'(x)$ so that $\hat{H}''(y) = g'(x)/F_r'(x)$. Since F_r is strictly increasing, we only need to evaluate $g'(x)$. This can be easily done by observing that

$$F_r'(x) = \frac{(\psi_r'\phi_r - \psi_r\phi_r')(x)}{(\phi_r)^2(x)} = W\frac{S'(x)}{(\phi_r)^2(x)} \quad \text{and} \quad g(x) = \frac{(H'\phi_r - H\phi_r')(x)}{W\,S'(x)}$$

from which we get

$$g'(x) = \frac{\phi_r(x)(S'H'' - S''H')(x)}{W\,(S')^2(x)} - \frac{H(x)(S'\phi_r'' - S''\phi_r')(x)}{W\,(S')^2(x)}.$$

Now we use that $S''(x) = -2\mu(x)S'(x)/\sigma^2(x)$ to obtain

$$g'(x) = \frac{2}{W\,\sigma^2(x)(S')(x)}\Big[\phi_r(x)\mathbb{L}_X H(x) - H(x)\mathbb{L}_X\phi_r(x)\Big] = \frac{2\phi_r(x)}{W\,\sigma^2(x)(S')(x)}(\mathbb{L}_X H - rH)(x),$$

where in the last equality we have used that $\mathbb{L}_X\phi_r = r\phi_r$. The last expression proves the claim and we remark that the result holds even if $r = r(x)$ is state dependent.

A.2 Some remarks on state dependent discounting

Here we illustrate the case of a state dependent discount rate $(r(X_t))_{t\ge0}$. In this setting the cost functionals (1.1) become:

$$\mathcal{J}_i(\tau_1,\tau_2;x) := \mathsf{E}_x\Big[e^{-\int_0^{\tau_i} r(X_t)dt}G_i(X_{\tau_i})\mathbb{1}_{\{\tau_i<\tau_j\}} + e^{-\int_0^{\tau_j} r(X_t)dt}L_i(X_{\tau_j})\mathbb{1}_{\{\tau_i\ge\tau_j\}}\Big], \quad i = 1,2,\ j \ne i. \tag{A-1}$$

In order to extend the methodology applied above, we make sufficient assumptions on r to ensure the existence of strictly monotonic and strictly positive fundamental solutions ϕ_r, ψ_r to the ODE

$$\tfrac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) - r(x)f(x) = 0, \quad x \in \mathcal{I}. \tag{A-2}$$

In particular we assume that $r(x) = r_0 + r_1(x)$ for $x \in \mathcal{I}$, with $r_0 > 0$ and $r_1(\cdot) \ge 0$, bounded from above and continuous in $\mathcal{I}$. In this case we again have

$$\mathsf{E}_x\big[e^{-\int_0^{\tau(y)} r(X_t)dt}\big] = \begin{cases} \frac{\psi_r(x)}{\psi_r(y)}, & x < y, \\ \frac{\phi_r(x)}{\phi_r(y)}, & x > y, \end{cases} \tag{A-3}$$

for $x, y \in \mathcal{I}$ and $\tau(y) := \inf\{t \geq 0 : X_t = y\}$ (see [10], Prop. 2.1). The limits at the endpoints of the domain $\mathcal{I}$ of functions ϕ_r, ψ_r, ϕ_r'/S' and ψ_r'/S' remain the same as in the previous sections, depending on whether $\underline{x}$ is natural, entrance or exit. Instead of the expressions(2.5) we must now consider their generalisation (see par. 9 and 10, Ch. 2 of [7])

$$\frac{\psi_r'(b)}{S'(b)} - \frac{\psi_r'(a)}{S'(a)} = \int_a^b r(y)\psi_r(y)m'(y)dy, \qquad \frac{\phi_r'(b)}{S'(b)} - \frac{\phi_r'(a)}{S'(a)} = \int_a^b r(y)\phi_r(y)m'(y)dy, \tag{A-4}$$

for $\underline{x} < a < b < \overline{x}$.

It is then easy to see that all the arguments that we have used for the construction of Nash equilibria in the above sections can be repeated for state dependent discounting and all the results carry over to this setting with no additional difficulties. In particular one should notice that positivity and boundedness of $r(\cdot)$ allow us to find bounds similar to those that led to some of our key inequalities (e.g. (3.7) and (3.10)); for example, setting $\overline{r} := \sup_{z\in\mathcal{I}} r(z)$ the last term in (3.9) can be bounded from below as follows

$$\int_x^{x_h-\delta} \phi_r(z)m'(z)dz \geq \frac{1}{\overline{r}}\int_x^{x_h-\delta} r(z)\phi_r(z)m'(z)dz = \frac{1}{\overline{r}}\Big(\frac{\phi_r'(x_h-\delta)}{S'(x_h-\delta)} - \frac{\phi_r'(x)}{S'(x)}\Big)$$

and the rest of the proof follows in the same way also with state dependent discounting.

We also remark that the argument used to infer convexity and concavity of the transformed functions $\hat{H}$ in Lemma 3.1 and 3.9 hold in the same form, i.e. $\hat{H}(y)$ is strictly convex if and only if $\frac{1}{2}\sigma^2(x)H''(x) + \mu(x)H'(x) - r(x)H(x) > 0$ with $y = F_r(x)$.

A.3 Two Useful Optimal Stopping Problems

In this section we study two optimal stopping problems in which a stopper aims at choosing an $\mathbb{F}$-stopping time minimising a given expected reward. These optimal stopping problems are the building blocks of our construction of Nash equilbria for the nonzero-sum optimal stopping game (2.6) (see Theorem 3.4). In contrast to [11] we characterise the value functions as the largest convex minorants of the reward functions with no conditions on their sign as in, for example, [27].

A.3.1 A First Optimal Stopping Problem

Recall Definition 2.3 and consider a function $G \in \mathcal{A}_2$. Denote by $\hat{x} \in \mathcal{I}$ the unique point at which $\mathbb{L}_X G - rG$ changes its sign and take $x_o \in \mathcal{I}$ with $x_o < \hat{x}$. Let us introduce the infinite time horizon optimal stopping problem with value function

$$V_o(x) := \inf_{\tau\in\mathcal{T}} \mathsf{E}_x\Big[e^{-r\tau}G(X_\tau)\mathbb{1}_{\{\tau<\tau_o\}} + \vartheta e^{-r\tau_o}\mathbb{1}_{\{\tau\geq\tau_o\}}\Big], \tag{A-5}$$

where $\vartheta < G(x_o)$ and $\tau_o := \inf\{t \geq 0 : X_t^x \leq x_o\}$.

Set $y_o := F_r(x_o)$, with $F_r(\cdot)$ as in (3.3), and define the function

$$Q(y) := \begin{cases} \frac{\vartheta}{\phi_r(y)}, & 0 < y \leq y_o, \\ \hat{G}(y), & y > y_o, \end{cases} \tag{A-6}$$

with $\hat{G}(y) := \big(G/\phi_r\big) \circ F_r^{-1}(y)$. For $\hat{y} := F_r(\hat{x})$ we argue as in Section A.1 of this appendix and obtain that $\hat{G}$ is strictly concave in $[0, \hat{y})$ and strictly convex in $(\hat{y}, \infty)$. Let us then assume that there is a straight line $r_o(\cdot)$ which passes through the point $(y_o, Q(y_o))$ and is tangent to Q at a point $y_* > \hat{y}$ (this can be easily proven but we do not need it here and we leave it to the reader). This line is expressed as

$$r_o(y) = my + q, \quad y > 0, \tag{A-7}$$

with

$$\begin{cases} m := \frac{Q(y_*)-Q(y_o)}{y_*-y_o}, \\ q := Q(y_o) - m y_o. \end{cases} \tag{A-8}$$

By the convexity of $\hat{G}$ (and therefore of Q) in $(\hat{y}, +\infty)$ the point y_* is determined as the unique $y > \hat{y}$ that solves the tangency equation

$$\frac{Q(y) - Q(y_o)}{y - y_o} = Q'(y). \tag{A-9}$$

Proposition A.1. *Let $G \in \mathcal{A}_2$. Assume there exists $y_* > \hat{y}$ solving* (A-9) *(which is then unique). Recall* (A-7), (A-8), *define $x_* := F_r^{-1}(y_*)$ and the functions*

$$W(y) := \begin{cases} \frac{\vartheta}{\phi_r(y)}, & 0 < y \le y_o \\ my + q, & y_o < y \le y_* \\ \hat{G}(y), & y \ge y_*, \end{cases} \tag{A-10}$$

and

$$\widetilde{V}_o(x) := \phi_r(x) W(F_r(x)) = \begin{cases} \vartheta, & \underline{x} < x \le x_o \\ m\psi_r(x) + q\phi_r(x), & x_o < x \le x_* \\ G(x), & x_* \le x < \overline{x}. \end{cases} \tag{A-11}$$

Then one has $\widetilde{V}_o \equiv V_o$ and $\tau_* := \inf\{t \ge 0 : X_t^x \ge x_*\}$ is optimal for problem (A-5).

Proof. If $x \le x_o$ there is clearly nothing to prove. Therefore, take $x > x_o$ and notice by (A-11) that $(\mathbb{L}_X - r)\widetilde{V}_o(x) = 0$ if $x \in (x_o, x_*)$. Moreover, by Section A.1 we also have that $(\mathbb{L}_X - r)\widetilde{V}_o(x) \ge 0$ if $x \in (x_*, \overline{x})$, since $y_* > \hat{y}$ and $\hat{G}$ is convex in $(\hat{y}, \infty)$. Also, by construction, $\widetilde{V}_o(x_*) = G(x_*)$, $\widetilde{V}_o^{'}(x_*) = G^{'}(x_*)$, $\widetilde{V}_o(x_o) = \vartheta$ and $\widetilde{V}_o \le G$, for any $x > x_o$. Since $\widetilde{V}_o \in W^{2,\infty}_{\text{loc}}((x_0, \overline{x}))$ we can apply Itô-Tanaka's formula to the process $(e^{-rt}\widetilde{V}_o(X_t^x))_{t\ge 0}$ on the time interval $[0, \tau \wedge \tau_o]$, for arbitrary $\tau \in \mathcal{T}$, and obtain

$$\widetilde{V}_o(x) \le \mathsf{E}_x\Big[e^{-r\tau\wedge\tau_o}\widetilde{V}_o(X_{\tau\wedge\tau_o})\Big] \le \mathsf{E}_x\Big[e^{-r\tau}G(X_\tau)\mathbb{1}_{\{\tau<\tau_o\}} + \vartheta e^{-r\tau_o}\mathbb{1}_{\{\tau\ge\tau_o\}}\Big]$$

and hence $\widetilde{V}_o \le V_o$. Then repeating the argument with $\tau = \tau_* := \inf\{t \ge 0 : X_t^x \ge x_*\}$ we find $\widetilde{V}_o(x) = \mathsf{E}_x\Big[e^{-r\tau_*}G(X_{\tau_*})\mathbb{1}_{\{\tau_*<\tau_o\}} + \vartheta e^{-r\tau_o}\mathbb{1}_{\{\tau_*\ge\tau_o\}}\Big]$ and therefore $\widetilde{V}_o = V_o$ and τ_* is optimal. □

A.4 A Second Optimal Stopping Problem

We take the same setup as in section A.3.1, with the modifications that $G \in \mathcal{A}_1$, $x_o > \hat{x}$, $\tau_o := \inf\{t \ge 0 : X_t^x \ge x_o\}$ and

$$Q(y) := \begin{cases} \hat{G}(y), & 0 < y < y_o, \\ \frac{\vartheta}{\phi_r(y)}, & y \ge y_o, \end{cases} \tag{A-12}$$

and taking y_* as the unique $y \in (0, \hat{y})$ (if it exists) solving (A-9). Arguments completely symmetric to those developed in the previous section then allow us to prove the following proposition.

Proposition A.2. *Let $G \in \mathcal{A}_1$. Assume there exists $y_* < \hat{y}$ solving* (A-9) *(which is then unique). Then, for $x_* := F_r^{-1}(y_*)$ the function*

$$\widetilde{V}_o(x) := \begin{cases} G(x), & \underline{x} < x < x_*, \\ m\psi_r(x) + q\phi_r(x), & x_* < x < x_o, \\ \vartheta, & x_o \le x < \overline{x}, \end{cases} \tag{A-13}$$

is such that $\widetilde{V}_o \equiv V_o$ and the stopping time $\tau_* := \inf\{t \ge 0 : X_t^x \le x_*\}$ is optimal for problem (A-5).

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