Semigroup methods for spatial birth-and-death processes

Dissertation
zur
Erlangung des Doktorgrades (Dr. math.)
der
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der
Universität Bielefeld

vorgelegt von
Martin Friesen

1. Betreuer: Prof. Dr. Yu. G. Kondratiev
2. Betreuer: Dr. O. Kutoviy

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Contents

Introduction

The theory of interacting particle systems is a fast growing area in modern probability and infinite dimensional analysis with various applications in, e.g., mathematical physics, theoretical biology, ecology, social sciences and economy. The aim is to describe the time evolution of a huge collection of interacting entities. Such entities are called (microscopic) particles and are considered, depending on the particular choice of model, as molecules, cells , plants or animals , humans and agents of a market. The collection of all particles, which is typically of order $10^4 - 10^{23}$, is called microscopic state. Each particle from this state can, in principle, be described by a physical/ecological/biological mechanism. A detailed understanding of such mechanism yields the possibility to describe the time evolution of the microscopic state by solutions to certain systems of equations. Nevertheless, the complex structure of each particle makes it practically impossible to determine all parameters involved. Moreover, due to the huge number of particles it is hopeless to solve or even provide reasonable simulations for such large systems of equations. As a simplification each particle is therefore modelled as a random process. The parameters of such processes should be chosen in such a way that they fit with the experimental data. Moreover, the huge number of particles is described by statistical properties such as expectations, correlations and particle densities. A mathematical realization of above ideas leads, in the simplest case, to the description of a microscopic state in terms of a Markov process.

In this thesis we study certain classes of particle systems in the framework of Markov processes and are mainly focused on their statistical description. The methods used in this work are at present already well-developed but still leave several challenging problems unsolved. The aim of this thesis is to broaden the collection of available techniques used for the analysis of interacting particle systems. We provide a complete and self-contained approach by semigroup methods which includes an extension to time-inhomogeneous Markov processes. This methods are used for the construction of so-called birth-and-death processes, where each particle from the microscopic state may randomly disappear and new particles may randomly appear.

Classical birth-and-death dynamics are described by a system of ordinary differential equations, also known as Kolmogorov's differential equations, and are usually studied by semigroup methods on (weighted) spaces of summable real-valued sequences, cf. Feller, Kato [Kat54, Fel68, Fel71, HP74]. More recent attempts study such equations on the spaces ℓ^p for $p \in [1, \infty)$, see Arlotti, Banasiak [BA06] and others [BLM06, TV06]. In contrast to many real world models, see e.g. the Bolker, Dieckmann, Law, Pacala model [BP97, BP99, DL00, DL05] (short BDLP), such equations do not include the positions of the described particles. Other models coming from ecology and the modelling of mutations can be found in [Neu01, BCF$^+$14, KM66, SEW05, FFH$^+$15] and references therein.

The simplest possibility to include spatial structure is to assign to each particle a fixed site of a graph (e.g. from the lattice $\mathbb{Z}^d$). This are the so-called lattice models. For such models a rigorous study by semigroup methods is adequate and a detailed presentation can be found in the classical book of Liggett [Lig05] and references therein. Several models, such as the BDLP model, require that the positions of the particles are not a priori fixed. This means that $\mathbb{Z}^d$ should be replaced by a continuous location space, e.g. $\mathbb{R}^d$.

For the modelling of birth-and-death processes in continuum the theory of pure point processes is commonly used. Such processes share several properties with the processes associated to lattice models, but also include numerous unexpected features and require essentially different techniques for their mathematical treatment. Taking into account that they describe real-world particles it leads to the natural assumption that all particles are indistinguishable and any two particles cannot occupy the same position in the location space, say for simplicity $\mathbb{R}^d$. A microscopic state γ is then, by definition, a linear combination of point-masses δ_x, where $x \in \mathbb{R}^d$ is the position of a particle in the system. Here we encounter two different cases which, as we shall see later on, have to be treated by different techniques. A microscopic state which is given by a finite linear combination of point masses is called finite state, see chapter 2. Microscopic states being linear combinations of point masses δ_x with infinitely many different positions $x \in \mathbb{R}^d$ are called infinite states and are considered in the chapters three and four. The Markov dynamics of finite states can be analysed by a measure-valued generalization of Kolmogorov's differential equations. This equations have been first analysed by Feller [Fel40] and have been afterwards further investigated in the next 60 years, cf. [FMS14] and many others. A summary with applications to interacting particle systems is provided in the book of Chen [Che04]. For the considerations in this thesis it is reasonable to identify γ with a subset of $\mathbb{R}^d$, i.e. we consider the microscopic state as a (finite or infinite) collection of positions $x \in \mathbb{R}^d$.

Stochastic birth-and-death processes in continuum are heuristically described by a Markov (pre-)generator on a proper set of functions F. The general form of such operator (in the one-component case) is given by the heuristic expression

$$(LF)(\gamma) = \sum_{x \in \gamma} d(x, \gamma \backslash x)(F(\gamma \backslash x) - F(\gamma)) + \int_{\mathbb{R}^d} b(x, \gamma)(F(\gamma \cup x) - F(\gamma))\mathrm{d}x, \qquad (1)$$

where $\gamma \in E$. The state space (= configuration space) E is assumed to be either the space of all finite configurations

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d \mid |\eta| < \infty\}$$

or the space of all locally finite configurations

$$\Gamma = \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \quad \text{for all compacts } K \subset \mathbb{R}^d\}.$$

Here and in the following we write $|A|$ for the number of elements in $A \subset \mathbb{R}^d$. For simplicity of notation we write $\gamma \backslash x$, $\gamma \cup x$ instead of $\gamma \cup \{x\}$ and $\gamma \backslash \{x\}$, respectively. The birth-and-death Markov process $(X_t)_{t \geq 0} \subset E$ associated to the operator L, provided it exists, therefore consists of two elementary events. Namely, the death of particles ($\gamma \longmapsto \gamma \backslash x$) and birth of particles ($\gamma \longmapsto \gamma \cup x$). The death intensity $d(x, \gamma) \geq 0$ determines the probability that a point $x \in \gamma$ disappears from the configuration γ. The birth intensity $b(x, \gamma) \geq 0$ determines the probability for a new point $x \in \mathbb{R}^d$ to appear. In general both intensities depend on the present microscopic state γ of the process.

Solutions to the (backward) Kolmogorov equation on functions $F : E \longrightarrow \mathbb{R}$

$$\frac{\partial F_t}{\partial t} = LF_t, \quad F_t|_{t=0} = F_0 \tag{2}$$

are related to the Markov process $(X_t)_{t \geq 0}$ by

$$F_t(\gamma) = \mathbb{E}_\gamma(F_0(X_t)), \quad \gamma \in E, \ t \geq 0,$$

where $\mathbb{E}_\gamma$ denotes the expectation w.r.t. the probability measure $\mathbb{P}_\gamma$ for which $\mathbb{P}_\gamma(X_0 = \gamma) = 1$ holds. Here we encounter a fundamental difference in the theory of finite birth-and-death systems ($E = \Gamma_0$) and infinite systems ($E = \Gamma$). For finite systems equation (2) can be solved in spaces of continuous bounded functions, see Kolokoltsov [Kol06] and chapter 2 of this thesis. Hence we are able to construct a birth-and-death Markov process starting from any initial point $\eta \in \Gamma_0$. However, for infinite systems we cannot expect to solve equation (2) in any space of continuous functions and hence obtain a process for any initial configuration $\gamma \in \Gamma$, cf. Kondratiev, Skorokhod [KS06]. Note that any stochastic process having càdlàg paths, is necessarily contained in a proper subspace of Γ.

The adjoint Cauchy problem

$$\frac{\partial \mu_t}{\partial t} = L^* \mu_t, \quad \mu_t|_{t=0} = \mu_0 \tag{3}$$

is known as the (forward) Kolmogorov equation and describes the distribution of the process X_t. Because of the Markovian property of the operator L we expect that solutions to (3) can be constructed in the class of probability measures on E. In the physical literature, (3) is referred to the Fokker-Planck equation and probability measures μ on E are called states of the system. Functions $F : E \longrightarrow \mathbb{R}$ are hence called observables and expectations

$$\langle F, \mu_t \rangle := \int\limits_E F(\gamma) \mathrm{d}\mu_t(\gamma)$$

are considered as measurable quantities of the particle system.

Dynamics of finite systems

The study of birth-and-death processes with state space $E = \Gamma_0$ has been initiated by Preston [Pre75]. In particular, it was shown that under some conditions the processes are temporally ergodic. Later on the problem for convergence to equilibrium was studied by Lotwick, Silvermann and Møller [LS81, Møl89]. A necessary condition for the existence of a process with state space Γ_0 is given by

$$q(\eta) := \sum_{x\in\eta} d(x, \eta\backslash x) + \int_{\mathbb{R}^d} b(x, \eta)\mathrm{d}x < \infty. \tag{4}$$

The first term describes the intensity that a particle from the configuration η dies, whereas the integral in the second term is the intensity for the birth of a new particle. The value $q(\eta)$ is the cumulative intensity of the process in the state $\eta \in \Gamma_0$. The corresponding transition function is given by

$$Q(\eta, A) = \sum_{x\in\eta} d(x, \eta\backslash x)\mathbb{1}_A(\eta\backslash x) + \int_{\mathbb{R}^d} b(x, \eta)\mathbb{1}_A(\eta \cup x)\mathrm{d}x,$$

where $\mathbb{1}_A(\eta) := \begin{cases} 1, & \eta \in A \\ 0, & \eta \notin A \end{cases}$ and the operator given by (1) can be rewritten to

$$(LF)(\eta) = \int_{\Gamma_0} (F(\xi) - F(\eta))Q(\eta, \mathrm{d}\xi), \quad \eta \in \Gamma_0. \tag{5}$$

Hence the process described by the operator L is a pure jump Markov process and techniques coming from the theory of Markov chains are applicable. Such approach has been investigated in the last 20 years, a comprehensive summary of the obtained results can be found in [Che04]. Such processes can be also obtained as unique solutions to certain stochastic equations, cf. Bezborodov [Bez15a, Bez15b].

More recent problems in the theory of birth-and-death processes on Γ_0 deal with various scaling limits. In this thesis we only consider the so-called mean-field limit, for which the particles in the limiting description are distributed according to a Poisson measure. The mean-field limit is also known as the mesoscopic limit and can be obtained by various kinds of scalings, e.g. Vlasov and Lebowitz-Penrose to mention the most common ones. The limiting equation, or kinetic equation,

$$\frac{\partial \rho_t}{\partial t}(x) = v(\rho_t)(x), \quad \rho_t|_{t=0} = \rho_0, \quad x \in \mathbb{R}^d$$

is then (in general) a non-linear integro-differential equation for the approximate density of the particle system. The solution operator associated to above equation, provided it exists, preserves in many cases positivity. Hence it may determine a non-linear Markov process, cf. Kolokoltsov [Kol07, Kol10, Kol13]. The BDLP model given by $d(x, \eta\backslash x) = m + \sum\limits_{y \in \eta\backslash x} a^-(x-y)$ and $b(x,\eta) = \sum\limits_{y\in\eta} a^+(x-y)$ yields the kinetic equation

$$\frac{\partial \rho_t}{\partial t}(x) = -m\rho_t(x) - \rho_t(x) \int\limits_{\mathbb{R}^d} a^-(x-y)\rho_t(y)\mathrm{d}y + \int\limits_{\mathbb{R}^d} a^+(x-y)\rho_t(y)\mathrm{d}y.$$

Here $m > 0$ is the mortality rate, $0 \leq a^- \in L^1(\mathbb{R}^d)$ the competition and $0 \leq a^+ \in L^1(\mathbb{R}^d)$ the dispersion kernel, see chapter 3. A detailed analysis of such type of equations can be found in [FKT15], see also the references therein.

Let $k, r \in \mathbb{N}$, $\Gamma_0^{(\leq k)} = \{\eta \subset \mathbb{R}^d \mid |\eta| \leq k\}$ and $K : \Gamma_0 \times \mathcal{B}(\Gamma_0^{(\leq k)}) \longrightarrow \mathbb{R}_+$ a transition kernel with $K(\eta, \Gamma_0^{(\leq k)}) < \infty$ for all $\eta \in \Gamma_0$. Eibeck, Wagner [EW01, EW03] discussed existence, uniqueness and in particular the mesoscopic scaling for the (pre-)generator given by

$$(LF)(\eta) = \sum_{\xi\subset\eta} \mathbb{1}_{|\xi|\leq r}(\xi) \int\limits_{\Gamma_0^{(\leq k)}} (F(\eta\backslash\xi \cup \zeta) - F(\eta))K(\xi, \mathrm{d}\zeta). \tag{6}$$

In the corresponding dynamics each group $\xi \subset \eta$ of at most r particles may disappear and simultaneously a new group of at most k particles $\zeta \in \Gamma_0^{(\leq k)}$ appear somewhere in $\mathbb{R}^d$. The distribution of the new particles and the intensity of this event are both described by the transition kernel $K(\xi, \mathrm{d}\zeta)$. The term $r = 0$ corresponds to the pure birth of a finite group of particles $\zeta \in \Gamma_0^{(\leq k)}$ whereas the part $k = 0$ corresponds to the death of the subgroup of particles $\xi \in \eta$. All other terms describe merging, splitting or jumps of groups of particles. Considering above generator only for the cases $k = 0$ and $r = 0$ yields

$$(LF)(\eta) = \sum_{\xi\subset\eta} \mathbb{1}_{|\xi|\leq r}(\xi)K(\xi,\varnothing)(F(\eta\backslash\xi) - F(\eta)) + \int\limits_{\Gamma_0^{(\leq k)}} (F(\eta\cup\zeta) - F(\eta))K(\varnothing, \mathrm{d}\zeta).$$

In contrast to (1) such operator does not include the BDLP model.

This problem was also studied by Belavkin, Kolokoltsov [BK03] and the research continued in the series of works [Kol03, Kol04a, Kol04b, Kol04c, Kol06] leading to satisfactory results for the mesoscopic scaling described by Markov (pre-)generators given by

$$(LF)(\eta) = \sum_{\xi\subset\eta} \mathbb{1}_{|\xi|\leq r}(\xi) \int\limits_{\Gamma_0^{(\leq k)}} (F(\eta\backslash\xi \cup \zeta) - F(\eta))K(\xi, \eta, \mathrm{d}\zeta). \tag{7}$$

It is worth to stress the important difference to (6) which lies in the appearance of the additional dependence on η in the transition function $K(\xi, \eta, \mathrm{d}\xi)$. Such dependence includes a wide class of interacting particle systems which could not be considered by (6).

Particular examples such as kinetic equations of statistical mechanics, namely the one of Landau and Vlasov but also the Boltzmann and Smoluchovski equations have been considered in [Kol06]. The Replicator dynamics from the theory of evolutionary games is also discussed. The BDLP model was treated by stochastic differential equations in [FM04].

Dynamics of infinite systems

The construction of birth-and-death processes with state space Γ and associated Markov (pre-)generator (1) is challenging task of modern probability and is only partially solved at present. One of the main difficulties lies in the necessity to control the number of particles in a bounded region of $\mathbb{R}^d$. Markov processes on Γ given by the operator L as in (1) have, in general, infinite intensity in the sense that (4) is not fulfilled. Hence the representation (5) is no longer valid and most of the developed techniques for birth-and-death processes on Γ_0 are not applicable in this case.

A pure probabilistic approach by stochastic differential equations has been developed by Garcia, Kurtz [GK06]. Namely, for $d(x, \gamma\backslash x) = 1$ and a birth intensity with

$$|b(x, \gamma \cup y) - b(x, \gamma)| \leq a(x, y), \quad x, y \in \mathbb{R}^d$$

such that a satisfies some additional continuity condition, existence and uniqueness has been established and under additional conditions ergodicity for the processes was shown. Unfortunately several models from mathematical biology and ecology, see eg. [FFH$^+$15, KK16], do not satisfy these conditions.

A different, functional analytic, approach to the construction of the processes is related to the construction of solutions to either (2) or (3), respectively. Trying to solve (2) one immediately arrives at serious obstacles. The reason is that any known perturbation theory for such operators is not applicable and (in general) any two different states on Γ are orthogonal. It was proposed to investigate instead the statistical dynamics, i.e. the Fokker-Planck equation (3), on the space of so-called sub-Poissonian probability measures, cf. [KK02, FKO09, KKM08, FKK10, FKO13]. The notion of correlation functions turned out to be adequate for the analysis of (3). Therefore most modern results obtained for infinite systems are mainly based on the study of correlation functions. One possible definition of a correlation function is given below, details can be found in chapters three and four.

A probability measure (state) μ on Γ is said to have correlation function $k_\mu = (k_\mu^{(n)})_{n=0}^\infty$ if for any symmetric bounded function $G^{(n)} : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ with compact support the relation

$$\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} G^{(n)}(x_1, \dots, x_n) \mathrm{d}\mu(\gamma) = \frac{1}{n!} \int_{\mathbb{R}^{dn}} G^{(n)}(x_1, \dots, x_n) k_\mu^{(n)}(x_1, \dots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{8}$$

is satisfied. The value $k_\mu^{(n)}(x_1,\dots,x_n)$ describes the (non-normalized) probability density to find particles in the positions $x_1,\dots,x_n$. Setting

$$k_\mu(\eta) := \begin{cases} k_\mu^{(n)}(x_1,\dots,x_n), & \eta = \{x_1,\dots,x_n\},\ |\eta| = n \\ 0, & \text{otherwise} \end{cases}$$

yields a measurable function $k_\mu : \Gamma_0 \longrightarrow \mathbb{R}_+$. Conversely, any measurable function $G : \Gamma_0 \longrightarrow \mathbb{R}$ can be decomposed into its components $G^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}$, $n \geq 0$, where $G^{(n)}$ is symmetric and measurable. Denote by $B_{bs}(\Gamma_0)$ the collection of bounded functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ such that there exist a compact $\Lambda \subset \mathbb{R}^d$, $N \in \mathbb{N}$ with $G(\eta) = 0$ whenever $|\eta| > N$ or $\eta \cap \Lambda^c \neq \varnothing$. This space can be identified with the collection of all finite sequences of bounded, measurable, symmetric functions $(G^{(n)})_{n=0}^N$ having compact support in $(\mathbb{R}^d)^n$. Definition (8) suggests to consider for any $G \in B_{bs}(\Gamma_0)$ the K-transform given by

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta) = \sum_{n=0}^{\infty} \sum_{\{x_1,\dots,x_n\} \subset \gamma} G^{(n)}(x_1,\dots,x_n), \quad \gamma \in \Gamma,$$

where $\Subset$ means that the summation is taken only over all finite subsets of γ. Such functions $F = KG$ are known as additive type observables in statistical mechanics, see [Bog62], and we call functions G in such a case quasi-observables. The function KG then satisfies

$$(KG)(\gamma) = (KG)(\gamma \cap \Lambda)$$

and $|(KG)(\gamma)| \leq A(1+|\gamma\cap\Lambda|)^N$ for some constant $A = A(G) > 0$, i.e. it is a polynomially bounded cylinder function. Denote by $\mathcal{FP}(\Gamma) := KB_{bs}(\Gamma_0)$ the image of the K-transform, then for any $F \in \mathcal{FP}(\Gamma)$

$$(K^{-1}F)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta\backslash\xi|} F(\xi)$$

is the inverse transformation to K. The Lebesgue-Poisson measure on Γ_0 is defined by

$$\lambda = \delta_\varnothing + \sum_{n=1}^{\infty} \frac{1}{n!} \mathrm{d}^{(n)}x,$$

where $\mathrm{d}^{(n)}x$ is the restriction of the Lebesgue measure to $\Gamma_0^{(n)} = \{\eta \in \Gamma_0 \mid |\eta| = n\}$, see chapter 3. Taking the sum from $n = 0$ to ∞ in (8) yields the equivalent definition of a correlation function

$$\int_\Gamma KG(\gamma)\mathrm{d}\mu(\gamma) = \int_{\Gamma_0} G(\eta)k_\mu(\eta)\mathrm{d}\lambda(\eta), \quad G \in B_{bs}(\Gamma_0).$$

By definition, the correlation function $k_\mu : \Gamma_0 \longrightarrow \mathbb{R}_+$ satisfies $k_\mu(\varnothing) = 1$ and is positive definite in the sense that for any $G \in B_{bs}(\Gamma_0)$ such that $KG \geq 0$ one has

$$\int_{\Gamma_0} G(\eta)k_\mu(\eta)\mathrm{d}\lambda(\eta) \geq 0.$$

The correlation function k_μ is said to be sub-Poissonian (with bound β) if it satisfies for some constant $C(\mu) > 0$ the Ruelle bound

$$k_\mu(\eta) \leq C(\mu)e^{\beta|\eta|}, \quad \eta \in \Gamma_0.$$

A state μ on Γ is said to be sub-Poissonian (with bound β) if it has a correlation function k_μ satisfying above Ruelle bound for β. A function $k : \Gamma_0 \longrightarrow \mathbb{R}_+$ satisfying the Ruelle bound is the correlation function of a unique measure μ on Γ if and only if k is positive definite and satisfies the normalisation condition $k(\varnothing) = 1$.

Suppose that the states μ_t in (3) are sub-Poissonian, then using relation (8) we can rewrite (3) into an initial value problem for correlation functions given by

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t, \quad k_t|_{t=0} = k_0, \tag{9}$$

see [FKO09, FKO13]. The operator L^Δ can be computed from L and we expect that solutions to (9) are positive definite and hence provide an evolution of states. Since any function $k : \Gamma_0 \longrightarrow \mathbb{R}$ can be decomposed into its symmetric components $(k^{(n)})_{n=0}^\infty$, the initial value problem (9) is simply a system of function-valued differential equations. Surprisingly, it is possible to apply for such equations different kinds of perturbation methods to study existence, uniqueness and properties of solutions. The Ruelle bound suggests to study equation (9) on a weighted space of bounded functions, but for technical reasons it is simpler to study first the "pre-dual" equation on the space of integrable quasi-observables. Below we give a brief description of this scheme, details can be found in chapter 3.

Let $\widehat{L} := K^{-1}LK$ be defined on $B_{bs}(\Gamma_0)$. For $G \in B_{bs}(\Gamma_0)$ and any function k which satisfies the Ruelle bound also

$$\int\limits_{\Gamma_0} (\widehat{L}G)(\eta)k(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0} G(\eta)(L^\Delta k)(\eta)\mathrm{d}\lambda(\eta)$$

holds. Therefore solutions to

$$\frac{\partial G_t}{\partial t} = \widehat{L}G_t, \quad G_t|_{t=0} = G_0 \tag{10}$$

provide by duality solutions to (9). Using perturbation theory for analytic semigroups Finkelshtein, Kondratiev, Kutoviy [FKK12] constructed solutions to (10) and hence to (9) for general birth-and-death intensities. The mesoscopic limit was also studied by semigroup methods. Particular examples of above approach can be found in [FKKZ14, FKKO15, FKKK15, FFH+15, KK16], see also the references therein. Ergodicity has been established for the Glauber dynamics in [KKM10], whereas ergodicity for the equilibrium Glauber process was studied in [KL05]. A solution to (9) in general does not need to be positive definite and hence provide a solution to (3). For such property additional analysis is required and was only achieved for a few models, see e.g. [KKP08, KKM08, KK16].

Description of results

Evolution equations in scales of Banach spaces

Solving evolution equations given by an unbounded (linear) operator $A(t)$ is an important but also challenging task of applied mathematics. For many models of interacting particle systems, e.g. birth-and-death processes on Γ, the operator $A(t)$ can be realized as a bounded linear operator on a suitable chosen scale of Banach spaces $\mathbb{B} = (\mathbb{B}_\alpha)_\alpha$ with $\mathbb{B}_{\alpha'} \subset \mathbb{B}_\alpha$ for $\alpha' < \alpha$. Namely, for all $\alpha' < \alpha$ and $t \geq 0$ we have $A(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$, see (1.2) for the precise definition. Here $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ denotes the space of all bounded linear operators from $\mathbb{B}_{\alpha'}$ to $\mathbb{B}_\alpha$. The aim of the first chapter is to develope methods for the study of the related evolution equations by means of semigroup techniques but also beyond such.

The first section deals with evolution equations associated to an operator $A(t)$ acting as a bounded linear operator in a scale of Banach spaces. We introduce the notion of forward and backward evolution systems in scales of Banach spaces and relate them with their "generator" through solutions to the forward equation

$$\frac{\partial}{\partial t}u(t) = A(t)u(t), \quad u(s) = u_s, \quad t \in [s, \infty)$$

and backward equation

$$\frac{\partial}{\partial s}v(s) = -A(s)v(s), \quad v(t) = v_t, \quad s \in [0, t],$$

respectively. It turns out that above equations are well-posed (in a scale of Banach spaces) if and only if there exists a forward and backward evolution system with generator $A(t)$. The assumption that $A(t)$ is a bounded linear operator in a scale of Banach spaces is sufficient to guarantee that the associated forward and backward evolution systems are continuous in the uniform operator topology on $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$, whenever $\alpha' < \alpha$. In particular, any strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A acting as a bounded linear operator in the scale of Banach spaces $\mathbb{B}$, is continuous in the uniform operator topology on $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ for all $\alpha' < \alpha$. Thus instead of working with unbounded operators in one Banach space, one can try to realize the involved operators as bounded linear operators in a suitable chosen scale of Banach spaces and use the methods developed in the first chapter. In such a case one does not need to take care of the domain of $A(t)$, which is in general hard to handle.

Based on the methods from [Paz83, Kat70, Kat73] we provide a sufficient condition for above equations to be well-posed. The construction of the evolution systems is based on an approximation by piecewise constant operators $A_n(t)$. Similar constructions in triples of Banach spaces can be found in [Kol13, KPA88].

Stability of the solutions, that is continuous dependence on initial conditions and on the generator $A(t)$, is proved and existence and uniqueness for the adjoint equations is investigated. The obtained results share some similarities with those provided in [Cap02].

The last two sections go beyond semigroup methods and study linear and non-linear perturbations of above equations by an operator $B(t)$ on $\mathbb{B}$. The linear case is treated in the second section. In such a case it is assumed that $B(t)$ is an Ovcyannikov-type operator, by which we mean that there exists $M > 0$ such that for all $\alpha' < \alpha$ and $t \geq 0$: $B(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ and its operator norm satisfies

$$\|B(t)\|_{\alpha'\alpha} \leq \frac{M}{\alpha - \alpha'}.$$

Rxistence, uniqueness and stability of (local) solutions is proved for the perturbed forward equation

$$\frac{\partial}{\partial t}u(t) = (A(t) + B(t))u(t), \quad u(s) = u_s, \ t \geq s$$

and backward equation

$$\frac{\partial}{\partial s}v(s) = -(A(s) + B(s))v(s), \quad v(t) = v_t, \ s \leq t$$

with initial conditions $u_s, v_t \in \mathbb{B}_{\alpha'}$. A time-independent version of this result can be found in [Fin15]. Applications to birth-and-death processes are considered e.g. in [FK13, BKKK13, KK16], whereas applications to partial differential equations with $A(t) = 0$ are well-studied and can be found in [Nir72, Nis77, Zab89, Tiğ08, Tığ11].

The striking point in the analysis of such equations is that the obtained solutions cannot be localized in one fixed Banach space. More precisely for any $\alpha > \alpha'$ we can find $T(\alpha', \alpha) > 0$ such that $u(t) \in \mathbb{B}_\alpha$ for any $t \in [s, T(\alpha', \alpha))$. Such property resembles some sort of worsening, see [Liu91], and is one of the main reasons why we cannot expect that methods by semigroups on Banach spaces are applicable. Nevertheless, it is possible to derive a criterion for which global solutions exist, i.e. $T(\alpha', \alpha)$ is unbounded in α. Having this in mind we prove a comparison principle on Banach lattices which can be used to prove for a certain class of birth-and-death models existence of global solutions with above mentioned worsening property. Such approach was used for the BDLP model in [KK16].

The generalization of above statements to the non-linear equation

$$\frac{\partial}{\partial t}u(t) = A(t)u(t) + B(t, u(t)), \quad u(0) = u_0$$

is considered in the last section. Here $B(t, u)$ is a time-dependent non-linear operator acting as a continuous operator in a scale of Banach spaces. Existence and uniqueness was established by Safonov [Saf95] in the case $A(t) = 0$. Stability of the solution wrt. $A(t)$ and $B(t, u)$ is proved in the last section of the first chapter and can be used for the Vlasov scaling, see "Epistatic mutation selection model" in chapter 3.

Dynamics of finite systems

The aim of the second chapter is to provide a complete and self-contained analysis of (one and two-component) birth-and-death processes with state space Γ_0 or Γ_0^2, respectively. We extend known results to time-dependent (pre-)generators given in the one-component case, see (7), by

$$(L(t)F)(\eta) = \sum_{\xi \subset \eta} \int_{\Gamma_0} (F(\eta \backslash \xi \cup \zeta) - F(\eta)) K_t(\xi, \eta, \mathrm{d}\zeta). \tag{11}$$

The transition function

$$Q(t, \eta, A) = \sum_{\xi \subset \eta} \int_{\Gamma_0} \mathbb{1}_A(\eta \backslash \xi \cup \zeta) K_t(\xi, \eta, \mathrm{d}\zeta), \quad t \geq 0, \ \eta \in \Gamma_0$$

yields for above operator a similar form as in (5) and hence several well-known results for pure jump processes and Markov chains can be applied.

Based on the classical works [Fel40, GS75, FMS14] we study in the first section the evolution system $U(s,t)$ on an abstract locally compact Polish space E associated to the (pre-)generator of a pure Markov jump process

$$(L(t)F)(x) = \int_E (F(y) - F(x)) Q(t, x, \mathrm{d}y), \quad x \in E, \ t \geq 0.$$

The operator $L(t)$ is assumed to satisfy a Foster-Lyapunov type condition, see [MT93]. We assume that the transition function $Q(t, x, \mathrm{d}y)$ is weakly continuous and satisfies some additional technical conditions. It is shown that in such a case there exists an associated conservative Feller evolution system $U(s,t)$ on the space of continuous bounded functions. Hence by [Cas11] $U(s,t)$ is associated to a (minimal) Hunt process with state space E. For a countable state space E such result was obtained by martingale techniques in [ZZ87]. We show that $U(s,t)$ provides existence and uniqueness of solutions to the Kolmogorov equations and establish the relation to the jump process by the associated Martingale problem.

Above results are applied in the second section to the operator $L(t)$ given by (11) and to several examples from ecology, i.e. the BDLP and Dieckmann-Law model with time-dependent and non-translation invariant interaction kernels. The adjoint evolution system $U^*(t,s)$ provides by $U^*(t,s)\mu$ for a certain class of initial states μ on Γ_0 an evolution of states and hence weak solutions to

$$\frac{\partial}{\partial t} \int_{\Gamma_0} F(\eta) \mu_t(\mathrm{d}\eta) = \int_{\Gamma_0} L(t) F(\eta) \mu_t(\mathrm{d}\eta),$$

where $F : \Gamma_0 \longrightarrow \mathbb{R}$ is continuous and bounded. Sufficient conditions for which $L^1(\Gamma_0, \mathrm{d}\lambda)$ is invariant for $U^*(t,s)$ are given. By construction, the restriction $U^*(t,s)|_{L^1(\Gamma_0,\mathrm{d}\lambda)}$ becomes strongly continuous.

Afterwards we turn to the analysis of the time-homogeneous case and provide an equivalent construction for the associated evolution of states. Such evolution is given by a strongly continuous semigroup on the space of finite Borel measures and is used to characterise the conservativeness property.

The second part of this chapter deals with the precise relation between states on Γ_0 and their correlation measures. Given a state μ on Γ_0, the correlation measure ρ_μ on Γ_0 is defined by

$$\rho_\mu(A) := \int\limits_{\Gamma_0} (K\mathbb{1}_A)(\eta)\mathrm{d}\mu(\eta).$$

In such a case we can rewrite the Fokker-Planck equation (3) to an equation for correlation measures

$$\frac{\partial \rho_t}{\partial t} = \widetilde{L}^\Delta \rho_t, \quad \rho_t|_{t=0} = \rho_0, \tag{12}$$

where $\widetilde{L}^\Delta$ can be formally constructed by the duality

$$\int\limits_{\Gamma_0} \widehat{L}G(\eta)\rho(\mathrm{d}\eta) = \int\limits_{\Gamma_0} G(\eta)\widetilde{L}^\Delta\rho(\mathrm{d}\eta).$$

If μ is given by a density function R, that is of the form $\mu(\mathrm{d}\eta) = R(\eta)\mathrm{d}\lambda(\eta)$. Then $\rho_\mu(\eta) = k_\mu(\eta)\mathrm{d}\lambda(\eta)$ and k_μ is the correlation function for the measure μ. In such a case the simple identity

$$k_\mu(\eta) = \int\limits_{\Gamma_0} R(\eta \cup \xi)\mathrm{d}\lambda(\xi)$$

holds. It will be shown that the evolution system $U(s,t)$ associated to the operator $L(t)$ provides solutions to

$$\frac{\partial G_t}{\partial t} = \widehat{L}(t)G_t, \quad G_t|_{t=0} = G_0$$

and by duality an evolution of correlation measures, i.e. a weak solution to (12).

The last part of the second chapter is devoted to the converse statement. Here we consider only the time-homogeneous case with operator L given by (1). We construct a semigroup $T^\Delta(t)$ associated to the operator L^Δ on a weighted space of integrable correlation functions. The most important step is to show that $T^\Delta(t)$ preserves positive definiteness, i.e. let k_μ be the correlation function of some state μ on Γ_0. We will show that $T^\Delta(t)k_\mu$ is again the correlation function of a state μ_t on Γ_0.

Dynamics of infinite systems

In the last two chapters of this thesis we develope semigroup methods for one and two-component birth-and-death processes on Γ and Γ^2, respectively. Below we describe for simplicity only the one-component case with generator L given by (1). Let π_β be the Poisson measure on Γ, which is defined as the unique measure having Laplace transform

$$\int\limits_{\Gamma} e^{\sum\limits_{x\in\gamma} f(x)} \mathrm{d}\pi_\beta(\gamma) = \exp\left(e^\beta \int\limits_{\mathbb{R}^d} (e^{f(x)} - 1)\mathrm{d}x \right)$$

for every continuous function f with compact support. Solutions to (2) are constructed on the Banach space $\mathcal{E}_\beta$ of functions F for which the series

$$F(\gamma) = \sum_{\eta \Subset \gamma} G(\eta) = (KG)(\gamma)$$

converges π_β-a.e.. This is the same as to demand that

$$\|F\|_{\mathcal{E}_\beta} := |G^{(0)}| + \sum_{n=1}^{\infty} \frac{e^{\beta n}}{n!} \int\limits_{\mathbb{R}^{dn}} |G^{(n)}(x_1, \dots, x_n)|\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int\limits_{\Gamma_0} |G(\eta)| e^{\beta|\eta|} \mathrm{d}\lambda(\eta)$$

is finite. Let $\mathcal{L}_\beta := L^1(\Gamma_0, e^{\beta|\cdot|}\mathrm{d}\lambda)$, it is shown that $(L, \mathcal{FP}(\Gamma))$ is a (pre-)generator on $\mathcal{E}_\beta$ if and only if $(\widehat{L}, B_{bs}(\Gamma_0))$ is a (pre-)generator on $\mathcal{L}_\beta$. Using similar methods to [FKK12], we are able to show that under some type of Lyapunov condition the latter operator is in fact a (pre-)generator on $\mathcal{L}_\beta$. As a consequence, the closure of $(L, \mathcal{FP}(\Gamma))$ is the generator of an analytic semigroup of contractions. Let $T(t)$ be the associated semigroup on $\mathcal{E}_\beta$ and $\widehat{T}(t)$ the semigroup on $\mathcal{L}_\beta$. Properties such as stability w.r.t. initial conditions F_0 in (2) and continuous dependence on the intensities $d(x, \gamma\backslash x)$ and $b(x, \gamma)$ are studied by standard semigroup methods. Further analysis is concerned with the construction and the properties of an associated evolution of states.

The space $\mathcal{E}_\beta$ is chosen in such a way that its dual space can be identified with the space of all sub-Poissonian functions, i.e. any functional $\ell : \mathcal{E}_\beta \longrightarrow \mathbb{R}$ is represented by a sub-Poissonian function k_ℓ via

$$\begin{aligned}\ell(F) &= \int\limits_{\Gamma_0} G(\eta) k_\ell(\eta) \mathrm{d}\lambda(\eta) \\ &= G^{(0)} k_\ell^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{\mathbb{R}^{dn}} G^{(n)}(x_1, \dots, x_n) k_\ell^{(n)}(x_1, \dots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n\end{aligned}$$

Solutions to (9) are then given by the action of the adjoint semigroup $\widehat{T}(t)^* k_\ell =: k_t$ and hence satisfy the Ruelle bound for some constant $C_t > 0$

$$k_t^{(n)}(x_1, \dots, x_n) \le C_t e^{\beta n}, \quad n \ge 0.$$

The adjoint semigroup $T(t)^*$ on $\mathcal{E}_\beta^*$ satisfies for any $KG = F \in \mathcal{E}_\beta$ the relation

$$T(t)^*\ell(F) = \int_{\Gamma_0} \widehat{T}(t)G(\eta)k_\ell(\eta)\mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\eta)k_t(\eta)\mathrm{d}\lambda(\eta)$$
$$= G^{(0)}k_\ell^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{dn}} G^{(n)}(x_1, \dots, x_n)k_t^{(n)}(x_1, \dots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Suppose k_ℓ is positive definite and hence corresponds to a state μ_0 on Γ. Then we are able to prove that also k_t is positive definite. This yields the existence and uniqueness of a solution $(\mu_t)_{t\geq 0}$ to the Fokker-Planck equation (3). Following the approach proposed in [KKM08] we see that for any initial state μ_0 with sub-Poissonian correlation function k_0 there exists a Markov function associated to the operator L. It is worth to mention, that we establish uniqueness to (3) in the class of weak solutions. It is shown that if the initial condition is regular enough, then $(\mu_t)_{t\geq 0}$ is in fact a strong solution and strong uniqueness holds.

Vlasov scaling is shown for one and two-component systems. The kinetic equation for the approximate densities is, by construction, a system of two coupled non-linear integro-differential equations. Using the results obtained in the first chapter we are able to extend above results to the case of time-dependent intensities. In such a case the associated evolution systems will act as bounded linear operators in a suitable chosen scale of Banach spaces.

For ergodicity we suppose that the cumulative death intensity $\sum\limits_{x\in\eta} d(x, \eta\backslash x)$ is bounded away from zero on $\Gamma_0\backslash\{\emptyset\}$. Under this condition we prove the existence of a unique invariant measure μ_{inv} such that the corresponding evolution of states is ergodic with exponential rate, i.e.

$$\|\mu_t - \mu_{\mathrm{inv}}\|_{\mathcal{E}_\beta^*} \leq Ce^{-\varepsilon t}\|\mu_0 - \mu_{\mathrm{inv}}\|_{\mathcal{E}_\beta^*}, \quad t \geq 0$$

holds for some constants $C, \varepsilon > 0$ and any (admissible) initial state μ_0.

Examples for the modelling of tumour growth are considered in the end of each chapter. In the one-component case (chapter 3) we consider first a model describing the (free) proliferation of tumour cells. We will construct the evolution of correlation functions and show that they are not sub-Poissonian. This model is exactly solvable and serves as a guiding example for further investigation. In the remaining parts we apply above results to the BDLP model and Glauber dynamics with time-dependent and space inhomogeneous potentials. We study also the Dieckmann-Law model with time-homogeneous intensities and provide for all such models ergodicity. The Epistatic mutation selection model is one particular example of a model with non-linear Kolmogorov operator L. We consider a generalization for time-dependent intensities and construct a (local) evolution of correlation functions. Several two-component interacting birth-and-death models are considered in the end of the last chapter.

Weak-coupling limits

The last section is devoted to a particular case of the so-called random evolution framework, cf. [Pin91, SHS02]. In such a framework one is typically interested in the description of a stochastic process, usually referred as the system, in the presence of another stochastic process. The latter process is seen as the driving process for the system and can be interpreted as the environment influencing the system. For the realization of above scheme we consider a two-component birth-and-death process with (pre-)generator $L = L^S + L^E$. In this work we consider two different cases.

In the first case we suppose that the system is given by a Markov process with state space Γ_0. Its generator L^S is assumed to be given by the heuristic form

$$(L^S F)(\gamma,\eta) = \sum_{\xi\subset\eta}\int_{\Gamma_0}(F(\gamma,\eta\backslash\xi\cup\zeta) - F(\gamma,\eta))K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\eta),$$

where $K : \Gamma\times\Gamma_0\times\Gamma_0\times\Gamma_0 \longrightarrow \mathbb{R}_+$ is measurable and integrable in ζ. The environment is, e.g., the equilibrium diffusion process with generator L^E and invariant Gibbs measure μ_{inv}. In such a case the operator L^E is symmetric on $L^2(\Gamma, \mathrm{d}\mu_{\mathrm{inv}})$ and there exists an associated ergodic Markov semigroup, cf. [AKR98a, AKR98b]. Solutions to the Fokker-Planck equation

$$\frac{\partial\rho_t}{\partial t} = (L^S)^*\rho_t + L^E\rho_t, \quad \rho_t|_{t=0} = \rho_0$$

on $L^1(\Gamma\times\Gamma_0, \mathrm{d}(\mu_{\mathrm{inv}}\otimes\lambda))$ describe the evolution of densities of the coupled particle system. Suppose that $K(\gamma,\xi,\eta,\zeta)$ is for any ξ,η integrable in (γ,ζ) w.r.t. $\mu_{\mathrm{inv}}\otimes\lambda$. The weak-coupling limit, in probability theory also known as averaging, is obtained from solutions to the scaled Fokker-Planck equation

$$\frac{\partial\rho_t^\varepsilon}{\partial t} = (L^S)^*\rho_t^\varepsilon + \frac{1}{\varepsilon}L^E\rho_t^\varepsilon, \quad \rho_t^\varepsilon|_{t=0} = \rho_0 \in L^1(\Gamma_0, \mathrm{d}\lambda)$$

by taking the limit $\rho_t^\varepsilon \longrightarrow \overline{\rho}_t$, $\varepsilon\to 0$. We show that such limit exists and $\overline{\rho}_t$ solves

$$\frac{\partial\overline{\rho}_t}{\partial t} = \overline{L}^*\overline{\rho}_t, \quad \overline{\rho}_t|_{t=0} = \rho_0$$

on $L^1(\Gamma_0, \mathrm{d}\lambda)$, where $\overline{L}$ is obtained from (7) with $r = \infty$ and $K(\xi,\eta,\zeta)$ replaced by

$$\overline{K}(\xi,\eta,\zeta) := \int_{\Gamma} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\mu_{\mathrm{inv}}(\gamma).$$

The mathematical realization of this scheme mainly relies on the results obtained in the second chapter. It is shown for the BDLP model how this abstract statement can can be applied.

The second case is devoted the extension of above scheme to infinite systems. We suppose that L^E is given by (1), i.e. independent of γ^+ and only acts on the variable γ^-. Likewise we assume that the operator L^S is given by

$$\begin{aligned}(L^S F)(\gamma) = &\sum_{x\in\gamma^+} d^S(x,\gamma^+\backslash x,\gamma^-)(F(\gamma^+\backslash x,\gamma^-) - F(\gamma))\\ &+ \int_{\mathbb{R}^d} b^S(x,\gamma^+,\gamma^-)(F(\gamma^+\cup x,\gamma^-) - F(\gamma))\mathrm{d}x\end{aligned}$$

and hence only acts on the variable γ^+. We are interested in the limit $\varepsilon \to 0$ of solutions to the Fokker-Planck equation

$$\frac{\partial}{\partial t}\int_{\Gamma^2} F(\gamma)\mathrm{d}\mu_t^\varepsilon(\gamma) = \int_{\Gamma^2}\left(L^S F(\gamma) + \frac{1}{\varepsilon}L^E F(\gamma)\right)\mathrm{d}\mu_t^\varepsilon(\gamma), \quad F \in \mathcal{FP}(\Gamma^2) \tag{13}$$

with $\gamma = (\gamma^+,\gamma^-) \in \Gamma^2$ and initial condition μ_0 having sub-Poissonian correlation function. Suppose that L^E and L^S satisfy the conditions for which an evolution of states has been constructed in the third chapter. Moreover, assume that $\sum\limits_{x\in\eta^-} d^E(x,\eta^-\backslash x)$ is bounded away from zero on $\Gamma_0\backslash\{\varnothing\}$. The environment process is then ergodic with exponential rate. Let μ_{inv} be its invariant measure and define averaged intensities by

$$\begin{aligned}\overline{d}(x,\gamma^+) &:= \int_\Gamma d^S(x,\gamma^+,\gamma^-)\mathrm{d}\mu_{\mathrm{inv}}(\gamma^-)\\ \overline{b}(x,\gamma^+) &:= \int_\Gamma b^S(x,\gamma^+,\gamma^-)\mathrm{d}\mu_{\mathrm{inv}}(\gamma^-).\end{aligned}$$

For such averaged intensities we define a new Markov (pre-)generator given by

$$(\overline{L}F)(\gamma^+) := \sum_{x\in\gamma^+}\overline{d}(x,\gamma^+\backslash x)(F(\gamma^+\backslash x) - F(\gamma^+)) + \int_{\mathbb{R}^d}\overline{b}(x,\gamma^+)(F(\gamma^+\cup x) - F(\gamma^+))\mathrm{d}x$$

This (pre-)generator acts on functions $F \in \mathcal{FP}(\Gamma)$. Let μ_0 be any initial state with sub-Poissonian correlation function and denote by μ_t^ε the solution to (13). Moreover, let μ_0^+ be the marginal of μ_0 onto its first component and $\overline{\mu}_t$ the solution to the Fokker-Planck equation associated to $\overline{L}$ with initial state μ_0^+. We will show that for any $F \in \mathcal{FP}(\Gamma)$

$$\int_{\Gamma^2} F(\gamma^+)\mathrm{d}\mu_t^\varepsilon(\gamma^+,\gamma^-) \longrightarrow \int_\Gamma F(\gamma^+)\mathrm{d}\overline{\mu}_t(\gamma^+), \quad \varepsilon \to 0$$

holds.

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Chapter 1

Evolution equations in scales of Banach spaces

Let $\mathbb{B} = (\mathbb{B}_\alpha, \|\cdot\|_\alpha)_{\alpha>\alpha_*}$ be a scale of Banach spaces, that is for any $\alpha', \alpha > \alpha_*$: $\alpha' < \alpha$

$$\mathbb{B}_{\alpha'} \subset \mathbb{B}_\alpha, \quad \|\cdot\|_\alpha \leq \|\cdot\|_{\alpha'}. \tag{1.1}$$

Denote by $i_{\alpha'\alpha} \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ the corresponding embedding operator. Here and in the following $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ stands for the space of all bounded linear operators from $\mathbb{B}_{\alpha'}$ to $\mathbb{B}_\alpha$. We let $x = y$, $x \in \mathbb{B}_{\alpha'}$, $y \in \mathbb{B}_\alpha$ stand for $i_{\alpha'\alpha}x = y$. A bounded linear operator L in the scale $\mathbb{B}$ is, by definition, a collection of bounded linear operators from $\mathbb{B}_{\alpha'}$ to $\mathbb{B}_\alpha$, i.e. $L = (L_{\alpha'\alpha})_{\alpha'<\alpha} \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$, satisfying for $\alpha' < \alpha < \alpha''$

$$L_{\alpha'\alpha''} = i_{\alpha\alpha''} L_{\alpha'\alpha} = L_{\alpha\alpha''} i_{\alpha'\alpha}. \tag{1.2}$$

By $L \in L(\mathbb{B})$ we indicate that L is a bounded linear operator in the scale $\mathbb{B}$. Let $(L_n)_{n\in\mathbb{N}} \subset L(\mathbb{B})$ be a sequence of operators in the scale $\mathbb{B}$ and $L \in L(\mathbb{B})$. We say that L_n converges to L in the strong topology if for all $\alpha' < \alpha$ and all $x \in \mathbb{B}_{\alpha'}$

$$(L_n)_{\alpha'\alpha} x \longrightarrow L_{\alpha'\alpha} x, \quad n \to \infty$$

holds. The sequence converges, by definition, in the uniform topology if $L_n \longrightarrow L$, $n \to \infty$ holds for any $\alpha' < \alpha$ in the uniform operator topology on $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$. A family of bounded linear operators $(L(t))_{t\geq 0} \subset L(\mathbb{B})$ is said to be strongly continuous if $t \longmapsto L(t)$ is continuous in the strong topology. We say that $(L(t))_{t\geq 0}$ is continuous (w.r.t. the uniform topology) if $t \longmapsto L(t)$ is continuous in the uniform topology on $L(\mathbb{B})$. For two operators $L, K \in L(\mathbb{B})$ the composition $LK \in L(\mathbb{B})$ is defined by

$$(LK)_{\alpha'\alpha} := L_{\beta\alpha} K_{\alpha'\beta}, \tag{1.3}$$

where $\beta \in (\alpha', \alpha)$. It is worth noting that definition (1.3) does not depend on β, see (1.2). In the following we omit the subscripts $\alpha'\alpha$ when no confusion can arise. For any two families of operators $(L(t))_{t\geq 0}, (K(t))_{t\geq 0} \subset L(\mathbb{B})$ the product $(L(t)K(t))_{t\geq 0}$ is (strongly) continuous, provided both factors are (strongly) continuous.

One of the aims of this chapter is to study existence, uniqueness and properties of solutions to the abstract Cauchy problem

$$\frac{\partial u}{\partial t}(t) = L(t)u(t), \quad u(s) = x \in \mathbb{B}_{\alpha'}, \quad t \geq s. \tag{1.4}$$

Here $(L(t))_{t\geq 0} \subset L(\mathbb{B})$ is assumed to be at least strongly continuous. We distinguish between two kinds of solutions to above Cauchy problem. First we construct solutions by means of evolution systems, in such a case the solution u is defined on $[s, \infty)$. In the second case, which is used for perturbations of the operator $(L(t))_{t\geq 0}$, u is said to be a solution in the scale $\mathbb{B}$ to (1.4) if $u \in \bigcap\limits_{\alpha > \alpha'} C^1([0, T(\alpha', \alpha)); \mathbb{B}_\alpha)$ for some continuous function $T(\alpha', \alpha) > 0$ and u satisfies for any $\alpha > \alpha'$ equation (1.4) on $\mathbb{B}_\alpha$ for $s \leq t < T(\alpha', \alpha)$. Such a solution is (in general) only defined on the interval $[s, s + T(\alpha', \alpha))$ and is said to depend continuously on its initial data if there exists $C(t, \alpha', \alpha) > 0$ such that for all $\alpha' < \alpha$

$$\|u(t)\|_\alpha \leq C(t, \alpha', \alpha)\|x\|_{\alpha'}, \quad s \leq t < T(\alpha', \alpha)$$

holds. Moreover, it is shown that if $L_n(t) \longrightarrow L(t)$, $n \to \infty$ is fulfilled for any $t \geq 0$ in the uniform topology, then for any $\alpha' < \alpha$, $x \in \mathbb{B}_{\alpha'}$ and $s \geq 0$

$$u_n(t) \longrightarrow u(t), \quad s \leq t < s + T(\alpha', \alpha)$$

holds in $\mathbb{B}_\alpha$. Above results are extended in the third section to the non-linear version of (1.4), i.e. to the non-linear Cauchy problem

$$\frac{\partial u}{\partial t}(t) = A(t)u(t) + B(t, u(t)), \quad u(0) = x \in \mathbb{B}_{\alpha'}, \tag{1.5}$$

where $(A(t))_{t\geq 0} \subset L(\mathbb{B})$ and $B(t, u)$ is a non-linear operator acting in the scale $\mathbb{B}$.

1.1 Linear evolution equations

Let $\mathbb{E} = (\mathbb{E}_\alpha)_{\alpha > \alpha_*}$ be a scale of Banach spaces such that for any $\alpha' < \alpha$

$$\mathbb{E}_\alpha \subset \mathbb{E}_{\alpha'}, \quad \|\cdot\|_{\alpha'} \leq \|\cdot\|_\alpha$$

is fulfilled and suppose that $\mathbb{E}$ has dense embeddings, i.e. $\mathbb{E}_\alpha \subset \mathbb{E}_{\alpha'}$ is dense for all $\alpha' < \alpha$. Such scale of Banach spaces serves as a pre-dual scale of Banach spaces and we are mainly interested in the Cauchy problem (1.4) formulated on the dual scale of Banach spaces. Hence we preserve the notation $\mathbb{B}$ for the dual scale of Banach spaces introduced later on. The notation for $L(\mathbb{E})$ and convergence of sequences in $L(\mathbb{E})$ are defined in the same way as for a scale with property (1.1). We denote by $\|\cdot\|_{L(\mathbb{E}_\alpha)}$ the operator norm on $L(\mathbb{E}_\alpha)$ and if $\alpha' < \alpha$ by $\|\cdot\|_{\alpha\alpha'}$ the norm in $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$. The following definition summarizes the main objects of investigation for this section.

Definition 1.1.1. *Fix $\alpha > \alpha_*$, a family of bounded linear operators $(U_\alpha(t,s))_{0\le s\le t}$ on $L(\mathbb{E}_\alpha)$ is said to be a forward evolution system if it satisfies the following properties:*

1. *For all $0 \le s \le r \le t$*

$$U_\alpha(s,s) = 1, \quad U_\alpha(t,r)U_\alpha(r,s) = U_\alpha(t,s).$$

2. *$U_\alpha(t,s)$ is strongly continuous on $\mathbb{E}_\alpha$.*

A family of bounded linear operators $(V_\alpha(s,t))_{0\le s\le t}$ on $L(\mathbb{E}_\alpha)$ is said to be a backward evolution system if it satisfies:

1. *For all $0 \le s \le r \le t$*

$$V_\alpha(s,s) = 1, \quad V_\alpha(s,r)V_\alpha(r,t) = V_\alpha(s,t).$$

2. *$V_\alpha(s,t)$ is strongly continuous on $\mathbb{E}_\alpha$.*

A forward evolution system $(U(t,s))_{s\le t}$ in the scale $\mathbb{E}$ is, by definition, a collection of forward evolution systems $(U_\alpha(t,s))_{\alpha>\alpha_*}$ such that for any $\alpha', \alpha > \alpha_*$ with $\alpha' < \alpha$ the space $\mathbb{E}_\alpha$ is invariant for $U_{\alpha'}(t,s)$ and

$$U_{\alpha'}(t,s)|_{\mathbb{E}_\alpha} = U_\alpha(t,s).$$

A backward evolution system $(V(s,t))_{s\le t}$ in the scale $\mathbb{E}$ is defined in the same way.

Here and in the following we omit the subscript α if no confusion may arise. The relation of forward and backward evolution systems $U(t,s)$, $V(t,s)$ in the scale $\mathbb{E}$ with an infinitesimal operator (generator) is described in the next definition.

Definition 1.1.2. *Let $A = (A(t))_{t\ge 0} \subset L(\mathbb{E})$ be strongly continuous. A forward evolution system $U(t,s)$ in the scale $\mathbb{E}$ is said to have generator A if for any $\alpha' < \alpha$ and $x \in \mathbb{E}_\alpha$ the evolution $U(t,s)x$ is continuously differentiable in $\mathbb{E}_{\alpha'}$ and satisfies*

$$\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x \tag{1.6}$$

$$\frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x \tag{1.7}$$

in $\mathbb{E}_{\alpha'}$. A backward evolution system $V(s,t)$ in the scale $\mathbb{E}$ is said to have generator A if for any $\alpha' < \alpha$ and $x \in \mathbb{E}_\alpha$ the evolution $V(s,t)x$ is continuously differentiable in $\mathbb{E}_{\alpha'}$ and satisfies

$$\frac{\partial}{\partial t}V(s,t)x = V(s,t)A(t)x$$

$$\frac{\partial}{\partial s}V(s,t)x = -A(s)V(s,t)x$$

in $\mathbb{E}_{\alpha'}$. The cases $s = t$ should be understood as right or left derivative correspondingly.

In applications the generator $A(t)$ is typically known on a subclass of elements $D_{\alpha'}(t) \subset \mathbb{E}_{\alpha'}$ and one studies the closure of $(A(t), D_{\alpha'}(t))$ in $\mathbb{E}_{\alpha'}$. Above definition implies $A(t)$ acts as a bounded linear operator from $\mathbb{E}_{\alpha}$ to $\mathbb{E}_{\alpha'}$ and hence $\mathbb{E}_{\alpha} \subset D_{\alpha'}(t)$ holds for all $\alpha' < \alpha$, $t \geq 0$. Strong continuity and the uniform boundedness principle imply that for any $\alpha > \alpha_*$ and $T > 0$

$$\sup_{0 \leq s \leq t \leq T} \|U(t,s)\|_{L(\mathbb{E}_{\alpha})} =: M_1(\alpha, T) \tag{1.8}$$

and for $\alpha_* < \alpha' < \alpha$

$$\sup_{0 \leq t \leq T} \|A(t)\|_{\alpha\alpha'} =: M_2(\alpha, \alpha', T) \tag{1.9}$$

are finite. The next lemma collects some basic properties for forward and backward evolution systems in the scale $\mathbb{E}$.

Lemma 1.1.3. *Let $U(t,s)$ and $V(s,t)$ be forward (backward) evolution systems in the scale $\mathbb{E}$. Denote by $A = (A(t))_{t \geq 0} \subset L(\mathbb{B})$ their generators. Then the following assertions hold:*

1. *$U(t,s)$ and $V(s,t)$ are uniquely determined by A.*
2. *The evolution systems are continuous in the uniform topology on $L(\mathbb{E})$.*
3. *Suppose $A = (A(t))_{t \geq 0}$ is continuous in the uniform topology. Then $U(t,s)$ and $V(s,t)$ are continuously differentiable in the uniform topology on $L(\mathbb{E})$.*
4. *Let $\widetilde{U}(t,s)$ be another forward evolution system with generator $\widetilde{A}$ and suppose that both operators A and $\widetilde{A}$ are continuous in the uniform topology. Then for any $\alpha' < \alpha$ and $T > 0$*

   $$\|U(t,s) - \widetilde{U}(t,s)\|_{\alpha\alpha'} \leq M(\alpha', T) N(\alpha, T) \int_s^t \|A(r) - \widetilde{A}(r)\|_{\alpha\alpha'} \mathrm{d}r$$

   is satisfied, where the constants are given by $M(\alpha', T) := \sup\limits_{0 \leq s \leq t \leq T} \|U(t,s)\|_{L(\mathbb{E}_{\alpha'})}$ and $N(\alpha, T) := \sup\limits_{0 \leq s \leq t \leq T} \|\widetilde{U}(t,s)\|_{L(\mathbb{E}_{\alpha})}$.
5. *Let $\widetilde{V}(s,t)$ be another backward evolution system with generator $\widetilde{A}$ and suppose that both operators A and $\widetilde{A}$ are continuous in the uniform topology. Then for any $\alpha' < \alpha$ and $T > 0$*

   $$\|V(s,t) - \widetilde{V}(s,t)\|_{\alpha\alpha'} \leq M(\alpha', T) N(\alpha, T) \int_s^t \|A(r) - \widetilde{A}(r)\|_{\alpha\alpha'} \mathrm{d}r$$

is satisfied, where the constants are given by $M(\alpha',T) := \sup_{0\le s\le t\le T} \|V(s,t)\|_{L(\mathbb{E}_{\alpha'})}$ *and* $N(\alpha,T) := \sup_{0\le s\le t\le T} \|\widetilde{V}(s,t)\|_{L(\mathbb{E}_\alpha)}$.

Proof. We are going to prove the assertions only for the forward evolution system, the proof for the backward evolution system can be done in the same manner.
1. Let $U(t,s)$ and $\widetilde{U}(t,s)$ be two forward evolution systems with the same generator A. For any $\alpha' < \alpha$ and $x \in \mathbb{E}_\alpha$ let $\alpha'' \in (\alpha',\alpha)$, then $\widetilde{U}(r,s)x \in \mathbb{E}_{\alpha''}$ is continuously differentiable. Hence the composition $U(t,r)\widetilde{U}(r,s)x$ belongs to $\mathbb{E}_{\alpha''}$ and is continuously differentiable in $\mathbb{E}_{\alpha'}$ with derivative given by

$$\frac{\partial}{\partial r}(U(t,r)\widetilde{U}(r,s)x) = 0, \quad 0 \le s \le r \le t.$$

Integrating from s to t yields $\widetilde{U}(t,s)x = U(t,s)x$, and since $\mathbb{E}$ has dense embeddings the assertion is proved.
2. Let $\alpha' < \alpha$, $x \in \mathbb{E}_\alpha$ and fix $T > 0$. Then for any $0 \le s \le t' \le t \le T$

$$U(t,s)x - U(t',s)x = \int_{t'}^{t} A(r)U(r,s)x\mathrm{d}r,$$

and for any $0 \le s' \le s \le t \le T$

$$U(t,s)x - U(t,s')x = -\int_{s'}^{s} U(t,r)A(r)x\mathrm{d}r$$

hold in $\mathbb{E}_{\alpha'}$. Hence by (1.8) and (1.9) we obtain

$$\|U(t,s)x - U(t',s)x\|_{\alpha'} \le M_2(\alpha,\alpha',T)M_1(\alpha,T)\|x\|_\alpha(t-t')$$

and

$$\|U(t,s)x - U(t,s')x\|_{\alpha'} \le M_1(\alpha',T)M_2(\alpha,\alpha',T)\|x\|_\alpha(s-s').$$

The assertion follows from

$$\|U(t,s) - U(t',s')\|_{\alpha\alpha'} \le \|U(t,s) - U(t',s)\|_{\alpha\alpha'} + \|U(t',s) - U(t',s')\|_{\alpha\alpha'}.$$

3. Fix $\alpha' < \alpha$ and let $x \in \mathbb{E}_\alpha$, then

$$U(t,s)x = x + \int_s^t A(r)U(r,s)x\mathrm{d}r = x + \int_s^t U(t,r)A(r)x\mathrm{d}r$$

holds in $\mathbb{E}_{\alpha'}$. The assumptions and part 2. imply that $A(r)U(r,s)$ and $U(t,r)A(r)$ are continuous in (r,s) and (r,t) w.r.t. the uniform topology on $L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$.
4. The equality

$$U(t,s) - \widetilde{U}(t,s) = \int_s^t U(t,r)(A(r) - \widetilde{A}(r))\widetilde{U}(r,s)\mathrm{d}r$$

holds in the uniform topology on $L(\mathbb{E})$ and by (1.8), (1.9) we obtain the assertion. □

In many applications it is important to check whether $A = (A(t))_{t\geq 0}$ is the generator of a forward or backward evolution system. For $\alpha > \alpha_*$ let $\mathbb{E}_{\alpha+} := \bigcup_{\alpha'>\alpha} \mathbb{E}_{\alpha'}$, then $(A(t), \mathbb{E}_{\alpha+})_{t\geq 0}$ is a family of (possibly unbounded) linear operators on $\mathbb{E}_\alpha$. A sufficient condition for the existence of forward and backward evolution systems on the scale $\mathbb{E}$ is given in the statement below. Its proof is based on the classical construction for a pair of Banach spaces presented in [Paz83].

Theorem 1.1.4. *Let $(A(t))_{t\geq 0} \subset L(\mathbb{E})$ be continuous in the uniform topology and suppose that the condition below is satisfied.*

(a) For any $\alpha > \alpha_*$ and $t \geq 0$ the operator $(A(t), \mathbb{E}_{\alpha+})$ is closable and the closure is the generator of a C_0-semigroup $(S_t^\alpha(s))_{s\geq 0}$ on $\mathbb{E}_\alpha$. For any $\alpha' < \alpha$, $s \geq 0$ and $t \geq 0$ the space $\mathbb{E}_\alpha$ is invariant for $S_t^{\alpha'}(s)$ and $S_t^{\alpha'}(s)|_{\mathbb{E}_\alpha} = S_t^\alpha(s)$ holds.

If the condition

(b) For all $\alpha > \alpha_*$ there exist constants $M(\alpha) \geq 1$ and $\omega(\alpha) \in \mathbb{R}$ such that

$$\|S_{t_n}^\alpha(s_n)\cdots S_{t_1}^\alpha(s_1)\|_{L(\mathbb{E}_\alpha)} \leq M(\alpha)e^{\omega(\alpha)\sum_{j=1}^n s_j}$$

holds, where $n \in \mathbb{N}$, $s_1, \ldots, s_n \geq 0$ and $0 \leq t_1 \leq \cdots \leq t_n$ are arbitrary.

is satisfied, then there exists a forward evolution system $U(t,s)$ in the scale $\mathbb{E}$ such that A is its generator and for all $\alpha > \alpha_*$ and $0 \leq s \leq t$

$$\|U(t,s)\|_{L(\mathbb{E}_\alpha)} \leq M(\alpha)e^{\omega(\alpha)(t-s)} \tag{1.10}$$

is satisfied. If instead the condition

(b') For all $\alpha > \alpha_*$ there exist constants $M(\alpha) \geq 1$ and $\omega(\alpha) \in \mathbb{R}$ with

$$\|S_{t_1}^\alpha(s_1)\cdots S_{t_n}^\alpha(s_n)\|_{L(\mathbb{E}_\alpha)} \leq M(\alpha)e^{\omega(\alpha)\sum_{j=1}^n s_j}$$

holds, where $n \in \mathbb{N}$, $s_1, \ldots, s_n \geq 0$ and $0 \leq t_1 \leq \cdots \leq t_n$ are arbitrary.

is satisfied, then there exists a backward evolution system $V(s,t)$ in the scale $\mathbb{E}$ such that A is its generator and for all $\alpha > \alpha_*$ and $0 \le s \le t$

$$\|V(s,t)\|_{L(\mathbb{E}_\alpha)} \le M(\alpha)e^{\omega(\alpha)(t-s)}, \quad \alpha > \alpha_*,\ 0 \le s \le t$$

is satisfied.

Lemma 1.1.5. *Condition (a) from Theorem 1.1.4 is equivalent to the existence of a family of C_0-semigroups $(S_t^\alpha(s))_{s\ge 0}$ having the following properties:*

1. *For any $\alpha' < \alpha$ and $t \ge 0$ the space $\mathbb{E}_\alpha$ is invariant for $S_t^{\alpha'}(s)$ and $S_t^{\alpha'}(s)|_{\mathbb{E}_\alpha} = S_t^\alpha(s)$ holds.*
2. *For any $\alpha' < \alpha$, $t \ge 0$ and $x \in \mathbb{E}_\alpha$ the evolution $S_t^{\alpha'}(s)x$ is continuously differentiable in $\mathbb{E}_{\alpha'}$ such that*
$$\frac{\partial}{\partial s}S_t^{\alpha'}(s)x = A(t)S_t^{\alpha'}(s)x = S_t^{\alpha'}(s)A(t)x$$
is fulfilled in $\mathbb{E}_{\alpha'}$.

Proof. Suppose that condition (a) holds and let $(A_{\alpha'}(t), D(A_{\alpha'}(t)))$ be the closure of $(A(t), \mathbb{E}_{\alpha'+})$ in $\mathbb{E}_{\alpha'}$ Then for $x \in \mathbb{E}_{\alpha'+} \subset D(A_{\alpha'}(t))$ we obtain that

$$\frac{\partial}{\partial s}S_t^{\alpha'}(s)x = A_{\alpha'}(t)S_t^{\alpha'}(s)x = S_t^{\alpha'}(s)A_{\alpha'}(t)x$$

holds in $\mathbb{E}_{\alpha'}$. Let $\alpha > \alpha'' > \alpha'$ be such that $x \in \mathbb{E}_\alpha$. Then $A_{\alpha'}(t)x = A(t)x \in \mathbb{E}_{\alpha'}$ and $S_t^{\alpha'}(s)A_{\alpha'}(t)x = S_t^{\alpha'}(s)A(t)x \in \mathbb{E}_{\alpha'}$ are fulfilled. Moreover, by $S_t^{\alpha'}(s)x = S_t^{\alpha''}(s)x \in \mathbb{E}_{\alpha''}$ it follows that

$$A_{\alpha'}(t)S_t^{\alpha'}(s)x = A(t)S_t^{\alpha'}(s)x \in \mathbb{E}_{\alpha'}$$

are fulfilled, i.e. property 2. holds. Conversely, let $(A_{\alpha'}(t), D(A_{\alpha'}(t)))$ be the generator of the semigroup $(S_t^{\alpha'}(s))_{s\ge 0}$ with the properties 1. and 2. Then by property 2. the operator $(A_{\alpha'}(t), D(A_{\alpha'}(t)))$ is an extension of $(A(t), \mathbb{E}_{\alpha'+})$ and by property 1. $\mathbb{E}_{\alpha'+}$ is invariant for $S_t^{\alpha'}(s)$. Since $\mathbb{E}_{\alpha'+} \subset \mathbb{E}_{\alpha'}$ is dense, it is also a core for $(A_{\alpha'}(t), D(A_{\alpha'}(t)))$. □

Now we are ready for the proof of Theorem 1.1.4

Proof. (Theorem 1.1.4)
Fix $T > 0$ and define for $n \in \mathbb{N}$ piecewise constant operators $A_n(t)$ by setting $t_k^n = \frac{k}{n}T$ and

$$\begin{cases} A_n(t) = A(t_k^n), & t_k^n \le t < t_{k+1}^n, \quad k = 0, \dots, n-1 \\ A_n(T) = A(T). \end{cases}$$

Moreover, for any $\alpha > \alpha_*$ let $U_n^\alpha(t,s)$ be given by

$$U_n^\alpha(t,s) := \begin{cases} S^\alpha_{t_j^n}(t-s), & t_j^n \le s \le t \le t_{j+1}^n \\ S^\alpha_{t_k^n}(t-t_k^n) S_n^\alpha(l,k) S^\alpha_{t_l^n}(t_{l+1}^n - s), & k > l,\ t_k^n \le t \le t_{k+1}^n,\ t_l^n \le s \le t_{l+1}^n \end{cases}, \tag{1.11}$$

where $S_n^\alpha(l,k) := S^\alpha_{t_{k-1}^n}\left(\frac{T}{n}\right) \cdots S^\alpha_{t_{l+1}^n}\left(\frac{T}{n}\right)$ is time ordered in such a way that smaller times stand to the right. From [Paz83, Chapter 5, Theorem 3.1] it follows that for any $\alpha > \alpha_*$ there exists a forward evolution system $U_\alpha(t,s)$ on $\mathbb{E}_\alpha$ such that

$$\lim_{n\to\infty} U_n^\alpha(t,s) = U_\alpha(t,s) \tag{1.12}$$

holds strongly in $\mathbb{E}_\alpha$ and uniformly on compacts. This evolution systems satisfies by Lemma 1.1.5 the properties (1.7), (1.10) and for $t = s$ (1.6). Property (1.6) for $s < t$ follows by [Paz83, Chapter 5, Theorem 4.3] if we show that for any $\alpha' < \alpha$ the space $\mathbb{E}_\alpha$ is invariant for $U_{\alpha'}(t,s)$ and the restriction is strongly continuous w.r.t. $\|\cdot\|_\alpha$. Thus let $\alpha' < \alpha$ and $x \in \mathbb{E}_\alpha$, then $U_n^{\alpha'}(t,s)x \longrightarrow U_{\alpha'}(t,s)x$ in $\mathbb{E}_{\alpha'}$ and $U_n^\alpha(t,s)x \longrightarrow U_\alpha(t,s)x$ in $\mathbb{E}_\alpha$ hold. By $U_n^{\alpha'}(t,s)x = U_n^\alpha(t,s)x$ and $\|\cdot\|_{\alpha'} \le \|\cdot\|_\alpha$ we see that $U_\alpha(t,s)x = U_{\alpha'}(t,s)x$ is fulfilled. Thus $\mathbb{E}_\alpha$ is invariant for $U_{\alpha'}(t,s)$ and by $U_\alpha(t,s)x = U_{\alpha'}(t,s)x$ it is also strongly continuous w.r.t. $\|\cdot\|_\alpha$. For the construction of the backward evolution system, let

$$V_n(s,t) := \begin{cases} S_{t_j^n}(t-s), & t_j^n \le s \le t \le t_{j+1}^n, \\ S_{t_l^n}(t_{l+1}^n - s) S_n^\alpha(l,k) S_{t_k^n}(t - t_k^n), & k > l,\ t_k^n \le t \le t_{k+1}^n,\ t_l^n \le s \le t_{l+1}^n, \end{cases} \tag{1.13}$$

where $S_n^\alpha(l,k) := S^\alpha_{t_{l-1}^n}\left(\frac{T}{n}\right) \cdots S^\alpha_{t_{k+1}^n}\left(\frac{T}{n}\right)$ is now time ordered in the opposite direction. Repeating above arguments including the ones in [Paz83, Chapter 5, Theorem 3.1] and [Paz83, Chapter 5, Theorem 4.3] yields the assertion. □

In the following we relate the constructed evolution systems to the Cauchy problems

$$\frac{\partial}{\partial t} u(t) = A(t)u(t), \quad u(s) = x \in \mathbb{E}_\alpha, \quad t > s \tag{1.14}$$

and

$$\frac{\partial}{\partial s} v(s) = A(s)v(s), \quad v(t) = x \in \mathbb{E}_\alpha, \quad 0 \le s < t. \tag{1.15}$$

For equation (1.14) we use the terminology of $\mathbb{E}_\alpha$-valued solutions and adapt such definition to equation (1.15). Let $\alpha' < \alpha$, a function u is said to be a $\mathbb{E}_\alpha$-valued solution to (1.14) if $u \in C([s,\infty); \mathbb{E}_\alpha) \cap C^1((s,\infty); \mathbb{E}_{\alpha'})$ and u satisfies (1.14) in $\mathbb{E}_{\alpha'}$, cf. [Paz83, Chapter 5, Theorem 4.3]. A function v is a $\mathbb{E}_\alpha$-valued solution to equation (1.15) if $v \in C([0,t]; \mathbb{E}_\alpha) \cap C^1((0,t); \mathbb{E}_{\alpha'})$ and v satisfies (1.15) in $\mathbb{E}_{\alpha'}$. The next theorem was proved in [Paz83] for the forward evolution system $U(t,s)$ on a pair of Banach spaces. The proof can be adapted to this case.

Theorem 1.1.6. *Suppose that the same conditions as in Theorem 1.1.4 are fulfilled. Then for every $x \in \mathbb{E}_\alpha$ equation* (1.14) *has a unique $\mathbb{E}_\alpha$-valued solution, given by $u(t) = U(t,s)x$ and equation* (1.15) *has a unique $\mathbb{E}_\alpha$-valued solution, given by $v(s) = V(s,t)x$.*

Note that the differentiability at $t = s$ follows from $A(t) \in L(\mathbb{E})$ and was not stated in [Paz83]. The notion of $\mathbb{E}_\alpha$-valued solutions depends a priori on the choice of $\alpha' < \alpha$. However, in the case of Theorem 1.1.6 this notion is satisfied for any such α', hence we may omit the subscript α' in the definition as above. The next statement relates the constructed forward and backward evolution systems to solutions of the dual Cauchy problems given below. Denote by $\mathbb{B} = (\mathbb{E}_\alpha^*)_{\alpha > \alpha_*} =: \mathbb{E}^*$ the dual scale of Banach spaces, i.e. $\mathbb{B}_\alpha = \mathbb{E}_\alpha^*$ where $\mathbb{E}_\alpha^*$ is the dual Banach space to $\mathbb{E}_\alpha$. For $x \in \mathbb{E}_\alpha$ and $x^* \in \mathbb{B}_\alpha$ let $\langle x, x^* \rangle = x^*(x)$ be the dual pairing and denote by $U(s,t)^*$ and $V(t,s)^*$ the adjoint operators defined on the scale $\mathbb{B}$. This operators satisfy

$$V(t,r)^* V(r,s)^* = V(t,s)^*, \quad U(s,r)^* U(r,t)^* = U(s,t)^*,$$

and hence $V(t,s)^*$ is a forward evolution system whereas $U(s,t)^*$ is a backward evolution system on the scale $\mathbb{B}$. Using (1.6) and (1.7), it follows that they satisfy for any $\alpha' < \alpha$ and $x^* \in \mathbb{B}_{\alpha'}$ the equations

$$\begin{aligned} \frac{\partial}{\partial s} \langle x, U(s,t)^* x^* \rangle &= -\langle A(s)x, U(s,t)^* x^* \rangle, \quad x \in \mathbb{E}_\alpha, \quad s \in [0,t) \\ \frac{\partial}{\partial t} \langle x, V(t,s)^* x^* \rangle &= \langle A(t)x, V(t,s)^* x^* \rangle, \quad x \in \mathbb{E}_\alpha, \quad t \in [s,\infty). \end{aligned}$$

Denote by $\sigma(\mathbb{B}_{\alpha'}, \mathbb{E}_{\alpha'})$ the smallest topology on $\mathbb{B}_{\alpha'}$ for which all linear functionals $x^* : \mathbb{E}_{\alpha'} \longrightarrow \mathbb{R}$ are continuous.

Theorem 1.1.7. *Suppose that the same conditions as for Theorem 1.1.4 are satisfied. Let $\alpha' < \alpha$ and $x^* \in \mathbb{B}_{\alpha'}$ be arbitrary. Then the following holds:*

1. *Let $t > 0$ and $(u^*(s))_{s \in [0,t]} \subset \mathbb{B}_{\alpha'}$ be continuous w.r.t. $\sigma(\mathbb{B}_{\alpha'}, \mathbb{E}_{\alpha'})$ such that*

$$\frac{\partial}{\partial s} \langle x, u^*(s) \rangle = -\langle A(s)x, u^*(s) \rangle, \quad u^*(t) = x^*, \quad x \in \mathbb{E}_\alpha, \ s \in [0,t) \tag{1.16}$$

   is satisfied. Then $u^*(s) = U(s,t)^* x^*$ holds for any $s \in [0,t]$.

2. *Let $s \geq 0$ and $(v^*(t))_{t \in [s,\infty)}$ be continuous w.r.t. $\sigma(\mathbb{B}_{\alpha'}, \mathbb{E}_{\alpha'})$ such that*

$$\frac{\partial}{\partial t} \langle x, v^*(t) \rangle = \langle A(t)x, v^*(t) \rangle, \quad v^*(s) = x^*, \quad x \in \mathbb{E}_\alpha, \ t > s \tag{1.17}$$

   is satisfied. Then $v^*(t) = V(t,s)^* x^*$ holds for any $t \in [s,\infty)$.

Proof. Uniqueness for (1.17) was proved in [Kol13], so let us prove uniqueness for (1.16). Let $u^*(s) \in \mathbb{B}_{\alpha'}$ be any solution to (1.16), fix $s \in [0,t)$ and $x \in \mathbb{E}_\alpha$. For $r \in [s,t]$ let $g(r) := \langle U(r,s)x, u^*(r)\rangle$. Then for $\delta > 0$ sufficiently small and $r \in [s,t)$ we obtain

$$\frac{g(r+\delta)-g(r)}{\delta} = \left\langle \frac{U(r+\delta,s)x - U(r,s)x}{\delta}, u^*(r) \right\rangle + \left\langle U(r+\delta,s)x, \frac{u^*(r+\delta)-u^*(r)}{\delta} \right\rangle.$$

We have $U(r,s)x, U(r+\delta,s)x \in \mathbb{E}_\alpha$ and hence $A(r)U(r,s)x \in \mathbb{E}_{\alpha'}$. The first term therefore tends to $\langle A(r)U(r,s)x, u^*(r)\rangle$, when $\delta \to 0$. For the second term we get

$$\begin{aligned}
&\left| \left\langle U(r+\delta,s)x, \frac{u^*(r+\delta)-u^*(r)}{\delta} \right\rangle + \langle A(r)U(r,s)x, u^*(r)\rangle \right| \\
&\leq \left| \left\langle U(r+\delta,s)x, \frac{u^*(r+\delta)-u^*(r)}{\delta} \right\rangle - \left\langle U(r,s)x, \frac{u^*(r+\delta)-u^*(r)}{\delta} \right\rangle \right| \qquad (1.18) \\
&\quad + \left| \left\langle U(r,s)x, \frac{u^*(r+\delta)-u^*(r)}{\delta} \right\rangle + \langle A(r)U(r,s)x, u^*(r)\rangle \right|.
\end{aligned}$$

The second term tends to zero, since u^* is a solution to (1.16). The first term (1.18) is bounded by

$$\left\| \frac{u^*(r+\delta)-u^*(r)}{\delta} \right\|_{\mathbb{B}_{\alpha'}} \|U(r+\delta,s)x - U(r,s)\|_{\mathbb{E}_{\alpha'}}. \qquad (1.19)$$

By (1.16) $\frac{u^*(r+\delta)-u^*(r)}{\delta}$ is bounded w.r.t. $\|\cdot\|_{\mathbb{B}_{\alpha'}}$ in δ. The strong continuity of $U(r,s)$ implies that (1.19) tends to zero. Altogether we have shown that $g'(r) = 0$, which readily implies the assertion by

$$\langle x, u^*(s)\rangle = g(s) = g(t) = \langle U(t,s)x, u^*(t)\rangle = \langle U(t,s)x, x^*\rangle.$$

□

The next statement is an immediate consequence of duality and previous considerations.

Theorem 1.1.8. *Let $(A(t))_{t\geq 0} \subset L(\mathbb{E})$ be a family of operators in the scale $\mathbb{E}$ and $(A(t)^*)_{t\geq 0} \subset L(\mathbb{B})$ the collection of adjoint operators in the scale $\mathbb{B}$. Then $t \longmapsto A(t)$ is continuous in the uniform topology if and only if $t \longmapsto A(t)^*$ is continuous in the uniform topology. In such a case the forward and backward evolution systems $V(t,s)^*$ and $U(s,t)^*$ are continuous w.r.t. the uniform topology in the scale $\mathbb{B}$.*

1.2 Perturbation by linear operators

Let $\mathbb{B} = (\mathbb{B}_\alpha)_{\alpha>\alpha_*}$ be any scale of Banach spaces with property (1.1). The aim of this section is to prove existence and uniqueness of solutions in the scale $\mathbb{B}$ to the Cauchy problem

$$\frac{\partial}{\partial t}u(t) = A^\Delta(t)u(t) + B^\Delta(t)u(t), \quad u(s) = x \in \mathbb{B}_{\alpha'}, \tag{1.20}$$

where $(A^\Delta(t))_{t\geq 0} \subset L(\mathbb{B})$ is continuous in the uniform topology and $(B^\Delta(t))_{t\geq 0} \subset L(\mathbb{B})$ is strongly continuous in the scale. A similar version for the time-homogeneous case can be found in [Fin15]. For this section we suppose that there exists a forward evolution system $(V^\Delta(t,s))_{0\leq s\leq t} \subset L(\mathbb{B})$ such that for any $x \in \mathbb{B}_{\alpha'}$ and $\alpha' < \alpha$ the evolution $V^\Delta(t,s)x \in \mathbb{B}_\alpha$ satisfies for all $0 \leq s \leq t$

$$\frac{\partial}{\partial t}V^\Delta(t,s)x = A^\Delta(t)V^\Delta(t,s)x, \tag{1.21}$$

$$\frac{\partial}{\partial s}V^\Delta(t,s)x = -V^\Delta(t,s)A^\Delta(s)x \tag{1.22}$$

in $\mathbb{B}_\alpha$.

Remark 1.2.1. *Above assumption is fulfilled if e.g. $\mathbb{B} = \mathbb{E}^*$ for some scale of Banach spaces as in the previous section, $A(t)$ satisfies the conditions of Theorem 1.1.4 and $A^\Delta(t) = A(t)^*$, see Theorem 1.1.8.*

We will prove that solutions to (1.20) determine for any $\alpha' < \alpha$ a collection of solution operators $(W_{\alpha'\alpha}(t,s))_{0\leq s\leq t<T(\alpha',\alpha)}$ on $L(\mathbb{B}_{\alpha'},\mathbb{B}_\alpha)$, where $T(\alpha',\alpha) > 0$ is continuous and given in the statement below. This operators satisfy, by construction, for any $\alpha' < \alpha < \alpha''$, $x \in \mathbb{B}_{\alpha'}$ and $0 \leq t-s < \min\{T(\alpha',\alpha''), T(\alpha,\alpha''), T(\alpha',\alpha)\}$

$$W_{\alpha'\alpha}(t,s)x = W_{\alpha'\alpha''}(t,s)x = W_{\alpha\alpha''}(t,s)x.$$

Hence we omit the subscripts $\alpha'\alpha$ below.

Theorem 1.2.2. *Suppose that there exist constants $A \geq 1$ and $\omega \in \mathbb{R}$ such that for all $\alpha' < \alpha$*

$$\|V^\Delta(t,s)\|_{\alpha'\alpha} \leq Ae^{\omega(t-s)}, \quad 0 \leq s \leq t \tag{1.23}$$

holds. Let $(B^\Delta(t))_{t\geq 0} \subset L(\mathbb{B})$ be strongly continuous in t such that there exists an increasing continuous function $M(\alpha)$ satisfying for all $\alpha' < \alpha$

$$\|B^\Delta(t)\|_{\alpha'\alpha} \leq \frac{M(\alpha)}{\alpha-\alpha'}, \quad t \geq 0. \tag{1.24}$$

Define $T(\alpha',\alpha) := \frac{\alpha-\alpha'}{2AeM(\alpha)}$. Then there exists a unique family of operators $(W(t,s))_{0\leq s\leq t}$ with the properties:

1. *$W(t,s) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ for any $\alpha' < \alpha$ and $0 \le t - s < T(\alpha', \alpha)$ such that*

$$\|W(t,s)\|_{\alpha'\alpha} \le e^{\omega(t-s)} \frac{T(\alpha',\alpha)}{T(\alpha',\alpha) - (t-s)}$$

is satisfied. Moreover, for any $x \in \mathbb{B}_{\alpha'}$ and $\alpha' < \alpha$, $(s,t) \longmapsto W(t,s)x \in \mathbb{B}_\alpha$ is continuous for $0 \le t - s < T(\alpha', \alpha)$.

2. *For any $\alpha' < \alpha$ and $x \in \mathbb{B}_{\alpha'}$, $W(t,s)x$ is continuously differentiable in $\mathbb{B}_\alpha$ such that for all $0 \le t - s < T(\alpha', \alpha)$*

$$\frac{\partial}{\partial t} W(t,s)x = (A^\Delta(t) + B^\Delta(t))W(t,s)x, \tag{1.25}$$
$$\frac{\partial}{\partial s} W(t,s)x = -W(t,s)(A^\Delta(s) + B^\Delta(s))x \tag{1.26}$$

hold in $\mathbb{B}_\alpha$.

3. *Fix $s \ge 0$, $\alpha' < \alpha$, $x \in \mathbb{B}_{\alpha'}$ and suppose that there exists $T > 0$ and a function $u \in C([s, s+T); \mathbb{B}_\alpha) \cap C^1((s, s+T); \mathbb{B}_\alpha)$ such that for all $s \le t < s + T$*

$$\frac{\partial}{\partial t} u(t) = (A^\Delta(t) + B^\Delta(t))u(t), \ u(s) = x \tag{1.27}$$

is satisfied. Then $u(t) = W(t,s)x$ holds for any $s \le t < s + \min\{T, T(\alpha', \alpha)\}$.

Proof. Define a sequence of operators $(W_n(t,s))_{0 \le s \le t} \subset L(\mathbb{B})$ by $W_0(t,s)x = V^\Delta(t,s)x$ and

$$W_{n+1}(t,s)x := \int_s^t V^\Delta(t,r)B^\Delta(r)W_n(r,s)x \mathrm{d}r \tag{1.28}$$

for $x \in \mathbb{B}_{\alpha'}$. Then for any $\alpha' < \alpha$, $n \ge 0$ and $x \in \mathbb{B}_{\alpha'}$ the function $W_n(t,s)x$ is continuous in $\mathbb{B}_\alpha$ and satisfies

$$\|W_n(t,s)x\|_\alpha \le \|x\|_{\alpha'} e^{\omega(t-s)} \left(\frac{t-s}{T(\alpha',\alpha)} \right)^n.$$

In fact, let $\alpha_j := \alpha' + j\frac{\alpha - \alpha'}{2n}$, $j = 0, \dots, 2n$ and for $s \le t_1 \le \dots \le t_n \le t$

$$Q_n(t, t_1, \dots, t_n, s)x := V^\Delta(t, t_1)B^\Delta(t_1) \cdots V^\Delta(t_{2n-2}, t_{2n-1})B^\Delta(t_{2n-1})V^\Delta(t_{2n}, s)x.$$

Then by (1.23) and (1.24) we obtain

$$\begin{aligned}
\|W_n(t,s)x\|_\alpha &\le \int\limits_s^t \|V^\Delta(t,r)B^\Delta(r)W_n(r,s)x\|_\alpha \mathrm{d}r \\
&\le \int\limits_s^t \cdots \int\limits_s^{t_{n-1}} \|Q_n(t,t_1,\dots,t_n,s)x\|_\alpha \mathrm{d}t_n \dots \mathrm{d}t_1 \\
&\le A^n e^{\omega(t-s)}\|x\|_{\alpha'} \frac{(2n)^n}{(\alpha-\alpha')^n} \int\limits_s^t \cdots \int\limits_s^{t_{n-1}} \prod_{j=0}^{n-1} M(\alpha_{2j+1}) \mathrm{d}t_n \dots \mathrm{d}t_1 \\
&\le \|x\|_{\alpha'} e^{\omega(t-s)} \frac{(t-s)^n}{n!} \frac{(2M(\alpha)nA)^n}{(\alpha-\alpha')^n} \le \|x\|_{\alpha'} e^{\omega(t-s)} \left(\frac{2eAM(\alpha)(t-s)}{\alpha-\alpha'}\right)^n,
\end{aligned}$$

where we have used a variant of the Stirling formula, namely

$$\frac{1}{n!} \le \left(\frac{e}{n}\right)^n, \quad n \ge 1.$$

Choose $q \in (0,1)$, then we obtain for any $0 \le t-s \le qT(\alpha',\alpha)$

$$\|W_n(t,s)\|_\alpha \le \|x\|_{\alpha'} e^{\omega(t-s)} q^n$$

and hence the series $\sum\limits_{n=0}^{\infty} W_n(t,s)x =: W(t,s)x$ converges uniform. Since q was arbitrary, it follows that $W(t,s)x$ is continuous in (t,s) with $t-s < T(\alpha',\alpha)$ and satisfies

$$\begin{aligned}
\|W(t,s)x\|_\alpha &\le \sum_{n=0}^{\infty} \|W_n(t,s)x\|_\alpha \le \|x\|_{\alpha'} e^{\omega(t-s)} \sum_{n=0}^{\infty} \left(\frac{t-s}{T(\alpha',\alpha)}\right)^n \\
&= \|x\|_{\alpha'} e^{\omega(t-s)} \frac{T(\alpha',\alpha)}{T(\alpha',\alpha)-(t-s)}.
\end{aligned}$$

Next we show that $W(t,s)$ is differentiable. Take $\alpha_j := \alpha' + j\frac{\alpha-\alpha'}{2(n+1)}$, $j = 0,\dots,2(n+1)$, then we obtain for $s \le r \le t$ that

$$\begin{aligned}
\|V^\Delta(t,r)B^\Delta(r)W_n(r,s)x\|_\alpha &\le e^{\omega(t-s)}(AM(\alpha))^{n+1} \frac{2(n+1)}{\alpha-\alpha'} \frac{(t-s)^n}{n!} \frac{(2(n+1))^n}{(\alpha-\alpha')^n} \|x\|_{\alpha'} \\
&\le e^{\omega(t-s)} \|x\|_{\alpha'} (t-s)^n \left(\frac{e}{n}\right)^n \frac{(AM(\alpha))^{n+1}}{(\alpha-\alpha')^{n+1}} (2(n+1))^{n+1} \\
&= e^{\omega(t-s)} \|x\|_{\alpha'} (t-s)^n \frac{2AM(\alpha)}{\alpha-\alpha'} n \left(\frac{2eAM(\alpha)}{\alpha-\alpha'}\right)^n \left(\frac{n+1}{n}\right)^{n+1} \\
&\le e^{\omega(t-s)} \|x\|_{\alpha'} \frac{4eAM(\alpha)}{\alpha-\alpha'} n \left(\frac{t-s}{T(\alpha',\alpha)}\right)^n
\end{aligned}$$

is satisfied. So for any $s \leq r \leq t$ and $q \in (0,1)$ such that $|t-s| \leq qT(\alpha', \alpha)$ the series

$$\sum_{n=0}^{\infty} V^{\Delta}(t,r)B^{\Delta}(r)W_n(r,s)x$$

is uniformly convergent. For $t-s < T(\alpha', \alpha)$ we find $\alpha'' \in (\alpha', \alpha)$ such that $t-s < T(\alpha', \alpha'')$, hence $W(r,s)x \in \mathbb{B}_{\alpha''}$ is continuous. Since $V^{\Delta}(t,s)B^{\Delta}(r) \in L(\mathbb{B}_{\alpha''}, \mathbb{B}_{\alpha})$ is strongly continuous it follows that

$$\begin{aligned}
W(t,s)x &= W_0(t,s)x + \sum_{n=1}^{\infty} W_n(t,s)x \\
&= V^{\Delta}(t,s)x + \sum_{n=1}^{\infty} \int_s^t V^{\Delta}(t,r)B^{\Delta}(r)W_{n-1}(r,s)x\mathrm{d}r \\
&= V^{\Delta}(t,s)x + \int_s^t V^{\Delta}(t,r)B^{\Delta}(r)W(r,s)x\mathrm{d}r
\end{aligned}$$

is fulfilled. Hence $W(t,s)x$ is differentiable w.r.t. t in $\mathbb{B}_{\alpha}$ and differentiating above equality, see (1.21), yields (1.25). The sequence $(W_n(t,s)x)_{n\in\mathbb{N}}$ also satisfies the relation

$$W_{n+1}(t,s)x = \int_s^t W_n(t,r)B^{\Delta}(r)V^{\Delta}(r,s)x\mathrm{d}r$$

and a repetition of above arguments, shows that $(W(t,s)x)_{0\leq s\leq t}$ also satisfies

$$W(t,s)x = V^{\Delta}(t,s)x + \int_s^t W(t,r)B^{\Delta}(r)V^{\Delta}(r,s)x\mathrm{d}r.$$

The integrand on the right-hand side is continuous w.r.t. (t,r,s) in $\mathbb{B}_{\alpha}$ and hence $W(t,s)x \in \mathbb{B}_{\alpha}$ is differentiable. Namely, for $t-s < T(\alpha', \alpha)$ there exists $\alpha'' \in (\alpha', \alpha)$ such that $t-s < T(\alpha'', \alpha) < T(\alpha', \alpha)$ holds. Repeating the arguments from above and differentiating the right-hand side yields (1.26). For the last assertion let $w(t) := W(t,s)x - u(t)$, where $s \leq t < s + \min\{T, T(\alpha', \alpha)\}$. Then $w(s) = 0$ and w solves (1.27). It is therefore sufficient to show that $w = 0$. Applying (1.27) for u yields that for $s \leq t < s + T$

$$u(t) = V^{\Delta}(t,s)x + \int_s^t V^{\Delta}(t,r)B^{\Delta}(r)u(r)\mathrm{d}r$$

holds in $\mathbb{B}_{\alpha''}$ and any $\alpha'' > \alpha$. Hence for any $s \le t < s + \min\{T, T(\alpha', \alpha)\}$

$$w(t) = \int_s^t V^{\Delta}(t,r) B^{\Delta}(r) w(r) \mathrm{d}r$$

holds in $\mathbb{B}_{\alpha''}$. Define $\alpha_j := \alpha + j\frac{\alpha''-\alpha}{2n}$, $j = 0, \dots, 2n$ and $C_\alpha := \sup\limits_{r\in[s,t]} \|W(r,s)x - u(r)\|_\alpha < \infty$. It follows for $s \le t_n \le \dots \le t_1 \le t$ and

$$Q(t, t_1, \dots, t_n) := V^{\Delta}(t, t_1) B^{\Delta}(t_1) \cdots V^{\Delta}(t_{n-1}, t_n) B^{\Delta}(t_n) \tag{1.29}$$

that

$$\|Q(t, t_1, \dots, t_n) w(t_n)\|_{\alpha''} \le A^n e^{\omega(t-t_n)} \frac{M(\alpha'')^n (2n)^n}{(\alpha'' - \alpha)^n} \|w(t_n)\|_\alpha \tag{1.30}$$

holds. Hence we obtain the estimate

$$\begin{aligned}
\|w(t)\|_{\alpha''} &\le \int_s^t \cdots \int_s^{t_{n-1}} \|Q(t, t_1, \dots, t_n, s) w(t_n)\|_{\alpha''} \mathrm{d}t_n \dots \mathrm{d}t_1 \\
&\le \left(\frac{AM(\alpha'')2n}{\alpha'' - \alpha} \right)^n \int_s^t \cdots \int_s^{t_{n-1}} e^{\omega(t-t_n)} \|w(t_n)\|_\alpha \mathrm{d}t_n \dots \mathrm{d}t_1 \\
&\le C_\alpha e^{\omega(t-s)} \frac{(t-s)^n}{n!} n^n \left(\frac{2AM(\alpha'')}{\alpha'' - \alpha} \right)^n \\
&\le C_\alpha e^{\omega(t-s)} \left(\frac{2eAM(\alpha'')(t-s)}{\alpha'' - \alpha} \right)^n,
\end{aligned}$$

where we have assumed w.l.g. that $\omega \ge 0$. This implies $w(t) = 0$ in $\mathbb{B}_{\alpha''} \hookrightarrow \mathbb{B}_\alpha$ for

$$s \le t < s + \min\left\{ T, T(\alpha', \alpha), \frac{\alpha'' - \alpha}{2eAM(\alpha'')} \right\}.$$

Applying above arguments to $\alpha'' = \alpha + 1$ shows for any $\alpha' < \alpha$, $x \in \mathbb{B}_{\alpha'}$ that (1.27) is unique on $[s, s + T_0(\alpha', \alpha)q]$ for any $q \in (0,1)$ and $T_0(\alpha', \alpha) := \min\{T(\alpha', \alpha), \frac{1}{2eAM(\alpha+1)}\}$. Changing s to $s + T_0(\alpha', \alpha)q$ and iterating this procedure yields the assertion. Such an iteration is possible since $w(s + qT_0(\alpha', \alpha)) = 0 \in \mathbb{B}_{\alpha'}$. □

For $\alpha' < \alpha$ and $s \le t$ let

$$\alpha(t, s, \alpha') := \inf \{ \beta \ge \alpha' \mid W(t,s)x \in \mathbb{B}_\beta \},$$

then $\alpha(s,s,\alpha') = \alpha'$ and if $s \le t < s + T(\alpha',\alpha)$ also $\alpha(t,s,\alpha') \le \alpha$ follows. The continuity of $T(\alpha',\alpha)$ implies that there exists $\beta \in (\alpha',\alpha)$ such that $s \le t < s + T(\alpha',\beta)$ and hence $W(t,s) \in \mathbb{B}_\beta$, which implies that $\alpha(t,s,\alpha') \le \beta < \alpha$ is fulfilled. Uniqueness therefore implies that for any $s \le r \le t$, $0 \le s \le r < s + T(\alpha',\alpha)$ and $t < \min\{s + T(\alpha',\alpha), r + T(\alpha(r,s,\alpha'),\alpha)\}$

$$W(t,s)x = W(t,r)W(r,s)x$$

holds for all $x \in \mathbb{B}_{\alpha'}$.

Remark 1.2.3. *Above proof shows that if B^Δ is continuous in the uniform topology, then $W(t,s)$ is also continuously differentiable in the uniform topology.*

A global solution to (1.20) is, by definition, a function $u : \mathbb{R}_+ \longrightarrow \bigcup\limits_{\alpha > \alpha'} \mathbb{B}_\alpha$ such that for all $T > 0$ there exists $\alpha > \alpha'$ and $u|_{[0,T]}$ is a solution to (1.20) in $\mathbb{B}_\alpha$.

Corollary 1.2.4. *Let $\alpha' > \alpha_*$ and suppose that there exists a sequence $(\alpha_j)_{j \ge 0}$ such that $\alpha_j < \alpha_{j+1}$, $\alpha_0 = \alpha'$ and*

$$\sum_{j=0}^{\infty} \frac{\alpha_{j+1} - \alpha_j}{M(\alpha_{j+1})} = \infty \tag{1.31}$$

is satisfied. Then for any $x \in \mathbb{B}_{\alpha'}$ there exists a unique global solution to (1.20) *given by $W(t,s)x$. In particular, if $M(\alpha)$ is bounded by $M^* > 0$, then the assertions of Theorem 1.2.2 hold for $T(\alpha',\alpha) = \frac{\alpha - \alpha'}{2eAM^*}$ and $W(t,s)x$ provides for every $x \in \mathbb{B}_{\alpha'}$, $\alpha' > \alpha_*$ the unique global solution to* (1.20).

Proof. Let $x \in \mathbb{B}_{\alpha'}$, then $W(t,s)x$ is the unique solution to (1.20) on $[s, s + T(\alpha_0,\alpha_1))$ in $\mathbb{B}_{\alpha_1}$. Fix $q \in (0,1)$, then $W(t, s + qT(\alpha_0,\alpha_1))W(s + qT(\alpha_0,\alpha_1), s)x$ yields the unique solution on $[s + qT(\alpha_0,\alpha_1), s + q(T(\alpha_0,\alpha_1) + T(\alpha_1,\alpha_2))]$ in $\mathbb{B}_{\alpha_2}$. By iteration we obtain the unique solution on $[s, s + q(T(\alpha_0,\alpha_1) + \cdots + T(\alpha_N,\alpha_{N+1}))]$ in $\mathbb{B}_{\alpha_N}$ for any $N \in \mathbb{N}$. Such iteration yields a global solution since $\sum\limits_{j=0}^{\infty} T(\alpha_j,\alpha_{j+1}) = \frac{1}{2eA} \sum\limits_{j=0}^{\infty} \frac{\alpha_{j+1}-\alpha_j}{M(\alpha_{j+1})} = \infty$. For the second assertion consider $\alpha_j = \alpha' + j$, then

$$\frac{\alpha_{j+1} - \alpha_j}{M(\alpha_{j+1})} \ge \frac{1}{M^*} > 0$$

implies (1.31). □

Below we provide stability of the evolution system $W(t,s)$ w.r.t. the operators $A^\Delta(t)$ and $B^\Delta(t)$. For any $n \in \mathbb{N}$, let $(A_n^\Delta(t))_{t \ge 0}$ be continuous in the uniform topology in the scale $\mathbb{B}$ and $(V_n^\Delta(t,s))_{0 \le s \le t}$ the associated forward evolution systems. Suppose that there exists constants $A \ge 1$ and $\omega \in \mathbb{R}$ such that

$$\|V_n^\Delta(t,s)\|_{\alpha'\alpha} \le Ae^{\omega(t-s)}, \quad 0 \le s \le t, \ \alpha' < \alpha, \quad n \in \mathbb{N} \tag{1.32}$$

holds. Let $(B_n^\Delta(t))_{t\geq 0} \subset L(\mathbb{B})$ be strongly continuous in t for any $n \in \mathbb{N}$ such that there exists an increasing continuous function $M(\alpha)$ independent of n and it satisfies

$$\|B_n^\Delta(t)\|_{\alpha'\alpha} \leq \frac{M(\alpha)}{\alpha - \alpha'}, \quad \alpha' < \alpha, \quad t \geq 0, \quad n \in \mathbb{N}. \tag{1.33}$$

Theorem 1.2.5. *Suppose that there exist operators $A^\Delta(t), V^\Delta(t,s), B^\Delta(t)$ which satisfy the conditions of Theorem 1.2.2 with $M(\alpha)$ as in (1.33). Assume that for any $T > 0$ and $\alpha' < \alpha$*

$$\sup_{t\in[0,T]} \|B_n^\Delta(t) - B^\Delta(t)\|_{\alpha'\alpha} \longrightarrow 0, \quad n \to \infty \tag{1.34}$$

and

$$\sup_{t\in[0,T]} \|A_n^\Delta(t) - A^\Delta(t)\|_{\alpha'\alpha} \longrightarrow 0, \quad n \to \infty \tag{1.35}$$

are satisfied. Let $T(\alpha', \alpha) := \frac{\alpha-\alpha'}{2eAM(\alpha)}$. Then for any $n \in \mathbb{N}$ there exist evolution systems $W^n(t,s)$ and $W(t,s)$ corresponding to $(A_n^\Delta(t), B_n^\Delta(t))$ and $(A^\Delta(t), B^\Delta(t))$ respectively, with the properties stated in Theorem 1.2.2. Moreover, for any $\alpha' < \alpha$, $x \in \mathbb{B}_{\alpha'}$ and $q \in (0,1)$ the convergence

$$W^n(t,s)x \longrightarrow W(t,s)x, \quad n \to \infty$$

holds in $\mathbb{B}_\alpha$ uniformly on compacts such that $0 \leq t - s \leq qT(\alpha', \alpha)$.

Proof. The same arguments as in the proof of Lemma 1.1.3 together with (1.35) show that

$$\|V_n^\Delta(t,s) - V^\Delta(t,s)\|_{\alpha'\alpha} \longrightarrow 0, \quad n \to \infty \tag{1.36}$$

holds uniformly on compacts for $0 \leq s \leq t$. Therefore without loss of generality we can assume that $V^\Delta(t,s)$ satisfies (1.32) with the same constants. Estimates (1.32) and (1.33) together with Theorem 1.2.2 imply that $W^n(t,s), W(t,s)$ exist and by (1.28) are given by $W(t,s) = \sum\limits_{k=0}^{\infty} W_k(t,s)$ and $W^n(t,s) = \sum\limits_{k=0}^{\infty} W_k^n(t,s)$, respectively. Moreover, from (1.32) and (1.33) it follows

$$\|W_k^n(t,s)\|_{\alpha'\alpha} \leq e^{\omega(t-s)} \left(\frac{t-s}{T(\alpha', \alpha)} \right)^k$$

and hence the series converges uniformly for $0 \leq t - s \leq qT(\alpha', \alpha)$ and w.r.t. n. Thus it suffices to show $W_k^n(t,s) \longrightarrow W_k(t,s)$, $n \to \infty$ in $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ for any $k \in \mathbb{N}$. For $k = 0$ this follows from (1.36) and for $k \geq 1$ by induction and (1.34). □

Clearly it is not necessary to assume that (1.34) and (1.35) hold for each $T > 0$. Since, in general, we only obtain the existence of a local solution it is enough to check the convergence on any interval $[s, s + qT(\alpha', \alpha)]$, $s \geq 0$ and $q \in (0, 1)$.

Suppose that $\mathbb{B}$ is a scale of Banach lattices. Namely, for each $\alpha > \alpha_*$ the space $\mathbb{B}_\alpha$ is a Banach lattice, that is $\mathbb{B}_\alpha$ is an ordered Banach space and the order is compatible with the norm. For convenience of the reader a precise definition is given in the appendix A.1. The order is also assumed to be compatible with the scale $\mathbb{B}$, i.e. for all $\alpha' < \alpha$ and $x, y \in \mathbb{B}_{\alpha'}$

$$x \leq_{\alpha'} y \Leftrightarrow x \leq_\alpha y\,,$$

where $\leq_\alpha$ denotes the order on $\mathbb{B}_\alpha$ and $\leq_{\alpha'}$ the order on $\mathbb{B}_{\alpha'}$. Thus we can omit the dependence on α. For details, additional properties and perturbation theory for semigroups on Banach lattices we refer to [BA06].

Given $C \in L(\mathbb{B})$, we say that C is positive if for each $\alpha' < \alpha$, $x \in \mathbb{B}_{\alpha'}$: $x \geq 0$ implies $Cx \geq 0$. The next theorem establishes a comparison principle for the constructed solutions. Such principle can be used to construct global solutions, see [KK16].

Theorem 1.2.6. *Suppose that $A^\Delta(t)$ and $V^\Delta(t,s)$ are given as in Theorem 1.2.2 and $V^\Delta(t,s)$ is positive. Let $(B_0^\Delta(t))_{t\geq 0}, (B_1^\Delta(t))_{t\geq 0} \subset L(\mathbb{B})$ be two positive operators. Assume that $t \longmapsto B_j^\Delta(t) \in L(\mathbb{B})$ are strongly continuous in the scale $\mathbb{B}$ for $j = 0, 1$ and there exit continuous increasing functions $M_0(\alpha), M_1(\alpha) > 0$ satisfying for all $\alpha' < \alpha$ and $t \geq 0$*

$$\|B_j^\Delta(t)\|_{\alpha'\alpha} \leq \frac{M_j(\alpha)}{\alpha - \alpha'}, \quad j = 0, 1.$$

Denote by $(W_0(t,s))_{0\leq s\leq t}$ the forward evolution system corresponding to $A^\Delta(t)+B_0^\Delta(t)$ and by $(W_1(t,s))_{0\leq s\leq t}$ the forward evolution system corresponding to $A^\Delta(t) + B_0^\Delta(t) - B_1^\Delta(t)$. Suppose that $W_1(t,s)$ is positive, then for any $\alpha' < \alpha < \alpha''$ and $0 \leq x \in \mathbb{B}_{\alpha'}$

$$W_1^\Delta(t,s)x \leq W_0(t,s)x \tag{1.37}$$

holds for all $s \leq t < s + \min\left\{\frac{\alpha-\alpha'}{2eA(M_0(\alpha)+M_1(\alpha))}, \frac{\alpha''-\alpha}{2eAM_1(\alpha'')}\right\}$.

Proof. The proof of Theorem 1.2.2 implies for $s \leq t < s + \frac{\alpha-\alpha'}{2eA(M_0(\alpha)+M_1(\alpha))}$ and $w(t) := W_0(t,s)x - W_1(t,s)x$ that

$$\begin{aligned} w(t) &= \int_s^t V^\Delta(t,r)B_0^\Delta(r)w(r)\mathrm{d}r + \int_s^t V^\Delta(t,r)B_1^\Delta(r)W_1(r,s)x\mathrm{d}r \\ &\geq \int_s^t V^\Delta(t,r)B_1^\Delta(r)w(r)\mathrm{d}r \end{aligned}$$

holds in $\mathbb{B}_{\alpha''}$, where we have used (1.25) and that all operators are positive. Iterating this inequality yields for any $n \in \mathbb{N}$ in $\mathbb{B}_{\alpha''}$

$$W_0(t,s)x - W_1(t,s)x \geq \int_s^t \cdots \int_s^{t_{n-1}} Q(t,t_1,\ldots,t_n,s)w(t_n)\mathrm{d}t_n \cdots \mathrm{d}t_1 =: I_n,$$

where $Q(t,t_1,\ldots,t_n,s) := V^{\Delta}(t,t_1)B_1^{\Delta}(t_1)\cdots V^{\Delta}(t_{n-1},t_n)B^{\Delta}(t_n)$. Let $\alpha_j := \alpha + j\frac{\alpha''-\alpha}{2n}$, $j = 0,\ldots,2n$, $C_\alpha := \sup\limits_{r\in[s,t]} \|w(r)\|_\alpha$, then by (1.30)

$$\|I_n\|_{\alpha''} \leq C_\alpha e^{\omega(t-s)} \left(\frac{2eAM_1(\alpha'')(t-s)}{\alpha''-\alpha}\right)^n.$$

Hence if $s \leq t < s + \min\left\{\frac{\alpha-\alpha'}{2eA(M_0(\alpha)+M_1(\alpha))}, \frac{\alpha''-\alpha}{2eAM_1(\alpha'')}\right\}$, then $I_n \longrightarrow 0$, $n \to \infty$ in $\mathbb{B}_{\alpha''}$. □

1.3 Perturbation by non-linear operators

In this section we prove existence, uniqueness and stability of solutions to the non-linear Cauchy problem (1.5), i.e. to

$$\frac{\partial}{\partial t}u(t) = A(t)u(t) + B(u(t),t), \quad u(0) = x.$$

Let $\mathbb{B} = (\mathbb{B}_\alpha)_{\alpha\in[\alpha_*,\alpha^*]}$ be a scale of Banach spaces with $\mathbb{B}_{\alpha'} \subset \mathbb{B}_\alpha$ and $\|\cdot\|_\alpha \leq \|\cdot\|_{\alpha'}$ for $\alpha' < \alpha$. It is worth noting that we have to consider here a scale for which the index α is also bounded from above.

Existence and uniqueness

Fix $x \in \mathbb{B}_{\alpha_*}$ and let $\lambda > 0$. First we are going to prove the existence of mild solutions. The following summarizes our main assumptions for this purpose.

A1. There exists an evolution system of bounded linear operators $(U(t,s))_{0\leq s\leq t<\frac{\alpha^*-\alpha_*}{\lambda}}$ in the scale $\mathbb{B}$ such that for $0 \leq s \leq r \leq t$ (in the sense of (1.3))

$$U(t,t) = 1, \quad U(t,r)U(r,s) = U(t,s)$$

and $(t,s) \longmapsto U(t,s) \in L(\mathbb{B}_{\alpha'},\mathbb{B}_\alpha)$ is strongly continuous for all $\alpha_* \leq \alpha' < \alpha \leq \alpha^*$.

A2. There exist constants $C_1 > 0$ and $\beta \in [0,\frac{1}{2})$ such that for all $\alpha_* \leq \alpha' < \alpha \leq \alpha^*$

$$\|U(t,s)\|_{\alpha'\alpha} \leq \frac{C_1}{(\alpha-\alpha')^\beta}, \quad t \geq s \geq 0.$$

A3. For all $\alpha \in (\alpha_*, \alpha^*]$ there exists $C(x,\alpha) > 0$ such that for all $0 \le s,t \le \frac{\alpha-\alpha_*}{\lambda}$

$$\|U(t,0)x - U(s,0)x\|_\alpha \le C(x,\alpha)|t-s|.$$

For the non-linear part $B(u,t)$ we suppose the following:

B1. There exists $r > 0$ such that for all α', α with $\alpha_* \le \alpha' < \alpha < \alpha^*$

$$B_r^{\alpha'}(x) \times \left[0, \frac{\alpha^* - \alpha_*}{\lambda}\right) \ni (u,t) \longmapsto B(u,t) \in \mathbb{B}_\alpha$$

is continuous, where $B_r^{\alpha'}(x) := \{y \in \mathbb{B}_{\alpha'} \mid \|x-y\|_{\alpha'} \le r\}$.

B2. There exists a constant $C_2 > 0$ such that for all α', α with $\alpha_* \le \alpha' < \alpha < \alpha^*$, $t \in [0, \frac{\alpha^*-\alpha_*}{\lambda})$ and any $u, v \in B_r^{\alpha'}(x)$

$$\|B(u,t) - B(v,t)\|_\alpha \le \frac{C_2}{(\alpha-\alpha')^{1-\beta}} \|u-v\|_{\alpha'}.$$

B3. There exists a constant $C_3 > 0$ such that for all $t \in [0, \frac{\alpha^*-\alpha_*}{\lambda})$ and all $\alpha \in (\alpha_*, \alpha^*]$

$$\|B(x,t)\|_\alpha \le \frac{C_3}{\alpha - \alpha_*}.$$

Remark 1.3.1. *Usually non-linearities $C(u,t)$ are considered to be locally Lipschitz continuous. The existence of maximal solutions is shown in this case, cf. [Paz83] and references therein. Such non-linearities can be also taken into account in our setting, if we define $\tilde{B}(u,t) := B(u,t) + C(u,t)$ and check that $\tilde{B}$ satisfies assumptions B1 – B3.*

Given $x \in \mathbb{B}_{\alpha_*}$ and $\alpha_0 \in [\alpha_*, \alpha^*)$, a solution to

$$u(t) = U(t,0)x + \int_0^t U(t,s)B(u(s),s)\mathrm{d}s. \tag{1.38}$$

in the scale $(\mathbb{B}_\alpha)_{\alpha\in[\alpha_0,\alpha^*]}$ is a function $u : \left[0, \frac{\alpha^*-\alpha_0}{\lambda}\right) \longrightarrow \mathbb{B}_{\alpha^*}$ such that for all $\alpha \in (\alpha_0, \alpha^*]$

$$u|_{[0,\frac{\alpha-\alpha_0}{\lambda})} \in C\left(\left[0, \frac{\alpha-\alpha_0}{\lambda}\right); \mathbb{B}_\alpha\right),$$

$\|u(t) - x\|_\alpha \le r$ and u solves (1.38) in $\mathbb{B}_\alpha$. The idea for the proof of the next statement is based on the work [Saf95] and provides a generalization of this work.

Theorem 1.3.2. *Under conditions A1 – A3 and B1 – B3 with $x \in \mathbb{B}_{\alpha_*}$, for each $\alpha_0 \in (\alpha_*, \alpha^*)$ there exists $\lambda_0 > 0$ and, provided $\lambda > \lambda_0$, a unique solution u_{α_0} in the scale $(\mathbb{B}_\alpha, \|\cdot\|_\alpha)_{\alpha\in[\alpha_0,\alpha^*]}$ to (1.38). Moreover, each two solutions u_{α_0} and u_{α_1} with $\alpha_* < \alpha_0 < \alpha_1 < \alpha^*$ and $\lambda > \max\{\lambda_0(\alpha_0), \lambda(\alpha_1)\}$ satisfy for any $\alpha \in [\alpha_1, \alpha^*)$*

$$u_{\alpha_0}(t) = u_{\alpha_1}(t), \quad 0 \leq t < \frac{\alpha - \alpha_1}{\lambda}.$$

Proof. Fix $x \in \mathbb{B}_{\alpha_*}$ and $\alpha_0 \in (\alpha_*, \alpha^*)$. For $\gamma \geq 0$ define S^γ as the Banach space of all functions $u : \left[0, \frac{\alpha^*-\alpha_0}{\lambda}\right) \longrightarrow \mathbb{B}_{\alpha^*}$ such that for each $\alpha \in (\alpha_0, \alpha^*]$ the restriction $u|_{[0,T(\alpha))} \in C([0,T(\alpha)); \mathbb{B}_\alpha)$ with $T(\alpha) = \frac{\alpha-\alpha_0}{\lambda}$ satisfies

$$\|u\|^{(\gamma)} = \sup_{\substack{0\leq t<T(\alpha)\\ \alpha\in(\alpha_0,\alpha^*]}} (\alpha - \alpha_0 - \lambda t)^\gamma \|u(t)\|_\alpha < \infty.$$

Define the non-linear integral operator

$$\mathcal{T}(u)(t) := \int_0^t U(t,s)B(u(s),s)\mathrm{d}s, \tag{1.39}$$

then we will show the existence of a unique solution to $u = U(\cdot,0)x + \mathcal{T}u$. Now fix $\gamma \in (\beta, 1-\beta)$, let

$$\lambda_0 := \max\left\{\frac{2^{2\gamma+1-\beta}C_1C_2}{\gamma-\beta}, \frac{4^{1-\beta}C_2(\alpha^*-\alpha_0)^\beta}{\gamma(1+\|x\|_{\alpha_*})} + \frac{2^{2+\gamma}C_1C_2}{\gamma}, \right.$$
$$\left. \frac{C(x)(\alpha^*-\alpha_0)}{r} + \frac{2^{\gamma-1+\beta}C_1(\frac{C_3}{\alpha_0-\alpha_*} + C(x))(\alpha^*-\alpha_0)^{1-\beta}(1+\|x\|_{\alpha_*})}{(1-\gamma)r}\right\}$$

where $C(x) := C(x, \alpha_0)$. Define

$$M(u) := \sup_{0\leq\tau\leq\frac{\alpha^*-\alpha_0}{\lambda}} \sup_{\substack{0\leq t<T(\alpha)\\ \alpha\in(\alpha_0,\alpha^*]}} (\alpha - \alpha_0 - \lambda t)^\gamma \|B(u(t),\tau)\|_\alpha,$$

and

$$S_x := \left\{u \in S^\gamma(\lambda) \;\middle|\; \|u - x\|^{(0)} < r, \quad M(u) \leq \left(\frac{C_3}{\alpha_0 - \alpha_*} + C(x)\right)(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*})\right\}. \tag{1.40}$$

Lemma 1.3.3. *$U(\cdot,0)x \in S_x$.*

Proof. Observe first that $U(\cdot,0)x \in S^\gamma$, then by A3 we see that for $\alpha \in (\alpha_0, \alpha^*)$ and $0 \le t < T(\alpha)$

$$\|U(t,0)x - x\|_\alpha \le \|U(t,0)x - x\|_{\alpha_0} \le C(x)t \le C(x)\frac{\alpha - \alpha_0}{\lambda}\,,$$

so

$$\|U(\cdot,0)x - x\|^{(0)} \le C(x)\frac{\alpha^* - \alpha_0}{\lambda} \le r\frac{\lambda_0}{\lambda} < r. \tag{1.41}$$

For $M(U(\cdot,0)x) \le (\frac{C_3}{\alpha_0-\alpha_*} + C(x))(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*})$ it suffices to show that for all $0 \le \tau \le \frac{\alpha^*-\alpha_0}{\lambda}$, $\alpha \in (\alpha_0, \alpha^*]$ and all $0 \le t < T(\alpha)$ we have

$$f(t) = \rho(t)^\gamma g(t) \le \left(\frac{C_3}{\alpha_0 - \alpha_*} + C(x)\right)(\alpha^* - \alpha_*)^\gamma(1 + \|x\|_{\alpha_*}),$$

where $\rho(t) := \alpha - \alpha_0 - \lambda t$ and $g(t) := \|B(U(t,0)x,\tau)\|_\alpha$. The assumptions on B imply that f is continuous in t and using

$$(Df)(t) = \limsup_{s\searrow t} \frac{f(s) - f(t)}{s - t}$$

we obtain

$$(Df)(t) = -\gamma\lambda\rho(t)^{\gamma-1}g(t) + \rho(t)^\gamma(Dg)(t).$$

Now set $\alpha = \alpha' + \frac{\rho(t)}{2}$, then $0 \le t < T(\alpha') < T(\alpha)$ and for $t < s < T(\alpha')$

$$g(s) - g(t) \le \|B(U(s,0)x,\tau) - B(U(t,0)x,\tau)\|_\alpha \le \frac{C_2}{(\alpha - \alpha')^{1-\beta}}\|U(s,0)x - U(t,0)x\|_{\alpha'}$$

and thus dividing by $s - t$ and letting $s \searrow t$ we conclude

$$(Dg)(t) \le \frac{C_2C(x)}{(\alpha - \alpha')^{1-\beta}} = \frac{2^{1-\beta}C_2C(x)}{\rho(t)^{1-\beta}}.$$

From

$$\begin{aligned}\rho(t)(Df)(t) &\le -\gamma\lambda f(t) + 2^{1-\beta}C_2C(x)\rho(t)^{\gamma+\beta}\\ &\le -\gamma\lambda f(t) + \gamma\lambda\left(\frac{C_3}{\alpha_0 - \alpha_*} + C(x)\right)(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*})\end{aligned}$$

we conclude that if $f(t) > (\frac{C_3}{\alpha_0-\alpha_*} + C(x))(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*})$, then $(Df)(t) < 0$. But since $M(x) \le (\frac{C_3}{\alpha_0-\alpha_*} + C(x))(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*})$ implies

$$f(0) \le \left(\frac{C_3}{\alpha_0 - \alpha_*} + C(x)\right)(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*}).$$

We conclude $(Df)(t) \le 0$, which shows

$$M(U(\cdot,0)x) \le \left(\frac{C_3}{\alpha_0-\alpha_*} + C(x)\right)(\alpha^*-\alpha_0)^\gamma(1+\|x\|_{\alpha_*}). \tag{1.42}$$

□

Lemma 1.3.4. *$U(\cdot,0)x + \mathcal{T}(u) \in S_x$ whenever $u \in S_x$.*

Proof. Take $u \in S_x$, fix $\alpha \in (\alpha_0, \alpha^*]$ and $0 \le t < T(\alpha)$. For $\alpha = \alpha' + \frac{\rho(t)}{2}$ with $\rho(t)$ as before we get $0 \le t < T(\alpha') < T(\alpha)$. Hence

$$\begin{aligned}
\|\mathcal{T}(u)(t)\|_\alpha &\le \int_0^t \|U(t,s)B(u(s),s)\|_\alpha \mathrm{d}s \le \frac{C_1}{(\alpha-\alpha')^\beta}\int_0^t \|B(u(s),s)\|_{\alpha'}\mathrm{d}s \\
&\le \frac{2^\beta C_1 M(u)}{\rho(t)^\beta}\int_0^t (\alpha'-\alpha_0-\lambda s)^{-\gamma}\mathrm{d}s \le \frac{2^\beta C_1 M(u)}{\rho(t)^\beta}\frac{(\alpha'-\alpha_0-\lambda t)^{1-\gamma}}{(1-\gamma)\lambda} \\
&\le 2^{\gamma-1+\beta}C_1\left(\frac{C_3}{\alpha_0-\alpha_*}+C(x)\right)(\alpha^*-\alpha_0)^\gamma(1+\|x\|_{\alpha_*})\frac{\rho(t)^{1-\gamma-\beta}}{(1-\gamma)\lambda} \\
&\le \frac{2^{\gamma-1+\beta}C_1}{(1-\gamma)\lambda}\left(\frac{C_3}{\alpha_0-\alpha_*}+C(x)\right)(\alpha^*-\alpha_0)^{1-\beta}(1+\|x\|_{\alpha_*})
\end{aligned}$$

yields

$$\|\mathcal{T}(u)\|^{(0)} \le \frac{2^{\gamma-1+\beta}C_1}{(1-\gamma)\lambda}\left(\frac{C_3}{\alpha_0-\alpha_*}+C(x)\right)(\alpha^*-\alpha_0)^{1-\beta}(1+\|x\|_{\alpha_*}).$$

Using (1.41) implies

$$\begin{aligned}
&\|U(\cdot,0)x+\mathcal{T}(u)-x\|^{(0)} \le \|U(\cdot,0)x-x\|^{(0)} + \|\mathcal{T}(u)\|^{(0)} \\
&\le \frac{C(x)(\alpha^*-\alpha_0)}{\lambda} + \frac{2^{\gamma-1+\beta}C_1}{(1-\gamma)\lambda}\left(\frac{C_3}{\alpha_0-\alpha_*}+C(x)\right)(\alpha^*-\alpha_0)^{1-\beta}(1+\|x\|_{\alpha_*}) \\
&\le r\frac{\lambda_0}{\lambda} < r.
\end{aligned}$$

For the second condition we have to show

$$M(U(\cdot,0)x+\mathcal{T}(u)) \le \left(\frac{C_3}{\alpha_0-\alpha_*}+C(x)\right)(\alpha^*-\alpha_*)^\gamma(1+\|x\|_{\alpha_*}).$$

Similar to the proof of Lemma 1.3.3 it suffices to show that for all $0 \le \tau \le T(\alpha^*)$, $\alpha \in (\alpha_0, \alpha^*]$ and $0 \le t < T(\alpha)$ we have

$$f(t) = \rho(t)^\gamma g(t) \le \left(\frac{C_3}{\alpha_0-\alpha_*}+C(x)\right)(\alpha^*-\alpha_*)^\gamma(1+\|x\|_{\alpha_*}),$$

where $g(t) := \|B(U(t,0)x + \mathcal{T}(u))(t), \tau)\|_\alpha$. Again we obtain

$$(Df)(t) = -\gamma\lambda\rho(t)^{\gamma-1}g(t) + \rho(t)^\gamma(Dg)(t).$$

Now set $\alpha = \alpha' + \frac{\rho(t)}{2}$ and $\alpha'' = \alpha' + \frac{\rho(t)}{4}$. We see that $\alpha = \alpha'' + \frac{\rho(t)}{4}$,

$$0 \leq t < T(\alpha') < T(\alpha'') < T(\alpha)$$

and hence for $t < s \in T(\alpha')$

$$\begin{aligned}
&g(s) - g(t)\\
&\leq \frac{C_2}{(\alpha-\alpha'')^{1-\beta}}\|U(s,0)x - U(t,0)x\|_{\alpha''} + \frac{C_2}{(\alpha-\alpha'')^{1-\beta}}\|\mathcal{T}(u)(s) - \mathcal{T}(u)(t)\|_{\alpha''}\\
&\leq \frac{4^{1-\beta}C_2}{\rho(t)^{1-\beta}}\|U(s,0)x - U(t,0)x\|_{\alpha'} + \frac{4^{1-\beta}C_2}{\rho(t)^{1-\beta}}\int_t^s \|(U(s,\tau) - U(t,\tau))B(u(\tau),\tau)\|_{\alpha''}\mathrm{d}\tau\\
&\leq \frac{4^{1-\beta}C_2}{\rho(t)^{1-\beta}}\|U(s,0)x - U(t,0)x\|_{\alpha'} + \frac{4C_1C_2}{\rho(t)}\int_t^s \|B(u(\tau),\tau)\|_{\alpha'}\mathrm{d}\tau.
\end{aligned}$$

We conclude that

$$\begin{aligned}
(Dg)(t) &\leq 4^{1-\beta}C_2\rho(t)^{-1+\beta}C(x) + \frac{4C_1C_2}{\rho(t)}\|B(u(t),t)\|_{\alpha'}\\
&\leq 4^{1-\beta}C_2C(x)\rho(t)^{-1+\beta} + 4C_1C_2\rho(t)^{-1}M(u)(\alpha' - \alpha_0 - \lambda t)^{-\gamma}\\
&\leq 4^{1-\beta}C_2C(x)\rho(t)^{-1+\beta}\\
&\quad + 2^{2+\gamma}C_1C_2\rho(t)^{-\gamma-1}\left(\frac{C_3}{\alpha_0 - \alpha_*} + C(x)\right)(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*})
\end{aligned}$$

holds and finally

$$\begin{aligned}
\rho(t)(Df)(t) &\leq -\gamma\lambda f(t) + 4^{1-\beta}C_2C(x)\rho(t)^{\beta+\gamma}\\
&\quad + 2^{2+\gamma}C_1C_2\left(\frac{C_3}{\alpha_0 - \alpha_*} + C(x)\right)(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*})\\
&\leq -\gamma\lambda f(t) + \gamma\lambda\left(\frac{C_3}{\alpha_0 - \alpha_*} + C(x)\right)(\alpha^* - \alpha_0)^\gamma(1 + \|x\|_{\alpha_*}).
\end{aligned}$$

The assertion follows immediately as in Lemma 1.3.3. □

Let $u \in S^{\gamma+1-\beta}(\lambda)$, fix $\alpha \in (\alpha_0, \alpha^*]$ and set $\alpha = \alpha' + \frac{\rho(t)}{2}$ with $\rho(t)$ as before, and

$0 \leq t < T(\alpha)$. Then

$$(\alpha-\alpha_0-\lambda t)^\gamma \left\| \int_0^t U(t,s)u(s)\mathrm{d}s \right\|_\alpha \leq \frac{C_1}{(\alpha-\alpha')^\beta}(\alpha-\alpha_0-\lambda t)^\gamma \int_0^t \|u(s)\|_{\alpha'}\mathrm{d}s$$
$$\leq 2^\beta C_1 \rho(t)^{\gamma-\beta} \int_0^t (\alpha'-\alpha_0-\lambda s)^{-(\gamma+1-\beta)}\mathrm{d}s \|u\|^{(\gamma+1-\beta)}$$
$$\leq \frac{2^\beta C_1}{(\gamma-\beta)\lambda}\rho(t)^{\gamma-\beta}\|u\|^{(\gamma+1-\beta)}(\alpha'-\alpha_0-\lambda t)^{-\gamma+\beta}$$
$$= \frac{2^\gamma C_1}{(\gamma-\beta)\lambda}\|u\|^{(\gamma+1-\beta)}$$

implies that

$$\left\| \int_0^\bullet U(\cdot,s)u(s)\mathrm{d}s \right\|^{(\gamma)} \leq \frac{2^\gamma C_1}{(\gamma-\beta)\lambda}\|u\|^{(\gamma+1-\beta)}. \tag{1.43}$$

Analogously to (1.43) one shows that for $u, v \in S_x$

$$\|B(u(\cdot),\cdot) - B(v(\cdot),\cdot)\|^{(\gamma+1-\beta)} \leq 2^{\gamma+1-\beta}C_2\|u-v\|^{(\gamma)}. \tag{1.44}$$

Now let $u, v \in S_x$, then by (1.43) and (1.44) we arrive at

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|^{(\gamma)} \leq \frac{2^\gamma C_1}{(\gamma-\beta)\lambda}\|B(u(\cdot),\cdot)-B(v(\cdot),\cdot)\|^{(\gamma+1-\beta)}$$
$$\leq \frac{2^{2\gamma+1-\beta}C_1C_2}{(\gamma-\beta)\lambda}\|u-v\|^{(\gamma)} \leq \frac{\lambda_0}{\lambda}\|u-v\|^{(\gamma)}.$$

Setting $u^{(0)} = U(\cdot,0)x \in S_x$ and $u^{(k+1)} = U(\cdot,0)x + \mathcal{T}(u^{(k)})$ for $k \in \mathbb{N}_0$, Lemma 1.3.3 and 1.3.4 imply $u^{(k)} \in S_x$. The estimate $\|u^{(k+1)} - u^{(k)}\|^{(\gamma)} \leq \left(\frac{\lambda_0}{\lambda}\right)^k \|u^{(1)} - u^{(0)}\|^{(\gamma)}$ shows that $(u^{(k)})_{k\geq 0}$ is a Cauchy-sequence and thus there is a limit $u = \lim\limits_{k\to\infty} u^{(k)}$ in S^γ. Since for any $\alpha \in (\alpha_0, \alpha^*]$, $t \in [0, T(\alpha))$ and $k \in \mathbb{N}$: $\|u^{(k)}(t) - x\|_\alpha < r$, passing to the limit $k \to \infty$ yields $\|u(t) - x\|_\alpha \leq r$. Hence u is a solution to (1.38) in the scale $(\mathbb{B}_\alpha)_{\alpha\in[\alpha_0,\alpha^*]}$.

For uniqueness let u, v be two solutions to (1.38) in the scale $(\mathbb{B}_\alpha)_{\alpha\in[\alpha_0,\alpha^*]}$. Then by

$$(\alpha-\alpha_0-\lambda t)^\gamma\|u(t)\|_\alpha \leq (\alpha-\alpha_0-\lambda t)^\gamma\|u(t)-x\|_\alpha + (\alpha-\alpha_0-\lambda t)^\gamma\|x\|_\alpha$$
$$\leq (\alpha^*-\alpha_0)^\gamma r + (\alpha^*-\alpha_0)^\gamma\|x\|_{\alpha_*}$$

and similar estimate for v we get $u, v \in S^\gamma$. By (1.43) and (1.44) we obtain

$$\|u-v\|^{(\gamma)} = \|\mathcal{T}(u)-\mathcal{T}(v)\|^{(\gamma)} \leq \frac{\lambda_0}{\lambda}\|u-v\|^{(\gamma)},$$

which implies $u = v$. The last claim is a consequence of the next lemma.

Lemma 1.3.5. *Let $\alpha_* < \alpha_0 < \alpha_1$, denote by u_{α_0} and u_{α_1} the corresponding unique solutions in the scales $(\mathbb{B}_\alpha)_{\alpha\in[\alpha_0,\alpha^*]}$ respectively $(\mathbb{B}_\alpha)_{\alpha\in[\alpha_1,\alpha^*]}$ with $\lambda > \max\{\lambda_0(\alpha_0), \lambda_0(\alpha_1)\}$. Then for each $\alpha \in (\alpha_1, \alpha^*]$, we have*

$$u_{\alpha_0}\big|_{[0,\frac{\alpha-\alpha_1}{\lambda})} = u_{\alpha_1}.$$

Proof. The collection of functions $u_{\alpha_0}|_{[0,\frac{\alpha-\alpha_1}{\lambda})}$, $\alpha \in (\alpha_1, \alpha^*]$ is a solution in the scale $(\mathbb{B}_\alpha)_{\alpha\in[\alpha_1,\alpha^*]}$ and applying the uniqueness property for u_{α_1} we obtain the assertion. □

□

Remark 1.3.6. *Suppose that $U(t, 0)$ satisfies the inequality*

$$\|U(t,0)x - U(s,0)x\|_\alpha \leq C(x)|t-s|$$

with a constant $C(x) > 0$ independent of α. Then we can choose $\alpha_0 = \alpha_*$ in the main statement.

Let us now show existence and uniqueness of classical solutions to equation (1.5). Therefore, let $(\mathbb{E}_\alpha, ||| \cdot |||_\alpha)_{\alpha\in[\alpha_*,\alpha^*]}$ be another scale of Banach spaces with $\mathbb{B}_\alpha \subset \mathbb{E}_\alpha$ continuously embedded and $||| \cdot |||_\alpha \leq \| \cdot \|_\alpha$. The next condition relates the evolution system to its infinitesimal generator $A(t)$.

A4. There exists a family of linear operators $(A(t))_{t\in[0,\frac{\alpha^*-\alpha_*}{\lambda})}$ such that for all $\alpha' < \alpha$

$$\left[0, \frac{\alpha^*-\alpha_*}{\lambda}\right) \ni t \longmapsto A(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{E}_\alpha)$$

is strongly continuous. Moreover, the map $(t, s) \longmapsto U(t, s) \in L(\mathbb{B}_{\alpha'}, \mathbb{E}_\alpha)$ is strongly continuously differentiable with derivatives

$$\frac{\partial U}{\partial t}(t,s) = A(t)U(t,s), \quad 0 \leq s \leq t < \frac{\alpha^*-\alpha_*}{\lambda}$$

and

$$\frac{\partial U}{\partial s}(t,s) = -U(t,s)A(s), \quad 0 \leq s \leq t < \frac{\alpha^*-\alpha_*}{\lambda}.$$

The case $s = t$ should be understood as right or left derivative correspondingly.

Note that, in the case $\mathbb{E}_\alpha = \mathbb{B}_\alpha$ conditions A1 and A4 imply condition A3.

Definition 1.3.7. *A function $u : [0, \frac{\alpha^*-\alpha_*}{\lambda}) \longrightarrow \mathbb{B}_{\alpha^*}$ is called classical $\mathbb{B}$-valued solution to (1.5) if for each $\alpha \in (\alpha_*, \alpha^*]$ the restriction*

$$u|_{[0,\frac{\alpha-\alpha_*}{\lambda})} \in C^1\left(\left[0, \frac{\alpha-\alpha_*}{\lambda}\right); \mathbb{E}_\alpha\right) \cap C\left(\left[0, \frac{\alpha-\alpha_*}{\lambda}\right); \mathbb{B}_\alpha\right) \tag{1.45}$$

satisfies $\|u(t) - x\|_\alpha \leq r$, and it is a classical solution to (1.5) in $\mathbb{E}_\alpha$.

The next lemma shows that in the framework of scales of Banach spaces the concepts of mild and classical $\mathbb{B}$-valued solutions coincide. A summary of the classical concepts of $\mathbb{B}$-valued solutions can be found in [Paz83].

Lemma 1.3.8. *Let $x \in \mathbb{B}_{\alpha_*}$ and $u : [0, \frac{\alpha^*-\alpha_*}{\lambda}) \longrightarrow \mathbb{B}_{\alpha^*}$. Then u is a classical $\mathbb{B}$-valued solution to* (1.5) *if and only if u is a solution to* (1.38) *in the scale $(\mathbb{B}_\alpha)_{\alpha \in [\alpha_*, \alpha^*]}$.*

Proof. Suppose that u is a classical $\mathbb{B}$-valued solution to (1.5). Then for all $0 \leq s < t < \frac{\alpha-\alpha_*}{\lambda}$ and each $\alpha \in (\alpha_*, \alpha^*]$

$$\frac{\partial}{\partial s}\left(U(t,s)u(s)\right) = U(t,s)B(u(s),s)$$

holds in $\mathbb{E}_\alpha$. Hence integrating over s yields (1.38). For the converse let v be given with (1.45). Fix $\alpha \in (\alpha_*, \alpha^*]$, $t \in [0, \frac{\alpha-\alpha_*}{\lambda})$ and let $\alpha' \in (\alpha_*, \alpha)$ such that $0 \leq t < \frac{\alpha'-\alpha_*}{\lambda} < \frac{\alpha-\alpha_*}{\lambda}$ Then $v(s) \in \mathbb{B}_{\alpha'}$ for $s \in [0,t]$ and we get that the mapping $(t,s) \longmapsto U(t,s)v(s) \in \mathbb{E}_\alpha$ is continuous and continuously differentiable in t for fixed $s \in [0,t]$. Thus

$$\left[0, \frac{\alpha-\alpha_*}{\lambda}\right) \ni t \longmapsto \int_0^t U(t,s)v(s)\mathrm{d}s \in \mathbb{B}_\alpha$$

is continuously differentiable with derivative $v(t) + A(t)\int_0^t U(t,s)v(s)\mathrm{d}s$. Let u solve (1.38), then applying above argumentation to $v(s) := B(u(s),s)$ yields differentiability and differentiating (1.38) yields (1.5). □

Corollary 1.3.9. *Assume that conditions A1 – A4 and B1 – B3 are satisfied and let $x \in \mathbb{B}_{\alpha_*}$. Then for any $\alpha_0 \in (\alpha_*, \alpha^*)$ the solution given by Theorem 1.3.2 yields a unique classical $\mathbb{B}$-valued solution in the scale $(\mathbb{E}_\alpha)_{\alpha \in [\alpha_0, \alpha^*]}$ to* (1.5)*.*

Stability with respect to parameters

For the whole section we suppose that the conditions below are satisfied.

1. There exist $x_n, x \in \mathbb{B}_{\alpha_*}$ with $x_n \to x$ as $n \to \infty$.

2. There exist evolution systems $(U(s,t))_{0 \leq s \leq t < \frac{\alpha^*-\alpha_*}{\lambda}}$ and $(U_n(t,s))_{0 \leq s \leq t < \frac{\alpha^*-\alpha_*}{\lambda}}$ for $n \in \mathbb{N}$ satisfying properties A1 and A2 with constants $C_1 > 0$ and $\beta \in [0, \frac{1}{2})$ independent of $n \in \mathbb{N}$.

3. For any $\alpha \in (\alpha_*, \alpha^*)$ there exist a constant $C(\alpha) > 0$ such that for all $0 \leq s, t \leq \frac{\alpha-\alpha_*}{\lambda}$

$$\|U_n(t,0)x_n - U_n(s,0)x_n\|_\alpha \leq C(\alpha)|t-s|.$$

4. There exist operators B and B_n satisfying properties B1 – B3 with constants $\lambda, r, C_2, C_3 > 0$ independent of $n \in \mathbb{N}$.

5. For all $\alpha' < \alpha$ and each $z \in \mathbb{B}_{\alpha'}$ we have

$$U_n(t,s)z \to U(t,s)z, \quad n \to \infty \tag{1.46}$$

in $\mathbb{B}_\alpha$ uniformly on compacts in (t,s). If in addition $\|z - x\|_{\alpha'} \leq r$, then we have

$$B_n(z,t) \to B(z,t), \quad n \to \infty$$

in $\mathbb{B}_\alpha$ uniformly on compacts in t.

If for instance $A_n(t) \to A(t)$ in $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ and A4 is satisfied for $A(t)$ and $A_n(t)$, $n \in \mathbb{N}$, then (1.46) holds.

Theorem 1.3.10. *For each $\alpha_0 \in (\alpha_*, \alpha^*)$ there exist $\lambda_0 > 0$ and provided $\lambda > \lambda_0$, there exist unique solutions u to* (1.38) *and u_n to*

$$u_n(t) = U_n(t,0)x_n + \int\limits_0^t U_n(t,s)B_n(u_n(s),s)\mathrm{d}s \tag{1.47}$$

in the scale $(\mathbb{B}_\alpha)_{\alpha \in [\alpha_0, \alpha^*]}$. Moreover, for any $\alpha \in (\alpha_0, \alpha^*]$ and $T \in (0, \frac{\alpha - \alpha_0}{\lambda})$

$$u_n(t) \to u(t), \quad n \to \infty \tag{1.48}$$

holds uniformly on $[0,T]$ in $\mathbb{B}_\alpha$.

The rest of this section is devoted to the proof. By definition of λ_0 in Theorem 1.3.2 we can chose λ_0 to be independent of $n \in \mathbb{N}$, which implies the first assertion. Denote by u, u_n the corresponding solutions and by $\mathcal{T}_n$ the non-linear integral operator given as in (1.39) with B and U replaced by B_n and U_n.

Lemma 1.3.11. *$\|\mathcal{T}_n(U_n(\cdot,0)x_n)\|^{(\gamma)}$ is uniformly bounded in $n \in \mathbb{N}$.*

Proof. An analogous estimate to (1.42) shows that there is $C > 0$ such that

$$\sup_{0 \leq \tau \leq \frac{\alpha^* - \alpha_0}{\lambda}} \sup_{\substack{0 \leq t < T(\alpha) \\ \alpha \in (\alpha_0, \alpha^*]}} (\alpha - \alpha_0 - \lambda t)^\gamma \|B_n(U_n(\cdot,0)x_n, \tau)\|_\alpha \leq C, \quad n \in \mathbb{N}.$$

Let $\alpha \in (\alpha_0, \alpha^*]$, $t \in [0, T(\alpha))$, $\rho(t) = \alpha - \alpha_0 - \lambda t$ and define α' by the relation $\alpha = \alpha' + \frac{\rho(t)}{2}$, then $t < \frac{\alpha' - \alpha_0}{\lambda}$. Hence the estimate

$$\begin{aligned}
&(\alpha - \alpha_0 - \lambda t)^\gamma \left\| \int_0^t U_n(t,s) B_n(U_n(s,0)x_n, s) \mathrm{d}s \right\|_\alpha \\
&\leq C_1 \frac{(\alpha - \alpha_0 - \lambda t)^\gamma}{(\alpha - \alpha')^\beta} \int_0^t \|B_n(U_n(s,0)x_n, s)\|_{\alpha'} \mathrm{d}s \leq C C_1 2^\beta \rho(t)^{\gamma - \beta} \int_0^t (\alpha' - \alpha_0 - \lambda s)^{-\gamma} \mathrm{d}s \\
&\leq \frac{C C_1 2^\beta \rho(t)^{\gamma-\beta}}{\lambda(1-\gamma)} (\alpha' - \alpha_0 - \lambda t)^{1-\gamma} \leq \frac{2^{\gamma - 1 + \beta} C C_1}{\lambda(1-\gamma)} (\alpha_0 - \alpha_*)^{1-\beta}
\end{aligned}$$

implies the assertion. □

Fix $\alpha \in (\alpha_0, \alpha^*]$, $T \in (0, T(\alpha))$ and denote by $(u^{(k)})_{k \in \mathbb{N}}$ the sequence defined by $u^{(0)} = U(\cdot, 0)x$, $u^{(k+1)} = U(\cdot, 0)x + \mathcal{T}(u^{(k)})$. Similarly let $(u_n^{(k)})_{k\in\mathbb{N}}$ be given by $u_n^{(0)} = U_n(\cdot, 0)x_n$ and $u_n^{(k+1)} = U_n(\cdot, 0)x_n + \mathcal{T}_n(u_n^{(k)})$. For $\gamma \in (\beta, 1-\beta)$ we obtain

$$\begin{aligned}
\|u^{(k)} - u\|^{(\gamma)} &\leq \sum_{j=k}^{\infty} \|u^{(j+1)} - u^{(j)}\|^{(\gamma)} \leq \sum_{j=k}^{\infty} \left(\frac{\lambda_0}{\lambda}\right)^j \|u^{(1)} - u^{(0)}\|^{(\gamma)} \\
&= \sum_{j=k}^{\infty} \left(\frac{\lambda_0}{\lambda}\right)^j \|\mathcal{T}(U(\cdot, 0)x)\|^{(\gamma)}
\end{aligned}$$

and

$$\|u_n^{(k)} - u_n\|^{(\gamma)} \leq \sum_{j=k}^{\infty} \left(\frac{\lambda_0}{\lambda}\right)^j \|\mathcal{T}_n(U_n(\cdot, 0)x_n)\|^{(\gamma)}.$$

By Lemma 1.3.11 and

$$\|u(t) - u_n(t)\|_\alpha \leq (\alpha - \alpha_0 - \lambda t)^{-\gamma} \left(\|u^{(k)} - u\|^{(\gamma)} + \|u^{(k)}(t) - u_n^{(k)}(t)\|_\alpha + \|u_n^{(k)} - u_n\|^{(\gamma)} \right),$$

which implies that the first and last term tend to zero uniformly in n as $k \to \infty$. Thus it suffices to show for each k

$$\|u^{(k)}(t) - u_n^{(k)}(t)\|_\alpha \to 0, \quad n \to \infty$$

uniformly on $[0, T]$. However this is a consequence of the below lemma.

Lemma 1.3.12. *Let $v_n, v : [0, \frac{\alpha^* - \alpha_0}{\lambda}) \longrightarrow \mathbb{B}_{\alpha^*}$ be two functions with the properties that for each $\nu \in (\alpha_0, \alpha^*]$ and all $n \in \mathbb{N}$:*

1. *$\|v_n(t) - x_n\|_\nu < r$ and $\|v(t) - x\|_\nu < r$ for all $0 \leq t < \frac{\nu - \alpha_0}{\lambda}$.*

2. $v_n|_{[0,\frac{\nu-\alpha_0}{\lambda})}, v|_{[0,\frac{\nu-\alpha_0}{\lambda})} \in C([0,\frac{\nu-\alpha_0}{\lambda});\mathbb{B}_\nu)$ *and* $v_n(0) = x_n$, $v(0) = x$.

3. For each $0 < T < \frac{\nu-\alpha_0}{\lambda}$ *the convergence*

$$\|v_n(t) - v(t)\|_\nu \to 0, \quad n \to \infty$$

holds uniformly on $[0, T]$.

Then for each $\nu \in (\alpha_0, \alpha^*]$ *and* $T \in (0, T(\nu))$

$$\|U_n(t,0)x_n + \mathcal{T}_n(v_n)(t) - U(t,0)x - \mathcal{T}(v)(t)\|_\nu \to 0, \quad n \to 0$$

uniformly on $[0, T]$.

Proof. Consider

$$\begin{aligned}
&\|U_n(t,0)x_n + \mathcal{T}_n(v_n)(t) - U(t,0)x - \mathcal{T}(v)(t)\|_\nu \\
&\leq \|U_n(t,0)x_n - U(t,0)x\|_\nu + \|\mathcal{T}_n(v_n)(t) - \mathcal{T}(v)(t)\|_\nu \\
&\leq \|(U_n(t,0) - U(t,0))x\|_\nu + \|U_n(t,0)(x_n - x)\|_\nu + \\
&+ \quad \|\mathcal{T}_n(v)(t) - \mathcal{T}(v)(t)\|_\nu + \|\mathcal{T}_n(v)(t) - \mathcal{T}_n(v_n)(t)\|_\nu \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Take $\varepsilon > 0$, then we find $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \in [0, T]$

$$I_1 = \|(U_n(t,0) - U(t,0))x\|_\nu \leq \varepsilon.$$

For I_2 and $n \geq n_1$, where $n_1 = n_1(\varepsilon)$ is sufficiently large, we obtain

$$I_2 = \|U_n(t,s)(x_n - x)\|_\nu \leq \frac{C_1}{(\nu - \alpha_*)^\beta}\|x_n - x\|_{\alpha*} \leq \frac{C_1}{(\nu - \alpha_*)^\beta}\varepsilon.$$

It remains to estimate I_3, I_4. For I_3 we get

$$\begin{aligned}
I_3 &\leq \int_0^t \|(U_n(t,s) - U(t,s))B(v(s),s)\|_\nu \mathrm{d}s + \int_0^t \|U_n(t,s)(B_n(v(s),s) - B(v(s),s))\|_\nu \mathrm{d}s \\
&= J_1 + J_2.
\end{aligned}$$

Take $\alpha_0 < \alpha' < \alpha'' < \nu$ such that $T < \frac{\alpha'-\alpha_0}{\lambda}$, then $v_n(t), v(t) \in \mathbb{B}_{\alpha'}$ for $t \in [0, T]$ and hence the set $K_T = \{B(v(s), s) \mid s \in [0, T]\} \subset \mathbb{B}_{\alpha''}$ is compact. For large n, i.e. for $n \geq n_2$ we obtain that

$$J_1 \leq \int_0^t \sup_{z \in K_T} \|(U_n(t,s) - U(t,s))z\|_\nu \mathrm{d}s < \varepsilon T.$$

Similarly for J_2 the set $K_T = \{v(s)|s \in [0,T]\} \subset \mathbb{B}_{\alpha'}$ is compact. Thus there exists $\delta \in (0,1)$ such that $\|v(t) - x\|_{\alpha'} < (1-\delta)r$ for $t \in [0,T]$. Let $n_3 \geq n_2$ be such that $\|x - x_n\| < \delta r$ for $n_3 \geq n_2$, then

$$\|v(t) - x_n\|_{\alpha'} \leq \|v(t) - x\|_{\alpha'} + \|x - x_n\|_{\alpha'} < (1-\delta)r + \delta r = r.$$

Hence by assumption 5. there exists $n_4 \geq n_3$ such that

$$\begin{aligned} J_2 &\leq \frac{C_2}{(\nu - \alpha'')^{1-\beta}} \int_0^t \|B_n(v(s), s) - B(v(s), s)\|_{\alpha''} \mathrm{d}s \\ &\leq \frac{C_2}{(\nu - \alpha'')^{1-\beta}} \int_0^t \sup_{z \in K_T} \|B_n(z, s) - B(z, s)\|_{\alpha''} \mathrm{d}s \leq \frac{C_2 T}{(\nu - \alpha'')^{1-\beta}} \varepsilon. \end{aligned}$$

Concerning I_4 we obtain

$$\begin{aligned} \|T_n(v_n)(t) - T_n(v)(t)\|_\nu &\leq \int_0^t \|U_n(t,s)(B_n(v_n(s), s) - B_n(v(s), s)\|_\nu \mathrm{d}s \\ &\leq \frac{C_1}{(\nu - \alpha'')^\beta} \frac{C_2}{(\alpha'' - \alpha')^{1-\beta}} \int_0^t \|v_n(s) - v(s)\|_{\alpha'} \mathrm{d}s \\ &= \frac{C_1}{(\nu - \alpha'')^\beta} \frac{C_2}{(\alpha'' - \alpha')^{1-\beta}} T \sup_{s \in [0,T]} \|v_n(s) - v(s)\|_{\alpha'}, \end{aligned}$$

which shows the assertion. □

Chapter 2

Markov evolutions on Γ_0

This chapter is devoted to the construction and study of birth-and-death Markov evolutions in continuum with the additional constraint that for any moment of time $t \geq 0$ the number of particles remains finite. For shorthand notation we call such Markov evolutions: finite evolution, finite process or simply finite system. In the first section we discuss Markov jump processes on arbitrary locally compact Polish spaces. These results are afterwards applied (sections two and three) to (finite) birth-and-death Markov evolutions with the location space $\mathbb{R}^d$.

2.1 General Markov jump processes

Let E be a locally compact Polish space and denote by $\mathcal{B}(E)$ the Borel-σ-algebra on E. Denote by $BM(E)$ the Banach space of all bounded measurable functions and by $C_b(E)$ the subspace of all continuous bounded functions. A pure jump process is determined by its (infinitesimal) transition function, i.e. a function $Q : \mathbb{R}_+ \times E \times \mathcal{B}(E) \longrightarrow \mathbb{R}_+$ with the following properties:

1. For all $t \geq 0$, $x \in E$, $A \longmapsto Q(t, x, A)$ is a finite Borel measure with $Q(t, x, \{x\}) = 0$.
2. For all $A \in \mathcal{B}(E)$, $(t, x) \longmapsto Q(t, x, A)$ is measurable.
3. For all $T > 0$ and all compacts $B \subset E$

$$\sup_{(t,x)\in[0,T]\times B} Q(t, x, E) < \infty. \tag{2.1}$$

Let $BM_{loc}(E)$ be the space of locally bounded measurable functions and $C(E)$ be the space of continuous functions. Define for any $F : E \longrightarrow \mathbb{R}$

$$(Q(t)F)(x) := \int\limits_E F(y)Q(t, x, \mathrm{d}y), \quad t \geq 0,\ x \in E,$$

whenever it makes sense, i.e. $\int_E |F(y)|Q(t,x,\mathrm{d}y) < \infty$ for all $t \geq 0$, $x \in E$. Then $Q(t) : BM(E) \longrightarrow BM_{loc}(E)$ is a well-defined positive linear operator and $q(t,x) := (Q(t)1)(x) = Q(t,x,E)$ is locally bounded. If $Q(\cdot)C_b(E) \subset C(\mathbb{R}_+ \times E)$, i.e. for any $F \in C_b(E)$ the function $Q(\cdot)F$ is jointly continuous in (t,x), then we say that Q is jointly continuous. This simply means that, by definition, $Q(t,x,\mathrm{d}y)$ is weakly continuous in (t,x). In such a case (2.1) is automatically satisfied.

We briefly recall the results obtained in [Fel40, FMS14]. Let Q be a transition function. For $0 \leq s \leq t, x \in E$ and $A \in \mathcal{B}(E)$ let $P^{(0)}(s,x;t,A) := \delta(x,A)e^{-\int_s^t q(r,x)\mathrm{d}r}$, and for $n \geq 1$

$$P^{(n+1)}(s,x;t,A) := \int_s^t e^{-\int_s^r q(\tau,x)\mathrm{d}\tau} \left(\int_E P^{(n)}(r,y;t,A)Q(r,x,\mathrm{d}y) \right) \mathrm{d}r. \tag{2.2}$$

Here $\delta(x,A) := \mathbb{1}_A(x) = \delta_x(A)$. Then $P(s,x;t,A) = \sum_{n=0}^{\infty} P^{(n)}(s,x;t,A)$ is a sub-Markov transition function. Moreover, for fixed $A \in \mathcal{B}(E)$ and $x \in E$ it is absolutely continuous in s and t, respectively such that $P(s,x;t,A) \to \delta(x,A)$ holds uniformly in $A \in \mathcal{B}(E)$ whenever $s \to t^-$ or $t \to s^+$. For any $A \in \mathcal{B}(E)$ it is a.e. differentiable in $s \in [0,t]$ and satisfies

$$\frac{\partial P(s,x;t,A)}{\partial s} = q(s,x)P(s,x;t,A) - \int_E P(s,y;t,A)Q(s,x,\mathrm{d}y). \tag{2.3}$$

Likewise, for any compact $A \subset E$ it is differentiable for a.a. $t \in [s,\infty)$ and satisfies

$$\frac{\partial P(s,x;t,A)}{\partial t} = -\int_A q(t,y)P(s,x;t,\mathrm{d}y) + \int_E Q(t,y,A)P(s,x,t,\mathrm{d}y). \tag{2.4}$$

It follows from [FMS14] that P is the minimal solution to (2.3) and (2.4). Moreover, if $P(s,x;t,E) = 1$, then this solution is also unique.

The main point of our interest is to study the (sub-)Markovian evolution system

$$U(s,t)F(x) := \int_E F(y)P(s,x;t,\mathrm{d}y), \quad 0 \leq s \leq t \tag{2.5}$$

on the space of bounded measurable functions and extensions of it. Such an evolution system is a family of positive bounded linear operators such that $U(s,s)F = F$ and $U(s,r)U(r,t)F = U(s,t)F$ for $0 \leq s \leq r \leq t$. In the framework of general linear evolution equations discussed in the first chapter, above evolution system satisfies the backward evolution property. For $F \in BM(E)$ let

$$L(t)F(x) = \int_E (F(y) - F(x))Q(t,x,\mathrm{d}y), \quad t \geq 0 \tag{2.6}$$

be the (formal) generator of $U(s,t)$. Since in general $U(s,t)F$ is not continuous w.r.t. the norm on $BM(E)$ or $C_b(E)$, we cannot expect that some extension of $L(t)$ is a generator. One possibility to overcome this problem is to restrict $U(s,t)$, provided it is possible, to the space of continuous functions vanishing at infinity. Another possibility is to characterize $U(s,t)$ by its strict generator, cf. [Cas11]. It is also possible to consider other topologies which leads, e.g., to the concept of π-semigroups (in our case to π-evolution systems), cf. [Pri99]. For our needs it is sufficient to consider only the weaker concept of pointwise generator, for the precise meaning see the Proposition below.

Denote by $\mathcal{C}$ the collection of compact sets on E and by $\mathcal{C}_1$ the collection of compacts in $\mathbb{R}_+ \times E$. For a given non-negative function $V \in C(E)$ let $\|F\|_V := \sup\limits_{x\in E} \frac{|F(x)|}{1+V(x)}$ and denote by $BM_V(E)$ the space of all measurable functions for which $\|F\|_V$ is finite. Denote by $C_V(E) := BM_V(E) \cap C(E)$ its closed subspace of continuous functions. Below we state the main result for this section.

Proposition 2.1.1. *Assume that there exists a continuous function $V : E \longrightarrow \mathbb{R}_+$ such that $(t,x) \longmapsto Q(t)F(x)$ is continuous for any $F \in C_V(E)$. Moreover, suppose that there exists a continuous function $c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the properties below are satisfied.*

1. *For all $T > 0$ there exists $a(T) > 0$ such that $q(t,x) \leq a(T)V(x)$ holds for all $t \in [0,T]$ and $x \in E$.*

2. *The Foster-Lyapunov estimate*

$$\int\limits_E V(y)Q(t,x,\mathrm{d}y) \leq c(t)V(x) + q(t,x)V(x), \quad t \geq 0,\ x \in E \tag{2.7}$$

   is satisfied.

3. *For all $\varepsilon > 0$, $B \in \mathcal{C}$ and $T > 0$ there exists $A \in \mathcal{C}$ such that*

$$\int\limits_0^T Q(r,x,A^c)\mathrm{d}r < \varepsilon, \quad x \in B \tag{2.8}$$

   is fulfilled.

Then $U(s,t)$ is a conservative Feller evolution system, i.e. $U(s,t)1 = 1$ and $(s,t,x) \longmapsto U(s,t)F(x)$ is continuous for any $F \in C_b(E)$. Moreover, $U(s,t)$ can be extended to $BM_V(E)$ so that

$$\|U(s,t)F\|_V \leq e^{\int_s^t c(r)\mathrm{d}r}\|F\|_V, \quad 0 \leq s \leq t. \tag{2.9}$$

The relation to the Kolmogorov equations is given by the statements below:

(a) *For any* $F \in BM(E)$*,* $t > 0$ *and* $x \in E$*,* $[0, t] \ni s \longmapsto U(s, t)F(x)$ *is continuously differentiable and a solution to*

$$\frac{\partial}{\partial s} U(s,t)F(x) = -L(s)U(s,t)F(x). \tag{2.10}$$

If in addition $F \in C_V(E)$*, then* $s \longmapsto U(s, t)F(x)$ *is absolutely continuous and satisfies* (2.10) *a.e.*

(b) *Let* $F \in BM(E)$*. Then for any* $x \in E$*,* $s \geq 0$*,* $[s, \infty) \ni t \longmapsto U(s, t)F(x)$ *is absolutely continuous and satisfies for a.a.* $t \geq s$

$$\frac{\partial}{\partial t} U(s,t)F(x) = U(s,t)L(t)F(x). \tag{2.11}$$

(c) *Let* $V(s, t)$ *be a Feller evolution system on* $C_b(E)$*. If for any* $F \in C_b(E)$*,* $V(s, t)F$ *is a solution to* (2.10) *or* (2.11)*, then* $V(s, t) = U(s, t)$ *holds.*

The time-homogeneous case was, e.g., treated in [Che04, Kol06]. Condition (2.7) can be reformulated to

$$\int_E (V(y) - V(x))Q(t, x, \mathrm{d}y) \leq c(t)V(x), \quad t \geq 0, \ x \in E.$$

A transition function Q with property (2.8) is said to have the localization property. Property (c) means that $U(s, t)$ is the unique Feller evolution system associated with the operator $L(t)$. The rest of this section is devoted to the proof of above statement.

Suppose from now on the conditions given in Proposition 2.1.1 to be satisfied and let $\alpha \in (0, 1)$. Applying the iteration (2.2) to $(q(t, x), \alpha Q(t, x, \mathrm{d}y))$ yields the sub-probability function given by

$$P_\alpha(s, x; t, \mathrm{d}y) = \sum_{n=0}^{\infty} \alpha^n P^{(n)}(s, x; t, \mathrm{d}y). \tag{2.12}$$

Let $U_n(s, t)F(x) := \int_E F(y)P^{(n)}(s, x; t, \mathrm{d}y)$, then $U_\alpha(s, t)F(x) := \sum_{n=0}^{\infty} \alpha^n U^{(n)}(s, t)F(x)$ defines an evolution system. We will call $U_\alpha(s, t)$ the regularized evolution system associated to Q. Clearly, above series converges uniformly in (s, t, x). The next lemma establishes the Feller property for $U_\alpha(s, t)$, whereas the limit $\alpha \to 1$ will be considered at the end of this section.

Lemma 2.1.2. $(U_\alpha(s, t))_{0 \leq s \leq t}$ *is a Feller evolution on* $C_b(E)$*.*

Proof. It suffices to show that for any $n \geq 0$ and each $F \in C_b(E)$ the function $U^{(n)}(s,t)F(x)$ is continuous in all variables. Since $U^{(0)}(s,t)F(x) = F(x)e^{-\int_s^t q(r,x)\mathrm{d}r}$, by Lemma A.2.1 this clearly holds for $n = 0$. Assume the assertion holds for some $n \geq 0$. By (2.2) we get

$$U^{(n+1)}(s,t)F(x) = \int_s^t \int_E e^{-\int_s^r q(\tau,x)\mathrm{d}\tau}(U^{(n)}(r,t)F)(y)Q(r,x,\mathrm{d}y)\mathrm{d}r. \tag{2.13}$$

By induction hypothesis $e^{-\int_s^r q(\tau,x)\mathrm{d}\tau}(U^{(n)}(r,t)F)(y)$ is continuous in all variables. Moreover, due to $|U^{(n)}(r,t)F(y)| \leq \|F\|_\infty$ this function is bounded and hence by Lemma A.2.2 we see that also

$$(s,t,x) \longmapsto \int_E e^{-\int_s^r q(\tau,x)\mathrm{d}\tau}(U^{(n)}(r,t)F)(y)Q(r,x,\mathrm{d}y)$$

is continuous. Thus Lemma A.2.1 yields the continuity of $U^{(n+1)}(s,t)F(x)$ in the variables (s,t,x). □

The next result studies stability of the Feller evolution $U_\alpha(s,t)$ with respect to Q. That is given a sequence of transition functions $(Q_j)_{j\in\mathbb{N}}$, we are interested in conditions such that $U_{\alpha,j}(s,t)F \longrightarrow U_\alpha(s,t)F$ as $j \to \infty$, where $U_{\alpha,j}(s,t)$ are the regularized evolution systems defined as in (2.12). For functions $f \in C_b(E \times E)$ let $(Q(t)f(x,\cdot))(y) := \int_E f(x,w)Q(t,y,\mathrm{d}w)$.

Lemma 2.1.3. *Let $(Q_j)_{j\in\mathbb{N}}$ be a family of transition functions and assume that Q_j is weakly continuous for any $j \in \mathbb{N}$. Moreover, suppose that the following conditions below are satisfied.*

1. *Let $q_j(t,x) := Q_j(t,x,E)$, then $\sup_{\substack{j\geq 1 \\ (t,x)\in B}} q_j(t,x) < \infty$ holds for all $B \in \mathcal{C}_1$.*

2. *For any $f \in C_b(E \times E)$ the convergence*

$$(Q_j(t)f(x,\cdot))(x) \longrightarrow (Q(t)f(x,\cdot))(x), \quad j \to \infty \tag{2.14}$$

is uniform in $(t,x) \in B$ for any $B \in \mathcal{C}_1$.

Then for any $0 \leq s \leq t$ and $F \in C_b(E)$

$$U_{\alpha,j}(s,t)F \longrightarrow U_\alpha(s,t)F, \quad j \to \infty \tag{2.15}$$

holds uniformly on compacts. If instead of (2.14) *the stronger convergence in the total variation norm holds, i.e.*

$$\sup_{(t,x)\in[0,T]\times B} \|Q_j(t,x,\cdot)-Q(t,x,\cdot)\| \to 0, \quad j\to\infty$$

for any $T>0$, then the convergence (2.15) *is uniform on any $A \in \mathcal{C}_1$ and on $\|F\|_\infty \le 1$.*

Proof. Since Q, Q_j are transition functions, it follows that $U_{\alpha,j}(s,t)$ and $U_\alpha(s,t)$ are Feller evolution systems on $C_b(E)$ obtained by

$$U_{\alpha,j}(s,t)F(x) = \sum_{n=0}^{\infty} \alpha^n U_j^{(n)}(s,t)F(x)$$

and

$$U_\alpha(s,t)F(x) = \sum_{n=0}^{\infty} \alpha^n U^{(n)}(s,t)F(x).$$

Since $|U^{(n)}(s,t)F(x)|, |U_j^{(n)}(s,t)F(x)| \le \|F\|_\infty$ the convergence of the series is also uniform in $j \ge 1$. As a consequence it is enough to show for any $0\le s\le t$, any compact $B\subset E$, $n\ge 0$ and $F\in C_b(E)$

$$\lim_{j\to\infty} \sup_{x\in B} |U_j^{(n)}(s,t)F(x) - U^{(n)}(s,t)F(x)| = 0. \tag{2.16}$$

For $n=0$ this follows from (2.14) and

$$|U_j^{(0)}(s,t)F(x) - U^{(0)}(s,t)F(x)| \le \|F\|_\infty \int_s^t |q_j(r,x)-q(r,x)|\mathrm{d}r.$$

Assume that (2.16) holds for one $n\ge 0$, proceeding by induction we obtain for $x\in B, 0\le s\le t$ and $F\in C_b(E)$

$$|U_j^{(n+1)}(s,t)F(x) - U^{(n+1)}(s,t)F(x)| \le I_1 + I_2 + I_3\ ,$$

where we have used (2.13) and

$$I_1 = \int_s^t \int_E \left| e^{-\int_s^r q_j(\tau,x)\mathrm{d}\tau} - e^{-\int_s^r q(\tau,x)\mathrm{d}\tau} \right| |U_j^{(n)}(r,t)F(y)| Q_j(r,x,\mathrm{d}y)\mathrm{d}r$$

$$I_2 = \left| \int_s^t \int_E (U_j^{(n)}(r,t)F(y) - U^{(n)}(r,t)F(y)) e^{-\int_s^r q(\tau,x)\mathrm{d}\tau} Q_j(r,x,\mathrm{d}y)\mathrm{d}r \right|$$

$$I_3 = \left| \int_s^t \int_E e^{-\int_s^r q(\tau,x)\mathrm{d}\tau} U^{(n)}(r,t)F(y)(Q_j(r,x,\mathrm{d}y) - Q(r,x,\mathrm{d}y))\mathrm{d}r \right|.$$

The first integral can be estimated by using $|U_j^{(n)}(r,t)F(y)| \le \|F\|_\infty$ and $q_j(r,x) \le q^* := \sup_{j\ge 1} \sup_{(\tau,x)\in[s,t]\times B} Q_j(\tau,x,E)$ for each $r \in [s,t]$, which yields

$$\begin{aligned} I_1 &\le \|F\|_\infty \int_s^t q_j(r,x) \left| \int_s^r (q_j(\tau,x) - q(\tau,x))\mathrm{d}\tau \right| \mathrm{d}r \\ &\le \|F\|_\infty (t-s)^2 q^* \sup_{(\tau,x)\in[s,t]\times B} |q_j(\tau,x) - q(\tau,x)|. \end{aligned}$$

To estimate I_2 we need the following lemma.

Lemma 2.1.4. *For any $\varepsilon > 0$, $T > 0$ there exists a compact $A \subset E$ and $j_0 \ge 1$ such that*

$$\int_0^T Q_j(t,x,A^c)\mathrm{d}t \le \varepsilon, \quad x \in B, \; j \ge j_0.$$

Proof. Since Q has the localization property we can find a compact $A_1 \subset E$ such that

$$\int_0^T Q(t,x,A_1^c)\mathrm{d}t \le \frac{\varepsilon}{2}, \quad x \in B.$$

Choose compacts $A, A_2 \subset E$ such that $A_1 \subset \mathring{A}_2 \subset A_2 \subset \mathring{A} \subset A$, since $(\mathring{A}_2)^c$ and $(\mathring{A})^c$ are closed there exists a continuous function φ with $\mathbb{1}_{(\mathring{A})^c} \le \varphi \le \mathbb{1}_{(\mathring{A}_2)^c}$. We obtain

$$\int_0^T Q_j(t,x,A^c)\mathrm{d}t \le \int_0^T Q_j(t,x,(\mathring{A})^c)\mathrm{d}t \le \int_0^T \int_E \varphi(y) Q_j(t,x,\mathrm{d}y)\mathrm{d}t$$

and by (2.14) there exists $j_0 \ge 1$ such that for $j \ge j_0$, $x \in B$ and $t \in [0,T]$

$$\int_E \varphi(y) Q_j(t,x,\mathrm{d}y) \le \frac{\varepsilon}{2T} + \int_E \varphi(y) Q(t,x,\mathrm{d}y).$$

Therefore the assertion follows from

$$\begin{aligned} \int_0^T \int_E \varphi(y) Q_j(t,x,\mathrm{d}y)\mathrm{d}t &\le \frac{\varepsilon}{2} + \int_0^T \int_E \varphi(y) Q(t,x,\mathrm{d}y)\mathrm{d}t \\ &\le \frac{\varepsilon}{2} + \int_0^T Q(t,x,(\mathring{A}_2)^c)\mathrm{d}t \le \frac{\varepsilon}{2} + \int_0^T Q(t,x,A_1^c)\mathrm{d}t \le \varepsilon. \end{aligned}$$

□

Take $A \subset E$ and $j_0 \geq 1$ as in above lemma, then for any $j \geq j_0$ and $x \in B$

$$I_2 \leq \int_s^t \int_A |U_j^{(n)}(r,t)F(y) - U^{(n)}(r,t)F(y)| Q_j(r,x,\mathrm{d}y)\mathrm{d}r + 2\|F\|_\infty \int_s^t Q_j(r,x,A^c)\mathrm{d}r$$

$$\leq q^* \int_s^t \sup_{y \in A} |U_j^{(n)}(r,t)F(y) - U^{(n)}(r,t)F(y)|\mathrm{d}r + 2\|F\|_\infty \int_s^t Q_j(r,x,A^c)\mathrm{d}r$$

$$\leq q^* \int_s^t \sup_{y \in A} |U_j^{(n)}(r,t)F(y) - U^{(n)}(r,t)F(y)|\mathrm{d}r + 2\|F\|_\infty \varepsilon.$$

The integrand tends for each fixed $r \in [s,t]$ to zero as $j \to \infty$ and since

$$\sup_{y \in A} |U_j^{(n)}(r,t)F(y) - U^{(n)}(r,t)F(y)| \leq 2\|F\|_\infty$$

also the integral tends to zero. Altogether this shows the assertion for I_2. For the last integral observe that $(r,x,y) \longmapsto e^{-\int_s^r q(\tau,x)\mathrm{d}\tau} U^{(n)}(r,t)F(y)$ is continuous and moreover bounded by $\|F\|_\infty$. Therefore by (2.14) for any $r \in [s,t]$

$$F_j(r,s,t) := \sup_{x \in B} \left| \int_E e^{-\int_s^r q(\tau,x)\mathrm{d}\tau} U^{(n)}(r,t)F(y)(Q_j(r,x,\mathrm{d}y) - Q(r,x,\mathrm{d}y)) \right| \to 0, \quad j \to \infty$$

and since $F_j(r,s,t) \leq 2\|F\|_\infty q^*$ we obtain the assertion by dominated convergence. The second assertion can be proved very similarly, here only I_3 should be estimated again. □

As a consequence we can show that $U_\alpha(s,t)$ satisfies a Chernoff product formula. That is $U_\alpha(s,t)$ can be approximated by evolution systems $U_{\alpha,n}(s,t)$ with piecewise constant (in time) transition functions Q_n. More precisely, take for any $n \in \mathbb{N}$ a sequence $0 = t_0^{(n)} \leq t_k^{(n)} < t_{k+1}^{(n)}$ with $\sup_{k \geq 0} (t_{k+1}^{(n)} - t_k^{(n)}) \to 0$ as $n \to \infty$ and $t_k^{(n)} \longrightarrow \infty$, $k \to \infty$ for all $n \in \mathbb{N}$. Define piecewise constant transition functions by

$$Q_n(t,x,\mathrm{d}y) = Q(t_k^{(n)},x,\mathrm{d}y), \quad t_k^{(n)} \leq t < t_{k+1}^{(n)}, \ k \geq 0,$$

then Q_n is weakly continuous in x for any fixed $t \geq 0$ and $n \geq 1$. Denote by $U_{\alpha,n}(s,t)$ the regularized Feller evolutions on $C_b(E)$ constructed above, cf. Theorem 2.1.2. For fixed $r \geq 0$ set $Q^r(x,\mathrm{d}y) := Q(r,x,\mathrm{d}y)$, then Q^r is a weakly continuous transition function and its associated regularized Feller evolution on $C_b(E)$ can be represented by a Feller semigroup $T_{\alpha,r}(t)$.

Lemma 2.1.5. *Let $F \in C_b(E)$ and for $0 \leq s \leq t$ choose $m_0, m_1 \geq 1$ such that*

$$t_{m_0}^{(n)} \leq s < t_{m_0+1}^{(n)} < \cdots t_{m_1}^{(n)} \leq t < t_{m_1+1}^{(n)}. \tag{2.17}$$

Then

$$U_{\alpha,n}(s,t)F(x) = T_{\alpha,t_{m_0}^{(n)}}(t_{m_0+1}^{(n)} - s) \cdots T_{\alpha,t_{m_1}^{(n)}}(t - t_{m_1}^{(n)})F \to U_\alpha(s,t)F, \quad n \to \infty \tag{2.18}$$

holds uniformly on compacts.

Proof. First observe that for any compact $B \subset E$, $T > 0$ and $f \in C_b(E \times E)$ Lemma A.2.2 implies that $F(r,x) := \int_E f(x,y)Q(r,x,\mathrm{d}y)$ is continuous. Therefore

$$\int_E f(x,y)Q_n(r,x,\mathrm{d}y) \longrightarrow \int_E f(x,y)Q(r,x,\mathrm{d}y), \quad n \to \infty$$

holds uniformly in $(x,r) \in B \times [0,T]$ and hence (2.14) follows. Applying Lemma 2.1.3 we obtain for all $F \in C_b(E)$ and $0 \leq s \leq t$: $U_{\alpha,n}(s,t)F \to U_\alpha(s,t)F$ as $n \to \infty$ uniformly on compacts. By the evolution system property it follows that

$$U_{\alpha,n}(s,t) = U_\alpha(s, t_{m_0+1}^{(n)}) \cdots U_\alpha(t_{m_1}^{(n)}, t)$$

holds. For each pair $r_0 < r_1$ with $t_m^{(n)} \leq r_0 < r_1 \leq t_{m+1}^{(n)}$ for some $m \geq 1$, by (2.2) and (2.13) it follows that $U_{\alpha,n}(r_0,r_1) = T_{\alpha,t_m^{(n)}}(r_1 - r_0)$ and hence

$$U_{\alpha,n}(s,t) = T_{\alpha,t_{m_0}^{(n)}}(t_{m_0+1}^{(n)} - s) \cdots T_{\alpha,t_{m_1}^{(n)}}(t - t_{m_1}^{(n)})$$

implies the assertion. □

As a corollary of Lemma 2.1.3 we can show a time-homogenization principle. Heuristically it states that if $Q(t,x,\mathrm{d}y) \to \overline{Q}(x,\mathrm{d}y)$ when $t \to \infty$, then $U_\alpha(s,t)F$ can be approximated by $\overline{T}_\alpha(t-s)F$ where $\overline{T}_\alpha(t)$ is the regularized Feller semigroup on $C_b(E)$ associated with $\overline{Q}(x,\mathrm{d}y)$.

Corollary 2.1.6. *For $\varepsilon > 0$ define rescaled Q-functions by*

$$Q_\varepsilon(t,x,\mathrm{d}y) = Q\left(\frac{t}{\varepsilon}, x, \mathrm{d}y\right), \quad t \geq 0,\ x \in E,\ \varepsilon > 0.$$

Denote by $(U_{\alpha,\varepsilon}(s,t))_{t \geq s \geq 0}$ the associated regularized evolution systems on $C_b(E)$. Assume that there exists a weakly continuous transition function $\overline{Q}(x,\mathrm{d}y)$ with the properties:

1. *For any compact $B \subset E$ one has $\sup_{t \geq 0} \sup_{x \in B} q(t,x) < \infty$.*

2. $\overline{Q}$ *has the localization property, that is for all* $\varepsilon > 0$ *and all* $B \in \mathcal{C}$ *there exists* $A \in \mathcal{C}$ *such that* $\left\|\mathbb{1}_B \overline{Q} \mathbb{1}_{A^c}\right\|_\infty < \varepsilon$ *is fulfilled.*

3. *For all* $f \in C_b(E \times E)$*, any compact* $B \subset E$*,* $T > 0$ *and* $\delta > 0$

$$\sup_{\frac{\delta}{\varepsilon} \leq t \leq \frac{T}{\varepsilon}} \sup_{x \in B} |(Q(t)f(x,\cdot))(x) - (\overline{Q}f(x,\cdot))(x)| \to 0, \quad \varepsilon \to 0$$

holds.

Denote by $(\overline{T}_\alpha(t))_{t \geq 0}$ *the regularized Feller semigroup on* $C_b(E)$ *constructed as above and associated with* $\overline{Q}(x, \mathrm{d}y)$*. Then for any compact* $B \subset E$*,* $F \in C_b(E)$ *and* $0 < s \leq t$

$$\sup_{x \in B} |U_{\alpha,\varepsilon}(s,t)F(x) - \overline{T}_\alpha(t-s)F(x)| \longrightarrow 0, \quad \varepsilon \to 0$$

holds.

Proof. If we assume in (2.14) instead of uniform convergence on $[0, T] \times B$, uniform convergence on $[\delta, T] \times B$ for any $\delta > 0$, then Lemma 2.1.3 still holds with (2.15) for any $0 < s \leq t$ and $F \in C_b(E)$. □

In the following we consider the limit $\alpha \to 1$ and deduce from that $U(s,t)1 = 1$.

Theorem 2.1.7. *The evolution system* $U(s,t)$ *is conservative and can be extended to* $BM_V(E)$ *so that*

$$\|U(s,t)F\|_V \leq e^{\int_s^t c(r)\mathrm{d}r} \|F\|_V, \quad 0 \leq s \leq t.$$

Proof. Denote by $T_{\alpha,r}(t)$ the regularized semigroups with piecewise constant (in the time variable) transition functions, see Theorem 2.1.5 and by $T_r(t)$ their counterparts with $\alpha = 1$. Then $T_{\alpha,r}(t)V(x) \leq T_r(t)V(x)$. The moment condition (2.7) and the results obtained in [Che04, Kol06] imply that for any $r \geq 0$, $x \in E$ and $t \geq 0$

$$T_r(t)V(x) \leq e^{c(r)t}V(x).$$

Now given $0 \leq s < t$ and $n \in \mathbb{N}$ we can find $m_0, m_1 \geq 0$ with (2.17). For $m \geq 0$ let $V_m(x) := V(x) \wedge m$, then $V_m \in C_b(E)$ and hence

$$\begin{aligned} U_{\alpha,n}(s,t)V_m(x) &= T_{\alpha,t_{m_0}^{(n)}}(t_{m_0+1}^{(n)} - s) \cdots T_{\alpha,t_{m_1}^{(n)}}(t - t_{m_1}^{(n)})V_m(x) \\ &\leq T_{t_{m_0}^{(n)}}(t_{m_0+1}^{(n)} - s) \cdots T_{t_{m_1}^{(n)}}(t - t_{m_1}^{(n)})V(x) \\ &\leq V(x) \exp\left(c(t_{m_1}^{(n)})(t - t_{m_1}^{(n)}) + \cdots + c(t_{m_0}^{(n)})(t_{m_0+1}^{(n)} - s)\right). \end{aligned}$$

Letting $n \to \infty$ yields

$$U_\alpha(s,t)V_m(x) \leq V(x)\exp\left(\int\limits_s^t c(r)\mathrm{d}r\right).$$

The sequence $(U_\alpha(s,t)V_m(x))_{m\in\mathbb{N}}$ is increasing and bounded, so by monotone convergence it follows that

$$\int\limits_E V(x)P_\alpha(s,x;t,\mathrm{d}y) \leq V(x)\exp\left(\int\limits_s^t c(r)\mathrm{d}r\right) \tag{2.19}$$

is satisfied. The right-hand side is increasing in α, hence taking the limit $\alpha \to 1$ yields that $U(s,t)$ can be extended to $BM_V(E)$. By [Fel40] the evolution system $U(s,t)$ is conservative if and only if for any $s < t$, $x \in E$

$$\int\limits_s^t\int\limits_E q(r,y)P^{(n)}(s,x;r,\mathrm{d}y)\mathrm{d}r \longrightarrow 0, \quad n \to \infty.$$

Let $T > 0$ such that $[s,t] \subset [0,T]$, then by $q(r,y) \leq a(T)V(y)$ for $r \in [s,t]$ and (2.19) with $\alpha = 1$

$$\int\limits_s^t\int\limits_E q(r,y)P(s,x;r,\mathrm{d}y)\mathrm{d}r \leq a(T)V(x)\int\limits_s^t \exp\left(\int\limits_s^r c(\tau)\mathrm{d}\tau\right)\mathrm{d}r < \infty$$

follows. The assertion follows from the representation $P(s,x;r,\mathrm{d}y) = \sum\limits_{n=0}^{\infty} P^{(n)}(s,x;r,\mathrm{d}y)$.

□

The next result shows that $U(s,t)$ is differentiable in s.

Theorem 2.1.8. *For any $F \in BM(E)$, $t > 0$ and $x \in E$, $[0,t] \ni s \longmapsto U(s,t)F(x)$ is continuously differentiable and a solution to*

$$\frac{\partial}{\partial s}U(s,t)F(x) = -L(s)U(s,t)F(x).$$

Moreover, for any $F \in C_V(E)$ the function $U(s,t)F(x)$ is absolutely continuous in s and solves above equation a.e.. Let $V(s,t)$ be a Feller evolution system on $C_b(E)$ and assume that $V(s,t)F$ is a solution to (2.10) for any $F \in C_b(E)$, then $V(s,t) = U(s,t)$ is fulfilled.

Proof. By (2.3) we obtain for any $A \in \mathcal{B}(E)$ and $0 \leq s \leq t$

$$P(s,x;t,A) = \delta(x,A) + \int_s^t q(r,x)P(r,x;t,A)\mathrm{d}r - \int_s^t \int_E P(r,y;t,A)Q(r,x,\mathrm{d}y)\mathrm{d}r$$

and hence for any $F \in BM(E)$ and $x \in E$

$$U(s,t)F(x) = F(x) + \int_s^t q(r,x)U(r,t)F(x)\mathrm{d}r - \int_s^t \int_E U(r,t)F(y)Q(r,x,\mathrm{d}y)\mathrm{d}r$$

follows. Clearly $q(r,x)U(r,t)F(x)$ and by Lemma A.2.2 also $\int_E U(r,t)F(y)Q(r,x,\mathrm{d}y)$ are continuous in r, which implies that $L(r)U(r,t)F(x)$ is continuous in (r,t). Therefore

$$U(s,t)F(x) = F(x) - \int_s^t L(r)U(r,t)F(x)\mathrm{d}r \tag{2.20}$$

implies (2.10). If $F \in C_V(E)$, then $U(s,t)F(x)$ is bounded and measurable in (s,t). Hence by (2.7) $L(r)U(r,t)F(x)$ is well-defined and integrable w.r.t. r. In view of (2.20) it follows that $s \longmapsto U(s,t)F(x)$ is absolutely continuous and satisfies (2.10) for any $x \in E$.
Now let $V(s,t)$ be a Feller evolution on $C_b(E)$ which satisfies (2.10). By [Cas11, Chapter 2, Theorem 2.9] $V(s,t)$ is given by

$$V(s,t)F(x) = \int_E F(y)\widetilde{P}(s,x;t,\mathrm{d}y), \quad x \in E, \quad 0 \leq s \leq t,$$

where $\widetilde{P}$ is a transition probability function. Moreover, this evolution system satisfies (2.20) for any $F \in C_b(E)$ and hence by approximation also for any $F \in BM(E)$. Therefore for any $F = \mathbb{1}_A$, $A \in \mathcal{B}(E)$ it solves equation (2.10) which is simply (2.3). The minimality of P implies $P \leq \widetilde{P}$ and hence $U(s,t)F \leq V(s,t)F$. Since $U(s,t)$ is conservative it follows that $P(s,x;t,\mathrm{d}y)$ is the unique solution to (2.3), i.e. $P(s,x;t,\mathrm{d}y) = \widetilde{P}(s,x;t,\mathrm{d}y)$. □

Theorem 2.1.9. *Let $F \in BM(E)$, then for any $x \in E$ and $s \geq 0$, $[s,\infty) \ni t \longmapsto U(s,t)F(x)$ is absolutely continuous and satisfies for a.a. $t \geq s$*

$$\frac{\partial}{\partial t}U(s,t)F(x) = U(s,t)L(t)F(x).$$

Let $V(s,t)$ be a Feller evolution system on $C_b(E)$ and assume that $V(s,t)F$ is for any $F \in C_b(E)$ a solution to (2.11), then $V(s,t) = U(s,t)$ holds.

Proof. For all $0 \leq s \leq r \leq t < T$

$$\int_E q(r,y)P(s,x;t,\mathrm{d}y) \leq a(T)\int_E V(y)P(s,x;t,\mathrm{d}y) \leq a(T)V(x)e^{\int_s^t c(r)\mathrm{d}r} \tag{2.21}$$

and (2.4) implies for any $0 \leq s \leq t$ and compact $A \subset E$

$$P(s,x;t,A) = \delta(x,A) - \int_s^t\int_A q(r,y)P(s,x;r,\mathrm{d}y)\mathrm{d}r + \int_s^t\int_E Q(r,y,A)P(s,x;r,\mathrm{d}y)\mathrm{d}r.$$

By (2.21) this implies

$$U(s,t)F(x) = F(x) - \int_s^t\int_E q(r,y)F(y)P(s,x;r,\mathrm{d}y)\mathrm{d}r + \int_s^t\int_E Q(r)F(y)P(s,x;r,\mathrm{d}y)\mathrm{d}r$$

and hence

$$U(s,t)F(x) = F(x) + \int_s^t\int_E L(r)F(y)P(s,x;r,\mathrm{d}y)\mathrm{d}r$$

holds. The first assertion is proved. Uniqueness follows by the same arguments as for (2.10). □

Remark 2.1.10. *It is worth noting that in the time-homogeneous case* (2.10) *and* (2.11) *are equivalent and less restrictive conditions are sufficient to show that $U(s,t)$ is an Feller evolution, see [Kol06].*

Since $U(s,t)$ is given by a transition probability function we see that for each $x \in E$ and $s \geq 0$ there exists a probability space $(\Omega, \mathcal{F}^s, \mathbb{P}_{s,x})$ and a conservative Markov process $(X(t))_{t\geq s}$ on this space such that

$$U(s,t)F(x) = \mathbb{E}_{s,x}(F(X(t)), \quad F \in C_b(E),\ t \geq s.$$

This process is considered w.r.t. its natural filtration defined by $\mathcal{F}_\tau^s = \sigma\left(X(t) \mid s \leq t \leq \tau\right)$ for $s \leq \tau$. Note that this process is, by construction, a pure jump process. The next statement completes the proof of Proposition 2.1.1.

Corollary 2.1.11. *The following statements are true:*

1. *Let $F \in BM(E)$. Then for any fixed $s \geq 0$*

$$M_{s,F}(t) := F(X(t)) - F(X(s)) - \int_s^t L(r)F(X(r))\mathrm{d}r, \quad t \geq s$$

is a martingale with respect to $(\mathcal{F}_t^s)_{t\geq s}$ and $\mathbb{P}_{s,x}$.

2. For any $a > 0$, $x \in E$ and $0 \leq s < T$

$$\mathbb{P}_{s,x}\left(\sup_{t\in[s,T]} V(X(t)) \geq a\right) \leq V(x)\frac{e^{\int_s^T c(r)\mathrm{d}r}}{a}$$

holds.

3. $U(s,t)$ is a Feller evolution system.

Proof. 1. Let $0 \leq s \leq \tau \leq t$, then by the Markov property we obtain

$$\begin{aligned}
&\mathbb{E}_{s,x}(M_{s,F}(t)|\mathcal{F}^s_\tau) - M_{s,F}(\tau) = \mathbb{E}_{s,x}(M_{\tau,F}(t)|\mathcal{F}^s_\tau) = \mathbb{E}_{\tau,X(\tau)}(M_{\tau,F}(t)) \\
&= \mathbb{E}_{\tau,X(\tau)}(F(X(t))) - \mathbb{E}_{\tau,X(\tau)}(F(X(s))) - \int_\tau^t \mathbb{E}_{\tau,X(\tau)}(L(r)F(X(r)))\mathrm{d}r \\
&= \mathbb{E}_{\tau,X(\tau)}(F(X(t))) - \mathbb{E}_{\tau,X(\tau)}(F(X(s))) - \int_\tau^t \frac{\partial}{\partial r}\mathbb{E}_{\tau,X(\tau)}(F(X(r)))\mathrm{d}r = 0.
\end{aligned}$$

Here we have used that

$$\mathbb{E}_{\tau,X(\tau)}(L(r)F(X(r))) = (U(\tau,r)L(r)F)(X(\tau)) = \frac{\partial}{\partial r}U(\tau,r)F(X(\tau)).$$

2. Let $E_n := \{x \in E \mid V(x) < n\}$, fix $s \geq 0$ and define a family of stopping times

$$\tau_n := \inf\{t \geq s \mid X_t \notin E_n\}.$$

Let $\varphi_n \in C(E)$ be such that $\mathbb{1}_{\overline{E}_n} \leq \varphi_n \leq \mathbb{1}_{\overline{E}_{n+1}}$ and define a new transition function by $Q_n(t,x,\mathrm{d}y) := \varphi_n(x)Q(t,x,\mathrm{d}y)$. Then

$$L_n(t)F(x) := -\varphi_n(x)q(t,x)F(x) + \int_E F(y)\varphi_n(x)Q(t,x,\mathrm{d}y) = \varphi_n(x)L(t)F(x)$$

determines a bounded linear operator on $C_b(E)$ and $BM_V(E)$. Hence there exists an associated conservative Feller evolution system $U_n(s,t)$ on $C_b(E)$. This evolution system can be extended to $BM_V(E)$. Let $(X^n_t)_{t\geq 0}$ be the corresponding Markov process, and denote by $(\mathcal{F}^s_{t,n})_{t\geq s}$ its associated natural filtration. By construction it follows for $x \in E_n$ and $n \geq 1$ that these processes satisfy

$$(X_t)_{t<\tau_n} = (X^n_t)_{t<\tau_n} \tag{2.22}$$

in the sense of finite dimensional distributions. Fix $s \geq 0$ and define $g(t,x) := e^{-\int\limits_s^t c(r)\mathrm{d}r} V(x)$. A short computation shows that

$$\frac{\partial}{\partial t} g(t,x) + L(t)g(t,x) \leq 0.$$

Then

$$M_n(s,t) := g(t, X^n(t)) - g(s, X^n(s)) - \int\limits_s^t \left(\frac{\partial}{\partial r} + L_n(r) \right) g(r, X^n(r))\mathrm{d}r, \quad t \geq s$$

is a $\mathcal{F}^s_{t,n}$-martingale w.r.t. $\mathbb{P}_{s,x}$. Fix $x \in E_n$, $n \geq 1$, hence by Dynkin's formula

$$\mathbb{E}_{s,x}(g(t \wedge \tau_n, X^n_{t\wedge\tau_n})) = g(s,x) + \mathbb{E}_{s,x}\left(\int\limits_s^{t\wedge\tau_n} \left(\frac{\partial}{\partial r} + L_n(r) \right) g(r, X^n_r)\mathrm{d}r \right) \leq g(s,x) \quad (2.23)$$

holds. Here $\frac{\partial}{\partial r}$ acts only on the first variable of g. Let $M^n_t := e^{-\int\limits_s^t c(\sigma)\mathrm{d}\sigma} V(X^n_t)\mathbb{1}_{t<\tau_n}$, we will show that $(M^n_t)_{t\geq s}$ is a supermartingale. Fix $s \leq r \leq t$. On $\{r \geq \tau_n\} \in \mathcal{F}^s_{r,n}$ we have $M^n_t = M^n_r = 0$ and hence obtain

$$\mathbb{E}_{s,x}(M^n_t | \mathcal{F}^s_{r,n}) = M^n_r = 0.$$

On $\{r < \tau_n\}$ we have by the Markov property and (2.23)

$$\begin{aligned}\mathbb{E}_{s,x}(M^n_t | \mathcal{F}^s_{r,n}) &= e^{-\int\limits_s^t c(\sigma)\mathrm{d}\sigma} \mathbb{E}_{r,X^n_r}(V(X^n_t)\mathbb{1}_{t<\tau_n}) \leq \mathbb{E}_{r,X^n_r}(g(t\wedge\tau_n, X^n_{t\wedge\tau_n})) \\ &\leq g(r, X^n_r) = g(r\wedge\tau_n, X^n_{r\wedge\tau_n}) = M^n_r.\end{aligned}$$

Applying Doob's inequality yields

$$\mathbb{P}_{s,x}(\sup_{\substack{s\leq t\leq T \\ t<\tau_n}} g(t, X(t)) \geq a) = \mathbb{P}_{s,x}(\sup_{s\leq t\leq T} M^n_t \geq a) \leq \frac{1}{a}\mathbb{E}_{s,x}(M^n_s) = \frac{V(x)}{a}.$$

As a consequence we obtain

$$\begin{aligned}\mathbb{P}_{s,x}\left(\sup_{\substack{s\leq t\leq T \\ t<\tau_n}} V(X(t)) \geq a \right) &\leq \mathbb{P}_{s,x}\left(\sup_{\substack{s\leq t\leq T \\ t<\tau_n}} g(t, X(t)) \geq a e^{-\int\limits_s^T c(r)\mathrm{d}r} \right) \\ &\leq V(x)\frac{e^{\int\limits_s^T c(r)\mathrm{d}r}}{a}.\end{aligned}$$

Since $(X_t)_{t\geq s}$ is conservative it follows $\tau_n \longrightarrow \infty$ when $n \to \infty$. The assertion follows by monotone convergence and $n \to \infty$.
3. For any $F \in C_b(E)$, $x \in E_n$ and $n \geq 1$ it follows by (2.22)

$$\begin{aligned}|\mathbb{E}_{s,x}(F(X_t)) - \mathbb{E}_{s,x}(F(X_t^n))| &= |\mathbb{E}_{s,x}(F(X_t)\mathbb{1}_{\tau_n\leq t}) - \mathbb{E}_{s,x}(F(X_t^n)\mathbb{1}_{\tau_n\leq t})| \\ &\leq 2\|F\|_\infty \mathbb{P}_{s,x}(\tau_n \leq t).\end{aligned}$$

By

$$\mathbb{P}_{s,x}(\tau_n \leq t) \leq \mathbb{P}_{s,x}\left(\sup_{s\leq r\leq t} V(X(r)) \geq n\right) \leq \frac{V(x)}{n} e^{\int_s^t c(r)\mathrm{d}r}.$$

and the continuity of V we see that $U_n(s,t)F(x) \longrightarrow U(s,t)F(x)$ uniformly on compacts which implies the assertion. $\square$

We close this section with the relation to the evolution of measures. Let $\mathcal{M}(E)$ be the space of all finite, signed Borel measures on E equipped with the total variation norm. Define bounded linear operators $(U^*(t,s))_{0\leq s\leq t}$ on $\mathcal{M}(E)$ by

$$U^*(t,s)\mu(\mathrm{d}x) = \int_E P(s,y;t,\mathrm{d}x)\mu(\mathrm{d}y).$$

Then $U^*(t,t) = \mathrm{id}_{\mathcal{M}(E)}$, $U^*(t,r)U^*(r,s) = U^*(t,s)$ holds for $0 \leq s \leq r \leq t$ and

$$\int_E F(y)U^*(t,s)\mu(\mathrm{d}y) = \int_E U(s,t)F(y)\mu(\mathrm{d}y), \quad F \in C_b(E),\ \mu \in \mathcal{M}(E).$$

Previous considerations show that $U^*(t,s)\mu$ is the unique weak solution to the Fokker-Planck equation

$$\frac{\partial}{\partial t}\int_E F(y)U^*(t,s)\mu(\mathrm{d}y) = \int_E L(t)F(y)U^*(t,s)\mu(\mathrm{d}y),$$

where $\mu \in \mathcal{M}(E)$ is such that $\int_E V(x)|\mu|(\mathrm{d}x) < \infty$.

2.2 Dynamics on the space of finite configurations

In this section we review known results for the space of finite subsets of $\mathbb{R}^d$. The aim is to provide a general framework for birth-and-death dynamics in continuum such that the corresponding equations can be studied in the remaining sections.

2.2.1 One-component case

The configuration space Γ_0 is the space of all finite subsets of $\mathbb{R}^d$, i.e.

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d \mid |\eta| < \infty\},$$

where $|\eta|$ denotes the number of elements in the set η. This space has a natural decomposition into n-particle spaces, $\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}$, where $\Gamma_0^{(n)} = \{\eta \subset \mathbb{R}^d \mid |\eta| = n\}, \quad n \geq 1$ and in the case $n = 0$ we set $\Gamma_0^{(0)} = \{\varnothing\}$. For a compact $\Lambda \subset \mathbb{R}^d$ let

$$\Gamma_\Lambda = \{\eta \in \Gamma_0 \mid \eta \subset \Lambda\}$$

and $\Gamma_\Lambda^{(n)} = \{\eta \in \Gamma_0^{(n)} \mid \eta \subset \Lambda\}$. Denote by $\widetilde{(\mathbb{R}^d)^n}$ the space of all sequences $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ with $x_i \neq x_j$ for $i \neq j$. $\Gamma_0^{(n)}$ can be identified with $\widetilde{(\mathbb{R}^d)^n}$ via the symmetrization map

$$\mathrm{sym}_n : \widetilde{(\mathbb{R}^d)^n} \longrightarrow \Gamma_0^{(n)}, \ (x_1, \ldots, x_n) \longmapsto \{x_1, \ldots, x_n\},$$

which defines a topology on $\Gamma_0^{(n)}$. Namely, a set $A \subset \Gamma_0^{(n)}$ is open if and only if $\mathrm{sym}_n^{-1}(A) \subset \widetilde{(\mathbb{R}^d)^n}$ is open. On Γ_0 we define the topology of disjoint unions, i.e. a set $A \subset \Gamma_0$ is open iff $A \cap \Gamma_0^{(n)}$ is open in $\Gamma_0^{(n)}$ for all $n \in \mathbb{N}$. Then Γ_0 is a locally compact Polish space. Let $\mathcal{B}(\Gamma_0)$ stand for the Borel-σ-algebra on Γ_0. With respect to this topology for each $f \in C_b(\mathbb{R}^d)$ the function

$$\eta \longmapsto \langle f, \eta \rangle := \sum_{x \in \eta} f(x)$$

is continuous. Therefore convergence of a sequence $(\eta_n)_{n \in \mathbb{N}} \subset \Gamma_0$ to $\eta \in \Gamma_0$ can be rewritten to: there exists $N \in \mathbb{N}$ such that for all $n \geq N$: $\eta_n = \{x_1^{(n)}, \ldots, x_l^{(n)}\}, \ \eta = \{x_1, \ldots, x_l\}$ and

$$x_j^{(n)} \longrightarrow x_j, \quad n \to \infty, \ \forall j \in \{1, \ldots, l\}$$

is fulfilled. For given $\delta > 0$, $N \in \mathbb{N}_0$ and a compact $\Lambda \subset \mathbb{R}^d$ the set

$$B = \{\eta \in \Gamma_\Lambda \mid \forall x \neq y, \ x, y \in \eta : \ |x - y| \geq \delta, \quad |\eta| \leq N\} \tag{2.24}$$

is compact. Conversely, for any compact set $A \subset \Gamma_0$ there exist δ, N, Λ such that A is contained in a compact B defined above. Denote by $\mathrm{d}x$ the Lebesgue measure on $\mathbb{R}^d$ and by $\mathrm{d}^{\otimes n}x$ the product measure on $(\mathbb{R}^d)^n$. The image measure of $\mathrm{d}^{\otimes n}x$ on $\Gamma_0^{(n)}$ via sym_n is then denoted by $\mathrm{d}^{(n)}x$. The Lebesgue-Poisson measure is defined by

$$\lambda = \delta_\varnothing + \sum_{n=1}^{\infty} \frac{1}{n!} \mathrm{d}^{(n)}x.$$

Given a measurable function $G : \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}$, then

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} G(\xi, \eta \backslash \xi) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\xi, \eta) \mathrm{d}\lambda(\xi) \mathrm{d}\lambda(\eta) \tag{2.25}$$

holds, provided one side of the equality is finite for $|G|$. Here and in the following we write $\eta \backslash x$, $\eta \cup x$, instead of $\eta \backslash \{x\}$ and $\eta \cup \{x\}$. The decomposition $\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}$ implies that any measurable function $G : \Gamma_0 \longrightarrow \mathbb{R}$ can be represented as a sequence of symmetric measurable functions $(G^{(n)})_{n=0}^{\infty}$, where $G^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}$. Such functions are uniquely determined on the off-diagonal part $\widetilde{(\mathbb{R}^d)^n}$ and integration w.r.t. to the Lebesgue-Poisson measure is simply determined by the identity

$$\int_{\Gamma_0} G(\eta) \mathrm{d}\lambda(\eta) = G^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \cdots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

A set $M \subset \Gamma_0$ is said to be bounded if there exists $N \in \mathbb{N}$ and a compact $\Lambda \subset \mathbb{R}^d$ such that $M \subset \{\eta \in \Gamma_\Lambda \mid |\eta| \leq N\}$. For any $\xi \in \Gamma_0$ the set $\{\eta \in \Gamma_0 \mid \eta \cap \xi \neq \varnothing\}$ belongs to $\mathcal{B}(\Gamma_0)$ and $\lambda(\{\eta \in \Gamma_0 \mid \eta \cap \xi \neq \varnothing\}) = 0$ holds. A function G is said to have bounded support if there exists $N \in \mathbb{N}$ and a compact $\Lambda \subset \mathbb{R}^d$ such that G is supported on a bounded set. Denote by $B_{bs}(\Gamma_0)$ the space of all (measurable) bounded functions having bounded support. For a given measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ the Lebesgue-Poisson exponential is defined by

$$e_\lambda(f; \eta) := \prod_{x \in \eta} f(x)$$

and satisfies the combinatorial formula

$$\sum_{\xi \subset \eta} e_\lambda(f; \xi) = e_\lambda(1 + f; \eta).$$

For computations we will use the identity

$$\int_{\Gamma_0} e_\lambda(f; \eta) \mathrm{d}\lambda(\eta) = \exp\left(\int_{\mathbb{R}^d} f(x) \mathrm{d}x \right),$$

whenever $f \in L^1(\mathbb{R}^d)$.

2.2.2 Two-component case

This part provides a short extension to the two-component configuration space Γ_0^2, see [Fin13, FKO13] and the references therein. We suppose that two different particles cannot occupy the same location $x \in \mathbb{R}^d$ and therefore define the two-component state space by

$$\Gamma_0^2 = \{(\eta^+, \eta^-) \in \Gamma_0 \times \Gamma_0 \mid \eta^+ \cap \eta^- = \varnothing\}.$$

Here and in the following we simply write η instead of $(\eta^+, \eta^-) \in \Gamma_0^2$ if no confusion may arise. Set operations $\xi \subset \eta$, $\xi \cup \eta$ and $\eta \backslash \xi$ are defined component-wise, i.e. by $\xi^{\pm} \subset \eta^{\pm}$, etc. For $\eta \in \Gamma_0^2$ we let $|\eta| := |\eta^+| + |\eta^-|$. The space Γ_0^2 has the natural decomposition

$$\Gamma_0^2 = \bigsqcup_{n,m=0}^{\infty} \Gamma_0^{(n,m)},$$

where $\Gamma_0^{(n,m)} = \{(\eta^+, \eta^-) \subset \mathbb{R}^d \times \mathbb{R}^d \mid \eta^+ \cap \eta^- = \varnothing,\ |\eta^+| = n,\ |\eta^-| = m\}$. The topology on $\Gamma_0^{(n,m)}$ and Γ_0^2 is defined in the same way as for $\Gamma_0^{(n)}$ and Γ_0. It is not difficult to see that this topology is the same as the subspace topology of the product topology on $\Gamma_0 \times \Gamma_0$. In particular Γ_0^2 is a Polish space. A set $A \subset \Gamma_0^2$ is compact if it is contained in a set of the form

$$B := \{\eta \in \Gamma_0^2 \mid \eta^+, \eta^- \subset \Lambda,\ |\eta| \leq N,\ \forall x \neq y,\ x, y \in \eta^+ \cup \eta^- :\ |x - y| \geq \delta\} \tag{2.26}$$

for a compact $\Lambda \subset \mathbb{R}^d$, $N \in \mathbb{N}$ and $\delta > 0$. Conversely, the set B defined as above is compact as well. The Lebesgue-Poisson measure λ^2 on Γ_0^2 is defined as the restriction of $\lambda \otimes \lambda$ to Γ_0^2. Since no confusion may arise we use the same notation λ for the Lebesgue-Poisson measure λ^2 on Γ_0^2 and λ on Γ_0. We see that

$$\lambda \otimes \lambda(\{(\eta^+, \eta^-) \in \Gamma_0 \times \Gamma_0 \mid \eta^+ \cap \eta^- \neq \varnothing\}) = 0$$

holds. Hence integrals w.r.t. integrable functions $G : \Gamma_0^2 \longrightarrow \mathbb{R}$ can be also written as

$$\int_{\Gamma_0^2} G(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\eta^+, \eta^-) \mathrm{d}\lambda(\eta^+) \mathrm{d}\lambda(\eta^-).$$

Similarly to (2.25) the two-component Lebesgue-Poisson measure satisfies for any measurable function $G : \Gamma_0^2 \times \Gamma_0^2 \longrightarrow \mathbb{R}$

$$\int_{\Gamma_0^2} \sum_{\xi \subset \eta} G(\xi, \eta \backslash \xi) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} \int_{\Gamma_0^2} G(\xi, \eta) \mathrm{d}\lambda(\xi) \mathrm{d}\lambda(\eta) \tag{2.27}$$

provided one side of the equality is finite for $|G|$. A set $M \subset \Gamma_0^2$ is called bounded if there exist a compact $\Lambda \subset \mathbb{R}^d$ and $N \in \mathbb{N}_0$ such that

$$M \subset \{(\eta^+, \eta^-) \in \Gamma_0^2 \mid \eta^{\pm} \subset \Lambda,\ |\eta| \leq N\}.$$

A function G is said to have bounded support if it is supported on a bounded set. Denote by $B_{bs}(\Gamma_0^2)$ the space of all bounded, measurable functions having bounded support. We say that $H : \Gamma_0^2 \longrightarrow \mathbb{R}$ is locally integrable if it is integrable for any bounded set. This is the same as regarding that the integral $\int_{\Gamma_0^2} G(\eta)|H(\eta)| \mathrm{d}\lambda(\eta)$ is finite for all non-negative functions $G \in B_{bs}(\Gamma_0^2)$.

2.2.3 Description of the dynamics

General Markov birth-and-death processes on Γ_0 or Γ_0^2 respectively are given by a Markov (pre-)generator of the form

$$(L(t)F)(\eta) = \sum_{\xi \subset \eta} \int_E (F(\eta \backslash \xi \cup \zeta) - F(\eta)) K_t(\xi, \eta, \mathrm{d}\zeta), \quad \eta \in E, \ t \geq 0, \tag{2.28}$$

where E is either Γ_0 or Γ_0^2. Such Kolmogorov operator includes death, birth and jumps of groups of particles. In this generality it is also possible that particles switch their type, that is elementary events of the form

$$(\eta^+, \eta^-) \longmapsto (\eta^+ \backslash x, \eta^- \cup x) \text{ and } (\eta^+, \eta^-) \longmapsto (\eta^+ \cup x, \eta^- \backslash x)$$

are also included in the dynamics described by $L(t)$. Then, under some conditions given in the next section, the operator $L(t)$ can be rewritten to

$$L(t)F(\eta) = \int_E (F(\xi) - F(\eta)) Q(t, \eta, \mathrm{d}\xi)$$

and hence should determine a pure jump process on E. The construction of such process is closely related to the construction of solutions $(F_t)_{t \geq 0} \subset C_b(E)$ to

$$\frac{\partial F_t}{\partial t} = L(t) F_t, \quad F_t|_{t=0} = F_0. \tag{2.29}$$

It is the same as to solve the Fokker-Planck equation

$$\frac{\partial \mu_t}{\partial t} = L(t)^* \mu_t, \quad \mu_t|_{t=0} = \mu_0 \tag{2.30}$$

on the space of probability measures on E. Here $L(t)^*$ is the adjoint operator w.r.t. the duality pairing

$$\langle F, \mu \rangle := \int_E F(\eta) \mathrm{d}\mu(\eta).$$

Functions F are called observables, whereas probability measures μ states of the systems. Therefore we will refer to solutions $(F_t)_{t \geq 0}$ and $(\mu_t)_{t \geq 0}$ as the evolution of observables or states, respectively.

2.3 Time-inhomogeneous dynamics

In this section we provide general conditions for K_t such that the operator $L(t)$ given by (2.28) is associated with a Feller evolution system. Afterwards we study solutions to the Fokker-Planck equation (2.30) and relate them to the so-called evolution of correlation functions, cf. [KK02, FKO09]. For simplicity of notation all considerations are formulated only for the one-component case ($E = \Gamma_0$ above). The extension to multi-component systems is straightforward and will be performed for particular examples in the last section.

2.3.1 Evolution of observables and states

Consider the Kolmogorov operator $L(t)$ given by (2.28), we say that K_t satisfies the usual conditions if the conditions given below are satisfied.

1. For all $\eta, \xi \in \Gamma_0$ and $t \geq 0$: $K_t(\xi, \eta, \cdot) \geq 0$ is a finite, non-atomic Borel measure.
2. For all $A \in \mathcal{B}(\Gamma_0)$, the map $(t, \xi, \eta) \longmapsto K_t(\xi, \eta, A)$ is measurable.

For $t \geq 0$, $\eta \in \Gamma_0$ and $A \in \mathcal{B}(\Gamma_0)$ define $Q(t, \eta, \mathrm{d}\omega)$ by

$$Q(t, \eta, A) = \sum_{\xi \subset \eta} \int_{\Gamma_0} \mathbb{1}_A(\eta \backslash \xi \cup \zeta) K_t(\xi, \eta, \mathrm{d}\zeta). \tag{2.31}$$

The cumulative intensity is defined by $q(t, \eta) := Q(t, \eta, \Gamma_0) = \sum\limits_{\xi \subset \eta} K_t(\xi, \eta, \Gamma_0)$. We will work with the following conditions:

(A) For any $\varepsilon > 0$, $T > 0$ and any compact $B \subset \Gamma_0$ there exists another compact $A \subset \Gamma_0$ such that

$$\int_0^T Q(r, \eta, A^c) \mathrm{d}r < \varepsilon, \quad \eta \in B$$

is satisfied.

(B) There exist continuous functions $V : \Gamma_0 \longrightarrow \mathbb{R}_+$ and $c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\sum_{\xi \subset \eta} \int_{\Gamma_0} V(\eta \backslash \xi \cup \zeta) K_t(\xi, \eta, \mathrm{d}\zeta) \leq c(t) V(\eta) + q(t, \eta) V(\eta), \quad t \geq 0, \ \eta \in \Gamma_0 \tag{2.32}$$

holds.

(C) For any $F \in C(\Gamma_0)$ with $\sup\limits_{\eta \in \Gamma_0} \frac{|F(\eta)|}{1 + V(\eta)} < \infty$

$$(t, \eta) \longmapsto \sum_{\xi \subset \eta} \int_{\Gamma_0} F(\eta \backslash \xi \cup \zeta) K_t(\xi, \eta, \mathrm{d}\zeta)$$

is continuous.

(D) For any $T > 0$ there exists $a(T) > 0$ such that $q(t, \eta) \leq a(T)V(\eta)$ holds for all $\eta \in \Gamma_0$ and $t \in [0, T]$.

(E) For any $T > 0$ there exists $b(T) > 0$ such that $q(t, \eta) \geq b(T)q(T, \eta)$ holds for all $\eta \in \Gamma_0$ and $t \in [0, T]$.

As in the previous section let $BM_V(\Gamma_0)$ stand for the Banach space of all measurable functions F equipped with the norm $\|F\|_V = \sup\limits_{\eta \in \Gamma_0} \frac{|F(\eta)|}{1+V(\eta)}$. Denote by $C_V(\Gamma_0)$ the closed subspace of all continuous functions for which $\|\cdot\|_V$ is finite. Then condition (D) simply states that for any $F \in C_V(\Gamma_0)$ the action $L(t)F$, cf. (2.28), is continuous in (t, η).

Proposition 2.3.1. *Let K_t be a transition function with the usual conditions and assume that conditions (A) – (D) hold. Then there exists a unique associated conservative Feller evolution $U(s,t)$ on $C_b(\Gamma_0)$. This evolution system can be extended to $BM_V(\Gamma_0)$ so that*

$$|U(s,t)F(\eta)| \leq \|F\|_V V(\eta) e^{\int_s^t c(r)\mathrm{d}r}. \tag{2.33}$$

Moreover the following assertions are true:

1. *For any $F \in BM(\Gamma_0)$, $t > 0$ and $\eta \in \Gamma_0$, $U(s,t)F(\eta)$ is a solution to*

$$\frac{\partial}{\partial s} U(s,t)F(\eta) = -L(s)U(s,t)F(\eta), \quad s \in [0, t).$$

2. *Let $F \in BM(\Gamma_0)$. Then for any $s \geq 0$ and $\eta \in \Gamma_0$, $U(s,t)F(\eta)$ is a solution to*

$$\frac{\partial}{\partial t} U(s,t)F(\eta) = U(s,t)L(t)F(\eta), \quad a.a.\ t \geq s.$$

Proof. For each $\eta \in \Gamma_0$ and $\xi \subset \eta$ the map $\zeta \longmapsto \eta \backslash \xi \cup \zeta$ is measurable, hence the integral in (2.31) is well-defined. For fixed A it is measurable as a combination of measurable operations. Clearly Q is σ-additive in the last argument. The assertion follows by Proposition 2.1.1 and the identity

$$Q(t)F(\eta) := \int_{\Gamma_0} F(\xi) Q(t, \eta, \mathrm{d}\xi) = \sum_{\xi \subset \eta} \int_{\Gamma_0} F(\eta \backslash \xi \cup \zeta) K_t(\xi, \eta, \mathrm{d}\zeta)$$

for any $F \in C_b(\Gamma_0)$. □

Recall that for any bounded measurable function F, i.e. $F \in BM(\Gamma_0)$, and any finite Borel measure $\mu \in \mathcal{M}(\Gamma_0)$ the duality is defined by

$$\langle F, \mu \rangle = \int_{\Gamma_0} F(\eta) \mathrm{d}\mu(\eta).$$

Let C be a bounded linear operator on $BM(\Gamma_0)$. The adjoint operator C^* on $\mathcal{M}(\Gamma_0)$ w.r.t. this duality is defined by

$$\langle CF, \mu \rangle = \langle F, C^* \mu \rangle, \quad F \in BM(\Gamma_0), \quad \mu \in \mathcal{M}(\Gamma_0),$$

provided, of course, it exists. Let C' be the norm-adjoint operator on $BM(\Gamma_0)^*$. Any $\mu \in \mathcal{M}(\Gamma_0)$ defines by $F \longmapsto \langle F, \mu \rangle$ an element in $BM(\Gamma_0)^*$. The adjoint operator C^* exists on $\mathcal{M}(\Gamma_0)$ if and only if C' leaves $\mathcal{M}(\Gamma_0)$ invariant. In such a case C^* is given by $C^* = C'|_{\mathcal{M}(\Gamma_0)}$. In particular, for any $\eta \in \Gamma_0$ and $A \in \mathcal{B}(\Gamma_0)$

$$(C^* \delta_\eta)(A) = \langle \mathbb{1}_A, C^* \delta_\eta \rangle = \langle C \mathbb{1}_A, \delta_\eta \rangle = (C \mathbb{1}_A)(\eta) \tag{2.34}$$

holds. The considerations of the first section imply that $U(s,t)F$ is given by a transition probability function $P(s, \eta; t, \mathrm{d}\omega)$, that is

$$U(s,t)F(\eta) = \int_{\Gamma_0} F(\omega) P(s, \eta; t, \mathrm{d}\omega) \tag{2.35}$$

holds. The adjoint evolution system on $\mathcal{M}(\Gamma_0)$ is given by

$$U^*(t,s)\mu(A) = \int_{\Gamma_0} P(s, \eta; t, A) \mathrm{d}\mu(\mathrm{d}\eta).$$

The action of the adjoint evolution $U^*(t,s)\mu$ provides a weak solution to the Fokker-Planck equation (2.30). In particular, if conditions (A) – (D) are satisfied, then $U(t,s)^*$ is unique with such property.

Here and in the following we identify the space of densities $L^1(\Gamma_0, \mathrm{d}\lambda)$ with its image in $\mathcal{M}(\Gamma_0)$ given by the (isometric) embedding

$$L^1(\Gamma_0, \mathrm{d}\lambda) \ni R \longmapsto R \mathrm{d}\lambda \in \mathcal{M}(\Gamma_0).$$

The next theorem states conditions for which $U^*(t,s)$ leaves the space of densities invariant and its restriction to $L^1(\Gamma_0, \mathrm{d}\lambda)$ is strongly continuous.

Theorem 2.3.2. *Assume that $K_t(\xi, \eta, \mathrm{d}\zeta)$ satisfies the usual conditions, is absolutely continuous with respect to the Lebesgue-Poisson measure and the conditions (A) – (E) hold. Then $U^*(t,s)$ leaves $L^1(\Gamma_0, \mathrm{d}\lambda) \subset \mathcal{M}(\Gamma_0)$ invariant and is strongly continuous on $L^1(\Gamma_0, \mathrm{d}\lambda)$.*

Proof. Denote by $K_t(\xi, \eta, \zeta) = \frac{\mathrm{d}K_t(\xi,\eta,\mathrm{d}\zeta)}{\mathrm{d}\lambda(\zeta)}$ and let $L^*(t)$ be the adjoint operator with respect to the duality of $BM(\Gamma_0)$ and $\mathcal{M}(\Gamma_0)$. Then $L^*(t)$ is given by $L^*(t) = -q(t, \cdot) + Q(t)$ with $(-q(t, \cdot)R)(\eta) = -q(t, \eta)R(\eta)$ and

$$Q(t)R(\eta) = \sum_{\xi \subset \eta} \int_{\Gamma_0} R(\eta \backslash \xi \cup \zeta) K_t(\zeta, \eta \backslash \xi \cup \zeta, \xi) \mathrm{d}\lambda(\zeta), \tag{2.36}$$

see (2.25). For $t \geq 0$ let

$$D(L^*(t)) = \{R \in L^1(\Gamma_0, \mathrm{d}\lambda) \mid q(t,\cdot)R \in L^1(\Gamma_0, \mathrm{d}\lambda)\}. \tag{2.37}$$

First observe that $W^*(t,s)R(\eta) = e^{-\int_s^t q(r,\eta)\mathrm{d}r} R(\eta)$ is a positive contraction operator and $Q(t)$ is positive. In order to apply [ALMK14, Theorem 2.1] it is enough to show that for a.a. $t > s$ and all $R \in L^1(\Gamma_0, \mathrm{d}\lambda)$: $W^*(t,s)R \in D(L^*(t))$ and

$$\int_s^t \|Q(r)W^*(r,s)R\|_{L^1} \mathrm{d}r \leq \|R\|_{L^1} - \|W^*(t,s)R\|_{L^1}.$$

The first property follows by property (E) from

$$\int_{\Gamma_0} q(t,\eta)|W^*(t,s)R(\eta)|\mathrm{d}\lambda(\eta) \leq \int_{\Gamma_0} q(t,\eta)e^{-b(t)(t-s)q(t,\eta)}|R(\eta)|\mathrm{d}\lambda(\eta) \leq \frac{\|R\|_{L^1}}{b(t)(t-s)e}.$$

For the second property let $R \in L^1(\Gamma_0, \mathrm{d}\lambda)$ and note that

$$\int_{\Gamma_0} |Q(r)R(\eta)|\mathrm{d}\lambda(\eta) \leq \int_{\Gamma_0} q(r,\eta)|R(\eta)|\mathrm{d}\lambda(\eta)$$

holds. Altogether this implies

$$\begin{aligned}\int_s^t \|Q(r)W^*(r,s)R\|_{L^1} \mathrm{d}r &\leq \int_s^t \int_{\Gamma_0} q(r,\eta)e^{-\int_s^r q(\tau,\eta)\mathrm{d}\tau}|R(\eta)|\mathrm{d}\lambda(\eta)\mathrm{d}r \\ &= -\int_s^t \int_{\Gamma_0} \frac{\partial}{\partial r} e^{-\int_s^r q(\tau,\eta)\mathrm{d}\tau}|R(\eta)|\mathrm{d}\lambda(\eta)\mathrm{d}r = \|R\|_{L^1} - \|W^*(t,s)R\|_{L^1}.\end{aligned}$$

Hence by [ALMK14, Theorem 2.1] there exists a strongly continuous evolution family $(V^*(t,s))_{0\leq s\leq t}$ on $L^1(\Gamma_0, \mathrm{d}\lambda)$. The construction of $V^*(t,s)$ coincides with the construction of $U^*(t,s)$ restricted to $L^1(\Gamma_0, \mathrm{d}\lambda)$, i.e. $U^*(t,s)R = \sum_{n=0}^{\infty} U_n^*(t,s)R$ with $U_0^*(t,s)R = e^{-\int_s^t q(r,\eta)\mathrm{d}r} R$ and

$$U_{n+1}^*(t,s)R = \int_s^t U_n^*(r,t)Q(r)W^*(r,s)R\mathrm{d}r,$$

cf. [Fel40, Section 3, Theorem 1]. □

Remark 2.3.3. *For the application of [ALMK14, Theorem 2.1] it is necessary to show that* $t \longmapsto Q(t)R \in L^1(\Gamma_0, \mathrm{d}\lambda)$ *is measurable. Since* $L^1(\Gamma_0, \mathrm{d}\lambda)$ *is separable, strong measurability and weak measurability coincide, which is the reason why we have to restrict the evolution to the space of densities.*

2.3.2 Evolution of quasi-observables and correlation measures

In [KK02, FKO09] an alternative approach for the study of birth-and-death dynamics with state space Γ, i.e. all locally finite configurations, has been proposed. In particular the notion of correlation functions and quasi-observables was introduced and the relation to the evolution of observables and states has been pointed out. For dynamics with state space Γ this relation is only informal and should be realized for particular models. In the following we prove such relations for the evolution of observables, states, quasi-observables and correlation functions on the state space Γ_0 given by the operator $L(t)$ defined in (2.28).

Define for any measurable function $G : \Gamma_0 \longrightarrow \mathbb{R}$ the K-transform by

$$K_0 G(\eta) := \sum_{\xi \subset \eta} G(\xi), \quad \eta \in \Gamma_0.$$

Its inverse is again defined for any measurable function and it is given by

$$K_0^{-1} G(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \backslash \xi|} G(\xi), \quad \eta \in \Gamma_0.$$

Let $\varphi : \Gamma_0 \longrightarrow [1, \infty)$ be continuous, define $\varphi_0 := \varphi$ and $\varphi_{n+1} := K_0 \varphi_n$, $n \geq 0$. Denote by $\mathcal{L}_n$ the Banach space $L^1(\Gamma_0, \varphi_n \mathrm{d}\lambda)$ equipped with the norm

$$\|k\|_{\mathcal{L}_n} := \int\limits_{\Gamma_0} |k(\eta)| \varphi_n(\eta) \mathrm{d}\lambda(\eta)$$

and by $\mathcal{M}_n$ the Banach space of all Borel measures ρ equipped with the norm

$$\|\rho\|_{\mathcal{M}_n} := \int\limits_{\Gamma_0} \varphi_n(\eta) |\rho|(\mathrm{d}\eta).$$

Here $|\rho|$ is the total variation of ρ, i.e. $|\rho| = \rho^+ + \rho^-$ in the Hahn-Jordan decomposition. The embeddings $\mathcal{L}_n \subset \mathcal{M}_n$ are continuous and since $\varphi_n \leq \varphi_{n+1}$ we obtain $\|\cdot\|_{\mathcal{M}_n} \leq \|\cdot\|_{\mathcal{M}_{n+1}}$ and hence $\mathcal{M}_{n+1} \subset \mathcal{M}_n$. Let $\mathcal{M}_\infty := \bigcap\limits_{n \geq 0} \mathcal{M}_n$ and equip it with the locally convex Hausdorff topology determined by the family of seminorms $(\|\cdot\|_{\mathcal{M}_n})_{n \geq 0}$. A linear operator $A : \mathcal{M}_\infty \longrightarrow \mathcal{M}_\infty$ is continuous if for any $n \geq 0$ there exist $m \geq 0$ and $c > 0$ such that

$$\|A\rho\|_{\mathcal{M}_n} \leq c \|\rho\|_{\mathcal{M}_m}, \quad \rho \in \mathcal{M}_\infty. \tag{2.38}$$

In above considerations we can replace $\mathcal{M}_n$ always by $\mathcal{L}_n$. Let $\mathcal{K}_n$ stand for the Banach space of all continuous functions with norm

$$\|G\|_{\mathcal{K}_n} := \sup_{\eta \in \Gamma_0} \frac{|G(\eta)|}{\varphi_n(\eta)}.$$

Then $\|\cdot\|_{\mathcal{K}_{n+1}} \leq \|\cdot\|_{\mathcal{K}_n}$ and hence $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ holds. In analogy to $\mathcal{M}_\infty$ and $\mathcal{L}_\infty$ define $\mathcal{K}_\infty := \bigcup\limits_{n\geq 0} \mathcal{K}_n$, then K_0, K_0^{-1} are linear operators from $\mathcal{K}_\infty$ to $\mathcal{K}_\infty$. For $G \in \mathcal{K}_n$, $\rho \in \mathcal{M}_n$ and $k \in \mathcal{L}_n$ denote by $\langle G, \rho \rangle := \int\limits_{\Gamma_0} G(\eta)\rho(\mathrm{d}\eta)$ and $\langle G, k \rangle := \int\limits_{\Gamma_0} G(\eta)k(\eta)\mathrm{d}\lambda(\eta)$ the associated dual pairings of functions with measures.

Lemma 2.3.4. *The following assertions are satisfied:*

(a) For any $n \geq 0$: $K_0, K_0^{-1} : \mathcal{K}_n \longrightarrow \mathcal{K}_{n+1}$ are bounded linear operators satisfying

$$\|K_0\|_{L(\mathcal{K}_n,\mathcal{K}_{n+1})} \leq 1 \text{ and } \|K_0^{-1}\|_{L(\mathcal{K}_n,\mathcal{K}_{n+1})} \leq 1.$$

(b) For any $n \geq 0$, $G \in \mathcal{K}_n$ and $\rho \in \mathcal{M}_{n+1}$ the operators

$$(K_0^*\rho)(A) := \int\limits_{\Gamma_0} \sum_{\xi\subset\eta} \mathbb{1}_A(\xi)\mathrm{d}\rho(\eta)$$

and

$$(K_0^{-1})^*\rho(A) := \int\limits_{\Gamma_0} \sum_{\xi\subset\eta} \mathbb{1}_A(\xi)(-1)^{|\eta\backslash\xi|}\mathrm{d}\rho(\eta)$$

are bounded linear operators $\mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ and satisfy

$$\langle K_0G, \rho\rangle = \langle G, K_0^*\rho\rangle,\ \langle K_0^{-1}G, \rho\rangle = \langle G, (K_0^{-1})^*\rho\rangle. \tag{2.39}$$

Moreover for any $\rho \in \mathcal{M}_{n+2}$: $(K_0^{-1})^*K_0^*\rho = \rho = K_0^*(K_0^{-1})^*\rho$ holds. The restrictions $K_0^*|_{\mathcal{M}_\infty}$ and $(K_0^{-1})^*|_{\mathcal{M}_\infty}$ are continuous as operators $\mathcal{M}_\infty \longrightarrow \mathcal{M}_\infty$.

(c) For any $n \geq 0$ the restrictions $K_0^*, (K_0^{-1})^* : \mathcal{L}_{n+1} \longrightarrow \mathcal{L}_n$ are given by

$$(K_0^*k)(\eta) = \int\limits_{\Gamma_0} k(\eta\cup\xi)\mathrm{d}\lambda(\xi)$$

and

$$(K_0^{-1})^*k(\eta) = \int\limits_{\Gamma_0} (-1)^{|\xi|}k(\eta\cup\xi)\mathrm{d}\lambda(\xi).$$

Proof. (a) Follows immediately by the definition of the norms $\|\cdot\|_{\mathcal{K}_n}$, $n \geq 1$.
(b) Formulas (2.39) follow from the definition of the operators K_0^*, $(K_0^{-1})^*$ and

$$(K_0^{-1})^*K_0^*\rho = \rho = K_0^*(K_0^{-1})^*\rho$$

is a simple computation. Let $\rho \in \mathcal{M}_{n+1}$, then by $|K_0^*\rho| \leq K_0^*|\rho|$ it follows that

$$\|K_0^*\rho\|_{\mathcal{M}_n} \leq \int_{\Gamma_0} \varphi_n(\eta) K_0^*|\rho|(\mathrm{d}\eta) = \int_{\Gamma_0} \varphi_{n+1}(\eta)|\rho|(\mathrm{d}\eta) = \|\rho\|_{\mathcal{M}_{n+1}}$$

holds and hence $K_0^* : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ is bounded. The continuity of $K_0^*|_{\mathcal{M}_\infty} : \mathcal{M}_\infty \longrightarrow \mathcal{M}_\infty$ follows by above estimate and (2.38). The same arguments apply to $(K_0^{-1})^*$.
(c) Again we show only the assertion for K_0^*. Take $k \in \mathcal{L}_{n+1}$, then for $\rho := k\mathrm{d}\lambda$

$$(K_0^*\rho)(A) = \int_{\Gamma_0} \sum_{\xi \subset \eta} \mathbb{1}_A(\xi) k(\eta) \mathrm{d}\lambda(\eta)$$

holds. If $\lambda(A) = 0$, then $\sum\limits_{\xi \subset \eta} \mathbb{1}_A(\xi) = 0$ for a.a. $\eta \in \Gamma_0$ and hence $K_0^*\rho(A) = 0$. The representation formula for K_0^* can be computed directly by (2.25), which yields

$$\begin{aligned}\langle K_0 G, k \rangle &= \int_{\Gamma_0} \sum_{\xi \subset \eta} G(\xi) k(\eta) \mathrm{d}\lambda(\eta) \\ &= \int_{\Gamma_0}\int_{\Gamma_0} G(\xi) k(\eta \cup \xi) \mathrm{d}\lambda(\xi) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\xi) \left(\int_{\Gamma_0} k(\eta \cup \xi) \mathrm{d}\lambda(\eta) \right) \mathrm{d}\lambda(\xi).\end{aligned}$$

By $\|\rho\|_{\mathcal{M}_n} = \|k\|_{\mathcal{L}_n}$ it follows that $K_0^* : \mathcal{L}_{n+1} \longrightarrow \mathcal{L}_n$ is continuous for all $n \geq 0$. The formula for $(K_0^{-1})^* k$ can be proved in the same way. □

As proposed in [FKO09] the evolution of observables, see (2.29), can be formally rewritten to the Cauchy problem

$$\frac{\partial G_t}{\partial t} = \widehat{L}(t) G_t, \quad G_t|_{t=0}, \tag{2.40}$$

where $\widehat{L}(t) := K_0^{-1} L(t) K_0$. The solution should be therefore given by

$$\widehat{U}(s,t)G(\eta) := K_0^{-1} U(s,t) K_0 G(\eta), \quad \eta \in \Gamma_0, \ 0 \leq s \leq t. \tag{2.41}$$

Here and in the following we will say that $\widehat{U}(s,t)$ is the evolution of quasi-observables. The next lemma states some basic properties for the operators $\widehat{U}(s,t)$ and $\widehat{L}(t)$.

Lemma 2.3.5. *Let $L(t)$ be the Kolmogorov operator given by (2.28). Suppose that K_t satisfies the usual conditions and there exists $n \geq 1$ such that conditions (A) – (D) are satisfied for $V := \varphi_n$. Then the following statements are true:*

(a) For any $G \in \mathcal{K}_n$, $\widehat{U}(s,t)K_0^{-1}G \in \mathcal{K}_{n+1}$ and

$$\|\widehat{U}(s,t)K_0^{-1}G\|_{\mathcal{K}_{n+1}} \leq \exp\left(\int_s^t c(r)\mathrm{d}r\right)\|G\|_{\mathcal{K}_n}, \quad 0 \leq s \leq t. \tag{2.42}$$

(b) $\widehat{U}(s,t) : \mathcal{K}_{n-1} \longrightarrow \mathcal{K}_{n+1}$ is a bounded linear operator with

$$\|\widehat{U}(s,t)G\|_{\mathcal{K}_{n+1}} \leq \exp\left(\int_s^t c(r)\mathrm{d}r\right)\|G\|_{\mathcal{K}_{n-1}}, \quad 0 \leq s \leq t.$$

Moreover, if $K_0 G \geq 0$, then $K_0\widehat{U}(s,t)G \geq 0$ holds.

(c) For any $G \in \mathcal{K}_{n-1}$, $0 \leq s \leq r \leq t$, $\widehat{U}(s,r)$ is well-defined on elements $\widehat{U}(r,t)G$ and satisfies

$$\widehat{U}(s,s)G = G, \quad \widehat{U}(s,r)\widehat{U}(r,t)G = \widehat{U}(s,t)G.$$

(d) Assume that there exist $n_* < n$ and $\kappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$q(t,\eta)\varphi_{n_*}(\eta) \leq \kappa(t)\varphi_n(\eta), \quad t \geq 0, \;\; \eta \in \Gamma_0 \tag{2.43}$$

holds. Then $L(t) : \mathcal{K}_{n_*} \longrightarrow \mathcal{K}_n$ and $\widehat{L}(t) : \mathcal{K}_{n_*-1} \longrightarrow \mathcal{K}_{n+1}$ are bounded linear operators.

Proof. (a) Observe that $\widehat{U}(s,t)K_0^{-1}G = K_0^{-1}U(s,t)G$, and since $U(s,t) : \mathcal{K}_n \longrightarrow \mathcal{K}_n$ is bounded, cf. Proposition 2.1.1, we see that $\widehat{U}(s,t)K_0^{-1} : \mathcal{K}_n \longrightarrow \mathcal{K}_{n+1}$ is bounded. Moreover, (2.9) implies (2.42).
(b) The first property follows immediately from (a) and the second from

$$K_0\widehat{U}(s,t)G = U(s,t)K_0 G.$$

(c) $\widehat{U}(s,s)G = G$ is obvious and for the second observe

$$\widehat{U}(s,t)G = K_0^{-1}U(s,t)K_0 G = K_0^{-1}U(s,r)U(r,t)K_0 G.$$

Then $U(r,t)K_0 G \in \mathcal{K}_n$ and hence by (a) $\widehat{U}(s,r)K_0^{-1}U(s,t)K_0 G \in \mathcal{K}_{n+1}$, which implies

$$\widehat{U}(s,t)G = \widehat{U}(s,r)\widehat{U}(r,t)G.$$

(d) Conditions (2.43) and (2.32) imply for all $F \in \mathcal{K}_{n_*}$

$$\begin{aligned}
|L(t)F(\eta)| &\leq q(t,\eta)|F(\eta)| + |Q(t)F(\eta)| \\
&\leq \|F\|_{\mathcal{K}_{n_*}} q(t,\eta)\varphi_{n_*}(\eta) + \|F\|_{\mathcal{K}_{n_*}} \sum_{\xi \subset \eta} \int_{\Gamma_0} \varphi_{n_*}(\eta\backslash\xi \cup \zeta) K_t(\xi,\eta,\mathrm{d}\zeta) \\
&\leq \|F\|_{\mathcal{K}_{n_*}} \left(q(t,\eta)\varphi_{n_*}(\eta) + c(t)\varphi_{n_*}(\eta) + q(t,\eta)\varphi_{n_*}(\eta)\right) \\
&\leq \|F\|_{\mathcal{K}_{n_*}} \varphi_n(\eta)\left(2\kappa(t) + c(t)\right).
\end{aligned}$$

□

Let $G \in \mathcal{K}_{n-1}$, then $K_0 G \in \mathcal{K}_n$ and by (2.32)

$$
\begin{aligned}
|L(t)K_0G(\eta)| &\leq q(t,\eta)|K_0G(\eta)| + |Q(t)K_0G(\eta)| \\
&\leq \|K_0G\|_{\mathcal{K}_n} q(t,\eta)\varphi_n(\eta) + \|K_0G\|_{\mathcal{K}_n} \sum_{\xi \subset \eta} \int_{\Gamma_0} \varphi_n(\eta\backslash\xi \cup \zeta) K_t(\xi,\eta,\mathrm{d}\zeta) \\
&\leq \|K_0G\|_{\mathcal{K}_n} \left(2q(t,\eta)\varphi_n(\eta) + c(t)\varphi_n(\eta)\right)
\end{aligned}
$$

holds. Hence $L(t)K_0G$ is well-defined on $\mathcal{K}_{n-1}$ and therefore $\widehat{L}(t)G = K_0^{-1}L(t)K_0G$ is well-defined. Similar arguments can be used to show that $\widehat{L}(t)K_0^{-1}G$ is well-defined for any $G \in \mathcal{K}_n$. The next Proposition shows that $\widehat{U}(s,t)G$ is in fact a solution to the Cauchy problem (2.40).

Proposition 2.3.6. *Suppose that the same conditions as for Lemma 2.3.5 are fulfilled. Then for any $\eta \in \Gamma_0$ and $G \in \mathcal{K}_{n-1}$ the evolution $\widehat{U}(s,t)G(\eta)$ is absolutely continuous in $s \geq 0$ and satisfies for a.a. $s \in [0,t)$*

$$
\frac{\partial}{\partial s}\widehat{U}(s,t)G(\eta) = -\widehat{L}(s)\widehat{U}(s,t)G(\eta). \tag{2.44}
$$

Proof. Take $G \in \mathcal{K}_{n-1}$, then $K_0G \in \mathcal{K}_n$ and hence by Proposition 2.3.1 $U(s,t)K_0G(\eta)$ is absolutely continuous in s. The definition of K_0^{-1} therefore implies that $\widehat{U}(s,t)G(\eta)$ is absolutely continuous in s and

$$
\frac{\partial}{\partial s}\widehat{U}(s,t)G(\eta) = -K_0^{-1}L(s)U(s,t)K_0G(\eta)
$$

holds. Previous considerations imply that $\widehat{L}(s)K_0^{-1}U(s,t)K_0G = \widehat{L}(s)\widehat{U}(s,t)G$ is well-defined and hence (2.44) holds. □

As it was proposed in [KK02, FKO09] the Cauchy problem (2.30) can be rewritten to the Cauchy problem

$$
\frac{\partial \rho_t}{\partial t} = L^{\Delta}(t)\rho_t, \quad \rho_t|_{t=0} = \rho_0 \tag{2.45}
$$

on correlation measures. The operator $L^{\Delta}(t)$ is determined by the relation

$$
\langle \widehat{L}(t)G, \rho \rangle = \langle G, L^{\Delta}(t)\rho \rangle
$$

or equivalently by

$$
L^{\Delta}(t) := K_0^* L(t)(K_0^*)^{-1}.
$$

Thus let us define the linear operator $U^{\Delta}(t,s)$ by

$$
U^{\Delta}(t,s) := K_0^* U^*(t,s)(K_0^*)^{-1}. \tag{2.46}
$$

By Lemma 2.3.4.(b) it follows that $U^{\Delta}(t,s) : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_{n-1}$ is bounded and for any $\rho \in \mathcal{M}_n$ we get $U^{\Delta}(t,s)K_0^*\rho = K_0^*U^*(t,s)\rho \in \mathcal{M}_{n-1}$. For $\rho \in \mathcal{M}_{n+1}$ we get

$$U^{\Delta}(t,s)\rho = K_0^*U^*(t,s)(K_0^*)^{-1}\rho = K_0^*U^*(t,r)U^*(r,s)(K_0^*)^{-1}\rho$$

and since $U^*(r,s)(K_0^*)^{-1}\rho \in \mathcal{M}_n$ it follows $U^{\Delta}(t,r)K_0^*U^*(r,s)(K_0^*)^{-1}\rho \in \mathcal{M}_{n-1}$. That is

$$U^{\Delta}(s,s)\rho = \rho \text{ and } U^{\Delta}(t,r)U^{\Delta}(r,s)\rho = U^{\Delta}(t,s)\rho.$$

If in addition $K_t(\xi,\eta,\mathrm{d}\zeta) = K_t(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)$ for some measurable function $K_t(\xi,\eta,\zeta) \geq 0$ and condition (E) holds, then $U^{\Delta}(t,s)$ is a bounded linear operator from $\mathcal{L}_{n+1}$ to $\mathcal{L}_{n-1}$.

2.3.3 Examples: Ecological models

In this part we study two particular models, which have applications in ecological sciences. To simplify the proofs we consider first the case of a Markov (pre-)generator describing only the death of particles.

Lemma 2.3.7. *Consider the operator $L(t)$ given by*

$$(L(t)F)(\eta) = \sum_{\xi \subset \eta}(F(\eta\backslash\xi) - F(\eta))D_t(\xi,\eta), \quad t \in I,$$

where $(t,\xi,\eta) \longmapsto D_t(\xi,\eta) \geq 0$ is assumed to be continuous. Then condition (A) and the usual conditions holds. Moreover, $(t,\eta) \longmapsto L(t)F(\eta)$ is continuous for any $F \in C(\Gamma_0)$.

Proof. The associated function is given by $K_t(\xi,\eta,\mathrm{d}\zeta) = D_t(\xi,\eta)\delta_{\varnothing}(\mathrm{d}\zeta)$ and thus satisfies the usual conditions. The characterization of convergence in Γ_0 and continuity of D_t imply that for each $F \in C(\Gamma_0)$ also $L(t)F(\eta)$ is continuous in (t,η). Concerning (A), fix $\varepsilon > 0$, $T > 0$ and a compact $B \subset \Gamma_0$. Then there exist $\delta_B > 0$, $N_B \in \mathbb{N}$ and a compact $\Lambda_B \subset \mathbb{R}^d$ such that for each $\eta \in B$

$$|\eta| \leq N_B, \ \eta \subset \Lambda_B, \ \forall x,y \in \eta, \ x \neq y : \ |x-y| \geq \delta_B \tag{2.47}$$

holds. Let $A \subset \Gamma_0$ be a compact of the form (2.24) with δ, N, Λ as in (2.47). Then for each $\eta \in B$ and $\xi \subset \eta$ we obtain that (2.47) also holds for $\eta\backslash\xi$ instead of η. Hence $\eta\backslash\xi \in A$ and thus $Q(t,\eta,A^c) = 0$ for any $t \in [0,T]$. □

The BDLP-model

In [BP97, BP99, DL00, DL05] the so called Bolker-Dieckmann-Law-Pacala model (short BDLP-model) was introduced to study spatial patterns for certain ecological systems. Elements $x \in \eta$ are interpreted as plants and the configuration $\eta \in \Gamma_0$ describes therefore the whole ecological system. The BDLP-model is based only on the two elementary events $\eta \longmapsto \eta \cup x$ (branching of plants) and $\eta \longmapsto \eta \backslash x$ (death of plants). The branching is assumed to be density independent, that is any plant at position $x \in \eta$ creates with intensity $0 \leq \lambda \in C(\mathbb{R}_+ \times \mathbb{R}^d)$ a new plant at position $y \in \mathbb{R}^d \backslash \eta$ and the spatial probability distribution for the new plant is given by $a^+(x,y)\mathrm{d}y$, where $a^+ \in C(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover, each plant at position $x \in \eta$ has an individual lifetime independent of the other plants. Such lifetime is described by the intensity $0 \leq m \in C(\mathbb{R}_+ \times \mathbb{R}^d)$. The competition between different plants is assumed to be of additive type and hence of the form $\sum\limits_{y \in \eta \backslash x} a^-(x,y)$, where $0 \leq a^- \in C(\mathbb{R}^d \times \mathbb{R}^d)$ is the competition kernel. Above description is summarized in the form of the following Markov (pre-)generator

$$(L(t)F)(\eta) = \sum_{x \in \eta} \left(m(t,x) + \sum_{y \in \eta \backslash x} a^-(x,y) \right) (F(\eta \backslash x) - F(\eta))$$
$$+ \sum_{x \in \eta} \lambda(t,x) \int\limits_{\mathbb{R}^d} a^+(x,y)(F(\eta \cup y) - F(\eta))\mathrm{d}y.$$

Such model has been analysed in the time-homogeneous case in [FM04]. In applications one is often interested in a^+ being of the form

$$a^+(x,y) \sim \frac{1}{|x-y|^{\alpha}}, \quad |x-y| \to \infty$$

or

$$a^+(x,y) \sim e^{-\nu|x-y|^{\alpha}}, \quad |x-y| \to \infty.$$

Theorem 2.3.8. *Suppose that m, λ, a^- are continuous and bounded, a^+ is continuous with $1 = \int\limits_{\mathbb{R}^d} a^+(x,y)\mathrm{d}y$ and for any compact $\Lambda \subset \mathbb{R}^d$ there exists $a^* \geq 0$ with $a^* \in L^1(\mathbb{R}^d)$ such that*

$$a^+(x,y) \leq a^*(y), \quad x \in \Lambda, \ y \in \mathbb{R}^d$$

holds. Then conditions (A) – (D) hold for $V(\eta) = |\eta| + |\eta|^2$.

Proof. Let $B \subset \Gamma_0$ be a compact and take $N_B \in \mathbb{N}$, $\Lambda_B \subset \mathbb{R}^d$ and $\delta_B > 0$ like in (2.24). Let $A \subset \Gamma_0$ be another compact defined by (2.24) with $N_A := N_B + 1$, $\Lambda_B \subset \Lambda_A$ and $\delta_A \in (0, \delta_B)$, then $B \subset A$ holds. We obtain for $x \in \Lambda_B$ and $\eta \in B$

$$\int\limits_{\mathbb{R}^d} \mathbb{1}_{A^c}(\eta \cup y) a^+(x,y)\mathrm{d}y \leq \int\limits_{\Lambda_A^c} a^+(x,y)\mathrm{d}y + \int\limits_{B_{\delta_A}(\eta)} a^+(x,y)\mathrm{d}y,$$

where $B_{\delta_A}(\eta) := \{w \in \mathbb{R}^d \mid \exists y \in \eta : \ |w-y| < \delta_A\}$. Since $\eta \in B$ and $\delta_B > \delta_A$ we obtain $B_{\delta_A}(\eta) = \bigsqcup\limits_{y \in \eta} B_{\delta_A}(y) \subset \Lambda_B^{\delta_B}$ where $\Lambda_B^{\delta_B} := \{w \in \mathbb{R}^d \mid d(w, \Lambda_B) \leq \delta_B\}$ with

$d(w, \Lambda_B) := \inf \{|w - u| \mid u \in \Lambda_B\}$. Let $c > 0$ be such that $a^+(x, y) \leq c$ for all $x \in \Lambda_B$ and $y \in \Lambda_B^{\delta_B}$, then

$$\int\limits_{\mathbb{R}^d} \mathbb{1}_{A^c}(\eta \cup y) a^+(x, y) \mathrm{d}y \leq \int\limits_{\Lambda_A^c} a^*(y) \mathrm{d}y + N_B c |B_{\delta_A}|$$

is satisfied, where $|B_{\delta_A}|$ is the Lebesgue volume of $B_{\delta_A} = \{w \in \mathbb{R}^d \mid |w| \leq \delta_A\}$. Condition (A) now follows from above estimate, Lemma 2.3.7 and $\lambda \in C_b(\mathbb{R}_+ \times \mathbb{R}^d)$. Condition (B) follows from

$$\begin{aligned} (L(t)V)(\eta) &= \sum_{x \in \eta} \lambda(t, x) + 2 \sum_{x \in \eta} \sum_{y \in \eta \backslash x} a^-(x, y) \\ &\quad + 2|\eta| \sum_{x \in \eta} (\lambda(t, x) - m(t, x)) - 2|\eta| \sum_{x \in \eta} \sum_{y \in \eta \backslash x} a^-(x, y) \\ &\leq \max\{\|\lambda\|_\infty, 2\|a^-\|_\infty + 2\|\lambda\|_\infty + 2\|m\|_\infty\} V(\eta). \end{aligned}$$

Condition (D) is fulfilled due to

$$\begin{aligned} q(t, \eta) &= \sum_{x \in \eta} m(t, x) + \sum_{x \in \eta} \lambda(t, x) + \sum_{x \in \eta} \sum_{y \in \eta \backslash x} a^-(x, y) \\ &\leq \max\{\|m\|_\infty + \|\lambda\|_\infty, \|a^-\|_\infty\} V(\eta). \end{aligned}$$

For condition (C) it is enough to show that for any continuous function F such that $|F(\eta)| \leq \|F\|_V (1 + |\eta| + |\eta|^2)$ also $(t, \eta) \longmapsto \sum\limits_{x \in \eta} \lambda(t, x) \int\limits_{\mathbb{R}^d} a^+(x, y) F(\eta \cup y) \mathrm{d}y$ is continuous. Since $\lambda(t, x)$ is continuous it is enough to show that the integral is continuous. But this follows from dominated convergence and the condition imposed on a^+. $\square$

Above statement implies the following a priori estimate for the evolution of states. Let μ be a probability measure with $\int\limits_{\Gamma_0} (1 + |\eta| + |\eta|^2) \mu(\mathrm{d}\eta) < \infty$. Then

$$\int\limits_{\Gamma_0} (1 + |\eta| + |\eta|^2) U^*(t, s) \mu(\mathrm{d}\eta) \leq e^{(t-s)c} \int\limits_{\Gamma_0} (1 + |\eta| + |\eta|^2) \mu(\mathrm{d}\eta)$$

holds, $c := \max\{\|\lambda\|_\infty, 2\|a^-\|_\infty + 2\|\lambda\|_\infty + 2\|m\|_\infty\}$. Such estimate has been used in [FM04] for a certain scaling which lead to the well-known mesoscopic equation

$$\frac{\partial \rho_t}{\partial t}(x) = -m(t, x)\rho_t(x) - \int\limits_{\mathbb{R}^d} a^-(y, x) \rho_t(y) \mathrm{d}y \rho_t(x) + \int\limits_{\mathbb{R}^d} a^+(y, x) \lambda(t, y) \rho_t(y) \mathrm{d}y,$$

see also chapter 3 for details.

Dieckmann-Law model

In contrast to the BDLP-model we discuss here one possible extension for which the branching mechanism includes interactions of the plants. For simplicity we suppose that all intensities are translation invariant. A plant at location $x \in \eta$ shall now have the modified branching intensity given by

$$\lambda(t) + \sum_{y \in \eta \backslash x} b^+(x-y), \quad t \geq 0,$$

where $0 \leq b^+ \in C_b(\mathbb{R}^d)$. The location of the offspring is described by the probability densitiy $a^+(x-y)$. The modified Markov (pre-)generator is therefore given by

$$\begin{aligned}(L(t)F)(\eta) = &\sum_{x \in \eta} \left(m(t) + \sum_{y \in \eta \backslash x} a^-(x-y) \right) (F(\eta \backslash x) - F(\eta)) \\ &+ \sum_{x \in \eta} \lambda(t) \int_{\mathbb{R}^d} (F(\eta \cup w) - F(\eta)) a^+(x-y) \mathrm{d}w \\ &+ \sum_{x \in \eta} \sum_{y \in \eta \backslash x} b^+(x-y) \int_{\mathbb{R}^d} (F(\eta \cup w) - F(\eta)) a^+(x-w) \mathrm{d}w,\end{aligned}$$

where $m, \lambda \in C(\mathbb{R}_+)$ and $a^- \in C_b(\mathbb{R}^d)$. We assume that $a^- - b^+$ is a stable potential. By definition this means that there exists a constant $b \geq 0$ such that

$$\sum_{x \in \eta} \sum_{y \in \eta \backslash x} (a^-(x-y) - b^+(x-y)) \geq -b|\eta|, \quad \eta \in \Gamma_0.$$

Let $E^+(\eta) = \sum_{x \in \eta} \sum_{y \in \eta \backslash x} b^+(x-y)$ and $E^-(\eta) = \sum_{x \in \eta} \sum_{y \in \eta \backslash x} a^-(x-y)$, that it above condition is equivalent to

$$E^+(\eta) \leq b|\eta| + E^-(\eta), \quad \eta \in \Gamma_0.$$

Theorem 2.3.9. *Suppose that for any compact $\Lambda \subset \mathbb{R}^d$ there exists $a^* \in L^1(\mathbb{R}^d)$ which satisfies*

$$a^+(x-w) \leq a^*(w), \quad x \in \Lambda, \ w \in \mathbb{R}^d.$$

Then conditions (A) – (D) are satisfied for $V(\eta) := |\eta| + |\eta|^2$. Moreover, for any $n \geq 1$ and state μ with $\int_{\Gamma_0} |\eta|^n \mu(\mathrm{d}\eta) < \infty$, the evolution of states satisfies $\int_{\Gamma_0} |\eta|^n U^*(t,s)\mu(\mathrm{d}\eta) < \infty$. If in addition $m(t), \lambda(t) > 0$ for all $t \geq 0$, then condition (E) holds and $U^*(t,s)$ leaves the space of densities invariant.

Proof. Condition (A) will be shown for a more general case later on. Concerning condition (B) we have

$$(L(t)|\cdot|)(\eta) \leq (b + \lambda(t) - m(t))|\eta|$$

and by $(|\eta|+1)^2 - |\eta|^2 = 2|\eta| - 1$, $(|\eta|-1)^2 - |\eta|^2 = -2|\eta|$ also

$$(L(t)|\cdot|^2)(\eta) \leq (2\lambda(t) + \|b^+\|_\infty + 2b - 2m(t))|\eta|^2 + (\lambda(t) - \|b^+\|_\infty)|\eta|.$$

Altogether this yields

$$L(t)V(\eta) \leq |\eta|(b + 2\lambda(t) - m(t) - \|b^+\|_\infty) + |\eta|^2(2\lambda(t) + \|b^+\|_\infty + 2b - 2m(t)),$$

i.e. (2.32) is satisfied. Since

$$\begin{aligned} q(t,\eta) &= (m(t) + \lambda(t))|\eta| + E^+(\eta) + E^-(\eta) \\ &\leq (\|a^-\|_\infty + \|b^+\|_\infty)|\eta|^2 + |\eta| \sup_{t\in[0,T]} (m(t) + \lambda(t)) \end{aligned}$$

also (D) holds. For property (C) it is enough to show that $x \longmapsto \int\limits_{\mathbb{R}^d} F(\eta \cup y)a^+(x-y)\mathrm{d}y$ is continuous for any continuous function F with $|F(\eta)| \leq c(1 + |\eta| + |\eta|^2)$, $\eta \in \Gamma_0$ and some constant $c > 0$. But this follows immediately by dominated convergence and the assumptions on a^+. Property (E) is a direct consequence of the continuity of m and λ. For the remaining assertion it suffices to show that for any $n \geq 1$ there exist a continuous function $c_n : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ such that

$$(L(t)|\cdot|^n)(\eta) \leq c_n(t)|\eta|^n, \quad t \geq 0.$$

We have $(|\eta|+1)^n - |\eta|^n = \sum\limits_{l=0}^{n-1} \binom{n}{l}|\eta|^l$, $(|\eta|-1)^n - |\eta|^n = \sum\limits_{l=0}^{n-1} \binom{n}{l}(-1)^{n-l}|\eta|^k \leq 0$ and since $(L(t)|\cdot|^n)(\emptyset) = 0$ we can assume w.l.g. that $|\eta| > 0$. Hence

$$\begin{aligned} (L(t)|\cdot|^n)(\eta) &\leq \lambda(t)\sum_{l=0}^{n-1}\binom{n}{l}|\eta|^{l+1} + \sum_{l=0}^{n-1}\binom{n}{l}|\eta|^l(E^+(\eta) + (-1)^{n-l}E^-(\eta)) \\ &= \lambda(t)\sum_{l=1}^{n}\binom{n}{l-1}|\eta|^l + \sum_{l=1}^{n}\binom{n}{l-1}|\eta|^{l-1}(E^+(\eta) - (-1)^{n-l}E^-(\eta)) \\ &\leq |\eta|^n\lambda(t)\sum_{l=1}^{n}\binom{n}{l-1} + \sum_{l=1}^{n-1}\binom{n}{l-1}|\eta|^n(\|b^+\|_\infty + \|a^-\|_\infty) \\ &\quad + \binom{n}{n-1}(E^+(\eta) - E^-(\eta))|\eta|^{n-1} \\ &\leq 2^n(\lambda(t) + \|b^+\|_\infty + \|a^-\|_\infty)n \cdot |\eta|^n + bn|\eta|^n \end{aligned}$$

implies the assertion. □

Remark 2.3.10. *The proof shows that $U^*(t,s)$ maps the space of probability measures with the constraint $\int_{\Gamma_0} |\eta|^n \mu(\mathrm{d}\eta) < \infty$ continuously on itself. Moreover, using Corollary 2.1.11 one can show that*

$$\int_{\Gamma_0} |\eta| U^*(t,s)\mu(\mathrm{d}\eta) \leq e^{b(t-s)} e^{\int_s^t (\lambda(r)-m(r))\mathrm{d}r} \int_{\Gamma_0} |\eta|\mu(\mathrm{d}\eta)$$

and

$$\begin{aligned}\int_{\Gamma_0} (|\eta| + |\eta|^2) U^*(t,s)\mu(\mathrm{d}\eta) \leq{}& e^{(b-\|b^+\|_\infty)(t-s)} e^{\int_s^t (2\lambda(r)-m(r))\mathrm{d}r} \int_{\Gamma_0} |\eta|\mu(\mathrm{d}\eta) \\ &+ e^{(\|b^+\|_\infty + 2b)(t-s)} e^{2\int_s^t (\lambda(r)-m(r))\mathrm{d}r} \int_{\Gamma_0} |\eta|^2 \mu(\mathrm{d}\eta)\end{aligned}$$

are valid.

Generalized Dieckmann-Law model

Assume that any plant at position $x \in \eta$ may create any number $k \in \mathbb{N}$ of new plants. Their locations are, for any fixed $t \geq 0$, distributed according to the probability measure

$$a^+(t, x-y_1)\cdots a^+(t,x-y_k)\mathrm{d}y_1 \cdots \mathrm{d}y_k.$$

Therefore the (pre-)generator is assumed to be given by

$$\begin{aligned}(L(t)F)(\eta) ={}& \sum_{x\in\eta} \left(m(t,x) + \sum_{y \in \eta\backslash x} a^-(t, x-y) \right) (F(\eta\backslash x) - F(\eta)) \\ &+ \frac{1}{e} \sum_{x\in\eta} \lambda(t,x) \int_{\Gamma_0\backslash\{\varnothing\}} (F(\eta\cup\zeta) - F(\eta)) e_\lambda(a^+(t,x-\cdot);\zeta)\mathrm{d}\lambda(\zeta) \\ &+ \frac{1}{e} \sum_{x\in\eta}\sum_{y\in\eta\backslash x} b^+(t,x-y) \int_{\Gamma_0\backslash\{\varnothing\}} (F(\eta\cup\zeta) - F(\eta)) e_\lambda(a^+(t,x-\cdot);\zeta)\mathrm{d}\lambda(\zeta).\end{aligned}$$

The factor $\frac{1}{e}$ is a normalization factor since we have

$$\int_{\Gamma_0} e_\lambda(a^+(t,x-\cdot);\zeta)\mathrm{d}\lambda(\zeta) = e.$$

Theorem 2.3.11. *Let $0 \leq m, \lambda, a^-, b^+ \in C_b(\mathbb{R}_+ \times \mathbb{R}^d)$ with $a^+(t,\cdot)$ being a probability density for all $t \geq 0$. Suppose that for any compact $\Lambda \subset \mathbb{R}^d$ and $T > 0$ there exists $a^* \in L^1(\mathbb{R}^d)$ which satisfies*

$$a^+(t, x-y) \leq a^*(y), \quad x \in \Lambda, \ t \in [0,T], \ y \in \mathbb{R}^d. \tag{2.48}$$

Moreover, assume that $b^+(t,x) \leq a^-(t,x)$ holds for all $x \in \mathbb{R}^d$ and $t \geq 0$. Then conditions (A) – (D) are satisfied for $V(\eta) = |\eta| + |\eta|^2$.

Proof. By $\int\limits_{\Gamma_0 \backslash \varnothing} |\zeta| e_\lambda(a^+(t, x - \cdot); \zeta) \mathrm{d}\lambda(\zeta) = e$ we obtain

$$\begin{aligned} L(t)V(\eta) &= \sum_{x\in\eta} \big((2-e^{-1})\lambda(t,x) - 2m(t,x)\big) \\ &\quad + \sum_{x\in\eta}\sum_{y\in\eta\backslash x} \big((2-e^{-1})b^+(t,x-y) - 2a^-(t,x-y)\big) \\ &\quad + 2|\eta| \sum_{x\in\eta} (\lambda(t,x) - m(t,x)) + 2|\eta| \sum_{x\in\eta}\sum_{y\in\eta\backslash x} (b^+(t,x-y) - a^-(t,x-y)) \\ &\leq 2(\|\lambda\|_\infty + \|m\|_\infty) V(\eta), \end{aligned}$$

which implies condition (B). Condition (D) follows from

$$\begin{aligned} q(t,\eta) &= \sum_{x\in\eta} m(t,x) + \frac{e-1}{e}\sum_{x\in\eta}\lambda(t,x) + \sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^-(t,x-y) + \frac{e-1}{e}\sum_{x\in\eta}\sum_{y\in\eta\backslash x} b^+(t,x-y) \\ &\leq (\|m\|_\infty + \|\lambda\|_\infty)|\eta| + (\|a^-\|_\infty + \|b^+\|_\infty)|\eta|^2. \end{aligned}$$

In order to see (C), observe that the assertion is clear for the contribution from the terms of the operator $L(t)$ describing the death of plants. Because λ and b^+ are continuous it suffices to show for any $F \in C_V(\Gamma_0), \eta_n \to \eta, t_n \to t$ and $x_n \in \eta_n, x \in \eta$ with $x_n \to x$

$$\int\limits_{\Gamma_0\backslash\varnothing} F(\eta_n \cup \zeta) e_\lambda(a^+(t_n, \cdot - x_n); \zeta) \mathrm{d}\lambda(\zeta) \to \int\limits_{\Gamma_0\backslash\varnothing} F(\eta \cup \zeta) e_\lambda(a^+(t, \cdot - x); \zeta) \mathrm{d}\lambda(\zeta), \quad n \to \infty.$$

Since the integrand is continuous it converges for each $\zeta \in \Gamma_0 \backslash \varnothing$ and by (2.48) with compacts $K = \{t_n \mid n \geq 1\} \cup \{t\}$, $B = \{x_n \mid n \geq 1\} \cup \{x\}$ we obtain by dominated convergence the assertion. Therefore it remains to show property (A). Take $T > 0$ and fix a compact $B \subset \Gamma_0$. Hence there exists $\Lambda_B \subset \mathbb{R}^d$ compact, $N_B \in \mathbb{N}$ and $\delta_B > 0$ such that for any $\eta \in B$ (2.47) holds. Condition (A) was shown for the death of plants, so let us focus on the terms contributing to the birth. Due to the continuity of λ, b^+ the sum

$$\sum_{x\in\eta}\left(\lambda(t,x) + \sum_{y\in\eta\backslash x} b^+(t,x-y)\right)$$

is uniformly bounded on $[0,T] \times B$. Hence it is enough to estimate the integral. Take a compact set $A \subset \Gamma_0$ with the characteristics $N_A > N_B$, $\delta_A < \delta_B$, $\Lambda_B \subset \Lambda_A$, i.e. (2.24) and set

$$B_{\delta_A}(\eta) = \left\{ \xi \in \Gamma_0 \mid \xi \subset \bigcup_{x \in \eta} B_{\delta_A}(x) \right\},$$

where $B_{\delta_A}(x) = \{y \in \mathbb{R}^d \mid |x-y| < \delta_A\}$. Then we obtain for $\eta \in B$ and $x \in \eta$, so $x \in \Lambda_B$

$$\int_{\Gamma_0 \backslash \varnothing} \mathbb{1}_{A^c}(\eta \cup \zeta) e_\lambda(a^+(t, x - \cdot); \zeta) \mathrm{d}\lambda(\zeta)$$

$$\leq \left(\int_{|\zeta| > N_A - N_B} + \int_{B_{\delta_A}(\eta) \backslash \varnothing} + \int_{\Gamma_{\Lambda_A^c} \backslash \varnothing} + \int_{C(\delta_A)} \right) e_\lambda(a^+(t, x - \cdot); \zeta) \mathrm{d}\lambda(\zeta) = I_1 + I_2 + I_3 + I_4$$

where $C(\delta_A) = \{\zeta \in \Gamma_0 \mid \exists w \neq z,\ w, z \in \zeta :\ |w - z| < \delta_A\}$. For the first integral we obtain uniformly in $t \in [0,T], \eta \in B$ and $x \in \eta$

$$I_1 \leq \int_{|\zeta| > N_A - N_B} e_\lambda(a^*; \zeta) \mathrm{d}\lambda(\zeta) = \sum_{n = N_A - N_B + 1}^{\infty} \frac{\left(\int_{\mathbb{R}^d} a^*(y) \mathrm{d}y \right)^n}{n!}$$

and similarly for the third

$$I_3 \leq \int_{\Gamma_{\Lambda_A^c} \backslash \varnothing} e_\lambda(a^*; \zeta) \mathrm{d}\lambda(\zeta) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_{\Lambda_A^c} a^*(y) \mathrm{d}y \right)^n = \exp\left(\int_{\Lambda_A^c} a^*(y) \mathrm{d}y \right) - 1.$$

This two terms tend uniformly in $\eta \in B$ and $t \in [0,T]$ to zero as $N_A \to \infty$ and $\Lambda_A \to \mathbb{R}^d$. Denote by $c > 0$ a constant for which

$$a^+(t, z - w) \leq c, \quad t \in [0,T],\ z \in \Lambda_B,\ w \in \Lambda_B^{\delta_B}$$

with $\Lambda_B^{\delta_B} = \{w \in \mathbb{R}^d \mid d(w, \Lambda_B) \leq \delta_B\}$ holds, where $d(w, \Lambda_B) := \inf\{d(w,u) \mid u \in \Lambda_B\}$. For I_2 we obtain with $|B_{\delta_A}|$ the Lebesgue volume of a ball with radius δ_A in $\mathbb{R}^d$, since for any $w, z \in \eta$ with $w \neq z$: $B_{\delta_A}(w) \cap B_{\delta_A}(z) = \varnothing$

$$I_2 = \int_{B_{\delta_A}(\eta) \backslash \varnothing} e_\lambda(a^+(t, x - \cdot); \zeta) \mathrm{d}\lambda(\zeta) \leq \left(\sum_{n=1}^{\infty} \frac{c^n |B_{\delta_A}|^n}{n!} \right)^{|\eta|} = (e^{c|B_{\delta_A}|} - 1)^{|\eta|}.$$

Finally due to $C(\delta_A) \to \varnothing$ as $\delta_A \to 0$ we have shown that for all $T > 0$, all compacts $B \subset E$ and $\varepsilon > 0$ there is a compact $A \subset E$ such that

$$Q(t, \eta, A^c) < \varepsilon, \quad t \in [0, T],\ \eta \in B,$$

which is stronger then (A). □

Remark 2.3.12. *Condition* (2.48) *is for instance satisfied if there exist strictly positive continuous functions* $\lambda, C > 0$ *and* $R > 0$, $\alpha > \frac{d}{2}$ *such that*

$$a^+(t, x) \le \frac{C(t)}{(\lambda(t) + |x|^2)^{\alpha(t)}}, \quad |x| \ge R$$

holds.

Remark 2.3.13. *In the time-homogeneous case weaker conditions are sufficient to prove the Feller property.*

2.4 Time-homogeneous dynamics

In this section we analyse the time-homogeneous case. Although the results obtained in the last section apply to this case, several technical steps can be avoided and additional (stronger) results can be proved.

Suppose from now on that $K(\xi, \eta, \mathrm{d}\zeta)$ is independent of $t \ge 0$ and satisfies the usual conditions. Let L be the Kolmogorov operator given by (2.28), i.e.

$$(LF)(\eta) = \sum_{\xi \subset \eta} \int_{\Gamma_0} (F(\eta \backslash \xi \cup \zeta) - F(\eta)) K(\xi, \eta, \mathrm{d}\zeta), \quad \eta \in \Gamma_0 \tag{2.49}$$

and let $Q(\eta, \mathrm{d}\omega)$ be the infinitesimal transition function given by (2.31). Then the results of the first section imply that there exists a semigroup $(T(t))_{t \ge 0}$ of bounded linear operators on $BM(\Gamma_0)$. In particular, $T(t)$ is represented by a (sub-)probability function $P(t, \eta, \mathrm{d}\eta)$ by

$$T(t)F(\eta) = \int_{\Gamma_0} F(\xi) P(t, \eta, \mathrm{d}\xi), \quad t \ge 0,\ F \in BM(\Gamma_0), \tag{2.50}$$

see (2.5) and (2.35). This semigroup satisfies for all $\eta \in \Gamma_0$

$$LF(\eta) = \lim_{t \to 0} \frac{T(t)F(\eta) - F(\eta)}{t}, \tag{2.51}$$

repeat e.g. the arguments in the proof of Theorem 2.1.8. The adjoint semigroup $T(t)^*$ on $\mathcal{M}(\Gamma_0)$ is thus given by

$$T(t)^* \mu(A) = \int_{\Gamma_0} P(t, \eta, A) \mathrm{d}\mu(\eta), \quad t \ge 0, \quad A \in \mathcal{B}(\Gamma_0). \tag{2.52}$$

2.4.1 Evolution of observables and states

We want to give a characterization of conservativeness for $T(t)$. For this purpose we first provide an equivalent construction of the semigroup $T(t)$ and its adjoint semigroup $T(t)^*$ on $\mathcal{M}(\Gamma_0)$. Let L^* be the adjoint operator on $\mathcal{M}(\Gamma_0)$. Then L^* is given by

$$(L^*\mu)(\mathrm{d}\eta) = -q(\eta)\mu(\mathrm{d}\eta) + Q\mu(\mathrm{d}\eta), \tag{2.53}$$

where for any measurable set $A \subset \Gamma_0$

$$(Q\mu)(A) = \int_{\Gamma_0} Q(\eta, A)\mathrm{d}\mu(\eta) = \int_{\Gamma_0} \sum_{\xi \subset \eta} \int_{\Gamma_0} \mathbb{1}_A(\eta \backslash \xi \cup \zeta) K(\xi, \eta, \mathrm{d}\zeta)\mathrm{d}\mu(\eta).$$

Both operators $-q$ and Q are well-defined on the domain, cf. (2.37),

$$D(L^*) = \left\{ \mu \in \mathcal{M}(\Gamma_0) \;\middle|\; \int_{\Gamma_0} q(\eta)|\mu|(\mathrm{d}\eta) < \infty \right\}.$$

Moreover, $(-q, D(L^*))$ is the generator of an analytic semigroup given by $(e^{-tq}\mu)(\mathrm{d}\eta) = e^{-tq(\eta)}\mu(\mathrm{d}\eta)$, that is

$$(e^{-tq}\mu)(A) = \int_A e^{-tq(\eta)}\mu(\mathrm{d}\eta), \quad A \in \mathcal{B}(\Gamma_0).$$

The operator L^* satisfies for any $0 \le \mu \in D(L^*)$ the relation $L^*\mu(\Gamma_0) = 0$, i.e.

$$\int_{\Gamma_0} q(\eta)\mu(\mathrm{d}\eta) = (Q\mu)(\Gamma_0)$$

holds. By [TV06, Theorem 2.1] there exists an extension $(G, D(G))$ of $(L^*, D(L^*))$ which is the generator of a sub-stochastic semigroup $(\widetilde{T}(t)^*)_{t \ge 0}$ on $\mathcal{M}(\Gamma_0)$. Namely, $\widetilde{T}(t)^*$ is a strongly continuous semigroup such that it leaves the cone of positive measures invariant and satisfies $\|\widetilde{T}(t)^*\mu\|_{\mathcal{M}(\Gamma_0)} \le \|\mu\|_{\mathcal{M}(\Gamma_0)}$ for any $0 \le \mu \in \mathcal{M}(\Gamma_0)$. This semigroup is minimal in the sense that, given any other sub-stochastic semigroup $U(t)^*$ with generator being an extension of $(L^*, D(L^*))$, then $\widetilde{T}(t)^* \le U(t)^*$.

Lemma 2.4.1. *The semigroup $\widetilde{T}(t)^*$ coincides with the semigroup given by* (2.52).

Let $R(\lambda; -q)$ be the resolvent operator for $(-q, D(L^*))$, it can be realized as a bounded linear operator on $BM(\Gamma_0)$ and likewise on $\mathcal{M}(\Gamma_0)$, i.e.

$$R(\lambda; -q)F(\eta) = \frac{F(\eta)}{\lambda + q(\eta)}, \quad F \in BM(\Gamma_0)$$

and

$$R(\lambda; -q)\mu(A) = \int\limits_A \frac{1}{\lambda + q(\eta)}\mu(\mathrm{d}\eta), \quad \mu \in \mathcal{M}(\Gamma_0)$$

hold. For simplicity we preserve the notation $R(\lambda, -q)$ for both realizations. Hence we obtain

$$QR(\lambda; -q)\mu(A) = \int\limits_{\Gamma_0} Q(\eta, A)(R(\lambda; -q)\mu)(\mathrm{d}\eta) = \int\limits_{\Gamma_0} Q(\eta, A)\frac{1}{\lambda + q(\eta)}\mu(\mathrm{d}\eta)$$

and

$$R(\lambda; -q)QF(\eta) = \frac{1}{\lambda + q(\eta)}\int\limits_{\Gamma_0} F(\omega)Q(\eta, \mathrm{d}\omega).$$

This implies the relations

$$\langle F, R(\lambda; -q)\mu\rangle = \langle R(\lambda; -q)F, \mu\rangle$$

and

$$\langle F, QR(\lambda, -q)\mu\rangle = \langle R(\lambda, -q)QF, \mu\rangle.$$

Note that we use the notation Q for the corresponding operator on functions F and measures μ at the same time.

Proof. (Lemma 2.4.1)
The construction of $\widetilde{T}(t)^*$, cf. [ALMK11, Theorem 2.1], shows that $(G, D(G))$ satisfies for any $\mu \in \mathcal{M}(\Gamma_0)$ and $\lambda > 0$

$$R(\lambda; G)\mu = \lim_{n\to\infty} R(\lambda, -q)\sum_{k=0}^{n}(QR(\lambda; -q))^k\mu \tag{2.54}$$

in the total variation norm. Fix $\lambda > 0$ and define on $\mathcal{M}(\Gamma_0)$ a bounded linear operator by

$$R(\lambda)\mu = \int\limits_0^{\infty} e^{-\lambda t}T(t)^*\mu\mathrm{d}t.$$

The semigroup $T(t)^*$ is continuous w.r.t. the topology $\sigma(\mathcal{M}(\Gamma_0), BM(\Gamma_0))$ and hence the integral is well-defined w.r.t. this topology. Then (2.52) yields

$$R(\lambda)\mu = \int\limits_{\Gamma_0} \widehat{P}(\lambda, \xi, \cdot)\mu(\mathrm{d}\xi), \tag{2.55}$$

where $\widehat{P}(\lambda,\xi,\cdot)=\int\limits_0^\infty e^{-\lambda t}P(t,\xi,\cdot)\mathrm{d}t$. Due to [Che04, Theorem 2.16] $\widehat{P}$ is the unique minimal solution to the equation

$$\widehat{P}(\lambda,\eta,A)=\frac{1}{\lambda+q(\eta)}\delta_\eta(A)+\frac{1}{\lambda+q(\eta)}\int\limits_{\Gamma_0}\widehat{P}(\lambda,\xi,A)Q(\eta,\mathrm{d}\xi).$$

Such a minimal solution can be constructed as follows, cf. [Che04, Theorem 2.21]. Set $\widehat{P}^{(0)}(\lambda,\eta,A)=\frac{1}{\lambda+q(\eta)}\delta_\eta(A)$ and for $n\geq 0$

$$\widehat{P}^{(n+1)}(\lambda,\eta,A)=\frac{1}{\lambda+q(\eta)}\int\limits_{\Gamma_0}\widehat{P}^{(n)}(\lambda,\xi,A)Q(\eta,\mathrm{d}\xi). \tag{2.56}$$

Then $\widehat{P}(\lambda,\eta,A)$ is given by $\widehat{P}(\lambda,\eta,A)=\sum\limits_{n=0}^\infty\widehat{P}^{(n)}(\lambda,\eta,A)$. Hence by (2.55) we get

$$R(\lambda)\mu(A)=\sum_{n=0}^\infty\int\limits_{\Gamma_0}\widehat{P}^{(n)}(\lambda,\eta,A)\mu(\mathrm{d}\eta)=\sum_{n=0}^\infty R^{(n)}(\lambda)\mu(A),$$

where $R^{(n)}(\lambda)\mu(A)=\int\limits_{\Gamma_0}\widehat{P}^{(n)}(\lambda,\eta,A)\mu(\mathrm{d}\eta)$. Therefore, in view of (2.54), it suffices to show for any $n\geq 0$, $\mu\in\mathcal{M}(\Gamma_0)$ and $A\in\mathcal{B}(\Gamma_0)$ that

$$R^{(n)}(\lambda)\mu(A)=R(\lambda;-q)(QR(\lambda;-q))^n\mu(A)$$

holds. For $n=0$ this follows from

$$R^{(0)}(\lambda)\mu(A)=\int\limits_{\Gamma_0}\frac{1}{\lambda+q(\eta)}\mathbb{1}_A(\eta)\mu(\mathrm{d}\eta)=R(\lambda;-q)\mu(A).$$

Assume that this assertion holds for some $n\geq 0$. The induction hypothesis and (2.34) imply the relation

$$\begin{aligned}\widehat{P}^{(n)}(\lambda,\eta,A)&=\int\limits_{\Gamma_0}\widehat{P}^{(n)}(\lambda,\xi,A)\delta_\eta(\mathrm{d}\xi)=(R^{(n)}(\lambda)\delta_\eta)(A)\\&=R(\lambda;-q)(QR(\lambda;-q))^n\delta_\eta(A)=(R(\lambda;-q)Q)^nR(\lambda;-q)\mathbb{1}_A(\eta).\end{aligned}$$

Finally by (2.34) and (2.56) this yields

$$\begin{aligned} R^{(n+1)}(\lambda)\mu(A) &= \int_{\Gamma_0} \frac{1}{\lambda + q(\eta)} \int_{\Gamma_0} \widehat{P}^{(n)}(\lambda, \xi, A) Q(\eta, \mathrm{d}\xi)\mu(\mathrm{d}\eta) \\ &= \int_{\Gamma_0} \frac{1}{\lambda + q(\eta)} \int_{\Gamma_0} (R(\lambda; -q)Q)^n R(\lambda; -q) \mathbb{1}_A(\xi) Q(\eta, \mathrm{d}\xi)\mu(\mathrm{d}\eta) \\ &= \int_{\Gamma_0} (R(\lambda; -q)Q)^{n+1} R(\lambda; -q) \mathbb{1}_A(\eta)\mu(\mathrm{d}\eta) \\ &= R(\lambda; -q)(QR(\lambda; -q))^{n+1}\mu(A). \end{aligned}$$

□

Theorem 2.4.2. *Suppose that $K(\xi, \eta, \mathrm{d}\zeta)$ satisfies the usual conditions. Then the following assertions are equivalent:*

1. *The operator $(G, D(G))$ is the closure of $(L^*, D(L^*))$.*
2. *The semigroup $(T(t)^*)_{t\geq 0}$ is stochastic, i.e.*

$$\|T(t)^*\mu\|_{\mathcal{M}(\Gamma_0)} = \|\mu\|_{\mathcal{M}(\Gamma_0)}, \quad 0 \leq \mu \in \mathcal{M}(\Gamma_0).$$

3. *The semigroup $(T(t))_{t\geq 0}$ on observables is conservative, i.e. $T(t)1 = 1$, $t \geq 0$.*
4. *The transition probability function satisfies $P(t, \eta, \Gamma_0) = 1$ for all $t \geq 0$ and $\eta \in \Gamma_0$.*

If in addition, $K(\xi, \eta, \mathrm{d}\zeta) = K(\xi, \eta, \zeta)\mathrm{d}\lambda(\zeta)$ for some measurable function $K(\xi, \eta, \zeta)$, then $T(t)^*$ leaves the space of densities $L^1(\Gamma_0, \mathrm{d}\lambda)$ invariant.

Proof. The equivalence of the last 3 assertions follows by (2.50) and (2.52). Assume that $(G, D(G))$ is the closure of $(L^*, D(L^*))$, then it is well-known that $\widetilde{T}(t)^*$ is stochastic, cf. [TV06]. Hence by Lemma 2.4.1 $T(t)^*$ is stochastic. Conversely, suppose that $T(t)^*$ is stochastic. Then $\|T(t)^*\mu\|_{\mathcal{M}(\Gamma_0)} = \|\mu\|_{\mathcal{M}(\Gamma_0)}$ for any $0 \leq \nu \in \mathcal{M}(\Gamma_0)$. Hence [ALMK11, Corollary 3.6] implies condition 1. in this case. If $K(\xi, \eta, \mathrm{d}\zeta) = K(\xi, \eta, \zeta)\mathrm{d}\lambda(\eta)$, then by $T(t)^* = \widetilde{T}(t)^*$ and $\widetilde{T}(t)^* L^1(\Gamma_0, \mathrm{d}\lambda) \subset L^1(\Gamma_0, \mathrm{d}\lambda)$ it leaves the space of densities invariant. □

Suppose now that $K(\xi, \eta, \mathrm{d}\zeta) = K(\xi, \eta, \zeta)\mathrm{d}\lambda(\zeta)$ holds. Then L^* restricted to densities is given by $L^*R(\eta) = -q(\eta)R(\eta) + QR(\eta)$, where

$$QR(\eta) = \sum_{\xi \subset \eta} \int_{\Gamma_0} R(\eta \backslash \xi \cup \zeta) K(\zeta, \eta \backslash \xi \cup \zeta, \xi) \mathrm{d}\lambda(\zeta).$$

Above theorem implies that $T(t)^*$ leaves the space of densities invariant. Therefore we are able consider the Cauchy problem on densities $L^1(\Gamma_0, \mathrm{d}\lambda)$

$$\frac{\partial R_t}{\partial t} = L^* R_t, \quad R_t|_{t=0} = R_0 \in D(L^*). \tag{2.57}$$

If one of the equivalent statements in Theorem 2.4.2 is satisfied, then for each $R_0 \in D(L^*)$ above Cauchy problem has a unique solution given by the semigroup $T(t)^* R_0 = R_t$. The following lemma is used later on to show that a given evolution $(R_t)_{t \geq 0}$ is a solution to the Cauchy problem (2.57).

Lemma 2.4.3. *Suppose that one of the equivalent statements in Theorem 2.4.2 is satisfied and let $(G^*, D(G^*))$ be the adjoint operator to $(G, D(G))$ on $L^\infty(\Gamma_0, \mathrm{d}\lambda)$. Then for any $F \in D(G^*)$*

$$LF = G^* F$$

holds, where LF is defined by (2.49).

Proof. Take $R \in D(L^*) \subset D(G)$ and $F \in D(G^*)$, then

$$\langle G^* F, R \rangle = \langle F, GR \rangle = \langle F, L^* R \rangle$$

holds. By

$$\sum_{\xi \subset \eta} \int_{\Gamma_0} |F(\eta \backslash \xi \cup \zeta) - F(\eta)| K(\xi, \eta, \mathrm{d}\zeta) \leq \|F\|_{L^\infty} 2q(\eta), \quad \eta \in \Gamma_0$$

and $R \in D(L^*)$, (2.25) is applicable which yields $\langle F, L^* R \rangle = \langle LF, R \rangle$. □

Therefore we see that $(G^*, D(G^*)) = (L, D(L))$ where

$$D(L) = \{F \in L^\infty(\Gamma_0, \mathrm{d}\lambda) \mid LF \in L^\infty(\Gamma_0, \mathrm{d}\lambda)\} \tag{2.58}$$

is the maximal domain for L.

2.4.2 Evolution of quasi-observables and correlation functions

The aim of this part is to provide another technique for the existence and uniqueness of the time-homogeneous Cauchy problems (2.29) and (2.30). Namely semigroups for the evolution of quasi-observables

$$\frac{\partial G_t}{\partial t} = \widehat{L} G_t, \quad G_t|_{t=0} = G_0$$

and evolution of correlation functions

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t, \quad k_t|_{t=0} = k_0. \tag{2.59}$$

The general form of the operators $\widehat{L}$ and L^Δ, cf. [FKO09] and [FKO13], suggests to consider the operator L^Δ given by

$$(L^\Delta k)(\eta) = -\sum_{\xi\subset\eta}\int_{\Gamma_0} k(\eta\cup\zeta)D(\xi,\eta\backslash\xi,\zeta)\mathrm{d}\lambda(\zeta) + \sum_{\xi\subset\eta}\int_{\Gamma_0} k(\eta\backslash\xi\cup\zeta)B(\xi,\eta\backslash\xi,\zeta)\mathrm{d}\lambda(\zeta),$$

where B, D are measurable, $B(\xi,\eta\backslash\xi,\xi) = 0$ and $D(\xi,\eta\backslash\xi,\varnothing) \geq 0$ for all $\eta \in \Gamma_0$ and $\xi \subset \eta$. For a continuous function $V : \Gamma_0 \longrightarrow (0,\infty)$ let $\mathcal{L}_V$ stand for the Banach space of equivalence classes of functions k with the norm

$$\begin{aligned}\|k\|_{\mathcal{L}_V} &:= \int_{\Gamma_0} |k(\eta)|V(\eta)\mathrm{d}\lambda(\eta)\\ &= |k^{(0)}|V^{(0)} + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{(\mathbb{R}^d)^n} |k^{(n)}(x_1,\dots,x_n)|V^{(n)}(x_1,\dots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n,\end{aligned}$$

where $k = (k^{(n)})_{n=0}^\infty$ and $V = (V^{(n)})_{n=0}^\infty$ is the decomposition into their components on $\widetilde{(\mathbb{R}^d)^n} \cong \Gamma_0^{(n)}$.

Remark 2.4.4. *Let $\Lambda \subset \mathbb{R}^d$ be a compact and $D_\Lambda > 0$ such that $V(\eta) \geq \frac{1}{D_\Lambda} > 0$ for $\eta \subset \Lambda$. Then for any $n \in \mathbb{N}$*

$$\begin{aligned}\|k\|_{\mathcal{L}_V} &\geq \frac{1}{n!}\int_{\Lambda^n} |k^{(n)}(x_1,\cdots,x_n)|V^{(n)}(x_1,\dots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n\\ &\geq \frac{1}{n!}\frac{1}{D_\Lambda^n}\int_{\Lambda^n}|k^{(n)}(x_1,\cdots,x_n)|\mathrm{d}x_1\cdots\mathrm{d}x_n.\end{aligned}$$

implies

$$\int_{\Lambda^n}|k^{(n)}(x_1,\cdots,x_n)|\mathrm{d}x_1\cdots\mathrm{d}x_n \leq n!D_\Lambda^n\|k\|_{\mathcal{L}_V}. \tag{2.60}$$

In particular, if V is bounded away from zero on Γ_0, say constant, then each $k \in \mathcal{L}_V$ is necessarily integrable and hence might correspond to a density of a measure on Γ_0.

We consider the Cauchy problem for L^Δ on the Banach space $\mathcal{L}_V$. Define $M(\eta) := \sum\limits_{\xi\subset\eta} D(\xi,\eta\backslash\xi,\varnothing) \geq 0$ and the domain

$$D(L^\Delta) := \{k \in \mathcal{L}_V \mid M\cdot k \in \mathcal{L}_V\}.$$

Then $(-M, D(L^\Delta))$ is the generator of an analytic semigroup (of angle $\frac{\pi}{2}$) given by

$$(e^{-tM}k)(\eta) = e^{-tM(\eta)}k(\eta), \quad \eta\in\Gamma_0,\ t\geq 0.$$

The operator L^Δ admits the decomposition $L^\Delta = -M + L_1^\Delta$, where

$$(L_1^\Delta k)(\eta) = -\sum_{\xi\subset\eta}\int_{\Gamma_0\backslash\varnothing} k(\eta\cup\zeta)D(\xi,\eta\backslash\xi,\zeta)\mathrm{d}\lambda(\zeta) + \sum_{\xi\subset\eta}\int_{\Gamma_0} k(\eta\backslash\xi\cup\zeta)B(\xi,\eta\backslash\xi,\zeta)\mathrm{d}\lambda(\zeta).$$

Define the auxiliary function $c(\eta)$ given by

$$c(\eta) = \frac{1}{V(\eta)}\sum_{\xi\subset\eta}V(\xi)\left(\sum_{\zeta\subset\xi}|D(\zeta,\xi\backslash\zeta,\eta\backslash\xi)|\right) + \frac{1}{V(\eta)}\sum_{\xi\subset\eta}\int_{\Gamma_0}|B(\zeta,\eta\backslash\xi,\xi)|V(\eta\backslash\xi\cup\zeta)\mathrm{d}\lambda(\zeta).$$

Theorem 2.4.5. *Suppose that there exists a constant $a\in(0,2)$ such that*

$$c(\eta) \le aM(\eta), \quad \eta\in\Gamma_0 \tag{2.61}$$

holds. Then $(L^\Delta, D(L^\Delta))$ is the generator of an analytic semigroup $(T^\Delta(t))_{t\geq 0}$ of contractions.

Proof. Define a new operator B_1^Δ on $D(L^\Delta)$ by

$$(B_1^\Delta k)(\eta) = \sum_{\xi\subset\eta}\int_{\Gamma_0\backslash\varnothing} k(\eta\cup\zeta)|D(\xi,\eta\backslash\xi,\zeta)|\mathrm{d}\lambda(\zeta) + \sum_{\xi\subset\eta}\int_{\Gamma_0} k(\eta\backslash\xi\cup\zeta)|B(\xi,\eta\backslash\xi,\zeta)|\mathrm{d}\lambda(\zeta),$$

then for $0\le k\in D(L^\Delta)$ we obtain by (2.25)

$$\begin{aligned}
&\int_{\Gamma_0} B_1^\Delta k(\eta)V(\eta)\mathrm{d}\lambda(\eta) = \\
&-\int_{\Gamma_0} M(\eta)k(\eta)V(\eta)\mathrm{d}\lambda(\eta) + \int_{\Gamma_0}\left(\sum_{\zeta\subset\eta}\sum_{\xi\subset\eta\backslash\zeta}|D(\xi,\eta\backslash\zeta\backslash\xi,\zeta)|V(\eta\backslash\zeta)\right)k(\eta)\mathrm{d}\lambda(\eta) \\
&+\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0} k(\eta\cup\zeta)|B(\xi,\eta,\zeta)|V(\eta\cup\xi)\mathrm{d}\lambda(\zeta)\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
=& -\int_{\Gamma_0} M(\eta)k(\eta)V(\eta)\mathrm{d}\lambda(\eta) + \int_{\Gamma_0} k(\eta)V(\eta)\left(\sum_{\zeta\subset\eta}\sum_{\xi\subset\eta\backslash\zeta}|D(\xi,\eta\backslash\xi\backslash\zeta,\zeta)|\frac{V(\eta\backslash\zeta)}{V(\eta)}\right)\mathrm{d}\lambda(\eta) \\
&+\int_{\Gamma_0} k(\eta)V(\eta)\left(\sum_{\zeta\subset\eta}\int_{\Gamma_0}|B(\xi,\eta\backslash\zeta,\zeta)|\frac{V(\eta\backslash\zeta\cup\xi)}{V(\eta)}\mathrm{d}\lambda(\xi)\right)\mathrm{d}\lambda(\eta) \\
=& \int_{\Gamma_0} k(\eta)V(\eta)(c(\eta)-M(\eta))\mathrm{d}\lambda(\eta) \le (a-1)\int_{\Gamma_0} k(\eta)V(\eta)M(\eta)\mathrm{d}\lambda(\eta),
\end{aligned}$$

where we have used the identity

$$\sum_{\zeta\subset\eta}\sum_{\xi\subset\eta\backslash\zeta}|D(\xi,\eta\backslash\xi\backslash\zeta,\zeta)|V(\eta\backslash\zeta) = \sum_{\zeta\subset\eta}V(\zeta)\left(\sum_{\xi\subset\zeta}|D(\xi,\zeta\backslash\xi,\eta\backslash\zeta)|\right).$$

This identity follows by the substitution $\zeta\longmapsto\eta\backslash\zeta$. Let $r\in(0,1)$ be such that $a<1+r<2$, then for any $0\leq k\in D(L^{\Delta})$

$$\int_{\Gamma_0}\left(-M(\eta)+\frac{1}{r}B_1^{\Delta}\right)k(\eta)V(\eta)\mathrm{d}\lambda(\eta)\leq 0$$

holds. By [TV06, Theorem 2.2] it follows that $(-M+B_1^{\Delta},D(L^{\Delta}))$ is the generator of a sub-stochastic semigroup $U^{\Delta}(t)$ on $\mathcal{L}_V$. Then by [AR91, Theorem 1.1] also $(L^{\Delta},D(L^{\Delta}))$ is the generator of an analytic C_0-semigroup $T^{\Delta}(t)$ and by [AR91, Theorem 1.2] this semigroup satisfies $|T^{\Delta}(t)k|\leq U^{\Delta}(t)|k|$. This shows that for any $t\geq 0$

$$\|T^{\Delta}(t)k\|_{\mathcal{L}_V}\leq\int_{\Gamma_0}U^{\Delta}(t)|k|(\eta)V(\eta)\mathrm{d}\lambda(\eta)\leq\int_{\Gamma_0}|k(\eta)|V(\eta)\mathrm{d}\lambda(\eta)=\|k\|_{\mathcal{L}_V}.$$

□

Let $\mathcal{K}_V$ be the dual space to $\mathcal{L}_V$. This space can be identified with the collection of all equivalence classes of functions G equipped with the norm

$$\|G\|_{\mathcal{K}_V}=\operatorname*{ess\,sup}_{\eta\in\Gamma_0}\frac{|G(\eta)|}{V(\eta)}.$$

In the following we want to give sufficient conditions so that $T^{\Delta}(t)$ provides an evolution of densities. For this purpose suppose that the operator L is given by

$$(LF)(\eta)=\sum_{x\in\eta}d(x,\eta\backslash x)(F(\eta\backslash x)-F(\eta))+\int_{\mathbb{R}^d}b(x,\eta)(F(\eta\cup x)-F(\eta))\mathrm{d}x$$

with measurable intensities $d(x,\eta\backslash x),b(x,\eta)\geq 0$ and

$$\int_{\mathbb{R}^d}b(x,\eta)\mathrm{d}x<\infty,\quad \eta\in\Gamma_0$$

holds. This operator is a particular example of (2.49) with

$$K(\xi,\eta,\zeta)=\mathbb{1}_{\Gamma^{(0)}}(\zeta)\mathbb{1}_{\Gamma^{(1)}}(\xi)\sum_{x\in\xi}d(x,\eta\backslash x)+\mathbb{1}_{\Gamma^{(0)}}(\xi)\mathbb{1}_{\Gamma^{(1)}}(\zeta)\sum_{x\in\zeta}b(x,\eta).$$

It is not difficult to see that $K(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)$ satisfies the usual conditions. Hence there exists a (minimal) semigroup $(T(t))_{t\geq 0}$ on $BM(\Gamma_0)$ associated with the operator L and the adjoint semigroup $(T(t)^*)_{t\geq 0}$ is strongly continuous on $L^1(\Gamma_0,\mathrm{d}\lambda)$. The same computations as in [FKK12] yield

$$L^{\Delta}k(\eta) = -\sum_{x\in\eta}\int_{\Gamma_0} k(\eta\cup\eta)(K_0^{-1}d(x,\cdot\cup\eta\backslash x))(\zeta)\mathrm{d}\lambda(\zeta)$$
$$+\sum_{x\in\eta}\int_{\Gamma_0} k(\zeta\cup\eta\backslash x)(K_0^{-1}b(x,\cdot\cup\eta\backslash x))(\zeta)\mathrm{d}\lambda(\zeta).$$

For any $G\in B_{bs}(\Gamma_0)$ we have

$$\langle G, L^{\Delta}k\rangle = \langle \widehat{L}G, k\rangle$$

and $\widehat{L} := K_0^{-1}LK_0$ is given by

$$(\widehat{L}G)(\eta) = -\sum_{\xi\subset\eta} G(\xi)\sum_{x\in\xi}(K_0^{-1}d(x,\cdot\cup\eta\backslash x))(\eta\backslash\xi)$$
$$+\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi\cup x)(K_0^{-1}b(x,\cdot\cup\xi))(\eta\backslash\xi)\mathrm{d}x.$$

Thus condition (2.61) can be restated to

$$\sum_{\xi\subset\eta} V(\xi)\sum_{x\in\xi}|K_0^{-1}d(x,\cdot\cup\eta\backslash x)|(\eta\backslash\xi) + \sum_{\xi\subset\eta}\int_{\mathbb{R}^d} V(\xi\cup x)|K_0^{-1}b(x,\cdot\cup\xi)|(\eta\backslash\xi)\mathrm{d}x$$
$$\leq a\cdot\sum_{x\in\eta} d(x,\eta\backslash x)V(\eta).$$

The cumulative intensity is given by

$$q(\eta) := \sum_{x\in\eta} d(x,\eta\backslash x) + \int_{\mathbb{R}^d} b(x,\eta)\mathrm{d}x.$$

Theorem 2.4.6. *Suppose that* (2.61) *is satisfied for* $V(\eta) = K_0\varphi(\eta)$ *with* $\varphi:\Gamma_0\longrightarrow[1,\infty)$. *Moreover, assume that there exists a constant* $C>0$ *for which*

$$3^{|\eta|}(1+q(\eta)) \leq C\varphi(\eta), \quad \eta\in\Gamma_0$$

holds and assume that one of the equivalent conditions of Theorem 2.4.2 is fulfilled. Then for any $k_0\in\mathcal{L}_V$

$$(K_0^*)^{-1}T^{\Delta}(t)k_0 = T(t)^*(K_0^*)^{-1}k_0, \quad t\geq 0 \tag{2.62}$$

holds. In particular let $k_0 \in \mathcal{L}_V$ be such that $R_0 := (K_0^*)^{-1}k_0$ is a probability density on Γ_0, then

$$R_t := (K_0^*)^{-1}T^\Delta(t)k_0$$

is again a probability density on Γ_0.

Proof. By Lemma 2.3.4 the function $(K_0^*)^{-1}T^\Delta(t)k_0 \in \mathcal{L}_\varphi$ is continuous in $t \geq 0$. Since $\mathcal{L}_\varphi \subset L^1(\Gamma_0, \mathrm{d}\lambda)$ is continuously embedded it is also continuous on $L^1(\Gamma_0, \mathrm{d}\lambda)$ in $t \geq 0$. Moreover,

$$\begin{aligned}\int\limits_{\Gamma_0} q(\eta)|(K_0^*)^{-1}T^\Delta(t)k_0(\eta)|\mathrm{d}\lambda(\eta) &\leq \int\limits_{\Gamma_0}\int\limits_{\Gamma_0} q(\eta)|T^\Delta(t)k_0(\eta\cup\xi)|\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta)\\ &= \int\limits_{\Gamma_0} |T^\Delta(t)k_0(\eta)|\sum_{\xi\subset\eta} q(\xi)\mathrm{d}\lambda(\eta)\\ &\leq C\int\limits_{\Gamma_0} |T^\Delta(t)k_0(\eta)|\sum_{\xi\subset\eta}\varphi(\xi)\mathrm{d}\lambda(\eta) = C\|T^\Delta(t)k_0\|_{\mathcal{L}_V} < \infty\end{aligned}$$

implies that $(K_0^*)^{-1}T^\Delta(t)k_0 \in D(L^*)$. If we show for any $F \in D(L)$, see (2.58), the identity

$$\langle F, (K_0^*)^{-1}T^\Delta(t)k_0\rangle = \langle F, (K_0^*)^{-1}k_0\rangle + \int\limits_0^t \langle LF, (K_0^*)^{-1}T^\Delta(s)k_0\rangle \mathrm{d}s, \quad t \geq 0, \tag{2.63}$$

then $(K_0^*)^{-1}T^\Delta(t)k_0$ is a weak solution to the Cauchy problem (2.57) and hence (2.62) is proved, cf. [Bal77]. So let $F \in D(L)$, then we can find a function G such that $F = K_0G$ and $|G(\eta)| \leq c2^{|\eta|}$ for some constant $c = c(G) > 0$. Fix any $k \in \mathcal{L}_V$, then $(K_0^*)^{-1}k \in \mathcal{L}_\varphi$ and $\sum\limits_{\xi\subset\eta}|G(\xi)| \leq c(G)3^{|\eta|}$. Therefore we obtain

$$\begin{aligned}\int\limits_{\Gamma_0}\int\limits_{\Gamma_0} |k(\eta\cup\xi)|\sum_{\zeta\subset\eta}|G(\zeta)|\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) &\leq c(G)\int\limits_{\Gamma_0}\int\limits_{\Gamma_0} 3^{|\eta|}|k(\eta\cup\xi)|\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta)\\ &= c(G)\int\limits_{\Gamma_0}\sum_{\xi\subset\eta}3^{|\xi|}|k(\eta)|\mathrm{d}\lambda(\eta)\\ &\leq Cc(G)\|k\|_{\mathcal{L}_V}\end{aligned}$$

and hence by (2.25)

$$\langle G, k\rangle = \langle K_0G, (K_0^*)^{-1}k\rangle \tag{2.64}$$

holds. Using (2.25) we obtain

$$\langle G, L^{\Delta}k\rangle = \langle \widehat{L}G, k\rangle = \langle K_0\widehat{L}G, (K_0^*)^{-1}k\rangle = \langle LK_0G, (K_0^*)^{-1}k\rangle. \tag{2.65}$$

Since

$$K_0\widehat{L}G(\eta) = LK_0G(\eta) = -\sum_{x\in\eta} d(x,\eta\backslash x)(K_0G(\cdot\cup x))(\eta\backslash x) + \int_{\mathbb{R}^d} b(x,\eta)(K_0G(\cdot\cup x))(\eta)\mathrm{d}x$$

identity (2.25) is applicable provided

$$\int_{\Gamma_0}\int_{\Gamma_0} |k(\eta\cup\xi)| l(G)(\eta)\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) < \infty$$

is satisfied, where

$$l(G)(\eta) := \sum_{x\in\eta} d(x,\eta\backslash x)(K_0|G|(\cdot\cup x))(\eta\backslash x) + \int_{\mathbb{R}^d} b(x,\eta)(K_0|G|(\cdot\cup x))(\eta)\mathrm{d}x.$$

But this follows from $l(G)(\eta) \leq 2c(G)3^{|\eta|}q(\eta)$ and

$$\begin{aligned}\int_{\Gamma_0}\int_{\Gamma_0} |k(\eta\cup\xi)| l(G)(\eta)\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) &\leq 2c(G)\int_{\Gamma_0} |k(\eta)| \sum_{\xi\subset\eta} 3^{|\xi|}q(\xi)\mathrm{d}\lambda(\eta)\\ &\leq 2c(G)C\|k\|_{\mathcal{L}_V}.\end{aligned}$$

Applying (2.64) and (2.65) to $k = T^{\Delta}(t)k_0$ yields (2.63) and hence the assertion. □

2.4.3 Examples: Tumour development models

The aim is to describe the development of brain tumours. Reasonable models, including effects like increased speed of propagation of tumour cells, require to introduce at least two type of cells and study the interactions between these cells, see [FFH$^+$15] and references therein. Here we assume for simplicity that the tumour cells have only two possible states and consider therefore Γ_0^2 as the state space of the Markov dynamics. A configuration $\eta = (\eta^+, \eta^-) \in \Gamma_0^2$ is then considered as the collection of tumour cells. The cells η^- are said to be in the so-called proliferating state. The Markov evolution for this type of cells is assumed to be given by the Markov (pre-)generator

$$\begin{aligned}(L_-F)(\eta) = &\sum_{x\in\eta^-} m(x)(F(\eta^+, \eta^-\backslash x) - F(\eta))\\ &+ \sum_{x\in\eta^-} \lambda(x)\int_{\mathbb{R}^d} a^+(x,y)(F(\eta^+, \eta^-\cup y) - F(\eta))\mathrm{d}y\end{aligned}$$

with continuous bounded non-negative functions m, λ and $a^+ \geq 0$ continuous with $1 = \int_{\mathbb{R}^d} a^+(x,y)\mathrm{d}y$ for all $x \in \mathbb{R}^d$. Since proliferation is a local interaction we may assume that $a^+(x,\cdot)$ is fast decaying or even has compact support for any $x \in \mathbb{R}^d$. The rate $m(x)$ is typically small compared with $\lambda(x)$ and hence the number of cells η^- will grow exponentially in $t \geq 0$. This reflects the steady growth of the number of tumour cells. Due to competition and other biological effects such cells have the possibility to change their type, i.e. a cell $x \in \eta^-$ becomes an element of η^+ and vice versa. The corresponding Markov operator for this elementary events is given by the general form

$$\begin{aligned}(AF)(\eta) = &\sum_{x\in\eta^-} p(x,\eta^+,\eta^-\backslash x)(F(\eta^+ \cup x, \eta^-\backslash x) - F(\eta))\\ &+ \sum_{x\in\eta^+} q(x,\eta^+\backslash x,\eta^-)(F(\eta^+\backslash x, \eta^- \cup x) - F(\eta)).\end{aligned}$$

Here $p(x,\eta^+,\eta^-\backslash x), q(x,\eta^+,\eta^-\backslash x) \geq 0$ are assumed to be continuous and bounded. For the dynamics of the cells η^+ we assume that each cell moves according to a random walk independently of each other. Such motion is described by the operator

$$(L_+F)(\eta) = \sum_{x\in\eta^+} \varkappa(x) \int_{\mathbb{R}^d} c(x,y)(F(\eta^+\backslash x \cup y, \eta^-) - F(\eta))\mathrm{d}y$$

with $\varkappa \geq 0$ continuous and bounded and $c(x,y) \geq 0$ continuous such that $1 = \int_{\mathbb{R}^d} c(x,y)\mathrm{d}y$ for all $x \in \mathbb{R}^d$. In comparison to a^+ we may assume that $c(x,\cdot)$ has only polynomial decay when $|y| \to \infty$. This resembles the observations that small tumour patters can be observed far away from the main tumour pattern. The overall Markov dynamics is then described by the sum of above operators, i.e. let

$$L := L_- + L_+ + A.$$

The interplay of this two types of cells can be described heuristically in the following way. A cell $x \in \eta^-$ has two options. On the one-hand side it will produce several new cells and then die due to its natural death rate $m(x) > 0$. On the other-hand it may also change its type and start immediately moving within the brain. With high probability this jumps will be far compared to the distance of proliferation. After a certain time this moving cell will reach a substantially less dense region and hence will change its type back to the proliferating state. Such microscopic dynamics may cause the creation of new tumour patterns for which the distance to the old pattern is large compared to proliferation length. An important technical obstacle is related to real measurements of tumour cells. Namely, it is only possible to observe tumour patters larger then some minimal size. Such minimal size is related to the technical equipment being used. Since the moving cells η^+ form only a small part of the tumour, the treatment is essentially restricted to the treatment of the proliferating cells η^-. One goal is to determine the front wave propagation, derive reasonable extremal statistics, and consequently predict the size and possible locations of a significantly wider amount of tumour cells. We expect that this kind of insights will lead to a better understanding of the microscopic structure of tumours and hence to new therapeutic treatments of tumours. Applying [Kol06] for the Lyapunov function $V(\eta) = |\eta^+| + |\eta^-|$ yields.

Theorem 2.4.7. *Suppose that for any compact $\Lambda \subset \mathbb{R}^d$ there exists $a^* \in L^1(\mathbb{R}^d)$ for which*

$$a^+(x,y),\ c(x,y) \leq a^*(y), \quad x \in \Lambda,\ y \in \mathbb{R}^d$$

holds. Then there exist a conservative Feller semigroup $(T(t))_{t\geq 0}$ with property (2.50)*. This semigroup is related to the operator L by the identity* (2.51)*. The adjoint semigroup $(T(t)^*)_{t\geq 0}$ on $\mathcal{M}(\Gamma_0^2)$ leaves the space of densities invariant and is given by* (2.52)*.*

Chapter 3

Markov evolutions on Γ

In this chapter we first present the main results for one-component birth-and-death Markov evolutions and study afterwards various applications in mathematical biology, in particular models describing the stochastic behaviour of cells within an organism.

3.1 Preliminaries

3.1.1 Harmonic analysis on Γ

Let Γ be the space of all locally finite configurations on $\mathbb{R}^d$, that is

$$\Gamma = \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap \Lambda| < \infty, \quad \forall \Lambda \subset \mathbb{R}^d \text{ compact } \},$$

where $|\gamma \cap \Lambda|$ denotes the number of points inside Λ. The topology on Γ is defined as the smallest topology such that all maps

$$\gamma \longmapsto \sum_{x \in \gamma} f(x)$$

are continuous, where f is continuous with compact support, cf. [AKR98a]. This topology is metrizable in such a way that Γ becomes separable and complete, i.e. Γ is a Polish space, cf. [KK06] and the references therein. Let $\mathcal{B}(\Gamma)$ stand for the Borel-σ-algebra on Γ. Then $\mathcal{B}(\Gamma)$ is generated by sets of the form

$$\{\gamma \in \Gamma \mid |\gamma \cap \Lambda| = n\},$$

where $n \geq 0$ and $\Lambda \subset \mathbb{R}^d$ runs over all compacts. For a compact $\Lambda \subset \mathbb{R}^d$ define

$$\Gamma_\Lambda := \{\gamma \in \Gamma \mid \gamma \subset \Lambda\} = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)},$$

where $\Gamma_\Lambda^{(n)} = \{\gamma \in \Gamma_\Lambda \mid |\gamma \cap \Lambda| = n\}$ for $n \geq 1$ and $\Gamma_\Lambda^{(0)} = \{\varnothing\}$. Define $p_\Lambda : \Gamma \longrightarrow \Gamma_\Lambda$ by $p_\Lambda(\gamma) := \gamma_\Lambda := \gamma \cap \Lambda$ and sets $A \in \mathcal{B}(\Gamma)$ of the form $A = p_\Lambda^{-1}(A')$ for some $A' \in \mathcal{B}(\Gamma_\Lambda)$ are called cylinder sets. Let $\mathcal{B}_{\text{cyl}}(\Gamma)$ be the algebra of cylinder sets, i.e.

$$\mathcal{B}_{\text{cyl}}(\Gamma) = \bigcup_\Lambda p_\Lambda^{-1}(\mathcal{B}(\Gamma_\Lambda)),$$

where the union runs over all compacts $\Lambda \subset \mathbb{R}^d$. The Poisson measure π_β is defined for $\beta \in \mathbb{R}$ as the unique Borel probability measure on Γ having Laplace transform

$$\int_\Gamma e^{\sum_{x\in\gamma} f(x)} \mathrm{d}\pi_\beta(\gamma) = \exp\left(e^\beta \int_{\mathbb{R}^d} (e^{f(x)} - 1)\mathrm{d}x \right)$$

for all continuous functions f with compact support. In the following we recall basic notions of harmonic analysis on the configuration space Γ. For more detailed information and proofs we refer to [KK02].

A function $F : \Gamma \longrightarrow \mathbb{R}$ is called cylinder function if $F(\gamma) = F(\gamma_\Lambda)$ holds for all $\gamma \in \Gamma$ and some compact $\Lambda \subset \mathbb{R}^d$. Therefore F is a cylinder function if and only if it is measurable w.r.t. $\mathcal{B}_{\text{cyl}}(\Gamma)$. Let μ be a Borel probability measure on Γ, μ is said to be locally absolutely continuous w.r.t. to π_β if for each compact $\Lambda \subset \mathbb{R}^d$ the measure $\mu^\Lambda := \mu p_\Lambda^{-1}$ defined on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ is absolutely continuous w.r.t. $\pi_\beta^\Lambda := \pi_\beta p_\Lambda^{-1}$. This definition is in fact independent of β, therefore we will simply say that μ is locally absolutely continuous w.r.t. to the Poisson measure. The measure μ is said to have finite local moments if for all compacts $\Lambda \subset \mathbb{R}^d$ and $n \geq 1$

$$\int_\Gamma |\gamma \cap \Lambda|^n \mathrm{d}\mu(\gamma) < \infty.$$

Define for any $G \in B_{bs}(\Gamma_0)$ the K-transform by

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta),$$

here $\eta \Subset \gamma$ means that the sum runs only over all finite subsets η of γ. Then KG is a polynomially bounded cylinder function, i.e. there exists a compact $\Lambda \subset \mathbb{R}^d$ with $(KG)(\gamma) = (KG)(\gamma \cap \Lambda)$ and constants $C > 0$ and $N \in \mathbb{N}$ with

$$|(KG)(\gamma)| \leq C(1 + |\gamma \cap \Lambda|)^N, \quad \gamma \in \Gamma.$$

The K-transform $K : B_{bs}(\Gamma) \longrightarrow \mathcal{FP}(\Gamma) := K(B_{bs}(\Gamma_0))$ is a positivity preserving isomorphism with inverse given by

$$(K^{-1}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0.$$

For any Borel probability measure μ, which has finite local moments and is locally absolutely continuous w.r.t. the Poisson measure, we define the correlation function $k_\mu : \Gamma_0 \longrightarrow \mathbb{R}_+$ by the relation

$$\int_\Gamma KG(\gamma)\mathrm{d}\mu(\gamma) = \int_{\Gamma_0} G(\eta)k_\mu(\eta)\mathrm{d}\lambda(\eta).$$

Above relation is assumed to hold for all functions $G \in B_{bs}(\Gamma_0)$. The correlation function is uniquely determined by above relation and is locally integrable. Conversely suppose that for a given measure μ there exits a (locally integrable) correlation function k_μ. Then μ is locally absolutely continuous w.r.t. the Poisson measure and has finite local moments. For such a measure μ and correlation function k_μ the K-transform can be uniquely extended to a linear contraction operator $K : L^1(\Gamma_0, k_\mu \mathrm{d}\lambda) \longrightarrow L^1(\Gamma, \mathrm{d}\mu)$ such that

$$KG(\gamma) = \sum_{\eta \Subset \gamma} G(\eta)$$

holds for μ-a.a. $\gamma \in \Gamma$ and any $G \in L^1(\Gamma_0, k_\mu \mathrm{d}\lambda)$. Here and in the following we use for simplicity the notation $L^1(\Gamma_0, k_\mu \mathrm{d}\lambda) =: \mathcal{L}_{k_\mu}$ and if $k_\mu(\eta) = e^{\beta|\eta|}$, then we also write $\mathcal{L}_\beta$ instead of $\mathcal{L}_{e^{\beta|\cdot|}}$. The next statement was proved, e.g., in [KK02] and establishes the precise relation between correlation functions and Borel probability measures on Γ.

Theorem 3.1.1. *The following two assertions hold:*

1. *Let μ be a Borel probability measure on Γ with correlation function k_μ. Then $k_\mu(\varnothing) = 1$ and k_μ is positive definite, i.e. for any $G \in B_{bs}(\Gamma_0)$ with $KG \geq 0$*

$$\int_{\Gamma_0} G(\eta)k_\mu(\eta)\mathrm{d}\lambda(\eta) \geq 0$$

   holds.

2. *Conversely, let $k : \Gamma_0 \longrightarrow \mathbb{R}_+$ be positive definite such that $k(\varnothing) = 1$ holds. Suppose that there exist $\beta \in \mathbb{R}$ and a constant $C(k) > 0$ such that*

$$k^{(n)}(x_1, \dots, x_n) \leq C(k)e^{\beta n}$$

   holds. Then there exists a unique probability measure μ on Γ with k as its correlation function.

Denote by $\mathcal{P}_\beta$ the space of all probability measures μ such that for each μ there exists a correlation function k_μ and this function satisfies for some constant $C(\mu) > 0$

$$k_\mu(\eta) \leq C(\mu)e^{\beta|\eta|}, \quad \eta \in \Gamma_0. \tag{3.1}$$

Let $\mathcal{K}_\beta$ be the Banach space of equivalence classes of functions $k : \Gamma_0 \longrightarrow \mathbb{R}$ equipped with the norm

$$\|k\|_{\mathcal{K}_\beta} = \operatorname*{ess\,sup}_{\eta \in \Gamma_0} \frac{|k(\eta)|}{e^{\beta|\eta|}}.$$

Then we can identify $\mathcal{L}_\beta^*$ with $\mathcal{K}_\beta$ and the duality is given by

$$\langle G, k \rangle = \int_{\Gamma_0} G(\eta)k(\eta)\mathrm{d}\lambda(\eta),$$

where $G \in \mathcal{L}_\beta$ and $k \in \mathcal{K}_\beta$. The main part of the construction of an evolution of states is related to the proof that a given function k is in fact positive definite.

3.1.2 Markov dynamics on Γ

Let L be a Markov (pre-)generator on Γ, the precise form of L will be specified in the next section. The aim is to construct a semigroup $T(t)$ associated to the (backward) Kolmogorov equation on observables $F : \Gamma \longrightarrow \mathbb{R}$

$$\frac{\partial F_t}{\partial t} = LF_t, \quad F_t|_{t=0} = F_0. \tag{3.2}$$

The adjoint semigroup $T(t)^*$ then yields solutions to the forward Kolmogorov equation, in the physical literature also known as the Fokker-Planck equation

$$\frac{\partial}{\partial t} \int_\Gamma F(\gamma)\mathrm{d}\mu_t(\gamma) = \int_\Gamma (LF)(\gamma)\mathrm{d}\mu_t(\gamma), \quad \mu_t|_{t=0} = \mu_0, \tag{3.3}$$

where $F \in \mathcal{FP}(\Gamma)$. In [KK02, FKO09] it was proposed to study above Cauchy problems in terms of the operators $\widehat{L} := K_0^{-1}LK_0$ and L^Δ defined by the relation

$$\int_\Gamma \widehat{L}G(\eta)k(\eta)\mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\eta)L^\Delta k(\eta)\mathrm{d}\lambda(\eta), \quad G \in B_{bs}(\Gamma_0). \tag{3.4}$$

Solutions to the Cauchy problem

$$\frac{\partial G_t}{\partial t} = \widehat{L}G_t, \quad G_t|_{t=0} = G_0 \tag{3.5}$$

are then called quasi-observables (evolution of quasi-observables). Solutions to (3.2) are formally related to (3.5) by the relation $F_t = KG_t$. We expect that solutions to the Cauchy problem

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t, \quad k_t|_{t=0} = k_0 \tag{3.6}$$

are positive definite and hence determine uniquely a family of probability measures $(\mu_t)_{t\geq 0}$ such that k_t is the correlation function for μ_t. In such a case $(\mu_t)_{t\geq 0}$ should be a solution to (3.3). This general scheme will be realized for a particular choice of the operator L.

3.1.3 General description of Vlasov scaling

The subsequent overview is a short summary of the general scheme proposed in [FKK10], for particular examples see also [FKK11, FKK13b, BKK15] and references therein. The aim is to construct for a given Markov (pre-)generator L on Γ a certain scaling L_n, such that the following scheme holds. Let $T_n^\Delta(t) = e^{tL_n^\Delta}$ be the heuristic representation of the scaled evolution of correlation functions, see (3.6). The particular choice of $L \to L_n$ should preserve the order of singularity, that is the limit

$$n^{-|\eta|}T_n^\Delta(t)n^{|\eta|}k \longrightarrow T_V^\Delta(t)k, \quad n \to \infty \tag{3.7}$$

should exist and the evolution $T_V^\Delta(t)$ should preserve Lebesgue-Poisson exponentials. Namely, if $r_0(\eta) = e_\lambda(\rho_0; \eta)$, then $T_V^\Delta(t)r_0(\eta) = e_\lambda(\rho_t; \eta)$ holds. The function ρ_t solves in such a case the non-linear integro-differential equation

$$\frac{\partial \rho_t}{\partial t} = v(\rho_t). \tag{3.8}$$

For many particular models $v(\rho)$ can be computed explicitly. Equation (3.8) is the so-called mesoscopic limit or the kinetic description for the density of the particle system. Instead of investigating the limits (3.7), we define renormalized operators $L_{n,\text{ren}}^\Delta := n^{-|\eta|}L_n^\Delta n^{|\eta|}$ and study the behaviour of its associated semigroups $T_{n,\text{ren}}^\Delta(t)$ when $n \to \infty$. In such a case one can compute a limiting operator

$$L_{n,\text{ren}}^\Delta \longrightarrow L_V^\Delta, \quad n \to \infty \tag{3.9}$$

and show that L_V^Δ is associated to a semigroup $T_V^\Delta(t)$. This semigroup should satisfy

$$T_{n,\text{ren}}^\Delta(t) \longrightarrow T_V^\Delta(t) := e^{tL_V^\Delta}. \tag{3.10}$$

3.2 Main results

We present here the main results for general birth-and-death Markov evolutions. The proofs will be given (for the two-component case) in the next chapter.

3.2.1 Description of model

Consider a birth-and-death Markov (pre-)generator L given by

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\backslash x)(F(\gamma\backslash x) - F(\gamma)) + \int_{\mathbb{R}^d} b(x,\gamma)(F(\gamma\cup x) - F(\gamma))\mathrm{d}x. \tag{3.11}$$

Here $d(x,\gamma\backslash x) \in [0,\infty]$ is the so-called death-intensity and $b(x,\gamma) \in [0,\infty]$ the birth intensity of the birth-and-death process given by the operator L. For such intensities we suppose that the following condition is satisfied.

(A) There exists a measurable set $\Gamma_\infty \subset \Gamma$ such that for all $x \in \mathbb{R}^d$

$$\mathbb{R}^d \times \Gamma_\infty \ni (x,\gamma) \longmapsto d(x,\gamma\backslash x), b(x,\gamma) \in [0,\infty) \tag{3.12}$$

are measurable and for any compact $\Lambda \subset \mathbb{R}^d$ and bounded set $M \subset \Gamma_0$

$$\int_\Lambda \int_M (d(x,\eta) + b(x,\eta))\, \mathrm{d}\lambda(\eta)\mathrm{d}x < \infty \tag{3.13}$$

is fulfilled. Moreover, any measure $\mu \in \mathcal{P}_\beta$ is supported on Γ_∞, i.e. $\mu(\Gamma_\infty) = 1$.

3.2.2 Evolution of observables

For any function $F \in \mathcal{FP}(\Gamma)$ there exists a unique element $G \in B_{bs}(\Gamma_0)$ such that $F = KG$. For such F and G define the norm

$$\|F\|_{\mathcal{E}_\beta} := \|G\|_{\mathcal{L}_\beta} = \int_{\Gamma_0} |G(\eta)| e^{\beta|\eta|} \mathrm{d}\lambda(\eta),$$

which then satisfies

$$\|F\|_{L^1(\Gamma, \mathrm{d}\pi_\alpha)} \leq \int_\Gamma K|G|(\gamma)\mathrm{d}\pi_\beta(\gamma) = \int_{\Gamma_0} |G(\eta)|e^{\beta|\eta|}\mathrm{d}\lambda(\eta) = \|F\|_{\mathcal{E}_\beta}.$$

Let $\mathcal{E}_\beta$ stand for the completition of $\mathcal{FP}(\Gamma)$ w.r.t. the norm $\|\cdot\|_{\mathcal{E}_\beta}$. This space can be identified with the range of the K-transform on $\mathcal{L}_\beta$, i.e.

$$\mathcal{E}_\beta \cong \mathrm{Ran}(K) = \{KG \in L^1(\Gamma, \mathrm{d}\pi_\beta) \mid G \in \mathcal{L}_\beta\}$$

holds. A sequence $KG_n \in \mathcal{E}_\beta$ converges to KG if and only if $G_n \longrightarrow G$ in $\mathcal{L}_\beta$ as $n \to \infty$. For any $F \in \mathcal{E}_\beta$ we can associate a unique function $G \in \mathcal{L}_\beta$. This is expressed by $F = KG$. A similar construction has been used in [FKKZ12]. Let $\beta' < \beta$, then $\mathcal{L}_\beta \subset \mathcal{L}_{\beta'}$ is dense and hence by

$$\|F\|_{\mathcal{E}_{\beta'}} = \|G\|_{\mathcal{L}_{\beta'}} \leq \|G\|_{\mathcal{L}_\beta} = \|F\|_{\mathcal{E}_\beta}$$

this implies that $\mathcal{E}_\beta \hookrightarrow \mathcal{E}_{\beta'}$ is continuously and dense embedded.

Remark 3.2.1. *Decompose $KG \in \mathcal{E}_\beta$ into its positive and negative part, that is $KG = F_+ - F_-$ with $F_+, F_- \geq 0$, Then $F_\pm$ do not need to belong to $\mathcal{E}_\beta$, i.e. be of the form $F_\pm = KG_\pm$ for some $G_\pm \in \mathcal{L}_\beta$. Therefore $\mathcal{E}_\beta$ is not a vector lattice w.r.t. the natural order on functions.*

Define the cumulative death intensity by

$$M(\eta) := \sum_{x \in \eta} d(x, \eta \backslash x)$$

and introduce $c(L, \beta; \eta) = c(\eta)$ given by

$$c(\eta) = \sum_{x \in \eta} \int_{\Gamma_0} e^{\beta|\xi|} |K_0^{-1} d(x, \cdot \cup \eta \backslash x)|(\xi) \mathrm{d}\lambda(\xi) + e^{-\beta} \sum_{x \in \eta} \int_{\Gamma_0} e^{\beta|\xi|} |K_0^{-1} b(x, \cdot \cup \eta \backslash x)|(\xi) \mathrm{d}\lambda(\xi).$$

Note that $c(L, \beta; \eta)$ is sub-linear in the operator L. Define on $B_{bs}(\Gamma_0)$ a new operator $\widehat{L} := K_0^{-1} L K_0$ and denote by $\mathbb{1}^*$ the function given by

$$\mathbb{1}^*(\eta) := 0^{|\eta|} = \begin{cases} 1, & |\eta| = 0 \\ 0, & \text{otherwise} \end{cases}.$$

The next statement shows that the Cauchy problems (3.2) on $\mathcal{E}_\beta$ and (3.5) on $\mathcal{L}_\beta$ are in fact equivalent.

Theorem 3.2.2. *Assume (A) and that $c(\beta; \eta)$ is locally integrable, then the following assertions are equivalent:*

(a) The closure $(L, D(L))$ of $(L, \mathcal{FP}(\Gamma))$ is the generator of an analytic semigroup $(T(t))_{t \geq 0}$ on $\mathcal{E}_\beta$ such that $T(t)1 = 1$ holds and $T(t)$ is a contraction operator for each $t \geq 0$.

(b) The closure $(\widehat{L}, D(\widehat{L}))$ of $(\widehat{L}, B_{bs}(\Gamma_0))$ is the generator of an analytic semigroup $(\widehat{T}(t))_{t \geq 0}$ on $\mathcal{L}_\beta$ such that $\widehat{T}(t)\mathbb{1}^* = \mathbb{1}^*$ holds and $\widehat{T}(t)$ is a contraction operator for each $t \geq 0$.

This semigroups are for any $KG \in \mathcal{E}_\beta$ related by

$$T(t)KG = K\widehat{T}(t)G, \quad t \geq 0$$

and the corresponding generators are related by

$$D(L) = KD(\widehat{L}) = \{KG \in \mathcal{E}_\beta \mid G \in D(\widehat{L})\}$$

and $LKG = K\widehat{L}G$ for $G \in D(\widehat{L})$. The next proposition provides existence and uniqueness of solutions to the Kolmogorov equation (3.2).

Proposition 3.2.3. *Suppose that the intensities satisfy (A) and there exists $\beta \in \mathbb{R}$ and a constant $a = a(L, \beta) \in (0, 2)$ such that*

$$c(L, \beta; \eta) \leq a(L, \beta) M(\eta), \quad \eta \in \Gamma_0 \tag{3.14}$$

holds. Then following assertions are true:

(a) Condition (b) and therefore (a) of previous theorem are satisfied.

(b) Suppose that there exists $\beta_* < \beta^*$ with $\beta \in (\beta_*, \beta^*)$ such that for all $\beta' \in (\beta_*, \beta^*)$ condition (3.14) *is satisfied. For $\beta' \in (\beta_*, \beta)$ let $(T_{\beta'}(t))_{t\geq 0}$ be the semigroup generated by the closure of $(L, \mathcal{FP}(\Gamma))$ on $\mathcal{L}_{\beta'}$. Then $\mathcal{E}_{\beta'}$ is invariant for $T(t)$ and $T(t) = T_{\beta'}(t)|_{\mathcal{E}_\beta}$ holds.*

It should be noted that the upper bound 2 for $a(L, \beta)$ in (3.14) is the best possible. Namely, there exists a model such that $a(\beta) > 2$ and equation (3.5) has for every $G \in \mathcal{L}_\beta$ a unique solution, but such solutions do not form a strongly continuous semigroup on $\mathcal{L}_\beta$.

Remark 3.2.4. *Let d_1, b_1 and d_2, b_2 be two pairs of birth-and-death intensities for which condition (A) holds and denote by L_1 and L_2, respectively their associated generators. Then*

$$c(L_1 + L_2, \beta; \eta) \leq c(L_1, \beta; \eta) + c(L_2, \beta; \eta)$$

holds and hence if condition (3.14) *is satisfied for L_1 and L_2, it is also satisfied for the sum $L_1 + L_2$.*

Concerning continuous dependence on the intensities $d(x, \gamma\backslash x)$ and $b(x, \gamma)$ we can prove the following. Let $d_n(x, \gamma\backslash x), d(x, \gamma\backslash x), b_n(x, \gamma), b(x, \gamma) \in [0, \infty]$ be given and assume that they satisfy condition (A). In such a case there exists a common set Γ_∞ (independent of $n \in \mathbb{N}$) such that condition (A) holds for above intensities. Denote by L_n and L the associated Markov (pre-)generators and set

$$\begin{aligned} c_n(\beta; \eta) := &+ \sum_{x\in\eta} \int_{\Gamma_0} e^{\beta|\xi|} |K_0^{-1} d(x, \cdot \cup \eta\backslash x) - K_0^{-1} d_n(x, \cdot \cup \eta\backslash x)|(\xi) \mathrm{d}\lambda(\xi) \\ &+ e^{-\beta} \sum_{x\in\eta} \int_{\Gamma_0} e^{\beta|\xi|} |K_0^{-1} b(x, \cdot \cup \eta\backslash x) - K_0^{-1} b_n(x, \cdot \cup \eta\backslash x)|(\xi) \mathrm{d}\lambda(\xi) \end{aligned}$$

and $M_n(\eta) := \sum\limits_{x\in\eta} d_n(x, \eta\backslash x) \geq 0$.

Theorem 3.2.5. *Suppose that the conditions below are fulfilled.*

1. *There exists $\beta \in \mathbb{R}$ and a constant $a(\beta) \in (0, 2)$ such that*

$$c(L_n, \beta; \eta) \leq a(\beta) M_n(\eta), \quad \eta \in \Gamma_0, \ n \geq 1$$

   holds.

2. *There exist constants $A > 0$, $N \in \mathbb{N}$ and $\tau \geq 0$ such that*

$$d_n(x, \eta) \leq A(1 + |\eta|)^N e^{\tau|\eta|}, \quad \eta \in \Gamma_0, \ x \in \mathbb{R}^d$$

   holds.

3. $c_n(\beta;\eta) \longrightarrow 0$, $n \to \infty$ *holds for all* $\eta \in \Gamma_0$.

Then (3.14) *is satisfied, let* $T(t), T_n(t)$ *be the semigroups on* $\mathcal{E}_\beta$ *associated to* L *and* L_n*, respectively. Then for any* $F \in \mathcal{E}_\beta$

$$T_n(t)F \longrightarrow T(t)F, \quad n \to \infty$$

holds uniformly on compacts in $t \geq 0$.

3.2.3 Evolution of states

Suppose that conditions (A) and (3.14) are fulfilled. Let $(L, D(L))$ be the closure of $(L, \mathcal{FP}(\Gamma))$ in $\mathcal{E}_\beta$. Denote by $T(t)$ the semigroup generated by $(L, D(L))$. We suppose to show that under additional conditions the adjoint semigroup preserves positivity. Let $\mathcal{E}_\beta^*$ be the dual space to $\mathcal{E}_\beta$, then each functional $\ell \in \mathcal{E}_\beta^*$ can be represented by $k_\ell \in \mathcal{K}_\beta$, i.e.

$$\ell(KG) = \langle G, k_\ell \rangle$$

and $\|\ell\|_{\mathcal{E}_\beta^*} = \|k_\ell\|_{\mathcal{K}_\beta}$ holds. Let $(T(t)^*)_{t\geq 0}$ be the adjoint semigroup on $\mathcal{E}_\beta^*$ and $(\widehat{T}(t)^*)_{t\geq 0}$ be the adjoint semigroup on $\mathcal{K}_\beta$. Likewise we see that

$$(T(t)^*\ell)(KG) = \langle G, \widehat{T}(t)^* k_\ell \rangle, \quad KG \in \mathcal{E}_\beta, \ t \geq 0 \tag{3.15}$$

and $\|T(t)^*\ell\|_{\mathcal{E}_\beta^*} = \|\widehat{T}(t)^* k_\ell\|_{\mathcal{K}_\beta}$ are satisfied. Since $T(t)1 = 1$, $t \geq 0$, it follows that

$$(T(t)^*\ell)(1) = \ell(1) = k_\ell(\varnothing)$$

holds, which resembles the preservation of mass property. Thus we restrict all further considerations to the case $k_\ell(\varnothing) = 1$. The general case can be obtained by normalization. Let us start with the notion of solutions to the Fokker-Planck equation (3.3).

Definition 3.2.6. *A family of Borel probability measures* $(\mu_t)_{t\geq 0} \subset \mathcal{P}_\beta$ *is said to be a weak solution to* (3.3) *if for any* $F \in \mathcal{FP}(\Gamma)$*:* $t \longmapsto \langle LF, \mu_t \rangle$ *is locally integrable and satisfies*

$$\langle F, \mu_t \rangle = \langle F, \mu_0 \rangle + \int_0^t \langle LF, \mu_s \rangle \mathrm{d}s, \quad t \geq 0. \tag{3.16}$$

Remark 3.2.7. *Let* $(\mu_t)_{t\geq 0} \subset \mathcal{P}_\beta$*, then for any* $F \in \mathcal{FP}(\Gamma)$ *and* $t \geq 0$ *we get* $F \in L^1(\Gamma, \mathrm{d}\mu_t)$ *and by* (3.1)

$$\int_\Gamma |LF(\gamma)| \mathrm{d}\mu_t(\gamma) \leq \int_{\Gamma_0} |\widehat{L}G(\eta)| k_{\mu_t}(\eta) \mathrm{d}\lambda(\eta) \leq C(\mu_t) \int_{\Gamma_0} |\widehat{L}G(\eta)| e^{\beta|\eta|} \mathrm{d}\lambda(\eta)$$

also $LF \in L^1(\Gamma, \mathrm{d}\mu_t)$*, where we have used* $F = KG$*,* $G \in B_{bs}(\Gamma_0) \subset D(\widehat{L})$.

Uniqueness is stated in the next theorem.

Theorem 3.2.8. *(Uniqueness)*
Suppose that (A) and (3.14) *are satisfied. Then equation* (3.3) *has at most one solution* $(\mu_t)_{t\geq 0} \subset \mathcal{P}_\beta$ *such that its correlation functions* $(k_t)_{t\geq 0}$ *satisfy*

$$\sup_{t\in[0,T]} \|k_t\|_{\mathcal{K}_\beta} < \infty, \quad \forall T > 0.$$

Let us now focus on existence of solutions to (3.3). For a given initial state $\mu_0 \in \mathcal{P}_\beta$ with correlation function k_{μ_0}, the evolution $T(t)^*\mu_0 =: \mu_t \in \mathcal{E}_\beta^*$ is uniquely determined by $\widehat{T}(t)^* k_{\mu_0} =: k_{\mu_t} \in \mathcal{K}_\beta$, see (3.15). For existence it suffices to show that k_{μ_t} is positive definite. This resembles in proving that $T(t)^*$ is positivity preserving. For this purpose additional conditions are needed.

(B) There exist constants $A > 0$, $\tau \geq 0$ and $N \in \mathbb{N}$ such that

$$b(x, \eta) + d(x, \eta) \leq A(1 + |\eta|)^N e^{\tau|\eta|}, \quad x \in \mathbb{R}^d, \ \eta \in \Gamma_0. \tag{3.17}$$

(C) There exists β' with $\beta' + \tau < \beta$ such that there exists a constant $a(\beta') > 0$ with

$$c(\beta'; \eta) \leq a(\beta')M(\eta), \quad \eta \in \Gamma_0.$$

The crucial step in proving the positivity preservation property is identifying it with a certain evolution of states. For such reason we approximate L by operators L_δ which fit into the setting of the second chapter. Let $(R_\delta)_{\delta>0}$ be a sequence of continuous integrable functions with $0 < R_\delta \leq 1$ and $R_\delta(x) \nearrow 1$ as $\delta \to 0$ for all $x \in \mathbb{R}^d$. We will call such sequence of functions "localization sequence". Define a new birth intensity by $b_\delta(x, \eta) := R_\delta(x)b(x, \eta)$ for all $x \in \mathbb{R}^d$ and $\eta \in \Gamma_0$. Then by (3.17) this intensities satisfy for all $\delta > 0$

$$\int_{\mathbb{R}^d} b_\delta(x, \eta)\mathrm{d}x < \infty, \quad \eta \in \Gamma_0.$$

The considerations of the second chapter imply for each $\eta \in \Gamma_0$ the existence of an associated (minimal) birth-and-death process $(\eta_t)_{t\geq 0}$ starting from η with state space Γ_0. The following is our last assumption for existence of an evolution of states.

(D) There exists a localization sequence $(R_\delta)_{\delta>0}$ such that the associated (minimal) birth-and-death process is conservative, i.e. has no explosion starting from any initial point $\eta \in \Gamma_0$.

The next proposition is the main result for this section. Note that, $\mathcal{P}_{\beta'} \subset \mathcal{P}_\beta \subset \mathcal{E}_\beta^*$.

Proposition 3.2.9. *(Existence)*
Suppose that (A) – (D) and (3.14) *are fulfilled. Then $T(t)^*\mathcal{P}_{\beta'} \subset \mathcal{P}_\beta$. In particular for any $\mu_0 \in \mathcal{P}_{\beta'}$ there exists exactly one solution $(\mu_t)_{t\geq 0} \subset \mathcal{P}_\beta$ to* (3.3) *given by $T(t)^*\mu_0 = \mu_t$. If conditions (B) and (C) hold for all $\tau > 0$, then $T(t)^*\mathcal{P}_\beta \subset \mathcal{P}_\beta$ is satisfied.*

Continuity with respect to initial data establishes in the following estimate

$$\|T(t)^*\mu_0\|_{\mathcal{E}_\beta^*} = \|k_{\mu_t}\|_{\mathcal{K}_\beta} \leq \|k_{\mu_0}\|_{\mathcal{K}_\beta} = \|\mu_0\|_{\mathcal{E}_\beta^*}, \quad t \geq 0,$$

where $\widehat{T}(t)^*k_{\mu_0} = k_{\mu_t} \in \mathcal{K}_\beta$ is the correlation function corresponding to the evolution of states $\mu_t = T(t)^*\mu_0 \in \mathcal{P}_\beta$. Continuity in $t \geq 0$ (in general) only holds in the topology $\sigma(\mathcal{E}_\beta^*, \mathcal{E}_\beta)$. However, if we suppose that $\mu_0 \in \mathcal{E}_{\beta'}^*$ holds, then

$$\|T(t)^*\mu_0 - \mu_0\|_{\mathcal{E}_\beta^*} = \|k_{\mu_t} - k_{\mu_0}\|_{\mathcal{K}_\beta}, \quad t \geq 0$$

and the evolution k_{μ_t} is in fact continuous in the norm. Because $\mathcal{E}_\beta$ is not a Banach lattice w.r.t. the natural order on functions we are not able to show that $T(t)$ is positivity preserving. Above statement only implies for all $0 \leq F \in \mathcal{E}_\beta$ and $\mu_0 \in \mathcal{P}_{\beta'}$ that

$$\int\limits_\Gamma T(t)F(\gamma)\mathrm{d}\mu_0(\gamma) \geq 0$$

holds. The construction of the Markov function has been proposed in [KKM08].

Corollary 3.2.10. *Suppose that* (3.14) *and (A) – (D) hold for any $\tau > 0$ in* (3.17)*. Then for any $\mu \in \mathcal{P}_\beta$ there exists a Markov function $(X_t^\mu)_{t\geq 0}$ on the configuration space Γ with the initial distribution μ associated with the generator L.*

3.2.4 Ergodicity

For a given measure $\mu \in \mathcal{P}_\beta$ let $\langle F \rangle_\mu := \int\limits_\Gamma F(\gamma)\mathrm{d}\mu(\gamma)$. The next statement provides ergodicity for the semigroups $(T(t))_{t\geq 0}$ and $(T(t)^*)_{t\geq 0}$.

Proposition 3.2.11. *Suppose that conditions (A) – (D),* (3.14) *and $\inf\limits_{|\eta|\geq 1} M(\eta) > 0$ are fulfilled. Then there exists a unique invariant measure $\mu_{\mathrm{inv}} \in \mathcal{P}_\beta$, i.e.*

$$\int\limits_\Gamma LF(\gamma)\mathrm{d}\mu_{\mathrm{inv}}(\gamma) = 0, \quad \forall F \in \mathcal{FP}(\Gamma)$$

and $T(t)^*\mu_{\mathrm{inv}} = \mu_{\mathrm{inv}}$ hold for all $t \geq 0$. Moreover, there exist constants $C, \varepsilon > 0$ such that the assertions below are satisfied:

1. *For each* $F \in \mathcal{E}_\beta$

$$\|T(t)F - \langle F \rangle_{\mu_{\rm inv}}\|_{\mathcal{E}_\beta} \leq Ce^{-\varepsilon t}\|F - \langle F \rangle_{\mu_{\rm inv}}\|_{\mathcal{E}_\beta}, \quad t \geq 0$$

   holds.

2. *For any* $\mu_0 \in \mathcal{P}_{\beta'}$ *let* $\mu_t = T(t)^*\mu_0 \in \mathcal{P}_\beta$*, then*

$$\|\mu_t - \mu_{\rm inv}\|_{\mathcal{E}_\beta^*} \leq Ce^{-\varepsilon t}\|\mu_0 - \mu_{\rm inv}\|_{\mathcal{E}_\beta^*}, \quad t \geq 0$$

   holds. If conditions (B) and (C) hold for any $\tau > 0$*, then above claim also holds for* $\mu \in \mathcal{P}_\beta$*.*

The aggregation model is one particular example for which the cumulative death intensity is not bounded away from zero, i.e. the condition $\inf\limits_{|\eta| \geq 1} M(\eta) > 0$ is not satisfied, cf. [FKKZ14].

3.2.5 Vlasov scaling

Suppose we have given scaled intensities $d_n(x,\gamma), b_n(x,\gamma) \in [0,\infty]$ which all satisfy condition (A). Let

$$L_nF(\gamma) = \sum_{x \in \gamma} d_n(x,\gamma\backslash x)(F(\gamma\backslash x) - F(\gamma)) + n\int\limits_{\mathbb{R}^d} b_n(x,\gamma)(F(\gamma \cup x) - F(\gamma))\mathrm{d}x,$$

define $\widehat{L}_n := K_0^{-1}L_nK_0$ and the operator $\widehat{L}_{n,\mathrm{ren}} := R_n\widehat{L}_nR_{n^{-1}}$ with $R_\alpha G(\eta) := \alpha^{|\eta|}G(\eta)$. Introduce for $n \geq 1$

$$\begin{aligned} c_n(\beta;\eta) = &+ \sum_{x \in \eta}\int\limits_{\Gamma_0} |K_0^{-1}d_n(x, \cdot \cup \eta\backslash x)|(\xi)n^{|\xi|}e^{\beta|\xi|}\mathrm{d}\lambda(\xi) \\ &+ e^{-\beta}\sum_{x \in \eta}\int\limits_{\Gamma_0} |K_0^{-1}b_n(x, \cdot \cup \eta\backslash x)|(\xi)n^{|\xi|}e^{\beta|\xi|}\mathrm{d}\lambda(\xi) \end{aligned}$$

and $M_n(\eta) := \sum\limits_{x \in \eta} d_n(x,\eta\backslash x)$. For passing to the limit $n \to \infty$ we need the following conditions given below:

(V1) There exists $a(\beta) \in (0,2)$ such that

$$c_n(\beta;\eta) \leq a(\beta)M_n(\eta), \quad \eta \in \Gamma_0, \; n \in \mathbb{N}$$

holds.

(V2) For all $\xi \in \Gamma_0$ and $x \in \mathbb{R}^d$ the following limits exist in $\mathcal{L}_\beta$ and are independent of ξ

$$\lim_{n\to\infty} n^{|\cdot|}(K_0^{-1}d_n(x,\cdot\cup\xi)) = \lim_{n\to\infty} n^{|\cdot|}(K_0^{-1}d_n(x,\cdot)) =: D_x^V$$
$$\lim_{n\to\infty} n^{|\cdot|}(K_0^{-1}b_n(x,\cdot\cup\xi)) = \lim_{n\to\infty} n^{|\cdot|}(K_0^{-1}b_n(x,\cdot)) =: B_x^V$$

(V3) Let $M_V(\eta) := \sum\limits_{x\in\eta^+} D_x(\varnothing)$, then there exists $\sigma > 0$ such that either

$$M_n(\eta) \le \sigma M_V(\eta), \quad \eta \in \Gamma_0,\ n \in \mathbb{N}$$

or

$$M_n(\eta) \ge \sigma M_V(\eta), \quad \eta \in \Gamma_0,\ n \in \mathbb{N}$$

are satisfied.

Remark 3.2.12. *A collection of particular examples satisfying condition (V2) can be found in [FKK10, FFH*$^{+}$*15]. For many particular models M_n is monotone in $n \in \mathbb{N}$ and hence condition (V3) is satisfied.*

The next statements realizes the general approach for Vlasov scaling on the level of quasi-observables and correlation functions. It is a refinement of the result proved in [FKK12] where only strong solutions have been considered and is stated here only for completeness.

Theorem 3.2.13. *Suppose that conditions (V1) – (V3) are fulfilled. Then the following assertions hold:*

(a) For any $n \ge 1$ the closure $(\widehat{L}_{n,\mathrm{ren}}, D(\widehat{L}_{n,\mathrm{ren}}))$ of $(\widehat{L}_{n,\mathrm{ren}}, B_{bs}(\Gamma_0))$ is the generator of an analytic semigroup $(\widehat{T}_{n,\mathrm{ren}}(t))_{t\ge0}$ of contractions on $\mathcal{L}_\beta$.

(b) There exists an analytic semigroup $(\widehat{T}^V(t))_{t\ge0}$ of contractions on $\mathcal{L}_\beta$ such that for any $G \in \mathcal{L}_\beta$

$$\widehat{T}_{n,\mathrm{ren}}(t)G \longrightarrow \widehat{T}^V(t)G, \quad n \to \infty$$

holds uniformly on compacts in $t \ge 0$. The space $B_{bs}(\Gamma_0)$ is a core for the generator $(\widehat{L}_V, D(\widehat{L}_V))$ of $(\widehat{T}^V(t))_{t\ge0}$.

(c) For any $r_0 \in \mathcal{K}_\beta$ the unique weak solution to

$$\frac{\partial}{\partial t}\langle G, k_{t,n}\rangle = \langle \widehat{L}_{n,\mathrm{ren}}G, k_{t,n}\rangle, \quad k_{t,n}|_{t=0} = r_0, \quad G \in B_{bs}(\Gamma_0)$$

is given by $k_{t,n} = \widehat{T}_{n,\mathrm{ren}}(t)^* r_0$, and the unique weak solution to

$$\frac{\partial}{\partial t}\langle G, r_t\rangle = \langle \widehat{L}_V G, r_t\rangle, \quad r_t|_{t=0} = r_0, \quad G \in B_{bs}(\Gamma_0) \tag{3.18}$$

is given by $r_t = \widehat{T}^V(t)^* r_0$.

(d) Let $r_0(\eta) = \prod_{x\in\eta} \rho_0(x)$ *and* $\rho_0 \in L^\infty(\mathbb{R}^d)$ *with* $\|\rho_0\|_{L^\infty} \leq e^\beta$. *Assume that* $\rho_t \in L^\infty(\mathbb{R}^d)$ *with* $\|\rho_t\|_{L^\infty} \leq e^\beta$ *is a classical solution to*

$$\frac{\partial \rho_t}{\partial t}(x) = -\int_{\Gamma_0} e_\lambda(\rho_t;\xi) D_x^V(\xi)\mathrm{d}\lambda(\xi)\rho_t(x) + \int_{\Gamma_0} e_\lambda(\rho_t;\xi) B_x^V(\xi)\mathrm{d}\lambda(\xi) \tag{3.19}$$

and initial condition $\rho_t|_{t=0} = \rho_0$. *Then* $r_t(\eta) := \prod_{x\in\eta} \rho_t(x)$ *is a weak solution to* (3.18).

Remark 3.2.14. *For many particular models we also can show the convergence in (V2) in the operator norm of* $L(\mathcal{L}_\beta, \mathcal{L}_{\beta'})$, *cf. [FFH+ 15]. In such a case similar statements hold without condition (V3). Condition (V3) can also be replaced by*

$$d_n(x,\eta) \leq A(1+|\eta|)^N e^{\tau|\eta|}, \quad \eta \in \Gamma_0,\ x \in \mathbb{R}^d$$

for all $n \in \mathbb{N}$ *and some constants* $A > 0$, $N \in \mathbb{N}$ *and* $\tau \geq 0$.

Property (d) is known as the Chaos preservation property and the integro-differential equation for ρ_t is the same as in (3.8). The last statement also provides uniqueness of solutions to the integro-differential equation (3.19). Namely, for any initial condition $\rho_0 \in L^\infty(\mathbb{R}^d)$ with $\|\rho_0\|_{L^\infty} \leq e^\beta$ there exists at most one classical solution $\rho_t \in L^\infty(\mathbb{R}^d)$ with $\|\rho_t\|_{L^\infty} \leq e^\beta$. It is also possible to rewrite above result in terms of observables and states, the precise statement is given below.

Proposition 3.2.15. *Suppose that conditions (V1) – (V3) are satisfied. Then the following holds:*

(a) For $F = KG \in \mathcal{E}_\beta$ *the relations*

$$T_{n,\mathrm{ren}}(t)KG := K\widehat{T}_{n,\mathrm{ren}}(t)G, \quad t \geq 0,\ n \in \mathbb{N}$$

and

$$T^V(t)KG := \widehat{T}^V(t)G, \quad t \geq 0$$

define analytic semigroups of contractions on $\mathcal{E}_\beta$. *The generators are given by* $(K\widehat{L}_{n,\mathrm{ren}}, KD(\widehat{L}_{n,\mathrm{ren}}))$ *and* $(K\widehat{L}_V, KD(\widehat{L}_V))$.

(b) Above semigroups satisfy for any $F \in \mathcal{E}_\beta$

$$T_{n,\mathrm{ren}}(t)F \longrightarrow T^V(t)F, \quad n \to \infty$$

holds uniformly on compacts in $t \geq 0$.

(c) Let r_0 and r_t be as in Theorem 3.2.13, then for any $F \in \mathcal{E}_\beta$

$$\int\limits_{\Gamma} T_{n,\text{ren}}(t)F(\gamma)\mathrm{d}\pi_{r_0}(\gamma) \longrightarrow \int\limits_{\Gamma} T^V(t)F(\gamma)\mathrm{d}\pi_{r_0}(\gamma) = \int\limits_{\Gamma} F(\gamma)\mathrm{d}\pi_{r_t}(\gamma), \quad n \to \infty$$

holds uniformly on compacts in $t \geq 0$.

For $F = KG$ it follows that

$$\int\limits_{\Gamma} T_{n,\text{ren}}(t)F(\gamma)\mathrm{d}\pi_{r_0}(\gamma) = \int\limits_{\Gamma_0} G(\eta)(\widehat{T}_{n,\text{ren}}(t)r_0)(\eta)\mathrm{d}\lambda(\eta),$$

but $\widehat{T}_{n,\text{ren}}(t)r_0$ does not need to be positive definite and hence correspond to a probability measure on Γ. In fact, we can expect for chaotic initial conditions only that the evolution r_t is positive definite.

3.2.6 Extension to time-inhomogeneous intensities

For $t \geq 0$ let $d(t,x,\gamma), b(t,x,\gamma) \in [0,\infty]$ be given and suppose that there exists Γ_∞ (independent of $t \geq 0$) such that condition (A) is satisfied for any fixed $t \geq 0$. We are going to apply the results obtained in the first chapter for which we suppose that the following conditions hold:

(H1) There exist $\beta_* < \beta^*$ such that for all $\beta \in (\beta_*, \beta^*)$ and $t \geq 0$ there exists a constant $a(L(t);\beta) \in (0,2)$ satisfying

$$c(L(t),\beta;\eta) \leq a(L(t),\beta)M(t,\eta), \quad \eta \in \Gamma_0,\ t \geq 0,$$

where $M(t,\eta) = \sum\limits_{x\in\eta} d(t,x,\eta\backslash x)$.

(H2) There exist constants $A > 0$ and $N \in \mathbb{N}$ such that

$$d(t,x,\eta) \leq A(1+|\eta|)^N, \quad \eta \in \Gamma_0,\ x \in \mathbb{R}^d,\ t \geq 0$$

holds.

(H3) For any $\beta', \beta \in (\beta_*, \beta^*)$ with $\beta' < \beta$ the operator $t \longmapsto L(t) \in L(\mathcal{E}_\beta, \mathcal{E}_{\beta'})$ is continuous in the uniform operator topology.

Note that by (H1) and (H2) it follows that $L(t) \in L(\mathcal{E})$ where $\mathcal{E} = (\mathcal{E}_\beta)_{\beta\in(\beta_*,\beta^*)}$ is a scale of Banach spaces. Property (H3) states that $L = (L(t))_{t\geq 0}$ is continuous in the uniform topology in the scale $\mathcal{E}$.

Theorem 3.2.16. *Suppose that conditions (H1) – (H3) are fulfilled. Then there exist a forward evolution system* $(U(t,s))_{0\le s\le t}$ *and a backward evolution system* $(V(s,t))_{0\le s\le t}$ *in the scale* $\mathcal{E}$ *having generator* $(L(t))_{t\ge 0} \in L(\mathcal{E})$.

Above statement implies that the corresponding forward and backward evolution equations are well-posed on any $\mathcal{E}_\beta$, see Theorem 1.1.6.

Theorem 3.2.17. *Suppose that conditions (H1), (H3) and*

$$b(t,x,\eta) + d(t,x,\eta) \le A(1+|\eta|)^N, \quad \eta \in \Gamma_0,\ t \ge 0 \tag{3.20}$$

hold. Moreover, assume that for any fixed $t \ge 0$ *condition (D) holds for the operator* $L(t)$. *Then* $U^*(s,t)$ *and* $V^*(t,s)$ *are both positivity preserving.*

In the case of above statement the adjoint evolution systems $U^*(s,t)$ and $V^*(t,s)$ provide for each $\mu \in \mathcal{P}_\beta$ unique solutions to the time-dependent Fokker-Planck equations

$$\frac{\partial}{\partial s}\int_\Gamma F(\gamma)U^*(s,t)\mu(\mathrm{d}\gamma) = -\int_\Gamma L(s)F(\gamma)U^*(s,t)\mu(\mathrm{d}\gamma), \quad F \in \mathcal{FP}(\Gamma)$$

and

$$\frac{\partial}{\partial t}\int_\Gamma F(\gamma)V^*(t,s)\mu(\mathrm{d}\gamma) = \int_\Gamma L(t)F(\gamma)V^*(t,s)\mu(\mathrm{d}\gamma), \quad F \in \mathcal{FP}(\Gamma),$$

see Theorem 1.1.7. The next statement provides Vlasov scaling. For any $n \ge 1$ let $d_n(t,x,\gamma\backslash x), b_n(t,x,\gamma) \in [0,\infty]$ be given and define

$$c_n(t,\beta;\eta) = +\sum_{x\in\eta}\int_{\Gamma_0} |K_0^{-1}d_n(t,x,\cdot\cup\eta\backslash x)|(\xi)n^{|\xi|}e^{\beta|\xi|}\mathrm{d}\lambda(\xi)$$
$$+ e^{-\beta}\sum_{x\in\eta}\int_{\Gamma_0} |K_0^{-1}b_n(t,x,\cdot\cup\eta\backslash x)|(\xi)n^{|\xi|}e^{\beta|\xi|}\mathrm{d}\lambda(\xi).$$

Instead of the conditions (V1) – (V3) we suppose that the conditions given below are satisfied.

(W1) There exist $\beta_* < \beta^*$ such that for any $\beta \in (\beta_*,\beta^*)$ and any $t \ge 0$ there exists $a(t,\beta) \in (0,2)$ satisfying

$$c_n(t,\beta;\eta) \le a(t,\beta)M_n(t,\eta), \quad \eta \in \Gamma_0,\ n \in \mathbb{N},$$

where $M_n(t,\eta) := \sum_{x\in\eta} d_n(t,x,\eta\backslash x)$.

(W2) There exist constants $A > 0$ and $N \in \mathbb{N}$ such that

$$d_n(t, x, \eta) \leq A(1 + |\eta|)^N, \quad t \geq 0, \ \eta \in \Gamma_0, \ x \in \mathbb{R}^d$$

holds.

(W3) For all $\xi \in \Gamma_0$ and $x \in \mathbb{R}^d$ the following limits exist in the operator norm $L(\mathcal{L}_\beta, \mathcal{L}_{\beta'})$ for any $\beta' < \beta$ with $\beta', \beta \in (\beta_*, \beta^*)$ and are independent of ξ.

$$\lim_{n\to\infty} n^{|\cdot|}(K_0^{-1} d_n(t, x, \cdot \cup \xi)) = \lim_{n\to\infty} n^{|\cdot|}(K_0^{-1} d_n(t, x, \cdot)) =: D_x^V(t, \cdot)$$
$$\lim_{n\to\infty} n^{|\cdot|}(K_0^{-1} b_n(t, x, \cdot \cup \xi)) = \lim_{n\to\infty} n^{|\cdot|}(K_0^{-1} b_n(t, x, \cdot)) =: B_x^V(t, \cdot).$$

Moreover, above limits are uniform on any compact in $t \geq 0$.

For $n \geq 1$ let $\widehat{L}_n(t) := K_0^{-1} L_n(t) K_0$, $\widehat{L}_{n,\text{ren}}(t) := R_n \widehat{L}_n(t) R_{n^{-1}}$ and denote by $\mathcal{L}$ the scale of Banach spaces given by $\mathcal{L} = (\mathcal{L}_\beta)_{\beta \in (\beta_*, \beta^*)}$.

Theorem 3.2.18. *Suppose that conditions (W1) – (W3) are satisfied and assume that the operators $\widehat{L}_{n,\text{ren}}(t)$ are continuous in the uniform topology on $L(\mathcal{L})$ in $t \geq 0$. Then the following statements are satisfied:*

(a) There exist forward and backward evolution systems $\widehat{U}_{n,\text{ren}}(t, s)$ and $\widehat{V}_{n,\text{ren}}(s, t)$, respectively having generator $\widehat{L}_{n,\text{ren}}(t) \in L(\mathcal{L})$.

(b) There exist forward and backward evolution systems $\widehat{U}^V(t, s)$ and $\widehat{V}^V(s, t)$, respectively such that

$$\widehat{U}_{n,\text{ren}}(t, s) \longrightarrow \widehat{U}^V(t, s), \quad n \to \infty$$

and

$$\widehat{V}_{n,\text{ren}}(s, t) \longrightarrow \widehat{V}^V(s, t), \quad n \to \infty$$

holds uniformly on compacts in $t \geq 0$ in the uniform topology on $L(\mathcal{L})$. The generators satisfy $\widehat{L}_{n,\text{ren}}(t) \longrightarrow \widehat{L}_V(t)$ as $n \to \infty$ w.r.t. the uniform operator topology on $L(\mathcal{L})$ and uniformly on compacts in $t \geq 0$.

(c) For any $r \in \mathcal{K}_\beta$ the unique weak solution to the backward equation with $s \in [0, t)$

$$\frac{\partial}{\partial s}\langle G, k_{s,n}\rangle = -\langle \widehat{L}_{n,\text{ren}}(s) G, k_{s,n}\rangle, \quad k_{s,n}|_{s=t} = r, \quad G \in B_{bs}(\Gamma_0)$$

is given by $k_{s,n} = \widehat{U}_{n,\text{ren}}(s, t)^* r$ and the unique weak solution to the forward equation with $t \in [s, \infty)$

$$\frac{\partial}{\partial t}\langle G, k_{t,n}\rangle = \langle \widehat{L}_{n,\text{ren}}(t) G, k_{t,n}\rangle, \quad k_{t,n}|_{t=s} = r, \quad G \in B_{bs}(\Gamma_0)$$

is given by $k_{t,n} = \widehat{V}_{n,\text{ren}}(t, s)^* r$. The same assertions hold with $\widehat{L}_{n,\text{ren}}(t)$ replaced by $\widehat{L}_V(t)$ and $\widehat{U}_{n,\text{ren}}(s, t)^*, \widehat{V}_{n,\text{ren}}(t, s)^*$ replaced by $\widehat{U}^V(s, t)^*, \widehat{V}^V(t, s)^*$

(d) Let $r(\eta) = \prod\limits_{x\in\eta} \rho(x)$ *and* $\rho \in L^\infty(\mathbb{R}^d)$ *with* $\|\rho\|_{L^\infty} \le e^\beta$. *Assume that* $\rho_s \in L^\infty(\mathbb{R}^d)$ *with* $\|\rho_s\|_{L^\infty} \le e^\beta$ *is a classical solution to the backward equation* $s \in [0,t)$

$$\frac{\partial \rho_s}{\partial s}(x) = \int\limits_{\Gamma_0} e_\lambda(\rho_s;\xi) D_x^V(s,\xi)\mathrm{d}\lambda(\xi)\rho_s(x) - \int\limits_{\Gamma_0} e_\lambda(\rho_s;\xi) B_x^V(s,\xi)\mathrm{d}\lambda(\xi)$$

and initial condition $\rho_s|_{s=t} = \rho$. *Then* $r_s(\eta) := \prod\limits_{x\in\eta} \rho_s(x)$ *is a weak solution to*

$$\frac{\partial}{\partial s}\langle G, r_s\rangle = -\langle \widehat{L}_V(s)G, r_s\rangle, \quad r_s|_{s=t} = r, \quad G \in B_{bs}(\Gamma_0).$$

Assume that $\rho_t \in L^\infty(\mathbb{R}^d)$ *with* $\|\rho_t\|_{L^\infty} \le e^\beta$ *is a classical solution to the forward equation with* $t \in [s,\infty)$

$$\frac{\partial \rho_t}{\partial t}(x) = -\int\limits_{\Gamma_0} e_\lambda(\rho_t;\xi) D_x^V(t,\xi)\mathrm{d}\lambda(\xi)\rho_t(x) + \int\limits_{\Gamma_0} e_\lambda(\rho_t;\xi) B_x^V(t,\xi)\mathrm{d}\lambda(\xi)$$

and initial condition $\rho_t|_{t=s} = \rho$. *Then* $r_t(\eta) := \prod\limits_{x\in\eta} \rho_t(x)$ *is a weak solution to*

$$\frac{\partial}{\partial t}\langle G, r_t\rangle = \langle \widehat{L}_V(t)G, r_t\rangle, \quad r_t|_{t=s} = r, \quad G \in B_{bs}(\Gamma_0).$$

3.3 Finite system in ergodic environment

The main aim for this section is to describe the behaviour of a system with state space Γ_0 evolving in the presence of an equilibrium, ergodic environment, which is described by a Markov process with the state space Γ and an associated invariant measure μ. This situation is a particular case of so-called random evolution framework, see e.g. [Pin91, SHS02]. Examples for such environments have been constructed e.g. in [AKR98a, AKR98b, KL05]. There (via the Dirichlet forms technique) the existence of a Markov semigroup $T^E(t)$ on $L^2(\Gamma, \mathrm{d}\mu)$ has been shown, where μ is the unique invariant measure and $T^E(t)$ is symmetric on $L^2(\Gamma, \mathrm{d}\mu)$. As a consequence this semigroup can be extended to all $L^p(\Gamma, \mathrm{d}\mu)$ with $1 \le p < \infty$ and for $p = 1$ this extension, also denoted by $T^E(t)$, gives the evolution of densities. More precisely, if $R \in L^1(\Gamma, \mathrm{d}\mu)$ and the environment is in the initial state $R\mathrm{d}\mu$, then the time evolution is given by $R_t\mathrm{d}\mu$, where $R_t = T^E(t)R$. Above extension $T^E(t)$ is ergodic on $L^1(\Gamma, \mathrm{d}\mu)$, i.e., $T^E(t)R \to \int\limits_\Gamma R(\gamma)\mathrm{d}\mu(\gamma),\ t \to \infty$ in $L^1(\Gamma, \mathrm{d}\mu)$. Denote by L^E its generator. We will study the evolution of a system described by the Kolmogorov operator

$$(L^S F)(\gamma,\eta) = \sum_{\xi\subset\eta} \int\limits_{\Gamma_0} (F(\gamma, \eta\backslash\xi \cup \zeta) - F(\gamma,\eta))K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta).$$

The kernel $K(\gamma, \xi, \eta, \zeta) \geq 0$ depends on the present microscopic state $\gamma \in \Gamma$ of the environment. Therefore, solutions to the Fokker-Planck equation

$$\frac{\partial \rho_t}{\partial t} = (L^S)^* \rho_t + L^E \rho_t, \quad \rho_t|_{t=0} = \rho_0,$$

on the space $L^1(\Gamma \times \Gamma_0, \mathrm{d}(\mu \otimes \lambda))$ describe the evolution of densities of the joint Markov process for the system and environment. Here $(L^S)^*$ stands for the adjoint operator on densities $\rho(\gamma, \eta)$, which depends on γ as a parameter but acts only on the variable η. Similarly, L^E acts only on the first variable γ. The weak-coupling limit is obtained via an approximation ρ_t^ε, where ρ_t^ε solves the rescaled version of the Fokker-Planck equation

$$\frac{\partial \rho_t^\varepsilon}{\partial t} = (L^S)^* \rho_t^\varepsilon + \frac{1}{\varepsilon} L^E \rho_t^\varepsilon, \quad \rho_t^\varepsilon|_{t=0} = \rho_0 \in L^1(\Gamma_0, \mathrm{d}\lambda).$$

Thus we will seek for the limit $\rho_t^\varepsilon \to \overline{\rho}_t$ when $\varepsilon \to 0$. In such a case we prove that $\overline{\rho}_t$ solves the Fokker-Planck equation for a finite system determined by the averaged (pre-)generator

$$\overline{L} F(\eta) = \sum_{\xi \subset \eta} \int_{\Gamma_0} (F(\eta \backslash \xi \cup \zeta) - F(\eta)) \overline{K}(\xi, \eta, \zeta) \mathrm{d}\lambda(\zeta),$$

where $\overline{K}(\xi, \eta, \zeta) = \int_\Gamma K(\gamma, \xi, \eta, \zeta) \mathrm{d}\mu(\gamma)$. The aim is to realize this approach and show for one specific example how this can be applied.

3.3.1 Weak-coupling limit

Let us start with the main assumption on the environment process on Γ:

(E) There exists a probability measure μ on Γ and a positive semigroup of contractions $T^E(t)$ on $L^1(\Gamma, \mathrm{d}\mu)$, which is assumed to be L^1-ergodic, i.e., for each $R \in L^1(\Gamma, \mathrm{d}\mu)$

$$\int_\Gamma |T^E(t) R - \langle R \rangle_\mu | \mathrm{d}\mu \to 0, \quad t \to \infty.$$

Here $\langle R \rangle_\mu = \int_\Gamma R \mathrm{d}\mu$ denotes the average of R with respect to μ.

Denote by $(L^E, D(L^E))$ its generator. It is well-known that $\mathcal{L}_\mu := L^1(\Gamma \to L^1(\Gamma_0, \mathrm{d}\lambda), \mathrm{d}\mu)$ can be identified with $L^1(\Gamma \times \Gamma_0, \mathrm{d}(\mu \otimes \lambda))$ and the subspace

$$D = \left\{ f = \sum_{k=1}^n R_k \rho_k \;\middle|\; n \in \mathbb{N}, R_k \in L^1(\Gamma, \mathrm{d}\mu), \rho_k \in L^1(\Gamma_0, \mathrm{d}\lambda) \right\} \subset \mathcal{L}_\mu$$

is dense. Since $T^E(t)$ is positive it can be uniquely extended to $\mathcal{L}_\mu$, cf. [Gra04], such that for $f \in D$

$$T^E(t)f = \sum_{k=1}^{n} (T^E(t)R_k)\rho_k.$$

One has $\|(T^E(t)f)(\cdot,\gamma)\|_{L^1(\Gamma_0,\mathrm{d}\lambda)} \leq T^E(t)\|f(\cdot,\gamma)\|_{L^1(\Gamma_0,\mathrm{d}\lambda)}$ for all $f \in D$, thus this extension will be a positive strongly continuous semigroup of contractions which shall be again denoted by $T^E(t)$. For convenience we also denote the generator of the extended semigroup by $(L^E, D(L^E))$. This generator can be characterized by the relation

$$L^E f = \sum_{k=1}^{n} (L^E R_k)\rho_k,$$

where $f \in D$ with $R_k \in D(L^E)$. For $f \in D$ we obtain

$$\|T^E(t)f - \langle f\rangle_\mu\|_{\mathcal{L}_\mu} \leq \sum_{k=1}^{n} \|T^E(t)R_k - \langle R_k\rangle_\mu\|_{L^1(\Gamma,\mathrm{d}\mu)}\|\rho_k\|_{L^1(\Gamma_0,\mathrm{d}\lambda)} \to 0, \quad t \to \infty$$

and since $T^E(t)$ is a semigroup of contractions and D dense this implies for each $f \in \mathcal{L}_\mu$

$$\|T^E(t)f - \langle f\rangle_\mu\|_{\mathcal{L}_\mu} \to 0, \quad t \to \infty$$

Note that $\langle f\rangle_\mu(\eta) := \int_\Gamma f(\gamma,\eta)\mathrm{d}\mu(\gamma)$ is simply the projection of $\mathcal{L}_\mu$ onto $L^1(\Gamma_0, \mathrm{d}\lambda)$.

For the description of the system process we suppose that K is measurable with respect to all variables and

$$\int_\Gamma \int_{\Gamma_0} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)\mathrm{d}\mu(\gamma) < \infty, \quad \forall \xi,\eta \in \Gamma_0 \tag{3.21}$$

holds. Let us outline the construction of the evolution of densities on $\mathcal{L}_\mu = L^1(\Gamma \times \Gamma_0, \mathrm{d}(\mu \otimes \lambda))$. First of all, the Markov (pre-)generator L^S is assumed to be given by

$$L^S F(\gamma,\eta) = \sum_{\xi \subset \eta} \int_{\Gamma_0} (F(\gamma, \eta\backslash\xi \cup \zeta) - F(\gamma,\eta))K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta). \tag{3.22}$$

It can be rewritten as

$$L^S F(\gamma,\eta) = \int_{\Gamma_0} (F(\gamma,\omega) - F(\gamma,\eta))Q(\gamma,\eta,\mathrm{d}\omega),$$

where

$$Q(\gamma,\eta,A) = \sum_{\xi \subset \eta} \int_{\Gamma_0} \mathbb{1}_A(\eta\backslash\xi \cup \zeta)K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta).$$

Define $q(\gamma,\eta) := Q(\gamma,\eta,\Gamma_0) = \sum\limits_{\xi\subset\eta}\int\limits_{\Gamma_0} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)$, then the adjoint operator on densities $\rho \in \mathcal{L}_\mu$ is given by

$$(L^S)^*\rho(\gamma,\eta) = -q(\gamma,\eta)\rho(\gamma,\eta) + (B^*\rho)(\gamma,\eta),$$

where

$$(B^*\rho)(\gamma,\eta) = \sum_{\xi\subset\eta}\int\limits_{\Gamma_0} \rho(\gamma,\eta\backslash\xi\cup\zeta)K(\gamma,\zeta,\eta\backslash\xi\cup\zeta,\xi)\mathrm{d}\lambda(\zeta). \tag{3.23}$$

We are interested in the asymptotic regime $\varepsilon \to 0$ for solutions ρ_t^ε to the Cauchy problems

$$\frac{\partial\rho_t^\varepsilon}{\partial t} = (L^S)^*\rho_t^\varepsilon + \frac{1}{\varepsilon}L^E\rho_t^\varepsilon, \quad \rho_t^\varepsilon|_{t=0} = \rho_0 \in L^1(\Gamma_0,\lambda) \subset \mathcal{L}_\mu \tag{3.24}$$

on $\mathcal{L}_\mu$. Typically, it is hard to construct solutions to (3.24) in this generality. Let us define approximations $(L_\delta^S)^*$ by setting $K_\delta(\gamma,\xi,\eta,\zeta) := e^{-\delta q(\gamma,\eta)}K(\gamma,\xi,\eta,\zeta)$. Then L_δ^S is defined by (3.22) with K replaced by K_δ and $(L_\delta^S)^*$ is its adjoint given by

$$(L_\delta^S)^*\rho(\gamma,\eta) = -q(\gamma,\eta)e^{-\delta q(\gamma,\eta)}\rho(\gamma,\eta) + (B_\delta^*\rho)(\gamma,\eta).$$

The operator B_δ^* is simply given by (cf. (3.23))

$$(B_\delta^*\rho) = \sum_{\xi\subset\eta}\int\limits_{\Gamma_0} \rho(\gamma,\eta\backslash\xi\cup\zeta)e^{-\delta q(\gamma,\eta\backslash\xi\cup\zeta)}K(\gamma,\zeta,\eta\backslash\xi\cup\zeta,\xi)\mathrm{d}\lambda(\zeta).$$

Because of

$$\begin{aligned}\|B_\delta^*\rho\|_{\mathcal{L}_\mu} &\le \int\limits_{\Gamma}\int\limits_{\Gamma_0}\int\limits_{\Gamma_0}\sum_{\xi\subset\eta}|\rho(\gamma,\eta\backslash\xi\cup\zeta)|e^{-\delta q(\gamma,\eta\backslash\xi\cup\zeta)}K(\gamma,\zeta,\eta\backslash\xi\cup\zeta,\xi)\mathrm{d}\lambda(\zeta)\mathrm{d}\lambda(\eta)\mathrm{d}\mu(\gamma)\\ &= \int\limits_{\Gamma}\int\limits_{\Gamma_0}|\rho(\gamma,\eta)|e^{-\delta q(\gamma,\eta)}q(\gamma,\eta)\mathrm{d}\lambda(\eta)\mathrm{d}\mu(\gamma)\\ &\le \frac{1}{\delta}\|\rho\|_{\mathcal{L}_\mu}\end{aligned}$$

the operator B_δ^* is bounded on $\mathcal{L}_\mu$ and hence so is $(L_\delta^S)^*$. Let us fix the notation for the limiting objects when $\varepsilon \to 0$ and $\delta \to 0$. Define the averaged functions $\overline{K}$ and $\overline{K}_\delta$ by

$$\overline{K}(\xi,\eta,\zeta) := \int\limits_{\Gamma} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\mu(\gamma) \tag{3.25}$$

and

$$\overline{K}_\delta(\xi,\eta,\zeta) := \int\limits_\Gamma e^{-\delta q(\gamma,\eta)} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\mu(\gamma). \tag{3.26}$$

The results obtained in the second chapter show that there exist semigroups $\overline{T}(t)$ and $\overline{T}_\delta(t)$ given by the associated transition probability functions $\overline{P}$ and $\overline{P}_\delta$ which are determined by

$$\overline{L}F(\eta) = \sum_{\xi\subset\eta}\int\limits_{\Gamma_0}(F(\eta\backslash\xi\cup\zeta) - F(\eta))\overline{K}(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)$$

and

$$\overline{L}_\delta F(\eta) = \sum_{\xi\subset\eta}\int\limits_{\Gamma_0}(F(\eta\backslash\xi\cup\zeta) - F(\eta))\overline{K}_\delta(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta),$$

cf. (2.50) and (2.51). The adjoint semigroups on $L^1(\Gamma_0,\mathrm{d}\lambda)$ are denoted by $\overline{T}(t)^*$ and $\overline{T}_\delta(t)^*$ respectively. The corresponding generators are simply given by

$$(\overline{L}^*_\delta\rho)(\eta) = -\overline{q}_\delta(\eta)\rho(\eta) + \sum_{\xi\subset\eta}\int\limits_{\Gamma_0}\rho(\eta\backslash\xi\cup\zeta)\overline{K}_\delta(\zeta,\eta\backslash\xi\cup\zeta,\xi)\mathrm{d}\lambda(\zeta)$$

where $\overline{q}_\delta(\eta) = \sum\limits_{\xi\subset\eta}\int\limits_{\Gamma_0}\overline{K}_\delta(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)$. The same holds for $\overline{L}^*$ with $\overline{K}_\delta$ replaced by $\overline{K}$.

Proposition 3.3.1. *Assume that condition* (3.21) *satisfied. Then for any* $\varepsilon > 0$ *the operator* $(L^S_\delta)^* + \frac{1}{\varepsilon}L^E$ *is the generator of a sub-stochastic semigroup* $T_{\varepsilon,\delta}(t)$ *on* $\mathcal{L}_\mu$. *For any* $\delta > 0$ *and any* $\rho \in L^1(\Gamma_0,\mathrm{d}\lambda)$

$$\lim_{\varepsilon\to 0} T_{\varepsilon,\delta}(t)\rho = \overline{T}_\delta(t)^*\rho \tag{3.27}$$

holds uniformly on compacts in $t \geq 0$. *Assume that* $\overline{T}(t)^*$ *is stochastic, then for any* $\rho \in L^1(\Gamma_0,\mathrm{d}\lambda)$

$$\lim_{\delta\to 0} \overline{T}_\delta(t)^*\rho = \overline{T}(t)^*\rho \tag{3.28}$$

holds uniformly on compacts in $t \geq 0$.

Above assumption for $\overline{T}(t)^*$ being stochastic has been characterized in Theorem 2.4.2.

Proof. The operator $\frac{1}{\varepsilon}L^E$ is for any $\varepsilon > 0$ the generator of the semigroup $T^E(\frac{t}{\varepsilon})$ on $\mathcal{L}_\mu$. Since $(L^S_\delta)^*$ is bounded on $\mathcal{L}_\mu$ also the sum $(L^S_\delta)^* + \frac{1}{\varepsilon}L^E$ is the generator of a semigroup $T_{\varepsilon,\delta}(t)$. Due to the Trotter product formula this semigroup is sub-stochastic. So let us show (3.27), which holds true if we can apply [Kur73, Theorem 2.1]. Therefore observe that for $\rho \in \mathcal{L}_\mu$ and $\lambda > 0$

$$\left\|\lambda \int_0^\infty e^{-\lambda t} T^E(t)\rho \mathrm{d}t - \langle\rho\rangle_\mu\right\|_{\mathcal{L}_\mu} \leq \int_0^\infty e^{-s} \left\|T^E\left(\frac{s}{\lambda}\right)\rho - \langle\rho\rangle_\mu\right\|_{\mathcal{L}_\mu} \mathrm{d}s.$$

Since $T^E(t)$ is ergodic on $\mathcal{L}_\mu$ it follows that for fixed $s \geq 0$ the integrand tends to zero as $\lambda \to 0$. Due to $\|\langle\rho\rangle_\mu\|_{\mathcal{L}_\mu} \leq \|\rho\|_{\mathcal{L}_\mu}$ and the contraction property of $T^E(t)$ the integrand is bounded by $2\|\rho\|_{\mathcal{L}_\mu} e^{-s}$ and hence dominated convergence implies for all $\rho \in \mathcal{L}_\mu$

$$P\rho := \lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t} T^E(t)\rho \mathrm{d}t = \langle\rho\rangle_\mu.$$

The operator P is a projection on $\mathcal{L}_\mu$ with range $\mathrm{Ran}(P) \cong L^1(\Gamma_0, \mathrm{d}\lambda)$. Following the notion of [Kur73] $C\rho := P(L_\delta^S)^*\rho = \overline{L}_\delta^*\rho$ is defined on $L^1(\Gamma_0, \mathrm{d}\lambda)$ and is additionally bounded, which implies (3.27). For the second assertion observe that by Theorem 2.4.2

$$\overline{\mathrm{Dom}} := \left\{\rho \in L^1(\Gamma_0, \mathrm{d}\lambda) \;\middle|\; \int_{\Gamma_0} \overline{q}(\eta)|\rho(\eta)|\mathrm{d}\lambda(\eta) < \infty\right\}$$

is a core for $\overline{T}(t)^*$, since $\overline{T}(t)^*$ is stochastic. For any $\rho \in \overline{\mathrm{Dom}}$ it holds

$$\begin{aligned}
&\|\overline{L}_\delta^*\rho - \overline{L}^*\rho\| \\
&\leq \int_{\Gamma_0} |\rho(\eta)||\overline{q}_\delta(\eta) - \overline{q}(\eta)|\mathrm{d}\lambda(\eta) \\
&\quad + \int_{\Gamma_0} \sum_{\xi \subset \eta} \int_{\Gamma_0} |\rho(\eta\backslash\xi \cup \zeta)||\overline{K_\delta}(\zeta, \eta\backslash\xi \cup \zeta, \zeta) - \overline{K}(\zeta, \eta\backslash\xi \cup \zeta, \xi)|\mathrm{d}\lambda(\zeta)\mathrm{d}\lambda(\eta)
\end{aligned}$$

and by (3.25) and (3.26) for any $\delta > 0$ we obtain

$$|\overline{K_\delta}(\zeta, \eta\backslash\xi \cup \zeta, \xi) - \overline{K}(\zeta, \eta\backslash\xi \cup \zeta, \zeta)| \leq \int_\Gamma |1 - e^{-\delta q(\gamma, \eta\backslash\xi\cup\zeta)}|K(\gamma, \zeta, \eta\backslash\xi \cup \zeta, \xi)\mathrm{d}\mu(\gamma).$$

Since the integrand is bounded by $2K(\gamma, \zeta, \eta\backslash\xi \cup \zeta, \xi)$ and tends to zero for any $\gamma \in \Gamma$, dominated convergence yields that $|\overline{K_\delta}(\zeta, \eta\backslash\xi \cup \zeta, \xi) - \overline{K}(\zeta, \eta\backslash\xi \cup \zeta, \zeta)| \to 0$ as $\delta \to 0$ for any $\eta \in \Gamma_0$, $\xi \subset \eta$ and $\zeta \in \Gamma_0$. Finally due to $|\overline{K_\delta}(\zeta, \eta\backslash\xi \cup \zeta, \xi) - \overline{K}(\zeta, \eta\backslash\xi \cup \zeta, \zeta)| \leq 2\overline{K}(\zeta, \eta\backslash\xi \cup \zeta, \xi)$ the second term tend to zero as $\delta \to 0$. For the first term observe

$$|\overline{q}_\delta(\eta) - \overline{q}(\eta)| \leq \sum_{\xi \subset \eta} \int_{\Gamma_0} |\overline{K}_\delta(\xi, \eta, \zeta) - \overline{K}(\xi, \eta, \zeta)|\mathrm{d}\lambda(\zeta),$$

then above argument implies $\overline{q}_\delta(\eta) \longrightarrow \overline{q}(\eta)$ for all $\eta \in \Gamma_0$ as $\delta \to 0$. The assertion follows from $\overline{q}_\delta \leq \overline{q}$ and dominated convergence. □

3.3.2 Example: Medical treatment of tumours

We aim to describe the (stochastic) behaviour of tumours cells influenced by an injection of a certain medicine. The distribution of the medicine within the organism is assumed to be diffusive and hence is modelled by an equilibrium diffusion process on Γ for a given invariant (Gibbs) measure μ. For the construction of equilibrium diffusions and ergodicity see [AKR98a, AKR98b]. The behaviour of the tumour cells is modelled by a birth-and-death process on Γ_0 with Markov (pre-)generator

$$\begin{aligned}(L^S F)(\gamma,\eta) &= \sum_{x\in\eta}\left(m(x,\gamma) + \sum_{y\in\eta\backslash x} a^-(x-y)\right)(F(\gamma,\eta\backslash x) - F(\gamma,\eta))\\ &\quad + \sum_{x\in\eta}\lambda(x,\gamma)\int_{\mathbb{R}^d} a^+(x-y)(F(\gamma,\eta\cup y) - F(\gamma,\eta))\mathrm{d}y.\end{aligned}$$

The statistical dynamics for such model (without the presence of an environment) has been analysed, e.g., in [FM04, FKK09, FKKK15, KK16] and in the second chapter. The proliferation of cells is described by the probability density a^+ and competition of tumour cells by the kernel $a^- \geq 0$. The influence of the medicine on the tumour enters through the mortality $m(x,\gamma) > 0$ and proliferation intensity $\lambda(x,\gamma) > 0$. After scaling the averaged dynamics will be given by the generator

$$\begin{aligned}(\overline{L}F)(\eta) &= \sum_{x\in\eta}\left(\overline{m}(x) + \sum_{y\in\eta\backslash x} a^-(x-y)\right)(F(\eta\backslash x) - F(\eta))\\ &\quad + \sum_{x\in\eta}\overline{\lambda}(x)\int_{\mathbb{R}^d} a^+(x-y)(F(\eta\cup y) - F(\eta))\mathrm{d}y,\end{aligned}$$

where $\overline{m}(x) = \int_\Gamma m(x,\gamma)\mathrm{d}\mu(\gamma)$ and $\overline{\lambda}(x) = \int_\Gamma \lambda(x,\gamma)\mathrm{d}\mu(\gamma)$ are the averaged intensities. Proceeding as in the previous section denote by $T_{\varepsilon,\delta}(t)$ the scaled semigroup on densities $\mathcal{L}_\mu$ and by $\overline{T}(t)^*$ and $\overline{T}_\delta(t)^*$ the semigroups on $L^1(\Gamma_0, \mathrm{d}\lambda)$ defined by the adjoint operator $\overline{L}^*$ of $\overline{L}$ respectively their counterparts scaled by $\delta > 0$. The next result states conditions for which these semigroups exist and (3.27) holds.

Theorem 3.3.2. *Assume that all intensities $a^\pm, m, \lambda$ are non-negative, measurable, that a^+ is a probability density and that $m(x,\cdot), \lambda(x,\cdot)$ are integrable with respect to μ for any $x \in \mathbb{R}^d$. Then the semigroups $T_{\varepsilon,\delta}(t)$, $\overline{T}_\delta(t)^*$ and $\overline{T}(t)^*$ exist and* (3.27) *holds.*

Proof. First of all

$$q(\gamma,\eta) = \sum_{x\in\eta} m(x,\gamma) + \sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^-(x-y) + \sum_{x\in\eta}\lambda(x,\gamma)$$
$$= \sum_{\xi\subset\eta}\int_{\Gamma_0} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)$$

for any $\eta\in\Gamma_0$ and hence

$$\int_{\Gamma}\int_{\Gamma_0} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)\mathrm{d}\mu(\gamma) \leq \int_{\Gamma} q(\gamma,\eta)\mathrm{d}\mu(\gamma) < \infty$$

implies (3.21). The existence of the semigroup $\overline{T}(t)$ and $\overline{T}_\delta(t)$ has been established in the previous chapter. The considerations of the previous sections imply the existence of the semigroups and property (3.27) follows from Proposition 3.3.1. □

The reader may wonder why such weak assumptions are sufficient for existence and convergence of the semigroups. The crucial point here is that we consider an approximation by bounded linear operators and hence for each $\delta > 0$ no additional conditions are needed. In order to pass to the limit $\delta \to 0$ additional assumptions are necessary, which are given below. This statement is a particular case of the BDLP-model considered in the second chapter, see also [Kol06].

Theorem 3.3.3. *Assume that the conditions of previous theorem are fulfilled. If $\overline{m}, \overline{\lambda}, a^-$ are bounded, then $\overline{T}(t)^*$ is stochastic and hence* (3.28) *holds. If $\overline{m}, \overline{\lambda}, a^-$ are locally bounded, then $\overline{T}(t)^*$ is still stochastic, provided there exists a continuous function $\varphi : \mathbb{R}^d \longrightarrow [1,\infty)$ with $\varphi(x)\to\infty$ when $|x|\to\infty$ and $c > 0$ such that*

$$\overline{\lambda}(x)(a^+ * \varphi)(x) \leq c\varphi(x) + \varphi(x)\overline{m}(x), \quad x\in\mathbb{R}^d \tag{3.29}$$

holds.

As a concrete case we can take $\mu = \pi_z$ that is the Poisson measure with intensity $z > 0$. Let us take for the interactions

$$m(x,\gamma) = m_0 + \sum_{y\in\gamma}\kappa(x-y)$$

and

$$\lambda(x,\gamma) = \lambda_0 + \sum_{y\in\gamma}\psi(x-y)$$

with $\lambda_0 > m_0$, $0 \leq \kappa,\psi \in L^1(\mathbb{R}^d)$ and $\langle\psi\rangle < \langle\kappa\rangle$. Then $\overline{m} = m_0 + z\int_{\mathbb{R}^d}\kappa(y)\mathrm{d}y = m_0 + z\langle\kappa\rangle$ and $\overline{\lambda} = \lambda_0 + z\int_{\mathbb{R}^d}\psi(y)\mathrm{d}y = \lambda_0 + \langle\psi\rangle$. Define

$$\beta(z) = (\lambda_0 + z\langle\psi\rangle - m_0 - z\langle\kappa\rangle),$$

then for the function $V(\eta) = 1 + |\eta|$ a short computation yields

$$(\overline{L}V)(\eta) \leq \beta(z)|\eta|$$

and therefore an a priori estimate on the evolution of densities, provided a^- is bounded. More precisely, let $0 \leq \rho \in L^1(\Gamma_0, \mathrm{d}\lambda)$ with $\int\limits_{\Gamma_0}(1+|\eta|)\rho(\eta)\mathrm{d}\lambda(\eta) < \infty$ and $\int\limits_{\Gamma_0} \rho(\eta)\mathrm{d}\lambda(\eta) = 1$, then the evolution of densities for the averaged system is given by $\rho_t = \overline{T}(t)^*\rho$ and by the Grö nwall inequality we have

$$\int\limits_{\Gamma_0} |\eta|\rho_t(\eta)\mathrm{d}\lambda(\eta) \leq e^{\beta(z)t} \int\limits_{\Gamma_0} |\eta|\rho(\eta)\mathrm{d}\lambda(\eta), \quad t \geq 0.$$

Without medical treatment, i.e. $z = 0$, the number of tumour cells will grow exponentially in time. But due to the influence of the medicine such growth may be prevented or even exponential decay may be observed.

3.4 Examples

In this section we apply the main results to several stochastic birth-and-death processes on Γ describing the behaviour of cells within organisms.

3.4.1 Free cell-proliferation

In this part we investigate a model for the proliferation of cells. It is assumed that each cell has an exponential distributed lifetime with parameter $m > 0$. Moreover, each cell has another exponential distributed time, the so-called proliferation time, with parameter $\lambda > 0$. The corresponding elementary event is the splitting of a cell at position $x \in \gamma$ into two new cells. The position of the new cells is determined by the probability distribution

$$a(x - y_1, x - y_2)\mathrm{d}y_1\mathrm{d}y_2$$

and $a \geq 0$ is assumed to be symmetric in both variables. The Markov (pre-)generator is hence assumed to be given by

$$\begin{aligned}(LF)(\gamma) = m &\sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) \\ &+ \lambda\sum_{x\in\gamma}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(x - y_1, x - y_2)(F(\gamma\backslash x \cup y_1 \cup y_2) - F(\gamma))\mathrm{d}y_1\mathrm{d}y_2\end{aligned}$$

This model is exactly solvable and we construct the evolution of correlation functions explicitly. The analysis of this model will serve as a guiding example. Above model is very similar to the contact model, cf. [KS06, KKP08].

Theorem 3.4.1. *For $G \in B_{bs}(\Gamma_0)$ the operator $\widehat{L} = \widehat{L}_V + \widehat{B}$ is given by*

$$(\widehat{L}_V G)(\eta) = -(m+\lambda)|\eta|G(\eta) + \lambda \sum_{x\in\eta} \int_{\mathbb{R}^d} b(x-y)G(\eta\backslash x \cup y)\mathrm{d}y \tag{3.30}$$

with $\widehat{B}$ given by

$$(\widehat{B}G)(\eta) = \lambda \sum_{x\in\eta} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1, x-y_2)G(\eta\backslash x \cup y_1 \cup y_2)\mathrm{d}y. \tag{3.31}$$

Here $b \geq 0$ describes the effective proliferation and is given by

$$b(x) = \int_{\mathbb{R}^d} a(x,y)\mathrm{d}y + \int_{\mathbb{R}^d} a(y,x)\mathrm{d}y.$$

For $k : \Gamma_0 \longrightarrow \mathbb{R}$ such that $|k(\eta)| \leq |\eta|!C^{|\eta|}$ for some constant $C > 0$ the operator L^Δ is given by

$$L^\Delta = L_V^\Delta + B^\Delta,$$

where L_V^Δ is given by the same expression as $\widehat{L}_V$ and B^Δ by

$$(B^\Delta k)(\eta) = \lambda \sum_{y_1\in\eta} \sum_{y_2\in\eta\backslash y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2)k(\eta \cup x\backslash y_1\backslash y_2)dx. \tag{3.32}$$

Proof. Using the $K-$transform we obtain for $x \in \gamma$

$$(KG)(\gamma\backslash x) - (KG)(\gamma) = -\sum_{\eta\Subset\gamma\backslash x} G(\eta \cup x)$$

and therefore for the first part

$$\begin{aligned} m\sum_{x\in\gamma}((KG)(\gamma\backslash x) - (KG)(\gamma)) &= -m\sum_{x\in\gamma}\sum_{\eta\Subset\gamma\backslash x} G(\eta\cup x) \\ &= -m\sum_{\eta\Subset\gamma}\sum_{x\in\eta} G(\eta) = -mK(|\cdot|G)(\gamma). \end{aligned}$$

Applying the inverse $K-$transform we arrive at the expression $-m|\eta|G(\eta)$ reflecting the natural death of each cell. For the cell-division we first note that for $x \in \gamma$ and $y_1, y_1 \notin \gamma$

$$\begin{aligned} &(KG)(\gamma\backslash x \cup y_1 \cup y_1) - (KG)(\gamma) \\ &= \sum_{\eta\Subset\gamma\backslash x} \left(G(\eta\cup y_1) + G(\eta\cup y_2) + G(\eta\cup y_1\cup y_2) - G(\eta\cup x)\right). \end{aligned}$$

Therefore the birth-part is given by

$$\sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1,x-y_2)\left(G(\eta\backslash x\cup y_1)+G(\eta\backslash x\cup y_2)+G(\eta\backslash x\cup y_1\cup y_2)-G(\eta)\right)\mathrm{d}y_1\mathrm{d}y_2.$$

In the first two terms of the second part the integration over y_1 and y_2 respectively can be carried out, which gives together with the substitution $y_1, y_2 \to y$

$$\begin{aligned}&\lambda\sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1,x-y_2)\left(G(\eta\backslash x\cup y_1)+G(\eta\backslash x\cup y_2)\right)\mathrm{d}y_1\mathrm{d}y_2\\ &=\lambda\sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)G(\eta\backslash x\cup y)\mathrm{d}y.\end{aligned}$$

Altogether we obtain formulas (3.30) and (3.31). For $G \in B_{bs}(\Gamma_0)$ and k as described above, the operator L^Δ is uniquely determined by the pairing

$$\int_{\Gamma_0}(\widehat{L}G)(\eta)k(\eta)\mathrm{d}\lambda(\eta)=\int_{\Gamma_0}G(\eta)(L^\Delta k)(\eta)\mathrm{d}\lambda(\eta).$$

The negative multiplication part will therefore not change and for the second part we get

$$\begin{aligned}&\lambda\int_{\Gamma_0}\sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)G(\eta\backslash x\cup y)\mathrm{d}yk(\eta)\mathrm{d}\lambda(\eta)\\ &=\lambda\int_{\Gamma_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b(x-y)G(\eta\cup y)k(\eta\cup x)\mathrm{d}y\mathrm{d}x\mathrm{d}\lambda(\eta)\\ &=\lambda\int_{\Gamma_0}\sum_{y\in\eta}\int_{\mathbb{R}^d} b(x-y)k(\eta\cup x\backslash y)\mathrm{d}xG(\eta)\mathrm{d}\lambda(\eta).\end{aligned}$$

Finally

$$\begin{aligned}&\int_{\Gamma_0}(\widehat{B}G)(\eta)k(\eta)\mathrm{d}\lambda(\eta)\\ &=\lambda\int_{\Gamma_0}\sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1,x-y_2)G(\eta\backslash x\cup y_1\cup y_2)dy_1dy_2k(\eta)\mathrm{d}\lambda(\eta)\\ &=\lambda\int_{\Gamma_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1,x-y_2)G(\eta\cup y_1\cup y_2)k(\eta\cup x)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\lambda(\eta)\\ &=\lambda\int_{\Gamma_0}\sum_{y_1\in\eta}\sum_{y_2\in\eta\backslash y_1}\int_{\mathbb{R}^d} a(x-y_1,x-y_2)k(\eta\cup x\backslash y_1\backslash y_2)dxG(\eta)\mathrm{d}\lambda(\eta),\end{aligned}$$

proves the assertion. □

The next statement shows that the results stated in the previous section are not applicable in this case.

Theorem 3.4.2. *The function $c(\alpha;\eta)$ is given by*

$$c(\alpha;\eta) = (m+3\lambda)|\eta| + \lambda e^{-\alpha} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2)\mathrm{d}x$$

If in addition the expression

$$\theta = \min\left\{ \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y,x)dx, \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} a(x,x-y)dx \right\} \tag{3.33}$$

is finite, then for each $\alpha' < \alpha$ the operator $\widehat{L}$ acts as a bounded operator from $\mathcal{L}_\alpha$ to $\mathcal{L}_{\alpha'}$ and L^Δ is bounded from $\mathcal{K}_{\alpha'}$ to $\mathcal{K}_\alpha$. In this case the estimate

$$\|\widehat{L}\|_{L(\mathcal{L}_\alpha,\mathcal{L}_{\alpha'})} = \|L^\Delta\|_{L(\mathcal{K}_{\alpha'},\mathcal{K}_\alpha)} \le \frac{m+3\lambda}{e(\alpha-\alpha')} + \frac{4\lambda\theta e^{-\alpha'}}{e^2(\alpha-\alpha')^2} \tag{3.34}$$

holds.

Proof. The function $c(\alpha;\eta)$ is given by $c(\alpha;\eta) = e^{-\alpha|\eta|}L^\Delta e^{\alpha|\cdot|}(\eta)$ which implies the particular form for $c(\alpha;\eta)$. For the second assertion observe that

$$\begin{aligned}\sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2)\mathrm{d}x &= \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} \int_{\mathbb{R}^d} a(x-(y_2-y_1), x)\mathrm{d}x \\ &= \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} \int_{\mathbb{R}^d} a(x, x-(y_2-y_1))\mathrm{d}x\end{aligned}$$

and hence

$$\sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2)\mathrm{d}x \le \theta|\eta|^2.$$

The assertion now follows from the estimates

$$|\eta|e^{-(\alpha-\alpha')|\eta|} \le \frac{1}{e(\alpha-\alpha')}$$

and

$$|\eta|^2 e^{-(\alpha-\alpha')|\eta|} \le \frac{4}{e^2(\alpha-\alpha')^2}$$

for any $\alpha' < \alpha$. □

The operator $\widehat{L}$ is a sum of a particle number preserving part $\widehat{L}_V$ and an upper diagonal part $\widehat{B}$. Rewrite this number preserving part $\widehat{L}_V$ in the form

$$(\widehat{L}_V G)(\eta) = -(m-\lambda)|\eta|G(\eta) + \lambda \sum_{x\in\eta} \int_{\mathbb{R}^d} b(x-y)\left(G(\eta\backslash x \cup y) - G(\eta)\right)\mathrm{d}y.$$

This operator is well-defined on the domain

$$D(\widehat{L}_V) = \{G \in \mathcal{L}_\alpha \mid |\cdot|G \in \mathcal{L}_\alpha\}$$

and satisfies by previous theorem $\|\widehat{L}_V\|_{\alpha\alpha'} \leq \frac{m+3\lambda}{e(\alpha-\alpha')}$. Let us construct solutions to the Cauchy problem

$$\frac{\partial G_t}{\partial t} = \widehat{L}_V G_t, \quad G_t|_{t=0} = G_0. \tag{3.35}$$

For any $0 \leq G \in D(\widehat{L}_V)$

$$\int_{\Gamma_0} \widehat{L}_V G(\eta) e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) = (\lambda - m)\int_{\Gamma_0} G(\eta)|\eta|e^{\alpha|\eta|}\mathrm{d}\lambda(\eta)$$

and hence if $\lambda < m$, then $(\widehat{L}_V, D(\widehat{L}_V))$ is the generator of an analytic semigroup of contractions on $\mathcal{L}_\alpha$. If $\lambda = m$, then $(\widehat{L}_V, D(\widehat{L}_V))$ admits an extension, which is the generator of an substochastic semigroup, cf. [TV06]. For this particular model it is also possible to construct solutions in the case $m < \lambda$ which shall be done in the following. Let $G = (G^{(n)})_{n=0}^\infty$ be the decomposition of a measurable function $G : \Gamma_0 \longrightarrow \mathbb{R}$ into its components and set for $n \in \mathbb{N}$

$$(D_n G^{(n)})(x_1, \dots, x_n) = -(m-\lambda)nG^{(n)}(x_1, \dots, x_n) + (A_n G)^{(n)}(x_1, \dots, x_n)$$

where

$$(A_n G)^{(n)}(x_1, \dots, x_n) = \lambda \sum_{k=1}^n \int_{\mathbb{R}^d} b(x_k - y)\left(G^{(n)}(x_1, \dots, \hat{x}_k, y, \dots, x_n) - G^{(n)}(x_1, \dots, x_n)\right)\mathrm{d}y.$$

Here $\hat{x}_k$ means that integration over the variable x_k should be omitted. For each $n \in \mathbb{N}_0$ the operator $\widehat{L}_V$ is diagonal, i.e. it acts only on $G^{(n)}$. The equation

$$\frac{\partial G_t^{(n)}}{\partial t} = D_n G_t^{(n)}, \quad G_t^{(n)}|_{t=0} = G_0^{(n)}$$

has a solution $G_t^{(n)} = e^{-(m-\lambda)nt}H_t^{(n)}$, were $H_t^{(n)}$ solves

$$\frac{\partial H_t^{(n)}}{\partial t} = A_n H_t^{(n)}, \quad H_t^{(n)}|_{t=0} = G_0^{(n)}.$$

The operator A_n describes for each cell a Random walk in continuous time. The jumping times are independent and exponentially distributed with parameter 2λ and the probability of a cell located at $x \in \mathbb{R}^d$ to jump in the region $\mathrm{d}y$ is given by

$$\frac{1}{2}b(x-y)\mathrm{d}y.$$

The next lemma was proved in [KKP08].

Lemma 3.4.3. *The operator D_n is a bounded linear operator on $L^1((\mathbb{R}^d)^n)$ and $L^\infty((\mathbb{R}^d)^n)$ for any $n \geq 1$ and the corresponding semigroup is a positive contraction semigroup.*

Let $G_0 = (G_0^{(n)})_{n\in\mathbb{N}}$ be measurable such that each component $G_0^{(n)}$ is integrable. Then $e^{-(m-\lambda)nt}e^{tA_n}G_0^{(n)} = e^{tD_n}G_0^{(n)}$ is well-defined and the vector $G_t = (e^{tD_n}G_0^{(n)})_{n=0}^{\infty}$ is the unique component-wise solution to (3.35). This solution, if $G_0 \in \mathcal{L}_\alpha$, evolves in the scale of Banach spaces $\mathcal{L}_\alpha$ with $\alpha(t) = \alpha + (m-\lambda)t$, i.e. $G_t \in \mathcal{L}_{\alpha(t)}$, which follows from

$$\begin{aligned}\|G_t\|_{\mathcal{L}_{\alpha(t)}} &= \sum_{n=0}^{\infty} \frac{e^{-(m-\lambda)nt}e^{\alpha(t)n}}{n!} \int\limits_{(\mathbb{R}^d)^n} |e^{tA_n}G_0^{(n)}(x_1,\dots,x_n)|\mathrm{d}x_1\dots\mathrm{d}x_n \\ &\leq \sum_{n=0}^{\infty} \frac{e^{\alpha n}}{n!} \int\limits_{(\mathbb{R}^d)^n} |G_0^{(n)}(x_1,\dots,x_n)|\mathrm{d}x_1\dots\mathrm{d}x_n = \|G_0\|_{\mathcal{L}_\alpha}.\end{aligned}$$

The presence of the perturbation $\widehat{B}$ implies that the solution cannot satisfy $G_t \in \mathcal{L}_{\alpha(t)}$ for $t > 0$ and any $\alpha(t)$. Since $\widehat{B}$ sends functions of $n+1$ variables to functions of n variables it is not helpful to discuss a solution formula, though it is possible. More precise results will be investigated in terms of correlation functions. Let $k_0 = (k_0^{(n)})_{n=0}^{\infty}$ be non-negative and measurable such that $k_0^{(n)} \in L^\infty((\mathbb{R}^d)^n)$, then $(e^{tD_n}k_0^{(n)})_{n=0}^{\infty}$ is the unique component-wise solution to

$$\frac{\partial k_t}{\partial t} = L_V^{\Delta}k_t, \quad k_t|_{t=0} = k_0.$$

Denote by B_n^{Δ} the operator given by (3.32) taking functions from n variables to functions with $n+1$ variables, i.e.

$$(B_{n+1}^{\Delta}k^{(n)})(x_1,\dots,x_{n+1}) = \lambda \sum_{k=1}^{n+1}\sum_{\substack{j=1\\ j\neq k}}^{n+1} \int\limits_{\mathbb{R}^d} a(x-x_k, x-x_j)k^{(n)}(x_1,\dots,\hat{x}_k,\hat{x}_j,x,\dots,x_{n+1})\mathrm{d}x.$$

The solution to (3.6) is then given by

$$k_t^{(n+1)} = e^{-(m-\lambda)(n+1)t}e^{tA_{n+1}}k_0^{(n+1)} + \int\limits_0^t e^{-(m-\lambda)(n+1)(t-s)}e^{(t-s)A_{n+1}}B_{n+1}^{\Delta}k_s^{(n)}\mathrm{d}s. \qquad (3.36)$$

The next statement establishes asymptotic clustering for the evolution of correlation functions constructed above.

Theorem 3.4.4. *For each $k_0 \geq 0$ measurable, such that $k_0^{(n)} \in L^\infty((\mathbb{R}^d)^n)$, there exist a unique solution $k_t \geq 0$, given recursively by formula (3.36). If θ is finite, then for each initial condition satisfying $k_0(\eta) \leq |\eta|!C^{|\eta|}$ for some constant $C > 0$, this solution obeys the bound*

$$k_t(\eta) \leq |\eta|!(C+t)^{|\eta|}(1+\theta)^{|\eta|}\kappa(t)^{|\eta|}e^{-(m-\lambda)|\eta|t}$$

with $\kappa(t) = \max\{1, \lambda, \lambda e^{(m-\lambda)t}\}$. If there exists $\delta > 0$ such that $a(x,y) \geq \alpha > 0$ for some $\alpha > 0$ and all $|x|, |y| \leq \delta$, then for each $k_0(\eta) = C^{|\eta|}$ the solution k_t satisfies for any $\eta \in \Gamma_0$ with

$$\forall x, y \in \eta,\ x \neq y: \ |x-y| < \delta$$

the estimate

$$k_t(\eta) \geq \beta^{|\eta|}e^{-(m-\lambda)|\eta|t}|\eta|! \quad t \geq 1,$$

where $\beta = \min\{C, 2|B_\delta|\lambda\alpha\tau\}$ with $\tau = \begin{cases} \frac{1}{\lambda-m} & ,\lambda > m \\ 1 & ,\lambda \leq m \end{cases}$ and $|B_\delta|$ is the Lebesgue volume of the ball B_δ of radius δ.

Proof. For the bound from above we proceed by induction on the number of cells $|\eta|$. The first correlation function is given by

$$k_t^{(1)} = e^{-(m-\lambda)t}e^{tA_1}k_0^{(1)}$$

and hence by positivity of $(e^{tA_1})_{t\geq 0}$ and $e^{tA_1}C = C$

$$k_t^{(1)} \leq e^{-(m-\lambda)t}C \leq (C+t)(1+\theta)\kappa(t)e^{-(m-\lambda)t}.$$

For $n \to n+1$ we get with $|\eta| = n+1$

$$\begin{aligned}
k_t^{(n+1)} &\leq e^{-(m-\lambda)(n+1)t}(n+1)!C^{n+1} + \int_0^t e^{-(m-\lambda)(n+1)(t-s)}e^{(t-s)A_{n+1}}B_{n+1}^{\Delta}k_s^{(n)}ds \\
&\leq e^{-(m-\lambda)(n+1)t}(n+1)!C^{n+1} \\
&\quad + (1+\theta)^{n+1}(n+1)!\lambda n\int_0^t e^{-(m-\lambda)(n+1)(t-s)}(C+s)^n\kappa(s)^n e^{-(m-\lambda)ns}\mathrm{d}s \\
&\leq e^{-(m-\lambda)(n+1)t}(n+1)!C^{n+1} \\
&\quad + (n+1)!\kappa(t)^{n+1}(1+\theta)^{n+1}\left((C+t)^{n+1} - C^{n+1}\right)e^{-(m-\lambda)(n+1)t} \\
&\leq (n+1)!(C+t)^{n+1}(1+\theta)^{n+1}\kappa(t)^{n+1}e^{-(m-\lambda)(n+1)t}.
\end{aligned}$$

Here we used the fact that for $s \leq t$ we have $\kappa(s) \leq \kappa(t)$. For the second part let $k_0^{(n)} = C^n$, then $e^{tA_n}k_0 = C^n$ and therefore $k_t^{(1)} = e^{-(m-\lambda)t}C \geq \beta e^{-(m-\lambda)t}$. For $n \to n+1$

and $t \geq 1$ we obtain

$$k_t^{(n+1)} \geq e^{-(m-\lambda)(n+1)t} C^{n+1} + 2|B_\delta|\lambda\alpha\beta^n \int_0^t e^{-(m-\lambda)(n+1)(t-s)}(n+1)ne^{-(m-\lambda)ns} n! \mathrm{d}s$$

$$\geq e^{-(m-\lambda)(n+1)t} \int_0^t e^{(m-\lambda)s} ds \cdot (n+1)! 2|B_\delta|\lambda\alpha\beta^n$$

$$\geq e^{-(m-\lambda)(n+1)t} \beta^{n+1}(n+1)!.$$

□

Above estimates show that if the probability distribution a has no hard core, i.e. $a(0) > 0$ for continuous distributions, then the system will consist of clusters. Appearance of such clusters is caused by properties of the operator B^Δ. The part L_V^Δ contains information about asymptotic behaviour, speed of propagation etc., whereas B^Δ contains information about correlations of the system. Assume for simplicity that in the cell-division the position of the new cells are independent of each other. Then we may write $a(x, y) = c(x)c(y)$ for some symmetric function $0 \leq c \in L^1(\mathbb{R}^d)$ normalized to 1. If for example c is continuous and non-vanishing, then previous assumptions are satisfied and we get the bound

$$\beta^n n! e^{-(m-\lambda)nt} \leq k_t^{(n)}.$$

The same results have been shown in [KKP08] for the case $a(x, y) = c(x)\delta(y)$, where each cell creates a new cell and its location is described by the kernel c. In contrast to this model, the old cell will not die. Clearly such models should have the same qualitative properties.

Vlasov Scaling

Following the general scheme of Vlasov scaling described before, we scale the potentials by $a \longmapsto \frac{1}{n}a$ and accelerate the birth by a factor n. Clearly, since the birth only consists of the a-part, this will not change the operator itself, i.e. $L_n = L$. The operator on quasi-observables is then given by $\widehat{L}_{n,\text{ren}} = R_n \widehat{L} R_{n^{-1}}$, where $R_n G(\eta) = n^{|\eta|}G(\eta)$. In this case we obtain $\widehat{L}_{n,\text{ren}} = \widehat{L}_V + \frac{1}{n}\widehat{B}$. and the operator can be defined on the same domain for all $n \geq 1$. The evolution of scaled correlation functions is then determined by the Cauchy problem

$$\frac{\partial k_{n,t}}{\partial t} = L_V^\Delta k_{n,t} + \frac{1}{n} B^\Delta k_{n,t}, \quad k_{n,t}|_{t=0} = r_0.$$

For every collection of L^∞-functions $(r_0^{(k)})_{k=0}^\infty$ above Cauchy problem has the solution $k_{n,t}$ given by its components

$$k_{n,t}^{(k+1)} = e^{-(m-\lambda)(k+1)t} e^{tA_{k+1}} r_0^{(k+1)} + \frac{1}{n} \int\limits_0^t e^{-(m-\lambda)(k+1)(t-s)} e^{(t-s)A_{k+1}} B_{k+1}^\Delta k_s^{(k)} \mathrm{d}s.$$

This solution satisfies for any $k \geq 1$

$$k_{n,t}^{(k)} \longrightarrow e^{-(m-\lambda)kt} e^{tA_k} r_0^{(k)}, \quad n \to \infty$$

in $L^\infty((\mathbb{R}^d)^k)$. In particular if $r_0^{(k)}(x_1, \ldots, x_k) = \rho_0(x_1) \cdots \rho_0(x_k)$, then

$$e^{-(m-\lambda)kt} e^{tA_k} r_0^{(k)}(x_1, \ldots, x_k) = \rho_t(x_1) \cdots \rho_t(x_k)$$

where ρ_t is the classical solution to

$$\frac{\partial \rho_t}{\partial t} = -(m+\lambda)\rho_t + b * \rho_t, \quad \rho_t|_{t=0} = \rho_0.$$

3.4.2 Local regulation of cell-proliferation

As we have seen for the free-proliferation model the correlation functions $k_t^{(n)}$ behave like $n!$, see Theorem 3.4.4, and hence the main results cannot be applied. From a cell-biological point of view it is reasonable to introduce some type of competition between cells. Such competition will regulate the local density of the cell-system and hence it will be reasonable to expect in such a case an evolution of correlation functions in the Banach space $\mathcal{K}_\beta$ for some $\beta > 0$. The regulation of the system can be achieved by introducing so-called fecundity or establishment effects, see [FKK13a]. Such effects resemble the needs of resources for proliferation. The intensity for the creation of a new cell therefore should depend on all neighbouring cells and be small in dense regions. In this work we follow an alternative approach and introduce additional competition, i.e. the death intensity depends on neighbouring cells and will be large in dense regions. Such competition is usually described by a pair interaction function $\varphi(x, y) \geq 0$ and hence the relative energy

$$E(x, \gamma) := \sum_{y \in \gamma} \varphi(x, y) \in [0, \infty].$$

The function φ is assumed to be non-negative and integrable in y. For the fulfilment of condition (A) it will be sufficient to find $\Gamma_\infty \subset \Gamma$ such that $\mu \in \mathcal{P}_\beta$ is supported on Γ_∞ and $E(x, \gamma)$ is finite for each $\gamma \in \Gamma_\infty$ and each $x \in \mathbb{R}^d$. If $\varphi(x, \cdot)$ is compactly supported for any $x \in \mathbb{R}^d$, then above condition is clearly satisfied. More generally suppose that for any $x \in \mathbb{R}^d$ there exists $C_x > 0$ such that

$$\varphi(x, y) \leq C_x g(y), \quad x, y \in \mathbb{R}^d \tag{3.37}$$

holds for a fixed integrable function $g : \mathbb{R}^d \longrightarrow \mathbb{R}_+$. Let

$$\Gamma_\infty = \left\{\gamma \in \Gamma \;\middle|\; \sum_{y\in\gamma} g(y) < \infty\right\}$$

and $\mu \in \mathcal{P}_\beta$ with correlation function $k_\mu \in \mathcal{K}_\beta$. Then

$$\int_\Gamma \sum_{x\in\gamma} g(y)\mathrm{d}\mu(\gamma) = \int_{\mathbb{R}^d} g(y)k_\mu^{(1)}(y)\mathrm{d}y \leq e^\beta \|k_\mu\|_{\mathcal{K}_\beta} \int_{\mathbb{R}^d} g(y)\mathrm{d}y$$

implies $\mu(\Gamma_\infty) = 1$. Here and in the following we always suppose that either $\varphi(x, \cdot)$ is compactly supported or condition (3.37) holds.

Time-inhomogeneous BDLP-model

Consider the Markov (pre-)generator given by

$$\begin{aligned}(L(t)F)(\gamma) = &\sum_{x\in\gamma}\left(m(t,x) + \lambda^-(t,x)\sum_{y\in\gamma\backslash x} a^-(x,y)\right)(F(\gamma\backslash x) - F(\gamma)) \\ &+ \sum_{x\in\gamma}\lambda^+(t,x)\int_{\mathbb{R}^d} a^+(x,y)(F(\gamma\cup y) - F(\gamma))\mathrm{d}y,\end{aligned} \tag{3.38}$$

where $a^\pm \geq 0$ are assumed to be bounded and for all $x \in \mathbb{R}^d$

$$1 = \int_{\mathbb{R}^d} a^+(x,y)\mathrm{d}y = \int_{\mathbb{R}^d} a^-(x,y)\mathrm{d}y$$

holds. The intensities $m, \lambda^+, \lambda^- > 0$ are supposed to be bounded and $t \longmapsto m(t,\cdot), \lambda^\pm(t,\cdot)$ are continuous w.r.t. the supremum norm. A short computation yields

$$\begin{aligned}(\widehat{L}(t)G)(\eta) = &-\sum_{x\in\eta} m(t,x)G(\eta) - \sum_{x\in\eta}\sum_{y\in\eta\backslash x}\lambda^-(t,x)a^-(x,y)G(\eta) \\ &-\sum_{x\in\eta}\sum_{y\in\eta\backslash x}\lambda^-(t,x)a^-(x,y)G(\eta\backslash x) \\ &+\sum_{x\in\eta}\lambda^+(t,x)\int_{\mathbb{R}^d} a^+(x,y)G(\eta\backslash x\cup y)\mathrm{d}y + \sum_{x\in\eta}\lambda^+(t,x)\int_{\mathbb{R}^d} a^+(x,y)G(\eta\cup y)\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}(L^{\Delta}(t)k)(\eta) = &-\sum_{x\in\eta} m(t,x)k(\eta) - \sum_{x\in\eta}\sum_{y\in\eta\setminus x}\lambda^-(t,x)a^-(x,y)k(\eta)\\ &-\sum_{x\in\eta}\int_{\mathbb{R}^d}\lambda^-(t,y)a^-(y,x)k(\eta\cup y)\mathrm{d}y\\ &+\sum_{x\in\eta}\int_{\mathbb{R}^d}\lambda^+(t,y)a^+(y,x)k(\eta\setminus x\cup y)\mathrm{d}y + \sum_{x\in\eta}\sum_{y\in\eta\setminus x}\lambda^+(t,y)a^+(y,x)k(\eta\setminus x).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}c(\beta;\eta) = &\sum_{x\in\eta}\left(m(t,x) + e^{\beta}\int_{\mathbb{R}^d}\lambda^-(t,y)a^-(y,x)\mathrm{d}y + \int_{\mathbb{R}^d}\lambda^+(t,y)a^+(y,x)\mathrm{d}y\right)\\ &+\sum_{x\in\eta}\sum_{y\in\eta\setminus x}\lambda^-(t,x)a^-(x,y) + e^{-\beta}\sum_{x\in\eta}\sum_{y\in\eta\setminus x}\lambda^+(t,y)a^+(y,x)\end{aligned}$$

and $M(\eta) = \sum_{x\in\eta} m(t,x) + \sum_{x\in\eta}\sum_{y\in\eta\setminus x}\lambda^-(t,x)a^-(x,y)$.

Theorem 3.4.5. *Suppose that there exists $b \geq 0$ and $\vartheta > 0$ such that for all $\eta \in \Gamma_0$, $t \geq 0$*

$$\sum_{x\in\eta}\sum_{y\in\eta\setminus x}\lambda^+(t,y)a^+(y,x) \leq b|\eta| + \vartheta\sum_{x\in\eta}\sum_{y\in\eta\setminus x}\lambda^-(t,x)a^-(x,y) \tag{3.39}$$

holds. Assume that there exists $q \in (0,1)$ such that

$$\vartheta < \inf_{\substack{x\in\mathbb{R}^d\\ t\geq 0}} \frac{qm(t,x) - \int_{\mathbb{R}^d}\lambda^+(t,y)a^+(y,x)\mathrm{d}y - \frac{b}{\vartheta}}{\int_{\mathbb{R}^d}\lambda^-(t,y)a^-(y,x)\mathrm{d}y} \tag{3.40}$$

holds. Let $\beta_* := \log(\vartheta)$ and

$$\beta^* := \log\left(\inf_{\substack{x\in\mathbb{R}^d\\ t\geq 0}} \frac{qm(t,x) - \int_{\mathbb{R}^d}\lambda^+(t,y)a^+(y,x)\mathrm{d}y - \frac{b}{\vartheta}}{\int_{\mathbb{R}^d}\lambda^-(t,y)a^-(y,x)\mathrm{d}y}\right),$$

then all conditions of Theorem 3.2.17 are satisfied.

See also [Rue70] for conditions of the form (3.39).

Proof. Condition (3.20) and (D) for any $t \geq 0$ are clearly satisfied. It is not difficult to see that for any $\beta \in (\beta_*, \beta^*)$ the operator $L(t)$ act in the scale of Banach space $\mathcal{L}$ and $t \longmapsto L(t) \in L(\mathcal{L})$ is continuous in the uniform topology, cf. [FK13]. For condition (H1) let $\beta \in (\beta_*, \beta^*)$, then

$$c(L(t), \beta; \eta) \leq \sum_{x \in \eta} \left(m(t,x) + e^{\beta} \int_{\mathbb{R}^d} \lambda^-(t,y) a^-(y,x) \mathrm{d}y + \int_{\mathbb{R}^d} \lambda^+(t,y) a^+(y,x) \mathrm{d}y + b e^{-\beta} \right)$$
$$+ (1 + \vartheta e^{-\beta}) \sum_{x \in \eta} \sum_{y \in \eta \backslash x} \lambda^-(t,x) a^-(x,y)$$

and $1 + \vartheta e^{-\beta} < 2$ holds by $\beta_* < \beta$. We get for the other term

$$\begin{aligned} &m(t,x) + e^{\beta} \int_{\mathbb{R}^d} \lambda^-(t,y) a^-(y,x) \mathrm{d}y + \int_{\mathbb{R}^d} \lambda^+(t,y) a^+(y,x) \mathrm{d}y + b e^{-\beta} \\ &\leq m(t,x) + \int_{\mathbb{R}^d} \lambda^+(t,y) a^+(y,x) \mathrm{d}y + \frac{b}{\vartheta} + q m(t,x) - \int_{\mathbb{R}^d} \lambda^+(t,y) a^+(y,x) \mathrm{d}y - \frac{b}{\vartheta} \\ &= (1+q) m(t,x) < 2 m(t,x) \end{aligned}$$

and hence (H1) holds with $a(\beta) := 1 + \max\{q, \vartheta e^{-\beta}\}$. □

Remark 3.4.6. *If instead of* (3.40) *the weaker condition*

$$\vartheta < \inf_{\substack{x \in \mathbb{R}^d \\ t \in [0,T]}} \frac{q m(t,x) - \int\limits_{\mathbb{R}^d} \lambda^+(t,y) a^+(y,x) \mathrm{d}y - \frac{b}{\vartheta}}{\int\limits_{\mathbb{R}^d} \lambda^-(t,y) a^-(y,x) \mathrm{d}y}$$

holds for all $T > 0$, then we still can construct the associated forward and backward evolution systems and show that their adjoints are positivity preserving. But in such a case we cannot choose β^* to be independent of $T > 0$.

Let us continue with the Vlasov scaling. The time-homogeneous case was considered in [FKK13b]. The renormalized operator is given by

$$\begin{aligned} (\widehat{L}_{n,\mathrm{ren}}(t) G)(\eta) = &- \sum_{x \in \eta} m(t,x) G(\eta) - \sum_{x \in \eta} \sum_{y \in \eta \backslash x} \lambda^-(t,x) a^-(x,y) G(\eta \backslash x) \\ &+ \sum_{x \in \eta} \lambda^+(t,x) \int_{\mathbb{R}^d} a^+(x,y) G(\eta \backslash x \cup y) \mathrm{d}y \\ &- \frac{1}{n} \sum_{x \in \eta} \sum_{y \in \eta \backslash x} \lambda^-(t,x) a^-(x,y) G(\eta) + \frac{1}{n} \sum_{x \in \eta} \lambda^+(t,x) \int_{\mathbb{R}^d} a^+(x,y) G(\eta \cup y) \mathrm{d}y. \end{aligned}$$

Given the same conditions as in the previous theorem it is not difficult to see that (W1) – (W3) are satisfied. The kinetic description for the density is therefore given for $t > 0$ by the the backward equation with $s \in [0,t)$, $x \in \mathbb{R}^d$

$$\frac{\partial \rho_s}{\partial s}(x) = m(s,x)\rho_s(x) + \rho_s(x) \int\limits_{\mathbb{R}^d} \lambda^-(s,y)a^-(y,x)\rho_s(y)\mathrm{d}y - \int\limits_{\mathbb{R}^d} \lambda^+(s,y)a^+(y,x)\rho_s(y)\mathrm{d}y$$

$$\rho_s\big|_{s=t} = \rho_t$$

and for $s \geq 0$ by the forward equation $t \in [s,\infty)$, $x \in \mathbb{R}^d$

$$\frac{\partial \rho_t}{\partial t}(x) = -m(t,x)\rho_t(x) - \rho_t(x) \int\limits_{\mathbb{R}^d} \lambda^-(t,y)a^-(y,x)\rho_t(y)\mathrm{d}y + \int\limits_{\mathbb{R}^d} \lambda^+(t,y)a^+(y,x)\rho_t(y)\mathrm{d}y$$

$$\rho_t\big|_{t=s} = \rho_s.$$

For the analysis of such equations we refer to [JZ09, Yag09, Gar11, FKT15] and references therein.

Remark 3.4.7. *Suppose that $a^{\pm}(x,y) = a^{\pm}(y,x)$ holds, then by*

$$\sum_{x \in \eta} \sum_{y \in \eta \backslash x} \lambda^+(t,y)a^+(x,y) = \sum_{x \in \eta} \sum_{y \in \eta \backslash x} \lambda^+(t,x)a^+(x,y)$$

we can rewrite condition (3.39) *to*

$$\sum_{x \in \eta} \sum_{y \in \eta \backslash x} \lambda^+(t,x)a^+(x,y) \leq b|\eta| + \vartheta \sum_{x \in \eta} \sum_{y \in \eta \backslash x} \lambda^-(t,x)a^-(x,y).$$

Regulation by Glauber-type death

Suppose that the Markov (pre-)generator is given by

$$\begin{aligned}(LF)(\gamma) = &\sum_{x \in \gamma} \left(m + e^{E(x,\gamma\backslash x)}\right)\left(F(\gamma\backslash x) - F(\gamma)\right) \\ &+ \sum_{x \in \gamma} \left(\sum_{y \in \gamma \backslash x} b^+(x-y) \right) \int\limits_{\mathbb{R}^d} a^+(x-w)(F(\gamma \cup w) - F(\gamma))\mathrm{d}w,\end{aligned} \tag{3.41}$$

where $E(x,\gamma\backslash x) = \sum\limits_{y \in \gamma\backslash x} \varphi(x-y)$, $\varphi, a^+, b^+ \geq 0$ are assumed to be symmetric, integrable and bounded and $m > 0$. Hence we obtain for $\gamma \cap (\eta\backslash x) = \varnothing$

$$d(x, \gamma \cup \eta\backslash x) = m + e^{E(x,\gamma)}e^{E(x,\eta\backslash x)}$$

and hence

$$(K_0^{-1}d(x,\cdot\cup\eta\backslash x))(\xi) = 0^{|\xi|}m + e^{E(x,\eta\backslash x)}e_\lambda(e^{\varphi(x-\cdot)}-1;\xi).$$

Likewise we obtain

$$\begin{aligned}b(x,\gamma\cup\eta\backslash x) = &\sum_{y\in\gamma}\sum_{w\in\gamma\backslash y} b^+(w-y)a^+(x-y) + \sum_{y\in\gamma}\sum_{w\in\eta\backslash x} b^+(w-y)a^+(x-y)\\ &+ \sum_{y\in\eta\backslash x}\sum_{w\in\gamma} b^+(w-y)a^+(x-y) + \sum_{y\in\eta\backslash x}\sum_{w\in\eta\backslash x\backslash y} b^+(w-y)a^+(x-y)\end{aligned}$$

and hence

$$\begin{aligned}(K_0^{-1}b(x,\cdot\cup\eta\backslash x))(\xi) = &\, 0^{|\xi|}\sum_{y\in\eta\backslash x}\sum_{w\in\eta\backslash x\backslash y} b^+(w-y)a^+(x-y)\\ &+ \mathbb{1}_{\Gamma^{(1)}}(\xi)\sum_{w\in\xi}\sum_{y\in\eta\backslash x} b^+(w-y)a^+(w-x)\\ &+ \mathbb{1}_{\Gamma^{(1)}}(\xi)\sum_{y\in\xi}\sum_{w\in\eta\backslash x} b^+(w-y)a^+(x-y)\\ &+ \mathbb{1}_{\Gamma^{(2)}}(\xi)\sum_{y\in\xi}\sum_{w\in\xi\backslash y} b^+(w-y)a^+(x-y).\end{aligned}$$

This implies for $\beta\in\mathbb{R}$

$$\begin{aligned}c(\beta;\eta) = &\,(m + e^{\beta}\langle b^+\rangle\langle a^+\rangle)|\eta| + \kappa(\beta,\varphi)\sum_{x\in\eta} e^{E(x,\eta\backslash x)}\\ &+ \langle b^+\rangle\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y) + \sum_{x\in\eta}\sum_{y\in\eta\backslash x}(b^+ * a^+)(x-y)\\ &+ e^{-\beta}\sum_{x\in\eta}\sum_{y\in\eta\backslash x}\sum_{w\in\eta\backslash x\backslash y} b^+(w-y)a^+(x-y)\end{aligned}$$

and $M(\eta) = m|\eta| + \sum_{x\in\eta} e^{E(x,\eta\backslash x)}$ where $\kappa(\beta,\varphi) := \exp\left(e^{\beta}\int_{\mathbb{R}^d}(e^{\varphi(x)}-1)\mathrm{d}x\right)$.

Theorem 3.4.8. *Suppose that there exist constants $0\le b < m+1$, $\vartheta > 0$ and $c > 0$ such that*

$$\langle b^+\rangle\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y) + \sum_{x\in\eta}\sum_{y\in\eta\backslash x}(b^+ * a^+)(x-y) \le b|\eta| + \vartheta\sum_{x\in\eta}\sum_{y\in\eta\backslash x}\varphi(x-y)$$

holds and $\varphi - b^+$ is a stable potential. Moreover, assume that for all $\eta\in\Gamma_0$ and all $x,y,w\in\mathbb{R}^d$

$$b^+(w-y)a^+(x-y) \le c\varphi(x-w)\varphi(x-y)$$

is fulfilled. If in addition there exists $\beta < \log\left(\frac{m+1-b}{\langle b^+ \rangle\langle a^+ \rangle}\right)$ *with*

$$\max\left\{1, \vartheta, 2e^{-\beta}c\right\} + \kappa(\beta, \varphi) < 2, \tag{3.42}$$

then (3.14) *and (A) – (D) hold for* $\tau = \|\varphi\|_\infty$.

Proof. We obtain

$$c(\beta;\eta) \leq (m + e^\beta \langle b^+ \rangle\langle a^+ \rangle)|\eta| + \kappa(\beta,\varphi)\sum_{x\in\eta} e^{E(x,\eta\backslash x)}$$
$$+ b|\eta| + \vartheta \sum_{x\in\eta}\sum_{y\in\eta\backslash x}\varphi(x-y) + e^{-\beta}c\sum_{x\in\eta}\sum_{y\in\eta\backslash x}\sum_{w\in\eta\backslash x\backslash y}\varphi(x-y)\varphi(w-x).$$

By

$$\sum_{x\in\eta} e^{E(x,\eta\backslash x)} \geq |\eta| + \sum_{x\in\eta}\sum_{y\in\eta\backslash x}\varphi(x-y) + \frac{1}{2}\sum_{x\in\eta}\sum_{y\in\eta\backslash x}\sum_{w\in\eta\backslash x\backslash y}\varphi(x-y)\varphi(x-w)$$

it follows that

$$c(\beta;\eta) \leq (m + e^\beta\langle b^+ \rangle\langle a^+ \rangle + (b-1))|\eta| + \left(\kappa(\beta,\varphi) + \max\{1,\vartheta,2e^{-\beta}c\}\right)\sum_{x\in\eta} e^{E(x,\eta\backslash x)}$$

and hence (C) holds. Condition (3.14) is satisfied for

$$a(\beta) := \max\left\{1 + e^\beta\frac{\langle b^+ \rangle\langle a^+ \rangle}{m} + \frac{b-1}{m}, \kappa(\beta,\varphi) + \max\{1,\vartheta,2e^{-\beta}c\}\right\}.$$

Condition (D) clearly holds since all potentials are assumed to be positive and bounded (take e.g. $V(\eta) = 1 + |\eta|$). Condition (B) follows from

$$d(x,\eta) + b(x,\eta) = m + e^{E(x,\eta\backslash x)} + \sum_{y\in\eta}\sum_{w\in\eta\backslash x} b^+(y-w)a^+(x-y)$$
$$\leq m + e^{\|\varphi\|_\infty|\eta|} + \|a^+\|_\infty\|b^+\|_\infty|\eta|(|\eta|-1).$$

□

For the Vlasov scaling we scale the potentials by $a^+ \longmapsto \frac{1}{n}a^+$, $b^+ \longmapsto \frac{1}{n}b^+$, $\varphi \longmapsto \frac{1}{n}\varphi$ and the birth part by n. This leads to

$$c_n(\beta;\eta) = (m + e^\beta\langle b^+ \rangle\langle a^+ \rangle)|\eta| + \kappa_n(\beta,\varphi)\sum_{x\in\eta} e^{\frac{1}{n}E(x,\eta\backslash x)}$$
$$+ \frac{1}{n}\langle b^+ \rangle\sum_{x\in\eta}\sum_{y\in\eta\backslash x} a^+(x-y) + \frac{1}{n}\sum_{x\in\eta}\sum_{y\in\eta\backslash x}(b^+ * a^+)(x-y)$$
$$+ \frac{e^{-\beta}}{n^2}\sum_{x\in\eta}\sum_{y\in\eta\backslash x}\sum_{w\in\eta\backslash x\backslash y} b^+(y-w)a^+(x-y)$$

and $M_n(\eta) = m|\eta| + \kappa_n(\beta,\varphi) \sum\limits_{x\in\eta} e^{\frac{1}{n}E(x,\eta\backslash x)}$ where $\kappa_n(\beta,\varphi) = \exp\left(ne^{\beta} \int\limits_{\mathbb{R}^d} (e^{\frac{1}{n}\varphi(x)} - 1)\mathrm{d}x \right)$.

Theorem 3.4.9. *Suppose that the same conditions as in previous theorem with*

$$\max\{1, \vartheta, 2e^{-\beta}c\} + \exp\left(e^{\|\varphi\|_\infty + \beta}\langle\varphi\rangle\right) < 2$$

instead of (3.42) *are satisfied. Then conditions (V1) – (V3) are satisfied and the kinetic equation for the cell density is given by*

$$\frac{\partial \rho_t}{\partial t}(x) = -m\rho_t(x) - \rho_t(x)e^{(\varphi*\rho_t)(x)} + (a^+ * (b^+ * \rho_t))(x), \quad \rho_t|_{t=0} = \rho_0.$$

Proof. Condition (V1) can be shown in the same way as in the case $n = 1$. Let us show condition (V2) for the death. Observe that after taking the limit $n \to \infty$ we arrive at

$$D_x^V(\eta) = m0^{|\eta|} + e_\lambda(\varphi(x - \cdot);\eta).$$

Observe that

$$\begin{aligned}&\left|e^{\frac{1}{n}E(x,\eta\backslash x)} e_\lambda\left(\left(e^{\frac{1}{n}\varphi(x-\cdot)} - 1\right)n;\xi\right) - e_\lambda(\varphi(x-\cdot);\xi)\right|\\ &\leq e^{\frac{1}{n}E(x,\eta\backslash x)}\left|e_\lambda\left(\left(e^{\frac{1}{n}\varphi(x-\cdot)} - 1\right)n;\xi\right) - e_\lambda(\varphi(x-\cdot);\xi)\right|\\ &\quad + \left|1 - e^{\frac{1}{n}E(x,\eta\backslash x)}\right| e_\lambda(\varphi(x-\cdot);\xi).\end{aligned}$$

The second term tends to zero in $\mathcal{L}_\beta$ w.r.t. ξ, so let us consider the first term. The estimates $\left(e^{\frac{\varphi(x-y)}{n}} - 1\right)n \leq \varphi(x-y)e^{\varphi(x-y)}$, $\varphi(x-y) \leq \varphi(x-y)e^{\varphi(x-y)}$ and

$$\left|\left(e^{\frac{\varphi(x-y)}{n}} - 1\right)n - \varphi(x-y)\right| \leq \frac{1}{n}\varphi(x-y)^2 e^{\varphi(x-y)}$$

imply for all $x, y \in \mathbb{R}^d$

$$\begin{aligned}&\left|e_\lambda\left(\left(e^{\frac{\varphi(x-\cdot)}{n}} - 1\right)n;\xi\right) - e_\lambda(\varphi(x-\cdot);\xi)\right|\\ &\leq \sum_{y\in\xi}\left|\left(e^{\frac{\varphi(x-y)}{n}} - 1\right)n - \varphi(x-y)\right| e_\lambda(\varphi(x-\cdot)e^{\varphi(x-\cdot)};\xi\backslash y)\\ &\leq \frac{e^{\|\varphi\|_\infty}\|\varphi\|_\infty}{n}\sum_{y\in\xi}\varphi(x-y)e_\lambda(\varphi(x-\cdot)e^{\varphi(x-\cdot)};\xi\backslash y).\end{aligned}$$

This shows that

$$\begin{aligned}&\int\limits_{\Gamma_0}\left|e^{\frac{1}{n}E(x,\eta\backslash x)} e_\lambda\left(\left(e^{\frac{1}{n}\varphi(x-\cdot)} - 1\right)n;\xi\right) - e_\lambda(\varphi(x-\cdot);\xi)\right| e^{\beta|\xi|}\mathrm{d}\lambda(\xi)\\ &\leq \frac{\|\varphi\|_\infty e^{\|\varphi\|_\infty}}{n}\langle\varphi\rangle\exp\left(e^{\beta+\|\varphi\|_\infty}\langle\varphi\rangle\right).\end{aligned}$$

Convergence for the birth can be shown by similar estimates to [FK13], which yields

$$B_x^V(\eta) = \mathbb{1}_{\Gamma^{(2)}}(\eta) \sum_{y\in\eta} \sum_{w\in\eta\backslash y} b^+(w-y)a^+(x-y).$$

This implies $M_V(\eta) = m|\eta| \leq M_n(\eta)$ and hence condition (V3) holds. □

3.4.3 Ergodic cell-systems

The two previous models satisfied the condition $\inf\limits_{|\eta|\geq 1} M(\eta) > 0$ and hence were ergodic. However, the unique invariant measure for this models was $\mu_{\text{inv}} = \delta_{\varnothing}$. In this part we discuss models with non-trivial invariant measures.

Time-inhomogeneous Glauber dynamics

Suppose that the Markov (pre-)generator is given by

$$(L(t)F)(\gamma) = \sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) + z(t)\int\limits_{\mathbb{R}^d} e^{-E_t(x,\gamma)}(F(\gamma\cup x) - F(\gamma))\mathrm{d}x,$$

where $0 \leq z \in C_b(\mathbb{R}_+)$ and $E_t(x,\gamma) = \sum\limits_{y\in\gamma}\varphi(t,x-y)$ with $\varphi(t,x-y) = \varphi(t,y-x) \geq 0$ such that $t \longmapsto \varphi(t,\cdot)$ is continuous in the supremum norm $\|\cdot\|_\infty$ and L^1-norm $\|\cdot\|_1$. A short computation yields

$$c(L(t),\beta;\eta) = |\eta| + z(t)\kappa(t,\beta)e^{-\beta}\sum_{x\in\eta} e^{-E_t(x,\eta)}$$

and $M(\eta) = |\eta|$ where $\kappa(t,\beta) = \exp\left(e^{\beta}\int\limits_{\mathbb{R}^d}|e^{-\varphi(t,x)} - 1|\mathrm{d}x\right)$. The next statement provides one possible sufficient condition for the evolution of states.

Theorem 3.4.10. *Suppose that* $C_\varphi := \sup\limits_{t\geq 0}\int\limits_{\mathbb{R}^d}\varphi(t,x)\mathrm{d}x$ *is finite and there exist* $\beta \in \mathbb{R}$ *such that*

$$\|z\|_\infty e^{-\beta}\exp\left(e^{\beta}C_\varphi\right) < 1$$

holds. Then there exist $\beta_* < \beta^*$ *with* $\beta \in (\beta_*,\beta^*)$ *and condition (H1) holds for all* $\beta' \in (\beta_*,\beta^*)$*. Moreover, condition (H3) and* (3.20) *are satisfied.*

Proof. Because of $c(L(t),\beta;\eta) \leq (1 + \|z\|_\infty e^{-\beta}\exp\left(e^{\beta}C_\varphi\right))|\eta|$ the first assertion follows by the continuous dependence of $a(\beta) = 1 + \|z\|_\infty e^{-\beta}\exp\left(e^{\beta}C_\varphi\right)$ on β. Condition (3.20) follows readily by $\varphi \geq 0$ and (H3) was proved in [FK13]. □

Given above conditions it is not difficult to see that after scaling $\varphi \longmapsto \frac{1}{n}\varphi$ also conditions (W1) – (W3) are satisfied. The backward equation for the cell density is for $s \in [0, t)$ given by

$$\frac{\partial \rho_s}{\partial s}(x) = \rho_s(x) - z(s)e^{-(\varphi_s * \rho_s)(x)}, \quad \rho_s|_{s=t} = \rho_t$$

and the forward equation for $t \in [s, \infty)$

$$\frac{\partial \rho_t}{\partial t}(x) = -\rho_t(x) + z(t)e^{-(\varphi_t * \rho_t)(x)}, \quad \rho_t|_{t=s} = \rho_s.$$

In order to obtain ergodicity we suppose that φ and z do not depend on the time $t \geq 0$.

Theorem 3.4.11. *Suppose that there exist $\beta \in \mathbb{R}$ such that*

$$ze^{-\beta} \exp\left(e^{\beta} \int_{\mathbb{R}^d} |e^{-\varphi(x)} - 1| \mathrm{d}x \right) < 1 \tag{3.43}$$

holds. Then there exists a unique invariant (Gibbs) measure and the evolution of states is ergodic with exponential rate.

Remark 3.4.12. *The assumption, φ is integrable and bounded, is only necessary for the Vlasov scaling. For the evolution of states, it suffices to assume that $\int_{\mathbb{R}^d} |1 - e^{-\varphi(x)}| \mathrm{d}x$ is finite.*

Above statement was proved in [KKM10] for the stronger condition

$$ze^{-\beta} \exp\left(e^{\beta} \int_{\mathbb{R}^d} |1 - e^{-\varphi(x)}| \mathrm{d}x \right) < \frac{1}{\sqrt{2}}.$$

Taking $e^{-\beta} = \int_{\mathbb{R}^d} (1 - e^{-\varphi(x)}) \mathrm{d}x$ yields by (3.43) the well-known condition

$$z < \frac{1}{\int_{\mathbb{R}^d} (1 - e^{-\varphi(x)}) \mathrm{d}x}.$$

Ergodicity for individual based models

Consider the evolution of cells within the organism described by the generator L_0 either given by (3.38) or (3.41) respectively. Suppose that new cells are created by an external source, e.g. produced by undifferentiated cells. The distribution of the new cells is assumed to be uniformly in the space $\mathbb{R}^d$. Hence the stochastic dynamics can be described by the Markov (pre-)generator

$$(LF)(\gamma) = (L_0F)(\gamma) + z\int_{\mathbb{R}^d}(F(\gamma\cup x) - F(\gamma))\mathrm{d}x,$$

where $z > 0$.

Theorem 3.4.13. *Suppose that the conditions of Theorem 3.4.5 or Theorem 3.4.8 respectively for the generators L_0 are fulfilled. Then there exists $z_0 > 0$ such that for $z < z_0$ the conditions* (3.14) *and (A) – (D) are satisfied. In particular there exists a unique invariant measure $\mu_{\mathrm{inv}} \neq \delta_\emptyset$ and the evolution of states is ergodic with exponential rate.*

Proof. Clearly it is enough to show that condition (3.14) holds. We obtain

$$c(L,\beta;\eta) \leq c(L_0,\beta;\eta) + ze^{-\beta} \leq a(L_0,\beta) + ze^{-\beta}$$

and hence condition (3.14) holds for all $z < z_0$ with $z_0 := (2 - a(L_0,\beta))e^{\beta} > 0$. □

3.4.4 Epistatic mutation-selection balance model

In [KM66] a model for the dynamics of mutation-selection for an infinite-population was proposed. The mathematical analysis, in the language of interacting particle systems in continuum, can be found in the recent works [SEW05, KKO08, KKMP13]. Below we consider a generalization to time-dependent coefficients.

Let X be a complete, separable metric space and σ be a σ-finite Borel measure on X. Elements of X describe potential mutations and for $A \subset X$ the value $\sigma(A)$ is the rate at which spontaneously a mutant allele arises from A. Such allele is characterized by its position $x \in A$. The space of genotypes Γ_X is identified with the space of all locally finite subsets of X, i.e. $\gamma = \{x_n \mid n \in \mathbb{N}\} \in \Gamma_X$ if for any compact $K \subset X$: $\gamma \cap K$ contains only finitely many potential mutations. The topology is defined as the weakest topology such that

$$\gamma \longmapsto \sum_{x\in\gamma} f(x)$$

is continuous for any continuous function f having compact support. For additional properties see [AKR98a].

For each genotype $\gamma \in \Gamma_X$ we assign a "selection cost" functional

$$\Phi(t,\gamma) = \sum_{x\in\gamma} h(t,x) + \frac{1}{2}\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}\psi(t,x,y)$$

with non-negative, measurable functions $h,\psi \geq 0$ and $\psi(t,x,y) = \psi(t,y,x)$.
Denote by μ_t the state of the population at time t, that is μ_t is a Borel probability

measure on Γ_X. Then μ_t shall satisfy for a suitable collection of functions $F : \Gamma_X \longrightarrow \mathbb{R}$ the equation

$$\frac{\partial}{\partial t}\langle F, \mu_t\rangle = \langle L(t)F, \mu_t\rangle - \langle F\Phi(t, \cdot), \mu_t\rangle + \langle F, \mu_t\rangle\langle \Phi(t, \cdot), \mu_t\rangle \tag{3.44}$$

with initial condition $\mu_t|_{t=0} = \mu_0$ and the Markov (pre-)generator

$$(L(t)F)(\gamma) = \int_X (F(\gamma \cup x) - F(\gamma))a(t, x)\sigma(\mathrm{d}x), \quad \gamma \in \Gamma_X.$$

In the particular case $a = 1$ and Φ independent of t in [SEW05] a solution was constructed by the Feynmann-Kac formula and its behaviour for $t \to \infty$ was studied. Uniqueness of the solution μ_t could only be proved for the case $\psi = 0$. For particular functions $\psi = \psi(x, y)$, the results obtained in [KKO08] show that the limiting measure will be a Gibbs measure with energy $\Phi(\gamma)$. In this work we provide existence and uniqueness of (local) solutions to the associated hierarchical equations of correlation functions and derive its kinetic description. Therefore our existence and uniqueness result extends the one from [SEW05] and the kinetic description was not analysed by the authors there. We suppose from now on that the following conditions are fulfilled:

1. $h \geq$ is continuous from $\mathbb{R}_+$ to $L^1(X, \sigma) \cap L^\infty(X, \sigma)$ and $\psi \geq 0$ is continuous from $\mathbb{R}_+$ to $L^1(X^2, \sigma^{\otimes 2}) \cap L^\infty(X^2, \sigma^{\otimes 2})$.
2. $a \geq 0$ is continuous and bounded in its arguments (t, x).
3. For any $T > 0$
$$\sup_{(t,x)\in[0,T]\times X} \int_X \psi(t, x, y)\mathrm{d}\sigma(y) < \infty.$$

Correlation functions k_μ can be defined in the same way as for $X = \mathbb{R}^d$ where $\Gamma_{0,X} = \{\eta \subset X \mid |\eta| < \infty\}$, cf. [KK02, FKO09]. Let μ_t be a solution to (3.44) and assume that it has correlation functions k_t. Then k_t satisfies for any $G \in B_{bs}(\Gamma_{0,X})$

$$\frac{\partial}{\partial t}\langle G, k_t\rangle = \langle G, L^\Delta(t, k)k\rangle, \tag{3.45}$$

where $L^{\Delta}(t,k)k = -A_0^{\Delta}(t)k + A_1^{\Delta}(t)k + B^{\Delta}(t,k)k$ is given by

$$\begin{aligned}
A_0^{\Delta}(t)k(\eta) &= \Phi(t,\eta)k(\eta) + \int_X h(t,x)k(\eta\cup x)\sigma(\mathrm{d}x) \\
&\quad + \frac{1}{2}\int_X\int_X \psi(t,x,y)k(\eta\cup x\cup y)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \\
A_1^{\Delta}(t)k(\eta) &= -\sum_{x\in\eta}\int_X \psi(t,x,y)k(\eta\cup y)\sigma(\mathrm{d}y) + \sum_{x\in\eta} a(t,x)k(\eta\backslash x) \\
B^{\Delta}(t,k) &= \int_X h(t,x)k^{(1)}(x)\sigma(\mathrm{d}x) + \frac{1}{2}\int_X\int_X \psi(t,x,y)k^{(2)}(x,y)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y).
\end{aligned}$$

Conversely, let k_t be a solution to (3.45) and assume that there exist probability measures μ_t such that k_t is the correlation function to μ_t. Then $(\mu_t)_t$ solves also (3.44). As for $X = \mathbb{R}^d$, let $\mathcal{K}_\alpha$ be the Banach space of all equivalence classes of functions k with finite norm

$$\|k\|_{\mathcal{K}_\alpha} = \operatorname*{ess\ sup}_{\eta\in\Gamma_{0,X}} |k(\eta)|e^{-\alpha|\eta|}, \quad \alpha \geq 0.$$

Theorem 3.4.14. *For any $0 < \alpha_* < \alpha^*$ and $\varepsilon \in (0,\alpha_*)$ there exist $\lambda(\alpha^*,\alpha_*,k_0) = \lambda > 0$ such that for any $k_0 \in \mathcal{K}_{\alpha_*-\varepsilon}$ there exists a unique classical $\mathcal{K}$-valued solution k_t to*

$$\frac{\partial k_t}{\partial t} = L^{\Delta}(t,k_t)k_t, \quad k_t|_{t=0} = k_0.$$

with $0 \leq t < \frac{\alpha^*-\alpha_*}{\lambda}$.

Proof. Since $A_0^{\Delta}(t)$ is a sum of a multiplication operator and a bounded operator, it is not difficult to see that for any $\alpha \geq 0$ there exists a unique evolution family $(U_\alpha(t,s))_{0\leq s\leq t} \subset L(\mathcal{K}_\alpha)$ with the properties:

1. $U(t,s)$ satisfies $U_\alpha(t,s)|_{\mathbb{B}_{\alpha'}} = U_{\alpha'}(t,s)$ whenever $\alpha' < \alpha$.
2. For $\varkappa(r) := e^{\alpha}\int_X h(r,x)\sigma(\mathrm{d}x) + \frac{e^{2\alpha}}{2}\int_X\int_X \psi(r,x,y)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)$

$$\|U_\alpha(t,s)\|_{L(\mathcal{K}_\alpha)} \leq \exp\left(\int_s^t \varkappa(r)\mathrm{d}r\right).$$

3. For any $T > 0$ and $\alpha' < \alpha$ there exists $C(\alpha',\alpha,T) > 0$ such that

$$\|U_\alpha(t,0)k - U_\alpha(s,0)k\|_{\mathcal{K}_\alpha} \leq C(\alpha',\alpha,T)\|k\|_{\mathcal{K}_{\alpha'}}, \quad 0 \leq s,t \leq T.$$

Moreover, it is strongly continuously differentiable in $L(\mathcal{K}_{\alpha'}, \mathcal{K}_\alpha)$ with strong derivatives

$$\frac{\partial}{\partial t} U_\alpha(t,s)k = -A_0^\Delta(t) U_\alpha(t,s)k$$

and

$$\frac{\partial}{\partial s} U_\alpha(t,s) = U_\alpha(t,s) A_0^\Delta(s)k.$$

Therefore conditions A1 – A4 hold with $\beta = 0$ and any $\lambda > 0$. Now let $C(k,t) := A_1^\Delta(t)k + B^\Delta(t,k)k$, then it can be easily checked that $C(k,t)$ satisfies B1 – B3 with $r > 0$ arbitrary,

$$\begin{aligned} C_2 &= 2r(\alpha^* - \alpha_*) \sup_{0 \leq t \leq \frac{\alpha^* - \alpha_*}{\lambda}} e^{\alpha^*} \int_X h(t,x)\sigma(\mathrm{d}x) \\ &+ 2r(\alpha^* - \alpha_*) \sup_{0 \leq t \leq \frac{\alpha^* - \alpha_*}{\lambda}} \frac{e^{2\alpha^*}}{2} \int_X \int_X \psi(t,x,y)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \\ &+ e^{-\alpha_*} e^{-1} \|a\|_\infty + e^{-1} e^{\alpha^*} \sup_{x \in X} \sup_{0 \leq t \frac{\alpha^* - \alpha_*}{\lambda}} \int_X \psi(t,x,y)\sigma(\mathrm{d}y) \end{aligned}$$

and $C_3 := C_2$. The assertion now follows from Theorem 1.3.2 and Corollary 1.3.9 with $\mathbb{E}_\alpha = \mathcal{K}_\alpha$. □

Rescale the potentials $\psi \to \varepsilon^2 \psi, h \to \varepsilon h$ and $a \to \varepsilon^{-1} a$, where $\varepsilon > 0$ and denote by $L_\varepsilon^\Delta(t, \cdot)$ the associated operator on correlation functions. Afterwards define the renormalized operator

$$L_{\varepsilon,ren}^\Delta(t,k)k(\eta) := \varepsilon^{|\eta|} L_\varepsilon^\Delta(t, \varepsilon^{-|\cdot|}k)(\varepsilon^{-|\cdot|}k)(\eta) = -A_{0,\varepsilon}^\Delta(t)k + A_{1,\varepsilon}^\Delta(t)k + B^\Delta(t,k)k$$

with $\Phi_\varepsilon(t,\eta) = \varepsilon \sum\limits_{x \in \eta} h(t,x) + \varepsilon^2 \sum\limits_{x \in \eta} \sum\limits_{y \in \eta \backslash x} \psi(t,x,y)$ and

$$\begin{aligned} A_{0,\varepsilon}^\Delta(t)k(\eta) &= \Phi_\varepsilon(t,\eta)k(\eta) + \int_X h(t,x)k(\eta \cup x)\sigma(\mathrm{d}x) \\ &\quad + \frac{1}{2} \int_X \int_X \psi(t,x,y)k(\eta \cup x \cup y)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \\ A_{1,\varepsilon}^\Delta(t)k(\eta) &= -\varepsilon \sum_{y \in \eta} \int_X \psi(t,x,y)k(\eta \cup x)\sigma(\mathrm{d}x) + \sum_{x \in \eta} a(t,x)k(\eta \backslash x). \end{aligned}$$

Denote by

$$L_V^{\Delta}(t,k)k(\eta) = -\int_X h(t,x)k(\eta\cup x)\sigma(\mathrm{d}x) - \frac{1}{2}\int_X\int_X \psi(t,x,y)k(\eta\cup x\cup y)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)$$
$$+\sum_{x\in\eta} a(t,x)k(\eta\backslash x) + B^{\Delta}(t,k)k(\eta)$$

the pointwise limit when $\varepsilon \to 0$. An application of Theorem 1.3.10 yields

Theorem 3.4.15. *There exist $\lambda > 0$ and for $\varepsilon \in (0,1]$ unique classical $\mathcal{K}$-valued solutions $k_{t,\varepsilon}$ and r_t to*

$$\frac{\partial k_{t,\varepsilon}}{\partial t} = L_{\varepsilon,ren}^{\Delta}(t,k_{t,\varepsilon})k_{t,\varepsilon}, \quad k_{t,\varepsilon}|_{t=0} = k_0 \in \mathcal{K}_{\alpha_*-\varepsilon}$$

and

$$\frac{\partial r_t}{\partial t} = L_V^{\Delta}(t,r_t)r_t, \quad r_t|_{t=0} = r_0 \in \mathcal{K}_{\alpha_*-\varepsilon}. \tag{3.46}$$

Moreover, for any $\alpha \in (\alpha_*, \alpha^*]$ and $T \in (0, \frac{\alpha-\alpha_*}{\lambda})$

$$k_{t,\varepsilon} \to r_t, \quad \varepsilon \to 0$$

in $\mathcal{K}_\alpha$ uniformly on $[0,T]$.

Take $k_0(\eta) = \prod_{x\in\eta}\rho_0(x)$, $\rho_0 \in L^\infty(X,\sigma)$, let $\rho_t(x) = \rho_0(t) + \int_0^t a(s,x)\mathrm{d}s$, then $r_t(\eta) := \prod_{x\in\eta}\rho_t(x)$ is the unique solution to (3.46).

Chapter 4

Markov evolutions on Γ^2

In this chapter we present and prove the main results for two-component Markov birth-and-death evolutions. Examples from mathematical biology are presented in the last section.

4.1 Preliminaries

4.1.1 Harmonic analysis on Γ^2

For two-component systems the state space is defined as the direct product of two copies of Γ

$$\Gamma^2 := \{(\gamma^+, \gamma^-) \in \Gamma \times \Gamma \mid \gamma^+ \cap \gamma^- = \varnothing\},$$

, cf. [FKO13]. For simplicity of notation we write $\gamma := (\gamma^+, \gamma^-)$ and if necessary write x instead of $\{x\}$, hence $\gamma^\pm \backslash x, \gamma^\pm \cup x$ are well-defined set-operations. Likewise we use for $\eta \in \Gamma_0^2$ the notation $\eta \subset \gamma$ and $\gamma \backslash \eta$ by which we mean that $\eta^+ \subset \gamma^+, \eta^- \subset \gamma^-$ and $\gamma^+ \backslash \eta^+$, $\gamma^- \backslash \eta^-$. The restriction of the product topology on $\Gamma \times \Gamma$ topologizes Γ^2 in such a way that it becomes a Polish space. Γ^2 equipped with this topology becomes a Polish space. The Poisson measure $\pi_{\alpha,\beta}$ is defined for $\alpha, \beta \in \mathbb{R}$ as the unique measure having the Laplace transform

$$\int\limits_{\Gamma^2} e^{\sum\limits_{x \in \gamma^+} f(x)} e^{\sum\limits_{x \in \gamma^-} g(x)} \mathrm{d}\pi_{\alpha,\beta}(\gamma) = \exp\left(e^{\alpha} \int\limits_{\mathbb{R}^d} (e^{f(x)} - 1) \mathrm{d}x \right) \exp\left(e^{\beta} \int\limits_{\mathbb{R}^d} (e^{g(x)} - 1) \mathrm{d}x \right),$$

where $f, g : \mathbb{R}^d \longrightarrow \mathbb{R}$ are continuous with compact support. Hence it is simply the restriction of $\pi_\alpha \otimes \pi_\beta$ to Γ^2. Notions of cylinder sets, local absolute continuity w.r.t. the Poisson measure and finite local moments are adapted to Γ^2 in the obvious way. For

$G \in B_{bs}(\Gamma_0^2)$ define the K-transform by

$$(\mathbb{K}G)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta), \tag{4.1}$$

where $\eta \Subset \gamma$ means that the sum only runs over all finite subsets η of γ. Then $\mathbb{K}G$ is a polynomially bounded cylinder function, i.e. there exists a compact $\Lambda \subset \mathbb{R}^d$ and constants $C > 0,\ N \in \mathbb{N}$ such that $(\mathbb{K}G)(\gamma^+, \gamma^-) = (\mathbb{K}G)(\gamma^+ \cap \Lambda, \gamma^- \cap \Lambda)$ and

$$|(\mathbb{K}G)(\gamma)| \leq C(1 + |\gamma^+ \cap \Lambda| + |\gamma^- \cap \Lambda|)^N, \quad \gamma \in \Gamma^2$$

holds. The K-transform $\mathbb{K} : B_{bs}(\Gamma_0^2) \longrightarrow \mathcal{FP}(\Gamma^2) := K(B_{bs}(\Gamma_0^2))$ is a positivity preserving isomorphism with inverse given by

$$(\mathbb{K}^{-1}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \backslash \xi|} F(\xi), \quad \eta \in \Gamma_0^2.$$

Denote by $\mathbb{K}_0$ the restriction of $\mathbb{K}$ determined by evaluating $\mathbb{K}G$ only on Γ_0^2 for $G \in B_{bs}(\Gamma_0^2)$. Its inverse is then denoted by $\mathbb{K}_0^{-1}$. Given a probability measure μ on Γ^2 with finite local moments the correlation function $k_\mu : \Gamma_0^2 \longrightarrow \mathbb{R}_+$ for μ is defined by the relation

$$\int_{\Gamma^2} \mathbb{K}G(\gamma) \mathrm{d}\mu(\gamma) = \int_{\Gamma_0^2} G(\eta) k_\mu(\eta) \mathrm{d}\lambda(\eta), \tag{4.2}$$

provided it exists. In such a case k_μ is locally integrable and μ is locally absolutely continuous w.r.t. the Poisson measure. Conversely, let k_μ be a locally integrable correlation function associated to a probability measure μ. Then μ has finite local moments and is locally absolutely continuous w.r.t. the Poisson measure. In such a case the K-transform can be uniquely extended to a bounded linear operator $\mathbb{K} : L^1(\Gamma_0^2, k_\mu \mathrm{d}\lambda) \longrightarrow L^1(\Gamma^2, \mathrm{d}\mu)$ such that $\|\mathbb{K}G\|_{L^1(\Gamma^2, \mathrm{d}\mu)} \leq \|G\|_{L^1(\Gamma_0^2, k_\mu \mathrm{d}\lambda)}$ and (4.1) holds for μ-a.a. $\gamma \in \Gamma^2$. Let $\mathcal{L}_{k_\mu} := L^1(\Gamma_0^2, k_\mu \mathrm{d}\lambda)$ and for $k_\mu(\eta) := e^{\alpha|\eta^+|} e^{\beta|\eta^-|}$ we also write $\mathcal{L}_{k_\mu} \equiv \mathcal{L}_{\alpha,\beta}$. The next statement shows a one-to-one correspondence between certain classes of probability measures on Γ^2 and correlation functions.

Theorem 4.1.1. *The following assertions are satisfied.*

1. *Let μ be a probability measure on Γ^2 having finite local moments and correlation function k_μ. Then $k_\mu(\emptyset) = 1$ and k_μ is positive definite, i.e. for any $G \in B_{bs}(\Gamma_0^2)$ with $\mathbb{K}G \geq 0$:*

$$\int_{\Gamma_0^2} G(\eta) k_\mu(\eta) \mathrm{d}\lambda(\eta) \geq 0$$

holds.

2. Conversely, let $k : \Gamma_0^2 \longrightarrow \mathbb{R}_+$ be locally integrable, positive definite and satisfies $k(\varnothing) = 1$. Suppose there exist $\alpha, \beta \in \mathbb{R}$ and $C(\mu) > 0$ such that for all $n, m \geq 0$

$$k^{(n,m)}(x_1, \ldots, x_n; y_1, \ldots y_m) \leq C(\mu) e^{\alpha n} e^{\beta m}, \quad x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathbb{R}^d \tag{4.3}$$

holds. Then there exists a unique probability measure μ on Γ^2 with k as its correlation function.

For given $\alpha, \beta \in \mathbb{R}$ let $\mathcal{P}_{\alpha,\beta}$ be the space of all probability measures μ such that for each μ there exists an associated correlation function k_μ and this function satisfies for some constant $C(\mu) > 0$

$$k_\mu(\eta) \leq C(\mu) e^{\alpha|\eta^+|} e^{\beta|\eta^-|}, \quad \eta \in \Gamma_0^2,$$

see (4.3). Let $\mathcal{K}_{\alpha,\beta}$ stand for the Banach space of all equivalence classes of functions $k : \Gamma_0^2 \longrightarrow \mathbb{R}$ equipped with norm

$$\|k\|_{\mathcal{K}_{\alpha,\beta}} = \operatorname*{ess\,sup}_{\eta \in \Gamma_0^2} |k(\eta)| e^{-\alpha|\eta^+|} e^{-\beta|\eta^-|}.$$

Working with the measure $\mu \in \mathcal{P}_{\alpha,\beta}$ it is often important to apply Fubini's theorem which yields for any $G \in \mathcal{L}_{\alpha,\beta}$

$$\int_{\Gamma^2} \mathbb{K}G(\gamma) \mathrm{d}\mu(\gamma) = \int_\Gamma \int_\Gamma \mathbb{K}G(\gamma^+, \gamma^-) \mathrm{d}\mu(\gamma^+, \gamma^-).$$

4.1.2 Markov dynamics on Γ^2

Let L be a a Markov (pre-)generator on Γ^2, the precise form will be given in the next section. The corresponding Markov process can be constructed by solving the (backward) Kolmogorov equation on observables $F \in \mathcal{FP}(\Gamma^2)$

$$\frac{\partial F_t}{\partial t} = LF_t, \quad F_t|_{t=0} = F_0. \tag{4.4}$$

Formally it is the same as investigating solutions to the forward Kolmogorov equation (Fokker-Planck equation)

$$\frac{\partial}{\partial t} \int_{\Gamma^2} F(\gamma) \mathrm{d}\mu_t(\gamma) = \int_{\Gamma^2} (LF)(\gamma) \mathrm{d}\mu_t(\gamma), \quad \mu_t|_{t=0} = \mu_0. \tag{4.5}$$

Here $(\mu_t)_{t \geq 0}$ is a flow of Borel probability measures on Γ^2. As in the one-component case it is possible and, indeed, it was proposed in [FKO13] to study above Cauchy problems in terms of the evolution of quasi-observables and correlation functions. Define for this purpose the operators $\widehat{L} := \mathbb{K}^{-1}L\mathbb{K}$ and L^{Δ} by the relation

$$\int_{\Gamma^2} \widehat{L}G(\eta)k(\eta)\mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} G(\eta)L^{\Delta}k(\eta)\mathrm{d}\lambda(\eta), \quad G \in B_{bs}(\Gamma_0^2). \tag{4.6}$$

In such a case we study the Cauchy problem for quasi-observables

$$\frac{\partial}{\partial t}G_t = \widehat{L}G_t, \quad G_t|_{t=0} = G_0 \tag{4.7}$$

and for correlation functions

$$\frac{\partial}{\partial t}k_t = L^{\Delta}k_t, \quad k_t|_{t=0} = k_0. \tag{4.8}$$

Solutions to (4.4) are formally related to (4.7) by the relation $F_t = \mathbb{K}G_t$ and we expect that solutions to the Cauchy problem (4.8) are positive definite and hence determine uniquely a family of probability measures $(\mu_t)_{t\geq 0}$ such that k_t is the correlation function for μ_t. As a result, $(\mu_t)_{t\geq 0}$ should be a solution to (4.5).

4.1.3 General description of Vlasov scaling

Let us briefly fix the notation for Vlasov scaling in the two-component case. Let L be a Markov (pre-)generator on Γ^2, the aim is to find a scaling L_n such that the following scheme holds. Let $T_n^{\Delta}(t) = e^{tL_n^{\Delta}}$ be the (heuristic) representation of the scaled evolution of correlation functions, see (4.8). The particular choice of $L \to L_n$ should preserve the order of singularity, that is the limit

$$n^{-|\eta|}T_n^{\Delta}(t)n^{|\eta|}k \longrightarrow T_V^{\Delta}(t)k, \quad n \to 0 \tag{4.9}$$

should exist and the evolution $T_V^{\Delta}(t)$ should preserve Lebesgue-Poisson exponentials, i.e. if $r_0(\eta) = e_{\lambda}(\rho_0^E, \eta^-)e_{\lambda}(\rho_0^S; \eta^+)$, then $T_V^{\Delta}(t)r_0(\eta) = e_{\lambda}(\rho_t^E, \eta^-)e_{\lambda}(\rho_t^S; \eta^+)$. In such a case ρ_t^E, ρ_t^S satisfy a system of non-linear integro-differential equations

$$\frac{\partial \rho_t^E}{\partial t} = v_E(\rho_t^E, \rho_t^S) \tag{4.10}$$

$$\frac{\partial \rho_t^S}{\partial t} = v_S(\rho_t^E, \rho_t^S). \tag{4.11}$$

The functionals v_E, v_S can be computed explicitly for a large class of models. Instead of investigating the limit (4.9), we define renormalized operators $L_{n,\mathrm{ren}}^{\Delta} := n^{-|\eta|}L_n^{\Delta}n^{|\eta|}$ and study the behaviour of the semigroups $T_{n,\mathrm{ren}}^{\Delta}(t)$ when $n \to \infty$. In such a case one can compute a limiting operator

$$L_{n,\mathrm{ren}}^{\Delta} \longrightarrow L_V^{\Delta} \tag{4.12}$$

and show that L_V^Δ is associated to a semigroup $T_V^\Delta(t)$. The limit (4.9) is then obtained by showing the convergence

$$T_{n,\mathrm{ren}}^\Delta(t) \longrightarrow T_V^\Delta(t) \tag{4.13}$$

in a proper sense.

4.1.4 Description of model

In this chapter we discuss a general two-component model given by the formal Kolmogorov operator

$$(LF)(\gamma) = (L^S F)(\gamma) + (L^E F)(\gamma),$$

where

$$\begin{aligned}(L^E F)(\gamma) = & \sum_{x\in\gamma^-} d^E(x,\gamma^+,\gamma^-\backslash x)(F(\gamma^+,\gamma^-\backslash x) - F(\gamma)) \\ & + \int_{\mathbb{R}^d} b^E(x,\gamma)(F(\gamma^+,\gamma^-\cup x) - F(\gamma))\mathrm{d}x\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}(L^S F)(\gamma) = & \sum_{x\in\gamma^+} d^S(x,\gamma^+\backslash x,\gamma^-)(F(\gamma^+\backslash x,\gamma^-) - F(\gamma)) \\ & + \int_{\mathbb{R}^d} b^S(x,\gamma)(F(\gamma^+\cup x,\gamma^-) - F(\gamma))\mathrm{d}x.\end{aligned} \tag{4.15}$$

The functions d^E, d^S are the so-called death intensities and $b^E, b^S \geq 0$ the birth intensities of the birth-and-death process given by the operator L. All intensities are assumed to be non-negative. As in the one-component case we suppose that above intensities satisfy the condition given below.

(A) There exists a measurable set $\Gamma_\infty \subset \Gamma^2$ such that for all $x \in \mathbb{R}^d$

$$\mathbb{R}^d \times \Gamma_\infty^2 \ni (x,\gamma) \longmapsto d(x,\gamma\backslash x),\ b(x,\gamma) \in [0,\infty) \tag{4.16}$$

are measurable and for any compact $\Lambda \subset \mathbb{R}^d$ and bounded set $M \subset \Gamma_0^2$

$$\int_\Lambda \int_M (d^S(x,\eta) + d^E(x,\eta) + b^S(x,\eta) + b^E(x,\eta))\mathrm{d}\lambda(\eta)\mathrm{d}x < \infty \tag{4.17}$$

holds. Moreover, any measure $\mu \in \mathcal{P}_{\alpha,\beta}$ is supported on Γ_∞^2.

4.2 Evolution of observables

Similar to the one-component case, let $\mathcal{E}_{\alpha,\beta}$ be the completition of $\mathcal{FP}(\Gamma^2)$ w.r.t. to the norm

$$\|F\|_{\mathcal{E}_{\alpha,\beta}} := \|G\|_{\mathcal{L}_{\alpha,\beta}} = \int_{\Gamma_0^2} |G(\eta)| e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta),$$

where $F = \mathbb{K}G \in \mathcal{FP}(\Gamma^2)$. Then $\|F\|_{L^1(\Gamma^2, \mathrm{d}\pi_{\alpha,\beta})} \leq \|F\|_{\mathcal{E}_{\alpha,\beta}}$ and each $F \in \mathcal{E}_{\alpha,\beta}$ is uniquely determined by an element $G \in \mathcal{L}_{\alpha,\beta}$ for which we use the notation $F = \mathbb{K}G$. For any $\alpha' < \alpha$ and $\beta' < \beta$ the dense embedding $\mathcal{L}_{\alpha,\beta} \subset \mathcal{L}_{\alpha',\beta'}$ implies that $\mathcal{E}_{\alpha,\beta}$ is continuously embedded into $\mathcal{E}_{\alpha',\beta'}$. Introduce the cumulative death intensity by

$$M(\eta) := \sum_{x \in \eta^-} d^E(x, \eta^+, \eta^- \backslash x) + \sum_{x \in \eta^+} d^S(x, \eta^+ \backslash x, \eta^-)$$

and set $c(L, \alpha, \beta; \eta) = c(\eta) = c(\alpha, \beta; \eta)$ by

$$\begin{aligned} c(L, \alpha, \beta; \eta) := & \sum_{x \in \eta^-} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1} d^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) \mathrm{d}\lambda(\xi) \\ & + \sum_{x \in \eta^+} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) \mathrm{d}\lambda(\xi) \\ & + e^{-\beta} \sum_{x \in \eta^-} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1} b^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) \mathrm{d}\lambda(\xi) \\ & + e^{-\alpha} \sum_{x \in \eta^+} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1} b^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) \mathrm{d}\lambda(\xi). \end{aligned}$$

Define on $B_{bs}(\Gamma_0^2)$ the operator $\widehat{L} := \mathbb{K}_0^{-1} L \mathbb{K}_0$, see (4.7). Denote by $\mathbb{1}^*$ the function given by $\mathbb{1}^*(\eta) := 0^{|\eta|} = \begin{cases} 1, & |\eta| = 0 \\ 0, & \text{otherwise} \end{cases}$. Using the methods proposed in [FKK12, FKO13] we can compute $\widehat{L}$. It has the form $\widehat{L} = A + B$. The latter operators are well-defined for functions $G \in B_{bs}(\Gamma_0^2)$ and are given by $(AG)(\eta) = -M(\eta)G(\eta)$, where

$$M(\eta) = \sum_{x \in \eta^-} d^E(x, \eta^+, \eta^- \backslash x) + \sum_{x \in \eta^+} d^S(x, \eta^+ \backslash x, \eta^-) \geq 0$$

and by

$$
\begin{aligned}
(BG)(\eta) = &- \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi^-} (\mathbb{K}_0^{-1} d^E(x, \cdot \cup \xi^+, \cdot \cup \xi^- \backslash x))(\eta \backslash \xi) \\
&- \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi^+} (\mathbb{K}_0^{-1} d^S(x, \cdot \cup \xi^+ \backslash x, \cdot \cup \xi^-))(\eta \backslash \xi) \\
&+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+, \xi^- \cup x)(\mathbb{K}_0^{-1} b^E(x, \cdot \cup \xi^+, \cdot \cup \xi^-))(\eta \backslash \xi) \mathrm{d}x \\
&+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-)(\mathbb{K}_0^{-1} b^S(x, \cdot \cup \xi^+, \cdot \cup \xi^-))(\eta \backslash \xi) \mathrm{d}x.
\end{aligned}
$$

Lemma 4.2.1. *Suppose that condition (A) is fulfilled and $c(\alpha, \beta; \eta)$ is locally integrable. Then $(L, \mathcal{FP}(\Gamma^2))$ is a well-defined operator on $\mathcal{E}_{\alpha,\beta}$ and $(\widehat{L}, B_{bs}(\Gamma_0^2))$ is a well-defined operator on $\mathcal{L}_{\alpha,\beta}$.*

Proof. Let $F = \mathbb{K}G \in \mathcal{FP}(\Gamma^2)$, then by

$$
\int_{\Gamma^2} |LF(\gamma)| \mathrm{d}\pi_{\alpha,\beta}(\gamma) \leq \int_{\Gamma^2} \mathbb{K}|\widehat{L}G|(\gamma) \mathrm{d}\pi_{\alpha,\beta}(\gamma) = \int_{\Gamma_0^2} |\widehat{L}G(\eta)| e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta)
$$

it is enough to show that $(\widehat{L}, B_{bs}(\Gamma_0^2))$ is a well-defined operator on $\mathcal{L}_{\alpha,\beta}$. For each $G \in B_{bs}(\Gamma_0^2)$ we have

$$
\begin{aligned}
&\int_{\Gamma_0^2} |G(\eta)| M(\eta) \mathrm{d}\lambda(\eta) \\
&= \int_{\mathbb{R}^d} \int_{\Gamma_0^2} |G(\eta^+ \cup x, \eta^-)| d^S(x, \eta) \mathrm{d}\lambda(\eta) \mathrm{d}x + \int_{\mathbb{R}^d} \int_{\Gamma_0^2} |G(\eta^+, \eta^- \cup x)| d^E(x, \eta) \mathrm{d}\lambda(\eta) \mathrm{d}x
\end{aligned}
$$

and in view of (4.17) the latter expression is finite. It remains to show that $(B, B_{bs}(\Gamma_0^2))$ is a well-defined operator on $\mathcal{L}_{\alpha,\beta}$. Define a new (positive) operator B' on $B_{bs}(\Gamma_0^2)$ by

$$
\begin{aligned}
(B'G)(\eta^+, \eta^-) := &\sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi^-} |\mathbb{K}_0^{-1} d^E(x, \cdot \cup \xi^+, \cdot \cup \xi^- \backslash x)|(\eta \backslash \xi) \\
&+ \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi^+} |\mathbb{K}_0^{-1} d^S(x, \cdot \cup \xi^+ \backslash x, \cdot \cup \xi^-)|(\eta \backslash \xi) \\
&+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+, \xi^- \cup x)|\mathbb{K}_0^{-1} b^E(x, \cdot \cup \xi^+, \cdot \cup \xi^-)|(\eta \backslash \xi) \mathrm{d}x \\
&+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-)|\mathbb{K}_0^{-1} b^S(x, \cdot \cup \xi^+, \cdot \cup \xi^-)|(\eta \backslash \xi) \mathrm{d}x.
\end{aligned}
$$

Then $|BG| \leq B'|G|$ and by property (2.27) it follows that for any $0 \leq G \in B_{bs}(\Gamma_0^2)$

$$\int_{\Gamma_0^2} B'G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} (c(\alpha,\beta;\eta) - M(\eta))G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) \tag{4.18}$$

holds. The assertion is therefore proved. □

Below we show that the closure of $(L, \mathcal{FP}(\Gamma^2))$ is the generator of a strongly continuous semigroup.

Theorem 4.2.2. *Suppose that (A) is satisfied and assume that $c(\alpha,\beta;\eta)$ is locally integrable. Then the following assertions are equivalent:*

(a) The closure $(L, D(L))$ of $(L, \mathcal{FP}(\Gamma^2))$ is the generator of an analytic semigroup $(T(t))_{t\geq 0}$ of contraction on $\mathcal{E}_{\alpha,\beta}$ such that $T(t)1 = 1$ holds.

(b) The closure $(\widehat{L}, D(\widehat{L}))$ of $(\widehat{L}, B_{bs}(\Gamma_0^2))$ is the generator of an analytic semigroup $(\widehat{T}(t))_{t\geq 0}$ of contractions on $\mathcal{L}_{\alpha,\beta}$ such that $\widehat{T}(t)\mathbb{1}^* = \mathbb{1}^*$ holds.

Proof. It holds that $\mathbb{1}^* \in B_{bs}(\Gamma_0^2)$, $\widehat{L}\mathbb{1}^* = 0$ and since $\mathbb{K}\mathbb{1}^* = 1$ also $1 \in \mathcal{FP}(\Gamma^2)$ and $L1 = 0$ hold.
$(b) \Rightarrow (a)$: Define on $\mathcal{E}_{\alpha,\beta}$ a family of operators $(T(t))_{t\geq 0}$ by the relation

$$T(t)\mathbb{K}G = \mathbb{K}\widehat{T}(t)G, \quad \mathbb{K}G \in \mathcal{E}_{\alpha,\beta} \tag{4.19}$$

and hence

$$\|T(t)\mathbb{K}G\|_{\mathcal{E}_{\alpha,\beta}} = \|\widehat{T}(t)G\|_{\mathcal{L}_{\alpha,\beta}} \leq \|G\|_{\mathcal{L}_{\alpha,\beta}}.$$

The strong continuity follows from

$$\|T(t)\mathbb{K}G - \mathbb{K}G\|_{\mathcal{E}_{\alpha,\beta}} = \|\widehat{T}(t)G - G\|_{\mathcal{L}_{\alpha,\beta}}$$

and since $(\widehat{T}(t))_{t\geq 0}$ satisfies the semigroup property, so does $(T(t))_{t\geq 0}$. Hence $T(t)$ is a C_0-semigroup on $\mathcal{E}_{\alpha,\beta}$. For a given pair of functions $\mathbb{K}G, \mathbb{K}h \in \mathcal{E}_{\alpha,\beta}$

$$\frac{T(t)\mathbb{K}G - \mathbb{K}G}{t} \longrightarrow \mathbb{K}h, \quad t \to 0$$

holds in $\mathcal{E}_{\alpha,\beta}$ if and only if

$$\frac{\widehat{T}(t)G - G}{t} \longrightarrow h, \quad t \to 0$$

holds in $\mathcal{L}_{\alpha,\beta}$. This is possible if and only if $G \in D(\widehat{L})$ and $\widehat{L}G = h$. Therefore the generator of $(T(t))_{t\geq 0}$ is given by $L\mathbb{K}G = \mathbb{K}\widehat{L}G$ and

$$D(L) = \mathbb{K}D(\widehat{L}) = \{\mathbb{K}G \in \mathcal{E}_{\alpha,\beta} \mid G \in D(\widehat{L})\}.$$

To show that $\mathcal{FP}(\Gamma^2) \subset \mathcal{E}_{\alpha,\beta}$ is a core it suffices to show that closure of $\mathcal{FP}(\Gamma^2)$ in $\mathcal{E}_{\alpha,\beta}$ with respect to the graph norm

$$\|\mathbb{K}G\|_L = \|\mathbb{K}G\|_{\mathcal{E}_{\alpha,\beta}} + \|L\mathbb{K}G\|_{\mathcal{E}_{\alpha,\beta}}$$

coincides with $D(L)$. So let $\mathbb{K}G \in D(L)$, since $B_{bs}(\Gamma_0^2)$ is a core for $D(\widehat{L})$ there exists a sequence $(G_n)_{n\in\mathbb{N}} \subset B_{bs}(\Gamma_0^2)$ such that $G_n \longrightarrow G$ and $\widehat{L}G_n \longrightarrow \widehat{L}G$. By definition of the norm in $\mathcal{E}_{\alpha,\beta}$ this implies $\mathbb{K}G_n \longrightarrow \mathbb{K}G$ and $L\mathbb{K}G_n \longrightarrow L\mathbb{K}G$ in $\mathcal{E}_{\alpha,\beta}$ and hence with respect to the graph norm $\|\cdot\|_L$. The resolvent $R(\lambda; L)$ for L is given by

$$R(\lambda; L)\mathbb{K}G = \int_0^\infty e^{-\lambda t} T(t)\mathbb{K}G \mathrm{d}t = \int_0^\infty e^{-\lambda t} \mathbb{K}\widehat{T}(t)G \mathrm{d}t = \mathbb{K}R(\lambda; \widehat{L})G$$

and hence $(T(t))_{t\geq 0}$ is analytic.
(a) $\Rightarrow$ *(b)* : Since $T(t)\mathbb{K}G \in \mathcal{E}_{\alpha,\beta}$ for all $G \in \mathcal{L}_{\alpha,\beta}$ there exists a linear operator $\widehat{T}(t)$ on $\mathcal{L}_{\alpha,\beta}$ such that $\mathbb{K}\widehat{T}(t)G = T(t)\mathbb{K}G$, $G \in \mathcal{L}_{\alpha,\beta}$, $t \geq 0$. The definition of the norm in $\mathcal{E}_{\alpha,\beta}$ and the same arguments as above imply the assertion. □

The next Proposition provides existence and uniqueness of solutions to the Kolmogorov equation (4.4).

Proposition 4.2.3. *Suppose that (A) is satisfied and assume that there exists $\beta \in \mathbb{R}$ and a constant $a = a(\alpha, \beta) \in (0, 2)$ such that*

$$c(\alpha, \beta; \eta) \leq a(\alpha, \beta)M(\eta), \quad \eta \in \Gamma_0^2 \tag{4.20}$$

holds. Then the following assertions are true:

(a) Condition (b) and therefore (a) of Theorem 4.2.2 are satisfied.

(b) Suppose that there exist $\alpha_* < \alpha^*$ and $\beta_* < \beta^*$ with $\alpha \in (\alpha_*, \alpha^*)$ and $\beta \in (\beta_*, \beta^*)$ such that for all $\alpha' \in (\alpha_*, \alpha^*)$ and $\beta' \in (\beta_*, \beta^*)$ condition (4.20) *is satisfied. Denote by $T_{\alpha',\beta'}(t)$ the associated semigroup on $\mathcal{E}_{\alpha',\beta'}$, then for any $\alpha > \alpha'$ and $\beta > \beta'$ the space $\mathcal{E}_{\alpha,\beta}$ is invariant for $T_{\alpha',\beta'}(t)$ and $T(t) = T_{\alpha',\beta'}(t)|_{\mathcal{E}_{\alpha,\beta}}$ holds.*

Proof. (a) Set $D(\widehat{L}) := \{G \in \mathcal{L}_{\alpha,\beta} \mid M \cdot G \in \mathcal{L}_{\alpha,\beta}\}$. Then, since $M \geq 0$, the operator $(A, D(\widehat{L}))$ is the generator of an analytic (of angle $\frac{\pi}{2}$), positive C_0-semigroup $(e^{-tM})_{t\geq 0}$ on $\mathcal{L}_{\alpha,\beta}$, see [EN00]. Let B' be given as in the proof of Lemma 4.2.1. Then, since $|BG| \leq B'|G|$, it is enough to show that $(A + B', D(\widehat{L}))$ is resolvent positive, cf. [AR91, Theorem 1.1]. To this end we show that $(A + B', D(\widehat{L}))$ is the generator of a positive semigroup. We will prove afterwards that $(\widehat{L}, D(\widehat{L}))$ is the closure of $(\widehat{L}, B_{bs}(\Gamma_0))$.

So fix $r \in (0,1)$, cf. (4.20), such that

$$a(\alpha,\beta) < 1 + r < 2.$$

For each $0 \leq G \in D(\widehat{L})$, see (4.18), we obtain

$$\int_{\Gamma_0^2} B'G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} (c(\alpha,\beta;\eta) - M(\eta))G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta)$$
$$\leq (a(\alpha,\beta)-1)\int_{\Gamma_0^2} M(\eta)G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) \leq r\int_{\Gamma_0^2} M(\eta)G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta)$$

and hence

$$\int_{\Gamma_0^2} \left(A + \frac{1}{r}B'\right) G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) \leq 0$$

holds. Therefore by [TV06, Theorem 2.2] the operator $(A + B', D(\widehat{L}))$ is the generator of a sub-stochastic semigroup $(U(s))_{s\geq 0}$ and by [AR91, Theorem 1.1, Theorem 1.2] also $(A + B, D(\widehat{L})) = (\widehat{L}, D(\widehat{L}))$ is the generator of an analytic semigroup $(\widehat{T}(s))_{s\geq 0}$ such that

$$|\widehat{T}(s)G| \leq U(s)|G|, \quad G \in \mathcal{L}_{\alpha,\beta}.$$

Since $U(s)$ is a contraction operator, so is $\widehat{T}(s)$. The next lemma completes the proof of assertion (a).

Lemma 4.2.4. *$B_{bs}(\Gamma_0^2)$ is a core for the generator $(\widehat{L}, D(\widehat{L}))$ on $\mathcal{L}_{\alpha,\beta}$.*

Proof. Let $G \in D(\widehat{L})$, $A_n \subset \Gamma_0^2$ an increasing sequence of bounded sets with $\bigcup_{n\geq 1} A_n = \Gamma_0^2$ and let $G_n(\eta) := \mathbb{1}_{A_n}(\eta)\mathbb{1}_{|G|\leq n}(\eta)G(\eta)$. Then $|G_n| \leq |G|$, $G_n \longrightarrow G$ a.e. and by dominated convergence also $\widehat{L}G_n \longrightarrow \widehat{L}G$ as $n \to \infty$ almost everywhere. Moreover, by

$$|\widehat{L}G_n| \leq M|G_n| + B'|G_n| \leq (M + B')|G| \in L^1(\Gamma_0^2, \mathrm{d}\lambda)$$

and dominated convergence we obtain $\widehat{L}G_n \longrightarrow \widehat{L}G$ in $\mathcal{L}_{\alpha,\beta}$. Therefore $B_{bs}(\Gamma_0^2) \subset D(\widehat{L})$ is dense in the graph norm. □

(b) Let $\alpha' < \alpha$ and $\beta' < \beta$ such that (4.20) also holds for (α',β'). Denote by $(\widehat{T}_{\alpha',\beta'}(s))_{s\geq 0}$ the corresponding semigroup on $\mathcal{L}_{\alpha',\beta'}$ constructed in (a). Let $(\widehat{L}, D_{\alpha',\beta'}(\widehat{L}))$ be the generator of $\widehat{T}_{\alpha',\beta'}(t)$. By previous construction it is simply given by the action of the operator $\widehat{L}$ on the domain

$$D_{\alpha',\beta'}(\widehat{L}) := \{G \in \mathcal{L}_{\alpha',\beta'} \mid M \cdot G \in \mathcal{L}_{\alpha',\beta'}\}.$$

We have to show that $\mathcal{L}_{\alpha,\beta}$ is invariant for $\widehat{T}_{\alpha',\beta'}(s)$ and

$$\widehat{T}(s)G = \widehat{T}_{\alpha',\beta'}(s)G, \quad G \in \mathcal{L}_{\alpha,\beta}, \; s \geq 0. \tag{4.21}$$

To this end we define a linear isomorphism

$$S : \mathcal{L}_{\alpha,\beta} \longrightarrow \mathcal{L}_{\alpha',\beta'}, \quad (SG)(\eta) = e^{(\alpha-\alpha')|\eta^+|}e^{(\beta-\beta')|\eta^-|}G(\eta)$$

with inverse S^{-1} given by

$$(S^{-1}G)(\eta) = e^{-(\alpha-\alpha')|\eta^+|}e^{-(\beta-\beta')|\eta|}G(\eta).$$

Define on $\mathcal{L}_{\alpha',\beta'}$ a new operator by $\widehat{L}_1 := S\widehat{L}S^{-1}$ equipped with the domain

$$D_{\alpha',\beta'}(\widehat{L}_1) = \{G \in \mathcal{L}_{\alpha',\beta'} \mid S^{-1}G \in D(\widehat{L})\} = \{G \in \mathcal{L}_{\alpha',\beta'} \mid MS^{-1}G \in \mathcal{L}_{\alpha,\beta}\}.$$

Since

$$\|MS^{-1}G\|_{\mathcal{L}_{\alpha,\beta}} = \int\limits_{\Gamma_0^2} e^{-(\alpha-\alpha')|\eta^+|}e^{-(\beta-\beta')|\eta|}M(\eta)|G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) = \|MG\|_{\mathcal{L}_{\alpha',\beta'}}$$

we obtain $D_{\alpha',\beta'}(\widehat{L}_1) = \{G \in \mathcal{L}_{\alpha',\beta'} \mid M \cdot G \in \mathcal{L}_{\alpha',\beta'}\} = D_{\alpha',\beta'}(\widehat{L})$. Let us show that $(\widehat{L}_1, D_{\alpha',\beta'}(\widehat{L}_1))$ is the generator of a C_0-semigroup on $\mathcal{L}_{\alpha',\beta'}$. The definition of S and S^{-1} implies $\widehat{L}_1 = A + B_1$ where A is the same as for $\widehat{L}$ and B_1 is given by

$$\begin{aligned}
&(B_1G)(\eta) = \\
&- \sum_{\xi \subsetneq \eta} G(\xi)e^{(\alpha-\alpha')|\eta^+\backslash\xi^+|}e^{(\beta-\beta')|\eta^-\backslash\xi^-|}\sum_{x\in\xi^-}(\mathbb{K}_0^{-1}d^E(x, \cdot \cup \xi^+, \cdot \cup \xi^-\backslash x))(\eta\backslash\xi) \\
&- \sum_{\xi \subsetneq \eta} G(\xi)e^{(\alpha-\alpha')|\eta^+\backslash\xi^+|}e^{(\beta-\beta')|\eta^-\backslash\xi^-|}\sum_{x\in\xi^+}(\mathbb{K}_0^{-1}d^S(x, \cdot \cup \xi^+\backslash x, \cdot \cup \xi^-))(\eta\backslash\xi) \\
&+ e^{-(\beta-\beta')}\sum_{\xi\subset\eta}\int\limits_{\mathbb{R}^d} G(\xi^+, \xi^- \cup x)e^{(\alpha-\alpha')|\eta^+\backslash\xi^+|}e^{(\beta-\beta')|\eta^-\backslash\xi^-|}(\mathbb{K}_0^{-1}b^E(x, \cdot \cup \xi))(\eta\backslash\xi)\mathrm{d}x \\
&+ e^{-(\alpha-\alpha')}\sum_{\xi\subset\eta}\int\limits_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-)e^{(\alpha-\alpha')|\eta^+\backslash\xi^+|}e^{(\beta-\beta')|\eta^-\backslash\xi^-|}(\mathbb{K}_0^{-1}b^S(x, \cdot \cup \xi))(\eta\backslash\xi)\mathrm{d}x.
\end{aligned}$$

Define analogously to B' the positive operator B_1' such that $|B_1G| \leq B_1'|G|$, then for any non-negative function $G \in D_{\alpha',\beta'}(\widehat{L}_1)$ we obtain

$$\int\limits_{\Gamma_0^2} B_1'G(\eta)e^{\alpha'|\eta^+|}e^{\beta'|\eta^-|}\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0^2}(c(\alpha,\beta;\eta) - M(\eta))G(\eta)e^{\alpha'|\eta^+|}e^{\beta'|\eta^-|}\mathrm{d}\lambda(\eta).$$

The same arguments as for the construction of $\widehat{T}(t)$ show that $(A + B_1', D_{\alpha',\beta'}(\widehat{L}_1))$ is the generator of a sub-stochastic semigroup and hence $(\widehat{L}_1, D_{\alpha',\beta'}(\widehat{L}_1))$ is the generator of a C_0-semigroup. Now [Paz83, Chapter 4, Theorem 5.5, Theorem 5.8] implies that $\mathcal{L}_{\alpha,\beta}$ is invariant for $\widehat{T}_{\alpha',\beta'}(t)$ and the restriction to $\mathcal{L}_{\alpha,\beta}$ is a C_0-semigroup given by

$$\widetilde{T}_{\alpha,\beta}(t) := \widehat{T}_{\alpha',\beta'}(t)|_{\mathcal{L}_{\alpha,\beta}}.$$

The generator of $\widetilde{T}_{\alpha,\beta}(t)$ is given by the part of $(\widehat{L}, D_{\alpha',\beta'}(\widehat{L}))$ in $\mathcal{L}_{\alpha,\beta}$, that is by

$$\begin{aligned} D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}} &:= \{G \in D_{\alpha',\beta'}(\widehat{L}) \cap \mathcal{L}_{\alpha,\beta} \mid \widehat{L}G \in \mathcal{L}_{\alpha,\beta}\} \\ &= \{G \in \mathcal{L}_{\alpha,\beta} \mid M \cdot G \in \mathcal{L}_{\alpha',\beta'},\ \widehat{L}G \in \mathcal{L}_{\alpha,\beta}\}. \end{aligned}$$

Condition (4.20) therefore implies $D(\widehat{L}) \subset D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}}$ and hence $(\widehat{L}, D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}})$ is an extension of $(\widehat{L}, D(\widehat{L}))$. Denote by $R(\lambda;\widehat{L})$ the resolvent for $(\widehat{L}, D(\widehat{L}))$ and by $\widetilde{R}(\lambda;\widehat{L})$ the resolvent for $(\widehat{L}, D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}})$. For sufficiently large $\lambda > 0$ it follows that $R(\lambda,\widehat{L})G \in D(\widehat{L}) \subset D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}}$ for any $G \in \mathcal{L}_{\alpha,\beta}$ and thus

$$\widetilde{R}(\lambda;\widehat{L})G - R(\lambda;\widehat{L})G = \widetilde{R}(\lambda;\widehat{L})((\lambda - \widehat{L}) - (\lambda - \widehat{L}))R(\lambda;\widehat{L})G = 0,$$

where we have used that for elements in $D(\widehat{L})$ the action of the generators is given by the formulas for $\widehat{L} = A + B$ and hence coincide. □

For one-component models, i.e. $b^E = 0 = d^E$, a similar construction was already done in [FKK12]. The main assumption was that each term in $c(\alpha,\beta;\eta)$ is bounded by $\frac{3}{2}M(\eta)$ and it was not clear whether $\widehat{T}(t)$ is a contraction operator for $t \geq 0$. The next example shows that the constant 2 in (4.20) is optimal.

Theorem 4.2.5. *The constant* 2 *in condition* (4.20) *is optimal in the sense that it cannot be increased.*

Proof. It suffices to find a model with $a(\alpha,\beta) > 2$ and show that the Cauchy problem (4.7) does not admit a solution in $\mathcal{L}_{\alpha,\beta}$. Take $d^E = 1$, $b^E = z > 0$ constant and $b^S = d^S = 0$, then condition (4.20) can be restated to $z < e^{\beta}$ and $\alpha \in \mathbb{R}$ is arbitrary. The evolution equation (4.7) is in this case exactly solvable and hence has for every initial condition G_0 the solution $(G_t)_{t\geq 0}$ given by

$$G_t(\eta) = e^{-t|\eta^-|} \int_{\Gamma_0} G(\eta^+, \eta^- \cup \xi^-) e_\lambda\left(z(1 - e^{-t}); \xi^-\right) \mathrm{d}\lambda(\xi^-), \quad \eta \in \Gamma_0^2,$$

see [Fin11] for the one-component case. If condition (4.20) is satisfied, then $G_t \in \mathcal{L}_{\alpha,\beta}$. Suppose that $a(\alpha,\beta) > 2$, i.e. $z > e^{\beta}$ and let $t_0 > 0$ such that $(1 - e^{-t})z > e^{\beta}$ for all $t \geq t_0$ and hence

$$z(1 - e^{-t})(e^{-\beta} + e^{-t}) \geq z(1 - e^{-t})e^{-\beta} > 1.$$

Take $0 \le G \in \mathcal{L}_{\alpha,\beta}$ such that $G \notin \mathcal{L}_{\alpha,\beta'}$ for any $\beta' > \beta$. The unique solution G_t is then positive and satisfies

$$\begin{aligned}\|G_t\|_{\mathcal{L}_{\alpha,\beta}} &= \int_{\Gamma_0^2}\int_{\Gamma_0} e^{-t|\eta^-|}G(\eta^+,\eta^-\cup\xi^-)e_\lambda(z(1-e^{-t});\xi^-)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\xi^-)\mathrm{d}\lambda(\eta)\\ &= \int_{\Gamma_0^2} e^{\alpha|\eta^+|}G(\eta^+,\xi^-)e_\lambda(z(1-e^{-t});\xi^-)\sum_{\eta^-\subset\xi^-} e^{-t|\eta^-|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta^+,\xi^-)\\ &= \int_{\Gamma_0^2} G(\eta^+,\xi^-)\left((e^{-\beta}+e^{-t})z(1-e^{-t})\right)^{|\xi^-|}e^{\alpha|\eta^+|}e^{\beta|\xi^-|}\mathrm{d}\lambda(\eta^+,\xi^-) = \infty.\end{aligned}$$

□

Let $d_n^S, d^S, d_n^E, d^E, b_n^S, b^S, b_n^E, b^E \in [0,\infty]$ be birth-and-death intensities which satisfy condition (A). As in the one-component case, introduce $c_n(\alpha,\beta;\eta)$ by

$$\begin{aligned}&\sum_{x\in\eta^-}\int_{\Gamma_0^2} e^{\alpha|\xi^+|}e^{\beta|\xi^-|}|\mathbb{K}_0^{-1}d^E(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x) - \mathbb{K}_0^{-1}d_n^E(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi)\mathrm{d}\lambda(\xi)\\ &+\sum_{x\in\eta^+}\int_{\Gamma_0^2} e^{\alpha|\xi^+|}e^{\beta|\xi^-|}|\mathbb{K}_0^{-1}d^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-) - \mathbb{K}_0^{-1}d_n^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi)\mathrm{d}\lambda(\xi)\\ &+e^{-\beta}\sum_{x\in\eta^-}\int_{\Gamma_0^2} e^{\alpha|\xi^+|}e^{\beta|\xi^-|}|\mathbb{K}_0^{-1}b^E(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x) - \mathbb{K}_0^{-1}b_n^E(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi)\mathrm{d}\lambda(\xi)\\ &+e^{-\alpha}\sum_{x\in\eta^+}\int_{\Gamma_0^2} e^{\alpha|\xi^+|}e^{\beta|\xi^-|}|\mathbb{K}_0^{-1}b^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-) - \mathbb{K}_0^{-1}b_n^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi)\mathrm{d}\lambda(\xi)\end{aligned}$$

and $M_n(\eta) = \sum\limits_{x\in\eta^-} d_n^E(x,\eta^+,\eta^-\backslash x) + \sum\limits_{x\in\eta^+} d_n^S(x,\eta^+\backslash x,\eta^-)$. Denote by L_n the Kolmogorov operators associated to the intensities $d_n^S, d_n^E, b_n^S, b_n^E$. The next statement implies the continuous dependence of the constructed semigroups $T_n(t)$ w.r.t. above intensities.

Theorem 4.2.6. *Suppose that the following conditions are satisfied:*

1. *There exist $\alpha,\beta\in\mathbb{R}$ and a constant $a(\alpha,\beta)\in(0,2)$ such that*

$$c(L_n,\alpha,\beta;\eta) \le a(\alpha,\beta)M_n(\eta), \quad \eta\in\Gamma_0^2,\ n\ge 1$$

   holds.

2. There exist constants $A > 0$, $N \in \mathbb{N}$ and $\tau \geq 0$ such that

$$d_n^E(x,\eta) + d_n^S(x,\eta) \leq A(1+|\eta|)^N e^{\tau|\eta|}, \quad \eta \in \Gamma_0^2,\ x \in \mathbb{R}^d$$

holds.

3. $c_n(\alpha,\beta;\eta) \longrightarrow 0$, $n \to \infty$ holds for all $\eta \in \Gamma_0^2$.

Then (4.20) is satisfied. Let $T(t), T_n(t)$ be the semigroups on $\mathcal{E}_{\alpha,\beta}$ associated to L and L_n respectively. Then for any $F \in \mathcal{E}_{\alpha,\beta}$

$$T_n(t)F \longrightarrow T(t)F, \quad n \to \infty$$

holds uniformly on compacts in $t \geq 0$.

Proof. Since $|c(L_n,\alpha,\beta;\eta) - c(L,\alpha,\beta;\eta)| \leq c_n(\alpha,\beta;\eta) \longrightarrow 0$, $n \to \infty$ and $|M_n(\eta) - M(\eta)| \leq c_n(\alpha,\beta;\eta) \longrightarrow 0$ it follows that

$$c(L,\alpha,\beta;\eta) = \lim_{n\to\infty} c(L_n,\alpha,\beta;\eta) \leq a(\alpha,\beta) \lim_{n\to\infty} M_n(\eta) = a(\alpha,\beta)M(\eta)$$

and hence (4.20) holds. Let $T(t)$ and $T_n(t)$ be the semigroups on $\mathcal{E}_{\alpha,\beta}$ generated by the closure of $(L_n, \mathcal{FP}(\Gamma^2))$ and $(L, \mathcal{FP}(\Gamma^2))$ respectively. By Trotter-Kato approximation it suffices to show $L_n F \longrightarrow LF$ for any $F \in \mathcal{FP}(\Gamma^2)$. This is equivalent to $\widehat{L}_n G \longrightarrow \widehat{L}G$ for any $G \in B_{bs}(\Gamma_0^2)$. Therefore

$$\|\widehat{L}_n G - \widehat{L}G\|_{\mathcal{L}_{\alpha,\beta}} \leq \int\limits_{\Gamma_0^2} c_n(\alpha,\beta;\eta)|G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta)$$

and the integrand tends to zero. Since $c_n(\alpha,\beta;\eta) \leq a(\alpha,\beta)(M(\eta) + M_n(\eta))$ and by $M(\eta) = \lim_{n\to\infty} M_n(\eta) \leq A|\eta|^{N+1}e^{\tau|\eta|}$ it follows that $c_n(\alpha,\beta;\eta) \leq 2a(\alpha,\beta)A|\eta|^{N+1}e^{\tau|\eta|}$ and hence the assertion is satisfied by dominated convergence. □

4.3 Evolution of correlation functions

Suppose that condition (A) and (4.20) are fulfilled. Denote by $\widehat{T}(t)^*$ the adjoint semigroup on $\mathcal{K}_{\alpha,\beta}$ and by $(\widehat{L}^*, D(\widehat{L}^*))$ its generator. This is, by definition, the adjoint operator to $(\widehat{L}, D(\widehat{L}))$, i.e. $\langle \widehat{L}G, k\rangle = \langle G, \widehat{L}^* k\rangle$ for $G \in D(\widehat{L})$ and $k \in D(\widehat{L}^*)$.

Remark 4.3.1. *Let $\alpha' < \alpha$ and $\beta' < \beta$ be such that condition (4.20) holds for α', β' and α, β. Let $(\widehat{T}_{\alpha',\beta'}(s))_{s\geq 0}$ be the analytic semigroup constructed in Theorem 4.2.3. Then by (4.21) for any $G \in \mathcal{L}_{\alpha,\beta} \subset \mathcal{L}_{\alpha',\beta'}$ and $k \in \mathcal{K}_{\alpha',\beta'} \subset \mathcal{K}_{\alpha,\beta}$ we obtain*

$$\langle G, \widehat{T}_{\alpha',\beta'}(t)^* k\rangle = \langle \widehat{T}_{\alpha',\beta'}(t)G, k\rangle = \langle \widehat{T}(t)G, k\rangle = \langle G, \widehat{T}(t)^* k\rangle$$

and hence $\widehat{T}(t)^* k = \widehat{T}_{\alpha',\beta'}(t)^* k$ holds.

Because of the relation (4.6) it is reasonable to consider the linear operator L^{Δ} given by

$$
\begin{aligned}
(L^{\Delta}k)(\eta) = &- \sum_{x\in\eta^-} \int_{\Gamma_0^2} k(\eta \cup \xi)(\mathbb{K}_0^{-1} d^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x))(\xi) \mathrm{d}\lambda(\xi) \\
&- \sum_{x\in\eta^+} \int_{\Gamma_0^2} k(\eta \cup \xi)(\mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-))(\xi) \mathrm{d}\lambda(\xi) \\
&+ \sum_{x\in\eta^-} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+, \eta^- \cup \xi^- \backslash x)(\mathbb{K}_0^{-1} b^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x))(\xi) \mathrm{d}\lambda(\xi) \\
&+ \sum_{x\in\eta^+} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+ \backslash x, \eta^- \cup \xi^-)(\mathbb{K}_0^{-1} b^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-))(\xi) \mathrm{d}\lambda(\xi).
\end{aligned}
$$

We will consider this operator on the maximal domain

$$
D(L^{\Delta}) = \{ k \in \mathcal{K}_{\alpha,\beta} \mid L^{\Delta} k \in \mathcal{K}_{\alpha,\beta} \}.
$$

Below we will need the additional conditions:

(B') There exist constants $A > 0$, $N \in \mathbb{N}$ and $\tau \geq 0$ such that

$$
d^S(x, \eta) + d^E(x, \eta) \leq A(1 + |\eta|)^N e^{\tau|\eta|}, \quad x \in \mathbb{R}^d, \ \eta \in \Gamma_0^2
$$

holds.

(C) There exist $\alpha', \beta' \in \mathbb{R}$ with $\alpha' + \tau < \alpha$, $\beta' + \tau < \beta$ and a constant $a(\alpha', \beta') > 0$ such that the condition below is satisfied

$$
c(\alpha', \beta'; \eta) \leq a(\alpha', \beta') M(\eta), \quad \eta \in \Gamma_0^2.
$$

Lemma 4.3.2. *Suppose that* (4.20) *and (A) are fulfilled, then* $(\widehat{L}^*, D(\widehat{L}^*)) = (L^{\Delta}, D(L^{\Delta}))$. *If in addition conditions (B') and (C) hold, then* L^{Δ} *considered as an operator* $\mathcal{K}_{\alpha',\beta'} \longrightarrow \mathcal{K}_{\alpha,\beta}$ *is bounded. In particular* $\mathcal{K}_{\alpha',\beta'} \subset D(L^{\Delta})$ *holds.*

Proof. It is not difficult to see that for any $G \in D(\widehat{L})$ and $k \in D(\widehat{L}^*)$

$$
\int_{\Gamma_0^2} G(\eta)(\widehat{L}^* k)(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} (\widehat{L} G)(\eta) k(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} G(\eta)(L^{\Delta} k)(\eta) \mathrm{d}\lambda(\eta),
$$

see (2.27). Thus $L^{\Delta}k = \widehat{L}^* k \in \mathcal{K}_{\alpha,\beta}$ and hence $D(\widehat{L}^*) \subset D(L^{\Delta})$. Conversely let $k \in D(L^{\Delta})$, then for any $G \in D(\widehat{L})$ above equality implies $k \in D(\widehat{L}^*)$. For the second assertion observe that for $k \in \mathcal{K}_{\alpha',\beta'}$

$$\begin{aligned}|L^{\Delta}k(\eta)| &\leq \|k\|_{\mathcal{K}_{\alpha',\beta'}} c(\alpha',\beta';\eta) e^{\alpha'|\eta^+|} e^{\beta'|\eta^-|} \\ &\leq \|k\|_{\mathcal{K}_{\alpha',\beta'}} a(\alpha',\beta') M(\eta) e^{-(\alpha-\alpha')|\eta^+|} e^{-(\beta-\beta')|\eta^-|} e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \\ &\leq \|k\|_{\mathcal{K}_{\alpha',\beta'}} a(\alpha',\beta') A|\eta|^{N+1} e^{-(\alpha-\alpha'-\tau)|\eta^+|} e^{-(\beta-\beta'-\tau)|\eta^-|} e^{\alpha|\eta^+|} e^{\beta|\eta^-|},\end{aligned}$$

hence the assertion follows by $(|\eta^+| + |\eta^-|)^{N+1} \leq 2^N (|\eta^+|^{N+1} + |\eta^-|^{N+1})$,

$$x^a e^{-bx} \leq \left(\frac{a}{b}\right)^a e^{-a}, \quad a, x \geq 0, \ b > 0$$

and

$$|\eta|^{N+1} e^{-(\alpha-\alpha'-\tau)|\eta^+|} e^{-(\beta-\beta'-\tau)|\eta^-|} \leq \frac{2^N (N+1)^{N+1} e^{-(N+1)}}{(\alpha-\alpha'-\tau)^{N+1}} + \frac{2^N (N+1)^{N+1} e^{-(N+1)}}{(\beta-\beta'-\tau)^{N+1}}.$$

□

Since $\mathcal{L}_{\alpha,\beta}$ is not reflexive, $\widehat{T}(t)^*$ does not need to be strongly continuous. In fact it is continuous only w.r.t. the topology $\sigma((\mathcal{L}_{\alpha,\beta})^*, \mathcal{K}_{\alpha,\beta}) = \sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$. Here $\sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$ is the smallest topology such that all linear functionals $\mathcal{L}_{\alpha,\beta} \ni G \longmapsto \langle G, k \rangle$ are continuous, where $k \in \mathcal{K}_{\alpha,\beta}$. It is well-known that $\widehat{T}(t)^*$ is strongly continuous on $\mathcal{K}_{\alpha,\beta}^{\odot} = \overline{D(L^{\Delta})}$ and its restriction $\widehat{T}(t)^{\odot} := \widehat{T}(t)^*|_{\mathcal{K}_{\alpha,\beta}^{\odot}}$ is a C_0-semigroup with generator $\widehat{L}^{\odot}k = L^{\Delta}k$,

$$D(\widehat{L}^{\odot}) = \{k \in D(L^{\Delta}) \mid L^{\Delta}k \in \mathcal{K}_{\alpha,\beta}^{\odot}\}.$$

Hence we obtain existence and uniqueness of strong solutions to (4.8) on the Banach space $\mathcal{K}_{\alpha,\beta}^{\odot}$. Unfortunately this space depends on the generator and does not provide uniqueness for the weak solutions. Another possibility is to change the topology on $\mathcal{K}_{\alpha,\beta}$. Let $\mathcal{C}(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta}) =: \mathcal{C}$ be the topology of uniform convergence on compact subsets of $\mathcal{L}_{\alpha,\beta}$. A basis of neighbourhoods around 0 is given by sets of the form

$$\{k \in \mathcal{K}_{\alpha,\beta} \mid \sup_{G \in K} |\langle G, k \rangle| < \varepsilon\},$$

with $\varepsilon > 0$ and compact $K \subset \mathcal{L}_{\alpha,\beta}$, see [WZ02, WZ06, Lem10] and the references therein. The semigroup $(\widehat{T}(t)^*)_{t \geq 0}$ becomes continuous w.r.t. $\mathcal{C}$ and its generator w.r.t. $\mathcal{C}$ is exactly the adjoint operator $(L^{\Delta}, D(L^{\Delta}))$, cf. [WZ06, Theorem 1.4]. The next theorem is our main result for this section, it provides existence, uniqueness and regularity of solutions to the Cauchy problem (4.8) on $\mathcal{K}_{\alpha,\beta}$.

Theorem 4.3.3. *Suppose that* (4.20) *and (A) are satisfied. Then for any* $k_0 \in \mathcal{K}_{\alpha,\beta}$ *the equation*

$$\langle G, k_t \rangle = \langle G, k_0 \rangle + \int_0^t \langle \widehat{L}G, k_s \rangle \mathrm{d}s, \quad G \in D(\widehat{L}) \tag{4.22}$$

has a unique solution given by $k_t = \widehat{T}(t)^* k_0$. *This means that* k_t *is continuous w.r.t. to the topology* $\mathcal{C}$ *and satisfies* (4.22). *Moreover,* $t \longmapsto \langle G, k_t \rangle$ *is continuously differentiable and solves the Cauchy problem*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle G, k_t \rangle = \langle \widehat{L}G, k_t \rangle, \quad k_t|_{t=0} = k_0, \quad G \in D(\widehat{L}). \tag{4.23}$$

Assume that (B') and (C) are fulfilled. Then the following assertions are true:

1. *If* $k_0 \in \mathcal{K}_{\alpha',\beta'}$*, then* k_t *is continuous w.r.t. to the norm in* $\mathcal{K}_{\alpha,\beta}$.
2. *If* $k_0 \in \mathcal{K}_{\alpha',\beta'}$ *and* $\alpha' + 2\tau < \alpha$, $\beta' + 2\tau < \beta$*, then* k_t *is also continuously differentiable w.r.t. to the norm in* $\mathcal{K}_{\alpha,\beta}$ *and the unique classical solution to* (4.8).

Proof. Existence and uniqueness for the Cauchy problem (4.22) follows from [WZ06, Theorem 2.1] and Theorem 4.2.3. A direct proof can be achieved by the arguments provided in the proof of Theorem 1.1.7. Since $\widehat{T}(t)^*$ is continuous w.r.t. $\sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$, $t \longmapsto \langle \widehat{L}G, k_t \rangle$ is continuous and hence (4.22) implies (4.23).
1. If $k_0 \in \mathcal{K}_{\alpha',\beta'}$, then by Lemma 4.3.2 $L^{\Delta} k_0 \in \mathcal{K}_{\alpha,\beta}$ and hence $k_0 \in D(L^{\Delta}) \subset \mathcal{K}_{\alpha,\beta}^{\odot}$ which implies the assertion.
2. Suppose that $\alpha' + 2\tau < \alpha$, $\beta' + 2\tau < \beta$ and let $\alpha'' \in (\alpha', \alpha)$, $\beta'' \in (\beta', \beta)$ be such that $\alpha' + \tau < \alpha''$, $\alpha'' + \tau < \alpha$ and $\beta' + \tau < \beta''$, $\beta'' + \tau < \beta$. By Lemma 4.3.2 the operator L^{Δ} is bounded as $\mathcal{K}_{\alpha',\beta'} \to \mathcal{K}_{\alpha'',\beta''}$ and $\mathcal{K}_{\alpha'',\beta''} \to \mathcal{K}_{\alpha,\beta}$. Therefore $k_0 \in D(L^{\Delta})$ and $L^{\Delta} k_0 \in \mathcal{K}_{\alpha'',\beta''} \subset D(L^{\Delta})$. Thus $k_0 \in D(\widehat{L}^{\odot})$ implies that k_t is continuously differentiable w.r.t. the norm in $\mathcal{K}_{\alpha,\beta}$ and it is a classical solution to (4.8). □

We close this section with one sufficient condition for the evolution $\widehat{T}(t)^* k_0$ to satisfy the generalized Ruelle bound given below. Let $E : \Gamma_0^2 \longrightarrow \mathbb{R}_+$ be measurable such that

$$E(\eta) + E(\xi) \leq E(\eta \cup \xi), \quad \eta \cap \xi = \varnothing, \ \eta, \xi \in \Gamma_0^2 \tag{4.24}$$

holds. In particular this implies $E(\xi) \leq E(\eta)$ for $\xi \subset \eta \in \Gamma_0^2$. In applications such function is chosen to be growing at infinity, e.g. for non-negative potentials ϕ_1, ϕ_2, ϕ_3 of the form

$$E(\eta) = \sum_{x \in \eta^+} \sum_{y \in \eta^+ \backslash x} \phi_1(x - y) + \sum_{x \in \eta^-} \sum_{y \in \eta^- \backslash x} \phi_2(x - y) + \sum_{x \in \eta^+} \sum_{y \in \eta^-} \phi_3(x - y).$$

We will say that the correlation function $k \in \mathcal{K}_{\alpha,\beta}$ satisfies the generalized Ruelle bound if

$$|k(\eta)| \le Ce^{\alpha|\eta^+|}e^{\beta|\eta^-|}e^{-E(\eta)}, \quad \eta \in \Gamma_0^2$$

holds for some constant $C > 0$. The grand canonical Gibbs measure with activity $z > 0$ and pair potential ϕ is an example of a measure with such decay of correlations, cf. [KKK04].

Remark 4.3.4. *Suppose that k_μ is the correlation function for some probability measure μ on Γ^2 an k_μ satisfies the generalized Ruelle bound. For $i \in \mathbb{Z}^d$ let*

$$Q_i = \{r \in \mathbb{R}^d \mid i_k - \frac{1}{2} < r_k \le i_k + \frac{1}{2},\ k = 1, \dots, d\},$$

define $|\gamma_i| := |\gamma \cap Q_i|$ and set

$$U_n := \{\gamma \in \Gamma^2 \ ||\gamma_i^\pm| \le n(\max\{1, \log(\|i\|_\infty)\})^{\frac{1}{2}},\ \forall i \in \mathbb{Z}^d\}.$$

Suppose that the functional E is of the form

$$\begin{aligned} E(\eta) &= \sum_{x \in (\eta^+ \cup \eta^-)} \sum_{y \in (\eta^+ \cup \eta^-)\backslash x} \phi(x-y) \\ &= \sum_{x \in \eta^+} \sum_{y \in \eta^+ \backslash x} \phi(x-y) + 2 \sum_{x \in \eta^+} \sum_{y \in \eta^-} \phi(x-y) + \sum_{y \in \eta^-} \sum_{y \in \eta^- \backslash x} \phi(x-y), \end{aligned}$$

where ϕ is symmetric, integrable and superstable in the sense of Ruelle, cf. [Rue70]. Then in [KKK04] (for the one-component case) it was shown that $\mu(\bigcup_{n \ge 1} U_n) = 1$. In fact, it should be not difficult to adapt such result for the two-component case.

Define for $\alpha, \beta \in \mathbb{R}$ a function $c_{\mathrm{dec}}(\alpha, \beta; \eta)$ by

$$\begin{aligned} &c_{\mathrm{dec}}(\alpha, \beta; \eta) = \\ &+ \sum_{x \in \eta^-} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) e^{\alpha|\xi^+|} e^{\beta|\xi^-|} e^{-E(\xi)} \mathrm{d}\lambda(\xi) \\ &+ \sum_{x \in \eta^+} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) e^{\alpha|\xi^+|} e^{\beta|\xi^-|} e^{-E(\xi)} \mathrm{d}\lambda(\xi) \\ &+ e^{-\beta} \sum_{x \in \eta^-} e^{E(\eta) - E(\eta^+, \eta^- \backslash x)} \int_{\Gamma_0} |\mathbb{K}_0^{-1} b^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) e^{\alpha|\xi^+|} e^{\beta|\xi^-|} e^{-E(\xi)} \mathrm{d}\lambda(\xi) \\ &+ e^{-\alpha} \sum_{x \in \eta^+} e^{E(\eta) - E(\eta^+ \backslash x, \eta^-)} \int_{\Gamma_0} |\mathbb{K}_0^{-1} b^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) e^{\alpha|\xi^+|} e^{\beta|\xi^-|} e^{-E(\xi)} \mathrm{d}\lambda(\xi). \end{aligned}$$

Denote by $\mathcal{B}_{\alpha,\beta,E}$ the Banach space of functions G with norm

$$|||G|||_{\alpha,\beta,E} := \int_{\Gamma_0^2} |G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}e^{-E(\eta)}\mathrm{d}\lambda(\eta).$$

We identity the dual Banach space $\mathcal{B}^*_{\alpha,\beta,E}$ with the space of functions k having finite norm

$$|||k|||_{\alpha,\beta,E} := \operatorname*{ess\,sup}_{\eta\in\Gamma_0^2} |k(\eta)|e^{-\alpha|\eta^+|}e^{-\beta|\eta^-|}e^{E(\eta)}.$$

Then k satisfies the generalized Ruelle bound if $k \in \mathcal{B}^*_{\alpha,\beta,E}$. The next theorem gives one sufficient condition that the evolution $k_t = \widehat{T}(t)^*k_0$ satisfies the generalized Ruelle bound.

Theorem 4.3.5. *Let (A) be satisfied and suppose that there exists $a_{\mathrm{dec}}(\alpha,\beta) \in (0,2)$ such that*

$$c_{\mathrm{dec}}(\alpha,\beta;\eta) \leq a_{\mathrm{dec}}(\alpha,\beta)M(\eta), \quad \eta \in \Gamma_0^2$$

holds. Then $(\widehat{L}, D^{\mathcal{B}}(\widehat{L}))$ is the generator of an analytic semigroup of contractions $(\widehat{T}^{\mathcal{B}}(t))_{t\geq 0}$ on $\mathcal{B}_{\alpha,\beta,E}$, where

$$D^{\mathcal{B}}(\widehat{L}) = \{G \in \mathcal{B}_{\alpha,\beta,E} \mid M\cdot G \in \mathcal{B}_{\alpha,\beta,E}\}.$$

This semigroup satisfies similar statements to Theorem 4.2.3 and 4.3.3.
Suppose that (4.20) *holds. Then for every $k_0 \in \mathcal{B}^*_{\alpha,\beta,E}$ the unique weak solution to* (4.22) *is given by $\widehat{T}^{\mathcal{B}}(t)^*k_0 = k_t = \widehat{T}(t)^*k_0$ and hence satisfies $k_t \in \mathcal{B}^*_{\alpha,\beta,E}$.*

Proof. Denote by B' the positive operator defined in the proof of Theorem 4.2.3. For every $0 \leq G \in D^{\mathcal{B}}(\widehat{L})$ by property (4.24) and a short computation we see that

$$\int_{\Gamma_0^2} B'G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}e^{-E(\eta)}\mathrm{d}\lambda(\eta) \leq \int_{\Gamma_0^2} (c_{\mathrm{dec}}(\alpha,\beta;\eta) - M(\eta))G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}e^{-E(\eta)}\mathrm{d}\lambda(\eta)$$

is satisfied. The same arguments as for the proof of Theorem 4.2.3 show that $(\widehat{L}, D^{\mathcal{B}}(\widehat{L}))$ is the generator of an analytic semigroup $\widehat{T}^{\mathcal{B}}(t)$ on $\mathcal{B}_{\alpha,\beta}$ and Theorems 4.2.3 and 4.3.3 hold for this semigroup. Fix $k_0 \in \mathcal{B}^*_{\alpha,\beta,E}$, then $\widehat{T}^{\mathcal{B}}(t)^*k_0$ is the unique weak solution to (4.22) in $\mathcal{B}^*_{\alpha,\beta,E}$. Since $k_0 \in \mathcal{B}^*_{\alpha,\beta,E} \subset \mathcal{K}_{\alpha,\beta}$ is continuously embedded $\widehat{T}(t)^*k_0$ is the unique weak solution to (4.22) in $\mathcal{K}_{\alpha,\beta}$. Let us show that $\widehat{T}^{\mathcal{B}}(t)k_0$ is also a weak solution in $\mathcal{K}_{\alpha,\beta}$. Because of $\mathcal{L}_{\alpha,\beta} \subset \mathcal{B}_{\alpha,\beta,E}$ we see that $(\widehat{L}, D^{\mathcal{B}}(\widehat{L}))$ is an extension of $(\widehat{L}, D(\widehat{L}))$ and $\widehat{T}^{\mathcal{B}}(t)^*k_0 \in \mathcal{B}^*_{\alpha,\beta,E} \subset \mathcal{K}_{\alpha,\beta}$ is continuous w.r.t. $\sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$. Because of

$$|\widehat{T}^{\mathcal{B}}(t)k_0(\eta)| \leq e^{-E(\eta)}e^{\alpha|\eta^+|}e^{\beta|\eta^-|}|||\widehat{T}^{\mathcal{B}}(t)^*k_0|||_{\alpha,\beta,E} \leq e^{\alpha|\eta^+|}e^{\beta|\eta^-|}|||k_0|||_{\alpha,\beta,E}$$

we get by [WZ06, Lemma 1.10] that it is also continuous w.r.t. $\mathcal{C}(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$. As a consequence $\widehat{T}^{\mathcal{B}}(t)^*k_0$ is also a weak solution to (4.22) in $\mathcal{K}_{\alpha,\beta}$ and uniqueness implies $\widehat{T}(t)^*k_0 = \widehat{T}^{\mathcal{B}}(t)^*k_0$. □

4.4 Evolution of states

Suppose that (A) and (4.20) are satisfied. Let $T(t)$ be the semigroup on $\mathcal{E}_{\alpha,\beta}$ generated by the closure of $(L, \mathcal{FP}(\Gamma^2))$. Let $\mathcal{E}_{\alpha,\beta}^*$ be the dual Banach space to $\mathcal{E}_{\alpha,\beta}$ and $T(t)^*$ the adjoint semigroup on $\mathcal{E}_{\alpha,\beta}^*$. A functional $\ell \in \mathcal{E}_{\alpha,\beta}^*$ is called positive if for any $0 \leq \mathbb{K}G \in \mathcal{E}_{\alpha,\beta}$ the action satisfies $\ell(\mathbb{K}G) \geq 0$. Let $\mathcal{K}_{\alpha,\beta}^+ \subset \mathcal{K}_{\alpha,\beta}$ stand for the cone of all positive definite functions in $\mathcal{K}_{\alpha,\beta}$.

Lemma 4.4.1. *For any linear functional $\ell \in \mathcal{E}_{\alpha,\beta}^*$ there exists a unique function $k_\ell \in \mathcal{K}_{\alpha,\beta}$ such that*

$$\ell(\mathbb{K}G) = \langle G, k_\ell \rangle, \quad \mathbb{K}G \in \mathcal{E}_{\alpha,\beta} \tag{4.25}$$

and $\|\ell\|_{\mathcal{E}_{\alpha,\beta}^*} = \|k_\ell\|_{\mathcal{K}_{\alpha,\beta}}$ hold. The functional ℓ is positive if and only if $k_\ell \in \mathcal{K}_{\alpha,\beta}^+$. In such a case ℓ is given by

$$\ell(\mathbb{K}G) = \langle G, k_\ell \rangle = k_\ell(\emptyset)\langle \mathbb{K}G, \mu_\ell \rangle, \quad \mathbb{K}G \in \mathcal{FP}(\Gamma^2)$$

with $\mu_\ell \in \mathcal{P}_{\alpha,\beta}$ associated to the correlation function $\frac{1}{k_\ell(\emptyset)} k_\ell$.

Proof. Let $\ell \in \mathcal{E}_{\alpha,\beta}^*$, then $\widehat{\ell}(G) := \ell(\mathbb{K}G)$ defines an element in $\mathcal{L}_{\alpha,\beta}^* \cong \mathcal{K}_{\alpha,\beta}$. Hence there exists a unique element $k_\ell \in \mathcal{K}_{\alpha,\beta}$ such that $\widehat{\ell}(G) = \langle G, k_\ell \rangle$ and

$$\|\ell\|_{\mathcal{E}_{\alpha,\beta}^*} = \sup_{\|\mathbb{K}G\|_{\mathcal{E}_{\alpha,\beta}}=1} |\ell(\mathbb{K}G)| = \sup_{\|G\|_{\mathcal{L}_{\alpha,\beta}}=1} |\langle G, k_\ell \rangle| = \|k_\ell\|_{\mathcal{K}_{\alpha,\beta}}$$

holds. For $\mathbb{K}G \geq 0$ we get $\ell(\mathbb{K}G) = \langle G, k_\ell \rangle \geq 0$ if and only if k_ℓ is positive definite. The last assertion is a consequence of Theorem 4.1.1. □

As a consequence for any $\ell \in \mathcal{E}_{\alpha,\beta}^*$ the action $T(t)^*\ell$ is represented by $\widehat{T}(t)^* k_\ell \in \mathcal{K}_{\alpha,\beta}$, i.e. for any $\mathbb{K}G \in \mathcal{E}_{\alpha,\beta}$

$$(T(t)^*\ell)(\mathbb{K}G) = \ell(T(t)\mathbb{K}G) = \langle \widehat{T}(t)G, k_\ell \rangle = \langle G, \widehat{T}(t)^* k_\ell \rangle.$$

holds. From $\widehat{T}(t)\mathbb{1}^* = \mathbb{1}^*$ we obtain $\widehat{T}(t)^* k_0(\emptyset) = k_0(\emptyset)$ and by $\mathbb{K}\mathbb{1}^* = 1 \in \mathcal{E}_{\alpha,\beta}$

$$(T(t)^*\ell)(1) = \widehat{T}(t)^* k_\ell(\emptyset) = k_\ell(\emptyset). \tag{4.26}$$

Therefore the semigroup $(T(t)^*)_{t \geq 0}$ is conservative on $\mathcal{E}_{\alpha,\beta}^*$. Let us start with the notion of solutions to the Fokker-Planck equation (4.5).

Definition 4.4.2. *A flow of Borel probability measures $(\mu_t)_{t\geq 0} \subset \mathcal{P}_{\alpha,\beta}$ is said to be a weak solution to* (4.5) *if for any $F \in \mathcal{FP}(\Gamma^2)$, $t \longmapsto \langle LF, \mu_t \rangle$ is locally integrable and satisfies*

$$\langle F, \mu_t \rangle = \langle F, \mu_0 \rangle + \int_0^t \langle LF, \mu_s \rangle \mathrm{d}s, \quad t \geq 0. \tag{4.27}$$

Uniqueness is stated in the next theorem, its proof is achieved by showing that any solution to the Fokker-Planck equation (4.5) yields a weak solution to (4.8).

Theorem 4.4.3. *(Uniqueness) Suppose that (A) and* (4.20) *are fulfilled. Then equation* (4.5) *has at most one solution* $(\mu_t)_{t\geq 0} \subset \mathcal{P}_{\alpha,\beta}$ *such that its correlation functions* $(k_t)_{t\geq 0}$ *satisfy*

$$\sup_{t\in[0,T]} \|k_t\|_{\mathcal{K}_{\alpha,\beta}} < \infty, \quad \forall T > 0.$$

Proof. Let $(\mu_t)_{t\geq 0} \subset \mathcal{P}_{\alpha,\beta}$ be a solution to (4.5), and denote by $(k_t)_{t\geq 0} \subset \mathcal{K}_{\alpha,\beta}$ the associated correlation functions. Let $F \in \mathcal{FP}(\Gamma^2)$ and $G \in B_{bs}(\Gamma_0^2) \subset D(\widehat{L})$ such that $F = \mathbb{K}G$. Then by $k_t(\eta) \leq \|k_t\|_{\mathcal{K}_{\alpha,\beta}} e^{\alpha|\eta^+|}e^{\beta|\eta^-|}$ it follows that $G, \widehat{L}G \in \mathcal{L}_{\alpha,\beta} \subset \mathcal{L}_{k_t}$. Since $\mathbb{K} : \mathcal{L}_{k_t} \longrightarrow L^1(\Gamma^2, \mathrm{d}\mu_t)$ is continuous it follows that $F = \mathbb{K}G, L\mathbb{K}G = \mathbb{K}\widehat{L}G$ belong to $L^1(\Gamma^2, \mathrm{d}\mu_t)$ for any $t \geq 0$. Moreover,

$$\langle LF, \mu_t\rangle = \langle \mathbb{K}\widehat{L}G, \mu_t\rangle = \langle \widehat{L}G, k_t\rangle$$

and hence $t \longmapsto \langle \widehat{L}G, k_t\rangle$ is locally integrable. This show for any $G \in B_{bs}(\Gamma_0^2)$

$$\langle G, k_t\rangle = \langle G, k_0\rangle + \int_0^t \langle \widehat{L}G, k_s\rangle \mathrm{d}s, \quad t \geq 0.$$

Hence k_t is continuous w.r.t. $\sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$ and since k_t is norm-bounded on $[0, T]$ [WZ06, Lemma 1.10] implies that k_t is also continuous w.r.t. the topology $\mathcal{C}$. It remains to show that $(k_t)_{t\geq 0}$ solves (4.22) for any $G \in D(\widehat{L})$. To this end let $G \in D(\widehat{L})$, then there exists $G_n \in B_{bs}(\Gamma_0^2)$ such that $G_n \longrightarrow G$ and $\widehat{L}G_n \longrightarrow \widehat{L}G$ in $\mathcal{L}_{\alpha,\beta}$. Passing in

$$\langle G_n, k_t\rangle = \langle G_n, k_0\rangle + \int_0^t \langle \widehat{L}G_n, k_s\rangle \mathrm{d}s$$

to the limit $n \to \infty$ shows (4.22). As a consequence $(k_t)_{t\geq 0}$ is a weak solution to (4.22). □

Remark 4.4.4. *Let* $k_0 \in \mathcal{K}_{\alpha,\beta}$ *be positive definite and suppose that* $k_t := \widehat{T}(t)^* k_0 \in \mathcal{K}_{\alpha,\beta}$ *is positive definite. Then* $(k_t)_{t\geq 0}$ *is a weak solution to* (4.22) *and for each* $t \geq 0$ *there exists a unique* $\mu_t \in \mathcal{P}_{\alpha,\beta}$ *having correlation function* k_t*. By* $\langle G, k_t\rangle = \langle F, \mu_t\rangle$ *and* $\langle \widehat{L}G, k_t\rangle = \langle LF, \mu_t\rangle$ *it follows that* $(\mu_t)_{t\geq 0}$ *is a weak solution to* (4.5)*.*

Above considerations show that for existence of weak solutions to (4.5), it suffices to show that $\widehat{T}(t)^*$ preserves the cone of positive definite functions. The main idea for the proof of positive definiteness is to approximate the evolution $k_t = \widehat{T}(t)^* k_0$ by an auxiliary evolution $\widehat{T}_\delta(t)^* k_0$ and prove that $\widehat{T}_\delta(t)^* k_0$ is positive definite. Such idea was proposed in [KK16], where the authors proved for the BDLP-model positive-definiteness of a local evolution. Let $(R_\delta)_{\delta>0}$ be a sequence of continuous integrable functions with $0 < R_\delta \leq 1$ and $R_\delta(x) \nearrow 1$ as $\delta \to 0$ for all $x \in \mathbb{R}^d$. Define new birth intensities by $b^S_\delta(x,\eta) := R_\delta(x) b^S(x,\eta)$ and $b^E_\delta(x,\eta) := R_\delta(x) b^E(x,\eta)$ for all $x \in \mathbb{R}^d$ and $\eta \in \Gamma^2_0$. In the following we simply say that $(R_\delta)_{\delta>0}$ is a localization sequence. In such a case the overall birth intensity is finite, i.e. for any $\eta \in \Gamma^2_0$ and $\delta > 0$

$$\int\limits_{\mathbb{R}^d} \left(b^S_\delta(x,\eta) + b^E_\delta(x,\eta) \right) \mathrm{d}x < \infty \tag{4.28}$$

holds. The considerations of the second chapter imply for each $\eta \in \Gamma^2_0$ the existence of an associated (minimal) birth-and-death process $(\eta_t)_{t\geq 0}$ associated to L_δ starting from η with the state space Γ^2_0. Here L_δ is obtained from L by replacing b^S, b^E with b^S_δ, b^E_δ. The following are the main assumptions for the existence of weak solutions to the Fokker-Planck equation.

(B) There exist constants $A > 0$, $\tau \geq 0$ and $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$ and $\eta \in \Gamma^2_0$

$$d^S(x,\eta) + d^E(x,\eta) + b^S(x,\eta) + b^E(x,\eta) \leq A(1+|\eta|)^N e^{\tau|\eta|}. \tag{4.29}$$

(D) There exists a localization sequence $(R_\delta)_{\delta>0}$ such that the (minimal) birth-and-death process associated to L_δ is conservative, i.e. has no explosion starting from any initial point $\eta \in \Gamma^2_0$.

The next proposition is the main result. It provides positivity of the semigroups constructed above. Note that $\mathcal{P}_{\alpha',\beta'} \subset \mathcal{P}_{\alpha,\beta} \subset \mathcal{E}^*_{\alpha,\beta}$.

Proposition 4.4.5. *(Existence) Suppose that (A) – (D) and* (4.20) *are fulfilled. Then* $T(t)^*\mathcal{P}_{\alpha',\beta'} \subset \mathcal{P}_{\alpha,\beta}$ *holds. In particular for any* $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ *there exists exactly one solution* $(\mu_t)_{t\geq 0} \subset \mathcal{P}_{\alpha,\beta}$ *to* (4.5) *given by* $T(t)^*\mu_0 = \mu_t$*. If conditions (B) and (C) hold for all* $\tau > 0$*, then* $T(t)^*\mathcal{P}_{\alpha,\beta} \subset \mathcal{P}_{\alpha,\beta}$*.*

Existence of an associated Markov function is stated in the next corollary.

Corollary 4.4.6. *Suppose that (A) – (D) hold for any* $\tau > 0$ *and assume that* (4.20) *holds. Then for any* $\mu \in \mathcal{P}_{\alpha,\beta}$ *there exists a Markov function* $(X^\mu_t)_{t\geq 0}$ *on the configuration space* Γ^2 *with the initial distribution* μ *associated with the generator* L*.*

The rest of this section is devoted to the proof of Proposition 4.4.5. Consider a linear operator $\mathcal{I}_\delta = -D_\delta + Q_\delta$ on $L^1(\Gamma^2_0, \mathrm{d}\lambda)$, where the first operator is a multiplication operator given by the function $D_\delta(\eta) = M(\eta) + \int\limits_{\mathbb{R}^d} b^S_\delta(x,\eta)\mathrm{d}x + \int\limits_{\mathbb{R}^d} b^E_\delta(x,\eta)\mathrm{d}x$. The integrals are finite due to (4.28). The operator Q_δ is given by

$$Q_\delta R(\eta) = \int_{\mathbb{R}^d} d^E(x,\eta)R(\eta^+,\eta^-\cup x)\mathrm{d}x + \int_{\mathbb{R}^d} d^S(x,\eta)R(\eta^+\cup x,\eta^-)\mathrm{d}x$$
$$+ \sum_{x\in\eta^-} b_\delta^E(x,\eta^+,\eta^-\backslash x)R(\eta^+,\eta^-\backslash x) + \sum_{x\in\eta^+} b_\delta^S(x,\eta^+\backslash x,\eta^-)R(\eta^+\backslash x,\eta^-).$$

The operator $\mathcal{I}_\delta$ is considered on the domain

$$D(\mathcal{I}_\delta) = \{R \in L^1(\Gamma_0^2,\mathrm{d}\lambda) \mid D_\delta R \in L^1(\Gamma_0^2,\mathrm{d}\lambda)\}.$$

The results of the second chapter imply that for any $R_0 \in D(\mathcal{I}_\delta)$ the Cauchy problem

$$\frac{\partial R_t^\delta}{\partial t} = \mathcal{I}_\delta R_t^\delta, \quad R_t^\delta|_{t=0} = R_0 \tag{4.30}$$

admits a minimal solution given by a C_0-semigroup $(S_\delta(t))_{t\geq 0}$ on $L^1(\Gamma_0^2,\mathrm{d}\lambda)$. Condition (D) implies that $(\mathcal{I}_\delta, D(\mathcal{I}_\delta))$ is closable and the closure is the generator of the C_0-semigroup $(S_\delta(t))_{t\geq 0}$. Therefore, for any $R_0 \in D(\mathcal{I}_\delta)$ there exists exactly one solution to (4.30) and this solution is given by $S_\delta(t)R_0$. For technical reasons we will also need the adjoint semigroup on $L^\infty(\Gamma_0^2,\mathrm{d}\lambda)$. Let $(\mathcal{J}_\delta, D(\mathcal{J}_\delta))$ be the adjoint operator to $(\mathcal{I}_\delta, D(\mathcal{I}_\delta))$ on $L^\infty(\Gamma_0^2,\mathrm{d}\lambda)$. The following lemma is proved in the same way as Lemma 2.4.3.

Lemma 4.4.7. *For any $F \in D(\mathcal{J}_\delta)$ it holds that $\mathcal{J}_\delta F = L_\delta F$.*

Lemma 4.4.8. *For any $\delta > 0$ Theorem 4.2.3 and 4.3.3 hold with L replaced by L_δ. Let $\widehat{T}_\delta(t)$ and $\widehat{T}_\delta(t)^*$ be the semigroups on $\mathcal{L}_{\alpha,\beta}$ and $\mathcal{K}_{\alpha,\beta}$, respectively. Then for any $G \in \mathcal{L}_{\alpha,\beta}$*

$$\widehat{T}_\delta(t)G \longrightarrow \widehat{T}(t)G, \quad \delta \to 0$$

is satisfied.

Proof. Let $\widehat{L}_\delta = \mathbb{K}_0^{-1}L_\delta\mathbb{K}_0 = A + B_\delta$, where A is given as before and B_δ is obtained from B by multiplication of the terms for the birth by $R_\delta(x)$. This operator is defined on $D(\widehat{L})$ for any $\delta > 0$ and since $R_\delta \leq 1$ Theorem 4.2.3 and 4.3.3 can be applied to $\widehat{L}_\delta$, which yields the first assertion. For the second assertion observe that for $G \in D(\widehat{L})$ and $0 \leq h_\delta(x) := 1 - R_\delta(x) \leq 1$ we obtain

$$\|\widehat{L}_\delta G - \widehat{L}G\|_{\mathcal{L}_{\alpha,\beta}} \leq$$
$$+ \int_{\Gamma_0^2} |G(\xi)| \sum_{x\in\xi^+} h_\delta(x) \int_{\Gamma_0^2} |\mathbb{K}_0^{-1}b^S(x,\cdot\cup\xi^+\backslash x,\cdot\cup\xi^-)|(\eta)e^{\alpha|\xi^+|}e^{\alpha|\eta^+|}e^{\beta|\xi^-|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta)\mathrm{d}\lambda(\xi)$$
$$+ \int_{\Gamma_0^2} |G(\xi)| \sum_{x\in\xi^-} h_\delta(x) \int_{\Gamma_0^2} |\mathbb{K}_0^{-1}b^E(x,\cdot\cup\xi^+,\cdot\cup\xi^-\backslash x)|(\eta)e^{\alpha|\xi^+|}e^{\alpha|\eta^+|}e^{\beta|\xi^-|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta)\mathrm{d}\lambda(\xi).$$

The integrand tends for each $\xi \in \Gamma_0^2$ to zero as $\delta \to 0$, hence by dominated convergence $\widehat{L}_\delta G \to \widehat{L}G$. Trotter-Kato approximation therefore implies $\widehat{T}_\delta(t) \to \widehat{T}(t)$ strongly on $\mathcal{L}_{\alpha,\beta}$. $\square$

Let $\mathcal{B}_{\alpha,\beta}$ be the Banach space of all equivalence classes of functions G with norm

$$\|\!|G|\!\|_{\mathcal{B}_{\alpha,\beta}} = \int\limits_{\Gamma_0^2} |G(\eta)| e_\lambda(R_\delta;\eta^+) e_\lambda(R_\delta;\eta^-) e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta).$$

Likewise let $\mathcal{B}_{\alpha,\beta}^*$ be the Banach space of all equivalence classes of functions k with norm

$$\|\!|k|\!\|_{\mathcal{B}_{\alpha,\beta}^*} = \operatorname*{ess\,sup}_{\eta\in\Gamma_0^2} \frac{|k(\eta)|}{e_\lambda(R_\delta;\eta^+) e_\lambda(R_\delta;\eta^-) e^{\alpha|\eta^+|} e^{\beta|\eta^-|}}.$$

The same arguments as for the proof of Theorem 4.2.3 and 4.3.3 show that we can replace $\mathcal{L}_{\alpha,\beta}, \mathcal{K}_{\alpha,\beta}$ also by $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{B}_{\alpha,\beta}^*$. Denote by $U_\delta(t)$ and $U_\delta(t)^*$ the corresponding semigroups on $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{B}_{\alpha,\beta}^*$, respectively. Let $(\widehat{L}_\delta, D^{\mathcal{B}}(\widehat{L}))$ be the generator of $U_\delta(t)$. The proofs of Theorem 4.2.3 and 4.3.3 show that

$$D^{\mathcal{B}}(\widehat{L}) = \{G \in \mathcal{B}_{\alpha,\beta} \mid M \cdot G \in \mathcal{B}_{\alpha,\beta}\}.$$

Thus the Cauchy problem

$$\frac{\partial}{\partial t}\langle G, u_t^\delta\rangle = \langle \widehat{L}_\delta G, u_t^\delta\rangle, \quad u_t^\delta|_{t=0} = u_0, \quad \forall G \in D^{\mathcal{B}}(\widehat{L}) \tag{4.31}$$

has for every $u_0 \in \mathcal{B}_{\alpha,\beta}^*$ a unique weak solution in $\mathcal{B}_{\alpha,\beta}^*$ given by $U_\delta(t)^* u_0$.

Lemma 4.4.9. *Let $k_0 \in \mathcal{B}_{\alpha,\beta}^*$, then $\widehat{T}_\delta(t)^* k_0 = U_\delta(t)^* k_0$.*

Proof. First observe that $\mathcal{B}_{\alpha,\beta}^* \subset \mathcal{K}_{\alpha,\beta}$ continuously and hence $k_0 \in \mathcal{K}_{\alpha,\beta}$. In particular $u_t^\delta := U_\delta(t)^* k_0$ and $k_t^\delta := \widehat{T}_\delta(t)^* k_0$ are well-defined. Moreover, since also $\mathcal{L}_{\alpha,\beta} \subset \mathcal{B}_{\alpha,\beta}$ continuously we obtain $D(\widehat{L}) \subset D^{\mathcal{B}}(\widehat{L})$, i.e. $(\widehat{L}_\delta, D^{\mathcal{B}}(\widehat{L}))$ is an extension of $(\widehat{L}_\delta, D(\widehat{L}))$. Therefore $(u_t^\delta)_{t\geq 0}$ is also a weak solution to (4.22) and thus by uniqueness $u_t^\delta = k_t^\delta$, $t \geq 0$. $\square$

Lemma 4.4.10. *Let $k_0 \in \mathcal{B}_{\alpha',\beta'}^*$ be positive definite. Denote by $u_t^\delta \in \mathcal{B}_{\alpha,\beta}^*$ the unique weak solution to* (4.31)*, then u_t^δ is positive definite for any $t \geq 0$.*

Proof. Define for any $u \in \mathcal{B}^*_{\alpha,\beta}$ a linear operator $\mathcal{H}u(\eta) := \int\limits_{\Gamma_0^2} (-1)^{|\xi|} u(\eta \cup \xi) \mathrm{d}\lambda(\xi)$. Then $\mathcal{H}u$ is well-defined and satisfies for any $C_+, C_- > 0$

$$\begin{aligned}
&\int\limits_{\Gamma_0^2} |\mathcal{H}u(\eta)| C_+^{|\eta^+|} C_-^{|\eta^-|} \mathrm{d}\lambda(\eta) \le \int\limits_{\Gamma_0^2}\int\limits_{\Gamma_0^2} |u(\eta \cup \xi)| C_+^{|\eta^+|} C_-^{|\eta^-|} \mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
&= \int\limits_{\Gamma_0^2} \sum_{\xi \subset \eta} C_+^{|\xi^+|} C_-^{|\xi^-|} |u(\eta)| \mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0^2} (1 + C_+)^{|\eta^+|} (1 + C_-)^{|\eta^-|} |u(\eta)| \mathrm{d}\lambda(\eta) \\
&\le \|u\|_{\mathcal{B}^*_{\alpha,\beta}} \int\limits_{\Gamma_0^2} (1 + C_+)^{|\eta^+|} (1 + C_-)^{|\eta^-|} e^{\alpha|\eta^+|} e^{\beta|\eta^-|} e_\lambda(R_\delta; \eta^+) e_\lambda(R_\delta; \eta^-) \mathrm{d}\lambda(\eta),
\end{aligned}$$

i.e. $\mathcal{H} : \mathcal{B}^*_{\alpha,\beta} \longrightarrow \mathcal{L}_{\log(C_+),\log(C_-)}$ is continuous. Let $G \in \mathcal{B}_{\alpha,\beta}$ be arbitrary, then for any $u \in \mathcal{B}^*_{\alpha,\beta}$ we get by Fubini's theorem and (2.27)

$$\begin{aligned}
\langle \mathbb{K}_0 G, \mathcal{H}u \rangle &= \int\limits_{\Gamma_0^2} \sum_{\xi \subset \eta} G(\xi) \int\limits_{\Gamma_0^2} (-1)^{|\zeta|} u(\eta \cup \zeta) \mathrm{d}\lambda(\zeta) \mathrm{d}\lambda(\eta) \\
&= \int\limits_{\Gamma_0^2}\int\limits_{\Gamma_0^2}\int\limits_{\Gamma_0^2} G(\xi)(-1)^{|\zeta|} u(\eta \cup \xi \cup \zeta) \mathrm{d}\lambda(\zeta)\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
&= \int\limits_{\Gamma_0^2} G(\xi) \int\limits_{\Gamma_0^2} \sum_{\zeta \subset \eta} (-1)^{|\zeta|} u(\eta \cup \xi) \mathrm{d}\lambda(\eta)\mathrm{d}\lambda(\xi) = \int\limits_{\Gamma_0^2} G(\xi) u(\xi) \mathrm{d}\lambda(\xi) = \langle G, u \rangle
\end{aligned}$$

and thus

$$\langle \mathbb{K}_0 G, \mathcal{H}u \rangle = \langle G, u \rangle \tag{4.32}$$

holds. We can apply Fubini's theorem and (2.27) since

$$\begin{aligned}
&\int\limits_{\Gamma_0^2}\int\limits_{\Gamma_0^2}\int\limits_{\Gamma_0^2} |G(\xi)||u(\eta \cup \xi \cup \zeta)| \mathrm{d}\lambda(\zeta)\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
&\le \|u\|_{\mathcal{B}^*_{\alpha,\beta}} e^{2e^{\alpha}\langle R_\delta \rangle} e^{2e^{\beta}\langle R_\delta \rangle} \int\limits_{\Gamma_0^2} |G(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} e_\lambda(R_\delta; \xi^+) e_\lambda(R_\delta; \xi^-) \mathrm{d}\lambda(\xi)
\end{aligned}$$

is satisfied, where $\langle R_\delta \rangle := \int\limits_{\mathbb{R}^d} R_\delta(x)\mathrm{d}x$. For the same u and $G \in D^{\mathcal{B}}(\widehat{L})$ we obtain by (4.32) and $\mathbb{K}_0 \widehat{L}_\delta G = L_\delta \mathbb{K}_0 G$

$$\langle \widehat{L}_\delta G, u \rangle = \langle \mathbb{K}_0 \widehat{L}_\delta G, \mathcal{H}u \rangle = \langle L_\delta \mathbb{K}_0 G, \mathcal{H}u \rangle. \tag{4.33}$$

Now let $U_\delta(t)^* k_0 = u_t^\delta \in \mathcal{B}^*_{\alpha,\beta}$, then

$$\langle G, u_t^\delta \rangle = \langle G, u_0 \rangle + \int_0^t \langle \widehat{L}_\delta G, u_s^\delta \rangle \mathrm{d}s, \quad G \in D^{\mathcal{B}}(\widehat{L}).$$

Observe that condition (B) implies $\mathcal{K}_{\log(2),\log(2)} \subset D^{\mathcal{B}}(\widehat{L})$. Hence by (4.32) and (4.33) it follows for $R_t^\delta := \mathcal{H} u_t^\delta \in L^1(\Gamma_0^2, \mathrm{d}\lambda)$, $t \geq 0$ that

$$\langle \mathbb{K}_0 G, R_t^\delta \rangle = \langle \mathbb{K}_0 G, R_0 \rangle + \int_0^t \langle L_\delta \mathbb{K}_0 G, R_s^\delta \rangle \mathrm{d}s, \quad G \in \mathcal{K}_{\log(2),\log(2)}$$

holds. For any $F \in D(\mathcal{J}_\delta) \subset L^\infty(\Gamma_0, \mathrm{d}\lambda)$ we get $|\mathbb{K}_0^{-1} F(\eta)| \leq \|F\|_{L^\infty} 2^{|\eta|}$ and hence $D(\mathcal{J}_\delta) \subset \mathbb{K}_0 \mathcal{K}_{\log(2),\log(2)}$. Thus we can find $G \in \mathcal{K}_{\log(2),\log(2)}$ such that $\mathbb{K}_0 G = F \in D(\mathcal{J}_\delta)$. Lemma 4.4.7 therefore implies

$$\langle F, R_t^\delta \rangle = \langle F, R_0 \rangle + \int_0^t \langle \mathcal{J}_\delta F, R_s^\delta \rangle \mathrm{d}s, \quad F \in D(\mathcal{J}_\delta).$$

Since $k_0 \in \mathcal{B}^*_{\alpha',\beta'}$ we get by Theorem 4.3.3.1) that u_t^δ is continuous in $t \geq 0$ w.r.t. the norm in $\mathcal{B}^*_{\alpha,\beta}$. Because $\mathcal{H} : \mathcal{B}^*_{\alpha,\beta} \longrightarrow L^1(\Gamma_0^2, \mathrm{d}\lambda)$ is continuous, $R_t^\delta = \mathcal{H} u_t^\delta$ is continuous w.r.t. $t \geq 0$ on $L^1(\Gamma_0^2, \mathrm{d}\lambda)$. Hence $(R_t^\delta)_{t \geq 0}$ is a weak solution to (4.30). The main result from [Bal77] therefore implies $R_t^\delta = S_\delta(t) R_0 \geq 0$. Finally, for any $G \in B_{bs}(\Gamma_0^2)$ with $\mathbb{K} G \geq 0$ we get

$$\langle G, u_t^\delta \rangle = \langle \mathbb{K}_0 G, R_t^\delta \rangle \geq 0, \quad t \geq 0.$$

□

We are now prepared to complete the proof of positive definiteness.

Proof. (Proposition 4.4.5) Let $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ with correlation function $k_0 \in \mathcal{K}_{\alpha',\beta'}$. Define

$$k_{0,\delta}(\eta) := k_0(\eta) e_\lambda(R_\delta; \eta^+) e_\lambda(R_\delta; \eta^-), \quad \delta > 0, \ \eta \in \Gamma_0^2,$$

then $k_{0,\delta} \in \mathcal{B}^*_{\alpha',\beta'}$ and it is positive definite, cf. [Fin13, Fin11]. By Lemma 4.4.9 we get $\widehat{T}_\delta(t)^* k_{0,\delta} = U_\delta(t)^* k_{0,\delta} \in \mathcal{B}^*_{\alpha,\beta}$ and by Lemma 4.4.10 the latter expression is positive definite. Let $G \in B_{bs}(\Gamma_0^2)$ be such that $\mathbb{K} G \geq 0$. Then it suffices to show that

$$\langle G, \widehat{T}_\delta(t)^* k_{0,\delta} \rangle \longrightarrow \langle G, \widehat{T}(t)^* k_0 \rangle, \quad \delta \to 0.$$

To this end observe that

$$\langle G, \widehat{T}_\delta(t)^* k_{0,\delta} \rangle = \langle \widehat{T}_\delta(t)^* G - \widehat{T}(t) G, k_{0,\delta} \rangle + \langle \widehat{T}(t) G, k_{0,\delta} \rangle.$$

The first term can be estimated by

$$\|\widehat{T}_\delta(t)G - \widehat{T}(t)G\|_{\mathcal{L}_{\alpha,\beta}} \|k_0\|_{\mathcal{K}_{\alpha,\beta}}$$

and hence tends by Lemma 4.4.8 to zero. The second term tends by dominated convergence to $\langle \widehat{T}(t)G, k_0\rangle = \langle G, \widehat{T}(t)^* k_0\rangle$, which implies that $\widehat{T}(t)^* k_0$ is positive definite.
If conditions (B) and (C) hold for all $\tau > 0$, then $k_{0,\delta}(\eta) := e^{-\delta|\eta|}k_0(\eta)$ belongs to $\mathcal{K}_{\alpha-\delta,\beta-\delta}$ for any $\delta > 0$. Consequently, above considerations imply that $\widehat{T}(t)^* k_{0,\delta} \in \mathcal{K}_{\alpha,\beta}$ is positive definite. Taking the limit $\delta \to 0$ yields the assertion. □

Remark 4.4.11. *Suppose instead of (B) the following to be satisfied: There exist $A > 0$, $N \in \mathbb{N}$ and $\nu_b \geq 0$, $\nu_1, \nu_2 \geq 0$ such that for all $x \in \mathbb{R}^d$ and $\eta \in \Gamma_0^2$:*

$$\begin{aligned} b^S(x,\eta) + b^E(x,\eta) &\leq A(1+|\eta|)^N e^{\nu_b|\eta|} \\ d^S(x,\eta) &\leq A(1+|\eta|)^N e^{\nu_1|\eta|} \\ d^E(x,\eta) &\leq A(1+|\eta|)^N e^{\nu_2|\eta|}. \end{aligned}$$

Then for any positive definite $k_0 \in \mathcal{K}_{\alpha',\beta'}$ the evolution $\widehat{T}(t)^* k_0$ is positive definite, provided (C) holds for $\alpha' + \nu_1 < \alpha$, $\beta' + \nu_2 < \beta$.

4.5 Ergodicity

Let $\mu \in \mathcal{P}_{\alpha,\beta}$ and denote by k_μ its correlation function. Then $\mathcal{L}_{\alpha,\beta} \subset \mathcal{L}_{k_\mu}$ and hence $\mathcal{E}_{\alpha,\beta} \subset L^1(\Gamma^2, \mathrm{d}\mu)$. Therefore, for any $F \in \mathcal{E}_{\alpha,\beta}$ we see that $\langle F\rangle_\mu := \int_{\Gamma^2} F(\gamma)\mathrm{d}\mu(\gamma) = \int_{\Gamma_0^2} G(\eta)k_\mu(\eta)\mathrm{d}\lambda(\eta)$ is well-defined. The next statement provides ergodicity for the semigroups $(T(t))_{t\geq 0}$ and $(T(t)^*)_{t\geq 0}$.

Proposition 4.5.1. *Suppose that (A) – (D), (4.20) and $\inf\limits_{|\eta|\geq 1} M(\eta) > 0$ are fulfilled. Then there exists a unique invariant measure $\mu_{\mathrm{inv}} \in \mathcal{P}_{\alpha,\beta}$. Namely, μ_{inv} satisfies*

$$\int_{\Gamma^2} LF(\gamma)\mathrm{d}\mu_{\mathrm{inv}}(\gamma) = 0, \quad F \in \mathcal{FP}(\Gamma^2) \tag{4.34}$$

and $T(t)^*\mu_{\mathrm{inv}} = \mu_{\mathrm{inv}}$ for all $t \geq 0$. Moreover, there exist constants $C > 0$ and $\varepsilon > 0$ such that the following assertions hold:

1. *For each $F \in \mathcal{E}_{\alpha,\beta}$*

$$\|T(t)F - \langle F\rangle_{\mu_{\mathrm{inv}}}\|_{\mathcal{E}_{\alpha,\beta}} \leq Ce^{-\varepsilon t}\|F - \langle F\rangle_{\mu_{\mathrm{inv}}}\|_{\mathcal{E}_{\alpha,\beta}}, \quad t \geq 0 \tag{4.35}$$

   holds.

2. *For any $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ let $\mu_t = T(t)^*\mu_0 \in \mathcal{P}_{\alpha,\beta}$, then*

$$\|\mu_t - \mu_{\text{inv}}\|_{\mathcal{E}^*_{\alpha,\beta}} \leq Ce^{-\varepsilon t}\|\mu_0 - \mu_{\text{inv}}\|_{\mathcal{E}^*_{\alpha,\beta}}, \quad t \geq 0$$

holds. If conditions (B) and (C) hold for each $\tau > 0$, then above claim is also true for $\mu_0 \in \mathcal{P}_{\alpha,\beta}$.

The rest of this section is devoted to the proof of above proposition. Let $\mathcal{K}^{\geq 1}_{\alpha,\beta} = \{k \in \mathcal{K}_{\alpha,\beta} \mid k^{(0)} = 0\}$, $\mathcal{K}^0_{\alpha,\beta} = \{k \in \mathcal{K}_{\alpha,\beta} \mid k = \kappa \mathbb{1}^*, \ \kappa \in \mathbb{R}\}$ and denote by $\mathbb{1}^*(\eta) = 0^{|\eta^+|+|\eta^-|}$. Multiplication by $\mathbb{1}^*$ respectively $1-\mathbb{1}^*$ defines projection operators $\mathbb{1}^* : \mathcal{K}_{\alpha,\beta} \longrightarrow \mathcal{K}^0_{\alpha,\beta}$ and $(1-\mathbb{1}^*) : \mathcal{K}_{\alpha,\beta} \longrightarrow \mathcal{K}^{\geq 1}_{\alpha,\beta}$. These projections are orthogonal in the sense that $\mathbb{1}^*(1-\mathbb{1}^*) = (1-\mathbb{1}^*)\mathbb{1}^* = 0$. Hence we obtain the decomposition

$$\mathcal{K}_{\alpha,\beta} = \mathcal{K}^0_{\alpha,\beta} \oplus \mathcal{K}^{\geq 1}_{\alpha,\beta}.$$

Define the linear operator $Sk(\emptyset) = 0$ and for $\eta \neq \emptyset$

$$\begin{aligned}(Sk)(\eta) = &- \frac{1}{M(\eta)} \sum_{x \in \eta^-} \int_{\Gamma_0^2 \backslash \{\emptyset\}} k(\eta \cup \xi)(\mathbb{K}_0^{-1} d^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x))(\xi) \mathrm{d}\lambda(\xi) \\ &- \frac{1}{M(\eta)} \sum_{x \in \eta^+} \int_{\Gamma_0^2 \backslash \{\emptyset\}} k(\eta \cup \xi)(\mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-))(\xi) \mathrm{d}\lambda(\xi) \\ &+ \frac{1}{M(\eta)} \sum_{x \in \eta^-} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+, \eta^- \cup \xi^- \backslash x)(\mathbb{K}_0^{-1} b^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x))(\xi) \mathrm{d}\lambda(\xi) \\ &+ \frac{1}{M(\eta)} \sum_{x \in \eta^+} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+ \backslash x, \eta^- \cup \xi^-)(\mathbb{K}_0^{-1} b^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-))(\xi) \mathrm{d}\lambda(\xi),\end{aligned}$$

i.e. $Sk(\eta) = \frac{1}{M(\eta)} L^{\Delta}k(\eta) + k(\eta)$. The next theorem provides existence and uniqueness of solutions to the equation $L^{\Delta}k = 0$, i.e. for correlation functions. For one-component systems a similar result was proved for the case $\kappa = 1$ in [FKO13].

Theorem 4.5.2. *The equation*

$$L^{\Delta}k_{\text{inv}} = 0, \quad k_{\text{inv}}(\emptyset) = 1 \tag{4.36}$$

has a unique solution $k_{\text{inv}} \in \mathcal{K}_{\alpha,\beta}$. This solution is given by

$$k_{\text{inv}} = \mathbb{1}^* + (1-S)^{-1}S\mathbb{1}^*, \tag{4.37}$$

where $S\mathbb{1}^*(\eta) = \mathbb{1}_{\Gamma_0^{(1)}}(\eta^+)0^{|\eta^-|} \sum\limits_{x \in \eta^+} \frac{b^S(x,\emptyset)}{d^S(x,\emptyset)} + \mathbb{1}_{\Gamma_0^{(1)}}(\eta^-)0^{|\eta^+|} \sum\limits_{x \in \eta^-} \frac{b^E(x,\emptyset)}{d^E(x,\emptyset)}$. The equation

$$L^{\Delta}k^{\kappa}_{\text{inv}} = 0, \quad k^{\kappa}_{\text{inv}}(\emptyset) = \kappa$$

has for every $\kappa \in \mathbb{R}$ exactly one solution given by $k^{\kappa}_{\text{inv}} = \kappa k_{\text{inv}}$.

Proof. Let $k \in \mathcal{K}_{\alpha,\beta}$, then

$$M(\eta)|Sk(\eta)| \leq (a(\alpha,\beta)-1)M(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\|k\|_{\mathcal{K}_{\alpha,\beta}}$$

and hence by (4.20) $\|S\|_{L(\mathcal{K}_{\alpha,\beta})} < 1$. Moreover, since $S : \mathcal{K}_{\alpha,\beta}^{\geq 1} \longrightarrow \mathcal{K}_{\alpha,\beta}^{\geq 1}$ it follows that $1-S$ is invertible in $\mathcal{K}_{\alpha,\beta}^{\geq 1}$. Any solution $k \in \mathcal{K}_{\alpha,\beta}$ to (4.36) is also the solution to $M(S-1)k = 0$. Letting $\widetilde{k} = k - 1\!\!1^*$ yields by $M 1\!\!1^* = 0$

$$0 = M(S-1)\widetilde{k} + MS1\!\!1^*$$

and hence (4.36) is equivalent to

$$(1-S)\widetilde{k} = S1\!\!1^*, \quad \widetilde{k} := k - 1\!\!1^*.$$

Since $1-S$ is invertible on $\mathcal{K}_{\alpha,\beta}^{\geq 1}$ we obtain

$$\widetilde{k} = (1-S)^{-1}S1\!\!1^*.$$

□

For individual-based models, i.e. $b^S(x,\emptyset) = b^E(x,\emptyset) = 0$, the invariant state is simply $k_{\text{inv}}(\eta) = 1\!\!1^*(\eta)$. Such correlation function corresponds to the probability measure $\mu_{\text{inv}} = \delta_{\{\emptyset\}}$ on Γ^2. The next step is to establish ergodicity for the semigroups $\widehat{T}(t)$ on quasi-observables and $\widehat{T}(t)^*$ on correlation functions. Such ergodicity has been established for the (one-component) Glauber dynamics, see [KKM10]. Our approach is based on the ideas of this work. Let $\mathcal{L}_{\alpha,\beta}^0 := \{G \in \mathcal{L}_{\alpha,\beta} \mid G = \kappa 1\!\!1^*, \ \kappa \in \mathbb{R}\}$ and

$$\mathcal{L}_{\alpha,\beta}^{\geq 1} := \{G \in \mathcal{L}_{\alpha,\beta} \mid G(\emptyset) = 0\}.$$

Then any $G \in \mathcal{L}_{\alpha,\beta}$ admits a unique decomposition $G = 1\!\!1^*G + (1-1\!\!1^*)G = G_0 + G_1$ where $G_0 \in \mathcal{L}_{\alpha,\beta}^0$ and $G_1 \in \mathcal{L}_{\alpha,\beta}^{\geq 1}$, i.e. $\mathcal{L}_{\alpha,\beta} = \mathcal{L}_{\alpha,\beta}^0 \oplus \mathcal{L}_{\alpha,\beta}^{\geq 1}$. The projection onto $\mathcal{L}_{\alpha,\beta}^0$ is given by the multiplication with the function $1\!\!1^*(\eta) = 0^{|\eta|}$, i.e.

$$1\!\!1^* : \mathcal{L}_{\alpha,\beta} \longrightarrow \mathcal{L}_{\alpha,\beta}^0, \quad G \longmapsto 1\!\!1^*G = 0^{|\eta|}G^{(0)}, \quad G^{(0)} \in \mathbb{R}.$$

Thus by $M(\emptyset) = 0$ and $\widehat{L} = \widehat{L}(1-1\!\!1^*)$ the action of the operator $\widehat{L} = A + B$ can be represented in the form

$$\widehat{L} = 1\!\!1^*B(1-1\!\!1^*) + A(1-1\!\!1^*) + (1-1\!\!1^*)B(1-1\!\!1^*).$$

Therefore $\mathcal{L}_{\alpha,\beta}^{\geq 1}$ is invariant for A and $(1-1\!\!1^*)B$. Note that

$$1\!\!1^*BG(\eta) = 1\!\!1^*(\eta)\int\limits_{\mathbb{R}^d} G(\emptyset,x)b^E(x,\emptyset)\mathrm{d}x + 1\!\!1^*(\eta)\int\limits_{\mathbb{R}^d} G(x,\emptyset)b^S(x,\emptyset)\mathrm{d}x$$

is a positive operator. Denote by $B_{01} : \mathcal{L}^{\geq 1}_{\alpha,\beta} \longrightarrow \mathcal{L}^{0}_{\alpha,\beta}$, $B_{01}G = \mathbb{1}^* BG$ and by $L_{11} : \mathcal{L}^{\geq 1}_{\alpha,\beta} \longrightarrow \mathcal{L}^{\geq 1}_{\alpha,\beta}$, $L_{11}G = AG + (1 - \mathbb{1}^*)BG$ the restrictions to $\mathcal{L}^{\geq 1}_{\alpha,\beta}$, therefore we obtain

$$\widehat{L}G = B_{01}(1 - \mathbb{1}^*)G + L_{11}(1 - \mathbb{1}^*)G, \quad G \in \mathcal{L}_{\alpha,\beta}. \tag{4.38}$$

Moreover, since $D(\widehat{L}) = \{G \in \mathcal{L}_{\alpha,\beta} \mid M \cdot G \in \mathcal{L}_{\alpha,\beta}\}$ and $\mathcal{L}^{0}_{\alpha,\beta} \subset D(\widehat{L})$ it follows that $D(L_{11}) = D(\widehat{L}) \cap \mathcal{L}^{\geq 1}_{\alpha,\beta}$. The next theorem provides information about $\ker(\widehat{L})$ and the resolvent set $\rho(\widehat{L})$ on $\mathcal{L}_{\alpha,\beta}$.

Theorem 4.5.3. *Let*

$$\omega_0 := \sup \left\{ \omega \in \left[0, \frac{\pi}{4}\right] \;\middle|\; a(\alpha,\beta) < 1 + \cos(\omega) \right\}, \tag{4.39}$$

then the following statements hold:

1. *The point $\lambda = 0$ is an eigenvalue for $(\widehat{L}, D(\widehat{L}))$ with eigenspace $\mathcal{L}^{0}_{\alpha,\beta}$ and eigenvector $\mathbb{1}^*$.*

2. *Let $\lambda_0 := (2 - a(\alpha,\beta))M_* > 0$, where $M_* := \inf_{|\eta|\geq 1} M(\eta) > 0$. Then*

$$I_1 := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > -\lambda_0\} \backslash \{0\}$$

and

$$I_2 := \left\{ \lambda \in \mathbb{C} \;\middle|\; |\arg(\lambda)| < \frac{\pi}{2} + \omega_0 \right\} \backslash \{0\}$$

belong to the resolvent set $\rho(\widehat{L})$ of $\widehat{L}$ on $\mathcal{L}_{\alpha,\beta}$.

Proof. Let $(A_1, D(L_{11}))$ be the restriction of $(A, D(\widehat{L}))$ to $\mathcal{L}^{\geq 1}_{\alpha,\beta}$ and denote by $\|\cdot\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}$ the norm on $\mathcal{L}^{\geq 1}_{\alpha,\beta}$. This restriction is simply given by $AG = A_1(1 - \mathbb{1}^*)G$. Moreover for any $\lambda = u + iw$, $u \geq 0$, $w \in \mathbb{R}$

$$\left| \frac{G}{\lambda + M(\eta)} \right| \leq \frac{|G|}{\sqrt{(u + M_*)^2 + w^2}} \leq |G| \min\left(\frac{1}{|\lambda|}, \frac{1}{\sqrt{M_*^2 + w^2}} \right)$$

implies that $\lambda \in \rho(A_1)$ and

$$\|R(\lambda; A_1)G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq \min\left(\frac{1}{|\lambda|}, \frac{1}{\sqrt{M_*^2 + w^2}} \right) \|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}. \tag{4.40}$$

Let us show that for $\lambda = u + iw$, $u \geq 0$, $w \in \mathbb{R}$ the operator $\lambda - L_{11}$ is invertible, i.e. $\lambda \in \rho(L_{11})$. Due to the decomposition

$$(\lambda - L_{11}) = (1 - (1 - \mathbb{1}^*)BR(\lambda; A_1))(\lambda - A_1) \tag{4.41}$$

it suffices to show that $(1-(1-\mathbb{1}^*)BR(\lambda;A_1))$ is invertible on $\mathcal{L}^{\geq 1}_{\alpha,\beta}$. We obtain therefore

$$R(\lambda;L_{11}) = R(\lambda;A_1)(1-(1-\mathbb{1}^*)BR(\lambda;A_1))^{-1}. \tag{4.42}$$

In fact $(1-(1-\mathbb{1}^*)BR(\lambda;A_1))$ is invertible provided for any $G \in \mathcal{L}^{\geq 1}_{\alpha,\beta}$

$$\|(1-\mathbb{1}^*)BR(\lambda;A_1)G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq q\|A_1 G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}$$

for some constant $q \in (0,1)$. But this simply means that $(1-\mathbb{1}^*)B$ is relatively bounded to A_1 with constant q. Now let B' be the positive operator defined in the proof of Theorem 4.2.3, then $|BG| \leq B'|G|$ and $B|G|(\varnothing) = \mathbb{1}^*B|G|(\varnothing) = \mathbb{1}^*B'|G|(\varnothing) \geq 0$. Therefore we obtain for $q := a(\alpha,\beta) - 1 < 1$

$$\begin{aligned}
\|(1-\mathbb{1}^*)BG\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} &= \int\limits_{\Gamma_0^2\setminus\{\varnothing\}} |BG(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) \\
&\leq \int\limits_{\Gamma_0^2} B'|G|(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) - B|G|(\varnothing) \\
&\leq \int\limits_{\Gamma_0^2} (c(\alpha,\beta;\eta) - M(\eta))|G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) \\
&\leq (a(\alpha,\beta)-1)\int\limits_{\Gamma_0^2} M(\eta)|G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) \\
&= q \int\limits_{\Gamma_0^2\setminus\{\varnothing\}} M(\eta)|G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) = q\|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}
\end{aligned}$$

and hence our claim. In particular we obtain for $\lambda = u + iw$, $u \geq 0$, $w \in \mathbb{R}$ by (4.42) and (4.40) for $\lambda \in \rho(L_{11})$,

$$\|R(\lambda;L_{11})G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq \frac{\min\left(\frac{1}{|\lambda|}, \frac{1}{\sqrt{M_*^2+w^2}}\right)}{2-a(\alpha,\beta)}\|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}$$

and for $\lambda = iw$, $w \in \mathbb{R}$

$$\|R(iw,L_{11})G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq \frac{\sqrt{M_*^2+w^2}^{-1}}{2-a(\alpha,\beta)}\|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}.$$

For $\lambda = u + iw$, $0 > u > -\lambda_0$ and $w \in \mathbb{R}$ write

$$(u + iw - L_{11}) = (1 + uR(iw;L_{11}))(iw - L_{11}).$$

Then by $|u| < \lambda_0$ and $\frac{|u|}{\sqrt{M_*^2+w^2}}\frac{1}{2-a} \leq \frac{|u|}{\lambda_0} < 1$ we obtain $\lambda \in \rho(L_{11})$ and

$$\|R(\lambda; L_{11})G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq \frac{\sqrt{M_*^2 + w^2}^{-1}}{2 - a(\alpha, \beta)} \left(1 - \frac{|u|}{\lambda_0}\right)^{-1} \|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}.$$

Therefore I_1 belongs to the resolvent set of L_{11}. For I_2 let $\lambda = u + iw \in I_2$ and $u < 0$, then there exists $\omega \in (0, \omega_0)$ such that $|\arg(\lambda)| < \frac{\pi}{2} + \omega$ and hence

$$|w| = |\tan(\arg(\lambda))||u| \geq \cot(\omega)|u|.$$

This implies for $\eta \neq \varnothing$

$$|\lambda + M(\eta)|^2 = (u + M(\eta))^2 + w^2 \geq (u + M(\eta))^2 + \cot(\omega)^2 u^2.$$

The right-hand side is minimal for the choice $u = -\frac{M(\eta)}{1+\cot(\omega)^2}$ which yields

$$\begin{aligned}|\lambda + M(\eta)|^2 &\geq M(\eta)^2 \left(\left(\frac{\cot(\omega)^2}{1 + \cot(\omega)^2}\right)^2 + \frac{\cot(\omega)^2}{(1 + \cot(\omega)^2)^2}\right) \\ &= M(\eta)^2 \frac{\cot(\omega)^2}{1 + \cot(\omega)^2} = M(\eta)^2 \cos(\omega)^2.\end{aligned}$$

Then by

$$\|(1 - \mathbb{1}^*)BR(\lambda; A_1)G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq q\|A_1 R(\lambda; A_1)G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq \frac{q}{\cos(\omega)}\|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}$$

and (4.39) $q = a(\alpha, \beta) - 1 < \cos(\omega)$. By (4.41) we obtain $I_2 \subset \rho(L_{11})$ and for each $\lambda = u + iw$ such that $\frac{\pi}{2} < |\arg(\lambda)| < \frac{\pi}{2} + \omega$, $\lambda \neq 0$ for some $\omega \in (0, \omega_0)$

$$\begin{aligned}\|R(\lambda; L_{11})G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} &\leq \frac{\sqrt{(u^2 + M_*)^2 + w^2}^{-1}}{1 - \frac{q}{\cos(\omega)}}\|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \\ &\leq \frac{(1 - \frac{q}{\cos(\omega)})^{-1}}{|w|}\|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq \sqrt{2}\frac{(1 - \frac{q}{\cos(\omega)})^{-1}}{|\lambda|}\|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}.\end{aligned}$$

where we have used $|w| \geq \frac{|\lambda|}{\sqrt{2}}$ in the last estimate. For the first claim let $\psi \in D(\widehat{L})$ be an eigenvector to the eigenvalue 0. The decomposition $\psi = \mathbb{1}^*\psi + (1 - \mathbb{1}^*)\psi = \psi_0 + \psi_1$ with $\psi_0 \in \mathcal{L}^0_{\alpha,\beta}$ and $\psi_1 \in \mathcal{L}^{\geq 1}_{\alpha,\beta} \cap D(\widehat{L}) = D(L_{11})$ yields by (4.38)

$$0 = \widehat{L}\psi = \mathbb{1}^*B\psi_1 + L_{11}\psi_1 \in \mathcal{L}^0_{\alpha,\beta} \oplus \mathcal{L}^{\geq 1}_{\alpha,\beta}.$$

Hence $L_{11}\psi_1 = 0$ and since $0 \in \rho(L_{11})$ also $\psi_1 = 0$. For the second statement let $\lambda \in I_1 \cup I_2$ and $H = H_0 + H_1 \in \mathcal{L}^0_{\alpha,\beta} \oplus \mathcal{L}^{\geq 1}_{\alpha,\beta}$. Then we have to find $G \in D(\widehat{L})$ such that

$$(\lambda - \widehat{L})G = H.$$

Using again the decomposition of $\widehat{L}$, above equation is equivalent to the system of equations

$$\begin{aligned} \lambda G_0 - \mathbb{1}^* B G_1 &= H_0 \\ (\lambda - L_{11}) G_1 &= H_1. \end{aligned}$$

Since $\lambda \in I_1 \cup I_2 \subset \rho(L_{11})$ the second equation has a unique solution on $\mathcal{L}^{\geq 1}_{\alpha,\beta}$ given by $G_1 = R(\lambda; L_{11}) H_1$. Therefore G_0 is given by

$$G_0 = \frac{1}{\lambda} \left(H_0 + \mathbb{1}^* B R(\lambda; L_{11}) H_1 \right).$$

□

Remark 4.5.4. *The proof shows that for any $\varepsilon > 0$ there exists $\omega = \omega(\varepsilon) \in (0, \frac{\pi}{2})$ such that*

$$\Sigma(\varepsilon) := \left\{ \lambda \in \mathbb{C} \;\middle|\; |\arg(\lambda + \lambda_0 - \varepsilon)| \leq \frac{\pi}{2} + \omega \right\} \subset I_1 \cup I_2 \cup \{0\}$$

and there exists $M(\varepsilon) > 0$ such that

$$\|R(\lambda; L_{11}) G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}} \leq \frac{M(\varepsilon)}{|\lambda|} \|G\|_{\mathcal{L}^{\geq 1}_{\alpha,\beta}}$$

for all $\lambda \in \Sigma(\varepsilon) \backslash \{0\}$. Moreover, $(L_{11}, D(L_{11}))$ is a sectorial operator of angle ω_0 on $\mathcal{L}^{\geq 1}_{\alpha,\beta}$. Denote by $\widetilde{T}(t)$ the bounded analytic semigroup on $\mathcal{L}^{\geq 1}_{\alpha,\beta}$ given by (in the uniform operator topology)

$$\widetilde{T}(t) = \frac{1}{2\pi i} \int_\sigma e^{\zeta t} R(\zeta; L_{11}) \mathrm{d}\zeta, \quad t > 0, \tag{4.43}$$

see [Paz83]. Here σ denotes any piecewise smooth curve in

$$\left\{ \lambda \in \mathbb{C} \;\middle|\; |\arg(\lambda)| < \frac{\pi}{2} + \omega_0 \right\} \backslash \{0\}$$

running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\theta \in (\frac{\pi}{2}, \frac{\pi}{2} + \omega_0)$.

The $\mathcal{L}^{\geq 1}_{\alpha,\beta}$ part of $\widehat{T}(t)$ is given by $(1 - \mathbb{1}^*) \widehat{T}(t) (1 - \mathbb{1}^*)$, hence has generator $(1 - \mathbb{1}^*) \widehat{L} (1 - \mathbb{1}^*) = L_{11}$ and therefore coincides with $(\widetilde{T}(t))_{t \geq 0}$. This yields the decomposition

$$\widehat{T}(t) = \mathbb{1}^* + \mathbb{1}^* \widehat{T}(t)(1 - \mathbb{1}^*) + \widetilde{T}(t)(1 - \mathbb{1}^*), \quad t \geq 0. \tag{4.44}$$

and by duality we see that the adjoint semigroup $(\widehat{T}(t)^*)_{t \geq 0}$ on $\mathcal{K}_{\alpha,\beta}$ admits the decomposition

$$\widehat{T}(t)^* = \mathbb{1}^* + (1 - \mathbb{1}^*) \widehat{T}(t)^* \mathbb{1}^* + \widetilde{T}(t)^* (1 - \mathbb{1}^*), \quad t \geq 0, \tag{4.45}$$

where $\widetilde{T}(t)^* \in L(\mathcal{K}^{\geq 1}_{\alpha,\beta})$ is the adjoint semigroup to $(\widetilde{T}(t))_{t \geq 0}$. The next lemma provides a construction of the limiting projection operators, when $t \to \infty$.

Lemma 4.5.5. *Define a linear operator* $\widehat{S}G := B\frac{1}{M}(1-\mathbb{1}^*)G$ *on* $\mathcal{L}_{\alpha,\beta}$*, then* $\widehat{S} : \mathcal{L}_{\alpha,\beta} \longrightarrow \mathcal{L}_{\alpha,\beta}$ *is bounded and for any* $G \in \mathcal{L}_{\alpha,\beta}$ *and* $k \in \mathcal{K}_{\alpha,\beta}$

$$\langle \widehat{S}G, k\rangle = \langle G, Sk\rangle. \tag{4.46}$$

The operators, cf. (4.37),

$$\widehat{P}^* := \mathbb{1}^* + (1-S)^{-1}S\mathbb{1}^*$$

and

$$\widehat{P} := \mathbb{1}^* + \mathbb{1}^*\widehat{S}(1-\widehat{S})^{-1}$$

are projections on $\mathcal{K}_{\alpha,\beta}$ *and* $\mathcal{L}_{\alpha,\beta}$ *respectively, such that*

$$\langle \widehat{P}G, k\rangle = \langle G, \widehat{P}^*k\rangle.$$

Proof. First observe that $M(\eta) > 0$ for any $\eta \neq \emptyset$ and since $B : D(\widehat{L}) \cap \mathcal{L}_{\alpha,\beta}^{\geq 1} \longrightarrow \mathcal{L}_{\alpha,\beta}$ is well-defined, so is $\widehat{S}$. The inequalities

$$\begin{aligned}
\int\limits_{\Gamma_0^2} |\widehat{S}G(\eta)| e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) &= \int\limits_{\Gamma_0^2} \left| B(1-\mathbb{1}^*)\frac{G}{M}(\eta)\right| e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) \\
&\leq \int\limits_{\Gamma_0^2\backslash\{\emptyset\}} (c(\alpha,\beta;\eta) - M(\eta))\frac{|G(\eta)|}{M(\eta)} e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta) \\
&\leq (a(\alpha,\beta)-1)\int\limits_{\Gamma_0^2} |G(\eta)| e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta)
\end{aligned}$$

imply that $\widehat{S}$ is bounded with norm $\|\widehat{S}G\|_{\mathcal{L}_{\alpha,\beta}} \leq (a-1)\|G\|_{\mathcal{L}_{\alpha,\beta}}$ and property (4.46) is a consequence of the definition of $\widehat{S}$ and a short computation. Because of (4.20) we have $a - 1 < 1$ and hence $\widehat{P}$ is well-defined. The second part is a consequence of (4.46) and the representation formulas for $\widehat{P}$ and $\widehat{P}^*$. □

Since $\widehat{P}$ projects, by definition, onto $\mathcal{L}_{\alpha,\beta}^0$ it follows that $\widehat{P}G = \langle G, \widetilde{k}\rangle \mathbb{1}^*$ for some $\widetilde{k} \in \mathcal{K}_{\alpha,\beta}$. Therefore we obtain

$$\langle \mathbb{1}^*, k\rangle \, \langle G, \widetilde{k}\rangle = \langle \widehat{P}G, k\rangle = \langle G, \widehat{P}^*k\rangle$$

and hence

$$k(\emptyset)\widetilde{k}(\eta) = \widehat{P}^*k(\eta) = \widehat{P}^*(\mathbb{1}^*k)(\eta).$$

The right-hand-side depends only on the value $k(\emptyset)$, hence we can divide by $k(\emptyset) \neq 0$ which yields

$$\widetilde{k} = \widehat{P}^*\mathbb{1}^* = k_{\mathrm{inv}}$$

and therefore
$$\widehat{P}G(\eta) = \langle G, k_{\text{inv}}\rangle \mathbb{1}^*(\eta), \quad \eta \in \Gamma_0^2.$$
The adjoint operator $\widehat{P}^*$ is then given by

$$\widehat{P}^*k(\eta) = k(\emptyset)k_{\text{inv}}(\eta) = (\mathbb{1}^*k)(\eta)k_{\text{inv}}(\eta), \quad \eta \in \Gamma_0^2. \tag{4.47}$$

By $\widehat{T}(t)\mathbb{1}^* = \mathbb{1}^*$ this formulas yield

$$\widehat{P} = \widehat{T}(t)\widehat{P} = \widehat{P}\widehat{T}(t)$$

and $\widehat{P}^2 = \widehat{P}$. In the same way $(\widehat{P}^*)^2 = \widehat{P}^*$ and

$$\widehat{T}(t)^*\widehat{P}^* = \widehat{P}^*\widehat{T}(t)^* = \widehat{P}^*. \tag{4.48}$$

Now we are prepared to prove Proposition 4.5.1, i.e. ergodicity of the semigroups $\widehat{T}(t)$ and $\widehat{T}(t)^*$.

Proof. (Proposition 4.5.1)
The spectral properties stated in Remark 4.5.4, the representation formula (4.43) and (4.44), (4.45) imply that for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for any $t \geq 0$

$$\|(1 - \mathbb{1}^*)\widehat{T}(t)G\|_{\mathcal{L}_{\alpha,\beta}} \leq C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\|G\|_{\mathcal{L}_{\alpha,\beta}}, \quad G \in \mathcal{L}_{\alpha,\beta}^{\geq 1}$$

and hence by duality

$$\|\widehat{T}(t)^*k\|_{\mathcal{K}_{\alpha,\beta}} \leq C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\|k\|_{\mathcal{K}_{\alpha,\beta}}, \quad k \in \mathcal{K}_{\alpha,\beta}^{\geq 1},$$

repeat e.g. the arguments in [KKM10]. Let $k \in \mathcal{K}_{\alpha,\beta}$, by (4.47) we obtain

$$k - \widehat{P}^*k = (1 - \mathbb{1}^*)k \cdot k_{\text{inv}} \in \mathcal{K}_{\alpha,\beta}^{\geq 1}.$$

Using (4.48) we see that

$$\|\widehat{T}(t)^*k - \widehat{P}^*k\|_{\mathcal{K}_{\alpha,\beta}} = \|\widehat{T}(t)^*(k - \widehat{P}^*k)\|_{\mathcal{K}_{\alpha,\beta}} \leq C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\|k - \widehat{P}^*k\|_{\mathcal{K}_{\alpha,\beta}} \tag{4.49}$$

holds. Let $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$, $\mu_t \in \mathcal{P}_{\alpha,\beta}$ the associated evolution of states and $k_{\mu_t} \in \mathcal{K}_{\alpha,\beta}$ its correlation function for $t \geq 0$. Then for any $t \geq 0$

$$\begin{aligned}\|\mu_t - \mu_{\text{inv}}\|_{\mathcal{E}^*_{\alpha,\beta}} &= \|k_{\mu_t} - k_{\text{inv}}\|_{\mathcal{K}_{\alpha,\beta}} \\ &\leq C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\|k_{\mu_0} - k_{\text{inv}}\|_{\mathcal{K}_{\alpha,\beta}} = C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\|\mu_0 - \mu_{\text{inv}}\|_{\mathcal{E}^*_{\alpha,\beta}}\end{aligned}$$

holds and hence $k_{\rm inv}$ is a limit of positive definite functions. Thus there exists a unique measure $\mu_{\rm inv} \in \mathcal{P}_{\alpha,\beta}$ having $k_{\rm inv}$ as its correlation function. It follows for any $G \in B_{bs}(\Gamma_0^2)$

$$0 = \int\limits_{\Gamma_0^2} G(\eta) L^{\Delta} k_{\rm inv}(\eta) \mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0^2} \widehat{L} G(\eta) k_{\rm inv}(\eta) \mathrm{d}\lambda(\eta) = \int\limits_{\Gamma^2} L \mathbb{K} G(\gamma) \mathrm{d}\mu_{\rm inv}(\gamma)$$

and hence (4.34). Since $\widehat{T}(t)^* k_{\rm inv} = k_{\rm inv}$ it follows that $T(t)^* \mu_{\rm inv} = \mu_{\rm inv}$. It remains to show the estimate (4.35). Observe that by duality and (4.49) we obtain

$$\|\widehat{T}_{\alpha,\beta}(t) G - \widehat{P} G\|_{\mathcal{L}_{\alpha,\beta}} \leq C(\varepsilon)^{-(\lambda_0 - \varepsilon)t} \|G - \widehat{P} G\|_{\mathcal{L}_{\alpha,\beta}}. \tag{4.50}$$

Because of $\mathbb{K}\mathbb{1}^* = 1$ this implies

$$\|T_{\alpha,\beta}(t) K G - \langle G \rangle_{k_{\rm inv}}\|_{\mathcal{E}_{\alpha,\beta}} = \|\widehat{T}_{\alpha,\beta}(t) G - \widehat{P} G\|_{\mathcal{L}_{\alpha,\beta}}$$

and hence by (4.50) the convergence (4.35). □

Remark 4.5.6. *Let $\ell \in \mathcal{E}_{\alpha,\beta}^*$ and take $k_\ell \in \mathcal{K}_{\alpha,\beta}$ determined by (4.25). Define $\ell_{\rm inv}$ by $\ell_{\rm inv}(\mathbb{K}G) = \langle G, k_{\rm inv} \rangle \ell(1)$, where $k_{\rm inv}$ is the unique correlation function associated to the invariant measure $\mu_{\rm inv}$. Then*

$$\|T(t)^* \ell - \ell_{\rm inv}\|_{\mathcal{E}_{\alpha,\beta}^*} \leq C(\varepsilon) e^{-(\lambda_0 - \varepsilon)t} \|\ell - \ell_{\rm inv}\|_{\mathcal{E}_{\alpha,\beta}^*}$$

holds. That is $(T(t)^*)_{t \geq 0}$ is ergodic on $\mathcal{E}_{\alpha,\beta}^*$.

4.6 Vlasov scaling

Consider for $n \in \mathbb{N}$ scaled intensities $d_n^S, d_n^E, b_n^S, b_n^E \geq 0$ and suppose they satisfy condition (A). Let $L_n = L_n^S + L_n^E$ where

$$\begin{aligned} L_n^E F(\gamma) = &\sum_{x \in \gamma^-} d_n^E(x, \gamma^+, \gamma^- \backslash x)(F(\gamma^+, \gamma^- \backslash x) - F(\gamma^+, \gamma^-)) \\ &+ n \int\limits_{\mathbb{R}^d} b_n^E(x, \gamma^+, \gamma^-)(F(\gamma^+, \gamma^- \cup x) - F(\gamma^+, \gamma^-)) \mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} L_n^S(t) F(\gamma) = &\sum_{x \in \gamma^+} d_n^S(x, \gamma^+ \backslash x, \gamma^-)(F(\gamma^+ \backslash x, \gamma^-) - F(\gamma^+, \gamma^-)) \\ &+ n \int\limits_{\mathbb{R}^d} b_n^S(x, \gamma^+, \gamma^-)(F(\gamma^+ \cup x, \gamma^-) - F(\gamma^+, \gamma^-)) \mathrm{d}x. \end{aligned}$$

Introduce

$$
\begin{aligned}
c_n(\alpha,\beta;\eta) := +& \sum_{x\in\eta^-}\int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d_n^E(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
& \sum_{x\in\eta^+}\int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d_n^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
& + e^{-\beta} \sum_{x\in\eta^-}\int_{\Gamma_0} |\mathbb{K}_0^{-1} b_n^E(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
& + e^{-\alpha} \sum_{x\in\eta^+}\int_{\Gamma_0} |\mathbb{K}_0^{-1} b_n^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi)
\end{aligned}
$$

and $M_n(\eta) := \sum\limits_{x\in\eta^-} d_n^E(x,\eta^+,\eta^-\backslash x) + \sum\limits_{x\in\eta^+} d_n^S(x,\eta^+\backslash x,\eta^-)$. We will suppose the following conditions to be satisfied:

(V1) There exists $a(\alpha,\beta)\in(0,2)$ such that for all $\eta\in\Gamma_0^2$ and $n\in\mathbb{N}$

$$
c_n(\alpha,\beta;\eta) \le a(\alpha,\beta) M_n(\eta)
$$

is satisfied.

(V2) For all $\xi\in\Gamma_0^2$ and $x\in\mathbb{R}^d$ the following limits exist in $\mathcal{L}_{\alpha,\beta}$ and are independent of ξ

$$
\begin{aligned}
\lim_{n\to\infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^E(x,\cdot\cup\xi)) &= \lim_{n\to\infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^E(x,\cdot)) =: D_x^{V,E} \\
\lim_{n\to\infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^S(x,\cdot\cup\xi)) &= \lim_{n\to\infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^S(x,\cdot)) =: D_x^{V,S} \\
\lim_{n\to\infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^E(x,\cdot\cup\xi)) &= \lim_{n\to\infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^E(x,\cdot)) =: B_x^{V,E} \\
\lim_{n\to\infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^S(x,\cdot\cup\xi)) &= \lim_{n\to\infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^S(x,\cdot)) =: B_x^{V,S}.
\end{aligned}
$$

(V3) Let $M_V(\eta) := \sum\limits_{x\in\eta^+} D_x^S(\varnothing) + \sum\limits_{x\in\eta^-} D_x^E(\varnothing)$, then there exists $\sigma > 0$ such that either

$$
M_n(\eta) \le \sigma M_V(\eta), \quad \eta\in\Gamma_0^2,\ n\in\mathbb{N}
$$

or

$$
M_n(\eta) \le \sigma M_V(\eta), \quad \eta\in\Gamma_0^2,\ n\in\mathbb{N}
$$

are satisfied.

Define $\widehat{L}_n := \mathbb{K}_0^{-1} L_n \mathbb{K}_0$ and the renormalized operators $\widehat{L}_{n,\mathrm{ren}} := R_n \widehat{L}_n R_{n^{-1}}$, where $R_\alpha G(\eta) = \alpha^{|\eta|} G(\eta)$. Then we get $\widehat{L}_{n,\mathrm{ren}} = A_n + B_n$ with $(A_n G)(\eta) = -M_n(\eta) G(\eta)$, where

$$M_n(\eta) = \sum_{x \in \eta^-} d_n^E(x, \eta^+, \eta^- \backslash x) + \sum_{x \in \eta^+} d_n^S(x, \eta^+ \backslash x, \eta^-) \geq 0$$

and

$$\begin{aligned}
(B_n G)(\eta) = &- \sum_{\xi \subsetneq \eta} G(\xi) n^{|\eta \backslash \xi|} \sum_{x \in \xi^-} (\mathbb{K}_0^{-1} d_n^E(x, \cdot \cup \xi^+, \cdot \cup \xi^- \backslash x))(\eta \backslash \xi) \\
&- \sum_{\xi \subsetneq \eta} G(\xi) n^{|\eta \backslash \xi|} \sum_{x \in \xi^+} (\mathbb{K}_0^{-1} d_n^S(x, \cdot \cup \xi^+ \backslash x, \cdot \cup \xi^-))(\eta \backslash \xi) \\
&+ \sum_{\xi \subset \eta} n^{|\eta \backslash \xi|} \int_{\mathbb{R}^d} G(\xi^+, \xi^- \cup x)(\mathbb{K}_0^{-1} b_n^E(x, \cdot \cup \xi^+, \cdot \cup \xi^-))(\eta \backslash \xi) \mathrm{d}x \\
&+ \sum_{\xi \subset \eta} n^{|\eta \backslash \xi|} \int_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-)(\mathbb{K}_0^{-1} b_n^S(x, \cdot \cup \xi^+, \cdot \cup \xi^-))(\eta \backslash \xi) \mathrm{d}x.
\end{aligned}$$

In analogy to L^Δ, cf. Lemma 4.3.2, define a linear operator $L^\Delta_{n,\mathrm{ren}}$ by

$$\begin{aligned}
(L^\Delta_{n,\mathrm{ren}} k)(\eta) = &- \sum_{x \in \eta^-} \int_{\Gamma_0^2} k(\eta \cup \xi) n^{|\xi|} (\mathbb{K}_0^{-1} d_n^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x))(\xi) \mathrm{d}\lambda(\xi) \\
&- \sum_{x \in \eta^+} \int_{\Gamma_0^2} k(\eta \cup \xi) n^{|\xi|} (\mathbb{K}_0^{-1} d_n^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-))(\xi) \mathrm{d}\lambda(\xi) \\
&+ \sum_{x \in \eta^-} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+, \eta^- \cup \xi^- \backslash x) n^{|\xi|} (\mathbb{K}_0^{-1} b_n^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x))(\xi) \mathrm{d}\lambda(\xi) \\
&+ \sum_{x \in \eta^+} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+ \backslash x, \eta^- \cup \xi^-) n^{|\xi|} (\mathbb{K}_0^{-1} b_n^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-))(\xi) \mathrm{d}\lambda(\xi).
\end{aligned}$$

We obtain for any $G \in B_{bs}(\Gamma_0^2)$, $k \in \mathcal{K}_{\alpha,\beta}$ and $n \in \mathbb{N}$

$$\langle \widehat{L}_{n,\mathrm{ren}} G, k \rangle = \langle G, L^\Delta_{n,\mathrm{ren}} k \rangle.$$

The next theorem provides existence and uniqueness of an evolution of quasi-observables and correlation functions for any fixed $n \in \mathbb{N}$.

Theorem 4.6.1. *Suppose that condition (V1) is satisfied. Then for any fixed $n \in \mathbb{N}$ the following assertions are true:*

1. The closure of $(\widehat{L}_{n,\mathrm{ren}}, B_{bs}(\Gamma_0^2))$ *is given by* $(\widehat{L}_{n,\mathrm{ren}}, D(\widehat{L}_{n,\mathrm{ren}}))$*, where*

$$D(\widehat{L}_{n,\mathrm{ren}}) = \{G \in \mathcal{L}_{\alpha,\beta} \mid M_n \cdot G \in \mathcal{L}_{\alpha,\beta}\}.$$

It is the generator of an analytic C_0*-semigroup* $(\widehat{T}_{n,\mathrm{ren}}(s))_{s\geq 0}$ *of contractions on* $\mathcal{L}_{\alpha,\beta}$*.*

2. Let $\widehat{T}_{n,\mathrm{ren}}(t)^*$ *be the adjoint semigroup. The generator is given by* $(L^{\Delta}_{n,\mathrm{ren}}, D(L^{\Delta}_{n,\mathrm{ren}}))$ *with the (maximal) domain*

$$D(L^{\Delta}_{n,\mathrm{ren}}) = \{k \in \mathcal{K}_{\alpha,\beta} \mid L^{\Delta}_{n,\mathrm{ren}} k \in \mathcal{K}_{\alpha,\beta}\}.$$

For any $n \in \mathbb{N}$ *and* $k_0 \in \mathcal{K}_{\alpha,\beta}$*, there exists a unique weak solution to*

$$\frac{\partial}{\partial t}\langle G, k_{t,n}\rangle = \langle \widehat{L}_{n,\mathrm{ren}} G, k_{t,n}\rangle, \quad k_{t,n}|_{t=0} = k_0, \quad G \in D(\widehat{L}_{n,\mathrm{ren}})$$

given by $k_{t,n} = \widehat{T}_{n,\mathrm{ren}}(t)^* k_0$*.*

The case $n = 1$ is covered by the results obtained in Theorem 4.3.3. Following the arguments there, it is not difficult to adopt the proofs to this case. In the next step we construct the limiting dynamics when $n \to \infty$. Condition (V2) suggests to consider the limit

$$\widehat{L}_{n,\mathrm{ren}} G \longrightarrow \widehat{L}_V G, \quad n \to \infty.$$

The operator $\widehat{L}_V := A_V + B_V$ is given by $A_V G(\eta) = -M_V(\eta) G(\eta)$, where

$$\begin{aligned} M_V(\eta) &= \sum_{x\in\eta^+} D_x^{V,S}(\varnothing) + \sum_{x\in\eta^-} D_x^{V,E}(\varnothing) \\ B_V G(\eta) &= -\sum_{\substack{\xi^+\subsetneq\eta^+\\ \xi^-\subsetneq\eta^-}} G(\xi) \sum_{x\in\xi^+} D_x^{V,S}(\eta\backslash\xi) - \sum_{\substack{\xi^+\subsetneq\eta^+\\ \xi^-\subsetneq\eta^-}} G(\xi) \sum_{x\in\xi^-} D_x^{V,E}(\eta\backslash\xi) \\ &\quad + \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} G(\xi^+\cup x, \xi^-) B_x^{V,S}(\eta\backslash\xi)\mathrm{d}x + \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} G(\xi^+, \xi^-\cup x) B_x^{V,E}(\eta\backslash\xi)\mathrm{d}x. \end{aligned}$$

In the next theorem we establish existence and uniqueness of the dynamics described by the limiting operator $\widehat{L}_V$. Therefore let $D(\widehat{L}_V) := \{G \in \mathcal{L}_{\alpha,\beta} \mid M_V \cdot G \in \mathcal{L}_{\alpha,\beta}\}$, define

$$\begin{aligned} &c_V(\alpha,\beta;\eta) := \\ &+ \sum_{x\in\eta^+} \int_{\Gamma_0^2} |D_x^{V,S}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) + e^{-\alpha} \sum_{x\in\eta^+} \int_{\Gamma_0^2} |B_x^{V,S}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\ &+ \sum_{x\in\eta^-} \int_{\Gamma_0^2} |D_x^{V,E}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) + e^{-\beta} \sum_{x\in\eta^-} \int_{\Gamma_0^2} |B_x^{V,E}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \end{aligned}$$

and finally

$$
\begin{aligned}
(L_V^{\triangle}k)(\eta) := &- \sum_{x\in\eta^+} \int_{\Gamma_0^2} k(\eta\cup\xi) D_x^{V,S}(\xi)\mathrm{d}\lambda(\xi) - \sum_{x\in\eta^-} \int_{\Gamma_0^2} k(\eta\cup\xi) D_x^{V,E}(\xi)\mathrm{d}\lambda(\xi) \\
&+ \sum_{x\in\eta^+} \int_{\Gamma_0^2} k(\eta^+\backslash x\cup\xi^+, \eta^+\cup\xi^+) B_x^{V,S}(\xi)\mathrm{d}\lambda(\xi) \\
&+ \sum_{x\in\eta^-} \int_{\Gamma_0^2} k(\eta^+\cup\xi^+, \eta^-\backslash x\cup\xi^-) B_x^{V,E}(\xi)\mathrm{d}\lambda(\xi).
\end{aligned}
$$

Theorem 4.6.2. *Assume that conditions (V1), (V2) are satisfied. Then the following assertions are true:*

1. *The operator $(\widehat{L}_V, D(\widehat{L}_V))$ is the generator of an analytic semigroup $(\widehat{T}^V(t))_{t\geq 0}$ of contractions on $\mathcal{L}_{\alpha,\beta}$.*

2. *Let $(\widehat{T}^V(t)^*)_{t\geq 0}$ be the adjoint semigroup on $\mathcal{K}_{\alpha,\beta}$, then for any $r_0 \in \mathcal{K}_{\alpha,\beta}$ there exists a unique solution $r_t = \widehat{T}^V(t)^* r_0$ to the Cauchy problem*

$$
\frac{\partial}{\partial t}\langle G, r_t\rangle = \langle \widehat{L}_V G, r_t\rangle, \quad r_t|_{t=0} = r_0, \quad G \in D(\widehat{L}_V). \tag{4.51}
$$

3. *Let $r_0(\eta) = \prod_{x\in\eta^+} \rho_0^S(x) \prod_{x\in\eta^-} \rho_0^E(x)$ and $\rho_0^S, \rho_0^E \in L^\infty(\mathbb{R}^d)$ with $\|\rho_0^S\|_{L^\infty} \leq e^\alpha$, $\|\rho_0^E\|_{L^\infty} \leq e^\beta$. Assume that (ρ_t^S, ρ_t^E) is a classical solution to*

$$
\begin{aligned}
\frac{\partial \rho_t^E}{\partial t}(x) = &- \int_{\Gamma_0^2} e_\lambda(\rho_t^S;\xi^+) e_\lambda(\rho_t^E;\xi^-) D_x^{V,E}(\xi)\mathrm{d}\lambda(\xi)\rho_t^E(x) \\
&+ \int_{\Gamma_0^2} e_\lambda(\rho_t^S;\xi^+) e_\lambda(\rho_t^E;\xi^-) B_x^{V,E}(\xi)\mathrm{d}\lambda(\xi) \\
\frac{\partial \rho_t^S}{\partial t}(x) = &- \int_{\Gamma_0^2} e_\lambda(\rho_t^S;\xi^+) e_\lambda(\rho_t^E;\xi^-) D_x^{V,S}(\xi)\mathrm{d}\lambda(\xi)\rho_t^S(x) \\
&+ \int_{\Gamma_0^2} e_\lambda(\rho_t^S;\xi^+) e_\lambda(\rho_t^E;\xi^-) B_x^{V,S}(\xi)\mathrm{d}\lambda(\xi)
\end{aligned}
$$

with initial conditions $\rho_t^S|_{t=0} = \rho_0^S$, $\rho_t^E|_{t=0} = \rho_0^E$ and $\|\rho_t^S\|_{L^\infty} \leq e^\alpha$ and $\|\rho_t^E\|_{L^\infty} \leq e^\beta$. Then $r_t(\eta) := \prod_{x\in\eta^+} \rho_t^S(x) \prod_{x\in\eta^-} \rho_t^E(x)$ is a weak solution to (4.51) in $\mathcal{K}_{\alpha,\beta}$.

Proof. By conditions (V1) and (V2) it follows

$$\begin{aligned} c_V(\alpha,\beta;\eta) &\leq \lim_{n\to\infty} c_n(\alpha,\beta;\eta) \\ &\leq a(\alpha,\beta) \lim_{n\to\infty} M_n(\eta) = a(\alpha,\beta) M_V(\eta). \end{aligned}$$

Define a positive operator B_V' on $D(\widehat{L}_V)$ by

$$\begin{aligned} B_V' G(\eta) = &\sum_{\substack{\xi^+ \subsetneq \eta^+ \\ \xi^- \subsetneq \eta^-}} G(\xi) \sum_{x\in\xi^+} |D_x^{V,S}(\eta\backslash\xi)| + \sum_{\substack{\xi^+ \subsetneq \eta^+ \\ \xi^- \subsetneq \eta^-}} G(\xi) \sum_{x\in\xi^-} |D_x^{V,E}(\eta\backslash\xi)| \\ &+ \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} G(\xi^+\cup x, \xi^-) |B_x^{V,S}(\eta\backslash\xi)| \mathrm{d}x + \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} G(\xi^+, \xi^-\cup x) |B_x^{V,E}(\eta\backslash\xi)| \mathrm{d}x. \end{aligned}$$

Then it is not difficult to see that for any $0 \leq G \in D(\widehat{L}_V)$

$$\begin{aligned} \int_{\Gamma_0^2} B_V' G(\eta) e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) &= \int_{\Gamma_0^2} (c_V(\alpha,\beta;\eta) - M_V(\eta)) G(\eta) e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \\ &\leq (a(\alpha,\beta) - 1) \int_{\Gamma_0^2} M_V(\eta) G(\eta) e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \end{aligned}$$

is fulfilled. The same arguments as in the proof of Theorem 4.6.1 yield existence, analyticity and the contraction property of the semigroup $\widehat{T}^V(t)$. For the last assertion we only show that r_t is continuous w.r.t. $\mathcal{C}$. The other assertions are simple computations, see e.g. [FKO13]. First observe that by $|r_t(\eta)| \leq e^{\alpha|\eta^+|} e^{\beta|\eta^-|}$ the function r_t is norm-bounded and hence it suffices to show that it is continuous w.r.t. $\sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$. But this function is continuous in $t \geq 0$ for any η and hence the assertion follows by dominated convergence. □

Theorem 4.6.3. *Suppose that conditions (V1) – (V3) are fulfilled. Then* $\widehat{T}_{n,\mathrm{ren}}(t) \to \widehat{T}^V(t)$ *holds strongly in* $\mathcal{L}_{\alpha,\beta}$ *and uniformly on compacts in* $t \geq 0$.

Proof. We are going to apply [FKK12, Lemma 4.3] and Trotter-Kato approximation. Fix $\lambda > 0$ and denote by $R(\lambda; A_n)$ and $R(\lambda, A_V)$ the resolvent for A_ε and A_V, respectively.

Then it follows that $\|R(\lambda; A_n)\|_{L(\mathcal{L}_{\alpha,\beta})},\ \|R(\lambda; A_V)\|_{L(\mathcal{L}_{\alpha,\beta})} \leq \frac{1}{\lambda}$,

$$\begin{aligned}\|B_n R(\lambda; A_n)G\|_{\mathcal{L}_{\alpha,\beta}} &= \int\limits_{\Gamma_0^2} |B_n R(\lambda; A_n)G(\eta)| e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \\ &\leq \int\limits_{\Gamma_0^2} (c_n(\alpha, \beta; \eta) - M_n(\eta)) |R(\lambda; A_n)G(\eta)| e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \\ &\leq (a(\alpha, \beta) - 1) \int\limits_{\Gamma_0^2} \frac{M_n(\eta)}{\lambda + M_n(\eta)} |G(\eta)| e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \\ &\leq (a(\alpha, \beta) - 1) \|G\|_{\mathcal{L}_{\alpha,\beta}}\end{aligned}$$

and likewise

$$\begin{aligned}\|B_V R(\lambda; A_V)G\|_{\mathcal{L}_{\alpha,\beta}} &\leq \int\limits_{\Gamma_0^2} (c_V(\alpha, \beta; \eta) - M_V(\eta)) |R(\lambda; A_V)G(\eta)| e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \\ &\leq (a(\alpha, \beta) - 1) \|G\|_{\mathcal{L}_{\alpha,\beta}}\end{aligned}$$

hold. Since $M_n \longrightarrow M_V$ as $n \to \infty$, it is easy to show by dominated convergence that $R(\lambda; A_n) \longrightarrow R(\lambda; A_V)$ holds strongly in $\mathcal{L}_{\alpha,\beta}$ as $n \to \infty$. Hence it remains to show the convergence

$$B_n R(\lambda; A_n)G \longrightarrow B_V R(\lambda; A_V)G, \quad n \to \infty. \tag{4.52}$$

To do so, suppose that $M_n(\eta) \leq \sigma M_V(\eta)$ holds, then we estimate by

$$\begin{aligned}&\|B_n R(\lambda; A_n)G - B_V R(\lambda; A_V)G\|_{\mathcal{L}_{\alpha,\beta}} \\ &\leq \|(B_n - B_V)R(\lambda; A_V)G\|_{\mathcal{L}_{\alpha,\beta}} + \|B_n(R(\lambda; A_n) - R(\lambda; A_V))G\|_{\mathcal{L}_{\alpha,\beta}}.\end{aligned}$$

For the first term we obtain

$$\begin{aligned}\|(B_n - B_V)R(\lambda; A_V)G\|_{\mathcal{L}_{\alpha,\beta}} &= \int\limits_{\Gamma_0^2} |(B_n - B_V)R(\lambda; A_V)G(\eta)| e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \\ &\leq \int\limits_{\Gamma_0^2} \frac{|G(\eta)|}{\lambda + M_V(\eta)} e^{\alpha|\eta^+|} e^{\beta|\eta^-|} H_n(\eta) \mathrm{d}\lambda(\eta)\end{aligned}$$

where

$$
\begin{aligned}
&H_n(\eta) = \\
&+ \sum_{x\in\eta^-} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d_n^E(x, \cdot\cup\eta^+, \cdot\cup\eta^- \backslash x)(\xi) n^{|\xi|} - D_x^{V,E}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
&+ \sum_{x\in\eta^+} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d_n^S(x, \cdot\cup\eta^+ \backslash x, \cdot\cup\eta)(\xi) n^{|\xi|} - D_x^{V,S}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
&+ e^{-\beta} \sum_{x\in\eta^-} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} b_n^E(x, \cdot\cup\eta^+, \cdot\eta^- \cup \backslash x)(\xi) n^{|\xi|} - B_x^{V,E}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
&+ e^{-\alpha} \sum_{x\in\eta^+} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} b_n^S(x, \cdot\cup\eta^+ \backslash x, \cdot\cup\eta^-)(\xi) n^{|\xi|} - B_x^{V,S}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi).
\end{aligned}
$$

The $\mathcal{L}_{\alpha,\beta}$ convergence in condition (V2) implies that H_n tends to zero and because of

$$
H_n(\eta) \le c_n(\alpha,\beta;\eta) + c_V(\alpha,\beta;\eta) \le a(\alpha,\beta)\left(M_n(\eta) + M_V(\eta)\right) \le a(\alpha,\beta)(1+\sigma)M_V(\eta)
$$

dominated convergence implies $\|(B_n - B_V)R(\lambda;A_V)G\|_{\mathcal{L}_{\alpha,\beta}} \longrightarrow 0$, $n\to\infty$. For the second term we obtain

$$
\begin{aligned}
&\|B_n(R(\lambda;A_n) - R(\lambda;A_V))G\|_{\mathcal{L}_{\alpha,\beta}} \\
&= \int_{\Gamma_0^2} |B_n(R(\lambda;A_n) - R(\lambda;A_V))G(\eta)| e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \\
&\le \int_{\Gamma_0^2} (c_n(\alpha,\beta;\eta) - M_n(\eta)) \frac{|M_V(\eta) - M_n(\eta)|}{(\lambda + M_n(\eta))(\lambda + M_V(\eta))} e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta) \\
&\le (a(\alpha,\beta) - 1) \int_{\Gamma_0^2} M_n(\eta) \frac{|M_V(\eta) - M_n(\eta)|}{(\lambda + M_n(\eta))(\lambda + M_V(\eta))} e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \mathrm{d}\lambda(\eta)
\end{aligned}
$$

and observe that by (V2) the integrand tends to zero. Because of

$$
M_n(\eta) \frac{|M_V(\eta) - M_n(\eta)|}{(\lambda + M_n(\eta))(\lambda + M_V(\eta))} \le \frac{M_V(\eta)}{\lambda + M_V(\eta)} + \frac{M_n(\eta)}{\lambda + M_V(\eta)} \le 1 + \sigma
$$

we can apply dominated convergence and obtain therefore the assertion in the case $M_n \le \sigma M_V$. For the other case we estimate by

$$
\begin{aligned}
&\|B_n R(\lambda;A_n)G - B_V R(\lambda;A_V)G\|_{\mathcal{L}_{\alpha,\beta}} \\
&\le \|(B_n - B_V)R(\lambda;A_n)G\|_{\mathcal{L}_{\alpha,\beta}} + \|B_V(R(\lambda;A_n) - R(\lambda;A_V))G\|_{\mathcal{L}_{\alpha,\beta}}
\end{aligned}
$$

and apply similar arguments to deduce the assertion. □

Remark 4.6.4. *The proof shows that condition (V3) can be replaced by*

$$d_n^E(x,\eta) + d_n^S(x,\eta) \leq A(1+|\eta|)^N e^{\tau|\eta|}, \quad x \in \mathbb{R}^d, \ \eta \in \Gamma_0^2, \ n \in \mathbb{N}$$

for some constants $A > 0$, $N \in \mathbb{N}$ and $\tau \geq 0$.

4.7 Extension to time-inhomogeneous intensities

For $t \geq 0$ let $d^S(t,x,\gamma), d^E(t,x,\gamma), b^S(t,x,\gamma), b^E(t,x,\gamma) \in [0,\infty]$ be given and suppose that there exists Γ_∞^2 (independent of $t \geq 0$) such that condition (A) is satisfied for any fixed $t \geq 0$. We suppose that the following conditions hold:

(H1) There exist $\alpha_* < \alpha^*$ and $\beta_* < \beta^*$ such that for all $\alpha \in (\alpha_*, \alpha^*)$, $\beta \in (\beta_*, \beta^*)$ and $t \geq 0$ there exists a constant $a(L(t),\alpha,\beta) \in (0,2)$ which satisfies

$$c(L(t),\alpha,\beta;\eta) \leq a(L(t),\alpha,\beta) M(t,\eta), \quad \eta \in \Gamma_0, \ t \geq 0,$$

where $M(t,\eta) = \sum\limits_{x \in \eta^-} d^E(t,x,\eta^+,\eta^- \backslash x) + \sum\limits_{x \in \eta^+} d^S(t,x,\eta^+ \backslash x, \eta^-)$.

(H2) There exist constants $A > 0$ and $N \in \mathbb{N}$ such that

$$d^S(t,x,\eta) + d^E(t,x,\eta) \leq A(1+|\eta|)^N, \quad \eta \in \Gamma_0^2, \ x \in \mathbb{R}^d, \ t \geq 0$$

holds.

(H3) For any $\alpha', \alpha \in (\alpha_*, \alpha^*)$, $\beta', \beta \in (\beta_*, \beta^*)$ with $\alpha' < \alpha$ and $\beta' < \beta$ the map $t \longmapsto L(t) \in L(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'})$ is continuous in the uniform operator topology.

Consider a scale of Banach spaces given by $\mathcal{E} = (\mathcal{E}_{\alpha,\beta})_{\substack{\alpha \in (\alpha_*,\alpha^*) \\ \beta \in (\beta_*,\beta^*)}}$ and extend the notions introduced in the first chapter to this case in the obvious way.

Theorem 4.7.1. *Suppose that conditions (H1) – (H3) are satisfied. Then there exist a forward evolution system $(U(t,s))_{0 \leq s \leq t}$ and a backward evolution system $(V(s,t))_{0 \leq s \leq t}$ in the scale $\mathcal{E}$ having generator $(L(t))_{t \geq 0} \in L(\mathcal{E})$.*

Proof. We are going to apply Theorem 1.1.4. Let $\alpha' < \alpha$, $\beta' < \beta$ and $F = \mathbb{K}G \in \mathcal{E}_{\alpha,\beta}$, then

$$
\begin{aligned}
\|L(t)F\|_{\mathcal{E}_{\alpha',\beta'}} &= \|\widehat{L}(t)G\|_{\mathcal{L}_{\alpha',\beta'}} \\
&\leq \int\limits_{\Gamma_0^2} c(L(t),\alpha',\beta';\eta)|G(\eta)|e^{\alpha'|\eta^+|}e^{\beta'|\eta^-|}\mathrm{d}\lambda(\eta) \\
&\leq a(L(t),\alpha',\beta')\int\limits_{\Gamma_0^2} M(t,\eta)|G(\eta)|e^{\alpha'|\eta^+|}e^{\beta'|\eta^-|}\mathrm{d}\lambda(\eta) \\
&\leq Aa(L(t),\alpha',\beta')\int\limits_{\Gamma_0^2} |\eta|^{N+1}|G(\eta)|e^{-(\alpha-\alpha')|\eta^+|}e^{-(\beta-\beta')|\eta^-|}e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta).
\end{aligned}
$$

Hence there exists a constant $A = A(t,\alpha,\alpha',\beta,\beta') > 0$ satisfying

$$\|L(t)F\|_{\mathcal{E}_{\alpha',\beta'}} \leq A\|F\|_{\mathcal{E}_{\alpha,\beta}}.$$

Thus $L(t)$ is bounded from $\mathcal{E}_{\alpha,\beta}$ to $\mathcal{E}_{\alpha',\beta'}$ and by (H3) it is also continuous in the uniform topology w.r.t. $t \geq 0$. Condition (a) follows from Proposition 4.2.3 and condition (b) from the contraction property of the semigroups. □

Note that Theorem 1.1.4 was proved for a one-parameter scale of Banach spaces. The generalization for two-parameter scales of Banach spaces (as used above) is a straightforward repetition of the arguments there. The next statement shows the positivity preservation property of the adjoint evolution systems.

Theorem 4.7.2. *Suppose that conditions (H1), (H3) are satisfied and assume that there exists $A > 0$ such that for any $\eta \in \Gamma_0^2$ and $t \geq 0$*

$$d^S(t,x,\eta) + d^E(t,x,\eta) + b^S(t,x,\eta) + b^E(t,x,\eta) \leq A(1+|\eta|)^N \tag{4.53}$$

holds. If for any fixed $t \geq 0$ condition (D) holds for the operator $L(t)$, then $U^*(s,t)$ and $V^*(t,s)$ are positivity preserving.

Proof. Let $U(t,s)$ and $V(s,t)$ be the evolution systems constructed in Theorem 4.7.1 and $\widehat{U}(t,s)$, $\widehat{V}(s,t)$ the associated evolution systems for quasi-observables. The adjoint evolution systems then satisfy for $F = \mathbb{K}G \in \mathcal{FP}(\Gamma^2)$

$$\langle F, U^*(s,t)\mu\rangle = \langle G, \widehat{U}^*(s,t)k_\mu\rangle$$

and

$$\langle F, V^*(t,s)\mu\rangle = \langle G, \widehat{V}^*(t,s)k_\mu\rangle,$$

where $\mu \in \mathcal{P}_{\alpha,\beta}$ has correlation function k_μ. Thus it suffices to show that $\widehat{U}^*(s,t)k_\mu$ and $\widehat{V}^*(t,s)k_\mu$ are positive definite. Let $\widehat{U}_n(t,s)$ and $\widehat{V}_n(s,t)$ be the approximations defined in the proof of Theorem 1.1.4, see 1.12. By Proposition 4.4.5 and (1.11), (1.13) it follows for $G \in B_{bs}(\Gamma_0^2)$ with $\mathbb{K}G \geq 0$

$$\langle \widehat{U}_n(t,s)G, k_\mu \rangle = \langle G, \widehat{U}_n^*(s,t)k_\mu \rangle \geq 0$$

and

$$\langle \widehat{V}_n(s,t)G, k_\mu \rangle = \langle G, \widehat{V}_n^*(t,s)k_\mu \rangle \geq 0.$$

Letting $n \to \infty$ yields

$$0 \leq \lim_{n\to\infty} \langle \widehat{U}_n(t,s)G, k_\mu \rangle = \langle \widehat{U}(t,s)G, k_\mu \rangle = \langle G, \widehat{U}^*(s,t)k_\mu \rangle$$

and

$$\langle G, \widehat{V}^*(t,s)k_\mu \rangle \geq 0.$$

□

The adjoint evolution systems $U^*(s,t)$ and $V^*(t,s)$ are positivity preserving and provide for each $\mu \in \mathcal{P}_{\alpha,\beta}$ unique solutions to the time-dependent Fokker-Planck equations

$$\frac{\partial}{\partial s} \int_{\Gamma^2} F(\gamma)U^*(s,t)\mu(\mathrm{d}\gamma) = -\int_{\Gamma^2} L(s)F(\gamma)U^*(s,t)\mu(\mathrm{d}\gamma), \quad F \in \mathcal{FP}(\Gamma^2)$$

and

$$\frac{\partial}{\partial t} \int_{\Gamma^2} F(\gamma)V^*(t,s)\mu(\mathrm{d}\gamma) = \int_{\Gamma^2} L(t)F(\gamma)V^*(t,s)\mu(\mathrm{d}\gamma), \quad F \in \mathcal{FP}(\Gamma^2).$$

The last statement provides Vlasov scaling. For any $n \geq 1$, let $d_n(t,x,\gamma\backslash x), b_n(t,x,\gamma) \in [0,\infty]$ be the scaled birth-and-death intensities. Define

$$\begin{aligned}
c_n(t,\alpha,\beta;\eta) := &\sum_{x\in\eta^-} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d_n^E(t,x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
&+ \sum_{x\in\eta^+} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d_n^S(t,x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
&+ e^{-\beta} \sum_{x\in\eta^-} \int_{\Gamma_0} |\mathbb{K}_0^{-1} b_n^E(t,x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi) \\
&+ e^{-\alpha} \sum_{x\in\eta^+} \int_{\Gamma_0} |\mathbb{K}_0^{-1} b_n^S(t,x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \mathrm{d}\lambda(\xi).
\end{aligned}$$

Instead of the conditions (V1) – (V3) we suppose that the conditions given below are fulfilled:

(W1) There exist $\alpha_* < \alpha^*$, $\beta_* < \beta^*$ such that for any $\alpha \in (\alpha_*, \alpha^*)$, $\beta \in (\beta_*, \beta^*)$ and any $t \geq 0$ there exists $a(t, \alpha, \beta) \in (0, 2)$ such that

$$c_n(t, \alpha, \beta; \eta) \leq a(t, \alpha, \beta) M_n(t, \eta), \quad \eta \in \Gamma_0, \; n \in \mathbb{N}$$

holds, where $M_n(t, \eta) := \sum\limits_{x \in \eta^-} d_n^E(t, x, \eta^+, \eta^- \backslash x) + \sum\limits_{x \in \eta^+} d_n^S(t, x, \eta^+ \backslash x, \eta^-)$.

(W2) There exist constants $A > 0$ and $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$d_n^E(t, x, \eta) + d_n^S(t, x, \eta) \leq A(1 + |\eta|)^N, \quad t \geq 0, \; \eta \in \Gamma_0^2, \; x \in \mathbb{R}^d$$

holds.

(W3) For all $\xi \in \Gamma_0^2$ and $x \in \mathbb{R}^d$ the following limits exist in the operator norm $L(\mathcal{L}_{\alpha,\beta}, \mathcal{L}_{\alpha',\beta'})$, for any $\alpha' < \alpha$ and $\beta' < \beta$ with $\alpha', \alpha \in (\alpha_*, \alpha^*)$, $\beta', \beta \in (\beta_*, \beta^*)$ such that

$$\begin{aligned}
\lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^E(t, x, \cdot \cup \xi)) &= \lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^E(t, x, \cdot)) =: D_x^{V,E}(t, \cdot) \\
\lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^S(t, x, \cdot \cup \xi)) &= \lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^S(t, x, \cdot)) =: D_x^{V,S}(t, \cdot) \\
\lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^E(t, x, \cdot \cup \xi)) &= \lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^E(t, x, \cdot)) =: B_x^{V,E}(t, \cdot) \\
\lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^S(t, x, \cdot \cup \xi)) &= \lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^S(t, x, \cdot)) =: B_x^{V,S}(t, \cdot).
\end{aligned}$$

Moreover, above limits are uniform on any compact in $t \geq 0$ and are independent of ξ.

For $n \geq 1$, let $\widehat{L}_n(t) := K_0^{-1} L_n(t) K_0$, $\widehat{L}_{n,\text{ren}}(t) := R_n \widehat{L}_n(t) R_{n^{-1}}$ and denote by $\mathcal{L}$ the scale of Banach spaces given by $\mathcal{L} = (\mathcal{L}_{\alpha,\beta})_{\substack{\alpha \in (\alpha_*, \alpha^*) \\ \beta \in (\beta_*, \beta^*)}}$.

Theorem 4.7.3. *Suppose that conditions (W1) – (W3) are satisfied and assume that the operators $(\widehat{L}_{n,\text{ren}}(t))_{t \geq 0}$ are continuous in the uniform topology on $L(\mathcal{L})$. Then the following statements are satisfied:*

(a) There exist forward and backward evolution systems $\widehat{U}_{n,\text{ren}}(t, s)$ and $\widehat{V}_{n,\text{ren}}(s, t)$, respectively having generator $\widehat{L}_{n,\text{ren}}(t) \in L(\mathcal{L})$.

(b) There exist forward and backward evolution systems $\widehat{U}^V(t, s)$ and $\widehat{V}^V(s, t)$, respectively such that

$$\widehat{U}_{n,\text{ren}}(t, s) \longrightarrow \widehat{U}^V(t, s), \quad n \to \infty \widehat{V}_{n,\text{ren}}(s, t) \qquad \longrightarrow \widehat{V}^V(s, t), \quad n \to \infty$$

hold uniformly on compacts in $t \geq 0$ in the uniform topology on $L(\mathcal{L})$. The generators satisfy $\widehat{L}_{n,\text{ren}}(t) \longrightarrow \widehat{L}_V(t)$ as $n \to \infty$ w.r.t. the uniform operator topology on $L(\mathcal{L})$ and uniformly on compacts in $t \geq 0$.

(c) For any $r \in \mathcal{K}_{\alpha,\beta}$ the unique weak solution to the backward equation

$$\frac{\partial}{\partial s}\langle G, k_{s,n}\rangle = -\langle \widehat{L}_{n,\mathrm{ren}}(s)G, k_{s,n}\rangle, \quad k_{s,n}|_{s=t} = r, \;\; s \in [0,t) \quad G \in B_{bs}(\Gamma_0^2)$$

is given by $k_{s,n} = \widehat{U}_{n,\mathrm{ren}}(s,t)^* r$ and the unique weak solution to the forward equation

$$\frac{\partial}{\partial t}\langle G, k_{t,n}\rangle = \langle \widehat{L}_{n,\mathrm{ren}}(t)G, k_{t,n}\rangle, \quad k_{t,n}|_{t=s} = r, \;\; t \in [s,\infty), \quad G \in B_{bs}(\Gamma_0^2)$$

is given by $k_{t,n} = \widehat{V}_{n,\mathrm{ren}}(t,s)^* r$. The same assertions hold with $\widehat{L}_{n,\mathrm{ren}}(t)$ replaced by $\widehat{L}_V(t)$ and $\widehat{U}_{n,\mathrm{ren}}(s,t)^*, \widehat{V}_{n,\mathrm{ren}}(t,s)^*$ replaced by $\widehat{U}^V(s,t)^*, \widehat{V}^V(t,s)^*$.

(d) Let $r(\eta) = \prod_{x \in \eta^-} \rho^E(x) \prod_{x \in \eta^+} \rho^S(x)$ and $\rho^S, \rho^E \in L^\infty(\mathbb{R}^d)$ with $\|\rho^E\|_{L^\infty} \le e^\beta$ and $\|\rho^S\|_{L^\infty} \le e^\alpha$. Assume that $\rho_s^S, \rho_s^E \in L^\infty(\mathbb{R}^d)$ with $\|\rho_s^E\|_{L^\infty} \le e^\beta$, $\|\rho_s^S\|_{L^\infty} \le e^\alpha$ is a classical solution to the backward equation with $0 \le s < t$

$$\begin{aligned}
\frac{\partial \rho_s^E}{\partial s}(x) &= \int_{\Gamma_0^2} e_\lambda(\rho_s^S;\xi^+) e_\lambda(\rho_s^E;\xi^-) D_x^{V,E}(s,\xi)\mathrm{d}\lambda(\xi)\rho_s^E(x) \\
&\quad - \int_{\Gamma_0^2} e_\lambda(\rho_s^S;\xi^+) e_\lambda(\rho_s^E;\xi^-) B_x^{V,E}(s,\xi)\mathrm{d}\lambda(\xi) \\
\frac{\partial \rho_s^S}{\partial s}(x) &= \int_{\Gamma_0^2} e_\lambda(\rho_s^S;\xi^+) e_\lambda(\rho_s^E;\xi^-) D_x^{V,S}(s,\xi)\mathrm{d}\lambda(\xi)\rho_s^S(x) \\
&\quad - \int_{\Gamma_0^2} e_\lambda(\rho_s^S;\xi^+) e_\lambda(\rho_s^E;\xi^-) B_x^{V,S}(s,\xi)\mathrm{d}\lambda(\xi)
\end{aligned}$$

and initial condition $\rho_s|_{s=t} = \rho$. Then $r_s(\eta) := \prod_{x \in \eta^+} \rho_s^S(x) \prod_{x \in \eta^-} \rho_s^E(x)$ is a weak solution to

$$\frac{\partial}{\partial s}\langle G, r_s\rangle = -\langle \widehat{L}_V(s)G, r_s\rangle, \quad r_s|_{s=t} = r, \;\; s \in [0,t), \quad G \in B_{bs}(\Gamma_0^2).$$

Assume that $\rho_t^S, \rho_t^E \in L^\infty(\mathbb{R}^d)$ with $\|\rho_t^E\|_{L^\infty} \le e^\beta$, $\|\rho_t^S\|_{L^\infty} \le e^\alpha$ is a classical solution *to the forward equation* $t \in [s, \infty)$

$$\frac{\partial \rho_t^E}{\partial t}(x) = -\int_{\Gamma_0^2} e_\lambda(\rho_t^S; \xi^+) e_\lambda(\rho_t^E; \xi^-) D_x^{V,E}(t, \xi) \mathrm{d}\lambda(\xi) \rho_t^E(x)$$
$$+ \int_{\Gamma_0^2} e_\lambda(\rho_t^S; \xi^+) e_\lambda(\rho_t^E; \xi^-) B_x^{V,E}(t, \xi) \mathrm{d}\lambda(\xi)$$
$$\frac{\partial \rho_t^S}{\partial t}(x) = -\int_{\Gamma_0^2} e_\lambda(\rho_t^S; \xi^+) e_\lambda(\rho_t^E; \xi^-) D_x^{V,S}(t, \xi) \mathrm{d}\lambda(\xi) \rho_t^S(x)$$
$$+ \int_{\Gamma_0^2} e_\lambda(\rho_t^S; \xi^+) e_\lambda(\rho_t^E; \xi^-) B_x^{V,S}(t, \xi) \mathrm{d}\lambda(\xi)$$

and initial condition $\rho_t|_{t=s} = \rho$*. Then* $r_t(\eta) := \prod\limits_{x \in \eta^+} \rho_t^S(x) \prod\limits_{x \in \eta^-} \rho_t^E(x)$ *is a weak solution to*

$$\frac{\partial}{\partial t}\langle G, r_t \rangle = \langle \widehat{L}_V(t) G, r_t \rangle, \quad r_t|_{t=s} = r, \ t \in [s, \infty), \quad G \in B_{bs}(\Gamma_0^2).$$

Proof. Assertion (a) follows from (W1), (W2) and Theorem 1.1.4. Conditions (W1) – (W3) imply $\widehat{L}_{n,\mathrm{ren}}(t) \longrightarrow \widehat{L}_V(t)$ uniformly on compacts in the uniform topology in the scale $\mathcal{L}$. Hence Theorem 1.1.4 implies the existence of the evolution systems $\widehat{U}^V(t, s)$ and $\widehat{V}^V(s, t)$ and in view of Lemma 1.1.3 assertion (b) is proved. Assertion (c) is an immediate consequence of Theorem 1.1.6. Finally, assertion (d) can be proved in the same way as in the time-homogeneous case. □

4.8 Weak-coupling limit

In this part we establish the weak-coupling limit for two coupled general birth-and-death dynamics. Let $L = L^S + L^E$ be the corresponding Markov (pre-)generator and suppose that L^E does not depend on γ^+, i.e. is given by

$$(L^E F)(\gamma) = \sum_{x \in \eta^-} d^E(x, \gamma^- \backslash x)(F(\gamma^+, \gamma^- \backslash x) - F(\gamma^+, \gamma^-)) + \int_{\mathbb{R}^d} b^E(x, \gamma^-)(F(\gamma^+, \gamma^- \cup x) - F(\gamma^+, \gamma^-)) \mathrm{d}x. \tag{4.54}$$

The dynamics of the system shall be given by the general form (4.15). We suppose that the birth-and-death intensities satisfy:

(A') Condition (A) holds for a set $\Gamma_\infty^2 = \Gamma_\infty^+ \times \Gamma_\infty^-$ such that any $\mu \in \mathcal{P}_\alpha$ is supported on Γ_∞^+ and any $\mu \in \mathcal{P}_\beta$ is supported on Γ_∞^-.

Let $\mu \in \mathcal{P}_{\alpha,\beta}$, the marginals μ^+ and μ^- on Γ are for $A \in \mathcal{B}(\Gamma)$ given by $\mu^+(A) := \mu((A \times \Gamma) \cap \Gamma^2)$ and $\mu^-(A) := \mu((\Gamma \times A) \cap \Gamma^2)$. This definitions are equivalent to

$$\int\limits_{\Gamma^2} F(\gamma^\pm)\mathrm{d}\mu(\gamma) = \int\limits_{\Gamma} F(\gamma^\pm)\mathrm{d}\mu^\pm(\gamma), \quad F \in \mathcal{FP}(\Gamma).$$

Let k_μ be the correlation function for μ. Then for any $G \in B_{bs}(\Gamma_0)$ let $(G \otimes \mathbb{1}^*)(\eta) := \mathbb{1}^*(\eta^-)G(\eta^+)$, we obtain in such a case

$$\begin{aligned}\int\limits_{\Gamma} KG(\gamma^+)\mathrm{d}\mu^+(\gamma^+) &= \int\limits_{\Gamma^2} \mathbb{K}(G \otimes \mathbb{1}^*)(\gamma)\mathrm{d}\mu(\gamma) \\ &= \int\limits_{\Gamma_0^2} G(\eta^+)\mathbb{1}^*(\eta^-)k_\mu(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0} G(\eta^+)k_\mu(\eta^+,\emptyset)\mathrm{d}\lambda(\eta^+).\end{aligned}$$

Therefore $k_\mu(\cdot,\emptyset)$ is the correlation function for the marginal μ^+. A similar argument shows that $k_\mu(\emptyset,\cdot)$ is the correlation function for the marginal μ^-. Introduce the functions

$$\begin{aligned}c_E(\beta;\eta^-) := &\sum_{x\in\eta^-}\int\limits_{\Gamma_0^2} |K_0^{-1}d^E(x,\cdot\cup\eta^-\backslash x)|(\xi^-)e^{\beta|\xi^-|}\mathrm{d}\lambda(\xi^-) \\ &+ e^{-\beta}\sum_{x\in\eta^-}\int\limits_{\Gamma_0^2} |K_0^{-1}b^E(x,\cdot\cup\eta^-\backslash x)|(\xi^-)e^{\beta|\xi^-|}\mathrm{d}\lambda(\xi^-).\end{aligned}$$

and

$$\begin{aligned}c_S(\alpha,\beta;\eta) := &\sum_{x\in\eta^+}\int\limits_{\Gamma_0^2} |\mathbb{K}_0^{-1}d^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi)e^{\alpha|\xi^+|}e^{\beta|\xi^-|}\mathrm{d}\lambda(\xi) \\ &+ e^{-\alpha}\sum_{x\in\eta^+}\int\limits_{\Gamma_0} |\mathbb{K}_0^{-1}b^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi)e^{\alpha|\xi^+|}e^{\beta|\xi^-|}\mathrm{d}\lambda(\xi).\end{aligned}$$

Suppose that the conditions given below are fulfilled.

(E1) There exists a constant $0 < a_E(\beta) < 2$ satisfying

$$c_E(\beta;\eta^-) \le a_E(\beta)M_E(\eta^-), \quad \eta^- \in \Gamma_0,$$

where $M_E(\eta^-) = \sum\limits_{x\in\eta^-} d^E(x,\eta^-\backslash x)$.

(E2) The death intensity is strictly bounded away from zero, i.e. $\inf_{|\eta^-| \geq 1} M_E(\eta^-) > 0$ holds.

(E3) There exist constants $A > 0$, $N \in \mathbb{N}$ and $\nu^E \geq 0$ such that for all $x \in \mathbb{R}^d$ and $\eta^- \in \Gamma_0$

$$d^E(x, \eta^-) + b^E(x, \eta^-) \leq A(1 + |\eta^-|)^N e^{\nu^E |\eta^-|}$$

holds.

(E4) There exists β' with $\beta' + \nu^E < \beta$ and $a_E(\beta') > 0$ satisfying

$$c_E(\beta'; \eta^-) \leq a_E(\beta') M_E(\eta^-), \quad \eta^- \in \Gamma_0.$$

(S1) There exists a constant $0 < a_S(\alpha, \beta) < 2$ such that

$$c_S(\alpha, \beta; \eta) \leq a_S(\alpha, \beta) M_S(\eta), \quad \eta \in \Gamma_0^2$$

holds, where $M_S(\eta) = \sum_{x \in \eta^+} d^S(x, \eta^+ \backslash x, \eta^-)$.

(S2) There exist constants $A > 0$, $N \in \mathbb{N}$ and $\nu^S \geq 0$ such that for all $x \in \mathbb{R}^d$ and $\eta \in \Gamma_0^2$

$$d^S(x, \eta) + b^S(x, \eta) \leq A(1 + |\eta|)^N e^{\nu^S |\eta|}.$$

(S3) There exists α' with $\alpha' + \nu^S < \alpha$, β' from (E4) satisfies $\beta' + \max\{\nu^S, \nu^E\} < \beta$ and there exists a constant $a_S(\alpha', \beta') > 0$ satisfying

$$c_S(\alpha', \beta'; \eta) \leq a_S(\alpha', \beta') M_S(\eta), \quad \eta \in \Gamma_0^2.$$

(L) There exists a localization sequence $(R_\delta)_{\delta > 0}$ such that the (minimal) birth-and-death process associated to L_δ^E and $L_\delta^S + \frac{1}{\varepsilon} L_\delta^E$ is conservative, i.e. it has no explosion for any starting point and any $\delta > 0$, $\varepsilon > 0$.

Here L_δ^S and L_δ^E are given by (4.15) and (4.56) with $b^S(x, \eta)$ and $b^E(x, \eta)$ replaced by $R_\delta(x) b^S(x, \eta)$ and $R_\delta(x) b^E(x, \eta)$. Above conditions and the ergodicity statement of the third chapter imply that the evolution of the environment is ergodic. This ergodicity can be extended to the two-component state space for which the precise statement is given below.

Theorem 4.8.1. *The closure $(L^E, D(L^E))$ of the operator $(L^E, \mathcal{FP}(\Gamma))$ is the generator of an analytic semigroup $(T^E(t))_{t \geq 0}$ of contractions on $\mathcal{E}_\beta$. The adjoint operator $(T^E(t)^*)_{t \geq 0}$ on $\mathcal{E}_\beta^*$ satisfies $T^E(t)^* \mathcal{P}_{\beta'} \subset \mathcal{P}_\beta$ and there exists a unique invariant measure μ^E such that for all $F \in \mathcal{E}_{\alpha,\beta}$*

$$\|T^E(t)F - \langle F \rangle_{\mu_0^+ \otimes \mu^E}\|_{\mathcal{E}_{\alpha,\beta}} \leq C e^{-\lambda t} \|F\|_{\mathcal{E}_{\alpha,\beta}}$$

for some constants $C, \lambda_0 > 0$ independent of F and $t \geq 0$.

Define for all $x \in \mathbb{R}^d$ and $\gamma^+ \in \Gamma$ new intensities $\overline{b}(x, \eta^+)$ and $\overline{d}(x, \eta^+)$ by

$$\overline{b}(x, \gamma^+) := \int_{\Gamma} b^S(x, \gamma^+, \gamma^-) \mathrm{d}\mu^E(\gamma^-) \tag{4.55}$$

$$\overline{d}(x, \gamma^+) := \int_{\Gamma} d^S(x, \gamma^+, \gamma^-) \mathrm{d}\mu^E(\gamma^-) \tag{4.56}$$

and let

$$\begin{aligned} \overline{c}(\alpha; \eta^+) := & \sum_{x \in \eta^+} \int_{\Gamma_0} |K_0^{-1} \overline{d}(x, \cdot \cup \eta^+ \backslash x)|(\xi^+) e^{\alpha|\xi^+|} \mathrm{d}\lambda(\xi^+) \\ & + e^{-\alpha} \sum_{x \in \eta^+} \int_{\Gamma_0} |K_0^{-1} \overline{b}(x, \cdot \cup \eta^+ \backslash x)|(\xi^+) e^{\alpha|\xi^+|} \mathrm{d}\lambda(\xi^+). \end{aligned}$$

Above intensities are well-defined for $\gamma^+ \in \Gamma_\infty^+$. Define for above intensities the averaged Kolmogorov operator

$$(\overline{L}F)(\gamma^+) = \sum_{x \in \gamma^+} \overline{d}(x, \gamma^+ \backslash x)(F(\gamma^+ \backslash x) - F(\gamma^+)) + \int_{\mathbb{R}^d} \overline{b}(x, \gamma^+)(F(\gamma^+ \cup x) - F(\gamma^+)) \mathrm{d}x \tag{4.57}$$

and the averaged cumulative death intensity by $\overline{M}(\eta^+) := \sum\limits_{x \in \eta^+} \overline{d}(x, \eta^+ \backslash x)$. The next statement is the main result for this section.

Proposition 4.8.2. *Suppose that conditions (A'), (E1) – (E4), (S1) – (S3), (L) are fulfilled and assume that the following conditions are satisfied:*

1. *There exists a constant $\overline{a}(\alpha) \in (0, 2)$ such that*

$$\overline{c}(\alpha; \eta^+) \leq \overline{a}(\alpha) \overline{M}(\eta^+), \quad \eta^+ \in \Gamma_0 \tag{4.58}$$

   holds.

2. *There exists a localization sequence such that the (minimal) birth-and-death process associated to the operator $\overline{L}_\delta$ is conservative.*

3. *There exist $A > 0$, $N \in \mathbb{N}$ and $\overline{\nu} \geq 0$ such that*

$$\overline{d}(x, \eta^+) + \overline{b}(x, \eta^+) \leq A(1 + |\eta^+|)^N e^{\overline{\nu}|\eta^+|}$$

   holds for all $x \in \mathbb{R}^d$ and $\eta^+ \in \Gamma_0$.

4. α' given in (S3) satisfies $\alpha' + \max\{\overline{\nu}, \nu^S\} < \alpha$ and there exists $\overline{a}(\alpha') > 0$ such that

$$\overline{c}(\alpha'; \eta^+) \leq \overline{a}(\alpha')\overline{M}(\eta^+), \quad \eta^+ \in \Gamma_0$$

holds.

Then the following assertions are true:

1. *For any $\varepsilon > 0$ the operator $(L^S + \frac{1}{\varepsilon}L^E, \mathcal{FP}(\Gamma^2))$ is closable and the closure is the generator of an analytic semigroup $(T^\varepsilon(t))_{t\geq 0}$ of contractions on $\mathcal{E}_{\alpha,\beta}$. The adjoint semigroup yields for any $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ the unique solution to the Fokker-Planck equation for the Kolmogorov operator $L^S + \frac{1}{\varepsilon}L^E$ given by $T^\varepsilon(t)^*\mu_0 = \mu_t^\varepsilon$.*

2. *The operator $(\overline{L}, \mathcal{FP}(\Gamma))$ is closable and the closure is the generator of an analytic semigroup $(U_\alpha(t))_{t\geq 0}$ of contractions on $\mathcal{E}_\alpha$. The adjoint semigroup yields for any $\mu \in \mathcal{P}_{\alpha'}$ the unique solution to the Fokker-Planck equation for the operator $\overline{L}$ given by $U_\alpha(t)^*\mu = \overline{\mu}_t$.*

3. *For any $F \in \mathcal{E}_\alpha$*

$$T^\varepsilon(t)F \longrightarrow U_\alpha(t)F, \quad \varepsilon \to 0 \tag{4.59}$$

   holds uniformly on compacts in $t \geq 0$.

4. *For any $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ let μ_0^+ be the marginal on its first component, let $\overline{\mu}_t = U_\alpha(t)^*\mu_0^+$ and $\mu_t^\varepsilon := T^\varepsilon(t)^*\mu_0$. Denote by $\mu_t^{\varepsilon,+}$ its marginal on its first component, then for any $F \in \mathcal{E}_\alpha$*

$$\int_\Gamma F(\gamma^+)\mathrm{d}\mu_t^{\varepsilon,+}(\gamma^+) \longrightarrow \int_\Gamma F(\gamma^+)\mathrm{d}\overline{\mu}_t(\gamma^+), \quad \varepsilon \to 0$$

   holds uniformly on compacts in $t \geq 0$.

Remark 4.8.3. *Instead of condition (A') we also can suppose that (A) holds for the operators $L^S + \frac{1}{\varepsilon}L^E$ and $\overline{L}$.*

The rest of this section is devoted to the proof of above statements. Let

$$0^+ \otimes \mathcal{L}_\beta := \{G \in \mathcal{L}_{\alpha,\beta} \mid G(\eta) = 0^{|\eta^+|}G(\eta) = 0^{|\eta^+|}G(\emptyset, \eta^-)\}$$

and

$$\mathcal{L}_\alpha \otimes 0^- := \{G \in \mathcal{L}_{\alpha,\beta} \mid G(\eta) = 0^{|\eta^-|}G(\eta) = 0^{|\eta^-|}G(\eta^+, \emptyset)\}.$$

be the closed subspaces of functions in one variable. Multiplication by $0^{|\eta^+|}$ and $0^{|\eta^-|}$, respectively defines projection operators on $\mathcal{L}_{\alpha,\beta}$. The range of these operators is precisely

$0^+ \otimes \mathcal{L}_\alpha$ and $\mathcal{L}_\alpha \otimes 0^-$. Moreover this spaces can be identified with $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ respectively, i.e.

$$\widehat{P}_+ : 0^+ \otimes \mathcal{L}_\beta \longrightarrow \mathcal{L}_\beta, \quad \widehat{P}_+ G(\eta^-) = G(\emptyset, \eta^-)$$

and

$$\widehat{P}_- : \mathcal{L}_\alpha \otimes 0^- \longrightarrow \mathcal{L}_\alpha, \quad \widehat{P}_- G(\eta^+) = G(\eta^+, \emptyset)$$

are isometric isomorphisms with inverses given by

$$\widehat{P}_+^{-1} G(\eta) = 0^{|\eta^+|} G(\eta^-)$$
$$\widehat{P}_-^{-1} G(\eta) = 0^{|\eta^-|} G(\eta^+).$$

Given a bounded linear operator C on $\mathcal{L}_{\alpha,\beta}$, we will say that C leaves $\mathcal{L}_\beta$-invariant if it leaves $0^+ \otimes \mathcal{L}_\beta$ invariant. In such a case the restriction to $\mathcal{L}_\beta$ is defined by

$$C|_{\mathcal{L}_\beta} := \widehat{P}_+ C \widehat{P}_+^{-1}. \tag{4.60}$$

The same notation shall be used for $\mathcal{L}_\alpha$ and $\widehat{P}_-$ respectively. Let

$$\mathcal{X} := \{G_1 \otimes G_2 \mid G_1 \in \mathcal{L}_\alpha, \ G_2 \in \mathcal{L}_\beta\} \subset \mathcal{L}_{\alpha,\beta}$$

where $(G_1 \otimes G_2)(\eta) := G_1(\eta^+)G_2(\eta^-)$. Then $\mathrm{lin}(\mathcal{X}) \subset \mathcal{L}_{\alpha,\beta}$ is dense, where lin denotes the linear span of a given subset of $\mathcal{L}_{\alpha,\beta}$. Given bounded linear operators A_1 on $\mathcal{L}_\alpha$ and A_2 on $\mathcal{L}_\beta$, the product $A_1 \otimes A_2$ on $\mathcal{L}_{\alpha,\beta}$ is defined as the unique linear extension of the operator

$$(A_1 \otimes A_2)G(\eta) = A_1 G_1(\eta^+) A_2 G_2(\eta^+), \quad G \in \mathcal{X}.$$

This definition satisfies $\|(A_1 \otimes A_2)G\|_{\mathcal{L}_{\alpha,\beta}} = \|A_1 G_1\|_{\mathcal{L}_\alpha} \|A_2 G_2\|_{\mathcal{L}_\beta}$ and hence such extension exists. For A_2 being the identity operator we use the notation $A_1 \otimes \mathbb{1}$ and for A_1 being the identity we use the notation $\mathbb{1} \otimes A_2$ respectively. The next statement extends above definition to strongly continuous semigroups.

Theorem 4.8.4. *The following assertions are satisfied:*

(a) Let $(A_\alpha, D(A_\alpha))$ be the generator of a C_0-semigroup $(T_\alpha(t))_{t\geq 0}$ on $\mathcal{L}_\alpha$ and define

$$\mathcal{D} := \{G_1 \otimes G_2 \mid G_1 \in D(A_\alpha), \ G_2 \in \mathcal{L}_\beta\}.$$

Then $T_\alpha(t) \otimes \mathbb{1}$ is a C_0-semigroup on $\mathcal{L}_{\alpha,\beta}$. Let $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ be its generator. Then $\mathrm{lin}(\mathcal{D}) \subset \mathcal{L}_{\alpha,\beta}$ is dense and a core for $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ where

$$A_{\alpha,\beta}(G_1 \otimes G_2) = A_\alpha G_1 \otimes G_2, \quad G_1 \otimes G_2 \in \mathcal{D}.$$

(b) Let $(T_{\alpha,\beta}(t))_{t\geq 0}$ be a C_0-semigroup on $\mathcal{L}_{\alpha,\beta}$ and $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ its generator. Suppose that $T_{\alpha,\beta}(t)$ leaves $\mathcal{L}_\alpha$ invariant and let $T_\alpha(t) := T_{\alpha,\beta}(t)|_{\mathcal{L}_\alpha}$. Then $(T_\alpha(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{L}_\alpha$ and its generator is given by $A_\alpha = \widehat{P}_- A_{\alpha,\beta} \widehat{P}_-^{-1}$ and

$$D(A_\alpha) = \{G \in \mathcal{L}_\alpha \mid \widehat{P}_-^{-1} G \in D(A_{\alpha,\beta}), \ A_{\alpha,\beta} \widehat{P}_-^{-1} G \in \mathcal{L}_\alpha \otimes 0^-\}.$$

A similar result holds true, if we exchange the components $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$. Above statement is not difficult to prove, the details can be found in the appendix. Define the operator $\widehat{L} = \mathbb{K}_0^{-1} L \mathbb{K}_0 = \widehat{L}^S + \widehat{L}^E$ on quasi-observables. Then $\widehat{L}^E = A_E + B_E$ is given by

$$(A_E G)(\eta) = -M_E(\eta^-)G(\eta)$$

with

$$M_E(\eta^-) = \sum_{x\in\eta^-} d^E(x, \eta^- \backslash x) \geq 0, \quad \eta^- \in \Gamma_0$$

and

$$\begin{aligned}(B_E G)(\eta) = &- \sum_{\xi^- \subsetneq \eta^-} G(\eta^+, \xi^-) \sum_{x\in\xi^-} (K_0^{-1} d^E(x, \cdot \cup \xi^- \backslash x))(\eta^- \backslash \xi^-) \\ &+ \sum_{\xi^- \subset \eta^-} \int_{\mathbb{R}^d} G(\eta^+, \xi^- \cup x)(K_0^{-1} b^E(x, \cdot \cup \xi^-))(\eta^- \backslash \xi^-) \mathrm{d}x.\end{aligned}$$

The generator for the system is similarly given by $\widehat{L}^S = A_S + B_S$, where

$$(A_S G)(\eta) = -M_S(\eta)G(\eta)$$

with

$$M_S(\eta) = \sum_{x\in\eta^+} d^S(x, \eta^-, \eta^+ \backslash x) \geq 0, \quad \eta \in \Gamma_0^2$$

and

$$\begin{aligned}(B_S G)(\eta) = &- \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x\in\xi^+} (\mathbb{K}_0^{-1} d^S(x, \cdot \cup \xi^+ \backslash x, \cdot \cup \xi^-))(\eta \backslash \xi) \\ &+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-)(\mathbb{K}_0^{-1} b^S(x, \cdot \cup \xi^+, \cdot \cup \xi^-))(\eta \backslash \xi) \mathrm{d}x.\end{aligned} \tag{4.61}$$

Assumption (E1) and Theorem 3.2.3 imply that $(\widehat{L}^E, D(\widehat{L}^E))$ is the generator of an analytic semigroup of contractions on $\mathcal{L}_{\alpha,\beta}$, where

$$D(\widehat{L}^E) = \{G \in \mathcal{L}_{\alpha,\beta} \mid M_E \cdot G \in \mathcal{L}_{\alpha,\beta}\}.$$

Here and in the following we will use the notation $(\widehat{T}^E(t))_{t\geq 0}$ for the semigroup generated by $(\widehat{L}^E, D(\widehat{L}^E))$. Define the operator $(\widehat{L}^E|_{\mathcal{L}_\beta}, D_\beta(\widehat{L}^E))$ by

$$D_\beta(\widehat{L}^E) = \{G \in \mathcal{L}_\beta \mid M_E \cdot G \in \mathcal{L}_\beta\},$$

$\widehat{L}^E|_{\mathcal{L}_\beta} = A_E|_{\mathcal{L}_\beta} + B_E|_{\mathcal{L}_\beta}$, $(A_E|_{\mathcal{L}_\beta}G)(\eta^-) = -M_E(\eta^-)G(\eta^-)$ and

$$(B_E|_{\mathcal{L}_\beta}G)(\eta) = -\sum_{\xi^-\subsetneq\eta^-} G(\xi^-)\sum_{x\in\xi^-}(K_0^{-1}d^E(x,\cdot\cup\xi^-\backslash x))(\eta^-\backslash\xi^-) \\ + \sum_{\xi^-\subset\eta^-}\int_{\mathbb{R}^d} G(\xi^-\cup x)(K_0^{-1}b^E(x,\cdot\cup\xi^-))(\eta^-\backslash\xi^-)\mathrm{d}x. \tag{4.62}$$

Then, using again (E1) and Theorem 3.2.3 it follows that $(\widehat{L}^E|_{\mathcal{L}_\beta}, D_\beta(\widehat{L}^E))$ is the generator of an analytic semigroup $(T_\beta^E(t))_{t\ge0}$ of contractions on $\mathcal{L}_\beta$.

Lemma 4.8.5. *Let*

$$\mathcal{D} := \{G_1\otimes G_2 \in \mathcal{X} \mid G_2 \in D_\beta(\widehat{L}^E)\},$$

then $\mathrm{lin}(\mathcal{D}) \subset D(\widehat{L}^E)$ *is a core and* $\widehat{T}^E(t) = \mathbb{1}\otimes\widehat{T}_\beta^E(t)$. *Here* $\mathbb{1}$ *denotes the identity operator on* $\mathcal{L}_\alpha$. *Moreover, for* $G\in\mathcal{D}$ *it holds that*

$$\widehat{L}^E G(\eta) = G_1(\eta^+)(\widehat{L}^E|_{\mathcal{L}_\beta}G_2)(\eta^+). \tag{4.63}$$

Proof. Property (4.63) is evident and by Theorem 4.8.4.(a) it is enough to show $\widehat{T}^E(t) = \mathbb{1}\otimes\widehat{T}_\beta^E(t)$. For any $G\in\mathcal{D}$ the action $(\mathbb{1}\otimes\widehat{T}_\beta^E(t))G = G_1\otimes\widehat{T}_\beta^E(t)G_2$ is a solution to the Cauchy problem

$$\frac{\partial}{\partial t}G_t = \widehat{L}^E G_t, \quad G_t|_{t=0} = G$$

on $\mathcal{L}_{\alpha,\beta}$, see (4.63). Since for $G\in\mathcal{D}\subset D(\widehat{L}^E)$ this Cauchy problem has the unique solution given by $G_t = \widehat{T}^E(t)G$, it follows that $(\mathbb{1}\otimes\widehat{T}_\beta^E(t))G = \widehat{T}^E(t)G$. Again by Theorem 4.8.4.(a) $\mathcal{D}\subset\mathcal{L}_{\alpha,\beta}$ is dense and hence $\widehat{T}^E(t) = \mathbb{1}\otimes\widehat{T}_\beta^E(t)$. □

Using the duality

$$\langle G, k\rangle = \int_{\Gamma_0} G(\eta)k(\eta)\mathrm{d}\lambda(\eta), \quad G\in\mathcal{L}_\beta,\ k\in\mathcal{K}_\beta$$

we can compute the adjoint operator to $\widehat{L}^E|_{\mathcal{L}_\beta}$, which is given by

$$L^{E,\Delta}|_{\mathcal{K}_\beta}k(\eta^-) = -\sum_{x\in\eta^-}\int_{\Gamma_0} k(\eta^-\cup\xi^-)(K_0^{-1}d^E(x,\cdot\cup\eta^-\backslash x))(\xi^-)\mathrm{d}\lambda(\xi^-) \\ + \sum_{x\in\eta^-}\int_{\Gamma_0} k(\eta^-\backslash x\cup\xi^-)(K_0^{-1}b^E(x,\cdot\cup\eta^-\backslash x))(\xi^-)\mathrm{d}\lambda(\xi^-).$$

The operator will be considered on the maximal domain

$$D_\beta(L^{E,\Delta}|_{\mathcal{K}_\beta}) = \{k\in\mathcal{K}_\beta \mid L^{E,\Delta}|_{\mathcal{K}_\beta}k\in\mathcal{K}_\beta\}$$

and by Lemma 4.3.2 it follows that $(\widehat{L}^E|^*_{\mathcal{L}_\beta}, D(\widehat{L}^E|^*_{\mathcal{L}_\beta})) = (L^{E,\Delta}|_{\mathcal{K}_\beta}, D_\beta(L^{E,\Delta}|_{\mathcal{K}_\beta}))$. A function $k_{\text{inv}} \in D_\beta(L^{E,\Delta}|_{\mathcal{K}_\beta})$ is called invariant if it satisfies the equation

$$L^{E,\Delta}|_{\mathcal{K}_\beta} k_{\text{inv}} = 0, \quad k_{\text{inv}}(\emptyset) = 1.$$

An application of Proposition 3.2.11 implies the next statement.

Lemma 4.8.6. *There exists a unique probability measure $\mu^E \in \mathcal{P}_\beta$ with*

$$\int_\Gamma L^E F(\gamma^-) \mathrm{d}\mu^E(\gamma^-) = 0, \quad F \in \mathcal{FP}(\Gamma).$$

The associated correlation function $k_{\text{inv}} \in \mathcal{K}_\beta$ is invariant and the semigroup $\widehat{T}^E_\beta(t)$ is ergodic on $\mathcal{L}_\beta$. Namely, there exist constants $\lambda_0 > 0$, $C > 0$ such that for all $G \in \mathcal{L}_\beta$

$$\|\widehat{T}^E_\beta(t)G - \langle G, k_{\text{inv}}\rangle 0^{|\cdot|}\|_{\mathcal{L}_\beta} \leq Ce^{-\lambda_0 t}\|G - \langle G, k_{\text{inv}}\rangle 0^{|\cdot|}\|_{\mathcal{L}_\beta}$$

holds.

Define a projection operator $\widehat{P} : \mathcal{L}_{\alpha,\beta} \longrightarrow \mathcal{L}_\alpha \otimes 0^-$ by

$$\widehat{P}G(\eta) = \int_{\Gamma_0} G(\eta^+, \xi^-) k_{\text{inv}}(\xi^-) \mathrm{d}\lambda(\xi^-) 0^{|\eta^-|}. \tag{4.64}$$

Then $\widehat{P}$ leaves $\mathcal{L}_\beta$ invariant and the restriction to $\mathcal{L}_\beta$ is given by $\widehat{P}|_{\mathcal{L}_\beta} G(\eta^-) = \langle G, k_{\text{inv}}\rangle 0^{|\eta^-|}$. This can also be rewritten to

$$\widehat{P} = \mathbb{1} \otimes \widehat{P}|_{\mathcal{L}_\beta}. \tag{4.65}$$

The next statement extends the ergodicity to the semigroup $\widehat{T}^E(t)$ defined on $\mathcal{L}_{\alpha,\beta}$, i.e. proves Theorem 4.8.1.

Theorem 4.8.7. *For any $G \in \mathcal{L}_{\alpha,\beta}$ the following estimate holds*

$$\|\widehat{T}^E(t)G - \widehat{P}G\|_{\mathcal{L}_{\alpha,\beta}} \leq Ce^{-\lambda_0 t}\|G\|_{\mathcal{L}_{\alpha,\beta}}. \tag{4.66}$$

Let $(T^E(t))_{t\geq 0}$ be the associated semigroup on $\mathcal{E}_{\alpha,\beta}$ and $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ with correlation function $k_0 \in \mathcal{K}_{\alpha',\beta'}$. Then there exists a unique solution $(\mu_t)_{t\geq 0}$ to the Fokker-Planck equation with generator L^E given by $T^E(t)^*\mu_0$. Let $\mu_0^+(\mathrm{d}\gamma^+)$ be the marginal of μ_0, then

$$\|T^E(t)^*\mu_0 - \mu_0^+ \otimes \mu^E\|_{\mathcal{E}^*_{\alpha,\beta}} \leq Ce^{-\lambda_0 t}\|\mu_0\|_{\mathcal{E}^*_{\alpha,\beta}}.$$

Proof. Since $\widehat{T}^E(t)$ is a contraction operator it is enough to show (4.66) only for the dense set of functions given by $\text{lin}(\mathcal{X})$. So let $G = \sum_{n=1}^{N} G_n^1 G_n^2 \in \text{lin}(\mathcal{X})$ with $N \in \mathbb{N}$. From (4.64), i.e. (4.65) it follows that

$$\widehat{P}G = \sum_{n=1}^{N} G_n^1 \cdot \widehat{P}|_{\mathcal{L}_\beta} G_n^2$$

and by Lemma 4.8.5

$$\widehat{T}^E(t)G = \sum_{n=1}^{n} G_n^1 \cdot \widehat{T}^E_\beta(t) G_n^2.$$

Thus we obtain

$$\begin{aligned}\|\widehat{T}^E(t)G - \widehat{P}G\|_{\mathcal{L}_{\alpha,\beta}} &\leq \sum_{n=1}^{N} \|G_n^1\|_{\mathcal{L}_\alpha} \|\widehat{T}^E_\beta(t)G_n^2 - \widehat{P}|_{\mathcal{L}_\beta} G_n^2\|_{\mathcal{L}_\beta} \\ &\leq 2Ce^{-\lambda_0 t} \sum_{n=1}^{N} \|G_n^1\|_{\mathcal{L}_\alpha} \|G_n^2\|_{\mathcal{L}_\beta}.\end{aligned}$$

Now observe that

$$\mathcal{L}_{\alpha,\beta} \cong \left\{ G = \sum_{n=1}^{\infty} G_n^1 \otimes G_n^2 \mid (G_n^1)_{n\in\mathbb{N}} \subset \mathcal{L}_\alpha,\ (G_n^2)_{n\in\mathbb{N}} \subset \mathcal{L}_\beta,\ \sum_{n=1}^{\infty} \|G_n^1\|_{\mathcal{L}_\alpha} \|G_n^2\|_{\mathcal{L}_\beta} < \infty \right\},$$

cf. [Rya02]. Using this representation we obtain

$$\|G\|_{\mathcal{L}_{\alpha,\beta}} = \inf \left\{ \sum_{n=1}^{\infty} \|G_n^1\|_{\mathcal{L}_\alpha} \|G_n^2\|_{\mathcal{L}_\beta} \mid \sum_{n=1}^{\infty} \|G_n^1\|_{\mathcal{L}_\alpha} \|G_n^2\|_{\mathcal{L}_\beta} < \infty,\ G = \sum_{n=1}^{\infty} G_n^1 \otimes G_n^2 \right\}.$$

Let $G = \sum_{n=1}^{\infty} G_n^1 \otimes G_n^2 \in \mathcal{L}_{\alpha,\beta}$ and set $G_N := \sum_{n=1}^{N} G_n^1 \otimes G_n^2 \in \text{lin}(\mathcal{X})$. Above estimate implies

$$\|\widehat{T}^E(t)G_N - \widehat{P}G_N\|_{\mathcal{L}_{\alpha,\beta}} \leq 2Ce^{-\lambda_0 t} \|G\|_{\mathcal{L}_{\alpha,\beta}}.$$

Taking the limit $N \to \infty$ yields (4.66). Denote by $\widehat{P}^*$ the adjoint operator to $\widehat{P}$ which is given by

$$\widehat{P}^* k(\eta) = k(\eta^+, \emptyset) k_{\text{inv}}(\eta^-).$$

Hence we obtain by (4.66) and duality for any $k \in \mathcal{K}_{\alpha,\beta}$

$$\|\widehat{T}^E(t)^* k - \widehat{P}^* k\|_{\mathcal{K}_{\alpha,\beta}} \leq Ce^{-\lambda_0 t} \|k - \widehat{P}^* k\|_{\mathcal{K}_{\alpha,\beta}}.$$

Given $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ with correlation function $k_0 \in \mathcal{K}_{\alpha',\beta'}$, Proposition 4.4.5 yields the existence of a unique solution $(\mu_t)_{t\geq 0}$ to the Fokker-Planck equation for L^E and μ_t has correlation function $k_t = \widehat{T}^E(t)^* k_0$. Because of $k_t \to \widehat{P}^* k_0$ when $t \to \infty$, $\widehat{P}^* k$ is positive definite and hence the correlation function for some probability measure. For any $G \in B_{bs}(\Gamma_0)$ it holds that

$$\int_\Gamma KG(\gamma^+)\mathrm{d}\mu_0^+(\gamma^+) = \int_{\Gamma^2} KG(\gamma^+)\mathrm{d}\mu_0(\gamma) = \int_{\Gamma_0^2} G(\eta^+)0^{|\eta^-|}k_0(\eta)\mathrm{d}\lambda(\eta) = \int_{\Gamma_0} G(\eta^+)k_0(\eta^+,\varnothing)\mathrm{d}\lambda(\eta^+)$$

and hence for any $G \in B_{bs}(\Gamma_0^2)$

$$\int_{\Gamma^2} \mathbb{K}G(\gamma)\mathrm{d}(\mu_0^+ \otimes \mu^E)(\gamma) = \int_\Gamma \int_\Gamma \mathbb{K}G(\gamma^+,\gamma^-)\mathrm{d}\mu_0^S(\gamma^+)\mathrm{d}\mu^E(\gamma^-) = \int_{\Gamma_0}\int_{\Gamma_0} G(\eta^+,\eta^-)k_0(\eta^+,\varnothing)k_{\mathrm{inv}}(\eta^-)\mathrm{d}\lambda(\eta^+)\mathrm{d}\lambda(\eta^-)$$

is satisfied. Therefore the probability measure $\mu_0^+ \otimes \mu^E$ has correlation function $\widehat{P}^* k_0$. □

The next lemma is one necessary condition for the application of [Kur73, Theorem 2.1], which shall be applied later on.

Lemma 4.8.8. *$D(\widehat{L}^S) \cap D(\widehat{L}^E)$ is a core for the operator $(\widehat{L}^E, D(\widehat{L}^E))$, where*

$$D(\widehat{L}^S) = \{G \in \mathcal{L}_{\alpha,\beta} \mid M_S \cdot G \in \mathcal{L}_{\alpha,\beta}\}.$$

Proof. Let $G \in D(\widehat{L}^E)$ and for $\lambda > 0$ define $G_\lambda := \lambda R(\lambda; A_S)G = \frac{\lambda}{\lambda + M_S} G \in D(\widehat{L}^S) \cap D(\widehat{L}^E)$. Then we have to show that $G_\lambda \to G$ and $\widehat{L}^E G_\lambda \to \widehat{L}^E G_\lambda$ when $\lambda \to \infty$. The convergence $G_\lambda \to G$ is evident and the second one follows from

$$\begin{aligned}\|\widehat{L}^E G_\lambda - \widehat{L}^E G\|_{\mathcal{L}_{\alpha,\beta}} &\leq \|A_E(\lambda R(\lambda;A_S) - 1)G\|_{\mathcal{L}_{\alpha,\beta}} + \|B_E(\lambda R(\lambda;A_S) - 1)G\|_{\mathcal{L}_{\alpha,\beta}} \\ &\leq a_E(\beta)\|M_E(\lambda R(\lambda;A_S) - 1)G\|_{\mathcal{L}_{\alpha,\beta}} \\ &= a_E(\beta)\int_{\Gamma_0^2} \frac{M_S(\eta)M_E(\eta^-)}{\lambda + M_S(\eta)}|G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\mathrm{d}\lambda(\eta)\end{aligned}$$

and dominated convergence. □

Condition (S1) implies that $(\widehat{L}^S, D(\widehat{L}^S))$ and is the generator of an analytic semigroup of contractions $(\widehat{T}^S(t))_{t\geq 0}$ on $\mathcal{L}_{\alpha,\beta}$, see Theorem 4.2.3 In the following we are going to construct a semigroup for the limiting dynamics, that is the dynamics after taking the limit $\varepsilon \to 0$. The definition of the K-transform applied for $k(\eta) = 0^{|\eta^+|}k_{\text{inv}}(\eta^-)$ yields for $x \notin \eta^+$ and $\eta^+ \in \Gamma_0$

$$\int\limits_{\Gamma} d^S(x, \eta^+, \gamma^-)\mathrm{d}\mu^E(\gamma^-) = \int\limits_{\Gamma_0} \sum_{\zeta^- \subset \xi^-} (-1)^{|\xi^- \setminus \zeta^-|} d^S(x, \eta^+, \zeta^-) k_{\text{inv}}(\xi^-)\mathrm{d}\lambda(\xi^-) \tag{4.67}$$

$$\int\limits_{\Gamma} b^S(x, \eta^+, \gamma^-)\mathrm{d}\mu^E(\gamma^-) = \int\limits_{\Gamma_0} \sum_{\zeta^- \subset \xi^-} (-1)^{|\xi^- \setminus \zeta^-|} b^S(x, \eta^+, \zeta^-) k_{\text{inv}}(\xi^-)\mathrm{d}\lambda(\xi^-). \tag{4.68}$$

We obtain by (4.67) and (4.68)

$$\begin{aligned} K_0^{-1}\overline{d}(x, \cdot \cup \eta^+ \setminus x)(\xi^+) &= \sum_{\zeta^+ \subset \xi^+} (-1)^{|\xi^+ \setminus \zeta^+|} \int\limits_{\Gamma} d^S(x, \zeta^+ \cup \eta^+ \setminus x, \gamma^-)\mathrm{d}\mu^E(\gamma^-) \\ &= \int\limits_{\Gamma_0} \sum_{\substack{\zeta^+ \subset \xi^+ \\ \zeta^- \subset \xi^-}} (-1)^{|\xi^+ \setminus \zeta^+| + |\xi^- \setminus \zeta^-|} d^S(x, \zeta^+ \cup \eta^+ \setminus x, \zeta^-) k_{\text{inv}}(\xi^-)\mathrm{d}\lambda(\xi^-) \\ &= \int\limits_{\Gamma_0} (\mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \setminus x, \cdot))(\xi) k_{\text{inv}}(\xi^-)\mathrm{d}\lambda(\xi^-) \end{aligned}$$

and likewise

$$K_0^{-1}\overline{b}(x, \cdot \cup \eta^+ \setminus x)(\xi^+) = \int\limits_{\Gamma_0} (\mathbb{K}_0^{-1} b^S(x, \cdot \cup \eta^+ \setminus x, \cdot))(\xi) k_{\text{inv}}(\xi^-)\mathrm{d}\lambda(\xi^-).$$

Therefore we obtain

$$\begin{aligned} \overline{c}(\alpha; \eta^+) \leq &+ \sum_{x \in \eta^+} \int\limits_{\Gamma_0^2} |\mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \setminus x, \cdot)|(\xi) e^{\alpha|\xi^+|} k_{\text{inv}}(\xi^-)\mathrm{d}\lambda(\xi) \\ &+ e^{-\alpha} \sum_{x \in \eta^+} \int\limits_{\Gamma_0^2} |\mathbb{K}_0^{-1} b^S(x, \cdot \cup \xi^+ \setminus x, \cdot)|(\xi) e^{\alpha|\xi^+|} k_{\text{inv}}(\xi^-)\mathrm{d}\lambda(\xi), \\ &\leq a_S(\alpha, \beta)\|k_{\text{inv}}\|_{\mathcal{K}_\beta} M_S(\eta^+, \varnothing) < \infty. \end{aligned}$$

Lemma 4.8.9. *The operator $(\widehat{P}\widehat{L}^S, \mathrm{Ran}(\widehat{P}) \cap D(\widehat{L}^S))$ is closable and the closure is the generator of an analytic semigroup of contractions on $\mathcal{L}_\alpha \otimes 0^-$, where $\widehat{P}$ was defined in* (4.64)*.*

Proof. The projection operator $\widehat{P}$ satisfies $\mathrm{Ran}(\widehat{P}) = \mathcal{L}_\alpha \otimes 0^-$. Given $G(\eta^+)0^{|\eta^-|}$ we obtain $M_S(\eta^+, \eta^-)G(\eta^+)0^{|\eta^-|} = M_S(\eta^+, \varnothing)G(\eta^+)0^{|\eta^-|}$ and hence

$$\mathrm{Ran}(\widehat{P}) \cap D(\widehat{L}^S) = \{G = G_1 \otimes 0^- \mid G_1 \in D_\alpha(M_S(\cdot, \varnothing))\} = \widehat{P}_-^{-1} D_\alpha(M_S(\cdot, \varnothing)), \tag{4.69}$$

where $D_\alpha(M_S(\cdot,\varnothing)) := \{G \in \mathcal{L}_\alpha \mid M_S(\cdot,\varnothing)G \in \mathcal{L}_\alpha\}$. For $G(\eta) = G_1(\eta^+)0^{|\eta^-|}$ with $G_1 \in D_\alpha(M_S(\cdot,\varnothing))$ we obtain

$$\begin{aligned}\widehat{L}^S G(\eta) = &- \sum_{\xi^+\subset\eta^+} G_1(\xi^+) \sum_{x\in\xi^+} (\mathbb{K}_0^{-1} d^S(x,\cdot\cup\xi^+\backslash x,\cdot))(\eta^+\backslash\xi^+,\eta^-) \\ &+ \sum_{\xi^+\subset\eta^+} \int_{\mathbb{R}^d} G_1(\xi^+\cup x)(\mathbb{K}_0^{-1} b^S(x,\cdot\cup\xi^+,\cdot))(\eta^+\backslash\xi^+,\eta^-)\mathrm{d}x.\end{aligned}$$

Applying the operator $\widehat{P}$ yields

$$\begin{aligned}&\widehat{P}\widehat{L}^S G(\eta) = \\ &- 0^{|\eta^-|} \sum_{\xi^+\subset\eta^+} G_1(\xi^+) \sum_{x\in\xi^+} \int_{\Gamma_0} (\mathbb{K}_0^{-1} d^S(x,\cdot\cup\xi^+\backslash x,\cdot))(\eta^+\backslash\xi^+,\xi^-)k_{\mathrm{inv}}(\xi^-)\mathrm{d}\lambda(\xi^-) \\ &+ 0^{|\eta^-|} \sum_{\xi^+\subset\eta^+} \int_{\mathbb{R}^d} G_1(\xi^+\cup x) \int_{\Gamma_0} (\mathbb{K}_0^{-1} b^S(x,\cdot\cup\xi^+,\cdot))(\eta^+\backslash\xi^+,\xi^-)k_{\mathrm{inv}}(\xi^-)\mathrm{d}\lambda(\xi^-)\mathrm{d}x.\end{aligned}$$

As a consequence by (4.55), (4.56), (4.67) and (4.68) we arrive at

$$\begin{aligned}\widehat{P}\widehat{L}^S G(\eta) = &- \overline{M}(\eta^+)G_1(\eta^+)0^{|\eta^-|} \\ &- 0^{|\eta^-|} \sum_{\xi^+\subsetneq\eta^-} G_1(\xi^+) \sum_{x\in\xi^+} (K_0^{-1}\overline{d}(x,\cdot\cup\xi^+\backslash x))(\eta^+\backslash\xi^+) \\ &+ 0^{|\eta^-|} \sum_{\xi^+\subset\eta^+} \int_{\mathbb{R}^d} G_1(\xi^+\cup x)(K_0^{-1}\overline{b}(x,\cdot\cup\xi^+))(\eta^+\backslash\xi^+)\mathrm{d}x.\end{aligned}$$

Similar arguments as for Theorem 4.2.3 together with the assumption (4.58) imply that $(\widehat{P}\widehat{L}^S, \widehat{P}_-^{-1}D(\widehat{L}^S|_{\mathcal{L}_\alpha}))$ is the generator of an analytic semigroup $(\widehat{U}_\alpha(t)\otimes 0^-)_{t\geq 0}$ of contractions on $\mathcal{L}_\alpha \otimes 0^-$, where $D(\widehat{L}^S|_{\mathcal{L}_\alpha}) = \{G \in \mathcal{L}_\alpha \mid \overline{M}\cdot G \in \mathcal{L}_\alpha\}$. Because of $\overline{c}(\alpha;\eta^+) \leq \|k_{\mathrm{inv}}\|_{\mathcal{K}_\beta} a_S(\alpha,\beta)M_S(\eta^+,\varnothing)$ we obtain

$$(\widehat{P}\widehat{L}^S, \widehat{P}_-^{-1}B_{bs}(\Gamma_0)) \subset (\widehat{P}\widehat{L}^S, \widehat{P}_-^{-1}D_\alpha(M_S(\cdot,\varnothing))) \subset (\widehat{P}\widehat{L}^S, \widehat{P}_-^{-1}D(\widehat{L}^S|_{\mathcal{L}_\alpha})).$$

Hence it is enough to show that $\overline{(\widehat{P}\widehat{L}^S, \widehat{P}_-^{-1}B_{bs}(\Gamma_0))} = \overline{(\widehat{P}\widehat{L}^S, \widehat{P}_-^{-1}D(\widehat{L}^S|_{\mathcal{L}_\alpha}))}$. However, this can be shown by the same arguments as in Lemma 4.2.4. $\square$

Let $\widehat{U}_\alpha(t) := (\widehat{U}_\alpha(t)\otimes 0^-)(t)|_{\mathcal{L}_\alpha}$, then it has the generator $(\widehat{L}^S|_{\mathcal{L}_\alpha}, D(\widehat{L}^S|_{\mathcal{L}_\alpha}))$, where

$$\begin{aligned}\widehat{L}^S|_{\mathcal{L}_\alpha} G(\eta) = &- \sum_{\xi^+\subset\eta^-} G(\xi^+) \sum_{x\in\xi^+} (K_0^{-1}\overline{d}(x,\cdot\cup\xi^+\backslash x))(\eta^+\backslash\xi^+) \\ &+ \sum_{\xi^+\subset\eta^+} \int_{\mathbb{R}^d} G(\xi^+\cup x)(K_0^{-1}\overline{b}(x,\cdot\cup\xi^+))(\eta^+\backslash\xi^+)\mathrm{d}x.\end{aligned}$$

The next statement establishes the weak-coupling limit for quasi-observables and correlation functions.

Theorem 4.8.10. *For every $\varepsilon > 0$ the operator $(\widehat{L}^S + \frac{1}{\varepsilon}\widehat{L}^E, D(\widehat{L}^S) \cap D(\widehat{L}^E))$ is the generator of an analytic semigroup of contractions on $\mathcal{L}_{\alpha,\beta}$. Let $\widehat{T}^\varepsilon(t)$ be the semigroup generated by $\widehat{L}^S + \frac{1}{\varepsilon}\widehat{L}^E$. Then for any $G \in \mathcal{L}_\alpha$*

$$\widehat{T}^\varepsilon(t)\widehat{P}_-^{-1}G \longrightarrow \widehat{P}_-^{-1}\widehat{U}_\alpha(t)G, \quad \varepsilon \to 0 \tag{4.70}$$

and for any $k \in \mathcal{K}_{\alpha,\beta}$

$$\int\limits_{\Gamma_0} G(\eta^+)(\widehat{T}^\varepsilon(t)^*k)(\eta^+, \varnothing)\mathrm{d}\lambda(\eta^+) \longrightarrow \int\limits_{\Gamma_0} G(\eta^+)\left(\widehat{U}_\alpha(t)^*k(\cdot, \varnothing)\right)(\eta^+)\mathrm{d}\lambda(\eta^+), \quad \varepsilon \to 0 \tag{4.71}$$

holds uniformly on compacts in $t \geq 0$.

Proof. For $\varepsilon > 0$ and $\eta \in \Gamma_0^2$ we get

$$\begin{aligned} c_S(\alpha,\beta;\eta^+) + \frac{1}{\varepsilon}c_E(\beta;\eta^-) &\leq a_S(\alpha,\beta)M_S(\eta) + \frac{a_E(\beta)}{\varepsilon}M_E(\eta^-) \\ &\leq \max\{a_E(\beta), a_S(\alpha,\beta)\}\left(M_S(\eta) + \frac{1}{\varepsilon}M_E(\eta^-)\right). \end{aligned}$$

Theorem 4.2.3 and conditions (E1), (S1) imply that $(\widehat{L}^S + \frac{1}{\varepsilon}\widehat{L}^E, D(\widehat{L}^S) \cap D(\widehat{L}^E))$ is the generator of an analytic semigroup of contractions on $\mathcal{L}_{\alpha,\beta}$. Applying [Kur73, Theorem 2.1] yields (4.70) and hence

$$\langle \widehat{P}_-^{-1}G, \widehat{T}^\varepsilon(t)^*k\rangle = \langle \widehat{T}^\varepsilon(t)\widehat{P}_-^{-1}G, k\rangle \longrightarrow \langle \widehat{P}_-^{-1}\widehat{U}_\alpha(t)G, k\rangle$$

holds uniformly on compacts in $t \geq 0$. The convergence (4.71) now follows from

$$\begin{aligned} \langle \widehat{P}_-^{-1}\widehat{U}_\alpha(t)G, k\rangle &= \int\limits_{\Gamma_0} \widehat{U}_\alpha(t)G(\eta^+)k(\eta^+, \varnothing)\mathrm{d}\lambda(\eta^+) \\ &= \int\limits_{\Gamma_0} G(\eta^+)\left(\widehat{U}_\alpha(t)^*k(\cdot, \varnothing)\right)(\eta^+)\mathrm{d}\lambda(\eta^+). \end{aligned}$$

□

In view of Proposition 4.4.5 the assertions of Proposition 4.8.2 are proved.

4.9 Examples

The birth-and-death intensities in the examples given below consist of terms

$$E(x,\gamma^{\pm}) = \sum_{y\in\gamma^{\pm}} \varphi(x-y), \quad x\in\mathbb{R}^d, \quad \gamma^{\pm}\in\Gamma.$$

Here φ is a symmetric, non-negative integrable function. To assure condition (A) or (A') respectively, we always suppose that either φ is compactly supported or condition (3.37) is satisfied.

4.9.1 Two interacting Glauber dynamics

Suppose that the death intensities are given by

$$\begin{aligned} d^E(x,\gamma^+,\gamma^-\backslash x) &= \exp\left(-sE_{\psi^S}(x,\gamma^+)\right) \\ d^S(x,\gamma^+\backslash x,\gamma^-) &= \exp\left(-sE_{\psi^E}(x,\gamma^-)\right), \end{aligned}$$

where $s\in[0,\frac{1}{2}]$ and ψ^S,ψ^E are symmetric, non-negative and integrable. The birth intensities are assumed to be of the form

$$\begin{aligned} b^E(x,\gamma) &= z^E\exp\left(-(1-s)E_{\psi^S}(x,\gamma^+)\right)\exp\left(-E_{\phi^E}(x,\gamma^-)\right) \\ b^S(x,\gamma) &= z^S\exp\left(-(1-s)E_{\psi^E}(x,\gamma^-)\right)\exp\left(-E_{\phi^S}(x,\gamma^+)\right), \end{aligned}$$

where $z^E,z^S>0$ and ϕ^E,ϕ^S are assumed to be non-negative, symmetric and integrable. For $f:\mathbb{R}^d\longrightarrow\mathbb{R}_+$ let $E_f(x,\gamma^{\pm}) := \sum\limits_{y\in\gamma^{\pm}} f(x-y)$ and

$$C(f) := \int\limits_{\mathbb{R}^d} |e^{-f(x)}-1|\mathrm{d}x. \tag{4.72}$$

Evolution of states

The cumulative death intensity is given by

$$M(\eta) = \sum_{x\in\eta^-}\exp\left(-sE_{\psi^S}(x,\eta^+)\right) + \sum_{x\in\eta^+}\exp\left(-sE_{\psi^E}(x,\eta^-)\right)$$

and we obtain

$$\begin{aligned} |\mathbb{K}_0^{-1}d^E(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi) &= 0^{|\xi^-|}\exp\left(-sE_{\psi^S}(x,\eta^+)\right)e_\lambda\left(|e^{-s\psi^S(x-\cdot)}-1|;\xi^+\right) \\ |\mathbb{K}_0^{-1}d^S(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi) &= 0^{|\xi^+|}\exp\left(-sE_{\psi^E}(x,\eta^-)\right)e_\lambda\left(|e^{-s\psi^E(x-\cdot)}-1|;\xi^-\right). \end{aligned}$$

For the birth intensities it follows that

$$
\begin{aligned}
&|\mathbb{K}_0^{-1} b^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) \\
&= z^E e^{-(1-s)E_{\psi^S}(x,\eta^+)} e^{-E_{\phi^E}(x,\eta^- \backslash x)} e_\lambda \left(|e^{-(1-s)\psi^S(x-\cdot)} - 1|; \xi^+ \right) e_\lambda \left(|e^{-\phi^E(x-\cdot)} - 1|; \xi^- \right)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathbb{K}_0^{-1} b^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) \\
&= z^S e^{-(1-s)E_{\psi^E}(x,\eta^-)} e^{-E_{\phi^S}(x,\eta^+ \backslash x)} e_\lambda \left(|e^{-(1-s)\psi^E(x-\cdot)} - 1|; \xi^- \right) e_\lambda \left(|e^{-\phi^S(x-\cdot)} - 1|; \xi^+ \right)
\end{aligned}
$$

hold. Hence we obtain for any $\alpha, \beta \in \mathbb{R}$

$$
\begin{aligned}
&c(\alpha, \beta; \eta) = \\
&+ \exp\left(e^\alpha C(s\psi^S)\right) \sum_{x \in \eta^-} e^{-sE_{\psi^S}(x,\eta^+)} + \exp\left(e^\beta C(s\psi^E)\right) \sum_{x \in \eta^+} e^{-sE_{\psi^E}(x,\eta^-)} \\
&+ e^{-\beta} z^E \exp\left(e^\alpha C((1-s)\psi^S)\right) \exp\left(e^\beta C(\phi^E)\right) \sum_{x \in \eta^-} e^{-(1-s)E_{\psi^S}(x,\eta^+)} e^{-E_{\phi^E}(x,\eta^- \backslash x)} \\
&+ e^{-\alpha} z^S \exp\left(e^\beta C((1-s)\psi^E)\right) \exp\left(e^\alpha C(\phi^S)\right) \sum_{x \in \eta^+} e^{-(1-s)E_{\psi^E}(x,\eta^-)} e^{-E_{\phi^S}(x,\eta^+ \backslash x)}.
\end{aligned}
$$

The next theorem provides an evolution of states.

Theorem 4.9.1. *Let $\phi^S, \phi^E, \psi^S, \psi^E$ be symmetric, non-negative and integrable and assume that the paramters satisfy the relations*

$$
e^{e^\alpha C(s\psi^S)} + e^{-\beta} z^E e^{e^\alpha C((1-s)\psi^S)} e^{e^\beta C(\phi^E)} < 2 \tag{4.73}
$$

$$
e^{e^\beta C(s\psi^E)} + e^{-\alpha} z^S e^{e^\beta C((1-s)\psi^E)} e^{e^\alpha C(\phi^S)} < 2. \tag{4.74}
$$

Then conditions (A) – (D) are satisfied for $\tau = 0$ and (4.20) *holds. If in addition $s = 0$, then there exists a unique invariant measure $\mu_{\text{inv}} \in \mathcal{P}_{\alpha,\beta}$ and the dynamics described by the operator L is ergodic with exponential rate.*

Proof. Conditions (A), (B), (D) are obvious and (C) follows from the representation for $c(\eta)$. In view of $s \in [0, \frac{1}{2}]$, (4.73) and (4.74) condition (4.20) holds for

$$
\begin{aligned}
a(\alpha, \beta) = \max\Big\{ &e^{e^\alpha C(s\psi^S)} + e^{-\beta} z^E e^{e^\alpha C((1-s)\psi^S)} e^{e^\beta C(\phi^E)}, \\
&e^{e^\beta C(s\psi^E)} + e^{-\alpha} z^S e^{e^\beta C((1-s)\psi^E)} e^{e^\alpha C(\phi^S)} \Big\}.
\end{aligned}
$$

If $s = 0$, then $M(\eta) = |\eta|$ which yields ergodicity. ◻

In the case $s = 0$ conditions (4.73) and (4.74) simplify to

$$z^E e^{e^{\alpha} C(\psi^S)} e^{e^{\beta} C(\phi^E)} < e^{\beta} \tag{4.75}$$

$$z^S e^{e^{\beta} C(\psi^E)} e^{e^{\alpha} C(\phi^S)} < e^{\alpha}. \tag{4.76}$$

Of particular interest is the special case $\phi^S = 0 = \phi^E$, also known as the Widom-Rowlinson model. The non-equilibrium dynamics for this model has recently been analysed in [FKKO15], but without conditions (4.75) and (4.76) only existence of a local evolution of correlation functions could have been shown. Conditions (4.75) and (4.76) are satisfied for $e^{-\alpha} = C(\psi^S)$ and $e^{-\beta} = C(\psi^E)$ if

$$z^E < \frac{1}{eC(\psi^E)} \text{ and } z^S < \frac{1}{eC(\psi^S)}$$

are satisfied.

Vlasov scaling

For simplicity we consider the case $s = 0$, hence the death intensities need not to be scaled, i.e. are given by

$$d^E(x, \gamma^+, \gamma^- \backslash x) = 1 = d^S(x, \gamma^+ \backslash x, \gamma^-).$$

The scaled birth intensities are given by

$$b_n^E(x, \gamma) = z^E \exp\left(-\frac{1}{n} E_{\psi^S}(x, \gamma^+)\right) \exp\left(-\frac{1}{n} E_{\phi^E}(x, \gamma^-)\right)$$
$$b_n^S(x, \gamma) = z^S \exp\left(-\frac{1}{n} E_{\psi^E}(x, \gamma^-)\right) \exp\left(-\frac{1}{n} E_{\phi^S}(x, \gamma^+)\right).$$

This yields for the death intensities

$$|\mathbb{K}_0^{-1} d^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) = 0^{|\xi|}$$
$$|\mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) = 0^{|\xi|}.$$

For the birth intensities we get

$$|\mathbb{K}_0^{-1} b_n^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi)$$
$$= z^E e^{-\frac{1}{n} E_{\psi^S}(x, \eta^+)} e^{-\frac{1}{n} E_{\phi^E}(x, \eta^- \backslash x)} e_\lambda\left(\left|e^{-\frac{1}{n}\psi^S(x - \cdot)} - 1\right|; \xi^+\right) e_\lambda\left(\left|e^{-\frac{1}{n}\phi^E(x - \cdot)} - 1\right|; \xi^-\right)$$

and

$$|\mathbb{K}_0^{-1} b_n^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi)$$
$$= z^S e^{-\frac{1}{n} E_{\psi^E}(x, \eta^-)} e^{-\frac{1}{n} E_{\phi^S}(x, \eta^+ \backslash x)} e_\lambda\left(\left|e^{-\frac{1}{n}\psi^E(x - \cdot)} - 1\right|; \xi^-\right) e_\lambda\left(\left|e^{-\frac{1}{n}\phi^S(x - \cdot)} - 1\right|; \xi^+\right).$$

Moreover we have $M(\eta) = |\eta| = |\eta^+| + |\eta^-|$. Therefore for $\alpha, \beta \in \mathbb{R}$ (where we put in addition the factor n in front of the birth-terms)

$$\begin{aligned}&c_n(\alpha,\beta;\eta) \le |\eta| \\ &+ e^{-\beta} z^E \exp\left(e^{\alpha}\langle\psi^S\rangle\right) \exp\left(e^{\beta}\langle\phi^E\rangle\right) |\eta^-| + e^{-\alpha} z^S \exp\left(e^{\beta}\langle\psi^E\rangle\right) \exp\left(e^{\alpha}\langle\phi^S\rangle\right) |\eta^+|\end{aligned}$$

Thus condition (V1) is satisfied with

$$a(\alpha,\beta) = 1 + \max\left\{ e^{-\beta} z^E e^{e^{\alpha}\langle\psi^S\rangle} e^{e^{\beta}\langle\phi^E\rangle}, e^{-\alpha} z^S e^{e^{\beta}\langle\psi^E\rangle} e^{e^{\alpha}\langle\phi^S\rangle} \right\}$$

provided we suppose that

$$\begin{aligned} z^E \exp\left(e^{\alpha}\langle\psi^S\rangle\right) \exp\left(e^{\beta}\langle\phi^E\rangle\right) &< e^{\beta} \\ z^S \exp\left(e^{\beta}\langle\psi^E\rangle\right) \exp\left(e^{\alpha}\langle\phi^S\rangle\right) &< e^{\alpha} \end{aligned}$$

holds. Suppose that $\psi^S, \psi^E, \phi^S, \phi^E$ are in addition bounded, then condition (V2) is not difficult to see, cf. [FK13, FFH$^+$15]. This yields

$$D_x^{V,E}(\eta) = 0^{|\eta|} = D_x^{V,S}(\eta)$$

and

$$\begin{aligned} z^E e_{\lambda}\left(-\psi^S(x-\cdot);\xi^+\right) e_{\lambda}\left(-\phi^E(x-\cdot);\xi^-\right) &=: B_x^{V,E}(\eta) \\ z^S e_{\lambda}\left(-\psi^E(x-\cdot);\xi^-\right) e_{\lambda}\left(-\phi^S(x-\cdot);\xi^+\right) &=: B_x^{V,S}(\eta), \end{aligned}$$

and hence also (V3) holds. Therefore all previous results can be applied and we obtain the mesoscopic limit equations, cf. (4.10) and (4.11), given by

$$\frac{\partial \rho_t^E}{\partial t}(x) = -\rho_t^E(x) + z^E e^{-(\phi^E * \rho_t^E)(x)} e^{-(\psi^S * \rho_t^S)(x)} \tag{4.77}$$

$$\frac{\partial \rho_t^S}{\partial t}(x) = -\rho_t^S(x) + z^S e^{-(\phi^S * \rho_t^S)(x)} e^{-(\psi^E * \rho_t^E)(x)}. \tag{4.78}$$

Here and in the following $*$ denotes the usual convolution of functions on $\mathbb{R}^d$.

Weak-coupling limit

Suppose that $s = 0$ and $\psi^S = 0$ holds, then we have

$$d^E(x,\gamma^+,\gamma^-\backslash x) = 1 = d^S(x,\gamma^+\backslash x,\gamma^-).$$

The birth intensities are given by

$$\begin{aligned} b^E(x,\gamma) &= z^E \exp\left(-E_{\phi^E}(x,\gamma^-)\right) \\ b^S(x,\gamma) &= z^S \exp\left(-E_{\psi^E}(x,\gamma^-)\right) \exp\left(-E_{\phi^S}(x,\gamma^+)\right). \end{aligned}$$

Therefore we obtain $M_E(\eta^-) = |\eta^-|$, $M_S(\eta) = |\eta^+|$ and for $\alpha, \beta \in \mathbb{R}$

$$c_E(\beta;\eta) \leq \left(1 + e^{-\beta} z^E \exp\left(e^{\beta} C(\phi^E)\right)\right) |\eta^-|$$
$$c_S(\alpha,\beta;\eta) \leq \left(1 + e^{-\alpha} z^S \exp\left(e^{\beta} C(\psi^E)\right) \exp\left(e^{\alpha} C(\phi^S)\right)\right) |\eta^+|$$

follows. Suppose that

$$z^E \exp\left(e^{\beta} C(\phi^E)\right) < e^{\beta}$$
$$z^S \exp\left(e^{\beta} C(\psi^E)\right) \exp\left(e^{\alpha} C(\phi^S)\right) < e^{\alpha} \tag{4.79}$$

are fulfilled. Then conditions (E1) – (E4) and (S1) – (S3) are satisfied. Because of $b^E(x,\eta) \leq z^E$ and $b^S(x,\eta) \leq z^E$ for all $\eta \in \Gamma_0^2$ condition (L) holds e.g. for $R_\delta(x) := e^{-\delta|x|^2}$. The unique invariant measure for the environment is given by the Gibbs measure μ_{inv} with activity z^E and potential ϕ^E. Let

$$\overline{d}(x,\eta^+) := \int_\Gamma d^S(x,\eta^+,\gamma^-) \mathrm{d}\mu_{\mathrm{inv}}(\gamma^-) = 1$$
$$\overline{b}(x,\eta^+) := \int_\Gamma b^S(x,\eta^+,\gamma^-) \mathrm{d}\mu_{\mathrm{inv}}(\gamma^-) = z^S \exp\left(-E_{\phi^S}(x,\gamma^+)\right) \int_\Gamma e^{-E_{\psi^E}(x,\gamma^-)} \mathrm{d}\mu_{\mathrm{inv}}(\gamma^-).$$

Then $\overline{b}(x,\eta^+) \leq 1$ and with $\overline{\lambda}(x) := \int_\Gamma e^{-E_{\psi^E}(x,\gamma^-)} \mathrm{d}\mu_{\mathrm{inv}}(\gamma^-) \leq 1$ we get

$$\overline{c}(\alpha;\eta^+) = |\eta^+| + e^{-\alpha} z^S \exp\left(e^{\alpha} C(\phi^S)\right) \sum_{x \in \eta^+} \exp\left(-E_{\phi^S}(x,\gamma^+)\right) \overline{\lambda}(x).$$
$$\leq \left(1 + e^{-\alpha} z^S \exp\left(e^{\alpha} C(\phi^S)\right)\right) |\eta^+|.$$

Thus condition (4.58) holds since by (4.79)

$$z^S \exp\left(e^{\alpha} C(\phi^S)\right) < e^{\alpha}.$$

Hence we have shown that Proposition 4.8.2 is applicable. The limiting dynamics is given by the averaged operator

$$(\overline{L}F)(\gamma^+) = \sum_{x \in \gamma^+} (F(\gamma^+ \backslash x) - F(\gamma^+)) + z^S \int_{\mathbb{R}^d} \overline{\lambda}(x) e^{-E_{\phi^S}(x,\gamma^+)} (F(\gamma^+ \cup x) - F(\gamma^+)) \mathrm{d}x.$$

That is by a Glauber dynamics with potential ϕ^S and activity $z^S\overline{\lambda}$. The mesoscopic equation is in such a case given by

$$\frac{\partial \overline{\rho}_t}{\partial t}(x) = -\overline{\rho}_t(x) + z^S \overline{\lambda}(x) e^{-(\phi^S * \overline{\rho}_t)(x)}.$$

This equation can be obtained from (4.77) and (4.78). Namely, suppose that ρ^E is a solution to the stationary version of equation (4.77), i.e.

$$\rho^E(x) = z^E e^{-(\phi^E * \rho^E)(x)}, \quad \text{a.a. } x \in \mathbb{R}^d$$

holds. By $e^{(-\psi^E * \rho^E)(x)} \equiv \overline{\lambda}(x)$ we obtain from (4.78) above averaged kinetic equation.

4.9.2 BDLP-dynamics in Glauber environment

Let us consider death intensities given by

$$\begin{aligned} d^E(x, \gamma^+, \gamma^- \backslash x) &= 1 \\ d^S(x, \gamma^+ \backslash x, \gamma^-) &= m^S + \sum_{y \in \gamma^+ \backslash x} a^-(x-y) + \sum_{y \in \gamma^-} \phi(x-y), \end{aligned}$$

where $m^S > 0$ and $0 \leq a^-, \phi \in L^1(\mathbb{R}^d)$ are symmetric. The birth intensities are assumed to be of the form

$$\begin{aligned} b^E(x, \gamma) &= z^E \exp\left(-E_\psi(x, \gamma^-)\right) \\ b^S(x, \gamma) &= \sum_{y \in \gamma^+} a^+(x-y) + \sum_{y \in \gamma^-} b^+(x-y), \end{aligned}$$

where $z^E > 0$ and $0 \leq \psi, a^+, b^+ \in L^1(\mathbb{R}^d)$ are symmetric.

Evolution of states

We have

$$\mathbb{K}_0^{-1} d^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)(\xi) = 0^{|\xi|}$$

and

$$\begin{aligned} \mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)(\xi) &= 0^{|\xi|} m^S + 0^{|\xi|} \sum_{y \in \eta^+ \backslash x} a^-(x-y) + 0^{|\xi|} \sum_{y \in \eta^-} \phi(x-y) \\ &+ 0^{|\xi^-|} \mathbb{1}_{\Gamma^{(1)}}(\xi^+) \sum_{y \in \xi^+} a^-(x-y) + 0^{|\xi^+|} \mathbb{1}_{\Gamma^{(1)}}(\xi^-) \sum_{y \in \xi^-} \phi(x-y). \end{aligned}$$

Likewise we obtain

$$\mathbb{K}_0^{-1} b^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)(\xi) = z^E e^{-E_\psi(x, \eta^- \backslash x)} e_\lambda\left(e^{-\psi(x-\cdot)} - 1; \xi^-\right) 0^{|\xi^+|}$$

and

$$\begin{aligned} \mathbb{K}_0^{-1} b^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)(\xi) &= 0^{|\xi|} \sum_{y \in \eta^+ \backslash x} a^+(x-y) + 0^{|\xi|} \sum_{y \in \eta^-} b^+(x-y) \\ &+ 0^{|\xi^-|} \mathbb{1}_{\Gamma^{(1)}}(\xi^+) \sum_{y \in \xi^+} a^+(x-y) + 0^{|\xi^+|} \mathbb{1}_{\Gamma^{(1)}}(\xi^-) \sum_{y \in \xi^-} b^+(x-y). \end{aligned}$$

This yields

$$c(\alpha,\beta;\eta) = |\eta^-| + z^E e^{-\beta} \sum_{x\in\eta^-} e^{-E_\psi(x,\eta^-\setminus x)} \exp\left(e^\beta C(\psi)\right)$$
$$+ \left(m^S + e^\alpha\langle a^-\rangle + e^\beta\langle\phi\rangle + \langle a^+\rangle + \langle b^+\rangle\right)|\eta^+|$$
$$+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^-(x-y) + e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^+(x-y)$$
$$+ \sum_{x\in\eta^+}\sum_{y\in\eta^-}\phi(x-y) + e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^-} b^+(x-y).$$

Theorem 4.9.2. *Suppose that $a^\pm, b^+, \phi$ are bounded and there exist $\theta \in (0, e^\alpha)$ and $b \geq 0$ such that*

$$\sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^+(x-y) \leq \theta \sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^-(x-y) + b|\eta^+| \tag{4.80}$$

is satisfied. Moreover, assume that for some $\vartheta \in (0, e^\alpha)$ and

$$\vartheta\phi \geq b^+ \tag{4.81}$$
$$e^\beta > z^E \exp\left(e^\beta C(\psi)\right) \tag{4.82}$$
$$m^S > e^\alpha\langle a^-\rangle + e^\beta\langle\phi\rangle + \langle a^+\rangle + \langle b^+\rangle + e^{-\alpha}b \tag{4.83}$$

hold. Then conditions (A) – (D) are satisfied with $\tau = 0$ and (4.20) holds. In particular the evolution of states is ergodic with exponential rate.

Proof. Above conditions imply

$$c(\alpha,\beta;\eta) \leq \left(1 + z^E e^{-\beta}\exp\left(e^\beta C(\psi)\right)\right)|\eta^-|$$
$$+ \left(m^S + e^\alpha\langle a^-\rangle + e^\beta\langle\phi\rangle + \langle a^+\rangle + \langle b^+\rangle + e^{-\alpha}b\right)|\eta^+|$$
$$+ \left(1+\theta e^{-\alpha}\right)\sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^-(x-y) + \left(1+\vartheta e^{-\alpha}\right)\sum_{x\in\eta^+}\sum_{y\in\eta^-}\phi(x-y).$$

Since $M_E(\eta^-) = |\eta^-|$ and

$$M_S(\eta) = m^S|\eta^+| + \sum_{x\in\eta^+}\sum_{y\in\eta^-\setminus x} a^-(x-y) + \sum_{x\in\eta^+}\sum_{y\in\eta^-}\phi(x-y),$$

condition (C) is satisfied. Condition (B) holds since all potentials are bounded and (D) is obvious. Condition (4.20) holds with

$$a(\alpha,\beta) = 1 + \max\left\{ z^E e^{-\beta} e^{e^\beta C(\psi)}, \theta e^{-\alpha}, \vartheta e^{-\alpha}, \frac{e^\alpha\langle a^-\rangle + e^\beta\langle\phi\rangle + \langle a^+\rangle + \langle b^+\rangle + e^{-\alpha}b}{m^S}\right\}.$$

Ergodicity now follows from $M(\eta) \geq |\eta^-| + m^S|\eta^+|$. □

Vlasov scaling

Suppose that $a^{\pm}, b^+, \phi, \psi$ are bounded and (4.80) – (4.83) with $z^E e^{e^{\beta}\langle\psi\rangle} < e^{\beta}$ instead of (4.82) hold. Scaling of the potentials by $\frac{1}{n}$ yields for the death

$$\begin{aligned} d^E(x,\gamma^+,\gamma^-\backslash x) &= 1 \\ d^S_n(x,\gamma^+\backslash x,\gamma^-) &= m^S + \frac{1}{n}\sum_{y\in\gamma^+\backslash x} a^-(x-y) + \frac{1}{n}\sum_{y\in\gamma^-}\phi(x-y). \end{aligned}$$

For the birth we obtain

$$\begin{aligned} b^E_n(x,\gamma) &= z^E \exp\left(-\frac{1}{n}E_{\psi}(x,\gamma^-)\right) \\ b^S_n(x,\gamma) &= \frac{1}{n}\sum_{y\in\gamma^+} a^+(x-y) + \frac{1}{n}\sum_{y\in\gamma^-} b^+(x-y). \end{aligned}$$

We have together with the factor n in front of the terms contributing to the birth

$$\begin{aligned} c_n(\alpha,\beta;\eta) \leq |\eta^-| &+ z^E e^{-\beta}\sum_{x\in\eta^-} e^{-\frac{1}{n}E_{\psi}(x,\eta^-\backslash x)}\exp\left(e^{\beta}\langle\psi\rangle\right) \\ &+ \left(m^S + e^{\alpha}\langle a^-\rangle + e^{\beta}\langle\phi\rangle + \langle a^+\rangle + \langle b^+\rangle\right)|\eta^+| \\ &+ \frac{1}{n}\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a^-(x-y) + \frac{1}{n}e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a^+(x-y) \\ &+ \frac{1}{n}\sum_{x\in\eta^+}\sum_{y\in\eta^-}\phi(x-y) + \frac{1}{n}e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^-} b^+(x-y) \end{aligned}$$

and

$$M_n(\eta) = |\eta^-| + m^S|\eta^+| + \frac{1}{n}\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} a^-(x-y) + \frac{1}{n}\sum_{x\in\eta^+}\sum_{y\in\eta^-}\phi(x-y).$$

Hence condition (V1) is satisfied. Concerning condition (V2) observe that

$$\mathbb{K}_0^{-1} d^E(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)(\xi) = 0^{|\xi|}$$

and

$$\begin{aligned} \mathbb{K}_0^{-1} d^S_n(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)(\xi) &= 0^{|\xi|}m^S + \frac{1}{n}0^{|\xi|}\sum_{y\in\eta^+\backslash x} a^-(x-y) + \frac{1}{n}0^{|\xi|}\sum_{y\in\eta^-}\phi(x-y) \\ + \frac{1}{n}0^{|\xi^-|}\mathbb{1}_{\Gamma^{(1)}}(\xi^+)\sum_{y\in\xi^+} a^-(x-y) &+ \frac{1}{n}0^{|\xi^+|}\mathbb{1}_{\Gamma^{(1)}}(\xi^-)\sum_{y\in\xi^-}\phi(x-y). \end{aligned}$$

Taking $n \to \infty$ yields $D_x^{V,E}(\eta) = 0^{|\eta|}$ and

$$D_x^{V,S}(\eta) = 0^{|\eta|} m^S + 0^{|\xi^-|} \mathbb{1}_{\Gamma^{(1)}}(\xi^+) \sum_{y \in \xi^+} a^-(x-y) + 0^{|\xi^+|} \mathbb{1}_{\Gamma^{(1)}}(\xi^-) \sum_{y \in \xi^-} \phi(x-y).$$

For the birth intensity of the environment we obtain

$$\mathbb{K}_0^{-1} b_n^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)(\xi) = z^E e^{-\frac{1}{n} E_\psi(x, \eta^- \backslash x)} e_\lambda \left(e^{-\frac{1}{n}\psi(x - \cdot)} - 1; \xi^- \right) 0^{|\xi^+|}$$

and hence

$$B_x^{V,E}(\eta) = z^E e_\lambda \left(-\psi(x - \cdot); \xi^- \right) 0^{|\xi^+|}.$$

Similarly for the birth intensity of the system

$$\mathbb{K}_0^{-1} b_n^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)(\xi) = \frac{1}{n} 0^{|\xi|} \sum_{y \in \eta^+ \backslash x} a^+(x-y) + \frac{1}{n} 0^{|\xi|} \sum_{y \in \eta^-} b^+(x-y)$$
$$+ \frac{1}{n} 0^{|\xi^-|} \mathbb{1}_{\Gamma^{(1)}}(\xi^+) \sum_{y \in \xi^+} a^+(x-y) + \frac{1}{n} 0^{|\xi^+|} \mathbb{1}_{\Gamma^{(1)}}(\xi^-) \sum_{y \in \xi^-} b^+(x-y)$$

yields

$$B_x^{V,S}(\eta) = 0^{|\xi^-|} \mathbb{1}_{\Gamma^{(1)}}(\xi^+) \sum_{y \in \xi^+} a^+(x-y) + 0^{|\xi^+|} \mathbb{1}_{\Gamma^{(1)}}(\xi^-) \sum_{y \in \xi^-} b^+(x-y).$$

This implies conditions (V2) and (V3). The kinetic equation is therefore given by

$$\frac{\partial \rho_t^E}{\partial t}(x) = -\rho_t^E(x) + z^E e^{-(\psi * \rho_t^E)(x)}$$
$$\frac{\partial \rho_t^S}{\partial t}(x) = -\left(m^S + (\phi * \rho_t^E)(x) \right) \rho_t^S(x) - \rho_t^S(x)(a^- * \rho_t^S)(x) + (a^+ * \rho_t^S)(x) + (b^+ * \rho_t^E)(x).$$

Weak-coupling limit

Suppose that the same as for the evolution of states are satisfied. By previous computations we get $M_E(\eta^-) = |\eta^-|$,

$$c_E(\beta; \eta^-) \le \left(1 + e^{-\beta} z^E \exp\left(e^\beta C(\psi) \right) \right) |\eta^-|$$

and hence (E1) – (E4) hold. For the system observe that

$$M_S(\eta) = m^S |\eta^+| + \sum_{x \in \eta^+} \sum_{y \in \eta^+ \backslash x} a^-(x-y) + \sum_{x \in \eta^+} \sum_{y \in \eta^-} \phi(x-y)$$

and

$$\begin{aligned}c_S(\alpha,\beta;\eta) = &\left(m^S + e^{\alpha}\langle a^-\rangle + e^{\beta}\langle\phi\rangle + \langle a^+\rangle + \langle b^+\rangle\right)|\eta^+| \\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^-(x-y) + e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^+(x-y) \\ &+ \sum_{x\in\eta^+}\sum_{y\in\eta^-}\phi(x-y) + e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^-} b^+(x-y).\end{aligned}$$

Then (S1) – (S3) are satisfied and (L) is not difficult to see. The unique invariant measure for the environment is the Gibbs measure μ_{inv} with activity z^E and potential ψ. The averaged intensities are therefore given by

$$\overline{d}(x,\eta^+) = m^S + \sum_{y\in\gamma^+\setminus x} a^-(x-y) + \int_{\Gamma}\sum_{y\in\gamma^-}\phi(x-y)\mathrm{d}\mu_{\text{inv}}(\gamma^-)$$

and

$$\overline{b}(x,\eta^+) = \sum_{y\in\gamma^+} a^+(x-y) + \int_{\Gamma}\sum_{y\in\gamma^-} b^+(x-y)\mathrm{d}\mu_{\text{inv}}(\gamma^-).$$

Let $\overline{\lambda}(x) := \int_{\Gamma}\sum_{y\in\gamma^-} b^+(x-y)\mathrm{d}\mu_{\text{inv}}(\gamma^-)$ and $\overline{m}(x) := \int_{\Gamma}\sum_{y\in\gamma^-}\phi(x-y)\mathrm{d}\mu_{\text{inv}}(\gamma^-)$, then

$$\overline{m}(x) \le z^E\int_{\Gamma}\int_{\mathbb{R}^d}\phi(x-y)e^{-E_{\psi}(x,\gamma^-)}\mathrm{d}y\mathrm{d}\mu_{\text{inv}}(\gamma^-) \le z^E\langle\phi\rangle$$

and $\overline{\lambda}(x) \le z^E\langle b^+\rangle$. It follows

$$\overline{M}(\eta^+) = m^S|\eta^+| + \sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^-(x-y) + \sum_{x\in\eta^+}\overline{m}(x)$$

and we only have to show (4.58). By (4.80)

$$\begin{aligned}\overline{c}(\alpha;\eta^+) = &\left(m^S + e^{\alpha}\langle a^-\rangle + \langle a^+\rangle\right)|\eta^+| + \sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^-(x-y) + e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^+(x-y) \\ &+ \sum_{x\in\eta^+}\overline{m}(x) + e^{-\alpha}\sum_{x\in\eta^+}\overline{\lambda}(x) \\ \le &\left(m^S + e^{\alpha}\langle a^-\rangle + \langle a^+\rangle + be^{-\alpha}\right)|\eta^+| + (1+\theta e^{-\alpha})\sum_{x\in\eta^+}\sum_{y\in\eta^+\setminus x} a^-(x-y) \\ &+ \sum_{x\in\eta^+}\overline{m}(x) + e^{-\alpha}\sum_{x\in\eta^+}\overline{\lambda}(x)\end{aligned}$$

condition (4.58) is satisfied, provided there exists $q \in (0,1)$ with

$$be^{-\alpha} + \overline{\lambda}(x)e^{-\alpha} + e^{\alpha}\langle a^- \rangle + \langle a^+ \rangle \leq q(m^S + \overline{m}(x))$$

for a.a. $x \in \mathbb{R}^d$. This is in particular satisfied if there exist $\kappa \in (0, e^{\alpha})$ with $\overline{\lambda} \leq \kappa \overline{m}$. The averaged (pre-)generator is given by

$$\begin{aligned}(\overline{L}F)(\gamma^+) = &\sum_{x \in \gamma^+} \left(m^S + \overline{m}(x) + \sum_{y \in \gamma^+ \backslash x} a^-(x-y) \right) (F(\gamma^+ \backslash x) - F(\gamma^+)) \\ &+ \sum_{x \in \gamma^+} \int_{\mathbb{R}^d} a^+(x-y)(F(\gamma^+ \cup y) - F(\gamma^+))\mathrm{d}y + \int_{\mathbb{R}^d} \overline{\lambda}(y)(F(\gamma^+ \cup y) - F(\gamma^+))\mathrm{d}y.\end{aligned}$$

The mesoscopic equation is in such a case given by

$$\frac{\partial \overline{\rho}_t}{\partial t}(x) = -(m^S + \overline{m}(x))\overline{\rho}_t(x) - \overline{\rho}_t(x)(a^- * \overline{\rho}_t)(x) + (a^+ * \overline{\rho}_t)(x) + \overline{\lambda}(x).$$

4.9.3 Density dependent branching in Glauber environment

Suppose that the death intensities are given by

$$\begin{aligned} d^E(x, \gamma^+, \gamma^- \backslash x) &= 1 \\ d^S(x, \gamma^+ \backslash x, \gamma^-) &= m^S \exp\left(E_{\phi^S}(x, \gamma^+ \backslash x) \right), \end{aligned}$$

where $m^S > 0$. The birth intensities are given by

$$\begin{aligned} b^E(x, \gamma) &= z^E \exp\left(-E_{\phi^E}(x, \gamma^-) \right) \\ b^S(x, \gamma) &= \sum_{y \in \gamma^+} \exp\left(-E_{\psi^E}(y, \gamma^-) \right) a^+(x-y) \end{aligned}$$

with $z^E > 0$ and $a^+, \phi^E, \phi^S, \psi^E$ symmetric, non-negative and integrable.

Evolution of states

Similar to previous models we obtain

$$\begin{aligned} |\mathbb{K}_0^{-1} d^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) &= 0^{|\xi|} \\ |\mathbb{K}_0^{-1} d^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) &= 0^{|\xi^-|} m^S e^{E_{\phi^S}(x, \eta^+ \backslash x)} e_\lambda\left(e^{\phi^S(x - \cdot)} - 1; \xi^+ \right) \\ |\mathbb{K}_0^{-1} b^E(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) &= 0^{|\xi^+|} z^E e^{-E_{\phi^E}(x, \eta^- \backslash x)} e_\lambda\left(|e^{-\phi^E(x - \cdot)} - 1|; \xi^- \right) \end{aligned}$$

and

$$
\begin{aligned}
&|\mathbb{K}_0^{-1} b^S(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) = \\
&+ \mathbb{1}_{\Gamma^{(1)}}(\xi^+) \sum_{y \in \xi^+} a^+(x-y) e^{-E_{\psi^E}(y,\eta^-)} e_\lambda \left(|e^{-\psi^E(y-\cdot)} - 1|; \xi^- \right) \\
&+ 0^{|\xi^+|} \sum_{y \in \eta^+ \backslash x} a^+(x-y) e_\lambda \left(|e^{-\psi^E(y-\cdot)} - 1|; \xi^- \right) e^{-E_{\psi^E}(y,\eta^-)}.
\end{aligned}
$$

The cumulative death intensity is given by $M(\eta) = |\eta^-| + m^S \sum\limits_{x \in \eta^+} e^{E_{\phi^S}(x,\eta^+ \backslash x)}$ and we obtain for $\alpha, \beta \in \mathbb{R}$

$$
\begin{aligned}
c(\alpha, \beta; \eta) = |\eta^-| &+ m^S \exp\left(e^\alpha C(-\phi^S)\right) \sum_{x \in \eta^+} e^{E_{\phi^S}(x,\eta^+ \backslash x)} \\
&+ z^E e^{-\beta} \exp\left(e^\beta C(\phi^E)\right) \sum_{x \in \eta^-} e^{-E_{\phi^E}(x,\eta^- \backslash x)} \\
&+ \exp\left(e^\beta C(\psi^E)\right) \sum_{x \in \eta^+} \int_{\mathbb{R}^d} a^+(x-y) e^{-E_{\psi^E}(y,\eta^-)} \mathrm{d}y \\
&+ e^{-\alpha} \exp\left(e^\beta C(\psi^E)\right) \sum_{x \in \eta^+} \sum_{y \in \eta^+ \backslash x} a^+(x-y) e^{-E_{\psi^E}(y,\eta^-)}.
\end{aligned}
$$

Theorem 4.9.3. *Suppose that $0 \neq \phi^S, a^+$ are bounded, there exist constants $\kappa > 0$ and $b \geq 0$ such that for all $\eta^+ \in \Gamma_0$*

$$
\sum_{x \in \eta^+} \sum_{y \in \eta^+ \backslash x} a^+(x-y) \leq \vartheta \sum_{x \in \eta^+} \sum_{y \in \eta^+ \backslash x} \phi^S(x-y) + b|\eta^+| \tag{4.84}
$$

and the parameters satisfy the relations

$$
\begin{aligned}
e^\beta &> z^E \exp\left(e^\beta C(\phi^E)\right) \\
2 &> e^{e^\alpha C(-\phi^S)} + \frac{\max\{\langle a^+ \rangle + b e^{-\alpha}, \vartheta e^{-\alpha}\}}{m^S} e^{e^\beta C(\psi^E)}.
\end{aligned}
$$

Then conditions (A) – (D) hold with $\tau = \|\phi^S\|_\infty$ and (4.20) is satisfied. The corresponding evolution of states is ergodic with exponential rate and the invariant measure is given by $\delta_\varnothing \otimes \mu^E$, where μ^E be the unique Gibbs measure with activity z^E and potential ϕ^E.

Proof. We obtain

$$\begin{aligned}
c(\eta) \leq &|\eta^-| \left(1 + z^E e^{-\beta} \exp\left(e^\beta C(\phi^E)\right)\right) + e^{-\alpha} \exp\left(e^\beta C(\psi^E)\right) \sum_{x\in\eta^+} \sum_{y\in\eta^+\backslash x} a^+(x-y) \\
&+ |\eta^+| \langle a^+\rangle \exp\left(e^\beta C(\psi^E)\right) + m^S \exp\left(e^\alpha C(-\phi^S)\right) \sum_{x\in\eta^+} e^{E_{\phi^S}(x,\eta^+\backslash x)} \\
\leq &|\eta^-| \left(1 + z^E e^{-\beta} \exp\left(e^\beta C(\phi^E)\right)\right) + |\eta^+| \left(\langle a^+\rangle + e^{-\alpha} b\right) \exp\left(e^\beta C(\psi^E)\right) \\
&+ m^S \exp\left(e^\alpha C(-\phi^S)\right) \sum_{x\in\eta^+} e^{E_{\phi^S}(x,\eta^+\backslash x)} + e^{-\alpha} \exp\left(e^\beta C(\psi^E)\right) \vartheta \sum_{x\in\eta^+} \sum_{y\in\eta^+\backslash x} \phi^S(x-y) \\
\leq &|\eta^-| \left(1 + z^E e^{-\beta} \exp\left(e^\beta C(\phi^E)\right)\right) \\
&+ \left(m^S e^{e^\alpha C(-\phi^S)} + \max\{\langle a^+\rangle + be^{-\alpha}, \vartheta e^{-\alpha}\} e^{e^\beta C(\psi^E)}\right) \sum_{x\in\eta^+} e^{E_{\phi^S}(x,\eta^+\backslash x)}.
\end{aligned}$$

This shows condition (C) with constant

$$a(\alpha,\beta) = \max\left\{1 + z^E e^{-\beta} e^{e^\beta C(\phi^E)}, e^{e^\alpha C(-\phi^S)} + \frac{\max\{\langle a^+\rangle + be^{-\alpha}, \vartheta e^{-\alpha}\}}{m^S} e^{e^\beta C(\psi^E)}\right\}.$$

Conditions (B) and (D) are not difficult to see. Finally for any cylinder function F

$$\int_{\Gamma^2} (LF)(\gamma) \mathrm{d}(\delta_\varnothing \otimes \mu^E)(\gamma) = \int_\Gamma (L^E F)(\varnothing, \gamma^-) \mathrm{d}\mu^E(\gamma^-) = 0$$

and hence $\delta_\varnothing \otimes \mu^E$ is the invariant measure. □

Vlasov scaling

Scaling all potentials by $\frac{1}{n}$ gives $d^E(x,\gamma^+,\gamma^-\backslash x) = 1$,

$$d^S(x,\gamma^+\backslash x,\gamma^-) = m^S \exp\left(\frac{1}{n} E_{\phi^S}(x,\gamma^+\backslash x)\right)$$

and for the birth intensities

$$\begin{aligned}
b^E(x,\gamma) &= z^E \exp\left(-\frac{1}{n} E_{\phi^E}(x,\gamma^-)\right) \\
b^S(x,\gamma) &= \frac{1}{n} \sum_{y\in\gamma^+} \exp\left(-\frac{1}{n} E_{\psi^E}(y,\gamma^-)\right) a^+(x-y).
\end{aligned}$$

Therefore, after scaling of the birth by n, $M_n(\eta) = |\eta^-| + m^S \sum_{x\in\eta^+} e^{\frac{1}{n}E_{\phi^S}(x,\eta^+\setminus x)}$ and

$$\begin{aligned} c_n(\alpha,\beta;\eta) \le |\eta^-| &+ m^S \exp\left(e^{\alpha}\langle\phi^S\rangle\right) \sum_{x\in\eta^+} e^{\frac{1}{n}E_{\phi^S}(x,\eta^+\setminus x)} \\ &+ z^E e^{-\beta} \exp\left(e^{\beta}\langle\phi^E\rangle\right) \sum_{x\in\eta^-} e^{-\frac{1}{n}E_{\phi^E}(x,\eta^-\setminus x)} \\ &+ \exp\left(e^{\beta}\langle\psi^E\rangle\right) \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a^+(x-y) e^{-\frac{1}{n}E_{\psi^E}(y,\eta^-)} \mathrm{d}y \\ &+ \frac{e^{-\alpha}}{n} \exp\left(e^{\beta}\langle\psi^E\rangle\right) \sum_{x\in\eta^+} \sum_{y\in\eta^+\setminus x} a^+(x-y) e^{-\frac{1}{n}E_{\psi^E}(y,\eta^-)}. \end{aligned}$$

Suppose that $0 \neq \phi^S, a^+, \phi^E, \psi^E, a^+$ are bounded, (4.84) holds and the parameters satisfy the stronger relations

$$\begin{aligned} e^{\beta} &> z^E \exp\left(e^{\beta}\langle\phi^E\rangle\right) \\ 2 &> e^{e^{\alpha}\langle\phi^S\rangle} + \frac{\max\{\langle a^+\rangle + be^{-\alpha}, \vartheta e^{-\alpha}\}}{m^S} e^{e^{\beta}\langle\psi^E\rangle}. \end{aligned}$$

Then conditions (V1) – (V3) are satisfied. This yields the kinetic equations

$$\begin{aligned} \frac{\partial\rho_t^E}{\partial t}(x) &= -\,\rho_t^E(x) + z^E e^{-(\phi^E*\rho_t^E)(x)} \\ \frac{\partial\rho_t^S}{\partial t}(x) &= -\,m^S\rho_t^S(x) e^{(\phi^S*\rho_t^S)(x)} + (a^+*\rho_t^S)(x) e^{-(\psi^E*\rho_t^E)(x)}. \end{aligned}$$

Weak-coupling limit

Suppose that the same conditions as for the evolution of states are fulfilled. Observe that $M_E(\eta^-) = |\eta^-|$, $M_S(\eta) = m^S \sum_{x\in\eta^+} e^{E_{\phi^S}(x,\eta^+\setminus x)}$. We have

$$\begin{aligned} c_E(\beta;\eta^-) &= |\eta^-| + z^E e^{-\beta} \exp\left(e^{\beta} C(\phi^E)\right) \sum_{x\in\eta^-} e^{-E_{\phi^E}(x,\eta^-\setminus x)} \\ c_S(\alpha,\beta;\eta) &= m^S \exp\left(e^{\alpha} C(-\phi^S)\right) \sum_{x\in\eta^+} e^{E_{\phi^S}(x,\eta^+\setminus x)} \\ &\quad + \exp\left(e^{\beta} C(\psi^E)\right) \sum_{x\in\eta^+} \int_{\mathbb{R}^d} a^+(x-y) e^{-E_{\psi^E}(y,\eta^-)} \mathrm{d}y \\ &\quad + e^{-\alpha} \exp\left(e^{\beta} C(\psi^E)\right) \sum_{x\in\eta^+} \sum_{y\in\eta^+\setminus x} a^+(x-y) e^{-E_{\psi^E}(y,\eta^-)}. \end{aligned}$$

Conditions (E1) – (E4), (S1) – (S3) and (L) can be checked in the same way as above. Let μ^E be the invariant measure for the environment, then

$$\overline{d}(x,\gamma^+) = m^S \exp\left(E_{\phi^S}(x,\gamma^+\backslash x)\right)$$

and with $\overline{\lambda}(y) := \int\limits_{\Gamma} \exp\left(-E_{\psi^E}(y,\gamma^-)\right) \mathrm{d}\mu^E(\gamma^-) \leq 1$

$$\overline{b}(x,\gamma^+) = \sum_{y\in\gamma^+} \overline{\lambda}(y) a^+(x-y).$$

Hence we obtain

$$\begin{aligned}\overline{c}(\alpha;\eta^+) = &+ \sum_{x\in\eta^+} \int\limits_{\mathbb{R}^d} \overline{\lambda}(y) a^+(x-y)\mathrm{d}y + e^{-\alpha} \sum_{x\in\eta^+} \sum_{y\in\eta^+\backslash x} \overline{\lambda}(y) a^+(x-y) \\ &+ m^S e^{e^{\alpha} C(-\phi^S)} \sum_{x\in\eta^+} e^{E_{\phi^S}(x,\eta^+\backslash x)}\end{aligned}$$

and $\overline{M}(\eta) = m^S \sum\limits_{x\in\eta^+} e^{E_{\phi^S}(x,\eta^+\backslash x)}$. It follows by (4.84) and $\overline{\lambda}(y) \leq 1$

$$\begin{aligned}\overline{c}(\alpha;\eta^+) &\leq \left(\langle a^+\rangle + be^{-\alpha}\right)|\eta^+| + \vartheta e^{-\alpha} \sum_{x\in\eta^+} \sum_{y\in\eta^+\backslash x} \phi^S(x-y) + m^S e^{e^{\alpha}C(-\phi^S)} \sum_{x\in\eta^+} e^{E_{\phi^S}(x,\eta^+\backslash x)} \\ &\leq \left(\frac{\max\left\{\langle a^+\rangle + be^{-\alpha}, \vartheta e^{-\alpha}\right\}}{m^S} + e^{e^{\alpha}C(-\phi^S)} \right) \overline{M}(\eta).\end{aligned}$$

Hence Proposition 4.8.2 is applicable and the averaged (pre-)generator is given by

$$\begin{aligned}(\overline{L}F)(\gamma^+) = m^S &\sum_{x\in\gamma^+} e^{E_{\phi^S}(x,\gamma^+\backslash x)} (F(\gamma^+\backslash x) - F(\gamma^+)) \\ &+ \sum_{x\in\gamma^+} \overline{\lambda}(x) \int\limits_{\mathbb{R}^d} a^+(x-y)(F(\gamma^+\cup y) - F(\gamma^+))\mathrm{d}y.\end{aligned}$$

The mesoscopic equation associated to this microscopic model is then

$$\frac{\partial \overline{\rho}_t}{\partial t}(x) = -m^S \overline{\rho}_t(x) e^{(\phi^S * \overline{\rho}_t)(x)} + \int\limits_{\mathbb{R}^d} \overline{\lambda}(y) a^+(x-y) \overline{\rho}_t(y)\mathrm{d}y.$$

4.9.4 Two interacting BDLP-models

Suppose that the death intensities are given by

$$\begin{aligned}d^E(x,\gamma^+,\gamma^-\backslash x) &= m^E + \sum_{y\in\gamma^-\backslash x} a^-(x-y) \\ d^S(x,\gamma^+\backslash x,\gamma^-) &= m^S + \sum_{y\in\gamma^+\backslash x} b^-(x-y) + \sum_{y\in\gamma^-} \varphi^-(x-y).\end{aligned}$$

The birth intensities are assumed to be given by

$$b^E(x,\gamma) = \sum_{y\in\gamma^-} a^+(x-y) + z$$
$$b^S(x,\gamma) = \sum_{y\in\gamma^+} b^+(x-y) + \sum_{y\in\gamma^-} \varphi^+(x-y).$$

Suppose that $z, m^S, m^E > 0$ and $a^\pm, b^\pm, \varphi^\pm$ are non-negative, symmetric and integrable. It can be shown that

$$\begin{aligned} c(\alpha,\beta;\eta) = &\left(m^S + e^\alpha\langle b^-\rangle + e^\beta\langle\varphi^-\rangle + \langle b^+\rangle + \langle\varphi^+\rangle\right)|\eta^+| \\ &+ \left(m^E + e^\beta\langle a^-\rangle + \langle a^+\rangle + ze^{-\beta}\right)|\eta^-| \\ &+ \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a^-(x-y) + \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} b^-(x-y) + \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi^-(x-y) \\ &+ e^{-\beta}\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a^+(x-y) + e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} b^+(x-y) \\ &+ e^{-\alpha}\sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi^+(x-y) \end{aligned}$$

and $M_E(\eta^-) = m^E|\eta^-| + \sum\limits_{x\in\eta^-}\sum\limits_{y\in\eta^-\backslash x} a^-(x-y)$,

$$M_S(\eta) = m^S|\eta^+| + \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} b^-(x-y) + \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi^-(x-y).$$

Theorem 4.9.4. *Suppose that $a^\pm, b^\pm, \varphi^\pm$ are bounded and there exist constants $b_1, b_2 \geq 0$ and $\vartheta_1, \vartheta_2, \vartheta_3 > 0$ such that*

$$\sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} b^+(x-y) \leq \vartheta_1 \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} b^-(x-y) + b_1|\eta^+|$$
$$\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a^+(x-y) \leq \vartheta_2 \sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a^-(x-y) + b_2|\eta^-|,$$

and $\varphi^+ \leq \vartheta_3\varphi^-$ hold. Moreover, assume that the parameters satisfy the relations $\vartheta_1, \vartheta_3 < e^\alpha$, $\vartheta_2 < e^\beta$,

$$m^S > e^\alpha\langle b^-\rangle + e^\beta\langle\varphi^-\rangle + e^{-\alpha}b_1 + \langle b^+\rangle + \langle\varphi^+\rangle$$
$$m^E > e^\beta\langle a^-\rangle + e^{-\beta}(b_2+z) + \langle a^+\rangle.$$

Then conditions (A) – (D) hold for $\tau = 0$ and (4.20) is fulfilled. The dynamics described by the operator L is ergodic with exponential rate and non-degenerated invariant measure.

Proof. It follows that

$$
\begin{aligned}
c(\alpha,\beta;\eta) \le & \left(m^S + e^{\alpha}\langle b^-\rangle + e^{\beta}\langle \varphi^-\rangle + \langle b^+\rangle + \langle \varphi^+\rangle + b_1 e^{-\alpha}\right)|\eta^+| \\
& + \left(m^E + e^{\beta}\langle a^-\rangle + \langle a^+\rangle + (z+b_2)e^{-\beta}\right)|\eta^-| \\
& + (1+\vartheta_1 e^{-\alpha}) \sum_{x\in\eta^+}\sum_{y\in\eta^+\backslash x} b^-(x-y) + (1+e^{-\beta}\vartheta_2)\sum_{x\in\eta^-}\sum_{y\in\eta^-\backslash x} a^-(x-y) \\
& + (1+\vartheta_3 e^{-\alpha}) \sum_{x\in\eta^+}\sum_{y\in\eta^-} \varphi^-(x-y).
\end{aligned}
$$

The same arguments as before imply (B) – (D) and (4.20). □

Suppose that the conditions given above are fulfilled. Then (V1) – (V3) are satisfied and after Vlasov scaling we arrive at the kinetic equations

$$
\begin{aligned}
\frac{\partial \rho_t^E}{\partial t}(x) =& - m^E \rho_t^E(x) - \rho_t^E(x)(a^- * \rho_t^E)(x) + (a^+ * \rho_t^E)(x) + z \\
\frac{\partial \rho_t^S}{\partial t}(x) =& - \left(m^S + (\varphi^- * \rho_t^E)(x)\right)\rho_t^S(x) - \rho_t^S(x)(b^- * \rho_t^S)(x) \\
& + (b^+ * \rho_t^S)(x) + (\varphi^+ * \rho_t^E)(x).
\end{aligned}
$$

The unique invariant measure for L^E is given by $\pi_{\frac{z}{m^E}}$ and hence the averaged intensities are given by

$$
\begin{aligned}
\overline{d}(x,\gamma^+) &= m^S + \frac{z}{m^E}\langle \varphi^-\rangle + \sum_{y\in\gamma^+\backslash x} b^-(x-y) \\
\overline{b}(x,\gamma^+) &= \sum_{y\in\gamma^+} b^+(x-y) + \langle \varphi^+\rangle \frac{z}{m^E}.
\end{aligned}
$$

Proposition 4.8.2 is applicable if

$$
e^{\alpha}\langle b^-\rangle + \left(\langle \varphi^+\rangle \frac{z}{m^E} + b_1\right)e^{-\alpha} + \langle b^+\rangle < m^S + \langle \varphi^-\rangle \frac{z}{m^E}.
$$

Applying the Vlasov scaling to the averaged system yields the kinetic equation

$$
\frac{\partial \rho_t}{\partial t}(x) = -\left(m^S + \frac{z}{m^E}\langle \varphi^-\rangle\right)\rho_t(x) - \rho_t(x)(b^- * \rho_t)(x) + (b^+ * \rho_t)(x) + \langle \varphi^+\rangle \frac{z}{m^E}.
$$

Such equation can be also obtained by simply setting $\rho_t^E = \frac{z}{m^E}$ in the coupled system of equations.

Appendix

A.1 Banach lattice

Here we give the definition of a Banach lattice, see e.g. [BA06]. Let X be a real vector space.

Definition A.1.1. *A partial order (simply order) on X is a relation $'' \leq ''$ on $X \times X$ such that the following are fulfilled for all $x, y, z \in X$*

1. $x \leq x$.
2. *If $x \leq y$ and $y \leq x$, then $x = y$.*
3. *If $x \leq y$ and $y \leq z$, then $x \leq z$.*

An upper bound for a set $A \subset X$ is an element $x^* \in X$ such that $a \leq x^*$ for all $a \in A$. A lower bound is then an element $x_* \in X$ such that $x_* \leq a$ for all $a \in A$. The supremum $\sup(A)$ of A is the last upper bound of A, i.e. given any other upper bound $x^* \in X$ of A, then $\sup(A) \leq x^*$. The infimum $\inf(A)$ of the set A is defined in the same way. We should emphasize that in general the supremum and infimum do not have to exist. A vector space X equipped with a partial order $'' \leq ''$ is called lattice if for every two elements $x, y \in X$ the supremum $\sup(\{x, y\})$ and infimum $\inf(\{x, y\})$ exist.

Definition A.1.2. *A vector space X equipped with an partial order $'' \leq ''$ is called ordered vector space if its vector structure is compatible with the order $'' \leq ''$, i.e.*

(a) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$.

(b) $x \leq y$ implies $\alpha x \leq \alpha y$ for all $\alpha \geq 0$ and $x \in X$.

If the ordered vector space X is also a lattice, then it is called vector lattice.

For a vector lattice X it is possible to define for any $x \in X$ its positive, negative part and absolute value by

$$x_+ := \sup\,\{x, 0\}, \quad x_- := \sup\,\{-x, 0\}$$

and $|x| := \sup\,\{x, -x\}$. By [BA06, Proposition 2.46] above operations satisfy the relations

$$x = x_+ - x_- \text{ and } |x| = x_+ + x_-.$$

Definition A.1.3. *A norm $\|\cdot\|$ on the vector space X is called lattice norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.*

A.2 Basic lemmas

Set $\Delta := \{(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid s \leq t\}$, the next two lemmas should be well-known and are included here only for convenience.

Lemma A.2.1. *Let $f_j : \Delta \times \mathbb{R}_+ \times E \longrightarrow \mathbb{R}$ be a family of measurable functions indexed by $j \in M$, where M is an arbitrary non-empty index set, such that*

1. *f_j is bounded on compacts uniformly in $j \in M$.*
2. *The map $(s,t,x) \longmapsto f_j(s,t,r,x)$ is continuous uniformly in $j \in M$ for fixed $r \in [s,t]$.*

Then $(s,t,x) \longmapsto \int_s^t f_j(s,t,r,x)\mathrm{d}r$ is continuous uniformly in $j \in M$.

Proof. Let $(s,t), (s_n,t_n) \in \Delta$ and $x, x_n \in E$ be such that $s_n \to s$, $t_n \to t$ and $x_n \to x$ as $n \to \infty$. We find $T > 0$ and a compact $B \subset E$ such that $s, s_n, t, t_n \in [0,T]$ and $x, x_n \in B$ for $n \in \mathbb{N}$. Let $f^* := \sup_{j \in M} \sup_{(t_1,t_2,t_3,x) \in \Delta \cap [0,T]^2 \times [0,T] \times B} f_j(t_1,t_2,t_3,x) < \infty$, then for any $n \in \mathbb{N}$ and $j \in M$

$$\begin{aligned}
&\left| \int_s^t f_j(s,t,r,x)\mathrm{d}r - \int_{s_n}^{t_n} f_j(s_n,t_n,r,x_n)\mathrm{d}r \right| \\
&\leq |s - s_n| f^* + |t - t_n| f^* + \int_0^T |f_j(s,t,r,x) - f_j(s_n,t_n,r,x_n)|\mathrm{d}r.
\end{aligned}$$

For each $r \in [0,T]$ the integrand on the right-hand-side tends to zero as $n \to \infty$, and since $|f_j(s,t,r,x) - f_j(s_n,t_n,r,x_n)| \leq 2f^*$ dominated convergence yields the assertion. □

The next lemma will show continuity in the case where instead of $\mathrm{d}r$ there is an arbitrary kernel $H(t,x,\mathrm{d}y)$. In such a case we will need that E is locally compact.

Lemma A.2.2. *Let E be a locally compact Polish space,*

$$f : \{(s,r,t) \in \mathbb{R}_+^3 \mid s \leq r \leq t\} \times E \times E \longrightarrow \mathbb{R}$$

be continuous and bounded, and let $H : I \times E \times \mathcal{B}(E) \longrightarrow \mathbb{R}_+$ be a weakly continuous kernel, i.e. for all $F \in C_b(E)$, $\mathbb{R}_+ \times E \ni (r,x) \longmapsto \int_E F(y)H(r,x,\mathrm{d}y)$ is continuous. Then

$$(s,r,t,x) \longmapsto \int_E f(s,r,t,x,y)H(r,x,\mathrm{d}y)$$

is continuous.

Proof. Let $s_n \le r_n \le t_n$ be such that $s_n \to s, r_n \to r, t_n \to t$ and $x_n \to x$ as $n \to \infty$. Fix $\varepsilon > 0$ and take $A \subset E$ compact with $H(r,x,A^c) < \varepsilon$. Since E is a locally compact space we can find another compact $A_1 \subset E$ with $A \subset \overset{\circ}{A_1} \subset A_1$. Portmanteau implies then $\limsup_{n\to\infty} H(r_n, x_n, (\overset{\circ}{A_1})^c) \le H(r,x,(\overset{\circ}{A_1})^c) \le H(r,x,A^c) < \varepsilon$. The function f restricted to the compact $\{(s_n,r_n,t_n) \mid n \in \mathbb{N}\} \cup \{(s,r,t)\} \times \{x_n \mid n \in \mathbb{N}\} \cup \{x\} \times A_1$ is uniformly continuous and hence we obtain for sufficiently large n

$$\begin{aligned}
&\left| \int_E f(s_n,r_n,t_n,x_n,y)H(r_n,x_n,\mathrm{d}y) - \int_E f(s,r,t,x,y)H(r,x,\mathrm{d}y) \right| \\
&\le \int_E |f(s_n,r_n,t_n,x_n,y) - f(s,r,t,x,y)| H(r_n,x_n,\mathrm{d}y) \\
&\quad + \left| \int_E f(s,r,t,x,y)H(r_n,x_n,\mathrm{d}y) - \int_E f(s,r,t,x,y)H(r,x,\mathrm{d}y) \right| \\
&\le H(r_n,x_n,A_1)\varepsilon + 2\|f\| H(r_n,x_n,(\overset{\circ}{A_1})^c) + \varepsilon \\
&\le H(r_n,x_n)\varepsilon + 2\|f\|\varepsilon + \varepsilon.
\end{aligned}$$

Due to the weak continuity of H the function $H(r,x) := H(r,x,E)$ is continuous and hence $H(r_n,x_n)$ is uniformly bounded in $n \in \mathbb{N}$, which shows the assertion. □

A.3 Proof of Theorem 4.8.4

(a) Clearly $T_\alpha(t) \otimes \mathbb{1}$ is a bounded linear operator on $\mathcal{L}_{\alpha,\beta}$. Let $M \ge 1$ and $\omega \in \mathbb{R}$ be such that $\|T_\alpha(t)\|_{L(\mathcal{L}_\alpha)} \le Me^{\omega t}$, then $\|T_\alpha(t) \otimes \mathbb{1}\|_{L(\mathcal{L}_{\alpha,\beta})} \le Me^{\omega t}$. For $G = G_1 \otimes G_2 \in \mathcal{X}$ we obtain

$$\|(T_\alpha(t) \otimes \mathbb{1})G - G\|_{\mathcal{L}_{\alpha,\beta}} = \|G_2\|_{\mathcal{L}_\beta} \|T_\alpha(t)G_1 - G_1\|_{\mathcal{L}_\alpha}$$

and hence it is strongly continuous on $\mathcal{X}$. Since $\mathcal{X}$ is dense in $\mathcal{L}_{\alpha,\beta}$ it follows that it is strongly continuous on the whole space $\mathcal{L}_{\alpha,\beta}$. Take $G = G_1 \otimes G_2 \in \mathcal{D}$, then

$$\frac{(T_\alpha(t) \otimes \mathbb{1})G - G}{t} = \left(\frac{T_\alpha(t)G_1 - G_1}{t} \right) \otimes G_2 \longrightarrow A_\alpha G_1 \otimes G_2, \quad t \to 0$$

shows that $\text{lin}(\mathcal{D}) \subset D(A_{\alpha,\beta})$ and $A_{\alpha,\beta}G = A_\alpha G_1 \otimes G_2$. For the last assertion it suffices to show that

$$\mathcal{G} := \left\{ G_1 \otimes G_2 \mid G_1 \in \bigcap_{n\geq 1} D(A_\alpha^n),\ G_2 \in \mathcal{L}_\beta \right\} \subset \mathcal{D}$$

is a core for $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$. For $G_1 \otimes G_2 \in \mathcal{G}$ we get $(T_\alpha(t)\otimes \mathbb{1})(G_1\otimes G_2) = T_\alpha(t)G_1\otimes G_2$ and hence $\text{lin}(\mathcal{G})$ is invariant for $T_\alpha(t)\otimes \mathbb{1}$. Moreover, we see that

$$\mathcal{X} \subset \overline{\text{lin}(\mathcal{G})} \subset \mathcal{L}_{\alpha,\beta}$$

and hence $\mathcal{L}_{\alpha,\beta} = \overline{\mathcal{X}} \subset \overline{\text{lin}(\mathcal{G})} \subset \mathcal{L}_{\alpha,\beta}$, which shows that $\text{lin}(\mathcal{G}) \subset \mathcal{L}_{\alpha,\beta}$ is dense.
(b) Let $M \geq 1$ and $\omega \in \mathbb{R}$ be such that $\|T_{\alpha,\beta}(t)\|_{L(\mathcal{L}_{\alpha,\beta})} \leq Me^{\omega t}$, then $T_\alpha(t)$ is clearly a bounded linear operator on $\mathcal{L}_\alpha$ and it holds that

$$\|T_\alpha(t)G\|_{\mathcal{L}_\alpha} \leq \|T_{\alpha,\beta}(t)\widehat{P}_-^{-1}G\|_{\mathcal{L}_{\alpha,\beta}} \leq Me^{\omega t}\|\widehat{P}_-^{-1}G\|_{\mathcal{L}_{\alpha,\beta}} = Me^{\omega t}\|G\|_{\mathcal{L}_\alpha}.$$

The semigroup property is evident and strong continuity follows from

$$\|T_\alpha(t)G - G\|_{\mathcal{L}_\alpha} = \|\widehat{P}_-(T_{\alpha,\beta}(t)\widehat{P}_-^{-1}G - \widehat{P}_-^{-1}G)\|_{\mathcal{L}_\alpha} \leq \|T_{\alpha,\beta}(t)\widehat{P}_-^{-1}G - \widehat{P}_-^{-1}G\|_{\mathcal{L}_{\alpha,\beta}}$$

and the strong continuity of $(T_{\alpha,\beta}(t))_{t\geq 0}$. Since $(T_{\alpha,\beta}(t))_{t\geq 0}$ leaves $\mathcal{L}_\alpha$ invariant it follows by definition that $T_{\alpha,\beta}(t)$ leaves $\mathcal{L}_\alpha \otimes 0^-$ invariant. The space $\mathcal{L}_\alpha \otimes 0^-$ is a closed subspace and hence the restriction $T_{\alpha,\beta}(t)|_{\mathcal{L}_\alpha\otimes 0^-}$ is the generator of a C_0-semigroup on $\mathcal{L}_\alpha \otimes 0^-$. The generator is in such a case given by the $\mathcal{L}_\alpha \otimes 0^-$-part of $A_{\alpha,\beta}$, that is by $(A_{\alpha,\beta}|_{\mathcal{L}_\alpha\otimes 0^-}, D(A_{\alpha,\beta})|_{\mathcal{L}_\alpha\otimes 0^-}$, where $A_{\alpha,\beta}|_{\mathcal{L}_\alpha\otimes 0^-}G = A_{\alpha,\beta}G$, $G \in D(A_{\alpha,\beta})|_{\mathcal{L}_\alpha\otimes 0^-}$ and

$$D(A_{\alpha,\beta})|_{\mathcal{L}_\alpha\otimes 0^-} = \{G \in D(A_{\alpha,\beta}) \cap \mathcal{L}_\alpha \otimes 0^- \mid A_{\alpha,\beta}G \in \mathcal{L}_{\alpha,\beta} \otimes 0^-\}.$$

Since $T_\alpha(t) = \widehat{P}_- T_{\alpha,\beta}(t)\widehat{P}_-^{-1}$ it follows that $A_\alpha = \widehat{P}_- A_{\alpha,\beta}|_{\mathcal{L}_\alpha\otimes 0^-}\widehat{P}_-^{-1}$ and

$$D(A_\alpha) = \widehat{P}_- D(A_{\alpha,\beta})|_{\mathcal{L}_\alpha\otimes 0^-} = \{G \in \mathcal{L}_\alpha \mid \widehat{P}_-^{-1}G \in D(A_{\alpha,\beta}), A_{\alpha,\beta}\widehat{P}_-^{-1}G \in \mathcal{L}_\alpha \otimes 0^-\}.$$

A.4 Eigenständigkeitserklärung

Hiermit erkläre ich, dass mir die geltende Promotionsordnung der Fakultät bekannt ist. Die vorliegende Arbeit ist eigenständig und ohne die Hilfe Dritter angefertigt und alles aus anderen Quellen und von anderen Personen übernommene Material, dass in der Arbeit verwendet wurde oder auf das direkt Bezug genommen wird, wurde als solches kenntlich gemacht. Die Arbeit enthält keine Passagen für welche Dritte weder unmittelbar noch mittelbar geldwerte Leistungen für Vermittlungstätigkeiten oder für Arbeiten erhalten haben. Die vorgelegte Arbeit wurde weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde zum Zweck einer Promotion oder eines anderen Prüfungsverfahrens vorgelegt.

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