The Ranking and the Value of Information

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Preface

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Contents

Chapter 1

Introduction

Risk is involved in almost all decision processes in real life. The riskiness may be related to unswayable issues, as for example uncertain weather conditions, or influenceable issues as for example the behavior of others. In economic environments, the uncertainty may relate to risky asset returns, uncertain production processes or innovations. The decision makers' imperfect knowledge plays a major role in determining the degree of uncertainty. Therefore, decision making under risk is inherently connected to available information at that point in time. Moreover, information might be more or less reliable, it reduces risk to greater or lesser extent. For decades economists study the impacts of information on economic behavior and economic outcomes.

The purpose of this work is to contribute to the understanding of 'better information' and how this can be formalized in economic models. In particular, two new informativeness criteria are defined and compared to some criteria which are frequently used in economic theory. Build on this, the work analyzes the demand for information and its impact on the equilibrium/economic outcomes.

In economic modeling it is distinguished between market risk and event risk. Market risk is related to the limited knowledge about endogenous variables as for instance other market participants' actions or risky equilibrium prices. In contrast, event risk is characterized by a probability space. In particular, this probability space consists of a set of states of the world and a corresponding state distribution. Each state fully and uniquely determines the decision maker's (economic) environment. The state distribution is often also called belief and reflects the decision makers imperfect knowledge about the state of the world. A belief may be objective or subjective, an individual or common assignment of probabilities to each state of the world. Intuitively it is clear that these assignments of probabilities to the states heavily depend on available information. Most parts of this work will focus on event risk rather than on market risk.

When deciding under risk, a decision maker cannot directly choose an action with corresponding outcome. Instead she chooses a random variable depending on the state of the world that maximizes her expected utility. Before choosing an action, the decision maker may reduce her uncertainty by acquiring additional information. Information acquisition can, for instance, be reading newspapers or asking an expert for advice. As mentioned above, the decision maker's belief depends on her available information. Therefore, the acquisition of additional information changes the decision maker's belief. In economic theory, information acquisition is frequently modeled by observing a signal. This signal is correlated to the state of the world and, therefore, contains information about it. For a given (prior) belief, the correlation of states and signals is determined by an information structure. Formally, an information system specifies for each state a conditional probability distribution on a set of possible signal realizations. After observing a signal realization, the decision maker update her prior belief via Bayes' rule.

Informativeness Criteria

Whether an information system is more precise than another one, i.e. induces a greater reduction of risk, has been extensively discussed in economic theory. Intuitively, the greater the correlation of states and signals the more informative the underlying information system. In spite of this clear intuition, up to today there is no clear answer how to compare informativeness of different information system. The reason for this is that informativeness criteria should fulfill some desirable properties. These properties might be natural ones as, for instance, reflexivity and transitivity, or desirable ones as for example robustness under certain state transformations.[1] Additionally, an information system might be quite precise for some signal realizations while it might be vague for some others. Moreover, some notions of informativeness seems to be appropriate in some economic environment while their applicability is questionable in others. Therefore, various informativeness concepts for different classes of information systems and economic situations have been developed.

Blackwell (1951, 1953) defined the possibly most famous informativeness criterion. He calls an information system more informative than another one if the former one can be stochastically transformed into the latter one in Blackwell's sense. Or put differently, an information system is more informative than another, if a signal observation of the latter is equal to a disturbed signal observation of the former. Blackwell shows that, if the set of possible actions is independent of the information revelation, every expected utility maximizer prefers an information system to another one if and only if the former is more informative than the latter.

Following the same intuition, Lehmann (1988)/Persico (1996) define an informativeness criterion for a subset of information systems. In contrast to Blackwell, for defining accuracy they use a certain state dependent signal transformation which transforms signals of the less accurate system into signals of the more accurate system.

Kim (1995) argue that Blackwell's criterion is not applicable in principal-agent models. Therefore, he defined a new criterion for the analysis of those problems: an information system is more reliable if its conditional signal distribution reacts more sensitive to changes in the state.

All of the three criteria mentioned so far compare conditional signal distributions in different ways. By contrast Eckwert and Zilcha (2008) formalize informativeness by looking at the updated distributions after a signal realization. In particular, an information system is more informative, if its posterior state distributions are more dispersed. Intuitively, the more the

[1] Of course, there are more natural and desirable properties for an informativeness criterion, but for reasons of readability these are introduced later in this work.

updated state distributions depent on the signal realizations, the more precise is the information system. Following the same idea, Ganuza and Penalva (2010) compare the dispersion of conditional state expectations.

As mentioned earlier, it is desirable that an informativeness criterion is invariant under certain state transformations. An example for such a transformation is an ordinal relabeling of states. Since such a relabeling is a one-to-one transformation of the states, the original state can be inferred without noise from each transformed state and vice versa. Therefore, the information about the transformed states that is revealed by an information system, is exactly the same as the information that it reveals about the original state. The informativeness criteria of Blackwell (1951, 1953), Lehmann (1988) and Kim (1995) meet this property while the criteria of Ganuza and Penalva (2010) does not.

Since the precision criteria of Ganuza and Penalva (2010) have other helpful properties, the first objective of this work is to define new informativeness criteria, weak and strong informativeness, that are invariant to ordinal relabelings of states and respect the informativeness criteria of Ganuza and Penalva (2010). The intuition of both criteria is that the more informative an information system the more spread out are the posterior conditional expectations. Moreover, as the name suggest, weak informativeness is weaker than strong informativeness in the sense that if an information system is strongly more informative than an alternative one, then it is also weakly more informative than the alternative one.

The Value of Information

Blackwell (1951) shows that all expected utility maximizers prefers 'better information' (in the sense of Blackwell's criterion) to 'worse information'. However, he assumes that the set decision maker's set of feasible actions does not vary with information revelation. Since signals may be observed by many people, this is not the case in most economic circumstances.

Hirshleifer (1971) have shown that better information can hurt agents in a pure exchange economy. Intuitively, better information destroys risk-sharing opportunities. In particular, Hirshleifer considers a pure exchange economy with one good (wealth) and two states of the world. The agents' are risk averse and possess risky state dependent endowments. If the agents trade state-contingent claims before the state is realized, they are able to share some of the risk. However, if the agents were be perfectly informed about the true state of the world before they start to trade with each other, then each agent would prefer to consume in the same state and they would not trade at all. Therefore, from ex ante perspective, each agent just consumes her endowment which delivers less (ex ante) expected utility than the risk sharing equilibrium without information.

In contrast, Eckwert and Zilcha (2000) examine conditions for restoring Blackwell's theorem in a production economy with productivity risk. The intuition of their result is that better information may improve the input allocation in the economy which may outweigh its negative effects on risk sharing. Nevertheless, they also show that in the presence of risk sharing markets the value of information still might be negative.

The second objective of this work is to examine the value of information if sets of feasible actions are not independent of the information system. This is done in two different economic environments.

First, the demand for costly private information is modeled. In this model action sets are independent of the signal realization but, since information is costly, depend on the choice of information system. The main results are driven by two simple effects. Each agent on the demand side plays a lottery where she has to guess the right state of nature. If she is right, then her resources will increase, otherwise they will remain constant. On the one hand better information leads to higher (ex ante) welfare as the chances of winning improve. But on the other hand better information leads to less budget available for consumption as it exhibits a higher price. Under these circumstances even risk neutral agents invest in information provided information is not too expensive. Furthermore, and perhaps more surprisingly, risk averse agents do not invest in information if their degree of risk aversion is sufficiently high. This is due to the fact that resources keep constant in the case of not winning the lottery.

And secondly, in a general equilibrium model which considers a many commodity production economy with risky endowments and efficient risk-sharing. There are two different types of agents, risk averse consumers and risk neutral firms. Both types of agents possess a risky endowment of commodities (inputs as well as outputs). At date 0, after receiving information but before observing the state of the world, the agents trade state-contingent claims in a competitive market. After the state realization, at date 1, the agents consume/produce according to their state-contingent claim. Therefore, since state-contingent claims are traded after the signal realization, the equilibrium prices, and therefore the sets of feasible actions, vary with the signal realization. Within this framework it is shown that weakly more informative (information) systems make every risk avers agent worse off. In particular, parts of the result by Schlee (2001) are generalized to a many commodity, production economy with complete risk sharing markets.

Asymmetric Information

Information is asymmetric if some market participants know more than others. A situation with asymmetric information is the Stackelberg game (Stackelberg (1934)). In his model Stackelberg describes a situation with asymmetric informations in which two firms compete with each other in a quantity competition. The information asymmetry is modeled by an information advantage of one firm. In particular, one firm, the leader, decides at first about its optimal supply (quantity) and the follower observes this before deciding about its own quantity. Therefore, given the action of the leader, the follower reacts to this by setting her best reaction. There exist a huge literature on these models for homogeneous as well as for horizontal differentiated commodities (see e.g., Amir and Jin (2001), Kreps and Scheinkman (1983), Vives (1985) and Vives (2005)). It is well established in the economic literature that for the commodities being perfect substitutes and the firms being quantity setters (Cournot competition), the leader is better off than the follower. The reason for this is that when deciding about her optimal quantity, the leader can take the followers behavior, i.e. her best reply, into account. The opposite is true for the firms being price setter (Bertrand competition).

Boyer and Moreaux (1987) studied the role of the strategy space under these circumstances. In particular, they allow both agents to choose whether to set a quantity or a price. Using a very restrictive demand structure they showed that it is always more profitable to be a quantity (price) setter if the goods are substitutes (complements). Consequently, uncertainty concerning the opponent's strategy space (market uncertainty) does not have any impact of a firms decision. Regarding total and consumers' surpluses they proofed that price competition is dominant for all degrees of product differentiation.

The third aim of this work is to generalize Boyer and Moreaux's results to a more general demand structure proposed by Dixit (1979) which allows different corss-effects and reservation prices for the goods.

Organization of the Work

Chapter 2 introduces the reader to the basic framework of decision making under risk. The concept of an information system and the information processing are explained.

Chapter 3 reviews some classic informativeness concepts from the economic literature, defines two new informativeness concepts, strong and weak informativeness, and discusses some of their properties and implication. Furthermore, these concepts are compared to some informativeness criteria used in the economics literature.

Chapter 4 studies the value of information in two different situations. First, the demand for costly information in a partial equilibrium model is analyzed. Furthermore, this model looks at the connection of the degree of risk aversion and the demand for information. And secondly, the impact of information on individual behavior and prices is studied in a general equilibrium model with production.

Chapter 5 then is a side step to industrial organization. This chapter studies the role of the strategy space in a Stackelberg game, i.e. a game with asymmetric information.

Finally, Chapter 6 summarizes the main results and gives some concluding remarks as well as some preview on further research.

All formal proofs have been relegated to the appendix.

Chapter 2

Information and Decision Making under Risk

Normally, the prediction of a future state of the world is uncertain. In economic theory this is often modeled by assuming that the state of the world is unknown at time of decision making. It is usual to assume that decision making under risk is rational in the sense of the expected utility rule by Morgenstern and von Neumann (1944). In particular, this means that agents choose an alternative that maximizes expected utility with respect to their belief about the future state of the world. Within the literature on economics this kind of modelling decision making under risk is called expected utility theory. Since these models play a major role later in this work, section 2.1 presents the basic framework of expected utility theory and introduces some notation.

Now imagine that the decision maker might get some additional information about the state of the world before the decision is to be made. Clearly, these additional information might change the agent's belief about the state of the world. Hence, the agent chooses an alternative that is optimal accoding to her new belief. In economic theory additional information are often modeled through the observation of random signals. These signals are correlated with the state of the world and, therefore, an signal observation reveals some information about the state of the world. In particular, after a signal observation a decision maker updates her belief accoding to Bayes' rule and then chooses an action which maximized her expected utility accoding to the updated belief. Hence, this behavior is called Baysian decision making and will formally be introduced in section 2.2.

Next, suppose that a decision maker observes more than just one information signal, say two of them. Then the decision maker will take both observations into account when updating her belief according to Bayes' rule. An opportunity how this can be formalized and modelled in economic theory will be given in section 2.3

For the comparison of different information signals, it is sometimes necessary to normalize them in a specific way. Therefore, section 2.4 shows how information signals can be normalized and explains why this is without loss of further generality.

2.1 Decision making under Risk

This section introduces the basic framework of decision making under risk. In economic theory a risky economic environment is typically modeled by a measurable space $(\Omega, \mathcal{F})$ consisting of a set of possible future states of the world Ω and a σ-algebra (of subsets in Ω). In general, Ω could be finite or infinite. Even though most of the statements are also true for finite sets of states, the following restricts attention to infinite sets of states. This is due to readability reasons. Therefore, unless explicitly mentioned otherwise, let $\Omega = (\underline{\omega}, \bar{\omega})$ be a convex subset of $\mathbb{R}$ and let $\mathcal{F}$ denote its (borelean) σ-algebra of subsets in Ω. At the time of decision making the future state of the world is unknown. In particular, the decision maker does not know which state $\omega \in \Omega$ will occur. Hence, the decision maker forms an subjective or objective probability measure μ_Ω on $\mathcal{F}$ representing his belief about the future state of the world. Consequently, $(\Omega, \mathcal{F}, \mu_\Omega)$ becomes a probability space. Throughout this work it is assumed that μ_Ω is characterized by an probability density function (or Lebesgue density)[2]

$$f_\Omega :\ \Omega \to \mathbb{R},\ \omega \mapsto f_\Omega(\omega)$$

with corresponding cumulative distribution function

$$F_\Omega :\ \Omega \to [0,1];\ \omega \mapsto F_\Omega(\omega) = \int\limits_{\underline{\omega}}^{\omega} f_\Omega(\omega')\mathrm{d}\omega' = \mathrm{Prob}\left(\tilde{\omega} \leq \omega\right). \tag{2.1}$$

Therefore, the term prior belief simultaneously denotes the agents belief about the state of the world μ_Ω, the corresponding density function f_Ω or cumulative distribution function F_Ω. Denote by $\Delta(\Omega)$ the set of all probability density functions $f_\Omega : \Omega \to \mathbb{R}$, i.e.

$$\Delta(\Omega) = \left\{ f_\Omega : \Omega \to \mathbb{R} \,\middle|\, f_\Omega(\omega) \geq 0\ \forall\ \omega \in \Omega \text{ and } \int\limits_{\Omega} f_\Omega(\omega)\mathrm{d}\omega = 1 \right\}.$$

In the following a measure μ_Ω always denotes the unique measure which is defined by the density function $f_\Omega \in \Delta(\Omega)$.

The decision maker has to choose an alternative a from a set of possible alternatives $\mathcal{A}$. In principle, an alternative could be anything, for instance, it could be an action or a consumption bundle. It is assumed that the decision maker's outcome depends on both, her action and the state of the world. Formally this is represented by a μ_Ω-measurable function

$$o :\ \mathcal{A} \times \Omega \to \mathcal{O},\ (a,\omega) \mapsto o(a,\omega),$$

where $\mathcal{O}$ denotes the set of possible outcomes. A widespread example for such a situation is a farmer. Her problem is to choose the right type of grain like corn or wheat. The set of outcomes consists of the farmer's possible incomes in the next year. The unknown state of the world might be the precipitation amount of the next year. Since the precipitation amount is essetial for the

[2]If Ω is finite, f_Ω denotes the corresponding probability mass function.

quantity and quality of the harvest in the next year, and therefore, for the farmer's income, the state of the world is relevant for the farmer's optimal decision.

The decision maker's preferences on the set of outcomes $\mathcal{O}$ are represented by an elementary utility function

$$u: \ \mathcal{O} \to \mathbb{R}, \ o \mapsto u(o).$$

Under certainty, i.e. when the state of the world is known, each alternative implements a certain outcome and the decision maker can choose an autcome which maximizes her utility. In contrast, under risk, the decision maker is unable to choose directly an outcome (through the choice of an action). Instead, each alternative $a \in \mathcal{A}$ induces a lottery on the set of outcomes. In particular this means, that choosing an alternative $a \in \mathcal{A}$ is the same as choosing a lottery $o(a, \tilde{\omega})$ on $\mathcal{O}$. Following Morgenstern and von Neumann (1944) it is assumed the decision maker now chooses an action for which the induced lottery $o(a, \tilde{\omega})$ maximizes her expected utility. This induces that the decision maker's preferences on the set of alternatives can be derived from her direct preferences on the set of outcomes $\mathcal{O}$ in the following way: the valuation of an alternative is determined as expected elementary utility of the lottery implemented by that alternative. Therefore, it is assumed that $u(o(a, \cdot))$ is μ_Ω-integrable for all $a \in \mathcal{A}$. Then the decision maker's preferences on $\mathcal{A}$ are represented by

$$U: \ \mathcal{A} \to \mathbb{R}, \ a \mapsto \mathbb{E}_\Omega \left[u(o(a, \tilde{\omega})) \right] := \int_\Omega u(o(a, \omega)) f_\Omega(\omega) \mathrm{d}\omega.$$

In order to simplify notation, denote by v the indirect elementary utility from a combination of an action and a state of the world. In particular, the indirect elementary utility is defined by

$$v: \ \mathcal{A} \times \Omega \to \mathbb{R}, \ (a, \omega) \mapsto u(o(a, \omega)). \tag{2.2}$$

The formal problem of a decision maker then is

$$\max_{a \in \mathcal{A}} U(a) = \max_{a \in \mathcal{A}} \mathbb{E}_{\Omega} \left[u(o(a, \tilde{\omega})) \right] = \max_{a \in \mathcal{A}} \mathbb{E}_{\Omega} \left[v(a, \tilde{\omega}) \right]. \tag{2.3}$$

The decision maker's behavior and her decision heavily depends on her attitude towards risk. A risk averse decision maker dislikes risks. Therefore, a decision maker is called risk averse if she prefers the expected outcome of an lottery over the lottery itsself.

Definition 2.1. *A decision maker is called*

$$\textit{risk} \left\{ \begin{array}{c} \textit{averse} \\ \textit{neutral} \\ \textit{affine} \end{array} \right\} \textit{ if and only if } u\left(\mathbb{E}_{\mathcal{O}}\left[\tilde{o}\right]\right) \begin{array}{c} > \\ = \\ < \end{array} \mathbb{E}_{\mathcal{O}}\left[u\left(\tilde{o}\right)\right]$$

for all lotteries $\tilde{o}$ on the set of outcomes.

From Jensens' inequality immediately follows that a decision maker with elementary utility function u is risk averse (neutral, affine) if and only if u is concave (linear, convex).

2.2 Decisions and Information

Decision making under risk is characterized by an unknown future state of the world. This state is determined by nature before anything else happens in the model. Therefore, this point in time represents the starting point of the model and is called "ex ante" stage. As argued above, the decision maker forms a probability distribution on the set of all possible states. At the next point in time, the "interim" stage which is placed after the ex ante stage and before the state of the world becomes observable, the decision maker can observe a random signal that is correlated to the state variable. If the signal is correlated to the state variable then it contains information about it. In particular, if the agent knows the common distribution of states and signals she can infer information about the future state and update her prior belief using Bayes' rule. Based on this updated belief the decision maker then choose an alternative which maximizes her expected utility. The last point in time is the "ex post" stage. At this stage the state variable becomes observable and the decision maker's gets her payoff according to her alternative and the state realization.

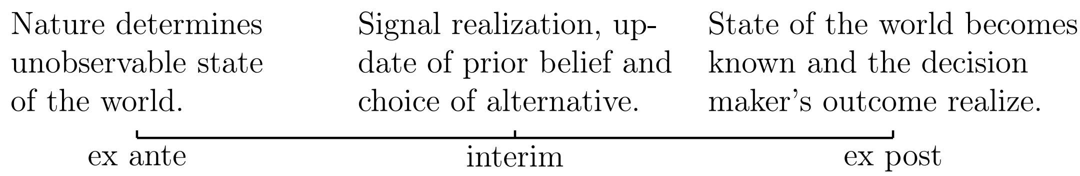

Figure 2.1: Timing of events.

2.2.1 Information Systems

In the traditional literature on information, the prior state distribution is typically kept fixed, and an information system is defined as a profile of signal distributions conditional on the state. In order to analyze the impact of the prior belief on the quality of information, the present study proceeds from a more general notion of informativeness by allowing for different priors and distinguishing between an information structure and an information system. For each state of the world $\omega \in \Omega$, an information structure defines conditional signal distributions on the set of signals S. An information system is a tuple consisting of an information structure and a prior belief. In order to define information structures and information systems formally, let $\tilde{\omega}$ denote the state variable. Denote by S the set of possible signals and by $\mathcal{S}$ the corresponding σ-algebra of subsets in S, i.e. $(S, \mathcal{S})$ is a measurable space. In principle, S can be finite or infinite but for readability reasons, attention is restricted to the case of infinite signal sets which are convex subsets of the real line, i.e. $S = [\underline{s}, \bar{s}]$. The discrete case will be presented in examples.

Definition 2.2. *(i) An <u>information structure</u> with corresponding state space Ω and signal space S is a family of conditional signal densities $f_{S|\Omega} =$*

$\{f_{S|\Omega}(s|\omega)\}_{s\in S,\omega\in\Omega}$.[3]

(ii) An <u>*information system*</u> *with corresponding state space Ω and signal space S is a tuple $(f_{S|\Omega}, f_\Omega)$, where $f_{S|\Omega}$ is an information structure and $f_\Omega \in \Delta(\Omega)$ is a probability density function on Ω.*

An information structure is a family of conditional signal distributions characterized by a family of conditional signal densities. Analogue to section 2.1 the conditional cumulative distribution function of the signals is

$$F_{S|\Omega} : S \times \Omega \to [0,1], s \mapsto \int_{\underline{s}}^{s} f_{S|\Omega}(s'|\omega)\mathrm{d}s'.$$

For simplicity, the following example 1 presents a finite version of the formulations above.

Example 1. Suppose $\Omega = \{\omega_1, \omega_2\}$ and $S = \{s_1, s_2\}$ with $\omega_1 < \omega_2$ and $s_1 < s_2$, respectively. Moreover, let the prior belief be given by the probability mass function $f_\Omega(\omega_1) = \mathrm{Prob}\,(\tilde{\omega} = \omega_1) = 1/2 = \mathrm{Prob}\,(\tilde{\omega} = \omega_2) = f_\Omega(\omega_2)$. Then the cumulative distribution is equal to

$$F_\Omega(\omega) = \begin{cases} 1/2 & \text{if } \omega = \omega_1 \\ 1 & \text{else.} \end{cases}$$

Moreover, the signal's conditional probability mass function is defined through the Markov-Matrix

$$f_{S|\Omega} = \begin{pmatrix} f(s_1|\omega_1) & f(s_2|\omega_1) \\ f(s_1|\omega_2) & f(s_2|\omega_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/4 & 3/4 \end{pmatrix}. \tag{2.4}$$

Then $f_{S|\Omega}$ defines an information structure. If the true state is ω_1 signal s_1 occurs with probability 1, while the signal s_2 will never occur. If the true state is ω_2, this information structure generates s_2 with probability $3/4$ and

[3] In particular, this means that $F_{S|\Omega}(s|\omega) := \int_{\underline{s}}^{s} f_{S|\Omega}(s'|\omega)\mathrm{d}s'$ is a cumulative distribution function and $F_{S|\Omega}(C|\cdot) = \int_C f_{S|\Omega}(s|\cdot)\mathrm{d}s$ is $\mathcal{F}$-measurable for all $C \subseteq S$.

while s_1 occurs with probability $1/4$. Last, $(f_{S|\Omega}, f_\Omega)$ defines an information system.

There are two extreme types of information structures: the fully uninformative one and the fully informative one. Following Nermuth (1982) an information structure is fully uninformative if the set of signals contains only one signal, say s^0, i.e. $S^0 = \{s^0\}$. This information structure is informationally equivalent to an information structure which conditional signal distributions are state independent, i.e. $f^0_{S|\Omega}(\cdot|\omega) = f^0_{S|\Omega}(\cdot|\omega')$ for all $\omega, \omega' \in \Omega$. Intuitively it is clear that the observation of a signal produced by this structure is equal to no observation at all, because no additional information can be inferred from a signal observation s. Denote all these fully uninfomative information structures by $f^0_{S^0|\Omega}$ where S^0 might be any convex subset of $\mathbb{R}$. It should be clear that the corresponding information systems $(f^0_{S^0|\Omega}, f_\Omega)$ are fully uninformative for all prior distributions f_Ω on Ω.

On the other extreme, an information system is called fully informative if the observation of an signal reveals the state with certainty. This means that every signal $s \in S^1$ there is a state $\omega \in \Omega$ such that the probability of observing s conditional on ω is strictly positive while this probability conditional on any other state $\omega' \neq \omega$ is zero. Denote such information structures by $f^1_{S^1|\Omega}$. Clearly, the corresponding information systems $(f^1_{S^1|\Omega}, f_\Omega)$ are fully informative for all prior distributions $f_\Omega \in \Delta(\Omega)$. Unfortunately, conditional signal distributions as described above are not representable by density functions $f^1_{S^1|\Omega}(\cdot|\omega)$. Hence, a fully informative information structure is not feasible in a setting, where attention is restricted to continuous information structures (i.e. to information structures which are defined by a family of conditional signal densities). However, for Ω finite, say $\Omega = \{\omega_1, \ldots, \omega_n\}$ with $n \in \mathbb{N}$, an fully informative information structure can be constructed by defining $S^1 = \Omega$ and

$$f^1_{\Omega|\Omega}(s|\omega) = \begin{cases} 1 & \text{if } s = \omega \\ 0 & \text{else.} \end{cases}$$

If the true state is ω, this structure produces the signal $s = \omega$ with probability one while any other signal will occur with probability zero. Hence, after the observation of an signal s the true state must be $\omega = s$, hence, the signal observation reveals the state with certainty.

In order to give an more concrete example consider $\Omega = \{\omega_1, \omega_2\}$ and $S = \{s_1, s_2\}$. The completely uninformative and the fully informative information structures are characterized by the Markov-Matrices

$$f^0_{S^0|\Omega} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad f^1_{S^1|\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2.2.2 Update of the Prior Belief

It is assumed that the decision maker knows the information system. Thus, the observation of a signal realization allows her to update her prior belief using Bayes' rule. Then the revised belief is used for the maximization of her expected utility. In particular, this means that the decision maker's problem, formally stated in equation (2.3), after a signal realization s becomes

$$\max_{a\in\mathcal{A}} \mathbb{E}_\Omega \left[v(a,\tilde{\omega})|s\right] = \max_{a\in\mathcal{A}} \mathbb{E}_\Omega \left[u(o(a,\tilde{\omega}))|s\right] = \int\limits_\Omega u(o(a,\omega)) f_{\Omega|S}(\omega|s)\mathrm{d}\omega, \qquad (2.5)$$

where $f_{\Omega|S}(\omega|s)$ denotes the posterior state distribution after a signal realization equal to s. The posterior state distribution is determined as follows: Consider an information system $(f_{S|\Omega}, f_\Omega)$. The joint probability density function of signals and states is given by

$$f_{S,\Omega} : \; S \times \Omega \to \mathbb{R}, \; (s,\omega) \mapsto f_{S|\Omega}(s|\omega) f_\Omega(\omega)$$

while the joint cumulative distribution function is

$$F_{S,\Omega} : \; S \times \Omega \to [0,1], \; (s,\omega) \mapsto \int\limits_{\underline{\omega}}^{\omega} \int\limits_{\underline{s}}^{s} f_{S,\Omega}(s',\omega')\mathrm{d}s'\mathrm{d}\omega' = \mathrm{Prob}(\tilde{s} \le s, \tilde{\omega} \le \omega).$$

The marginal distribution of the signals is characterized by the probability distribution function

$$f_S: \ S \to \mathbb{R}, \ s \mapsto \int_{\Omega} f_{S,\Omega}(s,\omega)\mathrm{d}\omega = \mathbb{E}_{\Omega}\left[f_{S|\Omega}(s|\tilde{\omega})\right]$$

with corresponding cumulative distribution function

$$F_S: \ S \to [0,1], \ s \mapsto \int_{\underline{s}}^{s} f_S(s')\mathrm{d}s' = \mathrm{Prob}(\tilde{s} \leq s). \tag{2.6}$$

Any signal with $f_S(s) = 0$ will never occur and, therefore, can be neglected in the revision of the prior belief. In particular, the posterior state distribution conditional on a redundant signal, (i.e. $f_S(s) = 0$) is not well defined and will not be computed. Of course, the marginal signal distribution depends on the prior belief. Therefore, for a fixed information structure but different prior, the corresponding information systems might have different redundant signals. Define the set of non-redundant signals of an information system $(f_{S|\Omega}, f_{\Omega})$ by $S(f_{S|\Omega}, f_{\Omega}) := \{s \in S | f_S(s) > 0\}$. An example for this can be found at the end of this subsection. Applying Bayes' rule for densities to non-redundant signals yields the posterior state (probability) distribution function after a signal realization equal to $s \in S(f_{S|\Omega}, f_{\Omega})$

$$f_{\Omega|S}: \ \Omega \times S(f_{S|\Omega}, f_{\Omega}), \ (\omega, s) \mapsto \frac{f_{S,\Omega}(s,\omega)}{f_S(s)} = \frac{f_{S|\Omega}(s|\omega)f_{\Omega}(\omega)}{f_S(s)}.$$

The corresponding posterior cumulative distribution function is

$$F_{\Omega|S}: \ \Omega \times S(f_{S|\Omega}, f_{\Omega}) \to [0,1], \ (\omega, s) = \int_{\underline{\omega}}^{\omega} f_{\Omega|S}(\omega'|s)\mathrm{d}\omega' = \mathrm{Prob}(\tilde{\omega} \leq \omega | s).$$

If Ω and S are finite, the procedure keeps the same, the only thing to change is to substitute all integrals by sums over the same sets. In order to make this more clear consider the continuation of example 1 below.

Example 1 (Continued)**.** Consider again $(f_{S|\Omega}, f_\Omega)$ as given in example 1 above. The joint probability distribution of signals and states is

$$f_{S,\Omega} = \begin{pmatrix} f_{S,\Omega}(s_1,\omega_1) & f_{S,\Omega}(s_2,\omega_1) \\ f_{S,\Omega}(s_1,\omega_2) & f_{S,\Omega}(s_2,\omega_2) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1/8 & 3/8 \end{pmatrix}$$

Therefore, the corresponding cumulative distribution function is

$$F_{S,\Omega} = \left(\sum_{s' \le s} \sum_{\omega' \le \omega} f_{S,\Omega}(s',\omega') \right)_{s \in S, \omega \in \Omega} = \begin{pmatrix} 1/2 & 1/2 \\ 5/8 & 1 \end{pmatrix}$$

while the marginal signal distribution is equal to

$$f_S : \ S \to [0,1], \ s \mapsto \sum_{\omega \in \Omega} f_{S,\Omega}(s,\omega) = \begin{cases} 5/8 & \text{if } s = s_1 \\ 3/8 & \text{if } s = s_2. \end{cases}$$

Hence, the set of non-redundant signals is equal to the whole signal set, i.e. $S(f_{S|\Omega}, f_\Omega) = S$. Clearly, the corresponding cdf is

$$F_S : \ S \to [0,1], \ s \mapsto \sum_{s' \le s} f_S(s') = \begin{cases} 5/8 & \text{if } s = s_1 \\ 1 & \text{if } s = s_2. \end{cases}$$

Applying Bayes' rule yields the posterior state distribution as

$$f_{\Omega|S} = \begin{pmatrix} f_{\Omega|S}(\omega_1|s_1) & f_{\Omega|S}(\omega_2|s_1) \\ f_{\Omega|S}(\omega_1|s_2) & f_{\Omega|S}(\omega_2|s_2) \end{pmatrix} = \begin{pmatrix} 4/5 & 1/5 \\ 0 & 1 \end{pmatrix}.$$

If s_1 is observed the updated probability for the state being ω_1 is $4/5$ while with probability $1/5$ the second state ω_2 is the true state. Signal s_2 is only observed if the true state is ω_2. Therefore, after observing s_2, the updated probability for the state being ω_1 is zero while ω_2 is the true state with probability 1. Consequently, the corresponding posterior cdf is

$$F_{\Omega|S} = \left(\sum_{\omega' \le \omega} f_{\Omega|S}(\omega'|s) \right)_{\omega \in \Omega, s \in S(f_{S|\Omega}, f_\Omega)} = \begin{pmatrix} 4/5 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theoretically also mixed forms with an infinite state space and a finite signal space or conversely are possible. For reasons of tractability these will be omitted here. Instead, an example for an information system with infinite signal space and finite state space can be found in the following section.

2.3 Combining Various Information Structures

If more than one information structure is available, a decision maker might decide to use multiple structures. That is, instead of observing a single signal from one structure, she might observe multiple signals of various information structures. An example for this would be the reading of various newspapers. If the conditional signals are perfectly correlated, the observation of an additional signal conveys no additional information about the state of the world (as an example consider newspapers which always publish equal articles). However, since one of the signals could be ignored, such a system never transmits less information than only one of the structures. If, in contrast, the signals are not perfectly correlated, then the observation of an additional signal conveys additional information about the state. For illustration of this idea, it is assumed that for any pair of information structures, $f_{S|\Omega}$ and $\bar{f}_{\bar{S}|\Omega}$, the conditional signals distribution are independent, i.e. $\tilde{s}|\omega$ and $\tilde{\bar{s}}|\omega$ are independent for all $\omega \in \Omega$. Therefore, for information structures $f^1_{S_1|\Omega}, \ldots, f^N_{S_N|\Omega}$, $N \in \mathbb{N}$, the joint conditional distribution of $(\tilde{s}_1, \ldots, \tilde{s}_N)$ is given by

$$f_{(S_1,\ldots,S_N)|\Omega} : \bigotimes_{i=1}^{N} S_i \times \Omega \to \mathbb{R}_+, (s_1, \ldots, s_N, \omega) \mapsto \prod_{i=1}^{N} f_{S_i|\Omega}(s_i|\omega). \qquad (2.7)$$

Then the family of conditional signal distributions $\left\{f_{(S_1,\ldots,S_N)|\Omega}(s|\omega)\right\}_{s\in\bigotimes_{i=1}^{N} S_i, \omega\in\Omega}$ defines an information structure $f_{(S_1,\ldots,S_N)|\Omega}$ with signal space $\bigotimes_{i=1}^{n} S_i$.[4] After

[4]Remark: In case of infinite signal spaces, this work restricted attention to signal spaces which are convex subsets of the real line. Clearly, this assumption has to be relaxed in order to allow for multiple signals. Therefore, it is assumed that signal spaces are convex

the observation of a signal $s \in \bigotimes_{i=1}^{N} S_i$, the revision of the prior belief is done along the same lines as described in the previous section 2.2.2. For a more concrete illustration consider Example 2 below.

Example 2. Consider again $f_{S|\Omega}$ as given in Example 1. Moreover, assume that this structure describes an experiment which is independently done twice. Therefore, two signals are observed, i.e. $N = 2$. Then the information structure $f_{(S,S)|\Omega}$ is defined through the joint conditional distribution of the two signals:

$$f_{(S,S)|\Omega} = \begin{pmatrix} f_{(S,S)|\Omega}(s_1, s_1|\omega_1) & f_{(S,S)|\Omega}(s_1, s_2|\omega_1) & f_{(S,S)|\Omega}(s_2, s_1|\omega_1) & f_{(S,S)|\Omega}(s_2, s_2|\omega_1) \\ f_{(S,S)|\Omega}(s_1, s_1|\omega_1) & f_{(S,S)|\Omega}(s_1, s_2|\omega_1) & f_{(S,S)|\Omega}(s_2, s_1|\omega_1) & f_{(S,S)|\Omega}(s_2, s_2|\omega_1) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/16 & 3/16 & 3/16 & 9/16 \end{pmatrix}.$$

The joint distribution of signals and states is

$$f_{S,S,\Omega} = \begin{pmatrix} f_{S,S,\Omega}(s_1, s_2, \omega_1) & f_{S,S,\Omega}(s_1, s_2, \omega_1) & f_{S,S,\Omega}(s_2, s_1, \omega_1) & f_{S,S,\Omega}(s_2, s_2, \omega_1) \\ f_{S,S,\Omega}(s_1, s_1, \omega_2) & f_{S,S,\Omega}(s_1, s_2, \omega_2) & f_{S,S,\Omega}(s_2, s_1, \omega_2) & f_{S,S,\Omega}(s_2, s_2, \omega_2) \end{pmatrix}$$
$$= \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 1/32 & 3/32 & 3/32 & 9/32 \end{pmatrix}.$$

Using Bayes' rule leads to the posterior state distribution:

$$f_{\Omega|(S,S)} = \begin{pmatrix} f_{\Omega|(S,S)}(\omega_1|(s_1, s_1)) & f_{\Omega|(S,S)}(\omega_2|(s_1, s_1)) \\ f_{\Omega|(S,S)}(\omega_1|(s_1, s_2)) & f_{\Omega|(S,S)}(\omega_2|(s_1, s_2)) \\ f_{\Omega|(S,S)}(\omega_1|(s_2, s_1)) & f_{\Omega|(S,S)}(\omega_2|(s_2, s_1)) \\ f_{\Omega|(S,S)}(\omega_1|(s_2, s_2)) & f_{\Omega|(S,S)}(\omega_2|(s_2, s_2)) \end{pmatrix} = \begin{pmatrix} 16/17 & 1/17 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

2.4 Normalization of the Signal Space

For some informativeness criteria and for graphical characterizations it is useful to normalize the signal space. This section shows that it is without further loss of generality to assume that the signals (ex ante) are uniformly

subsets of $\mathbb{R}^n$, whenever multiple signals are considered.

distributed on $[0,1]$. Consider an information system $(f_{S|\Omega}, f_\Omega)$. First, suppose $S = [\underline{s}, \bar{s}]$ and $S(f_{S|\Omega}, f_\Omega) = S$. Therefore, the marginal density of the (ex ante) signal distribution $f_S(s)$ is strictly positive on S. This implies that the corresponding cdf $F_S(s)$ is strictly increasing in s. Now consider the random variable $\tilde{s}^n := F_S(\tilde{s})$ where F_S is the cdf of the marginal signal distribution. Since $S = S(f_{S|\Omega}, f_\Omega)$ and since F_S is continuous and strictly increasing, $\tilde{s}^n$ is uniformly distributed on $[0,1]$. Moreover, since the transformation $F_S: S \rightarrow [0,1]$ is one-to-one, $\tilde{s}^n = F_S(\tilde{s})$ fully reveals the realizations of the original signal $\tilde{s}$ in terms of quantiles and, therefore, conveys exactly the same information as $\tilde{s}$. More precisely, the normalized information system $(f_{S^n|\Omega}, f_\Omega)$ defined by

$$f_{S^n|\Omega} = \left\{f_{S^n|\Omega}(s|\omega)\right\}_{s\in S^n=[0,1],\omega\in\Omega} = \left\{\frac{f_{S|\Omega}(F_S^{-1}(s)|\omega)}{f_S(F_S^{-1}(s))}\right\}_{s\in S^n=[0,1],\omega\in\Omega} \tag{2.8}$$

has ex ante uniformly distributed signals and is informationally equivalent to $(f_{S|\Omega}, f_\Omega)$. To see this, first compute the marginal probability distribution of $\tilde{s}^n$ as given by

$$f_{S^n}(s) = \int_{\underline{\omega}}^{\bar{\omega}} f^n_{S^n|\Omega}(s|\omega) f_\Omega(\omega)\mathrm{d}\omega = \frac{\int_{\underline{\omega}}^{\bar{\omega}} f_{S|\Omega}(F_S^{-1}(s)|\omega) f_\Omega(\omega)\mathrm{d}\omega}{f_S(F_S^{-1}(s))} = \frac{f_S(F_S^{-1}(s))}{f_S(F_S^{-1}(s))} = 1.$$

I.e. $\tilde{s}^n$ is (ex ante) uniform on $[0,1]$. Next, to see the 'informationally equivalence' of $(f_{S|\Omega}, f_\Omega)$ and $(f_{S^n|\Omega}, f_\Omega)$, have a closer look at Bayes' updating rule: The posterior conditional distribution after a signal realization $\hat{s}$ of information system $(f_{S^n|\Omega}, f_\Omega)$ is represented by the function

$$f_{\Omega|S^n}: \ \Omega \times [0,1], \ (\omega, \hat{s}) \mapsto \frac{f_{S^n}(\hat{s}|\omega) f_\Omega(\omega)}{f_{S^n}(\hat{s})} \overset{f^n_{S^n}(\hat{s})=1 \text{ for all } \hat{s}}{=} \frac{f_{S|\Omega}(F_S^{-1}(\hat{s})|\omega) f_\Omega(\omega)}{f_S(F_S^{-1}(\hat{s}))}.$$

Therefore, the observation of $\hat{s}$ when using $(f_{S^n|\Omega}, f_\Omega)$ implements exactly the same posterior conditional state distribution as the observation of $s = F_S^{-1}(\hat{s})$

when using $(f_{S|\Omega}, f_\Omega)$, i.e.

$$f_{\Omega|S^n}(\omega|\hat{s}) = f_{\Omega|S}(\omega|F_S^{-1}(\hat{s})) \text{ for all } (\hat{s}, \omega) \in [0,1] \times \Omega.$$

Moreover, since $F_S^{-1}(\tilde{s}^n)$ and $\tilde{s}$ are stochastically equal (i.e. have the same distributions), this implies that the random variables $f_{\Omega|S^n}(\omega|\tilde{s}^n)$ and $f_{\Omega|S}(\omega|\tilde{s})$ are stochastically equal. Hence, $(f_{S|\Omega}, f_\Omega)$ and $(f_{S^n|\Omega}, f_\Omega)$ convey exactly the same information about $\tilde{\omega}$ and, therefore, they are called informationally equivalent.

However, even if the marginal cdf F_S is not continuous or strictly increasing, as for example in case of finite S or $S(f_{S|\Omega}, f_\Omega) \subsetneq S$, Lehmann (1988) shows that it is always possible to define an informationally equivalent information system such that the corresponding marginal distribution is continuous and strictly increasing. This result by Lehmann (1988) implies that, even if the original signal set is finite, it is without loss of further generality to assume that the marginal distribution of the signals is uniform on $[0, 1]$. Or in other words, for all information systems $(f_{S|\Omega}, f_\Omega)$ considered in this work, there is an informationally equivalent system $(f_{S^n|\Omega}, f_\Omega)$ such that ex ante the signals are uniform on $[0, 1]$, i.e. $f_{S^n}(s) = 1$ for all $s \in [0, 1] =: S^n$.

The following example explains how an normalized information system can be achieved from an non-normalized one.

Example 1 (Continued)**.** Consider again $(f_{S|\Omega}, f_\Omega)$ as given above. Following Lehmann (1988), define the random variable $\tilde{s}^n$ by

$$\tilde{s}^n = s^n(\tilde{s}) = \begin{cases} f_S(s_1)\tilde{u} & = \frac{5}{8}\tilde{u} \quad \text{if } \tilde{s} = s_1 \\ f_S(s_1) + f_S(s_2)\tilde{u} & = \frac{5}{8} + \frac{3}{8}\tilde{u} \quad \text{if } \tilde{s} = s_2 \end{cases}$$

where $\tilde{u}$ is uniform on $[0, 1]$ and independent of $\tilde{s}$. Then $\tilde{s}^n$ is a random variable on $[0, 1]$. Moreover, if the realization $s \in [0, f_S(s_1)]$, then

$$s = f_S(s_1)u \leq s' \iff u \leq \frac{s'}{f_S(s_1)}.$$

Therefore, for $s \in [0, f_S(s_1)]$

$$\begin{aligned} F_{S^n}(s) = \text{Prob}(\tilde{s}^n \leq s) =& \text{Prob}(\tilde{s} = s_1)\text{Prob}\left(\tilde{u} \leq \frac{s}{f_S(s_1)}\right) \\ =& f_S(s_1)\frac{s}{f_S(s_1)} = s. \end{aligned} \tag{2.9}$$

For a realization $s \in (f_S(s_1), 1]$, then

$$s = f_S(s_1) + f_S(s_2)u \leq s' \iff u \leq \frac{s' - f_S(s_1)}{f_S(s_2)}$$

and, therefore,

$$\begin{aligned} F_{S^n}(s) = \text{Prob}(\tilde{s}^n \leq s) =& \text{Prob}(\tilde{s}^n \leq f_S(s_1)) + \text{Prob}\left(f_S(s_1) < \tilde{s}^n \leq s\right) \\ \overset{(2.9)}{=}& f_S(s_1) + \text{Prob}(\tilde{s} = s_2)\text{Prob}\left(\tilde{u} \leq \frac{s - f_S(s_1)}{f_S(s_2)}\right) \\ =& f_S(s_1) + f_S(s_2)\frac{s - f_S(s_1)}{f_S(s_2)} = s \end{aligned} \tag{2.10}$$

for $s \in (f_S(s_1), 1]$. Equations (2.9) and (2.10) imply that $\tilde{s}^n$ is uniform on $[0, 1]$. Last, the normalized system $(f_{S^n|\Omega}, f_\Omega)$ is given by

$$f^n_{S^n|\Omega} = \left\{\frac{f(F_S^{-1}(s)|\omega)}{f_S(F_S^{-1}(s))}\right\}_{s\in[0,1],\omega\in\{\omega_1,\omega_2\}} = \begin{cases} \frac{8}{5} & \text{if } (s,\omega) \in [0, {}^5\!/\!_8] \times \{\omega_1\} \\ 0 & \text{if } (s,\omega) \in ({}^5\!/\!_8, 1] \times \{\omega_1\} \\ \frac{2}{5} & \text{if } (s,\omega) \in [0, {}^5\!/\!_8] \times \{\omega_2\} \\ 2 & \text{if } (s,\omega) \in ({}^5\!/\!_8, 1] \times \{\omega_2\}. \end{cases}$$

2.5 Concluding Remarks

This chapter introduces the reader to the basic framework of decision making under risk. A formalization of a decision maker's attitude towards risk and two concepts to measure this are introduced. The concept of an information system formalizes the generation of random signals that contain information about the state of the world. The following chapter 3 deals with ordering of information systems in terms of informativeness.

Chapter 3

Information and Informativeness

The concept of an information system was introduced in the previous section. In contrast to the traditional literature on information, where the prior belief is typically kept fixed, this work considers a more general notion of an information system by allowing for different priors. Assume that a decision maker can choose between different information systems. What criterion can she use to order them in terms of informativeness? The purpose of this chapter is to provide various approaches, including two novel ones, to answer this question. As starting point, section 3.1 discusses natural requests and desirable properties of informativeness criteria. Building on this, section 3.2 presents Blackwell's informativeness criterion and the weaker informativeness concepts by Lehmann/Persico and Kim. Section 3.3 introduces two new informativeness criteria defined by Eckwert and Zilcha (2008) and Brandt et al. (2013, 2014).

Intuitively, the informativeness of an information system can be viewed as the statistical relatedness of the signals and the states. Increasing informativeness means 'adding correlation' between signals and states. Since an information system consists of an information structure $f_{S|\Omega}$ and a prior belief f_Ω, this can be achieved by modifying either the information structure or the prior. Thus, in such a setting, the informativeness of the system is jointly determined by the prior and the information structure.

In order to give an answer to the question above for fixed prior beliefs, Blackwell formalizes the intuitive idea that an information structure $f_{S|\Omega}$ is more informative than/sufficient for $\bar{f}_{\bar{S}|\Omega}$ (regardless of the prior belief), if the observation of signal $\bar{s}$ of the letter one is the same as a noisy observation of a signal s from the former one. Blackwell showed that every Bayesian decision maker prefers an information structure $f_{S|\Omega}$ to another structure $\bar{f}_{\bar{S}|\Omega}$ if and only if $\tilde{s}$ is more informative/sufficient for $\tilde{\bar{s}}$. This strong equivalence connects inforamtiveness (in the sense of Blackwell) with its value for the decision makers. Unfortunately, the strengh of Blackwell's theorem also shows that Blackell's criterion is quite restrictive (i.e. there are only a few information structures that are comparable with this criterion), because an information structure can only be more informative than another one, if it delivers higher ex ante expected utility for all expected utility maximizers. Since then various papers have proposed weaker criteria that can be applied to a broader set of information systems.

Lehmann (1988) and Persico (2000) use a criterion according to which all decision makers in a restricted class (those with single-crossing indirect utilities) prefer an information structure. This criterion has been successfully applied in auction theory.

Following similar lines, Kim (1995) proposes a criterion that is particularly useful for ranking information systems in an agency framework.

Following a different idea, namely that 'better' information implies more 'aggressive' Bayesian updating, Ganuza and Penalva (2010) equate more informativeness with (various kinds of) higher dispersion of posterior expectation. They show that an auctioneer provides too little information, and that both the socially efficient amount and the auctioneer's optimal choice of information increase with the number of bidders.

Similary, Li (2012) studies the effect of information and bias on NIH grant allocations. Reviewers who are related to applicants through citations are assumed to be better informed about the grant quality of a grant proposal. When estimating the model, her identifying assumption is that this difference in information ranks the dispersion of the reviewers' conditional expectations of quality.

Finally, Eckwert and Zilcha (2008) study screening mechanisms of individual skills in systems of higher education. Measuring the precision of a screening mechanism by a dispersion concept for conditional expectations, they show that better screening leads to more inequality in the distribution of actual incomes, but less inequality in the distribution of income opportunities.

3.1 Desirable Properties of Informativeness Criteria

Concerning the properties of an informativeness ranking, consider as a first step the two extreme information systems $(f^0_{S^0|\Omega}, f_\Omega)$ and $(f^1_{S^1|\Omega}, f_\Omega)$. No matter what the true state of the world is and regardless of the prior, the signals generated by the information system $(f^0_{S^0|\Omega}, f_\Omega)$ and the state variable are not correlated at all. Consequently, these signals do not contain any additional information about the state of the world. I.e. the application of $(f^0_{S^0|\Omega}, f_\Omega)$ is equivalent to applying of no information system at all, i.e. it is fully uninformative. Hence, no system $(f_{S|\Omega}, f_\Omega)$ should be ranked strictly smaller than $(f^0_{S^0|\Omega}, f_\Omega)$. On the other extreme, the observation of a signal generated by a fully informative system $(f^1_{S^1|\Omega}, f_\Omega)$ fully reveals the true state. Therefore, no system $(f_{S|\Omega}, f_\Omega)$ should be ranked strictly higher in terms of informativeness than $(f^1_{S^1|\Omega}, f_\Omega)$. In order to formulate these properties formally, the following notation is used for any informativeness ranking $\overset{\text{inf}}{\succsim}$:

- $(f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\succsim} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$ that means $(f_{S|\Omega}, f_\Omega)$ is more informative than $(\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$.
- $(f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\succ} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$ means that $(f_{S|\Omega}, f_\Omega)$ is strictly more informative than $(\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$,i.e. $(f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\succsim} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$ and $(\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega) \overset{\text{inf}}{\not\succsim} (f_{S|\Omega}, f_\Omega)$.
- $(f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\sim} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$ means that $(f_{S|\Omega}, f_\Omega)$ and $(\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$ are equally informative, i.e. $(f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\succsim} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$ and $(\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega) \overset{\text{inf}}{\succsim} (f_{S|\Omega}, f_\Omega)$.

With this notation, the two natural, minimal properties for any informativeness ranking $\overset{\text{inf}}{\succsim}$ can be formalized as follows:

$$(f^0_{S^0|\Omega}, \bar{f}_\Omega) \overset{\text{inf}}{\not\succ} (f_{S|\Omega}, f_\Omega) \tag{P0}$$
$$(f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\not\succ} (f^1_{S^1|\Omega}, \bar{f}_\Omega) \tag{P1}$$

for any information system $(f_{S|\Omega}, f_\Omega)$ and priors $\bar{f}_\Omega, f_\Omega \in \Delta(\Omega)$. Property (P0) means that an informativeness ranking should not rank any information system $(f_{S|\Omega}, f_\Omega)$ strictly smaller than the completely uninformative system $(f^0_{S^0|\Omega}, \bar{f}_\Omega)$ while $(P1)$ means that no system $f_{S|\Omega}, f_\Omega)$ should be ranked strictly higher than the fully informative $(f^1_{S^1|\Omega}, \bar{f}_\Omega)$. Clearly, these minimal requests can be tightend: desirable properties of an informativeness ranking are that $(f^0_{S^0|\Omega}, \bar{f}_\Omega)$ should be ranked weakly smaller than any information system $(f_{S|\Omega}, f_\Omega)$ and that $(f^1_{S^1|\Omega}, \bar{f}_\Omega)$ should be ranked weakly higher than any system $(f_{S|\Omega}, f_\Omega)$. I.e.

$$(f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\succsim} (f^0_{S^0|\Omega}, \bar{f}_\Omega) \tag{P0$'$}$$
$$\text{and } (f^1_{S^1|\Omega}, \bar{f}_\Omega) \overset{\text{inf}}{\succsim} (f_{S|\Omega}, f_\Omega) \tag{P1$'$}$$

for any information system $(f_{S|\Omega}, f_\Omega)$ and priors $\bar{f}_\Omega, f_\Omega \in \Delta(\Omega)$. Observe that (P0$'$) implies (P0) and that (P1$'$) implies (P1).

Further natural requests of an information ranking are transitivity and reflexivity, i.e.

$$\text{Transitivity: If } (f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\succsim} (\hat{f}_{\hat{S}|\Omega}, \hat{f}_\Omega) \text{ and } (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega) \overset{\text{inf}}{\succsim} (f_{S|\Omega}, f_\Omega)$$
$$\text{then } (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega) \overset{\text{inf}}{\succsim} (\hat{f}_{\hat{S}|\Omega}, \hat{f}_\Omega), \tag{P2}$$
$$\text{Reflexivity: } (f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\succsim} (f_{S|\Omega}, f_\Omega), \tag{P3}$$

for all information systems $(f_{S|\Omega}, f_\Omega)$, $(\hat{f}_{\hat{S}|\Omega}, \hat{f}_\Omega)$ and $(\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$. In particular this means, that an information ranking should define a preorder on the set of information systems.

Moreover, an information criterion should be invariant to injective transformations of the state space $t: \ \Omega \to \Omega'$. Since such transformations are one-to-one, each state ω can be inferred from $t(\omega)$ without noise, and vice versa. Hence, the information revealed by $(f_{S|\Omega}, f_\Omega)$ about Ω is the same as the information revealed by $(f_{S|t(\Omega)}, f_{t(\Omega)})$ about $t(\Omega)$, with $f_{S|t(\Omega)} = \left\{f_{S|t(\Omega)}(s|\omega')\right\}_{s\in S,\omega'\in t(\Omega)}$ defined by

$$f_{S|t(\Omega)}:\ S\times t(\Omega)\rightarrow\mathbb{R},\ (s,\omega')\mapsto f_{S|\Omega}(s|t^{-1}(\omega')) \tag{3.1}$$

and $f_{t(\Omega)}$ by

$$f_{t(\Omega)}:\ t(\Omega)\rightarrow[0,1],\ \omega'\mapsto\frac{f_\Omega(t^{-1}(\omega'))}{|t'(t^{-1}(\omega'))|}, \tag{3.2}$$

where $t^{-1}:\ \Omega'\rightarrow\Omega$ denotes the inverse function of t and t' its derivative with respect to ω (which is assumed to exist because of technical reason). Since the state space is ordered,[5] it is possible (and sometimes also needed) to weakening this property to a condition called *ordinality of states*. A ranking with this property is invariant to strictly increasing, $\mathcal{F}$-measurable transformations $t:\ \Omega\rightarrow\Omega'$ of the state space. Since such transformations are also one-to-one from Ω to $t(\Omega)$, the intuition keeps the same as for injective, $\mathcal{F}$-measurable transformation: the information revealed by $(f_{S|\Omega}, f_\Omega)$ about Ω is the same as the information revealed by $(f_{S|t(\Omega)}, f_{t(\Omega)})$ about $t(\Omega)$, where $f_{t(\Omega)}$ and $f_{S|t(\Omega)}$ are defined as above.

Definition 3.1. *(i) An information criterion $\overset{\text{inf}}{\succsim}$ satisfies the independence of state space property (IS), if*

$$(f_{S|\Omega}, f_\Omega)\overset{\text{inf}}{\succsim}(\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)\Rightarrow(f_{S|t(\Omega)}, f_{t(\Omega)})\overset{\text{inf}}{\succsim}(\bar{f}_{\bar{S}|t(\Omega)}, \bar{f}_{t(\Omega)}) \tag{IS}$$

for all injective $t:\ \Omega\rightarrow\Omega'$, $f_\Omega, \bar{f}_\Omega\in\Delta(\Omega)$ and all information structures $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega}$.

(ii) An information criterion $\overset{\text{inf}}{\succsim}$ satisfies the ordinality of states property

[5] Remember: $\Omega = [\underset{\sim}{\omega}, \bar{\omega}]\subseteq\mathbb{R}$. Moreover, if Ω would be finite, it is w.o.l.g. to assume that it is ordered.

(OS) if

$$(f_{S|\Omega}, f_\Omega) \overset{\text{inf}}{\succsim} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega) \Rightarrow (f_{S|t(\Omega)}, f_{t(\Omega)}) \overset{\text{inf}}{\succsim} (\bar{f}_{\bar{S}|t(\Omega)}, \bar{f}_{t(\Omega)}) \qquad \text{(OS)}$$

for all strictly increasing functions $t: \Omega \to \Omega'$, $f_\Omega, \bar{f}_\Omega \in \Delta(\Omega)$ *and all information structures* $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega}$.

The independence of state space property implies that the distance and the order of the states can be changed without changing informativeness ranking. Since any utility function $u : \Omega \to \mathbb{R}$ could be viewed as state transformation, the IS property additionally has the implication that all decision makers with different, injective vNM-prefecrences share a common view on the informativeness of a set of considered information systems. Unfortunately, IS is quite restrictive. Therefore, it is reasonable to impose the weaker condition of ordinality of states in some economic environments. Consider for instance an environment where the state space is a subset of the real line and different states represent different wealth levels. In such a setting one could argue that it suffices to restrict attention to strictly increasing state transformations as any other does not respect the fundamental underlying ranking of the state space. Moreover, in economic theory attention is frequently restricted to strictly increasing utility functions on the state space. Those utility functions constitute increasing state transformations. Thus, the OS property of information orders has the important implication that expected utility maximizers who have different strictly increasing vNM-preferences will share a common view on the informativeness of a set of considered information systems.

By the definitions of IS and OS it is clear that OS is weaker than IS, i.e. IS implies OS.

A decision maker cares about information only in so far as her wellbeing is affected. Hence, a rational decision maker will always choose that information system that delivers her the highest expected welfare. Hence, Bonnenblust et al. (1949) call an information system more valuable if it induces higher ex ante expected utility for all decision makers. In order to introduce the term 'more valuable' formally, define the optimal action after a signal observation equal to s by

$$a^* : S \to \mathcal{A}, s \mapsto \arg\max_{a \in \mathcal{A}} \mathbb{E}_\Omega \left[v(a, \tilde{\omega}) | s \right]. \tag{3.3}$$

Building on Bonnenblust et al. (1949), the value of information is defined as follows:

Definition 3.2. *For an arbitrary strategy $a : S \to \mathcal{A}, s \mapsto a(s)$, an information system $(f_{S|\Omega}, f_\Omega)$ and an indirect utility function $v : \mathcal{A} \times \Omega \to \mathbb{R}$ define the ex ante expected utility by*

$$V((f_{S|\Omega}, f_\Omega), a, v) := \mathbb{E}_S \left[\mathbb{E}_\Omega \left[v(a(s), \tilde{\omega}) | \tilde{s} \right] \right].$$

An information system $(f_{S|\Omega}, f_\Omega)$ is more valuable than the information system $(\bar{f}_{\bar{S}|\Omega}, f_\Omega)$ if for every expected utility maximizer with prior f_Ω holds that

$$V((f_{S|\Omega}, f_\Omega), a^*, v) \geq V((\bar{f}_{\bar{S}|\Omega}, f_\Omega), \bar{a}^*, v). \tag{3.4}$$

This concept induces a preorder on the set of information systems. It is unclear how this ordering relates to information orders. This will be studied in the following sections.

3.2 Blackwell's Sufficiency Criterion

Blackwell (1951) defines an criterion for the comparison of two information structures regardless of the prior. As already mentioned, Blackwell's criterion formalizes the intuitive idea that an information structure $f_{S|\Omega}$ is more informative than $\bar{f}_{\bar{S}|\Omega}$ regardless of the prior belief, if the observation of an signal $\bar{s}$ of the latter one is the same as an noisy observation of an signal s from the former. In particular, this means the following: Suppose the decision maker is not able to observe the signal s directly anymore and that there is a random transformation $\hat{f}_{\bar{S}|S}$ that randomly transforms an unobservable signal $s \in S$ into an observable signal $\bar{s} \in \bar{S}$. Therefore, $\hat{f}_{S|\bar{S}}$ can be interpreted as information structure with state space S and signal space $\bar{S}$.[6] Intuitively it is clear that the observation of the signal $\bar{s}$ is not more informative than the direct observation of s. The following definition etablishes Blackwell's sufficiency criterion for infinite signal spaces which easily can be modified for the case of a finite sets of signals. In contrast, whether Ω is finite or infinite does not play any role in the definition of Blackwell's criterion.

Definition 3.3 (Blackwell's Sufficiency Criterion)**.** *An information structure $f_{S|\Omega}$ is sufficient for the structure $\bar{f}_{\bar{S}|\Omega}$, $f_{S|\Omega} \overset{b}{\succsim} \bar{f}_{\bar{S}|\Omega}$, if and only if there exists an information structure $\hat{f}_{\bar{S}|S}$ such that $\hat{F}_{\bar{S}|S}(C|\cdot) = \int_{\bar{S}} 1_C(\bar{s})\bar{f}_{\bar{S}|S}(\bar{s}|\cdot)d\bar{s}$ is $\mathcal{S}$-measurable for all $C \in \bar{\mathcal{S}}$[7] and*

$$\bar{f}_{\bar{S}|\Omega}(\bar{s}|\omega) = \mathbb{E}_S\left[\hat{f}_{\bar{S}|S}(\bar{s}|\tilde{s})|\omega\right] = \int_S \hat{f}_{\bar{S}|S}(\bar{s}|s)f_{S|\Omega}(s|\omega)ds \text{ for all } \omega \in \Omega.$$

Important is that the information structure $\hat{f}_{\hat{S}|S}$ is independent of the state ω. This independence implies that $\hat{f}_{\bar{S}|S}$ is an information structure which convey information about $\tilde{s}$. Therefore, an observation of a signal from $\bar{f}_{\bar{S}|\Omega}$ does not contain more information about the state than a direct observation of an signal from $f_{S|\Omega}$. Additionally, this implies that if an information structure is more informative than another one, then the corresponding information systems are ranked in the same way for all prior distributions, i.e.

$$(f_{S|\Omega}, f_\Omega) \overset{b}{\succsim} (\bar{f}_{\bar{S}|\Omega}, f_\Omega) :\Leftrightarrow f_{S|\Omega} \overset{b}{\succsim} \bar{f}_{\bar{S}|\Omega},$$

for all prior beliefs $f_\Omega \in \Delta(\Omega)$. Hence, an information system $(f_{S|\Omega}, f_\Omega)$ is more informative than the system $(\bar{f}_{\bar{S}|\Omega}, f_\Omega)$ if and only if underlying information structure $f_{S|\Omega}$ is more informative than $\bar{f}_{\bar{S}|\Omega}$.

Next, observe that Blackwell's sufficiency criterion fulfill all natural requests and desirable properties of informativeness criteria stated in section

[6]Compare chapter 2.

[7]In particular, since $\hat{f}_{\bar{S}|S}(\cdot|s)$ is a probability distribution on $(\bar{S}, \bar{\mathcal{S}})$ this means that $\hat{F}_{\bar{S}|S}$ is a stochastic kernel.

3.1.

Lemma 3.1. *Blackwell's sufficiency criterion satisfies the basic properties (P0′), (P1′), (P2), (P3) and (IS).*

The properties (P2) and (P3) imply that Blackwell's sufficiency criterion defines a (partial) preorder on the set of information structures. Additionally, (P0′) and (P1′) imply that for any prior $f_\Omega \in \Delta(\Omega)$, the minimal element of this preorder is the system $(f^0_{S^0|\Omega}, f_\Omega)$ while the maximal element is $(f^1_{S^1|\Omega}, f_\Omega)$.

Moreover, Blackwell's criterion allows to compare information systems which signal spaces have different dimensions. This in turn allows for the comparison of combinations of independent information structures. As mentioned above, intuitively it is clear that the observation of additional signals reveal additional information about the state.

Proposition 3.1. *For all independent information structures $f_{S|\Omega}$ and $\bar{f}_{\bar{S}|\Omega}$, the information structure $f_{(S,\bar{S})|\Omega}$ with $f_{(S,\bar{S})|\Omega}(s,\bar{s}|\omega) := f_{S|\Omega}(s|\omega)\bar{f}_{\bar{S}|\Omega}(\bar{s}|\omega)$ is sufficient for $f_{S|\Omega}$ and $\bar{f}_{\bar{S}|\Omega}$, respectively.*

The combination of two information structures is always more informative than the underlying structures itsself. This observation is useful for the construction of information structures and systems with parametrized informativeness, which on their part are useful by modeling the demand and supply of information (compare chapter 4). As an example consider a framework in which only one (kind of) information structure is available, but it is possible to use several of these technologies simultaneously (compare Example 2). Proposition 3.1 implies that the informational content is increasing in the number of simultaneously used technologies. Therefore, this number parametrizes informational content.

As mentioned above, it is not so clear how informativeness criteria relate to individual ex ante expected utilities. However, Blackwell (1951, 1953) proofed that the preorder induced by his sufficency criterion is equivalent to the preorder induced by the 'more valuable' criterion by Bonnenblust et al. (1949).

Theorem 3.1 (Blackwell's theorem)**.** *For any fixed prior belief* $f_\Omega \in \Delta(\Omega)$ *is an information system* $(f_{S|\Omega}, f_\Omega)$ *more informative than a system* $(\bar{f}_{\bar{S}|\Omega}, f_\Omega)$ *if and only if it is more valuable all for expected utility maximizers.*

A more informative information system is always valuable for a single decision maker. Crucial for this result is, that the set of possible actions is independent of the signal realization. In most economic circumstances, as for example in a general equilibrium framework, this might not be the case. Hirshleifer (1971) was the first who demonstrated that in equilibrium information might make everybody worse off. He considers a small exchange economy with a single consumption good, risk avers agents and complete markets for state-contingent claims. Each agent is endowed with a risky endowment of the consumption good. The agents can share risks by trading state-contingent claims in complete markets before the state of the world is realized. If they were perfectly informed about the state before the markets for state-contingent claims are open, no trade at all will take place and the agents consume according to their endowments. Therefore, from an ex ante perspective, perfect information make the agents worse off by breaking down the risk sharing markets. A more detailed discussion of the value of information in equilibrium models will be provided in chapter 4.

3.2.1 Weaker criteria

Up to now this section studied Blackwell's sufficiency criterion. For fixed prior, an information system is sufficient for another one if and only if it delivers higher ex ante expected utility for all expected utility maximizers. In particular, a system is sufficient for another one if and only if it is more valuable in the sense of Definition 3.2. This powerful equivalence has the cost that only a few systems are comparable in terms of Blackwell's criterion. In order to relax this problem and to be able to compare more information systems with respect to their value, various information concepts has been developed which are applicable in different economic scenarios. This section will briefly introduce the definitions of two such criteria and relate them to the value of information for particular classes of decision makers.

The criterion by Lehmann and Persico

Lehmann (1988) and Persico (1996, 2000) define an information ranking, in this work reffered as accuracy, based on the intuition that the more informative a system, the more correlated are signals and states. By defining accuracy, they restrict attention to information structures with the monotone likelihood ratio property (MLRP). In order to avoid confusion, remember that attention is restricted to state and signals spaces which are convex subsets of the real line. Under these assumptions the MLRP is defined as follows:

Definition 3.4 (Monotone Likelihood Ratio Property)**.** *An information structure $f_{S|\Omega}$ has a monotone likelihood ratio if and only if $^{f_{S|\Omega}(s|\omega)}/_{f_{S|\Omega}(s|\omega')}$ is decreasing in s for all $\omega, \omega' \in \Omega$ such that $\omega' \geq \omega$.*

Milgrom (1981) relates this property of information structures with the slope of posterior conditional expectation (as a function of the signals realization s). He calls an information system $(f_{S|\Omega}, f_\Omega)$ monotone (in s) if for any $s, s' \in S$ with $s' \geq s$ it follows that $F_{\Omega|S}(\cdot|s')$ first order stochastically dominates $F_{\Omega|S}(\cdot|s)$.[8] Then Milgrom (1981) shows that an information system $(f_{S|\Omega}, f_\Omega)$ is monotone regardless of the prior if and only if the underlying structure, $f_{S|\Omega}$, has the MLRP:

Proposition 3.2. *An information system $(f_{S|\Omega}, f_\Omega)$ is monotone (in s) for all prior $f_\Omega \in \Delta(\Omega)$ if and only if $f_{S|\Omega}$ have the MLRP.*

Denote by $\mathcal{M}$ the set of information structures with MLRP. In order to give the definition of accuracy, define the quantile function of an conditional signal cdf, $F_{S|\Omega}$, by

$$F^{-1}_{S|\Omega}: \; [0,1] \times \Omega \to S, (p, \omega) \mapsto \inf \left\{ s \in S | F_{S|\Omega}(s|\omega) \geq p \right\}.$$

For fixed prior belief, Lehmann (1988) and Persico (1996) define accuracy as follows:

[8]Let F_Ω and $\bar{F}_\Omega$ denote two distributions of Ω. F_Ω first order stochastically dominates $\bar{F}_\Omega$ if and only if $F_\Omega(\omega) \leq \bar{F}_\Omega(\omega)$ for all $\omega \in \Omega$.

Definition 3.5. *Let $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega} \in \mathcal{M}$. Information structure $f_{S|\Omega}$ is more accurate than the structure $\bar{f}_{\bar{S}|\Omega}$, $f_{S|\Omega} \overset{a}{\succsim} \bar{f}_{\bar{S}|\Omega}$, if and only if*

$$T : \bar{S} \times \Omega \to S, (\bar{s}, \omega) \mapsto F^{-1}_{S|\Omega}(\bar{F}_{\bar{S}|\Omega}(\bar{s}|\omega)|\omega)$$

is nondecreasing in ω for all $\bar{s} \in \bar{S}$.

The function $T(\cdot, \omega) : \bar{S} \to S$ is a state dependent transformation of signals in $\bar{S}$ into signals in S. In order to get an intuition for this definition suppose for the moment $S = \bar{S}$. Since $T(\bar{s}, \omega)$ is nondecreasing in ω it adds correlation to the signal $\bar{s}$ in the following sense: If ω is low the transformed signal is lower than the original one and vice versa if ω is high. Hence, at least intuitively, the more accurate an information system, the more correlated are signals and states.

Similar to Blackwell's criterion, accuracy was originally defined for the comparison of information structures, however, throughout this work, an information system $(f_{S|\Omega}, f_\Omega)$ is more accurate than the system $(\bar{f}_{\bar{S}|\Omega}, f_\Omega)$ if and only of the underlying information structure $f_{S|\Omega}$ is more accurate then $\bar{f}_{\bar{S}|\Omega}$. Formally,

$$(f_{S|\Omega}, f_\Omega) \overset{a}{\succsim} (\bar{f}_{\bar{S}|\Omega}, f_\Omega) :\Leftrightarrow f_{S|\Omega} \overset{a}{\succsim} \bar{f}_{\bar{S}|\Omega}$$

for all $f_\Omega \in \Delta(\Omega)$. Moreover, since the transformation $T : \bar{S} \times \Omega \to S$ has to be increasing in ω for fixed $\bar{s} \in \bar{S}$ it follows that underlying ranking of the states (i.e. the ranking of the reals) is important for defining accuracy. Hence, it is obvious that accuracy cannot fulfill the independence of state property. However, it is easy to check that it has the ordinality of states property.

Proposition 3.3. *The accuracy ranking by Lehmann (1988) and Persico (1996) satisfies the basic properties (P0′), (P1′), (P2), (P3) and (OS).*

Within class of information structures with MLRP and for fixed prior belief Persico (1996, 2000) proofed that increasing accuracy is equivalent to increasing ex ante expected utility for all decision makers with single-crossing preferences. In order to formulate this result formally, it is necessary to define single crossing-preferences:

Definition 3.6 (Single-crossing preferences)**.** *Suppose the set of possible actions is a subset of the real line, i.e.* $\mathcal{A} \subseteq \mathbb{R}$*. A decision maker posses single crossing preferences if and only if her indirect elementary utility has the single-crossing property. That is, for all* $a, a' \in \mathcal{A}$ *such that* $a' \geq a$ *and for all* $\omega, \omega' \in \Omega$ *such that* $\omega' \geq \omega$ *it holds that*

$$v(a,\omega) - v(a',\omega) > 0 \;\Rightarrow\; v(a,\omega') - v(a',\omega') > 0.$$

The single crossing property implies that difference $v(a,\cdot) - v(a',\cdot)$ crosses 0 at most once and if so, it crosses from below. Therefore, loosely speaking a decision maker with single-crossing preferences want to coordinate small actions with small states and high actions with high states. For this class of decision problems Athey (2002) showed that the optimal action $a^*(s)$ is non-decreasing in the signal realization s if the underlying information structure posses the MLRP. The following theorem by Persico (1996, 2000) etablishes, as already mentioned above, a tight relationship between accuracy and the value of information for decision makers with single-crossing preferences.

Theorem 3.2. *Let* $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega} \in \mathcal{M}$*,* $f_\Omega \in \Delta(\Omega)$ *and* $\mathcal{A} \subseteq \mathbb{R}$ *compact. The information structure* $f_{S|\Omega}$ *is more accurate than the structure* $\bar{f}_{\bar{S}|\Omega}$ *if and only if* $V((f_{S|\Omega}, f_\Omega), a^*, v) \geq V((\bar{f}_{\bar{S}|\Omega}, f_\Omega), \bar{a}^*, v)$ *for all decision makers with single-crossing preferences.*

Similar to Blackwell's theorem which characterizes the ex ante expected utility for all decision makers in terms of sufficiency, theorem 3.2 characterizes the ex ante expected utility for all decision makers with single-crossing preferences in terms of accuracy. Of course, since the set of decsision makers with single-crossing preferences is a subset of all decision makers, the theorems 3.1 and 3.2 imply that if attention is restricted to information structures with MLRP, then accuracy is weaker than sufficiency.

Cororally 3.1. *Let* $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega} \in \mathcal{M}$*. Then,* $f_{S|\Omega} \overset{\text{b}}{\succsim} \bar{f}_{\bar{S}|\Omega} \Rightarrow f_{S|\Omega} \overset{\text{a}}{\succsim} \bar{f}_{\bar{S}|\Omega}$.

Kim's Criterion

As mentioned above, Blackwell's sufficiency criterion is quite restrictive. Additionally, Kim (1995) argues that it is not appropriate for the use in a principal-agent framework. As reason for this Kim mentions that sufficiency is based on forecasting the unobservable variable rather than on controlling it which is the objective in a principal-agent framework. Therefore, the crucial point is the (local) sensitivity of the signals with respect to a change in the state of the world rather than the correlation of signals and states. Kim introduces a new criterion which takes this objective into account. Kim's criterion measures the sensitivity of the conditional signal distribution, $f_{S|\Omega}(\cdot|\omega)$, for marginal changes in ω. In particular, the basic idea of Kim's criterion is the more sensitive the conditional signal distribution (for marginal changes in ω), the more informative is the corresponding information structure. In order to make this more clear, consider the fully uninformative information structure. Its conditional signal distributions are independent of the state variable and, hence, its sensitivity to marginal changes in ω is zero. If an information structure is partially informative, then the conditional signal distribution is not independent of the state and, hence, the sensitivity is different than zero.

Again, let Ω and S be convex subsets of the real line and assume that information structures are given by a family of conditional signal densities, $\left\{f_{S|\Omega}(\cdot|\omega)\right\}_{\omega\in\Omega}$, which are (twice) continuously differentiable. Denote the set of those information structure by $\mathcal{D}$. Kim measures the relative sensitivity of the conditional signal distribution by using the likelihood ratio

$$\frac{\partial f_{S|\Omega}/\partial\omega}{f_{S|\Omega}}(s|\omega) := \frac{\partial f_{S|\Omega}(s|\omega)/\partial\omega}{f_{S|\Omega}(s|\omega)}.$$

This ratio measures the relative change of the conditional signal density caused by a marginal change in the state variable. Define the cdf of the likelihood ratio as

$$L_{f_{S|\Omega}}(x,\omega) := Prob\left(\frac{\partial f_{S|\Omega}/\partial\omega}{f_{S|\Omega}}(\tilde{s}|\omega) \leq x\right).$$

This function determines the conditional probability that, given state ω, the information structure produces a signal which is less sensitive than x.

Kim compares the sensitivities of two information structures in terms of their likelihood ratio distributions by using the concept of a mean preserving spread (MPS).[9]

Definition 3.7. *Let $f_{S|\Omega},\ \bar{f}_{\bar{S}|\Omega} \in \mathcal{D}$. The information structure $f_{S|\Omega}$ is locally more informative than the structure $\bar{f}_{\bar{S}|\Omega}$, denoted by $f_{S|\Omega} \overset{\text{l-inf}}{\succsim} \bar{f}_{\bar{S}|\Omega}$, if and only if the likelihood ratio distribution $L_{f_{S|\Omega}}(\cdot,\omega)$ is a MPS of the likelihood ratio distribution $L_{\bar{f}_{\bar{S}|\Omega}}(\cdot,\omega)$ for all $\omega \in \Omega$.*

Intuitively, if the distribution $L_{f_{S|\Omega}}(\cdot,\omega)$ is a MPS of the distribution $L_{\bar{f}_{\bar{S}|\Omega}}(\cdot,\omega)$, then extremely sensitive signals (positive or negative) are more likeli to occur under $f_{S|\Omega}$ than under $\bar{f}_{\bar{S}|\Omega}$.

As for the criterea defined above, an information system $(f_{S|\Omega}, f_\Omega)$ is called locally more informative than the system $(\bar{f}_{\bar{S}|\Omega}, f_\Omega)$ if and only if the underlying information structures are ordered in the same way. I.e. for any $f_\Omega \in \Delta(\Omega)$ and $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega} \in \mathcal{D}$ set

$$(f_{S|\Omega}, f_\Omega) \overset{\text{l-inf}}{\succsim} (\bar{f}_{\bar{S}|\Omega}, f_\Omega) :\Leftrightarrow f_{S|\Omega} \overset{\text{l-inf}}{\succsim} \bar{f}_{\bar{S}|\Omega}.$$

Concerning the basic properties observe that, since the fully informative information structure is never continuously differentiable, it is not necessary (and not possible) to prove whether property $(P1)$ holds or not. Secondly, since $f_{S|t(\Omega)}$ has to be differentiable in ω, it suffices to show independence to increasing, differentiable state transformations.

Definition 3.8. *Let $f_{S|\Omega} \in \mathcal{D}$. An informativeness criterion $\overset{\text{inf}}{\succsim}$ satisfies*

[9] Let $\tilde{x}$ and $\tilde{y}$ be random variables with cumulative distribution functions F_X and F_Y, respectively. $\tilde{x}$ is a mean preserving spread (MPS) of $\tilde{y}$ iff $\mathbb{E}_X[\tilde{x}] = \mathbb{E}_Y[\tilde{y}]$ and $\int_{-\infty}^{x} F_X(x')\mathrm{d}x' \geq \int_{-\infty}^{x} F_Y(x')\mathrm{d}x'$ for all $x \in \mathbb{R}$. This is equivalent to the convex order for random variable with equal mean. For more details on this consider for instance Shaked and Shantbikumar (2007).

the weak ordinality of states property (wOS) if and only if

$$f_{S|\Omega} \overset{\text{inf}}{\succsim} \bar{f}_{\bar{S}|\Omega} \;\Rightarrow\; f_{S|t(\Omega)} \overset{\text{inf}}{\succsim} \bar{f}_{\bar{S}|t(\Omega)}$$

for all strictly increasing, differentiable state transformations $t: \; \Omega \to \mathbb{R}$.

The following proposition summarizes the basic properties of information ranking which are respected by Kim's criterion.

Proposition 3.4. *Local informativeness has the basic properties (P0), (P2), (P3) and (wOS).*

In order to relate local informativeness with the ex ante expected utility, consider a principal-agent framework. Assume that the principal is risk neutral while the agent is assumed to be risk avers. The agent chooses a level of effort $\omega \in \Omega$ which is unobservable for the principal. Instead, she observes a signal $s \in S$, e.g. the outcome of a production process, which is correlated to the agent's choice. The statistical dependence (or correlation) of signals and states is determined by an information structure $f_{S|\Omega}$. The conditional signal densities of this information structure are assumed to be twice continuously differentiable, i.e. $f_{S|\Omega} \in \mathcal{D}$. In order to implement a certain level of effort, the principal provides a bonus payment scheme $b: \; S \to \mathbb{R}$. In particular, this means that if s is observed, the agents gets a payoff in amount of $b(s)$. Moreover, assume that the agent's preferences for wealth and effort are additively separabel, increasing in wealth and decreasing in the effort level. More precisely, the agent's preferences are represented by $v(b, \omega) = u(b) - w(\omega)$ with $v' > 0$, $v'' < 0$ and $w' > 0$. Moreover, the reservation utility or outside option of the agent is $\bar{u}$ which is her utility in case of not signing a contract with the principal. Then, if the principal want to implement an effort level $\omega \in \Omega$ at minimum expected costs, she has to solve the following problem:

$$\min_{s(\cdot) \geq k} \int_{\underline{s}}^{\bar{s}} b(s) f_{S|\Omega}(s|\omega) \mathrm{d}s$$

$$\text{s. t.} \int_{\underline{s}}^{\bar{s}} u(b(s)) f_{S|\Omega}(s|\omega) \mathrm{d}\omega - w(\omega) \geq \bar{v} \tag{PC}$$

$$\int_{\underline{s}}^{\bar{s}} u(b(s)) {}^{\partial f_{S|\Omega}(s|\omega)}/_{\partial\omega}(s|\omega) \mathrm{d}s - w'(\omega) = 0 \tag{IC}$$

Condition (PC) is called *participation constraint* which assures that the agents gets at least her reservation utility $\bar{u}$ and, hence, assures that the agent is willing to sign the contract. (IC) is called incentive compatibility constraint. In particular, (IC) is equal to the first order condition (FOC) in the agents optimization problem and, hence, it guarantees that ω is in fact the optimal choice for the agent.[10] The lower bound of the agent's bonus payment k assures the existence of a solution to the principals problem, i.e. the existence of an optimal payment scheme $b^* : S \to \mathbb{R}$. Let

$$B : \Omega \times \mathcal{D}, (\omega, f_{S|\Omega}) \mapsto \int_S b^*(s) f_{S|\Omega}(s|\omega) \mathrm{d}s$$

denote the principal's ex ante expected cost for implementing ω under information structure $f_{S|\Omega}$. The following theorem, which was proven by Kim (1995) and Jewitt (1997), relates local informativeness with the ex ante expected implementing costs of some action ω.

Theorem 3.3. *Let $f_\Omega \in \Delta(\Omega)$ and $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega} \in \mathcal{D}$. Information structure $f_{S|\Omega}$ is locally more informative than the structure $\bar{f}_{\bar{S}|\Omega}$ if and only if $B(\omega, f_{S|\Omega}) \leq B(\omega, \bar{f}_{\bar{S}|\Omega})$ for all $\omega \in \Omega$.*

An information structure $f_{S|\Omega}$ is locally more informative than another, $\bar{f}_{\bar{S}|\Omega}$, if and only if the ex ante expected costs for implementing ω under the first system, $(f_{S|\Omega}, f_\Omega)$ are less or equal to those under the second system, $(\bar{f}_{\bar{S}|\Omega}, f_\Omega)$.

Since Grossman and Hart (1983) prove that the necessary part of Black-

[10] Given a payment scheme $b : S \to \mathbb{R}$ the agents optimization problem is: $\max_{\omega \in \Omega} \int_{\underline{s}}^{\bar{s}} u(b(s), \omega) \mathrm{d}s$ with corresponding FOC: $\int_{\underline{s}}^{\bar{s}} u(b(s)) {}^{\partial f_{S|\Omega}(s|\omega)}/_{\partial\omega}(s|\omega) \mathrm{d}s - w'(\omega) = 0$.

well's Theorem holds also in the principal-agent framework (i.e. $f_{S|\Omega} \succsim^{\mathrm{b}} \bar{f}_{\bar{S}|\Omega} \Rightarrow B(\omega, f_{S|\Omega}) \leq B(\omega, \bar{f}_{S|\Omega})$). Together with Theorem 3.3 this imply that local informativeness is weaker than sufficiency.

Cororally 3.2. *Let* $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega} \in \mathcal{D}$. *If* $f_{S|\Omega} \succsim^{\mathrm{b}} \bar{f}_{\bar{S}|\Omega}$ *then* $f_{S|\Omega} \succsim^{\text{l-inf}} \bar{f}_{\bar{S}|\Omega}$.

3.3 Information and the Dispersion of Posterior Expectations

Most parts of this section are based on Brandt et al. (2013) and Brandt et al. (2014). I am grateful to Prof. Dr. Bernhard Eckwert, Dr. Burkhard Drees and Dr. Felix Várdy for these collaborations.

The previous section discussed Blackwell's sufficiency criterion. Its intuition is than an information structure is more informative than another one, if a signal observation of the latter one is equal to an disturbed signal observation of the first one. Therefore, Blackwell's criterion is by definition independent of the prior belief.

This subsection introduces two different notions of informativeness, weak and strong informativeness, that follow a different intuition: the more informative an information system, the more 'aggressive' Bayesian updating and, hence, the more disperse the conditional expectations. Indeed, when signals are completely uninformative, beliefs are not updated at all. In that case, the posterior is equal to the prior, and the dispersion of the conditional expectation is zero. At the other extreme, perfectly informative signals fully reveal the state of the world. Thus, they induce 'complete' updating, which makes the dispersion of the posterior equal to the dispersion of the underlying states. In between, intermediate levels of informativeness lead to partial updating and, hence, tend to lead to intermediate levels of dispersion of posterior beliefs about the expected underlying state.

On the basis of this observation, some recent papers equate informativeness with the dispersion of conditional expectations. That is, they use dispersion orderings as full-fledged information concepts (compare Li (2012), Ganuza and Penalva (2010) and Eckwert and Zilcha (2008)). These applications illustrate the usefulness of dispersion orderings of conditional expectations as means to do economic analysis and, in particular, comparative statics excercises. However, an important question is whether these dispersion orderings really qualify as meaningful information criteria. As mentioned above, a desirable property for any information ranking is the ordinality of states property (OS).[11] Injective (increasing) transformations of the state space should not affect the informativeness and ranking of information systems, because the systems reveal the exact same information before and after the transformation. As shown above, the information concepts of Blackwell (1951), Lehmann (1988)/Persico (1996) and Kim (1995) satisfy this invariance property. However, the dispersion orders used by Ganuza and Penalva (2010) and others do not. A second problem with using dispersion orders as information criteria is that it is not clear what they mean in terms of the primitives of the model, i.e. the joint distribution of states and signals.

To remedy this problem and better understand the connection between information and the dispersion of conditional expectations, in this section, the weakest information criteria inducing some dispersion orders used in the literature will be derived. The starting points are the two dispersion concepts for conditional expectations of the state studied in Ganuza and Penalva (2010): supermodular dispersion and mean-preserving spread (MPS) dispersion. Then two information criteria are derived, each being compatible with one of these dispersion orders. The stronger criterion, which is compatible with supermodular dispersion, is denoted by 'strong informativeness' while the weaker criterion, which is compatible with MPS dispersion, is denoted by 'weak informativeness'.

More broadly, this section relates to the extensive literature on the effects of risk on individual behavior (see, e.g., Leland (1968); Sandmo (1971); Levhari and Weiss (1974)). In this context, Baker (2006) compares the effect of higher prior uncertainty versus a more informative signal on optimal de-

[11] The desirable, but more restrictive property of independence of states (IS), is not appropriate in this framework, since any non-monotone state transformation does not respect the underlying order of the state space. Hence, the monotonicity of information systems, a primitive of the model, gets lost under non-monotone state transformations.

cisions. For sequential decision problems, she identifies conditions such that the comparative statics of increasing informativeness are the same as those of increasing ex ante risk (dispersion of the prior). While Baker establishes a comparative statics similarity between higher risk and better information, she does not address the question whether risk and information are systematically related. Thus, both her focus and set-up are different from this work, even though the analyses are clearly related.

Closest to this section are the works by Ganuza and Penalva (2010) and by Eckwert and Zilcha (2008) suggesting that informativeness can simply be measured by the impact of (normalized) signals on the distribution of conditional expectations. This idea has also been applied in a recent paper by Li (2012) which tries to identify the effects of information and bias (relatedness between reviewer and applicant) on expert evaluations in the context of decisions about medical research grants. To separate these effects, one of her identifying assumptions ranks the variance of the expectations of posterior beliefs between 'unrelated' reviewers and 'related' reviewers.

The remainder of the subsection proceeds as follows: section 3.2.1 defines and discusses the two dispersion concepts for the expectations of posterior beliefs that are used as information orders by Ganuza and Penalva (2010). In section 3.2.2, two informativeness criteria are defined and graphically illustrated. Section 3.2.3 the information orders in section 3.2.2 are related to the dispersion orders in Section 3.2.1. In section 3.2.4 disentangles the effects of the prior and the information structure on informativeness and dispersion of posterior beliefs. Finally, section 3.2.5 concludes.

3.3.1 Supermodular- and MPS-Precision

Ganuza and Penalva (2010) define informativeness concepts for the class of information systems with monotone signals. They follow the intuition, that the more informative an information system, the more aggressive Bayesian updating and formalize this idea by equating informativeness with the dispersion of posterior state expectations. If, in addition, the information systems are monotone, a greater dispersion of posterior conditional expectation at least intuitively implies that signals and states are more correlated, i.e. small (high) signals are more likeli to co-occur in the state is small (high). it is meaningful to restrict attention to information systems which are monotone regardless of the prior (in the sense of Milgrom (1981)), i.e. to information systems with the MLRP. Moreover, since otherwise a comparison in terms of dispersion of posterior expectations is not senseful, the signal space as well as the marginal signal distributions must be equal across the set of information systems in consideration. Hence, attention is restricted to normalized information systems as introduced in section 2.4. Summing up, let $\Gamma(f_\Omega)$ denote the set of information structures with the monotone likelihood ratio property and normalized signals given the prior belief $f_\Omega \in \Delta(\Omega)$, i.e.

$$\Gamma(f_\Omega) := \left\{ f_{S|\Omega} : [0,1] \times \Omega \to \mathbb{R}_+ \;\middle|\; \begin{array}{l} f_{S|\Omega} \in \mathcal{M} \text{ with} \\ f_S(s) = \int\limits_\Omega f_{S|\Omega}(s|\omega) f_\Omega(\omega) \mathrm{d}\omega = 1 \ \forall \ s \in [0,1] \end{array} \right\}.$$

Moreover, define the set of all normalized information systems with MLRP by

$$\Gamma := \left\{ (f_{S|\Omega}, f_\Omega) | f_\Omega \in \Delta(\Omega) \text{ and } f_{S|\Omega} \in \Gamma(f_\Omega) \right\}. \tag{3.5}$$

Since signals are normalized, i.e. $S = \bar{S} = [0,1]$ for all $(f_{S|\Omega}, f_\Omega), (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega) \in \Gamma$, notation can be simplified: the signal set of an normalized information system is always denoted by $S := [0,1]$.

Building on Ganuza and Penalva (2010), supermodular- and mean preserving spread (MPS)-precision are defined as follows:

Definition 3.9. *Let* $(f_{S|\Omega}, f_\Omega), (\bar{f}_{S|\Omega}, \bar{f}_\Omega) \in \Gamma$.

(i) Information system $(f_{S|\Omega}, f_\Omega)$ *is more supermodular (SM) precise than* $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$*, denoted by* $(f_{S|\Omega}, f_\Omega) \stackrel{\text{sm}}{\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$*, iff* $\mathbb{E}_\Omega[\tilde{\omega}|s] - \bar{\mathbb{E}}_\Omega[\tilde{\omega}|s]$ *is non-decreasing in s for all* $s \in [0,1]$.

(ii) Information system $(f_{S|\Omega}, f_\Omega)$ *is more mean preserving spread (MPS) precise than* $(\bar{f}_{S|\Omega}, f_\Omega)$*, denoted by* $(f_{S|\Omega}, f_\Omega) \stackrel{\text{mps}}{\succsim} (\hat{f}_{S|\Omega}, f_\Omega)$*, iff* $\mathbb{E}_\Omega[\tilde{\omega}|\tilde{s}] - \mathbb{E}_\Omega[\tilde{\omega}]$ *is a MPS of* $\bar{\mathbb{E}}_\Omega[\tilde{\omega}|\tilde{s}] - \bar{\mathbb{E}}_\Omega[\tilde{\omega}]$.

Note that higher SM-dispersion uniformly raises the slope of the posterior state expectation as a function of the signal s. Therefore, SM precision implies MPS precision.

The dispersion of posterior expectations is related to informativeness in an intuitive sense: the greater the dispersion of posterior expectations the greater the correlation of signals and states. Indeed, Ganuza and Penalva (2010) use supermodular dispersion and MPS-dispersion as information concepts. Yet, while these concepts often provide convenient tools of analysis, they have two major shortcomings. First, the dispersion concepts are not based on primitives of the model, because they impose restrictions on the expectations of the posterior state distributions rather than on the information systems (i.e., the conditional signal distributions or the joint distribution of states and signals). And secondly, these concepts do not satisfy ordinality of states property. Even more problematic, a (ordinal) relabeling of the states that leaves all conditional signal distributions unchanged can turn around the ordering of information systems. This will be shown in the following example.

Example 2. Let $\Omega = [0,1]$ and $f_\Omega(\omega) = 2\omega$ for all $\omega \in [0,1]$. Consider the information systems $(f_{S|\Omega}, f_\Omega)$ and $(\bar{f}_{S|\Omega}, f_\Omega)$ given by $f_{S|\Omega}(s|\omega) = 1 + \frac{1}{2}(1-2s)(1-2\omega^2)$ and $\bar{f}_{S|\Omega}(s|\omega) = 1 + (1-2s)(1-\omega^2)(1-3\omega^2)$ for all $(s,\omega) \in [0,1]^2 = S \times \Omega$, respectively. Both information structures have the monotone likelihood ratio property (MLRP). Hence, by Milgrom (1981), $(f_{S|\Omega}, f_\Omega)$ and $(\bar{f}_{S|\Omega}, f_\Omega)$ are monotone for all prior f_Ω on Ω. Moreover, since

$$f_S(s) = \int_0^1 f_{S|\Omega}(s|\omega) f_\Omega(\omega) \mathrm{d}\omega = 1 = \int_0^1 \bar{f}_{S|\Omega}(s|\omega) f_\Omega(\omega) \mathrm{d}\omega = \bar{f}_S(s)$$

for all $s \in [0,1]$, follows $(f_{S|\Omega}, f_\Omega), (\bar{f}_{S|\Omega}, f_\Omega) \in \Gamma$. Computing the conditional state expectations yields that the difference of posterior, conditional state expectations,

$$\bar{\mathbb{E}}_\Omega\left[\tilde{\omega}|s\right] - \mathbb{E}_\Omega\left[\tilde{\omega}|s\right] = \int_0^1 \omega^2(6\omega^4 - 6\omega^2 + 1)(1-2s)\mathrm{d}\omega$$

$$= \frac{1-2s}{105}\left[(90\omega^4 - 126\omega^2 + 35)\omega^3\right]_{\omega=0}^{1} = \frac{2s-1}{105},$$

is (strictly) increasing in s. This implies $(\bar{f}_{S|\Omega}, f_\Omega) \overset{\text{sm}}{\succ} (f_{S|\Omega}, f_\Omega)$ and $(\bar{f}_{S|\Omega}, f_\Omega) \overset{\text{mps}}{\succ} (f_{S|\Omega}, f_\Omega)$.

Now consider the ordinal relabeling of states $t : [0,1] \to [0,1],\ \omega \mapsto \omega^4$ and define $\tilde{x} := t(\tilde{\omega})$. Then,

$$\bar{\mathbb{E}}_X\left[\tilde{x}|s\right] - \mathbb{E}_X\left[\tilde{x}|s\right] = \bar{\mathbb{E}}_\Omega\left[t(\tilde{\omega})|s\right] - \mathbb{E}_\Omega\left[t(\tilde{\omega})|s\right] = \int_0^1 \omega^5(6\omega^4 - 6\omega^2 + 1)(1-2s)\mathrm{d}\omega$$

$$= \frac{1-2s}{60}\left[(36\omega^4 - 45\omega^2 + 10)\omega^6\right]_{\omega=0}^{1} = \frac{1-2s}{60}$$

is (strictly) decreasing in s. Therefore, $(f_{S|t(\Omega)}, f_{t(\Omega)}) \overset{\text{sm}}{\succ} (\bar{f}_{S|t(\Omega)}, f_{t(\Omega)})$ and $(f_{S|t(\Omega)}, f_{t(\Omega)}) \overset{\text{mps}}{\succ} (\bar{f}_{S|t(\Omega)}, f_{t(\Omega)})$.

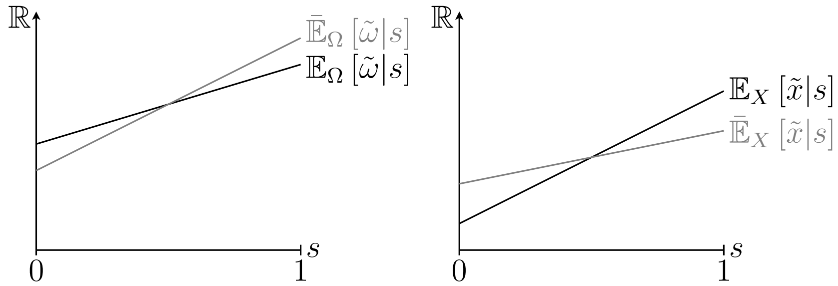

Figure 3.1: Conditional Expectations of $\tilde{\omega}$ and $\tilde{x} = t(\tilde{\omega})$.

The following proposition compares SM- and MPS-precision with the earlier approaches by Blackwell, Lehmann/Persico and Kim for the case of fixed priors. In particular, if the prior is fixed and the information structure has the MLRP, then MPS-precision is weaker than any other informativeness notion considered above. In contrast, sm-dispersion is neither weaker nor stronger than the informativeness concepts by Blackwell, Lehmann/Persico and Kim.

Proposition 3.5. *(i) Let $f_\Omega \in \Delta(\Omega)$. If the prior belief is fixed accross information systems, then MPS-precision is strictly weaker than sufficiency, accuracy and local informativeness. Formally,*

$$(f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega) \;\Rightarrow\; (f_{S|\Omega}, f_\Omega) \overset{mps}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega)$$

and

$$(f_{S|\Omega}, f_\Omega) \overset{mps}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega) \;\not\Rightarrow\; (f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega)$$

for all $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma(f_\Omega)$ if $x \in \{a,\ b\}$ and for all $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma(f_\Omega) \cap \mathcal{D}$ if $x =$ l-inf.

(ii) Let $f_\Omega \in \Delta(\Omega)$. Even if the prior is fixed accross information systems is SM-precision neither stronger nor weaker than sufficiency, accuracy and local informativeness. Formally,

$$(f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega) \;\not\Rightarrow\; (f_{S|\Omega}, f_\Omega) \overset{sm}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega)$$

and

$$(f_{S|\Omega}, f_\Omega) \overset{sm}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega) \;\not\Rightarrow\; (f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega)$$

for all $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma$ if $x \in \{a,\ b\}$ and for all $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma \cap \mathcal{D}$ if $x = li$.

This proposition is a combination of existing results.[12] Therefore, the proof is also a combination of those results: First, Ganuza and Penalva (2006) show that accuracy implies MPS-precision but does not imply SM-precision. Since sufficiency implies accuracy (see Cororally 3.1) and since accuracy and local informativeness are equivalent for information structures with MLRP (see Jewitt (1997)), this implies that both concepts, sufficiency

[12] See Hermelingmeier (2010) and below.

and local informativeness, imply MPS-precision. This in turn implies that the information structures in Example 2 are neither ordered in terms of sufficiency nor in terms of accuracy/local informativeness,[13] Consequently, neither MPS- nor SM-precision imply any of the other three information orders.

3.3.2 Strong and Weak Informativeness

In this section two information orders, strong and weak informativeness, are defined that satisfy the ordinality of states property. Additionally, they are linked to the dispersion orders of Definition 3.9. The approach to ranking the informational content about ω builds on the idea of Ganuza and Penalva (2010): the greater the dispersion of posterior conditional (state) expectation, the more correltated are signals and states and ,hence, the more informative is the information system. As mentioned above, the starting points are the two dispersion concepts studied in Ganuza and Penalva (2010): supermodular- and MPS-dispersion. As argued in the previous section 3.1, these concepts do not satisfy the OS property which causes problems when using them as informativeness criteria. This problem is solved by the following definition of weak and strong informativeness:

Definition 3.10 (Strong and Weak Informativeness)**.** *Let* $(f_{S|\Omega}, f_\Omega), (\bar{f}_{S|\Omega}, \bar{f}_\Omega) \in \Gamma$.

(i) Information system $(f_{S|\Omega}, f_\Omega)$ *is strongly more informative than* $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$*, denoted by* $(f_{S|\Omega}, f_\Omega) \succsim^{\text{s-inf}} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$*, iff*

$$\int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega') - \bar{f}_{S,\Omega}(s,\omega')d\omega'$$

[13]This follows from the fact that sufficiency and accuracy/local informativeness all imply MPS-precision for fixed but arbitrary prior belief (i.e. the prior is arbitrary but the same in both information systems). Since the systems considered in example 3 have equal prior and since their ordering in terms of MPS-precision depends on this prior belief it follows immediately that those systems cannot be ordered in terms of sufficiency or accuracy/local informativeness.

is non-increasing in $s \in [0,1]$ *for all* $\omega \in \Omega$.

(ii) Information system $(f_{S|\Omega}, f_\Omega)$ *is weakly more informative than* $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$*, denoted by* $(f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$*, iff*

$$F_{S,\Omega}(s,\omega) - sF_\Omega(\omega) \geq \bar{F}_{S,\Omega}(s,\omega) - s\bar{F}_\Omega(\omega)$$

for all $(s,\omega) \in [0,1] \times \Omega$.

The intuition for the weak informativeness criterion is as follows: Let the set $L_s := [0,s]$ corresponds to the information that the signal is smaller than s. According to Milgrom (1981) such information always represent bad news. Indeed, the conditional state distribution conditional on the information L_s is dominated by the prior state distribution, because $s\left[F_{\Omega|S}(\omega|L_s) - F_\Omega(\omega)\right] = F_{S,\Omega}(s,\omega) - sF_\Omega(\omega) \geq 0$.[14] In the same spirit, L_s is said to be better news under $(f_{S,\Omega}, f_\Omega)$ than under $(\bar{f}_{S,\Omega}, \bar{f}_\Omega)$, if

$$\begin{aligned} &F_{\Omega|S}(\omega|L_s) - F_\Omega(\omega) \leq \bar{F}_{\Omega|S}(\omega|L_s) - \bar{F}_\Omega(\omega) \;\; \forall\, \omega \in \Omega \\ \Leftrightarrow &F_{S,\Omega}(s,\omega) - sF_\Omega(\omega) \leq \bar{F}_{S,\Omega}(s,\omega) - s\bar{F}_\Omega(\omega) \;\; \forall\, \omega \in \Omega \end{aligned}$$

Hence, definition 3.10(ii) says that an information system $(f_{S|\Omega}, f_\Omega)$ is weakly more informative than $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$, iff any set of small signals, L_s, is worse news under the former system than under the latter one. Equivalently, all sets of high signals, $S\backslash L_s$, are better news under a weakly more informative systems.

To intuitively understand the strong informativeness criterion, notice that signal s is better news under $(f_{S|\Omega}, f_\Omega)$ than under $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$, iff $F_{\Omega|S}(\omega|s) - \bar{F}_{\Omega|S}(\omega|s) \leq 0 \;\forall\, \omega \in \Omega$. Likewise, an increase in s is a greater improvement of news under $(f_{S|\Omega}, f_\Omega)$ than under $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$, iff $F_{\Omega|S}(\omega|s) - \bar{F}_{\Omega|S}(\omega|s)$ is non-increasing in $s \;\forall\, \omega \in \Omega$. Now observe that $F_{\Omega|S}(\omega|s) - \bar{F}_{\Omega|S}(\omega|s) = \int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega') - \bar{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega'$, because signals are uniformly distributed on

[14] The inequality $F_{S,\Omega}(s,\omega) - sF_\Omega(\omega) \geq 0$ is implied by the MLRP: if $F_{S|\Omega}$ has the MLRP, then $F_{S,\Omega}(s,\omega)$ is a concave in s all $\omega \in \Omega$. Since $F_{S,\Omega}(s,\omega) = sF_\Omega(\omega)$ for $s = 0$ and $s = 1$ this implies $F_{S,\Omega}(s,\omega) \geq sF_\Omega(\omega)$ for all $(s,\omega) \in S \times \Omega$.

[0,1]. Hence, definition 3(i) says that an information system $(f_{S|\Omega}, f_\Omega)$ is strongly more informative than $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$, if and only if a rise in s constitutes a greater improvement of news under $(f_{S|\Omega}, f_\Omega)$ than under $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$. As their names suggest, the strong criterion implies the weak criterion. This follows from the definition of $(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$ and the fact that $\int_0^1 \int_{\underline{\omega}}^{\omega} [f_{S,\Omega}(s,\omega') - f_\Omega(\omega)] - \left[\bar{f}_{S,\Omega}(s,\omega') - \bar{f}_\Omega(\omega')\right] \mathrm{d}\omega' \mathrm{d}s = 0$.

For an illustration of weak and strong informativeness consider the following example which highlights their differences.

Example 3. Suppose $\Omega = [0,1]$ and fix $f_\Omega(\omega) = 1 \;\forall\; \omega$.

(i) First, consider the family of information systems $(f^\theta_{S|\Omega}, f_\Omega)_{\theta\in[0,1]}$ defined by

$$f^\theta_{S|\Omega}(s|\omega) = 1 + \theta(1-2s)(1-2\omega).$$

Then, $(\tilde{s},\tilde{\omega})_{\theta\in[0,1]}$ is distributed according to the Farlie-Gumbel-Morgenstern copula $F^\theta_{S,\Omega}(s,\omega) = C_\theta(s,\omega) = s\omega + \theta s\omega(1-s)(1-\omega)$. In particular, this implies $f^\theta_S(s) = 1$ for all $s \in [0,1]$ and $\theta \in [0,1]$. Moreover, since $\partial/\partial s \left(f^\theta_{S|\Omega}(s|\omega)/f^\theta_{S|\Omega}(s|\omega')\right) = 4\theta(\omega-\omega')/(f^\theta_{S|\Omega}(s|\omega'))^2 \leq 0$ for all $\omega' \geq \omega$, $f^\theta_{S|\Omega}$ has the MLRP and, hence, $(f^\theta_{S|\Omega}, f_\Omega) \in \Gamma$.

Next, observe that an increase in θ uniformly raises (lowers) the slope of $f^\theta_{S|\Omega}$ in s for ω high (low). Since $f^\theta_{S|\Omega}(s|\omega)$ is linear in s for all $\omega \in \Omega$, this implies for $\theta \geq \bar{\theta}$ that $f^\theta_{S|\Omega}(s|\omega) - f^{\bar{\theta}}_{S|\Omega}(s,\omega)$ is increasing (decreasing) in s for ω high (low). Consequently, $(f^\theta_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\succsim} (f^{\bar{\theta}}_{S|\Omega}, f_\Omega)$. Formally,

$$D_\omega(s) := \int_0^\omega f^\theta_{S,\Omega}(s,\omega') - f^{\bar{\theta}}_{S,\Omega}(s,\omega')\mathrm{d}\omega' = (\theta-\bar{\theta})(1-2s)(\omega-\omega^2)$$

is decreasing in s whenever $\theta \geq \bar{\theta}$. Hence, the family $(f^\theta_{S|\Omega}, f_\Omega)_{\theta\in[0,1]}$ is ordered in terms of strong informativeness: $(f^\theta_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\succsim} (f^{\bar{\theta}}_{S|\Omega}, f_\Omega) \Leftrightarrow\; \theta \geq \bar{\theta}$.

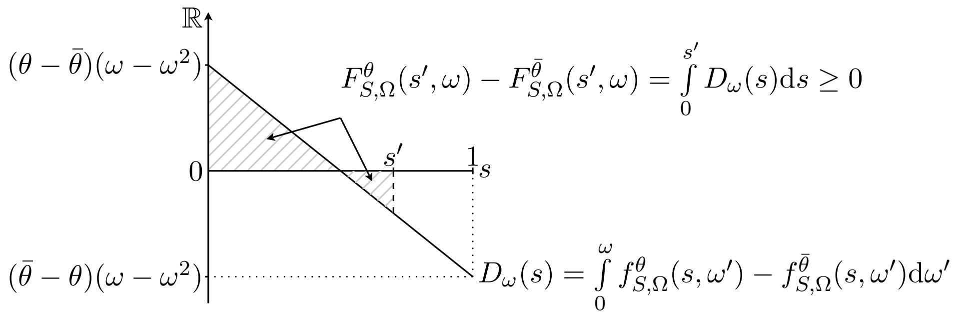

Figure 3.2: Farlie-Gumbel-Morgenstern Copula.

(ii) Now consider the family of information systems $((f^{\vartheta}_{S|\Omega},\Omega,S),f_\Omega)_{\vartheta\in[0,1]}$ with

$$f^{\vartheta}_{S|\Omega}(s|\omega)=\left[1+\vartheta(\vartheta\ln(s)\ln(1-\omega)-\ln(s(1-\omega))-1)\right]e^{-\vartheta\ln(s)\ln(1-\omega)}.$$

Then, $(\tilde{s},\tilde{\omega})_\vartheta$ are distributed according to the Gumbel-Barnett copula $F^{\vartheta}_{S,\Omega}(s,\omega)=C_\vartheta(s,\omega)=s-s(1-\omega)e^{-\vartheta\ln(s)\ln(1-\omega)}$. This implies that $f^{\vartheta}_S(s)=1$ for all $s\in[0,1]$ and $\vartheta\in[0,1]$. In order to proof that $f_{S|\Omega}$ has the MLRP consider

$$\frac{f^{\vartheta}_{S|\Omega}(s|\omega)}{f^{\vartheta}_{S|\Omega}(s|\omega')}=\underbrace{\frac{1+\vartheta(\vartheta\ln(s)\ln(1-\omega)-\ln(s(1-\omega))-1)}{1+\vartheta(\vartheta\ln(s)\ln(1-\omega')-\ln(s(1-\omega'))-1)}}_{=:k(s,\omega,\omega')}\underbrace{\frac{e^{-\vartheta\ln(s)\ln(1-\omega)}}{e^{-\vartheta\ln(s)\ln(1-\omega')}}}_{=:l(s,\omega,\omega')}.$$

Observe that for $\omega'\geq\omega$ it follows that

$$\frac{\partial k(s,\omega,\omega')}{\partial s}=\frac{\vartheta^3\left[\ln(1-\omega')-\ln(1-\omega)\right]}{\left[1+\vartheta(\vartheta\ln(s)\ln(1-\omega')-\ln(s(1-\omega'))-1)\right]^2 s}\leq 0$$

and that

$$\frac{\partial l(s,\omega,\omega')}{\partial s}=\frac{\vartheta\left[\ln(1-\omega')-\ln(1-\omega)\right]}{s}l(s,\omega,\omega')\leq 0.$$

Since $l(s,\omega,\omega')\geq 0$ and $k(s,\omega,\omega')\geq 0$ for all $(s,\omega,\omega')\in S\times\Omega^2$,[15] these

[15]Since $e^x\geq 0$ for all $x\in\mathbb{R}$, it is obvious that $l(s,\omega,\omega')\geq 0$ for all $(s,\omega,\omega')\in S\times\Omega^2$. Then, $k(s,\omega,\omega')\geq 0$ for all $(s,\omega,\omega')\in S\times\Omega^2$ follows from the fact that $k(s,\omega,\omega')$ is the product of the likelihood ratio $f^{\vartheta}_{S|\Omega}(s|\omega)/f^{\vartheta}_{S|\Omega}(s|\omega')\geq 0$ and $1/l(s,\omega,\omega')\geq 0$.

observations imply

$$\frac{\partial f^{\vartheta}_{S|\Omega}(s|\omega)/f^{\vartheta}_{S|\Omega}(s|\omega')}{\partial s} = k(s,\omega,\omega')\frac{\partial l(s,\omega,\omega')}{\partial s} + l(s,\omega,\omega')\frac{\partial k(s,\omega,\omega')}{\partial s} \leq 0$$

which shows that $f^{\vartheta}_{S|\Omega}$ has the MLRP and, hence, $(f^{\vartheta}_{S|\Omega}, f_\Omega) \in \Gamma$.

Concerning informativeness, observe that $C_\vartheta(s,\omega)$ is increasing in ϑ and, hence, $(f^{\vartheta}_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (f^{\bar{\vartheta}}_{S|\Omega}, f_\Omega) \Leftrightarrow \vartheta \geq \bar{\vartheta}$. However, in constrast to the information system considered in (i), $f^{\vartheta}_{S|\Omega}(s|\omega)$ is not linear in s or ω and, hence, it depends on ω and s whether an increase in ϑ increases or decreases the slope of $f^{\vartheta}_{S|\Omega}(s|\omega)$. Consequently, $(f^{\vartheta}_{S|\Omega}, f_\Omega)_{\vartheta\in[0,1]}$ can not be ordered by strong informativeness.[16]

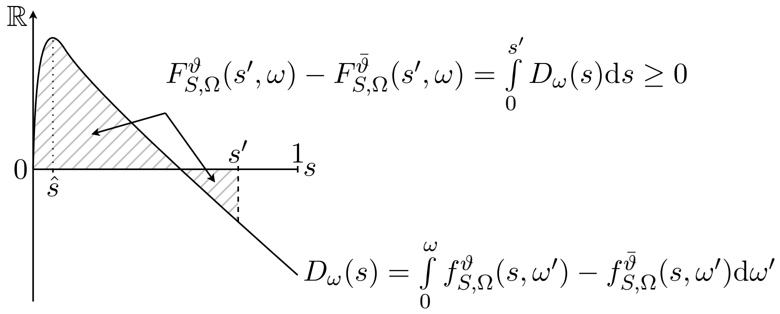

Figure 3.3: Gumbel-Barnett-Copula.

The following proposition establishes that none of the two information criteria rank an information system above the fully informative system or below the fully uninformative one.

Proposition 3.6. *The weak and the strong informativeness criterion satisfy (P0′), (P1)-(P3) and OS.*

Suppose for the moment that $f_\Omega = \bar{f}_\Omega$. Since signals are already normalized, this implies that the marginal distributions of states and signals are identical across information systems. Hence, the strong and weak information criteria can be expressed in terms of properties of the copulas

[16] For a formal treatment consider the continuation of this example below.

associated with $(\tilde{s},\tilde{\omega})$, $C_{S,\Omega}(s,v) = F_{S,\Omega}(s,F_\Omega^{-1}(v))$, $(s,v) \in [0,1]\times[0,1]$. Since $C_{S,\Omega}(s,F_\Omega(\omega)) = F_{S,\Omega}(s,\omega)$, definition 3.10 implies

$$(f_{S|\Omega},f_\Omega) \overset{\text{w-inf}}{\succsim} (\bar{f}_{S|\Omega},\bar{f}_\Omega) \Leftrightarrow C_{S,\Omega}(s,F_\Omega(\omega)) \geq \bar{C}_{S,\Omega}(s,\bar{F}_\Omega(\omega)) \text{ for all } (s,\omega)\in[0,1]\times\Omega,$$

while

$$(f_{S|\Omega},f_\Omega) \overset{\text{s-inf}}{\succsim} (\bar{f}_{S|\Omega},\bar{f}_\Omega) \Leftrightarrow C_{S,\Omega}(s,F_\Omega(\omega)) - \bar{C}_{S,\Omega}(s,\bar{F}_\Omega(\omega)) \text{ is concave in } s\ \forall\ \omega\in\Omega.$$

As examples for these characterizations of weak and strong informativeness in terms of properties of copulas consider again the Farlie-Gumbel-Morgenstern copula and the Gumbel-Barnett-Copula.

Example 4 (Continued)**.** (i) Again, consider the family of information systems $(f^\theta_{S|\Omega},f_\Omega)_{\theta\in[0,1]}$ as defined in Example 3. From the first part of Example 3 it is $(f^\theta_{S|\Omega},f_\Omega) \overset{\text{s-inf}}{\succsim} (f^{\bar{\theta}}_{S|\Omega},f_\Omega) \Leftrightarrow\ \theta\geq\bar{\theta} \Leftrightarrow \frac{\partial^2}{\partial s^2}\left[C_\theta(s,\omega)-C_{\bar{\theta}}(s,\omega)\right] = -2\omega(1-\omega)(\theta-\bar{\theta})\geq 0$.

(ii) Next, consider again the family $((f^\vartheta_{S|\Omega},\Omega,S),f_\Omega)_{\vartheta\in[0,1]}$ as defined above. From the first part of this example it is known that $(f^\vartheta_{S|\Omega},f_\Omega) \overset{\text{w-inf}}{\succsim} (f^{\bar{\vartheta}}_{S|\Omega},f_\Omega)$ $\Leftrightarrow \vartheta\geq\bar{\vartheta}$. Moreover, without a formal proof, it was claimed the $\vartheta\geq\bar{\vartheta} \not\Rightarrow$ $(f^\vartheta_{S|\Omega},f_\Omega) \overset{\text{s-inf}}{\succsim} (f^{\bar{\vartheta}},f_\Omega)$. With the characterization of strong informativeness in terms of copulas, the formal proof of the statement is quite easy: For $\vartheta>\bar{\vartheta}$ it is $\frac{\partial^2}{\partial s^2}(C_\vartheta(s,\omega)-C_{\bar{\vartheta}}(s,\omega)) \underset{\leq}{\overset{\geq}{}} 0 \Leftrightarrow\ s \underset{\geq}{\overset{\leq}{}} \hat{s} := \left(\frac{\bar{\vartheta}(\bar{\vartheta}\ln(1-\omega)-1)}{\vartheta(\vartheta\ln(1-\omega)-1)}\right)^{-\frac{1}{(\vartheta-\bar{\vartheta})\ln(1-\omega)}} \geq 0$. And hence, $(f^\vartheta_{S|\Omega},f_\Omega) \overset{\text{s-inf}}{\not\succsim} (f^{\bar{\vartheta}}_{S|\Omega},f_\Omega)$.

Relating strong and weak informativeness with the dispersion of posterior expectations

Definition 3.9 introduced two dispersion concepts for the comparison of posterior conditional (state) expectations. The next proposition characterizes the relationship between strong and weak informativeness and these dispersion concepts.

Proposition 3.7. *Let $f_{S|\Omega} \in \Gamma(f_\Omega)$ and $\bar{f}_{S|\Omega} \in \Gamma(\bar{f}_\Omega)$.*

(i) $(f_{S|\Omega}, f_\Omega) \succsim^{\text{s-inf}} (\bar{f}_{S|\Omega}, \bar{f}_\Omega) \Leftrightarrow (f_{S|t(\Omega)}, f_{t(\Omega)}) \succsim^{\text{sm}} (\bar{f}_{S|t(\Omega)}, \bar{f}_{t(\Omega)})$ *for all strictly increasing* $t: \Omega \to \mathbb{R}$.

(ii) $(f_{S|\Omega}, f_\Omega) \succsim^{\text{w-inf}} (\bar{f}_{S|\Omega}, \bar{f}_\Omega) \Leftrightarrow (f_{S|t(\Omega)}, f_{t(\Omega)}) \succsim^{\text{mps}} (\bar{f}_{S|t(\Omega)}, \bar{f}_{t(\Omega)})$ *for all strictly increasing* $t: \Omega \to \mathbb{R}$.

Proposition 3.7 establishes a tight relationship between information orders and dispersion orders of conditional expectations. Broadly speaking, under a more informative structure the posterior state densities react more sensitively to changes of signals and, hence, the conditional expectation of the state (or monotone function thereof) is more dispersed. In other words, conditional expectations are more dispersed when systems are more informative.

One may of a relabeling of states as a strictly increasing utility function defined on the state space. With this interpretaation in mind, Proposition 3.7 has the important implication that even when expected utility maximizers have different increasing vNM-preferences and different priors, they will nevertheless share a common view on dispersion comparisons with respect to conditional expected state utilities if the information systems in question can be ordered by strong or weak informativeness. Even stronger, regardless of preferences and priors,

- an information system becomes strongly more informative if and only if a higher signal induces a larger gain in expected utility.
- an information system becomes weakly more informative if and only if (normalized) conditional expected utilities become more MPS-dispered.

Comparison with other informativeness concepts

This part deals with the question how strong and weak informativeness relate to the other informativeness concepts considered in this work. Since all informativeness concepts in section 3.2 satisfies at least OS, the Propositions 3.5 and 3.7 imply that for equal but arbitrary prior, weak informativeness is weaker than the criteria by Blackwell, Lehmann/Persico and Kim.

Cororally 3.3. *Let* $f_\Omega \in \Delta(\Omega)$.

(i) $(f_{S|\Omega}, f_\Omega) \succsim^{x} (\bar{f}_{S|\Omega}, f_\Omega) \;\Rightarrow\; (f_{S|\Omega}, f_\Omega) \succsim^{\text{w-inf}} (\bar{f}_{S|\Omega}, f_\Omega)$ *for all* $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma(f_\Omega)$ *and* $x \in \{s\text{-}inf, a, b\}$.

(ii) $(f_{S|\Omega}, f_\Omega) \succsim^{\text{l-inf}} (\bar{f}_{S|\Omega}, f_\Omega) \;\Rightarrow\; (f_{S|\Omega}, f_\Omega) \succsim^{\text{w-inf}} (\bar{f}_{S|\Omega}, f_\Omega)$ *for all* $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma(f_\Omega) \cap \mathcal{D}$.

In contrast, for strong informativeness the same arguments (Propositions 3.5 and 3.7 plus OS of the informativeness concpets by Blackwell, Lehmann/Persico and Kim) implies that it is neither stronger nor weaker than the criteria by Blackwell, Lehmann/Persico and Kim.

The value of strong and weak informativeness

The next natural question is how weak and strong informativeness relate to a decision maker's ex ante expected utility. Indeed, weak informativeness plus equal priors characterizes higher ex ante expected welfare for all supermodular decision problems. In order to define supermodular decision problems, assume for the moment $\mathcal{A} \subseteq \mathbb{R}$. Then, a decsision problem is supermodular if the indirect elementary utility function $v(a, \omega) = u(o(a, \omega))$ is supermodular in (a, ω). An indirect elementary utility function $v : \mathcal{A} \times \Omega \to \mathbb{R}$ is supermodular in (a, ω) if and only if the incremental returns

$$r(\omega) := v(a', \omega) - v(a, \omega)$$

is non-decreasing in ω for all $a', a \in \mathcal{A}$ with $a' \geq a$.[17] Denote the class of supermodular indirect utility functions by $\mathcal{R}$. Simple examples for supermodular objective functions are the profit function of a firm with risky output or a coordination game in which one player is nature which randomly chooses an action. More examples can for instance be found in the books by Topkis (1998) and Cooper (1999).

[17]Remark: If $v : \mathcal{A} \times \Omega \to \mathbb{R}$ is twice differentiable in both arguments, supermodularity is also characterized by a positive cross-derivative, i.e. $\frac{\partial^2 v(a,\omega)}{\partial a \partial \omega} \geq 0$.

Theorem 3.4. *Let $f_{S|\Omega} \in \Gamma(f_\Omega)$ and $\bar{f}_{S|\Omega} \in \Gamma(\bar{f}_\Omega)$. The information system $(f_{S|\Omega}, f_\Omega)$ is more valuable than the system $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$ for all decision makers with supermodular utility functions if and only if $(f_{S|\Omega}, f_\Omega) \succsim^{\text{w-inf}} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$ and $F_\Omega(\omega) = \bar{F}_\Omega(\omega)\ \forall\ \omega \in \Omega$. Formally,*

$$\left[\begin{array}{l} V((f_{S|\Omega}, f_\Omega), a^*, v) \geq V((\bar{f}_{S|\Omega}, \bar{f}_\Omega), \bar{a}^*, v) \\ \textit{for all supermodular } v : \mathcal{A} \times \Omega \to \mathbb{R} \end{array}\right] \Leftrightarrow \left[\begin{array}{l} (f_{S|\Omega}, f_\Omega) \succsim^{\text{w-inf}} (\bar{f}_{S|\Omega}, \bar{f}_\Omega) \\ \textit{and } F_\Omega(\omega) = \bar{F}_\Omega(\omega)\ \forall\ \omega \in \Omega \end{array}\right].$$

An increase in weak informativeness plus equal priors increases the joint probability of signals smaller (greater) than s and states smaller (greater) than ω, $F_{S,\Omega}(s, \omega)$ $(1 - F_S(s) - F_\Omega(\omega) + F_{S,\Omega}(s, \omega))$. Hence, the ability to coordinate small (high) actions with small (high) states increases as weak informativeness increases and the prior keeps constant. If the indirect utility is supermodular in (a, ω), exactly this coordination of actions and states is the goal of the decision maker. Hence, weak informativeness plus equal priors characterize more valuable for all decision makers with supermodular payoff functions.

Remark. *If the prior is fixed the weak information criterion simplifies to $F_{S,\Omega}(s, \omega) \geq \bar{F}_{S,\Omega}(s, \omega)$ for all $(s, \omega) \in S \times \Omega$. This is equivalent to the MIO-ND condition in the theorem by Athey and Levin (2001). For fixed priors, their result is slightly more general than proposition 3.4. They characterize 'more valuable' for utility functions with various types of incremental returns (i.e. different curvatures of $r(\omega) = v(a, \omega) - v(a', \omega)$, $a, a' \in \mathcal{A}$, $a \geq a'$) in terms of their MIO condition. In particular, for the utility functions with non-decreasing incremental returns, i.e. supermodular utility functions, their MIO-ND condition is equal to weak informativeness for fixed priors.*

Now consider strong informativeness. Since weak informativeness is implied by strong informativeness, a direct consequence of Theorem 3.4 is that strong informativeness is sufficient to guarantee higher ex ante expected welfare for all decision makers with supermodular preferences. In contrast to this, the next Proposition is an impossibility result which shows that, if the prior is fixed, there exists no class of utility functions such that the strong information criterion is necessary for a comparison of information systems in terms of their ex ante value for this class of utilities. Formally,

Proposition 3.8. *Let $f_\Omega \in \Delta(\Omega)$ and $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma(f_\Omega)$. There exists no class of payoff functions $\mathcal{U}$ such that:*

$$V((f_{S|\Omega}, f_\Omega), a^*, u) \geq V((\bar{f}_{S|\Omega}, f_\Omega), \bar{a}^*, u) \; \forall \; u \in \mathcal{U} \;\Rightarrow\; (f_{S|\Omega}, f_\Omega) \stackrel{\text{s-inf}}{\succsim} (\bar{f}_{S,\Omega}, f_\Omega). \quad (3.6)$$

There is no class of utility functions such that the strong informativeness criterion is a necessary condition for an order of information systems with respect to their ex ante value. The reason is that strong informativeness is not weaker than sufficiency, which in turn is equivalent to an order of information systems in terms of their ex ante value for all expected utility maximizers. In other words, strong informativeness is too restrictive for being necessary for an order of information systems in terms of their ex ante value.

The Role of the Prior and the Information Structure

The informativeness and dispersion properties of an information system are determined jointly by the prior and the structure of an information system. In particular, the informativeness of an information system with a fixed structure typically varies under different priors. Informativeness depends on the statistical correlation between signals and states of nature which changes with the prior even when the information structure remains the same. Consider, for example, an information structure $f_{S|\Omega}$ that associates with all states $\omega \leq \omega_0$ the same conditional signal distribution, while it associates different conditional signal distributions with states $\omega \geq \omega_0$. If the prior f_Ω is concentrated on $[\underline{\omega}, \omega_0]$ then the information system $(f_{S|\Omega}, f_\Omega)$ is fully uninformative, while it becomes (partially) informative otherwise. Hence, it is important look at both - the impact of the prior on the informativeness of an information system while the structure is fixed (testing the structure for different prior) and, conversely, the impact of the structure on the informativeness of the system while keeping the prior fixed. For this purpose, the analysis first looks at the impact of a change in the prior while the information structure is fixed. Next, the prior is fixed and the analysis disentangles the impact of the information structure on the informativeness of the system.

Keeping the Information Structure fixed

Intuitively, if the prior becomes more dispersed, e.g., in the sense of a mean-preserving spread or fatter tails, then a high signal constitutes better news due to the greater upward potential offered by the prior distribution. Similarly, as the downward potential of the prior has also increased, a low signal constitutes worse news. If this intuition is correct, then a more dispersed prior should lead to higher dispersion of the information system. This can, however, be quite misleading, as is demonstrated by the following example. The example illustrates that under a fixed information structure, $f_{S|\Omega}$, a robust relationship between the dispersion of the prior, f_Ω, and the informativeness of the associated information system, $(f_{S|\Omega}, f_\Omega)$, does not exist. More precisely, a more dispersed prior does not necessarily lead to more dispersion of the conditional expectation, or higher informativeness of the system. Indeed, it will be shown that, for suitably chosen $(f_{S|\Omega}, f_\Omega)$, a mean preserving spread (MPS) of the prior raises the expectation of any monotone increasing transformation $t : \Omega \to \mathbb{R}$ of $\tilde{\omega}$ conditional on the lowest signal $s = 0$. This implies, of course, that an information system with structure $f_{S|\Omega}$ does not become more MPS-disperse or weakly more informative under a more MPS dispersed prior. A fortiori, the system does not become strongly more informative.

Example 5. Let $f_\Omega,\ \bar{f}_\Omega \in \Delta\Omega$ and assume that $\bar{f}_\Omega$ differs from f_Ω by a MPS, i.e. $\mathbb{E}_\Omega\left[\tilde{\omega}\right] = \bar{\mathbb{E}}_\Omega\left[\tilde{\omega}\right]$ and $\int_{\underline{\omega}}^{\omega} F_\Omega(\omega')\mathrm{d}\omega' \leq \int_{\underline{\omega}}^{\omega} \bar{F}_\Omega(\omega')\mathrm{d}\omega'$ for all $\omega \in \Omega$. Further assume that there exists $\hat{\omega} \in (\underline{\omega}, \bar{\omega})$ such that $F_\Omega : \Omega \to \mathbb{R}$ is strictly concave and $\bar{F}_\Omega$ is strictly convex on $[\underline{\omega}, \hat{\omega}]$ (cf. Figure 3.4 below).

Define $\rho(\omega) := {}^{F_\Omega(\omega)}/_{F_\Omega(\hat{\omega})}$ and $\bar{\rho}(\omega) := {}^{\bar{F}_\Omega(\omega)}/_{\bar{F}_\Omega(\hat{\omega})}$. Clearly, $\rho(\underline{\omega}) = \bar{\rho}(\underline{\omega}) = 0$ and $\rho(\hat{\omega}) = \bar{\rho}(\hat{\omega}) = 1$. Since $\rho(\omega)$ is strictly concave and $\bar{\rho}(\omega)$ is strictly convex on $[\underline{\omega}, \hat{\omega}]$ this implies

$$\rho(\omega) > \bar{\rho}(\omega) \ \ \forall\, \omega \in (\underline{\omega}, \hat{\omega}).$$

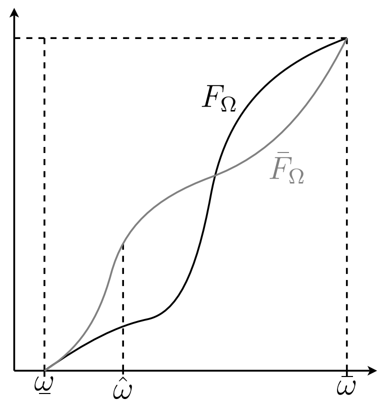

Figure 3.4: F_Ω and $\bar{F}_\Omega$.

The information structure is defined as follows: Let $\zeta : \Omega \to [0,1]$ with

$$\zeta(\omega) = \begin{cases} 0 & \text{if } \omega \in [\underline{\omega}, \hat{\omega}] \\ 1 & \text{if } \omega \in (\hat{\omega}, \bar{\omega}]. \end{cases}$$

Then the information structure is given by

$$f_{S|\Omega}(s|\omega) = 1 + \zeta(\omega)\beta(s), \tag{3.7}$$

where $\beta : [0,1] \to [-1,1]$ is an increasing function with $\beta(0) = -1$, $\beta(1) = 1$, and $\int_0^1 \beta(s)\mathrm{d}s = 0$. Therefore, the ratio

$$\frac{f_{S|\Omega}(s|\omega)}{f_{S|\Omega}(s|\omega')} = \frac{1 + \zeta(\omega)\beta(s)}{1 + \zeta(\omega')\beta(s)} = \begin{cases} 1 & \text{if } \omega, \omega' \in [\underline{\omega}, \hat{\omega}] \text{ or } \omega, \omega' \in (\hat{\omega}, \bar{\omega}] \\ \frac{1}{1+\beta(s)} & \text{if } \omega \in [\underline{\omega}, \hat{\omega}] \text{ and } \omega' \in (\hat{\omega}, \bar{\omega}] \end{cases}$$

is (weakly) decreasing in s for all $\omega \leq \omega'$ and, hence, the information structure has the MLRP. Finally, to make sure that the signals ex ante are uniformly distributed, the signals need to be normalized under both priors (compare chapter 2). For this purpose define

$$\tilde{s}^n := F_S(\tilde{s}) \text{ and } \tilde{\bar{s}}^n := \bar{F}_S(\tilde{s}),$$

where the marginal cdf's of the signals, $F_S : S \to [0,1]$ and $\bar{F}_S : S \to [0,1]$, are defined as in equation (2.6).

Now consider the normalized signal realization $s^n = 0 = \bar{s}^n$, which corresponds to $s = 0 = \bar{s}$. Equation (3.7) implies

$$F_{\Omega|S^n}(\omega|s^n = 0) = F_{\Omega|S}(\omega|s = 0) = \int_{\underline{\omega}}^{\omega} f_{S|\Omega}(\omega'|s = 0)\mathrm{d}\omega'$$

$$\overset{(3.7)}{=} \frac{1}{F_\Omega(\hat{\omega})} \int_{\underline{\omega}}^{\min\{\omega,\hat{\omega}\}} f_\Omega(\omega')\mathrm{d}\omega' = \begin{cases} \rho(\omega) & \text{if } \omega \leq \hat{\omega} \\ 1 & \text{else} \end{cases}$$

$$\geq \begin{cases} \bar{\rho}(\omega) & \text{if } \omega \leq \hat{\omega} \\ 1 & \text{else} \end{cases} = \frac{1}{\bar{F}_\Omega(\hat{\omega})} \int_{\underline{\omega}}^{\min\{\omega,\hat{\omega}\}} \bar{f}_\Omega(\omega')\mathrm{d}\omega'$$

$$\overset{(3.7)}{=} \int_{\underline{\omega}}^{\omega} \bar{f}_{S|\Omega}(\omega'|\bar{s} = 0)\mathrm{d}\omega' = \bar{F}_{\Omega|S}(\omega|\bar{s} = 0) = \bar{F}_{\Omega|S^n}(\omega|\bar{s}^n = 0).$$

Hence, $\bar{F}_{\Omega|S^n}(\cdot|\bar{s} = 0)$ strictly dominates $F_{\Omega|S^n}(\cdot|s^n = 0)$ in the sense of first-order stochastic dominance. This implies

$$\mathbb{E}_\Omega[\tilde{\omega}|s^n = 0] - \mathbb{E}_\Omega[\tilde{\omega}] < \bar{\mathbb{E}}_\Omega[\tilde{\omega}|\bar{s}^n = 0] - \bar{\mathbb{E}}_\Omega[\tilde{\omega}].$$

Thus, conditional on the lowest signal, the riskier distribution has a higher expectation than the less risky one. This shows that $\bar{\mathbb{E}}_\Omega[\tilde{\omega}|\tilde{\bar{s}}^n] - \bar{\mathbb{E}}_\Omega[\tilde{\omega}]$ is not a MPS of $\mathbb{E}_\Omega[\tilde{\omega}|\tilde{s}^n] - \mathbb{E}_\Omega[\tilde{\omega}]$ and, hence, $(f_{S|\Omega}, \bar{f}_\Omega) \not\succsim^{\text{w-inf}} (f_{S|\Omega}, f_\Omega)$.

The intuition behind the result is that, conditional on the lowest signal, the riskier distribution dominates the less risky distribution in the first-order sense. According to the information structure in (3.7), under the lowest signal all conditional probability density is shifted proportionally towards $[\underline{\omega}, \omega_1]$. By assumption, $\bar{f}_\Omega$ is increasing and f_Ω is decreasing on $[\underline{\omega}, \omega_1]$. Under $\bar{f}_\Omega$, therefore, high states benefit more from the conditional probability shift than low states; and under f_Ω, low states benefit more than high states. This explains why $\bar{F}_{\Omega|S^n}(\cdot|0)$ dominates $F_{\Omega|S^n}(\cdot|0)$ in terms of first-order stochastic dominance.

The example shows that a MPS of the prior does not necessarily result in more MPS-dispersion of the conditional expectations and, hence, does not result in an increase in terms of weak informativeness. The same holds true with respect to sm-dispersion and strong informativeness, because sm-dispersion and strong informativeness are stronger than MPS-dispersion and weak informativeness, respectively.

Randomizing the Prior

The above example has shown that a more dispersed prior does not necessarily translate into higher dispersion of the conditional expectation or higher informativeness of an information system. In this section it will be shown that some such transformations of the prior do have analogues in terms of dispersion and informativeness. Consider two information systems with the same structure $f_{S|\Omega}$, and assume that these systems can be ranked with respect to (weak or strong) informativeness. Below it is shown that 'randomization' of the priors yields a new information system with intermediate (weak or strong) informativeness and dispersion, respectively. To ensure that this is a meaningful exercise, first it is established that, for a fixed information structure, $f_{S|\Omega}$, the projection of Γ on the priors yields a convex set.

Lemma 3.2. *The set*

$$\Delta(f_{S|\Omega}) := \big\{ f_\Omega \in \Delta\Omega | (f_{S|\Omega}, f_\Omega) \in \Gamma \big\}$$

of all priors of information systems in Γ with structure $f_{S|\Omega}$ is convex.

For given $f_\Omega, \bar{f}_\Omega \in \Delta(f_{S|\Omega})$, $\hat{f}_\Omega := \alpha f_\Omega + (1-\alpha)\bar{f}_\Omega$ is called 'randomized' prior.[18] The next proposition shows that the process of randomizing the prior of two ordered information systems with the same structure leads to an intermediate level of informativeness.

Proposition 3.9. *Let $f_\Omega, \bar{f}_\Omega \in \Delta(f_{S|\Omega})$, $\alpha \in [0,1]$, and define $\hat{f}_\Omega := \alpha f_\Omega +$ $(1-\alpha)\bar{f}_\Omega$. If $(f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (f_{S|\Omega}, \bar{f}_\Omega)$ then $(f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (f_{S|\Omega}, \hat{f}_\Omega) \overset{x}{\succsim} (f_{S|\Omega}, \bar{f}_\Omega)$ for $x \in \{$w-inf, s-inf$\}$.*

[18] Such 'randomized' prior play an important role in the theory on Knigthian uncertainty.

In combination with Proposition 3.7, Proposition 3.9 implies that the system $(f_{S|\Omega}, \hat{f}_\Omega)$ with a randomized prior not only exhibits intermediate informativeness but also intermediate dispersion of posterior conditional expectations.

Keeping the Prior fixed

This section studies information structures that can be ordered in terms of informativeness for fixed priors. In economic applications, signals are often generated from the states of nature by adding a noise term, i.e., $\tilde{s} = \omega + \tilde{\epsilon}$. In such a setting, the conditional dispersion of the signal coincides with the dispersion of the noise term. The signal is fully informative, if it has zero conditional dispersion (i.e., when the noise term is a constant). Otherwise, higher conditional signal dispersion reduces the informativeness of the system, because it makes the signal noisier. Thus, with an additive information structure in mind, it seems that conditional signal dispersion is inversely related to the informativeness of the signal.

Yet, in more general informational settings, this intuition is no longer accurate. For instance, a system is uninformative whenever all states generate the same conditional signal distribution. In that case, conditional signal dispersion and informativeness are unrelated. The following considers suitably restricted classes of information structures within which informativeness and conditional signal dispersion are, in fact, inversely related. These classes include all randomizations (i.e., convex combinations) of pairs of information systems with identical priors that can be ranked in terms of informativeness.

Following similar lines as in the proof of lemma 3.2, it can be verified that $\Gamma(f_\Omega)$, the set of normalized information structures with MLRP, is a convex set.

Lemma 3.3. *For fixed prior $f_\Omega \in \Delta\Omega$, the set $\Gamma(f_\Omega)$ of all normalized information structures with MLRP is convex.*

By Lemma 3.3, if the prior is fixed then the set of normalized structures with MLRP is closed under 'randomization'. Moreover, randomizing the signals of any two systems (with structures in $\Gamma(f_\Omega)$) that can be ranked in terms of informativeness yields a new information system with intermediate informativeness and intermediate dispersion.

Proposition 3.10. *Let* $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma(f_\Omega)$, $\alpha \in [0,1]$, *and define* $\hat{f}_{S|\Omega} := \alpha f_{S|\Omega} + (1-\alpha)\bar{f}_{S|\Omega}$. *If* $(f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega)$ *then* $(f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (\hat{f}_{S|\Omega}, f_\Omega)$ *and* $(\hat{f}_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega)$ *for* $x \in \{$*w-inf, s-inf*$\}$.

In combination with Proposition 3.7, Proposition 3.10 implies that the system with the randomized information structure exhibits intermediate dispersion. Moreover, as any system $(f_{S|\Omega}, f_\Omega)$ is strongly more informative than the uninformative system $(f^0_{S^0|\Omega}, f_\Omega)$, it follows from Proposition 3.10 that randomizing the structure of any information system with the uninformative strutcture $f^0_{S^0|\Omega}$ reduces both informativeness and dispersion of the system. This holds true for both dispersion concepts in Definition 3.9 and both informativeness concepts in Definition 3.10.

3.4 Concluding Remarks

This chapter presented some fundamental and desirable properties of informativenss criteria and introduced different notions of informativeness. Blackwell's sufficiency criterion (Blackwell (1951, 1953)) is statistically motivated and it is based on the idea of that a signal observation of a less informative system is equal to the distorted observation of a more informative system. Sufficiency is linked with the value of information through the strong equivalence that an information system is sufficient for another one if and only if every expected utility maximizer is better of under the first system than under the latter one. Since sufficiency is very restrictive, Lehmann (1988)/Persico (1996, 2000) and Kim (1995) proposed weaker criteria which link informativeness with the value of information for smaller classes of expected utility maximizers. All these three criteria are from the traditional literature on economics where the prior belief is typically kept fixed. In contrast, in the second part of the chapter it is argued that informativeness is jointly determined by an information structure and the prior belief. In order to take this into account two new information criteria, weak and strong informativeness (Brandt et al. (2013, 2014)), are defined which are based on the idea, the more informative an information system the more spread out are the posterior conditional state expectations. It is shown that, if the prior is fixed again, these criteria are more valuable for all decision makers with supermodular preferences.

Chapter 4

The Value of Information

As mentioned earlier, a decision maker cares about an information system only in so far as his wellbeing is affected. This chapter studies the impact of information on individual ex ante expected utility in different economic frameworks. In particular, the current chapter deals with the *value of information*. Recall from Definition 3.2 that an information system is more valuable for a decision maker if it delivers him higher ex ante expected utility. In particular, for a decision maker with indirect utility $v : \mathcal{A} \times \Omega \to \mathbb{R}$ and prior belief f_Ω is an information system $(f_{S|\Omega}, f_\Omega)$ more valuable than an information system $(\bar{f}_{\bar{S}|\Omega}, f_\Omega)$ iff

$$V((f_{S|\Omega}, f_\Omega), a^*, v) \geq V((\bar{f}_{\bar{S}|\Omega}, f_\Omega), \bar{a}^*, v),$$

where $V((f_{S|\Omega}, f_\Omega), a^*, v)$ and $V((f_{S|\Omega}, f_\Omega), a^*, v)$ denote the ex ante expected utilities of an agent with indirect utility v under the systems $(f_{S|\Omega}, f_\Omega)$ and $(\bar{f}_{\bar{S}|\Omega}, f_\Omega)$, respectively.[19] This notion of 'more valuable' defines a preorder on the set of information structures.

It is quite intuitive that better information lead to higher welfare, because better information reduces the risk to a larger extent and, hence, improves decision making. Conversely, it is not so clear when more valuable (for a certain class of decision makers) implies more informative in the some sense. The current chapter studies this in two different economic frameworks.

[19] Compare Def. 3.2.

First, in section 4.1 the value of costly information is analyzed. In this framework a more precise information system reduces the decision makers budget. Therefore, increasing informativeness has two contrary effects: a precision effect and a budget effect. The precision effect is that ‘better’ information improves the decision maker’s choice and, therefore, increases his expected utility. The budget effect decreases the decision maker’s budget and, hence, has negative impact on the decision maker’s expected utility.

Secondly, section 4.2 studies the value of information in a complete risk sharing market. For this purpose, section 4.2.1 presents conditions under which risk avers consumers and firms fully insure. Building on this, section 4.2.2 shows that in the presence of efficient and complete risk sharing markets and if the the productivity is state-independent, then the value of information is negative for all risk avers decision makers. This is a generalization of the result by Schlee (2001) to an production economy with risky endowments.

These two environment have in common that the decision makers’ sets of feasible actions are not independent of the underlying information system. In the first framework information are costly and, hence, the choice of an information system reduces the decision makers’ budget which, in turn, reduces the decision makers’ possible payoffs. In the second framework the prices for state-contigent claims depent on both: the information system and the particular signal realization which, in turn, changes the decision makers’ set of feasible consumption bundles. Consequently, information has a direct impact on the decision makers’ budget and, hence, on his set of feasible payoffs/consumption bundles in both frameworks under consideration. Therefore, it is not clear whether the impact of information on individual ex ante expected utility is positive or negative.

However, in a more simple framework where the set of possible actions is independent of the information system itsself and the signal realizations, Blackwell’s Theorem (Theorem 3.1, Blackwell (1951, 1953)) etablishes that Blackwell’s sufficiency criterion and the order by Bonnenblust et al. (1949) are equivalent: An information structure is more sufficient for another if and only if the former one is more valuable than the latter one for all expected utility maximizers. For special classes of decision makers and information systems similar results are valid for the informativeness concepts of Lehmann (1988)/Persico (1996) and Kim (1995). Crucial for these results is, that the set of possible actions is independent of underlying information system and the signal realization. Eckwert and Zilcha (2000) relax this assumption and look at the value of information in production economies. In their model, better information not only limits the risk sharing opportunities, i.e. the set of possible actions, but also improves the input allocation in the economy. Therefore, the impact of better information on welfare is ambigious - it could be positive or negative. However, they show that in the absence of risk sharing markets Blackwell's Theorem remains valid in their framework.

In contrast, Hirshleifer (1971) was the first who demonstrated that in equilibrium information might make everybody worse off. He considers a small exchange economy with a single consumption good, risk avers agents and complete markets for state-contingent claims. Each agent is endowed with a risky endowment of the consumption good. The agents can share risks by trading state-contingent claims in complete markets before the state of the world is realized. If they were perfectly informed about the state before the markets for state-contingent claims are open, no trade at all will take place and the agents consume according to their endowments. Therefore, from an ex ante perspective, perfect information make the agents worse off by breaking down the risk sharing markets. Schlee (2001) generalized Hirshleifer's result to an exchange economy with one commodity and complete and competitive risk sharing markets. Green (1981) examines a model with futures markets (without production). And in partial-equilibrium models, the failure of Blackwell's result has been shown by Schlee (1996) for a monopoly with random demand and by Sulganik and Zilcha (1996) for an exporting firm in the presence of a futures market for currency. The reasom for failure of Blackwell's theorem is, as in Hirshleifer (1971), that the sets of feasible action are signal-dependent.

4.1 Endogenous Risk in an Economy with Information Markets

This section is based on Brandt and Szczutkowski (2012). I am grateful to Dr. Andreas Szczutkowski for this collaboration.

The purpose of this section is to study the market for information in a microeconomic framework. Information services are described by a set of possible signals which are correlated to the state of the world.[20] Agents demand information services in order to reduce the uncertainty they face in their individual decision problems. Information markets have special properties which create difficulties in describing them theoretically (see e.g. Arrow (1978, 1999, 2003); Varian (2000)):

First, on the demand side, the willingness to pay for information depends on its 'informational content', i.e. on how much the information accounts for the reduction of risk.[21] Agents who employ an information service derive utility solely via the correlation of signals and states.[22] A criterion is needed which describes the informational content of an information system which consists of an information service and a prior belief. In this section Blackwell's sufficiency criterion is used in order to compare the informativeness of different information systems.

Secondly, it is not clear how information are produced. Additionally, it is problematic that the law of diminishing returns does not hold for the production of information. Once an information is produced/known it can be copied arbitrarily often which leads to linear costs in the 'quantity of information'. In the current section this problem does not play a role because it is assumed that production costs (for information) are convex in informativeness (of the

[20] In the context of this section, it seems more appropriate to use the term information service instead of information structure. Hence, in the rest of this section an information service denotes an information structure.

[21] This is also true for firms which use information as a production factor (e.g. a newspaper). On the one hand information may be necessary for certain production processes, but on the other hand information will not be used up in the production process so that the law of diminishing returns does not hold.

[22] An agent with prior belief f_Ω who employs an information service $f_{S|\Omega}$, posses the information system $(f_{S|\Omega}, f_\Omega)$.

service) and each decision maker demands an individual information service which is useless for all other decision makers.[23]

Furthermore, information services often exhibit characteristics of a public good. There is no rivalry in consumption, i.e. many agents can observe information signals simultaneously. This can e.g. be due to the informational function of market prices, workers mobility or reverse engineering. This problem plays no role in the current analysis because, as mentioned before, each decision maker demands an individual information service which is useless for all other decision makers.

Marschak (1971) states two central problems which are still not yet completely resolved in the literature. The first problem is to understand the system of demand and supply of information goods. The second problem is the question how social welfare is affected by the manner in which resources are allocated to information goods or services. This section addresses the first question.

An example economy is presented where agents demand information services with prices which differ according to their informational content. This is modeled via a class of parametrized information systems which can be ordered by Blackwell's sufficiency criterion (Blackwell (1953)). The focus of the analysis lies on the demand for information, which is fully described by the agents' decision problems. In particular, the question is which is the demanded level of informativeness? Supply is modelled by firms which produce information services with costs of production which depend on the informational content of the information services. The public good character of information plays no role in the analysis as each agent's decision problem is assumed to be independent from the decision problems of others, i.e. state spaces differ and coordination via market prices only occurs on the market for information services.

Each agent on the demand side plays a lottery where he has to guess the right state of nature. If he is right, then his resources will increase, otherwise they will remain constant. As an example one can think of the agents as farmers in different regions who forecast future, local weather conditions.

[23] As an example for such a situation consider screenings for cancer.

Each state of the weather is related to a best farming strategy so that a correct forecast is rewarded by a high crop.

The main results are driven by two simple effects. On the one hand better information leads to higher (ex ante) welfare as the chances of winning improve ('precision effect'). But on the other hand better information leads to less budget available for consumption as it exhibits a higher price ('budget effect'). In an interior equilibrium these effects cancel out. It is shown that under these circumstances even risk neutral agents invest in information. Furthermore, and perhaps more surprisingly, risk averse agents do not invest in information if their degree of risk aversion is sufficiently high. The main result shows that the demand for information is negatively correlated to the degree of (relative) risk aversion for a broad range of parameters. In this case equilibrium risk in this economy is negatively linked to the agents' degree of risk aversion.

Closest to this study are the works of Kihlstrom (1974), Radner and Stiglitz (1984) and Chade and Schlee (2002).

Kihlstrom (1974) models the demand for information in a setting with normally distributed random variables and CES-utility. He shows, similar to the present findings, that the demand for information depends positively on the agents' income and negatively depends on the information price. In contrast to the present analysis, he restricts the analysis to the demand side and does not analyze comparative statics with respect to the agents attitude towards risk.

Radner and Stiglitz (1984) and Chade and Schlee (2002) analyze the value of information in a much more general setting where, contrary to this approach, decision problems have sets of possible actions which are independent of the realized information. Furthermore information is costless and a complete, separable metric action space is assumed. Different to this section's findings, small improvements of information (starting from no information) are not welfare improving in their analysis.

4.1.1 The Model

The model consists of an infinite set of agents $\mathcal{I}$ who demand information and an infinite set of producers on the supply side. Each agent $i \in \mathcal{I}$ faces uncertainty in his decision problem described by a probability distribution, given by a probability mass function f_{Ω_i}, over a finite and individual set of states of nature $\Omega_i = \{\omega_{1,i}, \ldots, \omega_{n,i}\}$ with $n > 1$. The assumption of an individual state space captures the idea of focussing on a market for private information where the public good-character of information plays no role. Even if an agent observes an information signal produced by an information system owned by a different agent this does not help him in reducing his own uncertainty because it is assumed that individual states are uncorrelated accross the decision makers. Formally, risk in this economy is described by the product space of all individual state spaces, i.e. $\Omega_{Econ} = \bigotimes_{i \in \mathcal{I}} \Omega_i$. Moreover, a priori the individual states are equally likely, i.e. $f_{\Omega_i}(\omega_i) = 1/n$ for $\omega_i \in \Omega_i$ and all $i \in \mathcal{I}$.

The supply of information

There is an infinite number of producers who are able to produce information services $f^{\epsilon}_{S_i|\Omega_i}$ with parameter levels $\epsilon \in [0, 1]$. An information service $f^{\epsilon}_{S_i|\Omega_i}$ is only applicable to state space Ω_i and produces a signal $s \in S_i = \Omega_i$.[24] The statistical relationship between states and signals is thereby given by a stochastic transformation which assigns a probability distribution over forecasts s to a given state of nature $\omega \in \Omega_i$. In this section a specification from Nermuth (1982) is employed which is given by

$$f^{\epsilon}_{S_i|\Omega_i}(s|\omega) \;=\; \begin{cases} 1 - \frac{n-1}{n}\epsilon & \text{if } s = \omega \\ \frac{1}{n}\epsilon & \text{else,} \end{cases}$$

where $f^{\epsilon}_{S_i|\Omega_i}(s|\omega)$ denotes the probability of a produced signal realization s given the state ω. $\frac{n-1}{n}\epsilon$ then is the probability of an erroneous forecast, i.e.

[24] If the information service is used by agent i, he combines $f^{\epsilon}_{S_i|\Omega_i}$ with his prior f_{Ω_i}. This results in the information system $(f^{\epsilon}_{S_i|\Omega_i}, f_{\Omega_i})$.

for a signal $s \neq \omega$. $\epsilon \in [0,1]$ parametrizes the 'accuracy' of the forecast. In case of $\epsilon = 0$ the forecast is always perfect whereas $\epsilon = 1$ stands for a useless signal that has no effect on the agents' prior beliefs over Ω_i. Nermuth (1982) shows that this family of information services is ordered by ϵ with respect to Blackwell's sufficiency criterion (Blackwell (1951), Blackwell (1953)). Higher values of ϵ correspond to lower informativeness according to this criterion and, hence, ϵ will be referred to as the 'error level' of an information structure $f^{\epsilon}_{S|\Omega_i}$.

The costs of production of an information service negatively depend on the error level ϵ by assumption. Since $\epsilon = 1$ corresponds to an fully uninformative service it senseful to set $c(1) = 0$. Furthermore it is assumed that $c'(\epsilon) < 0$ and that $c''(\epsilon) \geq 0$ for all error levels ϵ.

In this model the market for information services (for each given error level) is competitive so that the price for an information service with error level ϵ, $P(\epsilon)$, will be equal to $c(\epsilon)$ in equilibrium. Therefore, in the further analysis it is $P(1) = 0,\ P'(\epsilon) < 0$ and $P''(\epsilon) \geq 0$.

The demand for information

Let $\mathcal{I}$ be a infinite set of homogeneous, risk averse agents. As mentioned above, each agent $i \in \mathcal{I}$ faces risk described by a set $\Omega_i = \{\omega_{1,i}, \ldots, \omega_{n,i}\}$ of $n > 1$ future states of nature.

Each agent is endowed with a budget of $m > 0$ units of a consumption good. In the first stage (ex ante), this budget can be spend for exactly one information structure with price $P(\epsilon)$ in order to reduce the risk in the interim decision problem. Interim, the agent faces a (personal) lottery in which the rest of the budget $m - P(\epsilon)$ is invested and where the agent has to forecast the underlying state of nature. In particular, the set of possible actions is equal to set of states of the world, i.e. $\mathcal{A}_i = \Omega_i$. If the forecast is right, then the lottery pays out a multiple $\alpha(m - P(\epsilon))$ ($\alpha > 1$) of the investment $m - P(\epsilon)$. In case of a wrong forecast the lottery simply pays out the amount of the stake $m - P(\epsilon)$. Due to symmetry the parameter i is dropped in the further description of the lottery and analysis of the model.

In the following $\hat{\omega}$ denotes the agent's forecast and ω stands for the prevailing state of nature. Then each agent's payoff is formally given by

$$o(\hat{\omega}, \omega) \ := \ \begin{cases} \alpha(m - P(\epsilon)) & \text{if } \omega = \hat{\omega} \\ m - P(\epsilon) & \text{else} \end{cases},$$

where $\alpha > 1$. Each agent faces a different personal lottery so that the aggregate payoff (which is equal to the sum of individual payoffs) of the economy as a whole is risky. Better forecasts in the economy increase the individual expected payoffs and, therefore, the aggregate expected payoff in the economy.

Agents are Bayesian decision makers with preferences over consumption described by a von Neumann–Morgenstern utility function $u : \mathbb{R} \to \mathbb{R}$ with $u' > 0$. Therefore, the indirect utility of an agent is equal to $v : \mathcal{A} \times \Omega \to \mathbb{R}, (a, \omega) \mapsto u(o(a, \omega))$. As mentioned above, all agents have symmetric prior beliefs so that an agent ex ante assigns probability $f_\Omega(\omega) = 1/n$ to any $\omega \in \Omega$. Bayes' Theorem then implies that the marginal distribution of the signals produced by an information system is also given by $f_S^\epsilon(s) = 1/n$ for all $s \in S$. In order to forecast the correct state the agents calculate the posterior probablities $f_{\Omega|S}^\epsilon$ of the states for a given signal:

$$f_{\Omega|S}^\epsilon(\omega|s) = \begin{cases} 1 - \frac{n-1}{n}\epsilon & \text{if } \omega = s \\ \frac{1}{n}\epsilon & \text{else.} \end{cases}$$

Agent i's interim forecast problem for a given signal realization $s \in S$ then is:

$$\max_{a \in \mathcal{A} = \Omega} \sum_{\omega \in \Omega} f_{\Omega|S}^\epsilon(\omega|s) v(a, \omega) = \max_{a \in \mathcal{A} = \Omega} \sum_{\omega \in \Omega} f_{\Omega|S}^\epsilon(\omega|s) u\left(o(a, \omega)\right).$$

Since $f_{\Omega|S}^\epsilon(s|s) \geq f_{\Omega|S}^\epsilon(\omega|s)$ for any $\omega \neq s$ it is clear that

$$a^*(s) = s,$$

i.e. the agent always decides for the most probable state of nature.

According to Definition 3.2, the ex ante expected utility of an agent is

$$\begin{aligned} W(\epsilon) &:= V((f^{\epsilon}_{S|\Omega}, f_{\Omega}), a^*, v) = \sum_{s\in\Omega} f^{\epsilon}_S(s) \sum_{\omega\in\Omega} f^{\epsilon}_{\Omega|S}(\omega|s) u\left(o(a^*(s),\omega)\right) \\ &= \left(1 - \frac{n-1}{n}\epsilon\right) u\left(\alpha(m - P(\epsilon))\right) + \frac{n-1}{n}\epsilon u\left(m - P(\epsilon)\right). \end{aligned} \tag{4.1}$$

The agents now choose an information service with an error level which maximizes their ex ante welfare. Changing the error level ϵ has two effects which can be seen by calculating the first derivative with respect to ϵ:

$$\begin{aligned} W'(\epsilon) &= \underbrace{\frac{n-1}{n}\left[u\left(m - P(\epsilon)\right) - u\left(\alpha(m - P(\epsilon))\right)\right]}_{=:PE(\epsilon)<0} \\ &+ \underbrace{\left[-P'(\epsilon)\left[\left(1 - \frac{n-1}{n}\epsilon\right)\alpha u'\left(\alpha(m - P(\epsilon))\right) + \frac{n-1}{n}\epsilon u'\left(m - P(\epsilon)\right)\right]\right]}_{=:BE(\epsilon)>0}. \end{aligned}$$

The first term will be called the 'precision effect' $PE(\epsilon)$ which accounts for the negative welfare effect through a worse forecast induced by a higher error level ϵ and holding utilities constant. A change of ϵ changes the success and failure probabilities in a linear way.

The second term, $BE(\epsilon)$, will be called 'budget effect' and describes the positive effect of higher error levels through a reduced information price (when holding probabilities constant). Note that the budget effect is determined by marginal utilities and the shape of the price function.

This decision problem is formally given by

$$\max_{\epsilon\in[\bar{\epsilon},1]} W(\epsilon), \tag{4.2}$$

where $\bar{\epsilon} := P^{-1}(m)$ denotes the error level which leads to zero consumption and therefore is never optimal. Error levels below $\bar{\epsilon}$ do not lie in the agents' budget set.

In order to understand the structure of the optimal error level as a function of the model parameters it is instructive to first consider the case of risk neutral agents.

Risk neutrality

Let the agents' preferences be represented by $u(c) = c$. From (4.1) ex ante welfare then gets:

$$W(\epsilon) = \left(1 - \frac{n-1}{n}\epsilon\right)\alpha(m - P(\epsilon)) + \frac{n-1}{n}\epsilon(m - P(\epsilon))$$

and it follows that the agents' information demand is uniquely determined by an interior solution of the problem (4.2), i.e. even risk neutral agents demand information.

Proposition 4.1 (information demand of risk neutral agents)**.** *For finite $|P'(1)|$ and m (budget) sufficiently high, risk neutral agents always demand information.*

Why do even risk neutral agents invest in information? To get an intuition, note that the price for 'null' information (maximal error level $\epsilon = 1$) is zero and $|P'(1)|$ is small, that is, the budget effect is relatively small at $\epsilon = 1$. In other words, small information improvements from 'null' are comparably cheap. Furthermore, changes of the error level affect not only the risk related to the lottery but expected welfare as well due to its effect on the probability of success. This precision effect is maximal at $\epsilon = 1$. Hence, for m sufficiently large, the precision effect dominates the budget effect at $\epsilon = 1$. Consequently, improvements in informativeness when starting from no information ($\epsilon = 1$) increase the agents expected payoffs and, hence, $\epsilon = 1$ is not optimal. At the minimal error level $\bar{\epsilon}$, on the contrary, the precision effect is equal to zero as $P(\bar{\epsilon})$ equals m while the budget effect is maximal as $P''(\epsilon) \geq 0$. Hence, at $\epsilon = \bar{\epsilon}$ the budget effect dominates the precision effect. Consequently, reductions of informativeness when starting from full information ($\epsilon = 0$) and, hence, $\epsilon = \bar{\epsilon}$ is not optimal. Therefore the optimal error level is characterized by an interior solution.

Analyzing this interior solution shows that information always is a normal good under the given specifications:

Cororally 4.1. *The optimal error level ϵ^* depends negatively on the number of states n and the budget m.*

An increase in the number of states can be interpreted as an increase in risk. Therefore, it should be intuitively clear that an increase in the number of states, n, leads to an decrease of the optimal error level ϵ^*. The intuition for an increase of the budget m is similar: An increase in m increases the riskyness of the lottery $o(\cdot, \tilde{\omega})$. Therefore, an increase in m leads to an decrease of the optimal error level. Formally, an increase in the number of states or the budget strengthens the precision effect (i.e. ${}^{dPE(\epsilon)}/_{dn}, {}^{dPE(\epsilon)}/_{dm} < 0$) while it keeps the budget effect constant (i.e. ${}^{dBE(\epsilon)}/_{dn}, {}^{dBE(\epsilon)}/_{dm} = 0$) and, hence, decreases the optimal error level. Moreover, this means that information are normal goods in this economy.

Now consider a more general formulation of preferences accounting for risk aversion. This is natural as the central characteristic of an information signal is the reduction of uncertainty or indeterminacy. The main question is how information trade is related to the agents' attitude towards risk in such an economy.

Constant relative risk aversion

Assume that the agents' preferences are of the constant relative risk aversion type (CRRA). The utility representation is given by

$$u(c) := \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \text{for } \sigma > 0,\ \sigma \neq 1 \\ \ln(c) & \text{for } \sigma = 1. \end{cases}$$

This leads to the following ex ante welfare function:

$$W(\epsilon) = \left(1 - \frac{n-1}{n}\epsilon\right) \frac{(\alpha(m - P(\epsilon)))^{1-\sigma}}{1-\sigma} + \frac{n-1}{n}\epsilon \frac{(m - P(\epsilon))^{1-\sigma}}{1-\sigma}.$$

It follows that the demand for information is well defined for a broad range of parameters:

Proposition 4.2 (Interior solution, uniqueness).

(i) For $|P'(1)|$ finite and m sufficiently high, the agents demand information with error level $\epsilon^* \in (\bar{\epsilon}, 1)$.

(ii) The optimal error level ϵ^* is uniquely determined for $\sigma \in (0, 1]$.

(iii) For $\sigma > 1$ and $P''' = 0$ the optimal error level ϵ^* is uniquely determined.

Intuitively it should be clear that costly information will not be demanded if the agents are endowed with a very small budget m.

Uniqueness of ϵ^* is ensured if expected welfare is strictly concave in ϵ. This curvature depends on the shape of u as well as on the shape of P in a non-trivial way. It is easy to show that for relative risk aversion $\sigma \in (0, 1]$ the precision effect as well as the budget effect negatively depend on the error level ϵ implying that welfare is indeed strictly concave in ϵ. For $\sigma > 1$ this need not be true in general but e.g. in the case of quadratic P it is.

In order to obtain comparative statics results it is further assumed that the agents' information demand is characterized by an interior and uniquely determined solution ϵ^*.

The next result shows that corollary 4.1 holds true in the more general case of constant relative risk aversion, i.e. information is a normal good:

Proposition 4.3. *(i) The optimal error level ϵ^* depends negatively on the number of possible future states of nature n and negatively on the budget m.*

(ii) ϵ^* depends negatively on the premium α for a sufficiently high degree of relative risk aversion.

As argued above, increasing the number of possible states or increasing the budget increases the riskyness of the lottery $o(\cdot, \tilde{\omega})$ which naturally leads to a higher information demand.

A higher α leads to a higher spread in the payoffs which increases the riskiness of the lottery and strengthens the precision effect. On the other hand the impact on the budget effect is ambigous. For sufficiently high degrees of risk aversion the first effect dominates and leads to a lower error level in equilibrium.

In order to gain further intuition for the main result, consider the case of a linear price function P. The relation between the error level and the level of risk aversion may be non monotonic in this economy:

Proposition 4.4 (Non-monotonicity of the error level in the degree of relative risk aversion)**.** *If prices for information are linear, i.e.* $P(\epsilon) = b(1-\epsilon)$, $b \geq m$, *the following holds true:*

(i) For sufficiently high degrees of relative risk aversion, $\sigma \geq 2$, *the agents keep uninformed, i.e.* $\epsilon^* = 1$.

(ii) If α *is sufficiently high, then there is a degree of relative risk aversion* $0 < \bar{\sigma} < 2$ *such that*

$$\frac{d\epsilon^*(\sigma)}{d\sigma} \begin{cases} < 0 & \text{if } \sigma < \bar{\sigma} \\ = 0 & \text{if } \sigma = \bar{\sigma} \text{ or } \sigma \geq 2 \\ > 0 & \text{if } \bar{\sigma} < \sigma < 2 \end{cases} .$$

The chosen error level is convex in the degree of relative risk aversion, or, put differently, information demand is convex in the degree of relative risk aversion. For low levels of risk aversion the result is intuitive: higher risk aversion leads to a smaller error level which increases the chances of winning the prize $\alpha(m - P(\epsilon))$ and leads to higher expected payoff. But for higher degrees of agents' risk aversion (but not too high, i.e. $\sigma < 2$) this relation turns into a positive one. In order to get a better understanding of this effect it is instructive to reformulate the lottery in the following form:

$$o(\hat{\omega}, \omega) = \underbrace{m - P(\epsilon)}_{\text{fixed payment}} + \underbrace{\begin{cases} (\alpha - 1)(m - P(\epsilon)) & \text{if } \omega = \hat{\omega} \\ 0 & \text{else.} \end{cases}}_{\text{uncertain payment}}$$

So the lottery prize consists of a certain and an uncertain component. Investing in information decreases the certain component of the lottery. If risk aversion is sufficiently high but not too high, then the attractivity of the uncertain payment decreases with the level of relative risk aversion and, hence, the agents decrease their investment in information. This explains the positive relationship between the error level and risk aversion for medium degrees of relative risk aversion. If reletive risk aversion then becomes large enough, then the agents completely avoid the uncertain component and maximize their fixed payment, i.e. they do not invest in information at all.

The agents' information decisions endogenously determines the amount of risk in this economy. In this interpretation the result reads as follows: higher degrees of risk aversion may lead to higher risk.

4.1.2 Concluding Remarks

The uncertainty in this model is endogenously determined by the agents' information decisions. Information services are costly with an equilibrium price which depends on the informativeness. The main goal of the section is to demonstrate that the demand for private information can be negatively correlated to the level of risk aversion. Given a sufficiently high level of risk aversion agents do not demand any information at all. As the information is related to the risky outcome component of the lottery, this maximizes the certain component in the payoff structure.

This effect will also be present in more elaborated economies with risk sharing, goods markets or an additional public signal. These extensions would be valuable at the expense of possibly loosing the clear cut - nature of the results.

4.2 The Value of Information in Economies with Production

The effects of better information in production economies are not clear: On the one hand, as pointed out by Eckwert and Zilcha (2000), better information might increase ex ante expected welfare by improving the input allocation. On the other hand, information might lower ex ante expected welfare by destroying risk sharing opportunities.

The purpose of this section is to examine the value of information in a many commodity production economy with risky (input) endowments and efficient risk sharing. Compared to a pure exchange economy (with only one commodity) as analyzed by Schlee (2001), the risk sharing opportunities are much richer in a many commodity production economy: First, if there is only one commodity, individuals can share risks only through shifting consumption from one state to another. In contrast, if there are many commodities, then individuals can substitute consumption of some commodity in some state by consuming some (potentially) other commodities in some other states. Secondly, the introduction of production increases the risk sharing opportunities even more. In particular, production introduces risk sharing opportunities which are not present without production. For example, if the aggregate endowment of a commodity is zero in some state, this can not be insured without production. With production, it is possible to compensate this by producing the commodity from others. And third, as pointed out by Eckwert and Zilcha (2000), in production economies (with state dependent productivity) better information might improve the allocation of commodities used as inputs which might overcome its negative effect on risk sharing. Nevertheless, the main result is that if risk sharing markets are complete and efficient, then information are harmful for all risk avers agents. The intuition for this result is that since, in contrast to Eckwert and Zilcha (2000), the productivity of the firms is state-independent, better information does not lead to an improvement of the input allocation which might overcome its the negative effects on the risk sharing opportunities in the economy.

In particular, in this section a two-period model with many commodities which can be used for consumption and production is considered. There are two different types of agents, risk avers consumers and risk neutral firms. Both types of agents possess a risky endowment of commodities (inputs as well as outputs). At date 0, after receiving information but before observing the state of the world, the agents trade state-contingent claims in a competitive market. After the state realization, at date 1, the agents consume or produce according to their state-contingent claim. Moreover, since the firms' profits are determined by the trade of state-contingent claims which are traded before the state realization, the firms' profits independent of the particular state relization.

4.2.1 The Model

Consider a competitive economy with I consumers, J firms, $C \geq 2$ commodities and N states of the world. Denote by $\mathcal{I}$ the set of consumers, by $\mathcal{J}$ the set of firms, by $\mathcal{C} \subseteq \mathbb{R}_+^C$ the commodity space and by $\Omega = \{\omega_1, \ldots, \omega_N\}$ the state space. Let the prior belief be given by the probability mass function f_Ω and assume w.o.l.g. $\omega_1 < \omega_2 < \ldots < \omega_N$.[25] The space of commodity $c \in \{1, \ldots, C\}$ is denoted by $\mathcal{C}_c \subseteq \mathbb{R}_+$. Assume that $\mathcal{C}$ is convex and compact. A commodity can be a physical good or a service and it can be used for consumption as well as for production.

In the following an agent i can be a consumer $i \in \mathcal{I}$ or a firm $i \in \mathcal{J}$. In state $\omega \in \Omega$ agent i is endowed with $w_c(\omega, i)$ units of commodity c. The vector

$$w(\omega, i) := (w_1(\omega, i), \ldots, w_C(\omega, i)) \in \mathcal{C}$$

denotes agent i's endowment vector in state ω. The agent's tupel of endowment vectors for each state is denoted by

$$w(i) := (w(\omega_1, i), \ldots, w(\omega_N, i)) \in \mathcal{C}^N.$$

Moreover, let

$$w_c(\omega) := \sum_{i \in \mathcal{I} \cup \mathcal{J}} w_c(\omega, i)$$

be the aggregate endowment of commodity c in state ω and

$$w(\omega) := (w_1(\omega), \ldots, w_C(\omega)) \in \mathcal{C}$$

the corresponding aggregate endowment vector in state ω.

Each firm $j \in \mathcal{J}$ has a non-empty, state-independent technology, or pro-

[25] I.e. $Prob(\tilde{\omega} = \omega) = f_\Omega(\omega)$.

duction set, given by

$$Y_j := \left\{y \in \mathbb{R}^C | T_j \leq 0\right\} \subseteq \mathbb{R}^C,$$

where $T_j : \mathbb{R}^C \to \mathbb{R}$ denotes firm j's transformation function. Throughout this section T_j is assumed to be continuously differentiable, increasing in each component and convex. A *production vector*,

$$y(\omega, j) := (y_1(\omega, j), \ldots, y_C(\omega, j) \in Y_j,$$

describes firm j's net outputs of the c commodities in state ω. Positive numbers denote outputs while negative numbers denote inputs. Moreover, state-independency of the firms' technologies implies that the firms, as the consumers, are exposed to endowment risk rather than productivity risk. Since state-contingent commodity claims are traded before the state realization and, hence, the firms' profits are not risky, the firms are assumed to be pure profit maximizers.

Each consumer $i \in \mathcal{I}$ has a twice (continuously) differentiable utility function $u_i : \mathcal{C} \to \mathbb{R}$. u_i is assumed to be increasing in each commodity, i.e. $\partial u_i(x_1,\ldots,x_C)/\partial x_c \geq 0$ for all $c \in \{1, \ldots, C\}$. Moreover, each consumer owns a claim to a share $\theta_{ij} \in [0, 1]$ of the profits of firm j such that $\sum_{i \in \mathcal{I}} \theta_{ij} = 1$.

The model has two stages: stage 0 is before the state realization while stage 1 is after the state realization. Before the state realization at stage 0, the agents trade state-contingent commodity claims for each commodity in a competitive market. Such a claim of commodity c for state ω pays exactly one unit of commodity c in state ω and nothing in other states. In particular, for the consumers this results in state-contingent consumption plans while for firms this results in state-contingent production plans. Denote by

$$p := (p_{1,\omega_1}, \ldots, p_{C,\omega_1}, p_{1,\omega_2}, \ldots, p_{C,\omega_2}, \ldots, p_{1,\omega_N}, \ldots, p_{C,\omega_N}) \in \left(\mathbb{R}^C_+\right)^N$$

the price vector for state-contingent claims, where $p_{c,\omega}$ denotes the price for one unit of commodity c in state ω. After the state realization, the commodities are allocated according to the agents' state-contigent claims and production takes place according to the firms' state-contingent production plans. For an overview of the timing see figure 4.1 below.

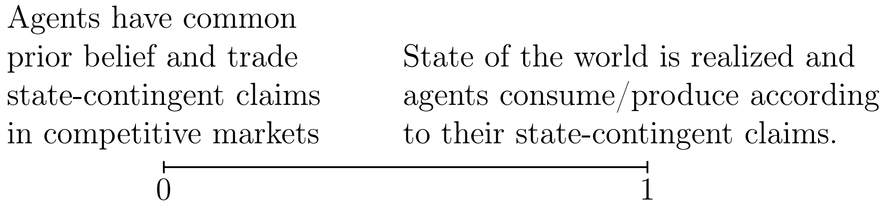

Figure 4.1: Timing of events.

Consumer i's consumption vector in state ω is denoted by $x(\omega, i) \in \mathcal{C}$. Equivalently, $y(\omega, j) \in Y_j$ denotes firm j's production vector in state ω. An *allocation in state* ω,

$$(x(\omega), y(\omega)) := (x(\omega, 1), \ldots, x(\omega, I), y(\omega, 1), \ldots, y(\omega, J)) \in \mathcal{C}^I \times \bigotimes_{j \in \mathcal{J}} Y_j$$

is a vector which assigns non-negative consumption levels $x_c(\omega, i)$ of each commodity $c \in \{1, \ldots, C\}$ to each consumer $i \in I$ and a production plan $y(\omega, j) \in Y_j$ to each firm $j \in \mathcal{J}$. An allocation in state ω is *feasible* if total consumption of each commodity does not exceed its total endowment plus its net output, i.e.

$$\sum_{i \in \mathcal{I}} x_c(\omega, i) \leq \sum_{i \in \mathcal{I}} w_c(\omega, i) + \sum_{j \in \mathcal{J}} (y_c(\omega, j) + w_c(\omega, j))$$

for all $\omega \in \Omega$ and $c \in \{1, \ldots, C\}$. Denote by

$$x(i) := (x(\omega_1, i), \ldots, x(\omega_N, i)) \in \mathcal{C}^N$$

consumer i's state-contingent consumption plan and by

$$y(j) := (y(\omega_1, j), \ldots, y(\omega_N, j)) \in Y_j^N$$

firm j's state-contingent production plan. An *allocation of contingent com-*

modities,

$$(x, y) := (x(1), \ldots, x(I), y(1), \ldots, y(J)) \in \left(\mathcal{C}^S\right)^I \times \bigotimes_{j\in\mathcal{J}} Y_j^N,$$

is a tupel which determines a commodity allocation for all states $\omega \in \Omega$. An allocation of contingent commodities is feasible if $(x(\omega), y(\omega))$ is feasible for all states $\omega \in \Omega$.

Using this, a Walrasian equilibrium for this economy is defined as follows:

Definition 4.1. *A Walrasian equilibrium with complete markets,* $(x^*, y^*, p^*) \in \left(\mathcal{C}^N\right)^I \times \bigotimes_{j\in\mathcal{J}} Y_j^N \times \left(\mathbb{R}_+^C\right)^N$, *consists of a feasible allocation of contingent claims* $(x^*, y^*) \in \left(\mathcal{C}^N\right)^I \times \bigotimes_{j\in\mathcal{J}} Y_j^N$ *and a price system* $p^* \in \left(\mathbb{R}_+^C\right)^N$ *such that for all firms* $j \in \mathcal{J}$

$$y^*(j) = \arg\max_{y\in Y_j^N} \left(p^*\right)^T \left(y + w(j)\right) \qquad \text{(profit maximization)}$$

and for all consumers $i \in \mathcal{I}$

$$x^*(i) = \arg\max_{x\in B_i} \mathbb{E}_\Omega \left[u_i(x(\tilde{\omega}, i))\right], \qquad \text{(utility maximization)}$$

where consumer i's budget set is defined by

$$B_i := \left\{ x \in \mathcal{C}^N | \left(p^*\right)^T \left(x^*(i) - w(i) - \sum_{j\in\mathcal{J}} \theta_{ij}(y^*(j) + w(j)) \right) \leq 0 \right\}.$$

Moreover, risk aversion is defined as usual: For any lottery on his consumption set, a risk averse consumer prefers the certainty equivalent of an lottery over the lottery itsself. Therefore, by Jensen's inequality, if an consumer is risk avers (neutral, affin) his utility is strictly concave (linear, convex) in $x \in \mathcal{C}$. Moreover, u_i is strictly concave (linear, convex) in each component. Since u_i is assumed to be twice differentiable this implies that the second derivatives with respect to some x_c, $c \in \{1, \ldots, C\}$, are negative (zero, positive).

If risk premia are zero it is intuitively clear that risk avers agents would like to insure fully. The next Proposition etablishes conditions under which a full insurance is possible and optimal.

Proposition 4.5. *Consider the competitive, private ownership economy described above. If all consumers are risk averse and if there is*

(i) either one risk neutral consumer who owns enough to insure all other agents,

(ii) or one firm with constant marginal rate of transformation (MRT) which owns enough to insure all others,

(iii) or no aggregate risk,

then in every Walrasian equilibrium with complete markets (if it exists) the (risk avers) consumers fully insure in terms of consumption and all firms (with non-constant MRT) fully insure in terms of production vectors. In particular,

$$x^*(\omega, i) = x^*(\omega', i) \text{ and } y^*(\omega, j) = y^*(\omega', j)$$

for all risk avers consumers i, all firms j with non-constant MRT and $\omega, \omega' \in \Omega$.

In equilibrium prices are fair. This means, the price for a state-contingent claim is equal to the commodity price in the 'certainty equivalent economy' multiplied with the state probability, i.e. $p^*_{c,\omega} = f_\Omega(\omega)\bar{p}^*_c$, where $\bar{p}^* \in \mathbb{R}^C_{>0}$ denotes the equilibrium price in the certainty equivalent economy.[26] In particular, this means that there are no risk premia. Hence, in the absence of aggregate risk (case (iii)), it is optimal for all agents to choose a full insurance.[27] In case (i) (or (ii)) the equilibrium price ratios are equal the risk neutral consumer's MRS (or to the MRT of that firm with constant MRT).

[26]The only difference of the certainty equivalent of this economy to this economy is that the risky endowments are replaced by their expectations.

[27]In more detail: In the absence of aggregate risk, the agents' optimal consumption/production bundles are feasible in all states. Since prices are fair it follows that the agents choose an full insurance which delivers the consumption bundle of the certainty equivalent economy.

Therefore, the risk neutral consumer (or the firm with constant MRT) is indifferent between consumption (production) of commodity c in state ω and commodity c' in state ω', i.e. these are perfect substitutes. Hence, the risk neutral consumer (firm with constant MRT) is willing and, by assumption, also able to insure the other agents. Together with the observation that risk premia are zero, this implies that it is optimal for risk averse consumers to smooth their consumption to the optimal consumption bundle of the certainty equivalent. Similary, as production sets are state independent and risk premia are zero, its optimal for firms (with non-constant MRT) to produce the same production vector in each state. In particular, this means that in all states, the optimal state-contingent production plan is equal to that in the certainty equivalent. Moreover, fair prices and state-independent production sets also imply that the optimal production plan and the firms profits are independent of the state distribution.

Cororally 4.2. *If any of the conditions (i)-(iii) of Proposition 4.5 hold, then the firm's equilibrium state-contingent production plan is independent of the state distribution.*

As argued above, equilibrium prices have no risk premia and they are determined by the (constant) MRS of the risk neutral consumer (or by the constant MRT of the firm with this). Hence, each firm's optimal state-contingent production plan is state independent and equal to its optimal production plan in the certainty equivalent economy. Therefore, also its profits are equal to its profits in the certainty equivalent.

4.2.2 The Model with Information

This section studies an extension of the previous model. In particular, it is additionally assumed that the agents may have access to some information before the market for state-contigent claims is open. Therefore, let $f_\Omega \in \Delta(\Omega)$ as before and assume that the agents (consumers and firms) may have access to a public information structure $f_{S|\Omega} \in \Gamma(f_\Omega)$ that produces a signal s from the signal set $S = [0,1]$.[28] See figure 4.2 for an overview of the timing in this model.

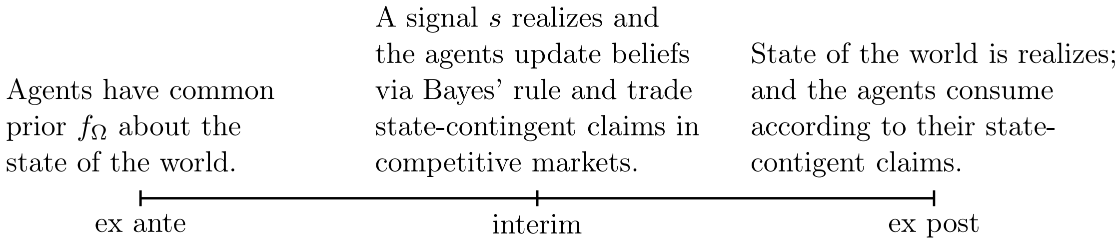

Figure 4.2: Timing with information.

The following analysis is based on a special informativeness concept. In order to compare as much as possible information systems, the weakest informativeness notion (introduced in this work) is chosen as information concept. By Proposition 3.3, this is, if the prior belief is fixed, the weak informativeness concept. Since the prior belief is fixed, the definition of weak informativeness becomes (compare Definition 3.10):

$$(f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega) \Leftrightarrow F_{S,\Omega}(s,\omega) \geq \bar{F}_{S,\Omega}(s,\omega) \text{ for all } (s,\omega) \in S \times \Omega.$$

The equilibrium allocation after observing s from information system $(f_{S|\Omega}, f_\Omega)$ solely depends on the (updated) posterior state distribution and, hence, on the information system $(f_{S|\Omega}, f_\Omega)$ and the signal realization s. In order to makes this clear let $(x^*((f_{S|\Omega}, f_\Omega), s), y^*((f_{S|\Omega}, f_\Omega), s), p^*((f_{S|\Omega}, f_\Omega), s))$ denote the equilibrium allocation after a signal realization s from information system $(f_{S|\Omega}, f_\Omega)$. Similar to Definition 3.2, the ex ante expected utility of consumer i is defined as follows:

$$V((f_{S|\Omega}, f_\Omega), x_i^*, u_i) := \mathbb{E}_S\left[\mathbb{E}_\Omega\left[u(x^*(\tilde{\omega}, i; (f_{S|\Omega}, f_\Omega), s))\right]\right],$$

[28] Recall that $\Gamma(f_\Omega)$ denotes the set of all monotone and for the prior f_Ω normalized information structures and that Γ denotes the set of all monotone and normalized information structures (compare page 49).

where, analogously to the previous section, $x^*(\omega, i; (f_{S|\Omega}, f_\Omega), s)$ denotes agent i's (optimal) consumption bundle in state ω under information system $(f_{S|\Omega}, f_\Omega)$ after a signal observation equal to s.

Firms are owned by consumers, therefore, the following definition of pareto efficiency (of information) restricts attention to consumers rather than to firms. Information are pareto superior (inferior) if 'better information' increases (decreases) the ex ante expected utility of at least one consumer without lowering (increasing) the ex ante expected utility of any other consumer. Formally,

Definition 4.2. *Better information are pareto superior (inferior) iff for* $(f_{S|\Omega}, f_\Omega), (\bar{f}_{S|\Omega}, f_\Omega) \in \Gamma$*:*

$$(f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (\bar{f}_{S|\Omega}, f_\Omega)$$
$$\Rightarrow \begin{bmatrix} V((f_{S|\Omega}, f_\Omega), x_i^*, u_i) \overset{(\leq)}{\geq} V((\bar{f}_{S|\Omega}, f_\Omega), x_i^*, u_i) \text{ for all } i \in \mathcal{I} \text{ and} \\ V((f_{S|\Omega}, f_\Omega), x_{i'}^*, u_{i'}) \overset{(<)}{>} V((\bar{f}_{S|\Omega}, f_\Omega), x_{i'}^*, u_{i'}) \text{ for at least one } i' \in \mathcal{I} \end{bmatrix}.$$

As seen in section 4.2.1, in equilibrium, the prices for state-contingent commodity claims depend on the state distribution. Therefore, from ex ante perspective, the improvement of information increases security price risk which makes risk averse consumers worse off.

Proposition 4.6. *If all consumers are risk avers and have monotone endowments and if there is*

(i) either one risk neutral consumer who owns enough to insure all other agents,

(ii) or one firm with constant marginal rate of substitution which owns enough to insure all others,

then information are pareto inferior in every Walrasian equilibrium with complete markets.

The intuition is similar to that of the result by Schlee (2001): Although better information reduces interim risk, from ex ante perspective, it also introduces or increases price risk making each risk averse consumer worse off. Since, in contrast to Eckwert and Zilcha (2000), production sets are state-independent, better information cannot improve the input allocation in such a way such that the improvement overcomes these negative effects.

In equilibrium, the price for a state-contingent claim for commodity c in state ω is $p^*_{c,\omega}((f_{S|\Omega}, f_\Omega), s) = f_{\Omega|S}(\omega|s)\bar{p}^*_c$, where $\bar{p}^* \in \mathbb{R}^C_+$ is the equilibrium price vector in the certainty equivalent economy (where each agent is endowed with his conditional expected endowment). This price vector is determined through the consumers' MRSs and the firms' MRTs which, in the certainty equivalent, are independent of the of the underlying state distribution, i.e. these are independent of the particular signal realization and the underlying information system. Hence, consumer i's conditional expected utility after a signal realization s from $(f_{S|\Omega}, f_\Omega)$ is equal his utility in the certainty equivalent economy, i.e.

$$\mathbb{E}_\Omega\left[u_i(x^*(\tilde{\omega}, i; (f_{S|\Omega}, f_\Omega), s))|s\right] = u_i(\bar{x}^*(i, \bar{p}^*, \bar{p}^{*T}\mathbb{E}_\Omega\left[w(\tilde{\omega}, i)|s\right])),$$

where $\bar{x}^*(, i, \bar{p}^*, \bar{p}^{*T}\mathbb{E}_\Omega\left[w(\tilde{\omega}, i)|s\right]) \in \mathcal{C}$ denotes agent i's optimal consumption bundle in the certainty equivalent economy.[29] Now, risk aversion implies that $u_i(\bar{x}^*(i, \bar{p}^*, m))$ is concave as a function of m and, hence, it follows that better information makes every risk avers consumer worse off.

The theorem by Schlee (2001) presents three conditions conditions under which information are pareto inferior in a single good, exchange economy with complete risk-sharing markets: first, all agents are risk avers and there is no aggregate risk. Secondly, all agents are risk averse and there exists one risk neutral who owns enough to insure all other. This condition is equivalent to the conditions (i) and (ii) above. And last, all agents are risk averse and the economy has a representative agents who satisfies the expected utility hypothesis with a concave, differentiable vNM-utility function. In contrast to the second condition, the first and the last conditions are not caputered here for the following reason: As under conditions (i) and (ii) of

[29] In particular, $\bar{x}^*(i, \cdot, \cdot) : \mathbb{R}^C_{>0} \times \mathbb{R}_{\geq 0} \to \mathcal{C}$ denotes agent i's Marshallian demand function, i.e. if agent i's budget is $m \geq 0$ and prices are $p \in \mathbb{R}^C_{>0}$ then agent i's optimal consumption bundle is $\bar{x}^*(i, p, m)$.

Prop. 4.5, consumer i's interim conditional expected utility is equal to his utility in the certainty equivalent economy. But under Schlee's first and third condition there might be no agent with constant MRSs or MRTs. Hence, it follows that the price system of the certainty equivalent economy might not be independent of the signal realization, i.e. $\bar{p}^*((f_{S|\Omega}, f_\Omega), s)$ might not be constant in s. Hence, the agents conditional expected utility,

$$\begin{aligned}&\mathbb{E}_\Omega\left[u_i(x^*(\tilde{\omega}, i; (f_{S|\Omega}, f_\Omega), s))|s\right]\\=&u_i(\bar{x}^*(i, \bar{p}^*((f_{S|\Omega}, f_\Omega), s), \bar{p}^{*T}((f_{S|\Omega}, f_\Omega), s)\mathbb{E}_\Omega\left[w(\tilde{\omega}, i)|s\right])),\end{aligned}$$

might not be concave in s which implies that the value of information might not necessarily be negative.

Cororally 4.2 implies that firm j's optimal, state-contingent production plan, $y^*(j; (f_{S|\Omega}, f_\Omega), s)$, is independent of the information system and the particular signal realization. More precisely, $y^*(\omega, j; (f_{S|\Omega}, f_\Omega), s)$ is equal to the optimal production plan in the certainty equivalent which in turn is independent of the underlying state distribution, i.e. $y^*(\omega, j; (f_{S|\Omega}, f_\Omega), s) = \bar{y}^*(j) \in Y_j$ for all $\omega \in \Omega$, where $\bar{y}^*(j) \in Y_j$ is the equilibrium production plan of firm j in the certainty equivalent economy. These two observations imply that the profit of firm j after a signal realization equal to s is equal to the firm's profit in the certainty equivalent economy:

$$\begin{aligned}\Pi_j((f_{S|\Omega}, f_\Omega), s) =&\sum_{\omega\in\Omega}\sum_{c=1}^{C} p^*_{c,\omega}((f_{S|\Omega}, f_\Omega), s)\left(y^*_c(j; (f_{S|\Omega}, f_\Omega), s) + w_c(\omega, j)\right)\\=&\sum_{c=1}^{C}\bar{p}^*_c\left(\bar{y}^*_c(j) + \mathbb{E}_\Omega\left[w_c(\tilde{\omega}, j)|s\right]\right),\end{aligned}$$

where $\bar{p}^*_c \in \mathbb{R}_+$ denotes the equilibrium price of commodity c in the certainty equivalent economy. Because of the law of iterated expectation this implies the the ex ante expected profits of each firm are independent of the underlying information system:

Proposition 4.7. *Under any of the condition (i)-(ii) of Proposition 4.6 and for any information system $(f_{S|\Omega}, f_\Omega) \in \Gamma$, the firms' ex ante expected profits* *are equal the their profits in the certainty equivalent economy. Formally,*

$$\mathbb{E}_S\left[\Pi_j((f_{S|\Omega}, f_\Omega), \tilde{s})\right] = (\bar{p}^*)^T \left(\bar{y}^*(j) + \mathbb{E}_\Omega\left[w(\tilde{\omega}, j)\right]\right) \text{ for all } (f_{S|\Omega}, f_\Omega) \in \Gamma.$$

On one hand, the firms' technologies are by assumption independent of the state of the world and the state distribution. Therefore, and by Corollary 4.2, the optimal production plans are independent of the underlying information system. On the other hand, prices depend on the state distribution and, therefore, on the signal realization and the underlying information system. Hence, the firms' interim profits depent on the signal realization and information system. But since prices are fair, the ex ante expected profits are always equal to that of the certainty equivalent independent of the underlying information structure.

4.2.3 Concluding Remarks

This section studied the value of information in a production economy with many commodities and complete risk sharing markets. Altough the risk sharing opportunities are much richer in an economy with many commodities and production possibilities than in a pure exchange economy with only one single good, it is shown that in the presence of complete risk sharing markets, information make every risk avers agent worse off. The intuition of this result is that from ex ante perspective better information increases price risk which makes risk avers agents worse off. This a generalization of the results by Hirshleifer (1971) and Schlee (2001).

Chapter 5

The Role of the Strategy Space in a Setting with Asymmetric Information

This chapter is based on a joint work with Dr. Dennis Heitmann.

Up to now this work has focused on event risk rather than on market risk which will be done in the current section. As mentioned in the introduction, market risk is related to the limited knowldge about endogenous variables as for instance other market participants' actions. Hence, market risk might origin from an asymmetric allocation of information. Information is asymmetric if some market participants know more than others. An example for such a situation is the Stackelberg game (Stackelberg (1934)). The Stackelberg game is a standard model in oligopoly theory, which is one of the most intensively discussed topics in mathematical economics and based on the pioneering works of Cournot (1838) and Betrand (1883). In the former one the firms simultaneously choose quantities while in the latter prices are the strategic variables. Despite these classical simultaneous move games, the model of Stackelberg (1934), as already mentioned above, describes a situation with asymmetric information in which one firm, the leader, decides at first and the follower observes this before deciding about an optimal strategy. There exist a huge literature on these models for homogeneous and as well as for horizontal differentiated goods (see e.g. Amir and Jin (2001), Dastidar (2004), Kreps and Scheinkman (1983), Vives (1985) and Vives (2005)). It is well established in the literature that for the goods being perfect substitutes and the firms being quantity setters, the leader is better off than the follower because the cross-effect is positive. The opposite is true for Bertrand competition.

In most of the literature on industrial organization the strategy space is exogenously given whereas the endogenous determination of roles and strategy spaces is rarely discussed. Based on a horizontally differentiated duopoly model by Dixit (1979), Singh and Vives (1984) considered a model in which the strategy space (price or quantity) is endogenously determined by the firms. The firms are allowed to offer two types of binding contracts to the consumers, i.e. a price or quantity contract in the first stage and in the second stage, the market stage, the firms compete simultaneously contingent on the type of contract. They showed that it is a dominant strategy for a firm to set strategically the quantity (price) if the goods are substitutes (complements). Boyer and Moreaux (1987) transferred the endogenously determined strategy spaces into the leader-follower model and compared consumer, producer and total surplus with the related values for the Nash equilibrium of the simultaneous move game. Using a very restrictive demand structure[30] they showed that it is always more profitable to be a quantity (price) setter if the goods are substitutes (complements). Concerning total and consumer surplus they proved that price competition is dominant for all degrees of product differentiation. Furthermore, they derived a unique ranking of the leader's and follower's prices, quantities and profits depending on the type of competition and the products being complements or substitutes.[31]

The purpose of this chapter is to provide these comparisons for a more general demand structure introduced by Dixit (1979) with different cross-effects and reservation prices for the goods. It is shown that some of Boyer and Moreaux's results are still valid in this more general framework, while

[30]In this setting the degree of product differentiation and reservation prices are correlated.

[31]For further details see Boyer and Moreaux (1987) Propositions 1 and 2.

others are not.

5.1 The Model

Consider an economy with a monopolistic sector and two firms, each one producing a horizontal differentiated good, and a competitive *numeraire* sector as introduced by Dixit (1979). Following Singh and Vives (1984) and Boyer and Moreaux (1987) assume that each firm can select whether to behave as a price or quantity setter. Furthermore, assume a duopoly with asymmetric information in which firm 1 is the market leader and firm 2 the follower, i.e. a Stackelberg setting with endogenous strategy space. Contingent on the strategy space decision, the price or quantity is chosen optimally. The game structure and some notations are summarized in figure 5.1. In particular, if firm 1 sets a price and firm 2 sets a quantity then π_i^{pq} (q_i^{pq}, p_i^{pq}) denotes firm i's profit (quantity, price). The subgame perfect equilibrium

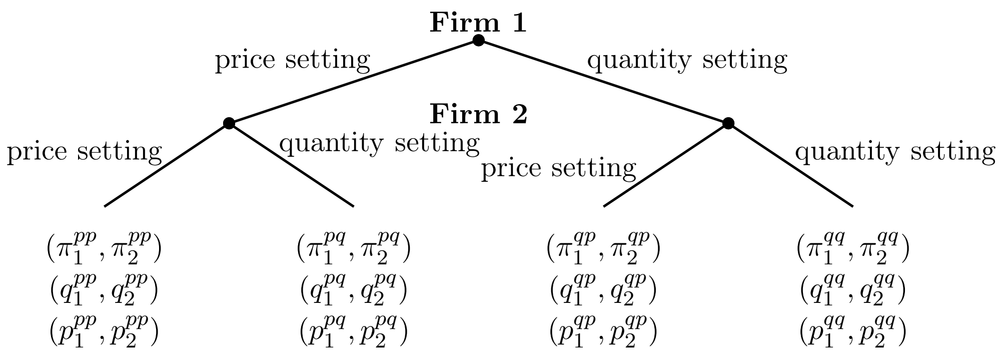

Figure 5.1: The game structure and notations.

of this two-stage game will be derived.

The utility function of the representative consumer is assumed to be quadratic and strictly concave and given by

$$u(q_1, q_2) = \alpha_1 q_1 + \alpha_2 q_2 - \frac{\beta_1 q_1^2 + 2\gamma q_1 q_2 + \beta_2 q_2^2}{2} - \sum_{i=1}^{2} p_i q_i$$

with $\alpha_i, \beta_i \in \mathbb{R}_+$, $i = 1, 2$, $\beta_1\beta_2 - \gamma^2 > 0$ (concavity condition) and $\alpha_i\beta_j -$

$\alpha_j\gamma > 0$ (positive market size). For arbitrary $\alpha_i, \beta_i \in \mathbb{R}_+$ this leads to the following domain of γ:

$$\left(-\sqrt{\beta_1\beta_2}, \min\left\{\sqrt{\beta_1\beta_2}, \frac{\alpha_1\beta_2}{\alpha_2}, \frac{\alpha_2\beta_1}{\alpha_1}\right\}\right).$$

Moreover, utility maximization of the representative consumer gives rise to a linear demand structure

$$\tilde{q}_i(p_i, p_j) = a_i - b_i p_i + c p_j, \quad i, j = 1, 2, \quad i \neq j, \tag{5.1}$$

with $a_i = \frac{\alpha_i\beta_j - \alpha_j\gamma}{\beta_1\beta_2 - \gamma^2} > 0$, $b_i = \frac{\beta_j}{\beta_1\beta_2 - \gamma^2} > 0$ and $c = \frac{\gamma}{\beta_1\beta_2 - \gamma^2}$. The corresponding inverse demand system is

$$\tilde{p}_i(q_i, q_j) = \alpha_i - \beta_i q_i - \gamma q_j, \quad i, j = 1, 2, \quad i \neq j. \tag{5.2}$$

The degree of product differentiation is determined by γ: The goods are complements, independent or substitutes according to whether $\gamma \lesseqqgtr 0$. Demand for good i is downward sloping in its own price and increasing (decreasing) in the competitor's price if the goods are substitutes (complements). The goods are perfect substitutes whenever $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 = \gamma$. Moreover, for $\alpha_1 = \alpha_2 = \frac{1}{1+\alpha}$ $\beta_1 = \beta_2 = \frac{1}{1-\alpha^2}$ and $\gamma = -\frac{\alpha}{1-\alpha^2}$, the demand structure is equal to that which is considered by Boyer and Moreaux (1987).

Firms have constant marginal costs, $C_1, C_2 \geq 0$. W.o.l.g. it is assumed that prices are net of marginal costs.[32] Then, profits of firm i are given by $\pi_i = p_i q_i$. In order to maximize profits, the firms can offer two different types of contracts with the consumers: a price and a quantity contract. If a firm chooses to offer the price contract, then the firm will have to supply that amount which the consumers demand at a predetermined price independently of the competitor's action. If a firm chooses to offer the quantity contract, then the firm have to supply a predetermined quantity independently of the competitor's action. Moreover, still following Singh and Vives (1984), it is assumed that the costs associated with changing the type of contract are

[32] Since $C_i \geq 0$ one may replace α_i and a_i by $\alpha_i - C_i$ and $a_i - b_i m_i + c m_j$, $i, j = 1, 2$, $i \neq j$, respectively.

extremely high such that firms make the decision about the type contract once and then stick to it. Hence, first the firms decide about the type of contract offered to the consumers, and afterwards they compete contingent on their chosen types of contract.

In case of pure quantity competition equation (5.2) is used for the profit maximization of the firms, whereas equation (5.1) is used in case of pure price competition. If one firm is a price setter and the other firm chooses the quantity, a third system is introduced, which easily can be derived by using equations (5.1) and (5.2):

$$\hat{q}_i(p_i, q_j) = \frac{\alpha_i - \gamma q_j - p_i}{\beta_i} \tag{5.3}$$
$$\hat{p}_j(p_i, q_j) = \frac{a_j + c p_i - q_j}{b_j}, \quad i, j = 1, 2, \quad i \neq j.$$

The demand system (5.3) is used if firm i sets the price and firm j chooses strategically the quantity.

Without loss of generality firm 1 is assumed to be the Stackelberg leader and before firm 1 decides about price setting or quantity setting, he computes all possible reactions of the follower firm 2. This leads to the following four cases:

Case 1: firm 1 sets a price

1. If firm 2 is also a price setter, this leads to the classical Stackelberg-Betrand competition and with equation (5.1) the profit of firm 2 is given by

$$\pi_2^{pp}(p_1, p_2) = p_2\ \tilde{q}_2(p_2, p_1).$$

   The first order condition $\frac{\partial \pi_2^{pp}}{\partial p_2} = 0$ leads to the standard Bertrand reaction function

$$p_2 = R_2^{pp}(p_1) = \frac{a_2 + c p_1}{2b_2} = \frac{\alpha_2 \beta_1 - \alpha_1 \gamma + \gamma p_1}{2\beta_2}. \tag{5.4}$$

2. If firm 2 sets quantity, its profits are derived by using equation (5.3). This yields

$$\pi_2^{pc}(p_1, q_2) = q_2\ \hat{p}_2(p_1, q_2).$$

The first order condition $\frac{\partial \pi_2^{pc}(p_1,q_2)}{\partial q_2} = 0$ leads to the following reaction function of firm 2

$$q_2 = R_2^{pq}(p_1) = \frac{a_2 + cp_1}{2} = \frac{\alpha_2\beta_1 - \gamma\alpha_1 + \gamma p_1}{2(\beta_1\beta_2 - \gamma^2)}. \tag{5.5}$$

Case 2: firm 1 sets a quantity

1. If firm 2 sets the price, then equation (5.3) implies for its profits

$$\pi_2^{qp}(q_1, p_2) = p_2\ \hat{q}_2(p_2, q_1)$$

which leads to the reaction function

$$p_2 = R_2^{qp}(q_1) = \frac{\alpha_2 - \gamma q_1}{2}. \tag{5.6}$$

2. For the case that firm 2 sets a quantity the firms compete in the standard Stackelberg quantity competition and the profit of firm 2 is given by

$$\pi_2^{qq}(q_1, q_2) = q_2\ \tilde{p}_2(q_2, q_1).$$

This leads to the standard Cournot reaction function

$$q_2 = R_2^{qq}(q_1) = \frac{\alpha_2 - \gamma q_1}{2\beta_2}. \tag{5.7}$$

Until now the followers best reply functions were derived. Now these are used for the derivation of the Stackelberg leader's optimal strategy.

Analysis of case 1 - the market leader sets a price

Case 1.a: firm 2 sets a price
In this case both firms, the leader and the follower, set prices. The leader uses the reaction function of the follower to maximize his profit given by

$$\begin{aligned}\tilde{\pi}_1^{pp}(p_1) :&= \pi_1^{pp}\big(p_1, R_2^{pp}(p_1)\big) = p_1\ \tilde{q}_1\big(p_1, R_2^{pp}(p_1)\big) \\ &= \frac{2a_2b_2p_1 + ca_2p_1 - 2b_1b_2p_1^2 + c^2p_1^2}{2b_2}.\end{aligned} \tag{5.8}$$

The first order condition $\frac{\partial \tilde{\pi}_1^{pp}}{\partial p_1} = 0$ gives rise to the optimal price

$$p_1^{pp} = \frac{2a_1b_2 + ca_2}{2(2b_1b_1 - c^2)} = \frac{2\beta_1\beta_2\alpha_1 - \beta_1\alpha_2\gamma - \alpha_1\gamma^2}{2(2\beta_1\beta_2 - \gamma^2)} > 0 \tag{5.9}$$

which induces the price of the other good that firm 2 selects by using equation (5.4)

$$p_2^{pp} = R_2^{pp}(p_1^{pp}) = \frac{(3\alpha_2\beta_1 - \alpha_1\gamma)(\beta_1\beta_2 - \gamma^2) + \beta_1\beta_2(\alpha_2\beta_1 - \alpha_1\gamma)}{4\beta_1(2\beta_1\beta_2 - \gamma^2)} > 0. \tag{5.10}$$

Using this prices it follows for the quantities and profits

$$\begin{aligned}
q_1^{pp} &= \frac{2\beta_1\beta_2\alpha_1 - \beta_1\alpha_2\gamma - \alpha_1\gamma^2}{4\beta_1(\beta_1\beta_2 - \gamma^2)} > 0,\\
q_2^{pp} &= \frac{(3\alpha_2\beta_1 - \alpha_1\gamma)(\beta_1\beta_2 - \gamma^2) + \beta_1\beta_2(\alpha_2\beta_1 - \alpha_1\gamma)}{4(\beta_1\beta_2 - \gamma^2)(2\beta_1\beta_2 - \gamma^2)} > 0,\\
\pi_1^{pp} &= \frac{(2\beta_1\beta_2\alpha_1 - \beta_1\alpha_2\gamma - \alpha_1\gamma^2)^2}{8\beta_1(\beta_1\beta_2 - \gamma^2)(2\beta_1\beta_2 - \gamma^2)} > 0,\\
\pi_2^{pp} &= \frac{((3\alpha_2\beta_1 - \alpha_1\gamma)(\beta_1\beta_2 - \gamma^2) + \beta_1\beta_2(\alpha_2\beta_1 - \alpha_1\gamma))^2}{16\beta_1(\beta_1\beta_2 - \gamma^2)(2\beta_1\beta_2 - \gamma^2)^2} > 0.
\end{aligned}$$

Case 1.b: firm 2 sets the quantity

In this case the leader sets prices while the follower reacts with quantity setting. The leader uses the reaction function of the follower in order to maximize his profit

$$\begin{aligned}\tilde{\pi}_1^{pq}\big(p_1\big) :&= \pi_1^{pq}\big(p_1, R_2^{pq}(p_1)\big) = p_1\, \hat{q}_1\big(p_1, R_2^{pq}(q_1)\big) \\ &= \frac{(2a_1 - 2b_1p_1 + 2c\alpha_2 + c\gamma a_1)p_1 - (2b_1 + c\gamma b_1)p_1^2}{2(1+\gamma c)}. \end{aligned} \tag{5.11}$$

Maximizing with respect to p_1 yields the optimal price for good 1 which coincides with p_1^{pq} in (5.9), i.e. $p_1^{pq} = p_1^{pp}$. By using the best reply of firm 2 given in (5.4) this implies $q_2^{pq} = q_2^{pp}$.

Now consider the case in which firm 1 is a quantity setter.

Analysis of case 2 - the market leader sets the quantity

Case 2.a: firm 2 sets the price

In this case the follower reacts with price setting and the profit of the leader is given by

$$\begin{aligned}\tilde{\pi}_1^{qp}\big(q_1\big) :&= \pi_1^{qp}\big(q_1, R_2^{qp}(q_1)\big) = q_1\, \hat{p}_1(q_1, R_2^{qp}(q_1)) \\ &= \frac{(2\alpha_1 - \gamma a_2 + c\alpha_1\gamma)q_1 - \beta_1(2+\gamma c)q_1^2}{2(1+\gamma c)}. \end{aligned} \tag{5.12}$$

The optimality condition $\frac{\partial \pi_1^{qp}}{\partial q_1} = 0$ leads to the optimal quantity of firm 1

$$q_1^{qp} = \frac{2\beta_2\alpha_1 - \alpha_2\gamma}{2(2\beta_1\beta_2 - \gamma^2)}.$$

And by using equation (5.6) it is

$$p_2^{qp} = R_2^{qp}(q_1^{qp}) = \frac{4\beta_1\beta_2\alpha_2 - \alpha_2\gamma^2 - 2\beta_2\alpha_1\gamma}{4(2\beta_1\beta_2 - \gamma^2)}. \tag{5.13}$$

The remaining values of the variables are given by

$$\begin{aligned}
p_1^{qp} &= \frac{2\beta_2\alpha_1 - \alpha_2\gamma}{4\beta_2},\\
q_2^{qp} &= \frac{4\beta_1\beta_2\alpha_2 - \alpha_2\gamma^2 - 2\beta_2\alpha_1\gamma}{4\beta_2(2\beta_1\beta_2 - \gamma^2)},\\
\pi_1^{qp} &= \frac{(2\beta_2\alpha_1 - \alpha_2\gamma)^2}{8\beta_2(2\beta_1\beta_2 - \gamma^2)},\\
\pi_2^{qp} &= \frac{(4\beta_1\beta_2\alpha_2 - \alpha_2\gamma^2 - 2\beta_2\alpha_1\gamma)^2}{16\beta_2(2\beta_1\beta_2 - \gamma^2)^2}.
\end{aligned}$$

Case 2.b: firm 2 sets the quantity

This is the standard Stackelberg competition and the profit of firm 1 is given by

$$\begin{aligned}
\tilde{\pi}_1^{qq}(q_1) :&= \pi_1^{qq}(q_1, R_2^{qq}(q_1)) = q_1\, \tilde{p}_1\big(q_1, R_2^{qq}(q_1)\big)\\
&= \frac{(2\beta_2\alpha_1 - \gamma\alpha_2)q_1 - (2\beta_1\beta_2 - \gamma^2)q_1^2}{2\beta_2}
\end{aligned} \tag{5.14}$$

from which the optimal quantity follows as $q_1^{qq} = q_1^{qp}$.

In order to solve for the leader's optimal choice, these four cases need to be compared. It follows that equilibrium prices, quantities and profits are predetermined by the leader's choice. In particular,

Proposition 5.1. *Independently of the follower's decision, the prices, quantities and profits of both firms are predetermined by the leader's decision. If the leader chooses to set a price then the following holds true:*

$$p_i^{pp} = p_i^{pq}, \quad q_i^{pp} = q_i^{pq} \quad \textit{and} \quad \pi_i^{pp} = \pi_i^{pq}, \quad i = 1, 2.$$

If the leader chooses to set a quantity, then the following holds true:

$$p_i^{qq} = p_i^{qp}, \quad q_i^{qq} = q_i^{qp} \quad \textit{and} \quad \pi_i^{qq} = \pi_i^{qp}, \quad i = 1, 2.$$

The economic interpretation is as follows: After the decision of the leader on the first stage, the follower acts as a monopolist on the remaining market and, hence, price and quantity setting by the follower yield the same outcome. This result also shows that the maximal profit of the follower is predetermined by the action of the market leader, who is able to anticipate the follower's best replies to the different types of contracts.

Proposition 5.1 implies that two distinct types of values for the variables are possible depending on the leader's choice. One is for the case in which firm 1 chooses price competition denoted by the upper index p. Whereas the upper index q denotes the case in which firm 1 chooses quantity competition. In particular,

$$\begin{aligned}
(\pi_1^p, \pi_2^p) &:= (\pi_1^{pp}, \pi_2^{pp}) = (\pi_1^{pq}, \pi_2^{pq}) \\
(q_1^p, q_2^p) &:= (q_1^{pp}, q_2^{pp}) = (q_1^{pq}, q_2^{pq}) \\
(p_1^p, p_2^p) &:= (p_1^{pp}, p_2^{pp}) = (p_1^{pq}, p_2^{pq}) \\
(\pi_1^q, \pi_2^q) &:= (\pi_1^{qq}, \pi_2^{qq}) = (\pi_1^{qp}, \pi_2^{qp}) \\
(q_1^q, q_2^q) &:= (q_1^{qq}, q_2^{qq}) = (q_1^{qp}, q_2^{qp}) \\
(p_1^q, p_2^q) &:= (p_1^{qq}, p_2^{qq}) = (p_1^{qp}, p_2^{qp}).
\end{aligned}$$

In order to solve for the leaders optimal decision on the type of contract, these two scenarios need to be compared. It follows that the leader's decision on the type of contract solely depends on the degree of product differentiation, i.e. wheter the goods are complements or substitutes. In particular, it is:

Proposition 5.2. *For the goods being substitutes (complements) the leader's price, quantity and corresponding profits are higher under quantity (price) setting than under price (quantity) setting. Under quantity (price) leadership also the follower's profit and price is higher than that under price (quantity) leadership of firm* 1*, while its quantity under quantity leadership is always lower than under price leadership.*

The comparison of all variables is summarized in figure 2.2. A direct implication of Proposition 5.2 is that if goods are substitutes (complements), then producers' surplus[33] is higher in the quantity (price) leader model than

[33] Producers' surplus in this case is defined as the sum of profits.

in the price (quantity) model. Only if goods are independent, i.e. $\gamma = 0$,and both firms are monopolists, the firms are indifferent between the strategic variables.

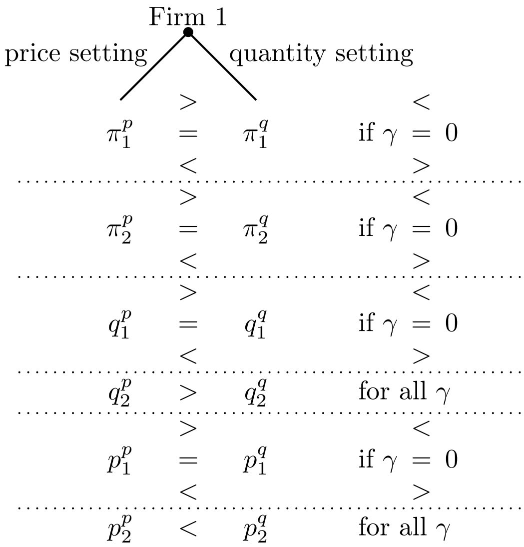

Figure 5.2: Comparison of case 1 and 2

As mentioned above, if $\alpha_1 = \alpha_2 = \frac{1}{1+\alpha}$, $\beta_1 = \beta_2 = \frac{1}{1-\alpha^2}$ and $\gamma = -\frac{\alpha}{1-\alpha^2}$ then the current demand structure is equal to that which was analyzed by Boyer and Moreaux (1987). Hence, their demand structure is a special case of that by Dixit (1979) which is also used in the current work. Moreover, it is easy to see that for the parameterization the reservation prices and the degree of product differentiation are correlated. Therefore, there exist unique ranking of the leader's and follower's prices, quantities and profits depending on the type of competition (price or quantity) and the property of the goods (complements or substitutes).[34] The following examples show that these rankings do not hold for the more general demand structure used here.

[34]For the price leader model Boyer and Moreaux (1987) show: $p_1^p > p_2^p$ and $q_1^p < q_2^p$ for all γ and $\pi_1^p \left\{ \begin{matrix} > \\ < \end{matrix} \right\} \pi_2^p$ if and only if $\gamma \left\{ \begin{matrix} < \\ > \end{matrix} \right\} 0$. And for the quantity leader model: $p_1^q < p_2^q$ and $q_1^q > q_2^q$ for all γ and $\pi_1^q \left\{ \begin{matrix} > \\ < \end{matrix} \right\} \pi_2^q$ if an only if $\gamma \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0$.

Example 6 (price leader)**.** Set $\alpha_1 = \beta_1 = 1$.

1. Prices: For $\beta_2 = 4$ and $\alpha_2 = 2$ it is $p_1^p < p_2^p \; \forall \; \gamma$.

2. Quantities: For $\beta_2 = 1$ and $\alpha_2 = \frac{1}{2}$ it is $q_1^p > q_2^p \; \forall \; \gamma$.

3. Profits: (a) For $\beta_2 = 1/4$ and $\alpha_2 = 1/2$ it is $\pi_1^p > \pi_2^p \; \forall \; \gamma$.

   (b) For $\beta_2 = 4$ and $\alpha_2 = 2$ it is $\pi_1^p < \pi_2^p \; \forall \; \gamma$.

Example 7 (quantity leader)**.** Set $\alpha_1 = \beta_1 = 1$.

1. Prices: For $\beta_2 = \frac{1}{4}$ and $\alpha_2 = \frac{1}{2}$ it is $p_1^q > p_2^q \; \forall \; \gamma$.

2. Quantities: For $\beta_2 = 1$ and $\alpha_2 = 2$ it is $q_1^q < q_2^q \; \forall \; \gamma$.

3. Profits: (a) For $\beta_2 = 1$ and $\alpha_2 = 2$ it is $\pi_1^q < \pi_2^q \; \forall \; \gamma$.

   (b) For $\beta_2 = 1$ and $\alpha_2 = \frac{1}{2}$ it is $\pi_1^q > \pi_2^q \; \forall \; \gamma$.

These examples suggest that the ordering of the reservation prices (or market size) is crucial for the ordering of the leader's and follower's profits.

The last result in this section compares the consumers' and total surplus under price and quantity leadership:[35]

Proposition 5.3. *Total and consumers' surplus are always higher in the price leader model than in the quantity leader model, whether the goods are complements or substitutes. Only if the goods are independent quantity competition is as good as price competition in terms of total and consumer surplus.*

This result confirms and generalizes the result of Boyer and Moreaux (1987) who showed that consumers' and total surplus are higher under price leadership than under quantity leadership independently of the goods beeing substitutes or complements. The same holds true in the simulanous move game by Singh and Vives (1984): Independent of the goods beeing substitutes or complements, consumers' and total surplus are always higher under price competition than under quantity competition.

[35] Here total surplus is equivalent to welfare, i.e. the sum of profits and consumer surplus.

5.2 Comparison between sequential and simultaneous move game

The previous section has shown that some but not all of the results by Boyer and Moreaux (1987) can be confirmed in the current model. In particular, the previous section compared the two possivle Stackelberg equilibria, given by p_i^p, q_i^p and π_i^p in case of price leadership and p_i^q, q_i^q and π_i^q under quantity leadership. Still following Boyer and Moreaux (1987) this section provides a comparison of these Stackelberg equilibria and the Nash equilibria of the underlying simultaneous move games as introduced by Singh and Vives (1984). Therfore, the analysis again has to distinguish between four different cases: both set quantities, both set prices and two cases in which firm i sets the price and firm j sets the quantity. Following Singh and Vives (1984) it is:

1. case: both firms are price setters
Using equation (5.1) the equilibrium prices and quantities of the simultaneous move game can be derived as

$$p_i^{BB} = \frac{2a_i b_j + c a_j}{4b_i b_j - c^2} = \frac{2\alpha_i\beta_i\beta_j - \alpha_i\gamma^2 - \beta_i\alpha_j\gamma}{4\beta_i\beta_j - \gamma^2}, \quad i,j = 1,2, \quad i \neq j$$

and

$$q_i^{BB} = b_i p_i^B = \frac{\beta_j\left(2\alpha_i\beta_i\beta_j - \beta_i\alpha_j\gamma - \alpha_i\gamma^2\right)}{4\beta_i^2\beta_j^2 - 5\beta_i\beta_j\gamma^2 + \gamma^4}, \quad i,j = 1,2, \quad i \neq j.$$

The profits can easily be derived as $\pi_i^{BB} = p_i^{BB} q_i^{BB}$.

2. case: both firms are quantity setters
Profit maximization under consideration of the inverse demand system (5.2) leads to the following equilibrium prices and quantities

$$q_i^{CC} = \frac{2\alpha_i\beta_j - \alpha_j\gamma}{4\beta_i\beta_j - \gamma^2}, \quad i,j = 1,2, \quad i \neq j$$

and

$$p_i^{CC} = \frac{\beta_i(2\alpha_i\beta_j - \alpha_j\gamma)}{4\beta_i\beta_j - \gamma^2}, \quad i,j = 1,2, \quad i \neq j$$

Clearly, the profits are $\pi_i^{CC} = p_i^{CC} q_i^{CC}$.

3. case: firm 1 sets the price and firm 2 sets the quantity
The equilibrium is characterized by

$$\begin{aligned} p_1^{BC} &= \frac{2a_1b_1 + a_2c}{4b_1b_2 - 3c^2} = \frac{2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma}{4\beta_1\beta_2 - 3\gamma^2}, \\ p_2^{BC} &= \frac{2a_2b_1 + a_1c - \frac{a_2c^2}{b_2}}{4b_1b_2 - 3c^2} = \frac{(2\alpha_2\beta_1 - \alpha_1\gamma)\left(\beta_1\beta_2 - \gamma^2\right)}{\beta_1\left(4\beta_1\beta_2 - 3\gamma^2\right)} \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} q_1^{BC} &= \tfrac{b_1b_2 - c^2}{b_2} p_1^{BC} = \tfrac{2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2}{\beta_1(4\beta_1\beta_2 - 3\gamma^2)}, \\ q_2^{BC} &= b_2 p_2^{BC} = \tfrac{2\alpha_2\beta_1 - \alpha_1\gamma}{4\beta_1\beta_2 - 3\gamma^2}, \end{aligned} \tag{5.16}$$

with corresponding profits $\pi_i^{BC} = p_i^{BC} q_i^{BC}$.

4. case: firm 1 sets the quantity and firm 2 sets the price
By arguments of symmetry just interchange indices i and j and the superscripts BC to CB in (5.15), (5.16) in order to get the equilibrium values.

The comparison if these equilibrium values of the simultanous move game and the equilibrium values of the Stackelberg-game confirms Boyer and Moreaux's result that the equilibrium prices and quantities of the Stackelberg-game are bounded from above and below by the equilibrium prices and quantities of the simulanuous move game, respectively. In particular,

Proposition 5.4. *1. Under price leadership in the Stackelberg-game the following holds*

$$p_1^{BB} < p_1^p < p_1^{BC} \text{ and } q_1^{BC} < q_1^p < q_1^{BB} \ \forall\ \gamma, \tag{5.17}$$

$$p_2^{BB} \overset{(>)}{<} p_2^p \overset{(>)}{<} p_2^{BC} \quad and \quad q_2^{BB} \overset{(>)}{<} q_2^p \overset{(>)}{<} q_2^{BC} \quad for \quad \gamma \overset{(<)}{>} 0 \tag{5.18}$$

$$\pi_1^{BC} < \pi_1^{BB} < \pi_1^p, \tag{5.19}$$

$$\pi_2^{BB} \overset{(>)}{<} \pi_2^p \overset{(>)}{<} \pi_2^{BC} \quad for \quad \gamma \overset{(<)}{>} 0. \tag{5.20}$$

2. *Quantity leadership implies*

$$q_1^{CC} < q_1^q < q_1^{CB} \; and \; p_1^{CB} < p_1^q < p_1^{CC}, \tag{5.21}$$

$$p_2^{CC} \overset{(<)}{>} p_2^q \overset{(<)}{>} p_2^{CB} \quad and \quad q_2^{CC} \overset{(<)}{>} q_2^q \overset{(<)}{>} q_2^{CB} \quad for \quad \gamma \overset{(<)}{>} 0 \tag{5.22}$$

$$\pi_1^{CB} < \pi_1^{CC} < \pi_1^q \; \forall \; \gamma, \tag{5.23}$$

$$\pi_2^{CB} \overset{(>)}{<} \pi_2^q \overset{(>)}{<} \pi_2^{CC} \quad for \quad \gamma \overset{(<)}{>} 0. \tag{5.24}$$

This result confirms Boyer and Moreaux (1987) but under weaker conditions in which no correlations between cross-effects and market size exist. Moreover, since the leader's profits are larger than the profits of the simulanuous move game (compare equation (5.19) and (5.23)), i.e. the (informational) advantage of beeing the leader (compared to the situation in the simulanuous move game) is also reflected the leader's profits.

For an illustration of Proposition 5.4 consider the following example.

Example 8. Consider $\alpha_1 = \alpha_2 = 4$, $\beta_1 = \beta_2 = 2$. This implies that $\frac{\gamma^2}{\beta_1\beta_2} = \frac{\gamma^2}{4} > 0$ measures the degree of product differentiation. For this parameter constellation the equations (5.17), (5.18), (5.19) and (5.20) [36] yield the following figures 5.3, 5.4 and 5.5.

[36] Examples for (5.21), (5.22), (5.23) and (5.24) is omitted.

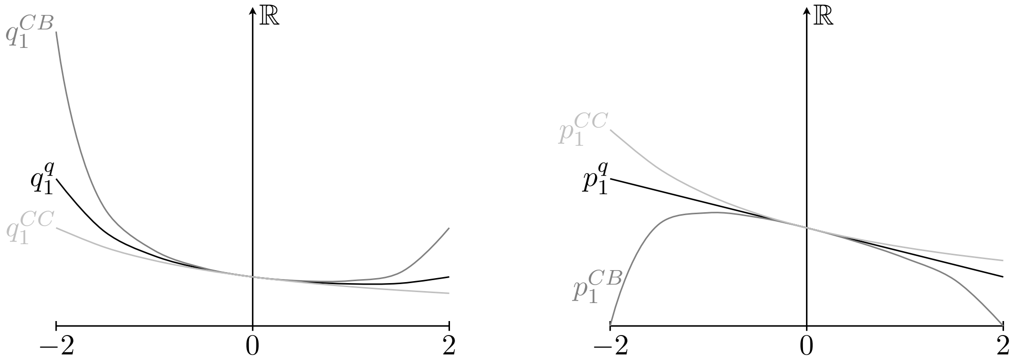

Figure 5.3: Quantity and price of firm 1.

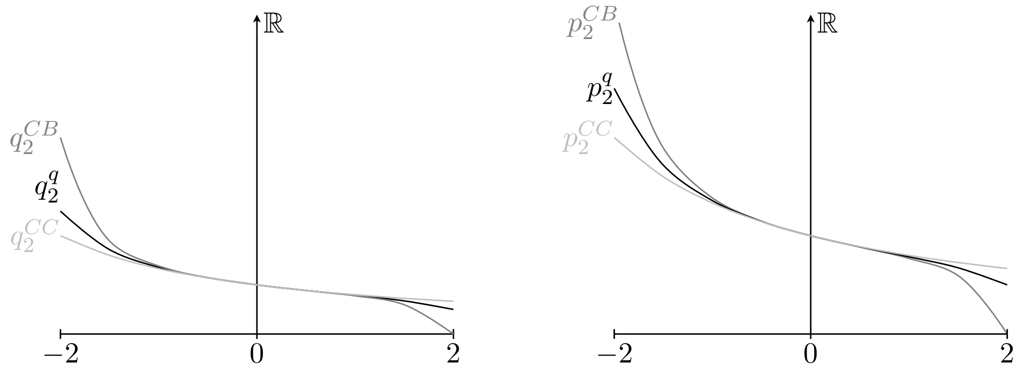

Figure 5.4: Quantity and price of firm 2.

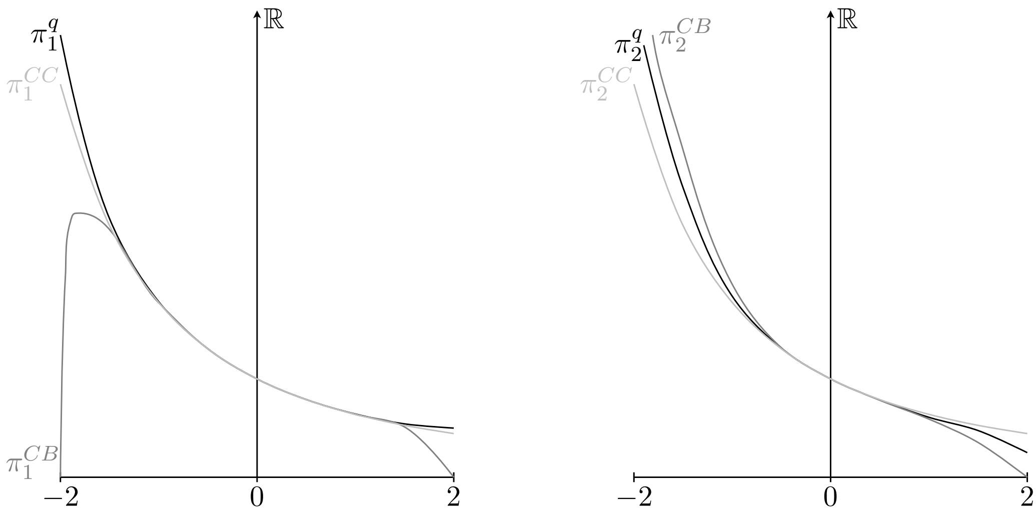

Figure 5.5: Profits of firm 1 and 2.

A direct consequence of Proposition 5.4 is that if the goods are substitutes (complements), then producers' surplus is higher in the Stackelberg-game under price (quantity) leadership than in the Bertrand (Cournot) equilibrium of the simultanuous move game.

Next, the last result compares total and consumers' surplus in the Stackelberg game with those of the simultanuous move game. Loosely speaking it is shown that consumers' and total surplus are increasing in the number of price-setting firms.

Proposition 5.5. *Total and consumer surplus is always highest in the Bertrand equilibrium and lowest in the Cournot equilibrium. In between the price (quantity) Stackelberg is always better (worse) than the mixed Nash in which firm 1 sets a price (quantity)*[37].

This generalizes the results of Boyer and Moreaux (1987), i.e. total and consumer surplus are increasing in the number of price-setting firms. Moreover, both, Proposition 5.5 and the result of Vives (1985), imply that simultaneous Bertrand competition is optimal in terms of welfare and consumer surplus.

5.3 Concluding Remarks

In this chapter the model by Boyer and Moreaux (1987) was generalized by using a less restrictive utility function which was introduced by Dixit (1979). The implications of this more general utility function are twofold: First, the demands for the two goods as a function of prices do not coincide and second the cross-effects are different. In contrast to Boyer and Moreaux (1987) it is shown that the leader's price (quantity) in the price Stackelberg model is not necessarily higher (lower) than the follower's one. Moreover, it is shown that price setting of at least one firm is preferable in terms of welfare and consumers' surplus independently of the game structure (simultaneous or sequential).

[37] The ranking of the mixed Nash in which firm 1 sets a price (quantity) and the quantity (price) Stackelberg depends on the exact parameter constelation.

Last, in contrast to Boyer and Moreaux (1987), the orderings of prices, quantities and profits of the different simultaneous move (Cournot, Bertrand, mixed) and leader-follower games by Boyer and Moreaux (1987) are verified only for firm 1. For firm 2 these orderings depend on both: the market size and the nature of the goods (substitutes or complements).

Chapter 6

Summary and Concluding Remarks

In general a decision maker's knowledge about future states of the world is imperfect and, hence, he faces risk when making a decision. However, the sourcing of additional information may increase the decision maker's knowledge and reduce the riskyness in the decision problem. A novel concept of information systems has been introduced in Chapter 2. In contrast to the traditional literature on economics where the prior is typically kept fixed, this new approach allows for different priors. Similar to the traditional literature, it formalizes the idea that information are revealed through the observation of a signal that is correlated with the state of the world. In particular, an information system consists of an information structure and a prior belief. Intuitively, the correlation of signals and states is jointly determined by the information structure and the prior belief.

Building on this, some fundamental and desirable properties of informativeness concepts have been discussed in Chapter 3 and two novel informativeness concepts, weak and strong informativeness, have been introduced. These two informativeness concepts take the impact of the prior on informativeness into account. Moreover, it is shown how these concepts relate to informativeness concepts of Blackwell (1951, 1953), Lehmann (1988)/Persico (1996) and Kim (1995): If the prior belief is fixed, then weak informativeness is weaker than any other informativeness concept considered in this work while strong informativeness is neither stronger nor weaker than the traditional concepts. Furthermore, an information system is more valuable for all decision makers with supermodular preferences if and only if the former information system is weakly more informative than the latter one and if additionally the information systems have equal underlying prior beliefs.

The following Chapter 4 analyzed the value of information in two different economic environments. The main result of the first model is that the demand for costly information is decreasing in the degree of (relative) risk aversion for broad range of parameters. The main result of the second model generalizes parts of the result by Schlee (2001) showing that the value of information is negative in a production economy with many commodities as well as complete and efficient risk-sharing markets.

Finally, Chapter 5 studies the role of information in a Stackelberg game. In particular, the value of information about the competitor's strategy space is zero. The intuition for this result is that after observing the leader's choice, the follower acts as a monopolist on the remaining market and, hence, it does not matter whether he sets a quantity or a price because his profits are predetermined by the leader's choice. Consequently, it does not matter for the leader whether the follower sets a price or a quantity and, hence, information about the follower's strategy space are worthless.

Some interesting research questions are not considered here. First, the production and supply of information. Suppose, for instance, the model of Chapter 4.1. How are the information systems produced? And, how does the production cost put together? Second, what about the equilibrium value of information in an economy where the consumers' preferences and/or the firms' production sets are state-dependent? And third, how are weak and strong informativeness characterized in terms of their value? Is it possible to characterize these concepts in terms of ex ante expected utility also for different prior beliefs? I am currently working on these questions in a joint project with Prof. Dr. Bernhard Eckwert and Dr. Felix Várdy. Our preliminary result is, loosely speaking, that a system is weakly more informative than another one if and only if it is more valuable for all decision makers with quasi-linear preferences who are indifferent (in a certain sense) without information.

Appendix A

Notation

Information systems: States and signals

- Ω - state space; $\mathcal{F}$ - σ-algebra of subsets in Ω;
  $\tilde{\omega}$ - state variable; ω - realization of state variable;
  F_Ω, f_Ω - cdf and pdf (or probability mass function) of prior belief on Ω;
  $\Delta(\Omega)$ - set of pdfs (or probability mass functions) on Ω.

- S - signal space; $\mathcal{S}$ - σ-algebra of subsets in S;
  $\tilde{s}$ - signal variable; s - signal realization;
  F_S, f_S - cdf and pdf (or probability mass function) of (marginal) signal distributions.

- $F_{S,\Omega}, f_{S,\Omega}$ - cdf and pdf of joint distribution of signals and states.

- $F_{\Omega|S}, f_{\Omega|S}$ - cdf and pdf of the posterior/conditional state distribution.

- $F^0_{S^0|\Omega}, f^0_{S^0|\Omega}$ - fully uninformative information structure;
  $F^1_{S^1|\Omega}, f^1_{S^1|\Omega}$ - full informative information structure.

- $\overset{\text{inf}}{\succsim}$ - arbitrary informativeness concept;
  $\overset{\text{b}}{\succsim}$ - Blackwell's sufficiency criterion;
  $\overset{\text{a}}{\succsim}$ - Accuracy;
  $\overset{\text{l-inf}}{\succsim}$ - Local informativeness;

$\succsim^{\text{mps}}$ - MPS-precision;
$\succsim^{\text{sm}}$ - SM-precision;
$\succsim^{\text{w-inf}}$ - weak informativeness;
$\succsim^{\text{s-inf}}$ - strong informativeness.

- $\mathcal{M}$ - set of information structures with MLRP;
  $\mathcal{D}$ - set of information structures which conditional signal densities are twice continuously differentiable with respect to ω.
  $\Gamma(f_\Omega)$ - set of information structures with MLRP and normalized signals given the prior f_Ω;
  Γ - set of all information systems with MLRP and normalized signals.

Decisions and utility

- $\mathcal{A}$ - action set.
- $\mathcal{O}$ - set of outcomes.
- u - (fundamental) utility;
  $U(a, \tilde{\omega})$ - expected utility;
  $v(a, \omega)$ - indirect utility (from an action and an state of world);
  $V((f_{S|\Omega}, f_\Omega), \cdot, v)$ - ex ante expected utility under information system $(f_{S|\Omega}, f_\Omega)$ for an agent with indirect or direct utility v.

Chapter 4

<u>Section 4.1</u>

- $\mathcal{I}$ - set of consumers;
- n - number of individual states;
- ϵ - error level;
- $f^\epsilon_{S_i|\Omega}$ - information service with error level ϵ;
- $c(\epsilon)$ - cost of producing an information service with error level ϵ;
- $P(\epsilon)$ - price for an information service with error level ϵ;

<u>Section 4.2</u>

- N - number of states;
- $\mathcal{J}$ - set of firms;
  J - number of firms;

- $\mathcal{I}$ set of consumers;
  I - number of consumers;

- $\mathcal{C} \subseteq \mathbb{R}_+^C$ - commodity space;
  C - number of different commodities;
  $\mathcal{C}_c \subseteq \mathbb{R}_+$ - space of commodity $c \in \{1, \ldots, C\}$.

- $w(\omega, i) \in \mathcal{C}$ - endowment vector of agent $i \in \mathcal{I} \cup \mathcal{J}$ in state ω;
  $w_c(\omega, i)$ - endowment of commodity c of agent i in state ω;
  $w(i) \in \mathcal{C}^N$ - agent i's tupel of endowment vectors for each state;
  $w_c(\omega) \in \mathcal{C}_c$ - aggregate endowment of commodity c in state ω;

- Y_j - firm j's technology/production set;
  $y(\omega, j) \in Y_j$ - firm j's production vector (positive=output; negative=input);
  $y(j)$ - firm j's state-contingent production plan;

- $x(\omega, i) \in \mathcal{C}$ - consumer i's consumption vector in state ω;
  $x(i) \in \mathcal{C}^N$ - consumer i's state-contingent consumption plan;
  B_i - consumer i's budget set;

- $(x(\omega), y(\omega)) \in \mathcal{C}^I \times \bigotimes_{j \in \mathcal{J}} Y_j$ - allocation in state ω;
  $(x, y) \in \left(\mathcal{C}^{\mathcal{S}}\right)^I \times \bigotimes_{j \in \mathcal{J}} Y_j^N$ - allocation of state-contingent commodities;

- $p \in \left(\mathbb{R}_+^C\right)^N$ - price vector for state-contingent claims.
  In particular, $p_{c,\omega}$ denotes the price for a claim which pays one unit of commodity c if the state is ω and nothing otherwise.

Chapter 5

- $\pi_i^{pq}(q_i^{pq}, p_i^{pq})$ - firm i's profit (quantity, price) if firm 1 sets a price and firm 2 sets a quantity;
  $\pi_i^{qp}(q_i^{pq}, p_i^{pq})$ - firm i's profit (quantity, price) if firm 1 sets a quantity and firm 2 sets a price;
  $\pi_i^{pp}(q_i^{pp}, p_i^{pp})$ - firm i's profit (quantity, price) if both firms set a price;
  $\pi_i^{qq}(q_i^{qq}, p_i^{qq})$ - firm i's profit (quantity, price) if both firms set quantities.

Appendix B

Proofs of Chapter 3

Proof of Lemma 3.1 (P0′): Define $\hat{f}^0_{S^0|S}:\ S^0 \times S,\ (s^0, s) \mapsto f^0_{S^0}(s^0)$. Then,

$$\mathbb{E}_S\left[\hat{f}^0_{S^0|S}(s^0|\tilde{s})|\omega\right] = \mathbb{E}_S\left[f^0_{S^0}(s^0)|\omega\right] = f^0_{S^0}(s^0) = f^0_{S^0|\Omega}(s^0|\omega)\ \ \forall\ s^0 \in S^0,\ \omega \in \Omega.$$

This implies $f_{S|\Omega} \overset{b}{\succsim} f^0_{S^0|\Omega}$ for all information structures $f_{S|\Omega}$.

(P1′): Consider $f^1_{S^1|\Omega}$. Define implicitly

$$\omega^1:\ S^1 \to \Omega,\ s^1 \mapsto \omega^1(s^1),$$

such that $s^1 \in C^1(\omega(s^1))$. This implies

$$s^1 \in C^1(\omega) \Rightarrow \omega^1(s^1) = \omega. \tag{B.1}$$

For an information structure $f_{S|\Omega}$ define

$$\hat{f}_{S|S^1}:\ S \times S^1,\ (s, s^1) \mapsto \hat{f}_{S|S^1}(s|s^1) = f_{S|\Omega}(s|\omega(s^1)). \tag{B.2}$$

Then

$$\mathbb{E}_{S^1}\left[\hat{f}_{S|S^1}(s|\tilde{s}^1)|\omega\right] \overset{(B.2)}{=} \mathbb{E}_{S^1}\left[f_{S|\Omega}(s|\omega^1(\tilde{s}^1))\right] \overset{(??),(B.1)}{=} f_{S|\Omega}(s|\omega)$$

for all $s \in S,\ \omega \in \Omega$. This implies $f^1_{S^1|\Omega} \overset{b}{\succsim} f_{S|\Omega}$ for all information structures $f_{S|\Omega}$.

(P2): Consider information structures $\bar{f}_{\bar{S}|\Omega}, f_{S|\Omega}$ and $\check{f}_{\check{S}|\Omega}$ such that $\bar{f}_{\bar{S}|\Omega} \overset{b}{\succsim} f_{S|\Omega}$ and $f_{S|\Omega} \overset{b}{\succsim} \check{f}_{\check{S}|\Omega}$. Therefore,

$$f_{S|\Omega}(s|\omega) = \bar{\mathbb{E}}_{\bar{S}} \left[\hat{\bar{f}}_{S|\bar{S}}(s|\tilde{\bar{s}})|\omega \right] \quad \forall\ s \in S,\ \omega \in \Omega \tag{B.3}$$

and

$$\check{f}_{\check{S}|\Omega}(\check{s}|\omega) = \mathbb{E}_{S} \left[\hat{\check{f}}_{\check{S}|S}(\check{s}|\tilde{s})|\omega \right] \quad \forall\ \check{s} \in \check{S},\ \omega \in \Omega. \tag{B.4}$$

Now define

$$\bar{\check{f}}_{\check{S}|\bar{S}} :\ \check{S} \times \bar{S} \to \mathbb{R}_+,\ (\check{s}, \bar{s}) \mapsto \mathbb{E}_S \left[\hat{\check{f}}_{\check{S}|S}(\check{s}|\tilde{s})|\bar{s} \right].$$

Then,

$$\begin{aligned} \bar{\mathbb{E}}_{\bar{S}} \left[\bar{\check{f}}_{\check{S}|\bar{S}}(\check{s}|\tilde{\bar{s}})|\omega \right] &= \bar{\mathbb{E}}_{\bar{S}} \left[\mathbb{E}_S \left[\hat{\check{f}}_{\hat{S}|S}(\check{s}|\tilde{s})|\tilde{\bar{s}} \right] |\omega \right] \\ = \quad & \int_{\bar{S}} \int_{S} \hat{\check{f}}_{\check{S}|S}(\check{s}|s) \hat{\bar{f}}_{S|\bar{S}}(s|\bar{s}) \bar{f}_{\bar{S}|\Omega}(\bar{s}|\omega) \mathrm{d}s \mathrm{d}\bar{s} \\ \overset{\text{Fubini},(B.3)}{=} & \int_{S} \hat{\check{f}}_{\check{S}|S}(\check{s}|s) f_{S|\Omega}(s|\omega) \mathrm{d}s \overset{(B.4)}{=} \check{f}_{\check{S}|\Omega}(\check{s}|\omega) \end{aligned}$$

for all $\check{s} \in \check{S},\ \omega \in \Omega$.

(P3): Define $\hat{f}_{S|S}(s|s') = 1_{\{s'\}}(s)$, where $\delta_{\{s'\}}$ denotes the indicator function of the set $\{s'\} \in \mathcal{S}$.[38] Then

$$\mathbb{E}_S \left[\hat{f}_{S|S}|\omega \right] = f_{S|\Omega}(s|\omega)\ \forall\ s \in S,\ \omega \in \Omega$$

shows $f_{S|\Omega} \overset{b}{\succsim} f_{S|\Omega}$ for all information structures $f_{S|\Omega}$.

(IS): Suppose $f_{S|\Omega} \overset{b}{\succsim} \bar{f}_{\bar{S}|\Omega}$ and let $t : \Omega \to \Omega'$ be bijective. Hence, there exists an information structure $\hat{f}_{\bar{S}|S}$ such that

$$\bar{f}_{\bar{S}|\Omega}(\bar{s}|\omega) = \mathbb{E}_S \left[\hat{f}_{\bar{S}|S}(\bar{s}|\tilde{s})|\omega \right]\ \forall\ \bar{s} \in \bar{S},\ \omega \in \Omega.$$

[38] Remember: $1_{\{s'\}}(s) = \begin{cases} 1 & \text{if } s = s' \\ 0 & \text{else.} \end{cases}$

Since $f_{S|t(\Omega)}(s|\omega') = f_{S|\Omega}(s|t^{-1}(\omega'))$ and $\bar{f}_{\bar{S}|t(\Omega)}(\bar{s}|\omega') = \bar{f}_{\bar{S}|\Omega}(\bar{s}|t^{-1}(\omega'))$ for all $s \in S, \bar{s} \in \bar{S}$ and $\omega' \in t(\Omega)$, this implies

$$\bar{f}_{S|t(\Omega)}(\bar{s}|\omega') = \bar{f}_{\bar{S}|\Omega}(\bar{s}|t^{-1}(\omega')) = \mathbb{E}_S\left[\hat{f}_{\bar{S}|S}(\bar{s}|\tilde{s})|t^{-1}(\omega')\right] \;\forall\; \bar{s} \in \bar{S},\; \omega' \in t(\Omega).$$

Hence, $f_{S|t(\Omega)} \overset{\mathrm{b}}{\succsim} \bar{f}_{\bar{S}|t(\Omega)}$. □

Proof of Proposition 3.1 Define

$$\gamma_{S|(\bar{S},S)} :\; S \times \bar{S} \times S, (s, \bar{s}, s') \mapsto \begin{cases} 1 & \text{if } s = s' \\ 0 & \text{else.} \end{cases}$$

Then,

$$\begin{aligned} \mathbb{E}_{(\hat{S},S)}\left[\gamma_{S|(\bar{S},S)}(s, \tilde{\hat{s}}, \tilde{s}')|\omega\right] &= \int\limits_{\bar{S}}\int\limits_{S} \gamma_{S|(\bar{S},S)}(s, \bar{s}, s') f_{(S,\bar{S})|\Omega}(\bar{s}, s'|\omega)\mathrm{d}s'\mathrm{d}\bar{s} \\ &= \int\limits_{\bar{S}} \bar{f}_{\bar{S}|\Omega}(\bar{s}|\omega) f_{S|\Omega}(s|\omega)\mathrm{d}\bar{s} = f_{S|\Omega}(s|\omega). \end{aligned}$$

□

The proof of Proposition 3.3 makes use of the following Lemma:

Lemma B.1. *(i)* $f_{S|\Omega} \in \mathcal{M} \implies F_{S|\Omega}(s|\omega) \geq F_{S|\Omega}(s|\omega')$ *for all* $\omega, \omega' \in \Omega$, $\omega' \geq \omega$, *and all* $s \in S$.

(ii) $f_{S|\Omega} \in \mathcal{M} \implies F^{-1}(p|\omega) \leq F^{-1}(p|\omega')$ *for all* $\omega, \omega' \in \Omega$ *and all* $p \in [0, 1]$.

Proof: (i) Let $f_{S|\Omega} \in \mathcal{M}$. The MLRP is equivalent to

$$f_{S|\Omega}(s'|\omega') f_{S|\Omega}(s|\omega) \geq f_{S|\Omega}(s|\omega') f_{S|\Omega}(s'|\omega) \text{ for all } \omega' \geq \omega \text{ and } s' \geq s. \quad \text{(B.5)}$$

Integration of (B.5) over s from $\underline{s}$ to s' delivers

$$\begin{aligned} & f_{S|\Omega}(s'|\omega')F_{S|\Omega}(s'|\omega) \geq f_{S|\Omega}(s'|\omega)F_{S|\Omega}(s'|\omega') \text{ for all } \omega' \geq \omega \text{ and } s' \in S \\ \Longleftrightarrow \ & \frac{f_{S|\Omega}(s'|\omega')}{f_{S|\Omega}(s|\omega)} \geq \frac{F_{S|\Omega}(s'|\omega')}{F_{S|\Omega}(s'|\omega)} \text{ for all } \omega' \geq \omega \text{ and } s' \in S. \end{aligned} \tag{B.6}$$

Next, integration of (B.5) over s' from s to $\bar{s}$ delivers

$$\begin{aligned} & (1 - F_{S|\Omega}(s|\omega'))f_{S|\Omega}(s|\omega) \geq f_{S|\Omega}(s|\omega')(1 - F_{S|\Omega}(s|\omega)) \text{ for all } \omega' \geq \omega \text{ and } s \in S \\ \Longleftrightarrow \ & \frac{1 - F_{S|\Omega}(s|\omega')}{1 - F_{S|\Omega}(s|\omega)} \geq \frac{f_{S|\Omega}(s|\omega')}{f_{S|\Omega}(s|\omega)} \text{ for all } \omega' \geq \omega \text{ and } s \in S. \end{aligned} \tag{B.7}$$

Combining equations (B.6) and (B.7) gives

$$\begin{aligned} & \frac{1 - F_{S|\Omega}(s|\omega')}{1 - F_{S|\Omega}(s|\omega)} \geq \frac{F_{S|\Omega}(s|\omega')}{F_{S|\Omega}(s|\omega)} \text{ for all } \omega' \geq \omega \text{ and } s \in S \\ \Longleftrightarrow \ & \frac{1 - F_{S|\Omega}(s|\omega')}{F_{S|\Omega}(s|\omega')} \geq \frac{1 - F_{S|\Omega}(s|\omega)}{F_{S|\Omega}(s|\omega)} \text{ for all } \omega' \geq \omega \text{ and } s \in S. \end{aligned}$$

Hence, $F_{S|\Omega}(s|\omega') \leq F_{S|\Omega}(s|\omega)$ for all $\omega \leq \omega'$ and $s \in S$.

(ii) Part (i) implies that $F_{S|\Omega}(s|\omega)$ is decreasing in ω for all $\omega \in \Omega$. Therefore, $F^{-1}_{S|\Omega}(p|\omega) := \inf \{ s \in S | F_{S|\Omega}(s|\omega) \geq p \}$ must be increasing in ω for all $p \in [0, 1]$. □

Proof of Proposition 3.3 (P0′): Let $f^0_{S^0|\Omega}, f_{S|\Omega} \in \mathcal{M}$. Since $f^0_{S^0|\Omega}$ is fully uninformative it follows $f^0_{S^0|\Omega}(s^0|\omega) = f^0_{S^0|\Omega}(s^0|\omega')$ for all $\omega, \omega' \in \Omega$ and all $s^0 \in S^0$. This implies that $F^0_{S^0|\Omega}(s^0|\omega)$ is constant in ω for all $s^0 \in S^0$. Since $f_{S|\Omega}$ has the MLRP, Lemma B.1 (ii) now implies that

$$T: \ S^0 \times \Omega, (s^0, \omega) \mapsto F^{-1}_{S|\Omega}(F^0_{S^0|\Omega}(s^0|\omega)|\omega)$$

is nondecreasing in ω for all $s^0 \in S^0$ which shows $f_{S|\Omega} \overset{a}{\succsim} f^0_{S^0|\Omega}$.

(P1′): Now consider $f^1_{S^1|\Omega}, f_{S|\Omega} \in \mathcal{M}$. First, observe that the definition of $f^1_{S^1|\Omega}$ implies $F^{1^{-1}}_{S^1|\Omega}(p|\omega) \in C^1(\omega)$ for all $p \in [0, 1]$. Next observe that for an information structure with MLRP, $f^1_{S^1|\Omega} \in \mathcal{M}$, the following holds: Let $s \in C^1(\omega)$ and $s' \in C^1(\omega')$. Since $C^1(\omega) \cap C^1(\omega')$, $\omega' \geq \omega$ implies $s \leq s'$. These two observations imply that

$$T: \ S \times \Omega \to S^1, \ (s, \omega) \mapsto F^{1^{-1}}_{S^1|\Omega}(F(s|\omega)|\omega)$$

is nondecreasing in ω for all $s \in S$.

<u>(P2):</u> Let $f_{S|\Omega}, \ \bar{f}_{S|\Omega}, \ \hat{f}_{\hat{S}|\Omega} \in \mathcal{M}$ such that $f_{S|\Omega} \overset{a}{\succsim} \bar{f}_{\bar{S}|\Omega}$ and $\bar{f}_{\bar{S}|\Omega} \overset{a}{\succsim} \hat{f}_{\hat{S}|\Omega}$. Hence,

$$\bar{T} : \bar{S} \times \Omega \to S, \ (\bar{s}, \omega) \mapsto F^{-1}_{S|\Omega}(\bar{F}_{\bar{S}|\Omega}(\bar{s}|\omega)|\omega)$$

and

$$\hat{\bar{T}} : \hat{S} \times \Omega \to \bar{S}, \ (\hat{s}, \omega) \mapsto \bar{F}^{-1}_{\bar{S}|\Omega}(\hat{F}_{\hat{S}|\Omega}(\hat{s}|\omega)|\omega)$$

are nondecreasing in ω for all $\bar{s} \in \bar{S}$ and $\hat{s} \in \hat{S}$, respectively. Since the composition of two nondecreasing functions is in turn nondecreasing, this implies that

$$\hat{T}: \ \hat{S} \times \Omega \to S, (\hat{s}, \omega) \mapsto F^{-1}_{S|\Omega}(\hat{F}_{\hat{S}|\Omega}(\hat{s}|\omega)|\omega) = \bar{T} \circ \hat{\bar{T}}(\hat{s}, \omega)$$

is nondecreasing in ω for all $\hat{s} \in \hat{S}$.

<u>(P3):</u> Let $f_{S|\Omega} \in \mathcal{M}$. Then,

$$T : S \times \Omega \to S, (s, \omega) \mapsto F^{-1}_{S|\Omega}(F_{S|\Omega}(s|\omega)|\omega) = s$$

is the projection of (s, ω) on s and, hence, constant in ω for all $s \in S$.

<u>(OS):</u> First, observe that for $f_{S|\Omega} \in \mathcal{M}$ and a strictly increasing state transformation $t : \ \Omega \to \mathbb{R}$ it is obvious that $f_{S|t(\Omega)} \in \mathcal{M}$.[39] Moreover, for

[39] In particular: $f_{S|\Omega}(s|\omega)/f_{S|\Omega}(s|\omega')$ is decreasing in s for all $\omega, \omega' \in \Omega$ such that $\omega' \geq \omega$ $\overset{t \text{ strictly increasing}}{\Longleftrightarrow}$ $f_{S|t(\Omega)}(s|\bar{\omega})/f_{S|t(\Omega)}(s|\bar{\omega}') = f_{S|\Omega}(s|t^{-1}(\bar{\omega}))/f_{S|\Omega}(s|t^{-1}(\bar{\omega}')$ is decreasing in s for all $\bar{\omega}, \bar{\omega}' \in t(\Omega)$ such that $\bar{\omega}' \geq \bar{\omega}$.

$f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega} \in \mathcal{M}$ and $t: \ \Omega \to \mathbb{R}$ strictly increasing, it is

$$T: \ \bar{S} \times t(\Omega) \to S, \ (s,\omega) \mapsto F^{-1}_{S|t(\Omega)}(\bar{F}_{\bar{S}|t(\Omega)}(\bar{s}|\omega)|\omega) = F^{-1}_{S|\Omega}(\bar{F}_{\bar{S}|\Omega}(\bar{s}|t^{-1}(\omega))|t^{-1}(\omega))$$

Since t is strictly increasing this implies that if $F^{-1}_{S|\Omega}(\bar{F}_{\bar{S}|\Omega}(\bar{s}|\omega)|\omega)$ is nondecreasing in ω for all $\bar{s} \in \bar{S}$, then $F^{-1}_{S|t(\Omega)}(\bar{F}_{\bar{S}|t(\Omega)}(\bar{s}|\omega')|\omega')$ is nondecreasing in ω' for all $\bar{s} \in \bar{S}$. $\square$

Proof of Proposition 3.4 (P0): Consider $f^0_{S^0|\Omega}, f_{S|\Omega} \in \mathcal{D}$. Since $f^0_{S^0|\Omega}$ is state independent it follows that $\partial f^0_{S^0|\Omega}(s^0|\omega)/\partial\omega = 0$ for all $s^0 \in S^0$ and $\omega \in \Omega$. Hence, $\mathbb{E}_S\left[\frac{\partial f^0_{S^0|\Omega}/\partial\omega}{f^0_{S^0|\Omega}}(\tilde{s}|\omega)|\omega\right] = 0$ for all $\omega \in \Omega$ and

$$L_{f^0_{S^0|\Omega}}(x,\omega) = \begin{cases} 0 & \text{, if } x < 0, \\ 1 & \text{, if } x \geq 0. \end{cases} \quad \text{for all } \omega \in \Omega.$$

- Case A: Suppose there is $\omega \in \Omega$ such that $\mathbb{E}_S\left[\frac{\partial f_{S|\Omega}/\partial\omega}{f_{S|\Omega}}(\tilde{s}|\omega)|\omega\right] \neq 0$. Then, whether $L_{f^0_{S^0|\Omega}}(\cdot,\omega)$ is a MPS of $L_{f_{S|\Omega}}(\cdot,\omega)$ nor $L_{f_{S|\Omega}}(\cdot,\omega)$ is a MPS of $L_{f^0_{S^0|\Omega}}(\cdot,\omega)$. Hence, $f_{S|\Omega} \not\succsim^{\text{l-inf}} \bar{f}_{\bar{S}|\Omega}$ and $\bar{f}_{\bar{S}|\Omega} \not\succsim^{\text{l-inf}} f_{S|\Omega}$.

- Case B: Suppose $\mathbb{E}_S\left[\frac{\partial f_{S|\Omega}/\partial\omega}{f_{S|\Omega}}(\tilde{s}|\omega)|\omega\right] = 0$ for all $\omega \in \Omega$. In particular, this implies the following: If $\frac{\partial f_{S|\Omega}}{\partial\omega}(s|\omega) \neq 0$ for some $\omega \in \Omega$ and $s \in S$ (i.e. $f_{S|\Omega}$ is not fully uninformative), then there exist $x < 0$ s.t. $L_{f_{S|\Omega}}(x,\omega) > 0$. Hence,

  $$\int_{-\infty}^{x} L_{f^0_{S^0|\Omega}}(x',\omega) - L_{f_{S|\Omega}}(x',\omega)\mathrm{d}x' = -\int_{-\infty}^{x} L_{f_{S|\Omega}}(x',\omega)\mathrm{d}x' < 0.$$

  This implies that $L_{f_{S|\Omega}}(\cdot,\omega)$ is not a MPS of $L_{f^0_{S^0|\Omega}}(\cdot,\omega)$, i.e. $f_{S|\Omega} \not\succsim^{\text{l-inf}} f^0_{S^0|\Omega}$.

(P2): Let $f_{S|\Omega}, \ \bar{f}_{\bar{S}|\Omega}, \ \hat{f}_{\hat{S}|\Omega} \in \mathcal{D}$ such that $f_{S|\Omega} \succsim^{\text{l-inf}} \bar{f}_{\bar{S}|\Omega}$ and $\bar{f}_{\bar{S}|\Omega} \succsim^{\text{l-inf}} \hat{f}_{\hat{S}|\Omega}$. In particular, this implies that $L_{f_{S|\Omega}}(\cdot,\omega)$ is a MPS of $L_{\bar{f}_{\bar{S}|\Omega}}(\cdot,\omega)$ for all $\omega \in \Omega$ and that $L_{\bar{f}_{\bar{S}|\Omega}}(\cdot,\omega)$ is a MPS of $L_{\hat{f}_{\hat{S}|\Omega}}(\cdot,\omega)$ for all $\omega \in \Omega$. Consequently, $L_{f_{S|\Omega}}(\cdot,\omega)$ and $L_{\hat{f}_{\hat{S}|\Omega}}(\cdot,\omega)$ have equal mean for all $\omega \in \Omega$. And since the convex order is transitive this implies $f_{S|\Omega} \succsim^{\text{l-inf}} \hat{f}_{\hat{S}|\Omega}$.

<u>(P3):</u> Obvious.

<u>(wOS):</u> This part of the proof makes use of the following equality: For any $f_{S|\Omega} \in \mathcal{D}$ and for any strictly increasing, (twice) continuously differentiable function $t : \Omega \to \mathbb{R}$ the following holds

$$\begin{aligned} L_{f_{S|t(\Omega)}}(x, \omega) =& Prob\left(\frac{\partial f_{S|t(\Omega)}/\partial\omega}{f_{S|t(\Omega)}}(\tilde{s}|\omega') \leq x\right) \\ =& Prob\left(\frac{\partial f_{S|\Omega}/\partial\omega}{f_{S|\Omega}}(\tilde{s}|t^{-1}(\omega')) \left(t^{-1}\right)'(\omega') \leq x\right) \\ =& L_{f_{S|\Omega}}\left(\frac{x}{\left(t^{-1}\right)'(\omega')}, t^{-1}(\omega')\right). \end{aligned} \tag{B.8}$$

Now consider $f_{S|\Omega}, \bar{f}_{\bar{S}|\Omega} \in \mathcal{D}$ such that $f_{S|\Omega} \succsim^{\text{l-inf}} \bar{f}_{\bar{S}|\Omega}$ and let $t : \Omega \to \mathbb{R}$ be strictly increasing and twice continuously differentiable. First, observe that this implies that $L_{f_{S|t(\Omega)}}(\cdot, \omega')$ and $L_{\bar{f}_{\bar{S}|t(\Omega)}}(\cdot, \omega)$ have equal mean for all $\omega' \in t(\Omega)$, i.e.

$$\begin{aligned} &\mathbb{E}_S\left[\frac{\partial f_{S|t(\Omega)}/\partial\omega}{f_{S|t(\Omega)}}(\tilde{s}|\omega')|\omega'\right] = \mathbb{E}_S\left[\frac{\partial f_{S|\Omega}/\partial\omega}{f_{S|\Omega}}(\tilde{s}|\omega')|t^{-1}(\omega')\right] \\ =&\mathbb{E}_S\left[\frac{\partial \bar{f}_{S|\Omega}/\partial\omega}{\bar{f}_{S|\Omega}}(\tilde{s}|\omega')|t^{-1}(\omega')\right] = \mathbb{E}_S\left[\frac{\partial \bar{f}_{\bar{S}|t(\Omega)}/\partial\omega}{\bar{f}_{\bar{S}|t(\Omega)}}(\tilde{s}|\omega')|\omega'\right] \end{aligned}$$

for all $\omega' \in t(\Omega)$. Then, the inequality

$$\begin{aligned} &\int_{-\infty}^{x} L_{f_{S|t(\Omega)}}(x', \omega')\mathrm{d}x' = \underbrace{\left(t^{-1}\right)'(\omega')}_{const.} \int_{-\infty}^{\frac{x}{(t^{-1})'(\omega')}} L_{f_{S|\Omega}}(x', t^{-1}(\omega'))\mathrm{d}x' \\ \leq& \underbrace{\left(t^{-1}\right)'(\omega')}_{const.} \int_{-\infty}^{\frac{x}{(t^{-1})'(\omega')}} L_{\bar{f}_{\bar{S}|\Omega}}(x', t^{-1}(\omega'))\mathrm{d}x' = \int_{-\infty}^{x} L_{\bar{f}_{\bar{S}|t(\Omega)}}(x', \omega')\mathrm{d}x' \end{aligned}$$

holds for all $x \in \mathbb{R}$ and $\omega' \in t(\Omega)$. The equalities follow from equation (B.8) and the transformation theorem while the inequality follows by assumption ($f_{S|\Omega} \overset{\text{l-inf}}{\succsim} \bar{f}_{\bar{S}|\Omega}$). Summing up, it follows $f_{S|t(\Omega)} \overset{\text{l-inf}}{\succsim} \bar{f}_{\bar{S}|t(\Omega)}$. □

Proof of Proposition 3.6 (P0′): Consider $f^0_{S|\Omega} \in \Gamma(f^0_\Omega)$, the normalized, fully uninfomative information structure, and $f_{S|\Omega} \in \Gamma(f_\Omega)$. Then the MLRP implies that

$$\int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega') - f^0_{S,\Omega}(s,\omega')\mathrm{d}\omega' = F_{\Omega|S}(\omega|s) - F_\Omega(\omega).$$

is decreasing in s and, hence, $(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\succsim} (f^0_{S|\Omega}, f^0_\Omega)$ and $(f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (f^0_{S|\Omega}, f^0_\Omega)$.

(P1): Let $f^1_{S|\Omega}$ denote the for the prior f^1_Ω normalized, fully informative information structure with MLRP (i.e. $f^1_{S|\Omega} \in \Gamma(f^1_\Omega)$ and $f^1_{S|\Omega}$ fully informative) and let $f_{S|\Omega} \in \Gamma(f_\Omega)$. Now consider

$$D(s,\omega) := \int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega') - f^1_{S,\Omega}(s,\omega')\mathrm{d}\omega' = \begin{cases} F_{\Omega|S}(\omega|s) & \text{if } \omega < \omega^1(s) \\ F_{\Omega|S}(\omega|s) - 1 & \text{if } \omega \geq \omega^1(s) \end{cases}$$

$$D(s,\omega) := \int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega') - f^1_{S,\Omega}(s,\omega')\mathrm{d}\omega' = \begin{cases} F_{\Omega|S}(\omega|s) & \text{if } \omega < \omega^1(s) \\ F_{\Omega|S}(\omega|s) - 1 & \text{if } \omega \geq \omega^1(s) \end{cases}$$

where $\omega^1(s) : S \to \Omega$ is implicitly defined by $s \in C^1(\omega^1(s))$. The MLRP implies that $\omega^1(s)$ is non-decreasing in s. Next fix $\hat{\omega} \in \Omega$ such that $F_{\Omega|S}(\hat{\omega}|\hat{s}) < 1$ where $\hat{s} := \inf(C^1(\omega'))$. If $f_{S,\Omega} \not\equiv f^1_{S,\Omega}$,[40] then exists $s' < \hat{s}$ with $\hat{\omega} < \omega^1(s')$ and $D(s',\hat{\omega}) = F_{\Omega|S}(\hat{\omega}|s') - 1 < 0$. Since for any $s'' \geq \hat{s}$ it is $\hat{\omega} \geq \omega^1(\hat{s})$, which implies $D(s'',\hat{\omega}) = F_{\Omega|S}(\hat{\omega}|s'') \geq 0$, it follows that $D(s,\hat{\omega})$ is increasing in $\hat{s}$. This shows $(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\not\succsim} (f^1_{S|\Omega}, f^1_\Omega)$ and $(f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\not\succsim} (f^1_{S|\Omega}, f^1_\Omega)$.

(P2): Consider $(f_{S|\Omega}, f_\Omega)$, $(\bar{f}_{S|\Omega}, \bar{f}_\Omega)$, $(\hat{f}_{S|\Omega}, \hat{f}_\Omega) \in \Gamma$ such that $(f_{S|\Omega}, f_\Omega) \overset{x}{\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$

[40] $f_{S,\Omega} \equiv f^1_{S,\Omega}$, then both systems are fully informative.

and $(\bar{f}_{S|\Omega}, \bar{f}_\Omega) \overset{\text{x}}{\succsim} (\hat{f}_{S|\Omega}, \hat{f}_\Omega)$ for $x \in \{$s-inf, w-inf$\}$. For $x =$ s-inf it follows that

$$\int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega') - \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega'$$
$$= \int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega') - \bar{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' + \int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega') - \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega'$$

is non-increasing in s for all $\omega \in \Omega$, i.e. $(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\succsim} (\hat{f}_{S|\Omega}, \hat{f}_\Omega)$.
$x =$ w-inf implies

$$F_{S,\Omega}(s,\omega) - sF_\Omega(\omega) \geq \bar{F}_{S,\Omega}(s,\omega) - s\bar{F}_\Omega(\omega) \geq \hat{F}_{S,\Omega}(s,\omega) - s\hat{F}_\Omega(\omega) \; \forall \; (s,\omega) \in S \times \Omega$$

and, hence, $(f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (\hat{f}_{S|\Omega}, \hat{f}_\Omega)$.

<u>(P3):</u> Obvious.

<u>(OS):</u> Consider a strictly increasing, differentiable state transformation $t: \Omega \to \mathbb{R}$ and define $\tilde{x} := t(\tilde{\omega})$. For an information system $(f_{S|\Omega}, f_\Omega) \in \Gamma$, equations (3.1) and (3.2) imply for the joint distribution of $\tilde{x} = t(\tilde{\omega})$ and s

$$f_{S,t(\Omega)}(s,x) = \frac{f_{S,\Omega}(s,t^{-1}(x))}{t'(t^{-1}(x))}.$$

Hence,

$$\int_{\underline{x}}^{x} f_{S,t(\Omega)}(s,x')\mathrm{d}x' = \int_{\underline{x}}^{x} \frac{f_{S,\Omega}(s,t^{-1}(x'))}{t'(t^{-1}(x'))}\mathrm{d}x' = \int_{\underline{\omega}}^{t^{-1}(x)} f_{S,\Omega}(s,\omega)\mathrm{d}\omega \qquad \text{(B.9)}$$

and

$$\begin{aligned}F_{S,t(\Omega)}(s,x) - sF_{t(\Omega)}(x) =& \int_0^s \int_{\underline{x}}^x f_{S,t(\Omega)}(s',x') - f_{t(\Omega)}(x')\mathrm{d}x'\mathrm{d}s' \\ =& \int_0^s \int_{\underline{\omega}}^{t^{-1}(x)} f_{S,\Omega}(s',\omega) - f_{\Omega}(\omega')\mathrm{d}\omega\mathrm{d}s' \\ =& F_{S,\Omega}(s,t^{-1}(x)) - sF_{\Omega}(t^{-1}(x)). \end{aligned} \tag{B.10}$$

The claim follow from Definition 3.10 and equations (B.9) and (B.10). □

The proof of Proposition 3.7 makes use of the following Lemma which is a variation of Theorem (3.A.5) in Shaked and Shantbikumar (2007).

Lemma B.2. *Let $\tilde{s}$ be uniformly distributed on $S = [0,1]$ and let $g_i : [0,1] \to \mathbb{R}$, $i = 1,2$, be integrable and increasing functions with $\mathbb{E}_S[g_1(\tilde{s})] = \mathbb{E}_S[g_2(\tilde{s})]$. Then $\tilde{x}_1 := g_1(\tilde{s})$ is a MPS of $\tilde{x}_2 := g_2(\tilde{s})$, iff*

$$\begin{aligned}&\int_0^s g_1(s') - g_2(s')ds' \leq 0 \quad \forall\ s \in [0,1] \\ \Longleftrightarrow &\int_s^1 g_1(s') - g_2(s')ds' \geq 0 \quad \forall\ s \in [0,1]\end{aligned}$$

Proof of Proposition 3.7 Integration by parts yields for any $f_{S|\Omega} \in \Gamma(f_\Omega)$, any $f_\Omega \in \Delta(\Omega)$ and any (strictly) increasing state transformation $t : \Omega \to \mathbb{R}$

$$\mathbb{E}_\Omega[t(\tilde{\omega})|s] = t(\bar{\omega}) - \int_\Omega \int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega')t'(\omega)\mathrm{d}\omega'\mathrm{d}\omega = t(\bar{\omega}) - \int_\Omega F_{\Omega|S}(\omega|s)t'(\omega)\mathrm{d}\omega. \tag{B.11}$$

First, consider the strong informativeness criterion. The weak information criterion will be dealt with subsequently.

(i) "$\Rightarrow$": Let $f_{S|\Omega} \in \Gamma(f_\Omega)$, $\bar{f}_{S|\Omega} \in \Gamma(\bar{f}_\Omega)$ and assume $(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$. Using equation B.11 yields

$$\begin{aligned}&\mathbb{E}_\Omega\left[t(\tilde{\omega})|s\right] - \bar{\mathbb{E}}_\Omega\left[t(\tilde{\omega})|s\right]\\ &= -\int\limits_\Omega \int\limits_{\underline{\omega}}^{\omega} \left[f_{S,\Omega}(s,\omega') - \bar{f}_{S,\Omega}(s,\omega')\right] t(\omega')\mathrm{d}\omega'\mathrm{d}\omega \end{aligned} \tag{B.12}$$

$(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$ implies that $\int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega) - \bar{f}_{S,\Omega}(s,\omega)$ is non-increasing in s for all $\omega \in \Omega$. Therefore, the LHS of B.12 is non-decreasing in s. Together with OS this shows $(f_{S|t(\Omega)}, f_{t(\Omega)}) \overset{\text{sm}}{\succsim} (\bar{f}_{S|t(\Omega)}, f_{t(\Omega)})$ for all (strictly) increasing state transformations $t : \Omega \to \mathbb{R}$.

"$\Leftarrow$": This direction is shown by contradiction: $(f_{S|\Omega}, f_\Omega) \overset{\text{sm}}{\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$ and $(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\not\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega) \Longrightarrow \exists\, t : \Omega \to \mathbb{R}$ s.t. $(f_{S|t(\Omega)}, f_{t(\Omega)}) \overset{\text{sm}}{\not\succsim} (\bar{f}_{S|t(\Omega)}, \bar{f}_{t(\Omega)})$.

Therefore, suppose $(f_{S|\Omega}, f_\Omega) \overset{\text{sm}}{\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$ and $(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\not\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$. Now define

$$D(s,\omega) := \int\limits_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega') - \bar{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega'. \tag{B.13}$$

Since $(f_{S|\Omega}, f_\Omega) \overset{\text{s-inf}}{\not\succsim} (\bar{f}_{S|\Omega}, \bar{f}_\Omega)$ for some $s', s'' \in S = S$, $s' > s''$, there exists $\Omega_0(s, s') \subset \Omega$ (with positive measure) such that

$$D(s',\omega) - D(s'',\omega) > 0 \;\text{ for any } \omega \in \Omega_0(s', s''). \tag{B.14}$$

Choose $\epsilon > 0$ and define a state transformation such that

$$t'(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega_0(s', s''), \\ \epsilon & \text{if } \omega \notin \Omega_0(s', s''). \end{cases} \tag{B.15}$$

This yields

$$\mathbb{E}_\Omega\left[t(\tilde{\omega})|s\right] - \bar{\mathbb{E}}_\Omega\left[t(\tilde{\omega})|s\right] \overset{\text{(B.12),(B.21)}}{=} -\int\limits_\Omega D(s,\omega)t'(\omega)\mathrm{d}\omega. \tag{B.16}$$

Thus,

$$\begin{aligned}
&\left(\mathbb{E}_\Omega(t(\tilde\omega)|s') - \bar{\mathbb{E}}_\Omega\left[t(\tilde\omega)|s'\right]\right) - \left(\mathbb{E}_\Omega(t(\tilde\omega)|s'') - \bar{\mathbb{E}}_\Omega\left[t(\tilde\omega)|s''\right]\right)\\
\overset{(B.15),(B.16)}{=} & -\int\limits_{\Omega_o(s',s'')} \left(D(s',\omega) - D(s'',\omega)\right) t'(\omega)\mathrm{d}\omega - \int\limits_{\Omega\setminus\Omega_0(s',s'')} \left(D(s',\omega) - D(s'',\omega)\right) t'(\omega)\mathrm{d}\omega\\
\overset{(B.15)}{\le} & -\int\limits_{\Omega_o(s',s'')} \left(D(s',\omega) - D(s'',\omega)\right)\mathrm{d}\omega + \epsilon \int\limits_{\Omega\setminus\Omega_0(s',s'')} \left|D(s',\omega) - D(s'',\omega)\right|\mathrm{d}\omega
\end{aligned}$$

which is by (B.14) negatice for ϵ sufficiently small. Hence, $(f_{S|t(\Omega)}, f_{t(\Omega)}) \not\succsim^{\text{sm}} (\bar f_{S|t(\Omega)}, \bar f_{t(\Omega)})$.

(ii) "$\Rightarrow$": Suppose $(f_{S|\Omega}, f_\Omega) \succsim^{\text{w-inf}} (\bar f_{S|\Omega}, \bar f_\Omega)$. From the Definition of the weak informativeness criterion follows

$$0 \ge \int_\Omega \left[\bar F_{S,\Omega}(s,\omega) - s\bar F_\Omega(\omega)\right] - \left[F_{S,\Omega}(s,\omega) - sF_\Omega(\omega)\right]\mathrm{d}\omega \tag{B.17}$$

$$\overset{\text{Fubini}}{=} \int\limits_0^s \int\limits_\Omega \left[\bar F_{\Omega|S}(\omega|s) - \bar F_\Omega(\omega)\right] - \left[F_{\Omega|S}(\omega|s) - F_\Omega(\omega)\right]\mathrm{d}\omega\mathrm{d}s \tag{B.18}$$

$$\overset{(B.11)}{=} \int\limits_0^s \left[\bar{\mathbb{E}}_{\Omega|S}\left[\tilde\omega|s\right] - \bar{\mathbb{E}}_\Omega\left[\tilde\omega\right]\right] - \left[\mathbb{E}_{\Omega|S}\left[\tilde\omega|s\right] - \mathbb{E}_\Omega\left[\tilde\omega\right]\right]\mathrm{d}s \tag{B.19}$$

Together with Lemma B.2 and OS this implies $(f_{S|t(\Omega)}, f_{t(\Omega)}) \succsim^{\text{mps}} (\bar f_{S|t(\Omega)}, \bar f_{t(\Omega)})$ for all strictly increasing $t : \Omega \to \mathbb{R}$.

"$\Leftarrow$": Along the same lines as the "$\Leftarrow$"-part of (i) above. □

Proof of Theorem 3.4 "$\Rightarrow$": First suppose there is $\omega_0 \in \Omega$ such that $F_\Omega(\omega_0) > \bar F_\Omega(\omega_0)$. Then there exists $t_1 : \Omega \to \mathbb{R}$, t_1 strictly increasing, such that $\mathbb{E}_\Omega\left[t_1(\tilde\omega)\right] > \bar{\mathbb{E}}_\Omega\left[t_1(\tilde\omega)\right]$. Now define $v_{t_1} : \mathcal{A}\times\Omega \to \mathbb{R}, (a,\omega) \mapsto -t_1(\omega)$. Obviously, v_{t_1} is (weakly) supermodular in (a,ω) and

$$V((f_{S|\Omega}, f_\Omega), a^*, v_{t_1}) = -\mathbb{E}_\Omega\left[t(\tilde\omega)\right] < -\bar{\mathbb{E}}_\Omega\left[t(\tilde\omega)\right] = V((\bar f_{\bar S|\Omega}, f_\Omega), \bar a^*, v_{t_1})$$

Secondly, suppose the opposite, there exists $\omega_0 \in \Omega$ such that $F_\Omega(\omega_0) <$

$\bar{F}_\Omega(\omega_0)$. Then there exists $t_2 : \Omega \to \mathbb{R}$, t_2 strictly increasing, such that $\mathbb{E}_\Omega[t_2(\tilde{\omega})] < \bar{\mathbb{E}}_\Omega[t_2(\tilde{\omega})]$. Now define $v_{t_2} : \mathcal{A} \times \Omega \to \mathbb{R}, (a, \omega) \mapsto t(\omega)$. Obviously, u_{t_2} is (weakly) supermodular in (a, ω) and

$$V((f_{S|\Omega}, f_\Omega), a^*, v_{t_2}) = \mathbb{E}_\Omega[t(\tilde{\omega})] < \bar{\mathbb{E}}_\Omega[t(\tilde{\omega})] = V((\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega), \bar{a}^*, v_{t_2})$$

This shows

$$\left[\begin{array}{l} V((f_{S|\Omega}, f_\Omega), a^*, v) \geq V((\bar{f}_{\bar{S}|\Omega}, f_\Omega), \bar{a}^*, v) \\ \text{for all supermodular } v : \mathcal{A} \times \Omega \to \mathbb{R} \end{array}\right] \Rightarrow F_\Omega(\omega) = \bar{F}_\Omega(\omega) \ \forall \ \omega \in \Omega.$$

Next suppose $(f_{S|\Omega}, f_\Omega) \not\succsim^{\text{w-inf}} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega)$ and $F_\Omega(\omega) = \bar{F}_\Omega(\omega) \ \forall \ \omega \in \Omega$.[41] Then, by Prop. 2 in Brandt et al. (2014), there exists a strictly increasing function $t : \Omega \to \mathbb{R}$ such that $\mathbb{E}_\Omega[t(\tilde{\omega})|\tilde{s}]$ is not a MPS of $\bar{\mathbb{E}}_\Omega[t(\tilde{\omega})|\tilde{s}]$.[42] Hence, Lemma 1 implies that there is $\hat{s} \in [0, 1]$ such that

$$\mathbb{E}_\Omega[t(\tilde{\omega})|\tilde{s} \geq \hat{s}] < \bar{\mathbb{E}}_\Omega[t(\tilde{\omega})|\tilde{s} \geq \hat{s}]. \tag{B.20}$$

For arbitrary $\hat{a} \in \mathcal{A}$ define

$$v_{t,\hat{s}}(a, \omega) := \begin{cases} C & \text{if } a < \hat{a} \\ t(\omega) & \text{if } a \geq \hat{a} \end{cases}$$

where $C := \mathbb{E}_\Omega[t(\tilde{\omega})|\hat{s}]$ is constant. For $a', a \in \mathcal{A}$ such that $a' \geq a$ it is

$$v(a', \omega) - v(a, \omega) = \begin{cases} 0 & \text{if } a \leq a' < \hat{a} \text{ or } \hat{a} \leq a \leq a' \\ t(\omega) & \text{if } a < \hat{a} \leq a'. \end{cases}$$

Since $t : \Omega \to \mathbb{R}$ is strictly increasing this implies that $v_{t,\hat{s}} : A \times \Omega \to \mathbb{R}$ is

[41] The assumption of equal priors, i.e. $F_\Omega(\omega) = \bar{F}_\Omega(\omega) \ \forall \, \omega \in \Omega$, is only for simplification. Along similar lines as follows it is possible to show the necessarity of weak informativeness without this restriction.

[42] $F_\Omega(\omega) = \bar{F}_\Omega(\omega) \ \forall \ \omega \in \Omega$ implies $\mathbb{E}_\Omega[t(\tilde{\omega})] = \bar{\mathbb{E}}_\Omega[t(\tilde{\omega})]$ for all $t : \Omega \to \mathbb{R}$.

supermodular in (a, ω). Moreover, for $a'' \geq \hat{a} > a'$ the MLRP implies

$$\mathbb{E}_\Omega\left[v(a'', \tilde{\omega})|s\right] = \mathbb{E}_\Omega\left[t(\tilde{\omega})|s\right] \underset{\leq}{\geq} \mathbb{E}_\Omega\left[t(\tilde{\omega})|\hat{s}\right] = \mathbb{E}_\Omega\left[v(a', \tilde{\omega})|s\right] \overset{\text{MLRP}}{\Leftrightarrow} s \underset{\leq}{\geq} \hat{s}.$$

Hence, under information system $(f_{S|\Omega}, f_\Omega)$ it is optimal to choose an action $a < \hat{a}$ if $s < \hat{s}$ and to choose an action $a \geq \hat{a}$ whenever $s \geq \hat{s}$.[43] Formally,

$$a^* : S \to \mathcal{A}, s \mapsto \begin{cases} a' & \text{if } s < s_1 \\ a'' & \text{if } s \geq s_1 \end{cases}$$

for some $a' < \hat{a} \leq a''$. The optimal ex ante expected utility under information system $(f_{S|\Omega}, f_\Omega)$ is given by

$$\begin{aligned} V((f_{S|\Omega}, f_\Omega), a^*, u_{t,\hat{s}}) = & \quad \hat{s}\mathbb{E}_\Omega\left[t(\tilde{\omega})|\hat{s}\right] + (1-\hat{s})\mathbb{E}_\Omega\left[t(\tilde{\omega})|\tilde{s} \geq \hat{s}\right] \\ \overset{(B.20)}{<} & \quad \hat{s}\mathbb{E}_\Omega\left[t(\tilde{\omega})|\hat{s}\right] + (1-\hat{s})\bar{\mathbb{E}}_\Omega\left[t(\tilde{\omega})|\tilde{s} \geq \hat{s}\right] \\ = & \quad V((\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega), a^*, u_{t,\hat{s}}) \\ \overset{\text{optimality}}{\leq} & \quad V((\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega), \bar{a}^*, u_{t,\hat{s}}) \end{aligned}$$

This implies that $V((f_{S|\Omega}, f_\Omega), a^*, v) \geq V((\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega), \bar{a}^*, v)$ does not hold for all supermodular functions $v : \mathcal{A} \times \Omega \to \mathbb{R}$ and, hence,

$$\left[\begin{array}{l} V((f_{S|\Omega}, f_\Omega), a^*, v) \geq V((\bar{f}_{\bar{S}|\Omega}, f_\Omega), \bar{a}^*, v) \\ \text{for all supermodular } v : \mathcal{A} \times \Omega \to \mathbb{R} \end{array}\right] \not\Rightarrow \left[\begin{array}{l} (f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega) \text{ and} \\ F_\Omega(\omega) = \bar{F}_\Omega(\omega)\ \forall\ \omega \in \Omega \end{array}\right].$$

"$\Leftarrow$": $(f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (\bar{f}_{\bar{S},\Omega}, \bar{f}_\Omega)$ plus $F_\Omega(\omega) = \bar{F}_\Omega(\omega)\,\forall\,\omega \in \Omega$ imply $F_{S,\Omega}(s, \omega) \geq \bar{F}_{S,\Omega}(s, \omega)$ for all $(s, \omega) \in S \times \Omega$. Since $F_\Omega(\omega) = \bar{F}_\Omega(\omega)\ \forall\ \omega \in \Omega$ and $F_S(s) = \bar{F}_{\bar{S}}(s)\ \forall\ s \in S = \bar{S} = [0, 1]$ this is the concordance order as proposed by Joe (1990). By Müller and Scarsini (2000) the concordance order

[43] Remark: This is one optimal strategy, there might be others.

is equivalent to the supermodular stochastic order.[44] Hence,

$$\left[\begin{array}{l}(f_{S|\Omega}, f_\Omega) \overset{\text{w-inf}}{\succsim} (\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega) \\ \text{and } F_\Omega(\omega) = \bar{F}_\Omega(\omega)\ \forall\ \omega \in \Omega\end{array}\right] \Longrightarrow \left[\begin{array}{l}\mathbb{E}_{S,\Omega}\left[u(a(\tilde{s}), \tilde{\omega})\right] \geq \bar{\mathbb{E}}_{\bar{S},\Omega}\left[u(a(\tilde{s}), \tilde{\omega})\right] \\ \text{for all supermodular } u \text{ and monotone} \\ \text{increasing } a : S \to \mathcal{A}.\end{array}\right] \tag{B.21}$$

Additionally, from Theorem 2 in Athey (2002) it follows that if $v : \mathcal{A}\times\Omega \to \mathbb{R}$ is supermodular and $f_{S|\Omega}$ has the MLRP, then the optimal strategy $a^* : S \to \mathcal{A}$ is non-decreasing in s. Therefore, if $v : \mathcal{A} \times \Omega \to \mathbb{R}$ is supermodular, then $v : S \times \Omega, (s, \omega) \mapsto v(a^*(s), \omega)$ is supermodular in (s, ω). Hence,

$$V((\bar{f}_{\bar{S}|\Omega}, \bar{f}_\Omega), \bar{a}^*, v) \overset{\text{(B.21)}}{\leq} V((f_{S|\Omega}, f_\Omega), \bar{a}^*, v) \overset{\text{optimality}}{\leq} V((f_{S|\Omega}, f_\Omega), a^*, v)$$

for all supermodular functions $v : \mathcal{A} \times \Omega \to \mathbb{R}$. □

Proof of Proposition 3.8 Proposition 1(ii) in Ganuza and Penalva (2010) together with Proposition 3.7 show that 'sufficiency' does not imply 'strongly more informative'. Now suppose equation (3.6) would hold for some arbitrary class of payoff functions $\mathcal{U}$. Then, by Blackwell's theorem, 'sufficiency' would imply 'strong informativeness' which is a contradiction to the observation above. □

Proof of Lemma 3.2 Let $(f_{S|\Omega}, \bar{f}_\Omega), (f_{S|\Omega}, \hat{f}_\Omega) \in \Gamma$ and for $\alpha \in [0, 1]$ define $\check{f}_\Omega := \alpha\bar{f}_\Omega + (1 - \alpha)\hat{f}_\Omega$. Since $f_{S|\Omega}$ has the MLRP, it suffices to show that the signals under $(f_{S|\Omega}, \check{f}_\Omega)$ are normalized, i.e. $\check{f}_S(s) = 1\ \forall\ s \in [0, 1]$:

$$\begin{aligned}\check{f}_S(s) = &\int_\Omega f_{S|\Omega}(s|\omega)\check{f}_\Omega(\omega)\mathrm{d}\omega = \int_\Omega f_{S|\Omega}(s|\omega)\left[\alpha\bar{f}_\Omega(\omega) + (1 - \alpha)\hat{f}_\Omega(\omega)\right]\mathrm{d}\omega \\ =&\alpha\bar{f}_S(s) + (1 - \alpha)\hat{f}_S(s) = 1\end{aligned}$$

□

[44] A random vector $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ is said to be smaller than the random vector $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ in the supermodular (stochastic) order iff $E[f(\tilde{x})] \leq E[f(\tilde{y})]$ for all supermodular functions $f : \mathbb{R}^n \to \mathbb{R}$. Remark: A function $\mathbb{R}^n \to \mathbb{R}$ is supermodular, iff $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$.

Proof of Proposition 3.9 Let $(f_{S|\Omega}, \bar{f}_\Omega), (f_{S|\Omega}, \hat{f}_\Omega) \in \Gamma$ such that $(f_{S|\Omega}, \bar{f}_\Omega) \overset{x}{\sim} (f_{S|\Omega}, \hat{f}_\Omega)$, $x \in \{\text{w-inf}, \text{ s-inf}\}$, and for $\alpha \in [0,1]$ define $\check{f}_\Omega := \alpha \bar{f}_\Omega + (1-\alpha)\hat{f}_\Omega$. Straightforward calculation shows

$$\int_{\underline{\omega}}^{\omega} \check{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' = \alpha \int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' + (1-\alpha) \int_{\underline{\omega}}^{\omega} \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' \tag{B.22}$$

and

$$\check{F}_{S,\Omega}(s,\omega) - s\check{F}_\Omega(\omega) = \alpha \left[\bar{F}_{S,\Omega}(s,\omega) - s\bar{F}_\Omega(\omega)\right] - (1-\alpha)\left[\hat{F}_{S,\Omega}(s,\omega) - s\hat{F}_\Omega(\omega)\right] \tag{B.23}$$

Equation (B.22) implies

$$\int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega') - \check{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' =(1-\alpha) \int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega') - \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' \tag{B.24}$$

$$\int_{\underline{\omega}}^{\omega} \check{f}_{S,\Omega}(s,\omega') - \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' =\alpha \int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega') - \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega'. \tag{B.25}$$

For $x =$ w-inf the claim follows from equation (B.23) while for $x =$ s-inf the claim follows from equations (B.24) and (B.25). □

Proof of Lemma 3.3 Let $f_\Omega \in \Delta(\Omega)$, $\bar{f}_{S|\Omega}, \hat{f}_{S|\Omega} \in \Gamma(f_\Omega)$ and for $\alpha \in [0,1]$ define $\check{f}_{S|\Omega} := \alpha \bar{f}_{S|\Omega} + (1-\alpha)\hat{f}_{S|\Omega}$.

(i) The signals under $(\check{f}_{S|\Omega}, f_\Omega)$ are normalized:

$$\begin{aligned}\check{f}_S(s) =& \int_\Omega \check{f}_{S|\Omega}(s|\omega) f_\Omega(\omega)\mathrm{d}\omega = \int_\Omega \left[\alpha \bar{f}_{S|\Omega}(s|\omega) + (1-\alpha)\hat{f}_{S|\Omega}(s|\omega)\right] f_\Omega(\omega)\mathrm{d}\omega \\ =& \alpha \bar{f}_S(s) + (1-\alpha)\hat{f}_S(s) = 1\end{aligned}$$

(ii) $\check{f}_{S|\Omega}$ has the MLRP: To see this consider the posterior, conditional cumulative state distribution under $(\check{f}_{S|\Omega}, f_\Omega)$. That is

$$\check{F}_{\Omega|S}(\omega|s) \overset{\check{f}_S(s)=1}{=} \int_{\underline{\omega}}^{\omega} f_{S,\Omega}(s,\omega')\mathrm{d}\omega' = \int_{\underline{\omega}}^{\omega} \alpha\bar{f}_{S,\Omega}(s,\omega') + (1-\alpha)\hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega'$$
$$=\alpha\bar{F}_{\Omega|S}(\omega|s) + (1-\alpha)\hat{F}_{\Omega|S}(\omega|s).$$

Since $\bar{f}_{S|\Omega}$ and $\hat{f}_{S|\Omega}$ have the MLRP, this together with proposition 3.2 imply

$$\begin{aligned}\check{F}_{\Omega|S}(\omega|s') =&\alpha\bar{F}_{\Omega|S}(\omega|s') + (1-\alpha)\hat{F}_{\Omega|S}(\omega|s') \\ \geq&\alpha\bar{F}_{\Omega|S}(\omega|s'') + (1-\alpha)\hat{F}_{\Omega|S}(\omega|s'') = \check{F}_{\Omega|S}(\omega|s'')\end{aligned}$$

for all $s', s'' \in S$, $s' \leq s''$, and for all $f_\Omega \in \Delta(\Omega)$. Hence, again by proposition 3.2, $\check{f}_{S|\Omega}$ has the MLRP.

□

Proof of Proposition 3.10 Let $f_\Omega \in \Delta(\Omega)$, $\bar{f}_{S|\Omega}, \hat{f}_{S|\Omega} \in \Gamma(f_\Omega)$ such that $(\bar{f}_{S|\Omega}, f_\Omega) \overset{x}{\precsim} (\hat{f}_{S|\Omega}, f_\Omega)$, $x \in \{\text{w-inf, s-inf}\}$, and for $\alpha \in [0,1]$ define $\check{f}_{S|\Omega} := \alpha\bar{f}_{S|\Omega} + (1-\alpha)\hat{f}_{S|\Omega}$. Straightforward calculation shows

$$\int_{\underline{\omega}}^{\omega} \check{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' = \alpha\int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' + (1-\alpha)\int_{\underline{\omega}}^{\omega} \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' \qquad \text{(B.26)}$$

and

$$\check{F}_{S,\Omega}(s,\omega) - s\check{F}_\Omega(\omega) = \alpha\left[\bar{F}_{S,\Omega}(s,\omega) - s\bar{F}_\Omega(\omega)\right] - (1-\alpha)\left[\hat{F}_{S,\Omega}(s,\omega) - s\hat{F}_\Omega(\omega)\right] \qquad \text{(B.27)}$$

Equation (B.26) implies

$$\int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega') - \check{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' = (1-\alpha)\int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega') - \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' \quad \text{(B.28)}$$

$$\int_{\underline{\omega}}^{\omega} \check{f}_{S,\Omega}(s,\omega') - \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega' = \alpha\int_{\underline{\omega}}^{\omega} \bar{f}_{S,\Omega}(s,\omega') - \hat{f}_{S,\Omega}(s,\omega')\mathrm{d}\omega'. \quad \text{(B.29)}$$

For $x =$ w-inf the claim follows from equation (B.27) while for $x =$ s-inf the claim follows from equations (B.28) and (B.29).

□

Appendix C

Proofs of Chapter 4

Proof of Proposition 4.1 Calculating the first and second order conditions for (4.2):

$$\begin{aligned} W'(\epsilon) =& \frac{n-1}{n}(1-\alpha)(m-P(\epsilon)) - P'(\epsilon)\left[\left(1-\frac{n-1}{n}\epsilon\right)\alpha + \frac{n-1}{n}\epsilon\right] \\ W''(\epsilon) =& 2\frac{n-1}{n}P'(\epsilon)(\alpha-1) - P''(\epsilon)\left[\left(1-\frac{n-1}{n}\epsilon\right)\alpha + \frac{n-1}{n}\epsilon\right] < 0. \end{aligned} \tag{C.1}$$

This implies that W is strictly concave with respect to ϵ. Regarding the limits it follows:

$$\begin{aligned} \lim_{\epsilon\to\bar{\epsilon}} W'(\epsilon) =& -P'(\bar{\epsilon})\left[\left(1-\frac{n-1}{n}\bar{\epsilon}\right)\alpha + \frac{n-1}{n}\bar{\epsilon}\right] \;>\; 0 \\ \lim_{\epsilon\to 1} W'(\epsilon) =& \frac{n-1}{n}(1-\alpha)m - P'(1)\frac{\alpha+(n-1)}{n} \;<\; 0, \end{aligned}$$

where last inequality holds for finite $|P'(1)|$ and m (or n) sufficiently high. Then there is a unique $\epsilon^* \in (\bar{\epsilon}, 1)$ which maximizes $W(\epsilon)$ on $[\bar{\epsilon}, 1]$. □

Proof of Cororally 4.1 $\frac{dW'(\epsilon)}{dn} = \frac{1-\alpha}{n^2}\left[m - P(\epsilon) - \epsilon P'(\epsilon)\right] < 0$ due to $P' < 0$ and $\alpha > 1$. Together with equation (C.1) this implies:

$$\frac{d\epsilon^*}{dn} = -\left.\frac{dW'(\epsilon)/dn}{W''(\epsilon)}\right|_{\epsilon=\epsilon^*} \;<\; 0.$$

Similarly: $\frac{dW'(\epsilon)}{dm} = \frac{n-1}{n}(1-\alpha) \ < \ 0$. Using (C.1):

$$\frac{d\epsilon^*}{dm} = -\frac{dW'(\epsilon)/dm}{W''(\epsilon)}\bigg|_{\epsilon=\epsilon^*} \ < \ 0.$$

□

Proof of Proposition 4.2 The proof for $\sigma = 1$ is straightforward and therefore omitted here. Now assume $\sigma \in \mathbb{R}_{>0,\neq 1}$.

(i) The FOC of the maximization problem is

$$\begin{aligned} W'(\epsilon) =& \frac{n-1}{n(1-\sigma)}(1-\alpha^{1-\sigma})(m-P(\epsilon))^{1-\sigma} \\ & - P'(\epsilon)(m-P(\epsilon))^{-\sigma}\left[\left(1-\frac{n-1}{n}\epsilon\right)\alpha^{1-\sigma} + \frac{n-1}{n}\epsilon\right] = 0 \end{aligned} \tag{C.2}$$

$$\iff \underbrace{-\frac{m-P(\epsilon)}{P'(\epsilon)}}_{=:LS(\epsilon)} = \underbrace{\frac{n(1-\sigma)}{\alpha^{1-\sigma}-1} \cdot \frac{\left(1-\frac{n-1}{n}\epsilon\right)\alpha^{1-\sigma} + \frac{n-1}{n}\epsilon}{n-1}}_{=:RS(\epsilon)}. \tag{C.3}$$

$RS(\epsilon)$ is affine linear in ϵ, i.e. $RS(\epsilon) = A(\sigma) + B(\sigma)\epsilon$ with

$$A(\sigma) := \frac{n\alpha^{1-\sigma}(1-\sigma)}{(\alpha^{1-\sigma}-1)(n-1)}, \quad B(\sigma) \ := \ \sigma - 1. \tag{C.4}$$

Moreover, we get

$$RS'(\epsilon) = \sigma - 1 \begin{cases} > 0 & \text{if } \sigma > 1 \\ < 0 & \text{if } \sigma < 1 \end{cases}$$

$$RS''(\epsilon) = 0$$

$$RS(\bar{\epsilon}) = \frac{n(1-\sigma)}{\alpha^{1-\sigma}-1} \cdot \frac{\left(1-\frac{n-1}{n}\bar{\epsilon}\right)\alpha^{1-\sigma} + \frac{n-1}{n}\bar{\epsilon}}{n-1} \ > \ 0 \tag{C.5}$$

$$RS(1) = \frac{n(1-\sigma)}{n-1}\left(\frac{1}{n} + \frac{1}{\alpha^{1-\sigma}-1}\right) \tag{C.6}$$

Similarly, for $LS(\epsilon)$:

$$LS'(\epsilon) = 1 + \frac{(m - P(\epsilon))\, P''(\epsilon)}{(P(\epsilon))^2} \; > \; 0$$

$$LS''(\epsilon) = (P'(\epsilon))^2 \left[-P'(\epsilon)P''(\epsilon) + (m - P(\epsilon))\, P'''(\epsilon) \right] \tag{C.7}$$

$$LS(\bar{\epsilon}) = 0 \tag{C.8}$$

$$LS(1) = -\frac{m}{P'(1)} \; > \; 0 \tag{C.9}$$

Using the equations (C.5), (C.6), (C.8) and (C.9) yields:

$$LS(\bar{\epsilon}) < RS(\bar{\epsilon})$$
$$LS(1) > RS(1) \text{ for } m \text{ sufficiently high (and } |P'(1)| \text{ small);}$$

which shows (i).

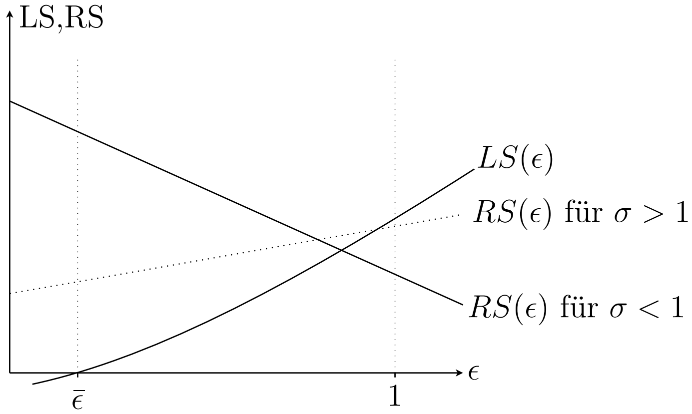

Figure C.1:

(ii) Follows from strict concavity:

$$\begin{aligned} W''(\epsilon) = & -2\frac{n-1}{n}(1 - \alpha^{1-\sigma})(m - P(\epsilon))^{-\sigma} P'(\epsilon) \\ & - \left[P''(\epsilon) + \sigma (P'(\epsilon))^2 \right] \left[\left(1 - \frac{n-1}{n}\epsilon \right) \alpha^{1-\sigma} + \frac{n-1}{n}\epsilon \right] (m - P(\epsilon))^{-\sigma-1} \\ & < 0 \text{ for } \sigma < 1 \end{aligned}$$

(iii) Equation (C.7) implies that $LS''(\epsilon)$ is convex for $P''' = 0$ which implies uniqueness. It should be clear at this point that this is true for many other cases as well. □

Proof of Proposition 4.3 The proof is straightforward for $\sigma = 1$. In the following consider the case $\sigma > 0,\ \sigma \neq 1$.

(i) Differentiating $W'(\epsilon)$ with respect to n gives:

$$\frac{dW'(\epsilon)}{dn} = \frac{(m-P(\epsilon))^{-\sigma}}{n^2}\Bigg[\underbrace{\frac{1-\alpha^{1-\sigma}}{1-\sigma}(m-P(\epsilon))}_{<\ 0} + \underbrace{P'(\epsilon)\epsilon(\alpha^{1-\sigma}-1)}_{<0 \text{ if } \sigma<1}\Bigg] < 0.$$

This shows the claim for $\sigma < 1$. To verify the claim for $\sigma > 1$ we rewrite the derivative in the following way:

$$\frac{dW'(\epsilon)}{dn} = \underbrace{\frac{(\alpha^{1-\sigma}-1)(m-P(\epsilon))^{-\sigma}}{(\sigma-1)n^2}}_{<\ 0}\Bigg[\underbrace{(m-P(\epsilon)) - P'(\epsilon)\epsilon(\sigma-1)}_{>0 \text{ if } \sigma>1}\Bigg] < 0.$$

Differentiating $W'(\epsilon)$ with respect to m yields:

$$\begin{aligned} &\left.\frac{dW'(\epsilon)}{dm}\right|_{\epsilon=\epsilon^*} \\ &= \frac{n-1}{n}\cdot(m-P(\epsilon^*))^{-\sigma}(1-\alpha^{1-\sigma}) \\ &\quad + \sigma P'(\epsilon)(m-P(\epsilon^*))^{-\sigma-1}\left[\left(1-\frac{n-1}{n}\epsilon^*\right)\alpha^{1-\sigma} + \frac{n-1}{n}\epsilon^*\right] \\ &= \underbrace{\frac{W'(\epsilon^*)}{m-P(\epsilon^*)}}_{=0} + (1+\sigma)P'(\epsilon)(m-P(\epsilon))^{-\sigma-1} \\ &\qquad \cdot\left[\left(1-\frac{n-1}{n}\epsilon^*\right)\alpha^{1-\sigma} + \frac{n-1}{n}\epsilon^*\right] <\ 0. \end{aligned}$$

(ii) Differentiating $W'(\epsilon)$ with respect to α gives:

$$\frac{dW'(\epsilon)}{d\alpha} = -\frac{n-1}{n}\left(m - P(\epsilon)\right)^{1-\sigma}\alpha^{-\sigma} \\ \underbrace{-P'(\epsilon)\left(m-P(\epsilon)\right)^{-\sigma}\left(1-\frac{n-1}{n}\epsilon\right)(1-\sigma)\alpha^{-\sigma}}_{>0 \text{ if } \sigma>1 \text{ since } P'(\epsilon)<0} < 0.$$

□

Proof of Proposition 4.4 (i) Determination of the minimal financiable error level $\bar{\epsilon}$: Since $m \leq b$ the minimal financiable error level solves $P(\bar{\epsilon}) = b(1-\bar{\epsilon}) = m \Leftrightarrow \bar{\epsilon} = 1 - \frac{m}{b}$. Hence, $\bar{\epsilon} := 1 - \frac{m}{b}$. Since $P(\epsilon)$ is linear it follows from Prop. 4.2 that ϵ^* is uniquely determined. Moreover, $\sigma \geq 2 > 1$ implies $\lim_{\epsilon \to \bar{\epsilon}} W(\epsilon) = -\infty$ and, hence, $\epsilon^* = \bar{\epsilon}$ cannot be optimal. Moreover, equations (C.3) and (C.4) imply that an interior optimal ϵ^* holds:

$$LS(\epsilon^*) \;=\; \frac{m - b(1-\epsilon^*)}{b} \stackrel{!}{=} A(\sigma) + B(\sigma)\epsilon^* \;=\; RS(\epsilon^*). \qquad \text{(C.10)}$$

This yields

$$\epsilon^* \;=\; \frac{A(\sigma) - LS(0)}{2-\sigma} \begin{cases} > \bar{\epsilon} & \text{if } \sigma < 2 \\ < \bar{\epsilon} & \text{if } \sigma > 2, \end{cases}$$

where the last inequalities hold because

$$\begin{aligned} & \frac{\alpha^{1-\sigma}}{\alpha^{1-\sigma}-1} \begin{cases} > 1 & \text{if } \sigma < 1 \\ < 0 & \text{if } \sigma > 1 \end{cases} \\ \stackrel{\bar{\epsilon}\in[0,1)}{\Longrightarrow} \; & \frac{n}{n-1}\frac{\alpha^{1-\sigma}}{\alpha^{1-\sigma}-1} \begin{cases} > \bar{\epsilon} & \text{if } \sigma < 1 \\ < \bar{\epsilon} & \text{if } \sigma > 1 \end{cases} \\ \Longrightarrow \; & A(\sigma) \; > (1-\sigma)\bar{\epsilon} \\ \Longrightarrow \; & \frac{A(\sigma)+\bar{\epsilon}}{2-\sigma} \begin{cases} > \bar{\epsilon} & \text{if } \sigma < 2 \\ < \bar{\epsilon} & \text{if } \sigma > 2. \end{cases} \end{aligned}$$

This implies that for $\sigma \geq 2$ there is no $\epsilon^* \in (\bar{\epsilon}, 1)$ which solves the optimality condition (C.10). This implies $\epsilon^* = 1$ whenever $\sigma \geq 2$. Moreover, ϵ^* can be characterized as a function of σ by

$$\epsilon^*(\sigma) = \begin{cases} \min\left\{\frac{A(\sigma)+\bar{\epsilon}}{2-\sigma}, 1\right\} & \text{if } \sigma < 2 \\ 1 & \text{if } \sigma \geq 2 \end{cases}, \tag{C.11}$$

which shows (i).

(ii) Differentiating $\epsilon^*(\sigma)$ as given in equation (C.11) with respect to σ gives (for $\sigma < 2$)

$$\frac{d\epsilon^*(\sigma)}{d\sigma} = \frac{A'(\sigma)(2-\sigma) + A(\sigma) + \bar{\epsilon}}{(2-\sigma)^2}.$$

For the limit follows

$$\lim_{\sigma \to 0} \frac{d\epsilon^*}{d\sigma} = \frac{1}{4}\left[\bar{\epsilon} + \underbrace{\frac{n}{n-1} \cdot \frac{\alpha(1-\alpha+2ln(\alpha))}{(\alpha-1)^2}}_{=:H(\alpha)}\right].$$

This expression is less than zero for α sufficiently high, because

$$\lim_{\alpha\to\infty,\sigma\to 0} \frac{d\epsilon^*(\sigma)}{d\sigma} = \frac{1}{4}\left(\bar{\epsilon} - \frac{n}{n-1}\right) = -\frac{1}{4} \cdot \frac{(n-1)m + nb}{(n-1)b} < 0$$

and $H'(\alpha) = \frac{n}{n-1} \cdot \frac{3(\alpha-1)-2ln(\alpha)(\alpha+1)}{(\alpha-1)^3} < 0$ for all $\alpha > 1$. To see that $H'(\alpha) < 0$ consider

$$h(\alpha) := 3(\alpha - 1) - 2ln(\alpha)(\alpha + 1).$$

Its first and second derivatives are

$$h'(\alpha) = 1 - \frac{2}{\alpha} - ln(\alpha), \quad h''(\alpha) = \frac{2(1-\alpha)}{\alpha^2} < 0.$$

It follows:

$$\begin{aligned} h'(\alpha) &< h'(1) = -1 < 0 \; \forall \alpha > 1 \\ \Longrightarrow h(\alpha) &< h(1) = 0 \; \forall \alpha > 1 \\ \Longrightarrow H'(\alpha) &< 0 \; \forall \; \alpha > 1. \end{aligned}$$

Now define $I(\sigma) := A'(\sigma)(2-\sigma) + A(\sigma) + \bar{\epsilon}$.[45] If $A''(\sigma) > 0$ for all $\sigma > 0$ (which will be shown below) the following holds true:

$$I'(\sigma) = A''(\sigma)(2-\sigma) \; > \; 0 \quad \text{for } \sigma < 2.$$

Since $\frac{d\epsilon^*(\sigma)}{d\sigma} = \frac{I(\sigma)}{(2-\sigma)^2}$ and since $\left.\frac{d\epsilon^*(\sigma)}{d\sigma}\right|_{\sigma=0} < 0$ (for α large enough) follows that $I(0) < 0$. In combination with $I'(\sigma) > 0$ for $\sigma < 2$ and $I(2) > 0$ this implies that there is a unique $\bar{\sigma} \in (0,2)$ such that

$$I(\sigma) \begin{cases} < 0 & \text{if } \sigma < \bar{\sigma} \\ = 0 & \text{if } \sigma = \bar{\sigma} \text{ or } \sigma \geq 2 \\ > 0 & \text{if } \bar{\sigma} < \sigma < 2 \end{cases},$$

which shows (ii).

It remains to show that $A''(\sigma) > 0$ for all $\sigma > 0$. To see this consider

$$A''(\sigma) = \frac{n}{n-1} \cdot \underbrace{\frac{\alpha^{1-\sigma} ln(\alpha)}{(\alpha^{1-\sigma}-1)^3}}_{=:g(\sigma)} \underbrace{\left[(1-\sigma)ln(\alpha)(\alpha^{1-\sigma}+1) + 2(1-\alpha^{1-\sigma})\right]}_{=:h(\sigma)}.$$

It is

$$g(\sigma), h(\sigma) \begin{cases} > 0 & \text{if } \sigma < 1 \\ < 0 & \text{if } \sigma > 1 \end{cases}$$

and hence $A''(\sigma) = \frac{n}{n-1} g(\sigma) h(\sigma) > 0 \; \forall \; \sigma$.

---

[45] I.e. $\frac{d\epsilon^*(\sigma)}{d\sigma} = \frac{I(\sigma)}{(2-\sigma)^2}$.

Details for the analysis of $h(\sigma)$:

$$h(\sigma_1) > h(1) = 0 > h(\sigma_2)$$

for $\sigma_1 < 1 < \sigma_2$ holds because

$$\begin{aligned} h'(\sigma) =& ln(\alpha)\left(\alpha^{1-\sigma} - 1 + (1-\sigma)(\alpha^{1-\sigma}+1) + 2(1-\alpha^{1-\sigma})\right) \\ h''(\sigma) =& ln(\alpha)(1-\sigma)ln(\alpha)^2\alpha^{1-\sigma} \begin{cases} > 0 & \text{if } \sigma < 1 \\ = 0 & \text{if } \sigma = 1 \\ < 0 & \text{if } \sigma > 1. \end{cases} \end{aligned} \tag{C.12}$$

By equation (C.12) follows:

$$\begin{aligned} h'(\sigma_2) \ &> h'(1) \ = \ 0 > h'(\sigma_1) \ \ \forall\ \sigma_1 < 1 < \sigma_2 \\ \Longrightarrow \ h(\sigma_1) \ &> h(1) \ = \ 0 \ > \ h(\sigma_2) \ \ \forall\ \sigma_1 < 1 < \sigma_2 \end{aligned}$$

□

Proof of Proposition 4.5

(i) In order to characterize the Walrasian equilibrium define $(\bar{x}^*, \bar{y}^*, \bar{p}^*) \in \mathcal{C}^I \times \bigotimes_{j\in\mathcal{J}} Y_j \times \mathbb{R}^C_+$ as the Walrasian equilibrium of the certainty equivalent economy (where each agent is endowed with his expected endowment). The FOC of the risk neutral consumer determines the equilibrium price ratios (in this certainty equivalent economy). In particular, ${}^{\bar{p}^*_c}\!/_{\bar{p}^*_{c'}} = {}^{a_c}\!/_{a_{c'}}$, where $a_c = {}^{\partial u_i(x)}\!/_{\partial x_c}$ is the risk neutral agent's marginal utility of commodity c. In equilibrium $\bar{x}^*(i)$ solves

$$\max_{x\in\mathcal{C}} u_i(x) \text{ s.t. } (\bar{p}^*)^T \left(x - \mathbb{E}_\Omega\left[w(\tilde{\omega}, i)\right] - \sum_{j\in\mathcal{J}} \theta_{ij}(\bar{y}^*(j) + \mathbb{E}_\Omega\left[w(\tilde{\omega}, j)\right]) \right) \leq 0 \tag{C.13}$$

while $\bar{y}^*(j)$ solves

$$\max_{y\in Y_j} (\bar{p}^*)^T \left(y + \mathbb{E}_\Omega\left[w(\tilde{\omega}, j)\right]\right). \tag{C.14}$$

Now turn back to the uncertain economy and define consumer i's state-contingent consumption vector by

$$x^*(\omega, i) := \bar{x}^*(i) \ \ \forall\, \omega \in \Omega,$$

each firm j's production plan by

$$y^*(\omega, j) := \bar{y}^*(j) \ \ \forall\, \omega \in \Omega$$

and the state-contingent commodity prices by

$$p^* := (f_\Omega(\omega_1)\bar{p}^*, \ldots, f_\Omega(\omega_N)\bar{p}^*) \in \mathbb{R}_+^{C \times N}$$

where f_Ω denotes the prior belief in the risky economy. Then (x^*, y^*, p^*) is a Walrasian equilibrium of the original economy with risky endowment and complete markets. To see this, first observe that the price ratios $p^*_{c,\omega}/p^*_{c',\omega'}$ are equal to the risk neutral consumer's MRS for consumption of commodity c in state ω and consumption of commodity c' in state ω', i.e.

$$\frac{p^*_{c,\omega}}{p^*_{c',\omega'}} = \frac{f_\Omega(\omega)a_c}{f_\Omega(\omega')a_{c'}} \text{ for all } \omega, \omega' \in \Omega \text{ and } c, c' \in \{1, \ldots, C\}\,. \tag{C.15}$$

This implies that the risk neutral consumer is indifferent between consumption of commodity c in state ω and consumption of commodity c' in state ω'.

Secondly, observe that $x^*(i)$ solves the (original) optimization problem of a risk avers consumer i. This is given by

$$\max_{x \in B_i} \mathbb{E}_\Omega \left[u_i(x(\tilde{\omega}, i))\right],$$

with $B_i := \left\{x \in \mathcal{C}^S | \, (p^*)^T \left(x(i) - w(i) - \sum_{j \in \mathcal{J}} \theta_{ij}(y^*(j) + w(j))\right) \leq 0\right\}$.[46]

[46]Compare Definition 4.1.

The corresponding FOC is

$$\frac{f_\Omega(\omega)\frac{\partial u_i(x(\omega,i))}{\partial x_c(\omega,i)}}{f_\Omega(\omega')\frac{\partial u_i(x(\omega',i))}{\partial x_{c'}(\omega',i)}} \overset{!}{=} \frac{p^*_{c,\omega}}{p^*_{c',\omega'}} \overset{(C.15)}{=} \frac{f_\Omega(\omega)a_c}{f_\Omega(\omega')a_{c'}} \;\forall\, \omega,\omega' \in \Omega;\; c,c' \in \{1,\ldots,N\}$$

$$\Longleftrightarrow \frac{\frac{\partial u_i(x(\omega,i))}{\partial x_c(\omega,i)}}{\frac{\partial u_i(x(\omega',i))}{\partial x_{c'}(\omega',i)}} \overset{!}{=} \frac{a_c}{a_{c'}} \text{ for all } \omega,\omega' \in \Omega;\; c,c' \in \{1,\ldots,N\}\,. \tag{C.16}$$

This is equal to the FOC of (C.13) and, hence, if $\bar{x}^*(i) \in \mathcal{C}$ is optimal in the certainty equivalent economy, then, if financiable, $x^*(i) = (\bar{x}^*(i),\ldots,\bar{x}^*(i)) \in \mathcal{C}^N$ is optimal in the risky economy. Financiability follows by plugging (x^*, y^*, p^*) into consumer i' budget constraint B_i:

$$\begin{aligned}
&(p^*)^T(x^*(i) - w(i) - \sum_{j\in\mathcal{J}}(\theta_{ij}y^*(j) + w(j)) \\
&\overset{\text{Def.}}{=} \sum_{\omega\in\Omega} f_\Omega(\omega)\,(\bar{p}^*)^T\left(\bar{x}^*(i) - w(\omega,i) - \sum_{j\in\mathcal{J}}(\theta_{ij}\bar{y}^*(j) + w(j))\right) \\
&= (\bar{p}^*)^T\left(\bar{x}^*(i) - \mathbb{E}_\Omega\left[w(\tilde{\omega},i)\right] - \sum_{j\in\mathcal{J}}(\theta_{ij}\bar{y}^*(j) + w(j))\right) \overset{(C.13)}{\leq 0}\,.
\end{aligned}$$

This implies $x^*(i) \in B_i$, i.e. $x^*(i)$ is financiable, and since the consumers' FOCs (C.13) and (C.16) of the certainty equivalent and the risky economy, respectively, coincide this implies $x^*(i) = \arg\max_{x\in B_i} \mathbb{E}_\Omega\left[x(\tilde{\omega},i)\right]$.

Next, observe that $y^*(j)$ solves firm j's optimization problem in the original, risky economy with complete markets. This is given by

$$\max_{y(j)\in Y_j^N} (p^*)^T\,(y(j) + w(j))$$

with corresponding FOC

$$\frac{f_\Omega(\omega)\frac{\partial T_j(y(\omega,j))}{\partial y_c(\omega,j)}}{f_\Omega(\omega')\frac{\partial T_j(y(\omega',j))}{\partial y_{c'}(\omega',j)}} \overset{!}{=} \frac{p^*_{c,\omega}}{p^*_{c',\omega'}} \overset{(C.15)}{=} \frac{f_\Omega(\omega)a_c}{f_\Omega(\omega')a_{c'}} \;\forall\, \omega,\omega' \in \Omega;\; c,c' \in \{1,\ldots,C\}\,.$$

$$\Longleftrightarrow \frac{\frac{\partial T_j(y(\omega,j))}{\partial y_c(\omega,j)}}{\frac{\partial T_j(y(\omega',j))}{\partial y_{c'}(\omega',j)}} = \frac{a_c}{a_{c'}} \;\forall\, \omega,\omega' \in \Omega;\; c,c' \in \{1,\ldots,C\}\,. \tag{C.17}$$

This is equal to the FOC of (C.14) and, hence, $y^*(j) = (\bar{y}^*(j), \ldots, \bar{y}^*(j)) = \arg \max_{y(j) \in Y_j^N} (p^*)^T (y(j) + w(j))$.

Last, by assumption, the risk neutral consumer owns enough to fully insure all others, i.e. (x^*, y^*) is feasible. This completes the proof of part (i).

(ii) Analogue to that of part (i). The only difference is that the equilibrium price ratios are determined by the firm with constant marginal rates of transformation. This firm is indifferent between production of commodity c in state ω or c' in state ω'. Additionally, by assumption, this firm also owns enough to insure the consumers and all other firms. Therefore, (x^*, y^*) as defined in part (i) is also feasible.

□

Proof of Cororally 4.2 The optimal production vector is determined by equating a firm's MRTs with the price ratios. Since prices are fair, the state distribution cancels out of these FOCs (compare (C.17)) and, hence, the optimal production plan is independent of the underlying state distribution. □

The proof of Proposition 4.6 makes use of the following Lemmata:

Lemma C.1. *Let* $(f_{S|\Omega}, f_\Omega), (\bar{f}_{S|\Omega}, f_\Omega) \in \Gamma$ *such that* $(f_{S|\Omega}, f_\Omega) \stackrel{\text{w-inf}}{\precsim} (\bar{f}_{S|\Omega}, f_\Omega)$ *and* $\Omega = \{\omega_1, \ldots, \omega_N\} \subseteq \mathbb{R}$ *with* $\omega_1 < \omega_2 < \ldots, \omega_N$*. Then the following is true for all weakly monotone functions* $t : \Omega \to \mathbb{R}$*:*

$$\int_0^s \mathbb{E}_\Omega \left[t(\tilde{\omega}) | s' \right] ds' \leq \int_0^s \bar{\mathbb{E}}_\Omega \left[t(\tilde{\omega}) | s' \right] ds'.$$

By Theorem (3.A.5) in Shaked and Shantbikumar (2007) this implies that $\mathbb{E}_\Omega \left[t(\tilde{\omega}) | \tilde{s} \right]$ *is a MPS of* $\bar{\mathbb{E}}_\Omega \left[t(\tilde{\omega}) | \tilde{s} \right]$*.*

Proof: Let $(f_{S|\Omega}, f_\Omega), (\bar{f}_{S|\Omega}, f_\Omega) \in \Gamma$ such that $(f_{S|\Omega}, f_\Omega) \underset{\sim}{\succ}^{\text{w-inf}} (\bar{f}_{S|\Omega}, f_\Omega)$ and $\Omega = \{\omega_1, \ldots, \omega_N\} \subseteq \mathbb{R}$ with $\omega_1 < \omega_2 < \ldots, \omega_N$. First, observe that for any weakly increasing function $t : \Omega \to \mathbb{R}$ the following holds:

$$\begin{aligned} \mathbb{E}_\Omega \left[t(\tilde{\omega}) | s \right] &= \sum_{n=1}^{N} f_{\Omega|S}(\omega_n|s) t(\omega_n) \\ &\overset{f_{\Omega|S}(\omega_{n+1}|s) = F_{\Omega|S}(\omega_{n+1}|s) - F_{\Omega|S}(\omega_n|s)}{=} t(\omega_N) - \sum_{n=1}^{N-1} F_{\Omega|S}(\omega_n|s)(t(\omega_{n+1}) - t(\omega_n)). \end{aligned} \tag{C.18}$$

Now, if $(f_{S|\Omega}, f_\Omega) \underset{\sim}{\succ}^{\text{w-inf}} (\bar{f}_{S|\Omega}, f_\Omega)$ it follows for $s \in [0, 1]$:

$$\begin{aligned} &\int_0^s \mathbb{E}_\Omega \left[t(\tilde{\omega}) | s' \right] - \bar{\mathbb{E}}_\Omega \left[t(\tilde{\omega}) | s' \right] \mathrm{d}s' \\ &\overset{\text{(C.18)}}{=} - \int_0^s \sum_{n=1}^{N-1} \left(F_{\Omega|S}(\omega_n|s') - \bar{F}_{\Omega|S}(\omega_n|s') \right) \left(t(\omega_{n+1}) - t(\omega_n) \right) \mathrm{d}s' \\ &= - \sum_{n=1}^{N-1} \left(t(\omega_{n+1}) - t(\omega_n) \right) \int_0^s F_{\Omega|S}(\omega_n|s') - \bar{F}_{\Omega|S}(\omega_n|s') \mathrm{d}s' \\ &= - \sum_{n=1}^{N-1} \underbrace{\left(t(\omega_{n+1}) - t(\omega_n) \right)}_{\geq 0} \underbrace{\left(F_{S,\Omega}(s, \omega_n) - \bar{F}_{S,\Omega}(s, \omega_n \right)}_{\geq 0} \leq 0 \end{aligned}$$

This implies that $\mathbb{E}_\Omega [t(\tilde{\omega})|\tilde{s}]$ is a MPS of $\bar{\mathbb{E}}_\Omega [t(\tilde{\omega})|\tilde{s}]$ for all weakly increasing functions $t : \Omega \to \mathbb{R}$. Theorem 3.A.12 in Shaked and Shantbikumar (2007) now implies that $\mathbb{E}_\Omega [-t(\tilde{\omega})|\tilde{s}]$ is a MPS of $\bar{\mathbb{E}}_\Omega [-t(\tilde{\omega})|\tilde{s}]$ for all weakly increasing functions $t : \Omega \to \mathbb{R}$, which proof the claim for all weakly decreasing function. □

Lemma C.2. *Let $f_{S|\Omega}, \bar{f}_{S|\Omega} \in \Gamma(f_\Omega)$. For $i = 1, \ldots, n$, $n \in \mathbb{N}$, let $g_i : \mathbb{R} \to \mathbb{R}$ be monotone (increasing or decreasing). If $\mathbb{E}_\Omega [g_i(\tilde{\omega})|\tilde{s}]$ is a MPS of $\bar{\mathbb{E}}_\Omega [g_i(\tilde{\omega})|\tilde{s}]$ for all $i = 1, \ldots, n$, then is $\sum_{i=1}^{n} \lambda_i \mathbb{E}_\Omega [g_i(\tilde{\omega})|\tilde{s}]$ a MPS of $\sum_{i=1}^{n} \lambda_i \mathbb{E}_\Omega [g_i(\tilde{\omega})|\tilde{s}]$ for all $\lambda_i \geq 0$.*

Proof: $\mathbb{E}_\Omega\left[g_i(\tilde{\omega})|\tilde{s}\right]$ is a MPS of $\bar{\mathbb{E}}_\Omega\left[g_i(\tilde{\omega})|\tilde{s}\right]$ implies

$$\int_0^s \mathbb{E}_\Omega\left[g_i(\tilde{\omega})|s'\right]\mathrm{d}s' \leq \int_0^s \bar{\mathbb{E}}_\Omega\left[g_i(\tilde{\omega})|s'\right]\mathrm{d}s'$$

$$\Longrightarrow \int_0^s \sum_{i=1}^n \lambda_i \mathbb{E}_\Omega\left[g_i(\tilde{\omega})|s'\right]\mathrm{d}s' \leq \int_0^s \sum_{i=1}^n \lambda_i \bar{\mathbb{E}}_\Omega\left[g_i(\tilde{\omega})|s'\right]\mathrm{d}s'$$

which shows the claim. □

Proof of Proposition 4.6 State-contingent claims are traded after the signal realization. Therefore, the equilibrium allocation (of state-contingent claims) depends on the posterior state distribution conditional on the signal realization from the information system $(f_{S|\Omega}, f_\Omega)$. Denote by

$$(x^*((f_{S|\Omega}, f_\Omega), s), y^*((f_{S|\Omega}, f_\Omega), s), p^*((f_{S|\Omega}, f_\Omega), s)) \in \left(\mathcal{C}^N\right)^I \times \bigotimes_{j\in\mathcal{J}} Y_j^N \times (\mathbb{R}_+^C)^N$$

the Walrasian equilibrium after a signal realization equal to s when the information system is $(f_{S|\Omega}, f_\Omega)$.

Next observe that under any of the conditions (i)-(ii) the equilibrium price system, $\bar{p}^*$, of the certainty equivalent economy (after a signal realization equal to s)[47] is independent of s.[48]

Moreover, Proposition 4.5 implies that under any of the conditions (i) or (ii), in equilibrium each risk avers consumer i smooth her consumption, i.e.

$$x^*(\omega, i; (f_{S|\Omega}, f_\Omega), s) = \bar{x}^*(i, \bar{p}^*, \bar{p}^{*T}\mathbb{E}_\Omega\left[w(\tilde{\omega}, i)|s\right]) \quad \forall\, i = 1, \ldots, I;$$

where $\bar{x}^*(i, \bar{p}^*, \bar{p}^{*T}\mathbb{E}_\Omega\left[w(\tilde{\omega}, i)\right])$ denotes agent i's optimal consumption bundle in the certainty equivalent economy.[49] This implies that the conditional

[47]I.e. that economy in which the agents random endowments are replaced by their expected endowments $\mathbb{E}\left[w(\tilde{\omega}, a)|s\right]$ for $a = 1, \ldots, I$ or $a = 1, \ldots, J$.

[48]In case (i) it is determined by the risk neutral consumer's MRSs. In case (ii) by the MRT of the firm with constant MRTs.

[49]In particular, $\bar{x}^*(i, \cdot, \cdot) : \mathbb{R}_{>0}^C \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ denotes agent i's Marshallian demand function, i.e. if agent i's budget is $m \geq 0$ and prices are $p \in \mathbb{R}_{>0}^C$ then agent i's optimal

expected utility after a signal realization equal to s is

$$\mathbb{E}_\Omega\left[u_i(x^*(\tilde{\omega}, i; (f_{S|\Omega}, f_\Omega), s))|s\right] = u_i(\bar{x}^*(i, \bar{p}^*, \bar{p}^{*T}\mathbb{E}_\Omega\left[w(\tilde{\omega}, i)|s\right])).$$

Risk aversion implies that $u_i(\bar{x}^*(i, \bar{p}, m))$ is concave as a function of $m \in \mathbb{R}_{\geq 0}$ for all price vectors $\bar{p} \in \mathbb{R}^C_+$.[50] Hence, by Lemma C.1 and Lemma C.2, it is

$$\begin{aligned} V((f_{S|\Omega}, f_\Omega), a^*, u_i) = & \quad \mathbb{E}_S\left[\mathbb{E}_\Omega\left[u_i(x^*(\tilde{\omega}, i; (f_{S|\Omega}, f_\Omega)))|\tilde{s}\right]\right] \\ = \mathbb{E}_S\left[u_i(\bar{x}^*(i, \bar{p}^*, \bar{p}^{*T}\mathbb{E}_\Omega\left[w(\tilde{\omega}, i)|\tilde{s}\right]))\right] \overset{\text{concavity \& MPS}}{\leq} & \quad \mathbb{E}_S\left[u_i(\bar{x}^*(i, \bar{p}^*, \bar{p}^{*T}\bar{\mathbb{E}}_\Omega\left[w(\tilde{\omega}, i)|\tilde{s}\right]))\right] \\ = \mathbb{E}_S\left[\bar{\mathbb{E}}_\Omega\left[u_i(x^*(\tilde{\omega}, i; (\bar{f}_{S|\Omega}, f_\Omega)))|\tilde{s}\right]\right] = & \quad V((\bar{f}_{S|\Omega}, f_\Omega), \bar{a}^*, u_i) \end{aligned}$$

which proofs the claim. □

Proof of Proposition 4.7 Cororally 4.2 implies

$$y^*(\omega, j; (f_{S|\Omega}, f_\Omega), s) = \bar{y}^*(j) \in Y_j \quad \forall\, \omega \in \Omega,$$

where $\bar{y}^*(j) \in Y_j$ is the equilibrium state-contingent production plan of firm j in the certainty equivalent economy. Since equilibrium prices are fair, i.e. $p^*_{c,\omega}((f_{S|\Omega}, f_\Omega), s) = f_{\Omega|S}(\omega|s)\bar{p}^*_c$, where $\bar{p}^*_c$ is the equilibrium price in the certainty equivalent, this implies the claim. □

consumption bundle is $\bar{x}^*(p, m)$.

[50] For a reference of this statement consider Quah (2000) on page 921.

Appendix D

Proofs and algebraic manipulations of Chapter 5

Proof of Proposition 5.2 Simple algebraic calculations lead to the following differences in prices, quantities and profits of case 1 and 2:

$$\Delta_{\pi_1} := \pi_1^p - \pi_1^q = \gamma^3 \frac{\overbrace{\beta_1\alpha_2^2\gamma + \beta_2\alpha_1^2\gamma - 2\beta_1\beta_1\alpha_1\alpha_2}^{<0}}{\underbrace{8\beta_1\beta_2(\beta_1\beta_2 - \gamma^2)(2\beta_1\beta_2 - \gamma^2)}_{>0}},$$

$$\Delta_{\pi_2} := \pi_2^p - \pi_2^q = \gamma^5 \frac{\overbrace{\beta_1\alpha_2^2\gamma + \beta_2\alpha_1^2\gamma - 2\beta_1\beta_1\alpha_1\alpha_2}^{<0}}{\underbrace{16\beta_1\beta_2(\beta_1\beta_2 - \gamma^2)(2\beta_1\beta_2 - \gamma^2)^2}_{>0}},$$

$$\Delta_{p_1} := p_1^p - p_1^q = \gamma^3 \frac{\overbrace{-\alpha_2}^{<0}}{\underbrace{4\beta_2(2\beta_1\beta_2 - \gamma^2)}_{>0}},$$

$$\Delta_{p_2} := p_2^p - p_2^q = \gamma^2 \frac{\overbrace{\alpha_1\gamma - 2\alpha_2\beta_1}^{<0}}{\underbrace{4\beta_1(2\beta_1\beta_2 - \gamma^2)}_{>0}},$$

$$\Delta_{q_1} := q_1^p - q_1^q = \gamma^3 \frac{\overbrace{\alpha_1\gamma - \alpha_2\beta_1}^{<0}}{\underbrace{\beta_1(\beta_1\beta_2 - \gamma^2)(2\beta_1\beta_2 - \gamma^2)}_{>0}},$$

$$\Delta_{q_2} := q_2^p - q_2^q = \gamma^2 \frac{\overbrace{2\beta_1\beta_2\alpha_2 - \beta_2\alpha_1\gamma - \alpha_2\gamma^2}^{>0}}{\underbrace{4\beta_2(2\beta_1\beta_2 - \gamma^2)(\beta_1\beta_2 - \gamma^2)}_{>0}}.$$

By using this it follows

$$\Delta_{\pi_1} \begin{cases} < 0 & \text{for} \quad \gamma > 0 \\ = 0 & \text{for} \quad \gamma = 0 \\ > 0 & \text{for} \quad \gamma < 0, \end{cases} \qquad \Delta_{\pi_2} \begin{cases} < 0 & \text{for} \quad \gamma > 0 \\ = 0 & \text{for} \quad \gamma = 0 \\ > 0 & \text{for} \quad \gamma < 0, \end{cases}$$

$$\Delta_{p_1} \begin{cases} < 0 & \text{for} \quad \gamma > 0 \\ = 0 & \text{for} \quad \gamma = 0 \\ > 0 & \text{for} \quad \gamma < 0, \end{cases} \qquad \Delta_{p_2} < 0 \quad \text{for all} \quad \gamma,$$

$$\Delta_{q_1} \begin{cases} < 0 & \text{for} \quad \gamma > 0 \\ = 0 & \text{for} \quad \gamma = 0 \\ > 0 & \text{for} \quad \gamma < 0, \end{cases} \qquad \Delta_{q_2} > 0 \quad \text{for all} \quad \gamma.$$

The results stated in Proposition 5.2 follow immediately. □

Proof of Proposition 5.3 Consumers' surplus: The idea of the proof is to show that the difference in consumers' surplus $u(\text{"price equilibrium"}) - u(\text{"quantity equilibrium"})$, as a function of γ has a global minimum at $\gamma = 0$.

Hence, for $l \in \{p, q\}$ define

$$u^l(\gamma) := u\left(q_1^l, q_2^l\right)\big|_{p_1=p_1^l,\ p_2=p_2^l}.$$

For $l = p$ ($l = q$), then $u^p(\gamma)$ ($u^p(\gamma)$) describes the consumer's utility in equilibrium as a function of γ if firm 1 acts as price (quantity) setter. Now, by using the equilibium values it follows

$$u^p(\gamma) = \frac{1}{32\beta_1(2\beta_1\beta_2-\gamma^2)^2(\beta_1\beta_2-\gamma^2)}\Big[-3\alpha_1^2\gamma^6 + 2\alpha_1\alpha_2\beta_1\gamma^5 + (16\alpha_1^2\beta_1\beta_2 + 5\alpha_2^2\beta_1^2)\gamma^4 - 4\alpha_1\alpha_2\beta_1^2\beta_2\gamma^3 - (28\alpha_1^2\beta_1^2\beta_2^2 + 20\alpha_2^2\beta_1^3\beta_2)\gamma^2 + 16(\alpha_1^2\beta_1^3\beta_2^3 + \alpha_2^2\beta_1^4\beta_2^2)\Big]$$

and

$$u^q(\gamma) = \frac{1}{32}\frac{1}{(2\beta_1\beta_2-\gamma^2)^2\beta_2}\Big[5\alpha_2^2\gamma^4 + 4\alpha_1\alpha_2\beta_2\gamma^3 - (20\alpha_2^2\beta_1\beta_2 + 12\alpha_1^2\beta_2^2)\gamma^2 + 16(\alpha_1^2\beta_1\beta_2^3 + \alpha_2^2\beta_1^2\beta_2^2)\Big].$$

The 'utility-difference-function' is defined by

$$\begin{aligned} d(\gamma) :=& u^p(\gamma) - u^q(\gamma) \\ =& \frac{\gamma^2}{32}\frac{1}{\beta_1\beta_2(2\beta_1\beta_2-\gamma^2)^2(\beta_1\beta_2-\gamma^2)}\Big[(5\alpha_2^2\beta_1 - 3\alpha_1^2\beta_2)\gamma^4 + 6\alpha_1\alpha_2\beta_1\beta_2\alpha^3 \\ &+ (4\alpha_1^2\beta_1\beta_2^2 - 20\alpha_2^2\beta_1^2\beta_2)\gamma^2 - 8\alpha_1\alpha_2\beta_1^2\beta_2^2\gamma + 16\alpha_2^2\beta_1^3\beta_2^2\Big]. \end{aligned} \tag{D.1}$$

The first and second derivatives are equal to

$$d'(\gamma) = \frac{\gamma}{16}\frac{1}{(\beta_1\beta_2-\gamma^2)^2(2\beta_1\beta_2-\gamma^2)^3}\Big[-3\alpha_1\alpha_2\gamma^7 + 11(\beta_1\alpha_1^2 - 5\beta_1\alpha_2^2)\gamma^6 - 9\beta_1\beta_2\alpha_1\alpha_2\gamma^5 + (38\beta_1^2\beta_2\alpha_2^2 - 26\beta_1\beta_2^2\alpha_1^2)\gamma^4 + 34\beta_1^2\beta_2^2\alpha_1\alpha_2\gamma^3 + (16\beta_1^2\beta_2^3\alpha_1^2 - 64\beta_1^3\beta_2^2\alpha_2^2)\gamma^2 - 24\beta_1^3\beta_2^3\alpha_1\alpha_2\gamma + 32\beta_1^4\beta_2^3\alpha_2^2\Big]$$

and

$$d''(\gamma) = \frac{1}{16(\beta_1\beta_2-\gamma^2)^3(2\beta_1\beta_2-\gamma^2)^4}\Big[6\alpha_1\alpha_2\gamma^{11} + (15\alpha_2^2\beta_1 - 33\alpha_1^2\beta_2)\gamma^{10} + 66\alpha_1\alpha_2\beta_1\beta_1\gamma^9$$
$$+ (53\alpha_1^2\beta_1\beta_2^2 - 155\alpha_2^2\beta_1^2\beta_2)\gamma^8 - 216\alpha_1\alpha_2\beta_1^2\beta_2^2\gamma^7 + (68\alpha_1^2\beta_1^2\beta_2^3 + 340\alpha_2^2\beta_1^3\beta_2^2)\gamma^6$$
$$+ 152\alpha_1\alpha_2\beta_1^3\beta_2^3\gamma^5 - (228\alpha_2^2\beta_1^4\beta_2^3 - 180\alpha_1^2\beta_1^3\beta_2^4)\gamma^4 + 80\alpha_1\alpha_2\beta_1^4\beta_2^4\gamma^3$$
$$+ (96\alpha_1^2\beta_1^4\beta_2^5 - 32\alpha_2^2\beta_1^5\beta_2^4)\gamma^2 - 96\alpha_1\alpha_2\beta_1^5\beta_2^5\gamma + 64\alpha_2^2\beta_1^6\beta_2^5\Big].$$

Since $d'(0) = 0$ and $d''(0) = \frac{\alpha_2^2}{4\beta_1\beta_2^2} > 0$ it follows that $d(0) = 0$ is a local minimum of $d(\gamma)$, i.e. $\exists \epsilon > 0$ s.t. $u^p(\gamma) > u^q(\gamma)\ \forall\ \gamma \in (-\epsilon, \epsilon)$.
Moreover, since $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{R}$ equation (D.1) implies

$$d(\gamma) = 0 \;\Leftrightarrow\; u^p(\gamma) = u^q(\gamma) \;\Leftrightarrow\; \gamma = 0.$$

This implies that at $\gamma = 0$ is also a global minimum of $d(\gamma)$ which proofs the claim.
Total surplus: Analog to the proof for consumers' surplus. □

Proof of Proposition 5.4 First, consider the equations (5.17), (5.18), (5.19) and (5.20). The profit of firm 1 for the case if firm 2 chooses also price competition is given by equation (5.8):

$$\begin{aligned}\tilde{\pi}_1^{pp}(p_1) &:= \pi_1^{pp}\big(p_1, R_2^{pp}(p_1)\big)\\ &= p_1\ \tilde{q}_1\big(p_1, R_2^{pp}(p_1)\big)\\ &= p_1\left(a_1 - b_1p_1 + cR_2^{pp}(p_1)\right)\end{aligned}$$

with $R_2^{pp}(p_1)$ given in equation (5.4). The first order condition can be written to

$$\begin{aligned}\frac{\partial\tilde{\pi}_1^{pp}}{\partial p_1} &= \underbrace{\frac{\partial\pi_1^{pp}}{\partial p_1}}_{Direct\ effect} + \underbrace{\frac{\partial\pi_1}{\partial p_2}\frac{\partial R_2^{pp}}{\partial p_1}}_{Strategic\ effect}\\ &= \frac{\partial\pi_1^{pp}}{\partial p_1} + \frac{\gamma}{\beta_1\beta_2-\gamma^2}\,\frac{\gamma}{2\beta_1}p_1.\end{aligned}$$

The strategic effect is positive for all feasible γ. In the equilibrium of the simultaneous Bertrand game on the second stage it holds that $\frac{\partial \pi_1^{pp}}{\partial p_1}(p_1^{BB}) = 0$ which implies that

$$\frac{\partial \tilde{\pi}_1^{pp}}{\partial p_1}(p_1^{BB}) = \frac{\gamma^2}{2\beta_1\left(\beta_1\beta_2 - \gamma^2\right)} p_1^{BB} > 0 \quad \text{for all} \quad \gamma.$$

As the profit function is concave it follows directly

$$p_1^{BB} < p_1^p. \tag{D.2}$$

The profit of firm 1 for the case if firm 2 chooses quantity competition is given by (5.11), i.e.

$$\begin{aligned}\tilde{\pi}_1^{pq}\big(p_1\big) :&= \pi_1^{pq}\big(p_1, R_2^{pq}(p_1)\big)\\ &= p_1\, \hat{q}_1\big(p_1, R_2^{pq}(q_1)\big)\\ &= p_1 \frac{\alpha_1 - \gamma R_2^{pq}(q_1) - p_1}{\beta_1}\end{aligned}$$

with $R_2^{pq}(p_1)$ given in (5.5). The first order condition can be written to

$$\begin{aligned}\frac{\partial \tilde{\pi}_1^{pq}}{\partial p_1} &= \underbrace{\frac{\partial \pi_1^{pq}}{\partial p_1}}_{Direct\ effect} + \underbrace{\frac{\partial \pi_1}{\partial q_2}\frac{\partial R_2^{pq}}{\partial p_1}}_{Strategic\ effect}\\ &= \frac{\partial \pi_1^{pq}}{\partial p_1} - \frac{\gamma p_1}{\beta_1}\,\frac{\gamma}{2(\beta_1\beta_2 - \gamma^2)}.\end{aligned}$$

The strategic effect is negative for all feasible γ. In the equilibrium of the simultaneous move it holds that $\frac{\partial \pi_1^{pq}}{\partial p_1}(p_1^{BC}) = 0$ which implies that

$$\frac{\partial \tilde{\pi}_1^{pq}}{\partial p_1}(p_1^{BC}) = -\frac{\gamma^2}{2\beta_1\left(\beta_1\beta_2 - \gamma^2\right)}\, p_1^{BC} < 0 \quad \text{for all} \quad \gamma.$$

It follows directly by concavity of the profit function that

$$p_1^{BC} > p_1^p \tag{D.3}$$

holds. Together with equation (D.2) this implies inquality (5.17). The other inequalities follow directly by using simple algebraic manipulations which are omitted here.

It remains to prove the equations (5.21), (5.22), (5.23) and (5.24). The profit of firm 1 for the case if both firms select quantity competition is given in equation (5.14):

$$\begin{aligned}\tilde{\pi}_1^{qq}(q_1) &:= \pi_1^{qq}(q_1, R_2^{qq}(q_1)) \\ &= q_1 \, \tilde{p}_1\big(q_1, R_2^{qq}(q_1)\big)\end{aligned}$$

with $R_2^{qq}(q_1)$ given in equation (5.7). The first order condition can be written to

$$\begin{aligned}\frac{\partial \tilde{\pi}_1^{qq}}{\partial q_1} &= \underbrace{\frac{\partial \pi_1^{qq}}{\partial q_1}}_{Direct\ effect} + \underbrace{\frac{\partial \pi_1}{\partial q_2}\frac{\partial R_2^{qq}}{\partial q_1}}_{Strategic\ effect} \\ &= \frac{\partial \pi_1^{qq}}{\partial q_1} + \frac{\gamma^2}{2\beta_2}\, q_1.\end{aligned}$$

The strategic effect is positive for all feasible γ. In the equilibrium of the simultaneous Cournot game it holds that $\frac{\partial \pi_1^{qq}}{\partial q_1}(q_1^{CC}) = 0$ which implies that

$$\frac{\partial \tilde{\pi}_1^{qq}}{\partial q_1}(q_1^{CC}) = \frac{\gamma^2}{2\beta_2}\, q_1^{CC} > 0 \quad \text{for all} \quad \gamma.$$

Concavity of the profit function implies

$$q_1^{CC} < q_1^{q}. \tag{D.4}$$

The profit of firm 1 for the case if firm 2 chooses price competition is given in equation (5.12)

$$\begin{aligned}\tilde{\pi}_1^{qp}\big(q_1\big) &:= \pi_1^{qp}\big(q_1, R_2^{qp}(q_1)\big) \\ &= q_1 \, \hat{p}_1(q_1, R_2^{qp}(q_1))\end{aligned}$$

with $R_2^{qp}(q_1)$ given in equation (5.6). The first order condition can be written to

$$\frac{\partial \tilde{\pi}_1^{qp}}{\partial q_1} = \underbrace{\frac{\partial \pi_1^{qp}}{\partial q_1}}_{Direct\ effect} + \underbrace{\frac{\partial \pi_1}{\partial p_2}\frac{\partial R_2^{qp}}{\partial q_1}}_{Strategic\ effect}$$
$$= \frac{\partial \pi_1^{qp}}{\partial q_1} - \frac{\gamma^2}{2\beta_2}\, q_1.$$

Obviously, the strategic effect is negative for all feasible γ. In the equilibrium of the simultaneous move it holds that $\frac{\partial \pi_1^{qp}}{\partial q_1}(p_1^{CB}) = 0$ which implies that

$$\frac{\partial \tilde{\pi}_1^{pq}}{\partial q_1}(q_1^{CB}) = -\frac{\gamma^2}{2\beta_2}\, q_1^{CB} < 0 \quad \text{for all} \quad \gamma.$$

It follows directly by concavity of the profit function that

$$q_1^{CB} > q_1^q. \tag{D.5}$$

Together with equation (D.4) this implies the second inequality of equation (5.17). The other inequalities follow analogoulsy. □

The explicit algebraic expressions of figures 5.3, 5.4 and 5.5 in Example 8:

$$q_1^q = \frac{16-4\gamma}{2(8-\gamma^2)}, \quad q_1^{CC} = \frac{16-4\gamma}{16-\gamma^2}, \quad q_1^{CB} = \frac{16-4\gamma}{16-3\gamma^2},$$

$$p_1^q = \frac{4-\gamma}{2}, \quad p_1^{CC} = \frac{8}{4+\gamma}, \quad p_1^{CB} = \frac{2(4-\gamma)(4-\gamma^2)}{16-3\gamma^2},$$

$$q_2^q = \frac{16-4\gamma-\gamma^2}{2(8-\gamma^2)},\ q_2^{CC} = \frac{16-4\gamma}{16-\gamma^2},\ q_2^{CB} = \frac{2(8-2\gamma-\gamma^2)}{16-3\gamma^2},$$

$$p_2^q = \frac{16 - 4\gamma - \gamma^2}{8 - \gamma^2}, \quad p_2^{CC} = \frac{8}{4 + \gamma}, \quad p_2^{CB} = \frac{4(8 - \gamma^2) - 8\gamma}{16 - 3\gamma^2},$$

$$\pi_1^q = \frac{(4 - \gamma)^2}{8 - \gamma^2}, \quad \pi_1^{CC} = \frac{32}{(4 + \gamma)^2}, \quad \pi_1^{CB} = \frac{8(4 - \gamma)^2(4 - \gamma^2)}{2(16 - 3\gamma^2)^2},$$

$$\pi_2^q = \frac{(4 - \gamma)^2}{8 - \gamma^2}, \quad \pi_2^{CC} = \frac{32}{(4 + \gamma)^2}, \quad \pi_2^{CB} = \frac{8(8 - 2\gamma - \gamma^2)^2}{(16 - 3\gamma^2)^2}.$$

Proof of Proposition 5.5 Analog to the proof of Proposition 5.3. □

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