On strict polynomial functors: monoidal structure and Cauchy filtration

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Contents

Introduction

Strict polynomial functors were first defined by Friedlander and Suslin in [14], using polynomial maps of finite dimensional vector spaces over a field k. They showed that the category of strict polynomial functors of a fixed degree d is equivalent to the category of modules over the Schur algebra $S_k(n,d)$ whenever $n \geq d$, as we recall in Proposition 2.9. These algebras, named after Issai Schur and originally used to describe the polynomial representations of the general linear group, have been extensively investigated.

We give an equivalent description of the category of strict polynomial functors, namely defining them as k-linear representations of the category $\Gamma^d \mathsf{P}_k$ of divided powers

$$\mathsf{Rep}\,\Gamma^d_k = \mathrm{Fun}_k(\Gamma^d \mathsf{P}_k, \mathbf{M}_k).$$

This provides an important structure on the category of strict polynomial functors and hence on modules over the Schur algebra: The tensor product on the category of divided powers induces in a natural way an internal tensor product in the category of strict polynomial functors (defined in [21]), which is defined for representable functors by

$$\Gamma^{d,V} \otimes_{\Gamma^d_k} \Gamma^{d,W} := \Gamma^{d,V\otimes W}$$

and can be extended to a general object, by recalling that every functor in $\mathsf{Rep}\,\Gamma^d_k$ can be obtained as a colimit of representable functors.

An important application of this tensor product is the *Ringel duality* for strict polynomial functors, that has been recently introduced by Touzé [32] and Chałupnik [3] and can be stated as the equivalence

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} - : D(\mathsf{Rep}\,\Gamma^d_k) \xrightarrow{\sim} D(\mathsf{Rep}\,\Gamma^d_k)$$

where, in particular, $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} \Lambda^d \cong S^d$.

Ringel duality yields relations between representation theory of Schur algebras and algebraic topology (cf. [32] for more details) and makes possible to translate results which are well known in one area in unknown results in the other one.

The aim of the first part of this thesis is to understand better the monoidal structure on the category $\mathsf{Rep}\,\Gamma^d_k$ given above. We follow the description and the notations given in [21].

In order to describe the internal tensor product more explicitly we make use of the monoidal structure on the category of representations of the symmetric group. In fact, observing that $\mathrm{End}_{\Gamma^d_k}(\Gamma^\omega) \cong k\mathfrak{S}_d$, for $\omega = (1, \ldots, 1)$, yields a functor

$$\mathcal{F} = \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\omega, -)\colon \mathsf{Rep}\,\Gamma^d_k \to k\mathfrak{S}_d\,\mathsf{Mod}$$

which allows us to compare both monoidal structures. It turns out that the structure is preserved under the functor $\mathcal{F}$. This yields explicit formulae for the tensor product of some particular polynomial functors. The following is the main result of the first part, and comes from a joint work with Rebecca Reischuk [2]. The functor

$$\mathcal{F} = \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\omega, -)\colon \mathsf{Rep}\,\Gamma^d_k \to k\mathfrak{S}_d\,\mathsf{Mod}$$

preserves the monoidal structure defined on strict polynomial functors, i.e.

$$\mathrm{Hom}_{\Gamma^d_k}(\Gamma^\omega, X \otimes_{\Gamma^d_k} Y) \cong \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\omega, X) \otimes_k \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\omega, Y) \tag{0.1}$$

If we identify $\mathsf{Rep}\,\Gamma^d_k$ with the category $\mathsf{Mod}\,S_k(n,d)$, the functor $\mathcal{F}$ is the equivalent of the *Schur functor* discussed by Green ([18], 6.1). In order to avoid confusion, we do not use this name for $\mathcal{F}$. We will namely call *Schur functors* the functors $\mathbb{S}_\lambda$ defined by Akin, Buchsbaum and Weyman in [1], which, together with their duals $\mathbb{W}_\lambda$ play an important role in the second part.

In the second part, we discuss the *Cauchy filtration* in the context of strict polynomial functors. We define functors F_λ and show that there is a filtration

$$0 = F_\infty \subset F_{(d)} \subset F_{(d-1,1)} \subset \cdots \subset F_{(2,1,\ldots,1)} \subset F_{(1,\ldots,1)} = \Gamma^d(V \otimes W) \tag{0.2}$$

The aim of the second part is to prove the following (Theorem 5.11).

For any partition λ, denote by λ^+ the partition successive to λ with respect to the lexicographic order. We have an isomorphism

$$\mathbb{W}_\lambda V \otimes_k \mathbb{W}_\lambda V \cong F_\lambda / F_{\lambda^+} \tag{0.3}$$

so that the associated graded object of (0.2) is

$$\bigoplus_\lambda \mathbb{W}_\lambda V \otimes \mathbb{W}_\lambda W$$

where λ runs over all partitions of weight d.

If k is a field of characteristic 0, all inclusions of the above filtration split and it yields a direct sum decomposition

$$\Gamma^d(V \otimes W) = \bigoplus_\lambda \mathbb{W}_\lambda V \otimes \mathbb{W}_\lambda W, \tag{0.4}$$

which we call *Cauchy Decomposition.* In the above situation we will say, following [1], that the decomposition (0.4) holds *up to filtration* in positive characteristic. From (0.4), we reobtain the *Cauchy Formula* for symmetric functions

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}), \tag{0.5}$$

by computation of characters (cf. Section 4.3.1). Hence, the Cauchy filtration can be seen as the *categorification* of the Cauchy Formula.

In the literature there are some statement of a dual result, a Cauchy filtration for the symmetric algebra functor, from which (0.2) can be obtained by duality, but this seems to be quite complicated. Moreover, the derived powers functors are very important in the study of the category $\mathsf{Rep}\,\Gamma^d_k$, since they give all projective objects. For this reason it is worth to give a more direct and easier proof.

The Cauchy filtration has a long history and has been stated in many contexts.

The formula (5.1) for symmetric functions can be already deduced from the work of Cauchy [4].

Akin, Buchsbaum and Weyman [1] prove a decomposition of the symmetric algebra in terms of Schur functors

$$S(V \otimes W) = \bigoplus_\lambda \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W),$$

which holds up to filtration in positive characteristic, from which (0.2) follows by duality, since $(\Gamma^d)^\circ \cong S^d$.

De Concini, Eisenbud and Procesi [7] give the same result, with a very different language: they prove a characteristic free decomposition of $R = k[X_{ij}]$, which holds up to filtration, into $G = \mathrm{GL}(n,k) \times \mathrm{GL}(m,k)$ modules

$$R \cong \sum L_\sigma \otimes_k {}_\sigma L,$$

where the action is given by $A^{-1} X_{ij} B$, for $(A,B) \in G$ and the modules L_σ are a characteristic-free version of the distinct polynomial irreducible representations of general linear groups, indexed by partitions and given by Schur in his thesis for k a field of characteristic zero.

Moreover, the Cauchy filtration is an important ingredient in the proof that $\mathsf{rep}\,\Gamma_k^d$ is an *highest weight category*, where the functors $\mathbb{W}_\lambda$ and $\mathbb{S}_\lambda$ give the costandard and standard objects (cf. [22], Theorem 7.1). Since we have an equivalence $\mathsf{rep}\,\Gamma_k^d \cong \mod S_k(n,d)$ between strict polynomial functors and modules over Schur algebras, it can equivalently be proved that $S_k(n,d)$ is a quasi-hereditary algebra [5]. This has been first done by S. Donkin [9], where an equivalent of the Cauchy filtration is given in terms of *good filtrations* of the injective modules over *generalized Schur algebras*.

We work over an arbitrary commutative ring k. We start by recalling some basic definitions concerning partitions and tableaux, which will be needed in the following chapters. In the second section, we collect some material about strict polynomial functors and the description of the internal tensor product given in [21].

In the third chapter we focus our attention on representations of the symmetric group $\mathfrak{S}_d$. In particular, we consider the $k\mathfrak{S}_d$-module structure on the d-th tensor power of a free k-module E and its decomposition into permutation modules. We calculate the tensor product of permutation modules and its decomposition. We show that the functor $\mathcal{F}$ maps certain important projective objects in the category of strict polynomial functors to the permutation modules. For this an essential ingredient is the parametrization of morphisms of these objects given by Totaro [30]. Finally we prove that $\mathcal{F}$ is a monoidal functor (cf. [26, XI.2] for the definition).

In the fourth section we assume that k is a field of characteristic 0. In this case, the functor $\mathcal{F}$ induces an equivalence between strict polynomial functors and representations of the symmetric group. Moreover we explain and use the connection to symmetric functions.

The last section is dedicated to the Cauchy filtration, that we state as a decomposition of the divided power functor, following [22] and [19]. This is the dual of the filtration given for symmetric powers by Akin, Buchsbaum and Weyman [1, Theorem III.1.4]. Hashimoto and Kurano extend the result of [1] to chain complexes and obtain a more general version of (0.2) as a corollary. Here we want to give a direct proof, following some of the methods used by [1] and exploiting the *universality* of the involved functors. We conclude by showing how the Cauchy Formula for symmetric functions can be obtained from (0.2) by computation of characters and discuss how the formula can be deduced from Cauchy's work.

Declaration: Parts of this thesis are based on joint work with Rebecca Reischuk and this has been published in [2]. This concerns Theorem 3.9, and I declare that I contributed substantially to the formulation and the proof of this result.

Sections 3.1.1 and 3.3.1 and Lemma 3.6, which are also are published in [2], are due to Rebecca Reischuk and have been included for the sake of completeness.

1 Preliminaries

A lot of the objets that we consider in this work are defined with the help of partitions and Young diagrams or indexed by them. In this first section we want to collect some material that will be useful in the following chapters.

1.1 Compositions and Partitions

Given two positive integers n, d, we will call an n-tuple of non negative integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\sum_{i=1}^{n} \lambda_i = d$ a *composition* of d in n parts. The set of all compositions of d in n parts will be denoted by $\Lambda(n, d)$. A composition $\lambda \in \Lambda(n, d)$ with the property $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ will be called, as usual, a *partition* of d. Let $\Lambda^+(n, d)$ denote the set of all partitions of d in n parts. Also write $\Lambda(d)$ and $\Lambda^+(d)$ for the sets of all compositions and of all partitions of d, respectively. Finally, one can consider also the sets of all compositions Λ and of all partitions Λ^+. If λ is a composition we denote by $|\lambda| = \sum_i \lambda_i$ its *weight*. For $\lambda \in \Lambda^+$, we denote by $l(\lambda)$ its *length*, i.e. the number of its nonzero terms. The *conjugate* partition $\tilde{\lambda}$ of a partition λ, is the partition whose jth part $\tilde{\lambda}_j$ equals the number of terms of λ which are greater than or equal to j.

We can consider the *union* of two partitions as follows. If $\lambda \in \Lambda^+(r, d)$ and $\mu \in \Lambda^+(s, e)$, then $\lambda \cup \mu \in \Lambda^+(r+s, d+e)$ is the unique partition obtained by reordering the elements of the composition $(\lambda_1, \lambda_2, \ldots, \lambda_r, \mu_1, \mu_2, \ldots, \mu_s)$.

For positive integers n, d, we consider on the set $\Lambda^+(n, d)$ two orderings:

- The *domincance order* is defined as follows. We have $\mu \trianglelefteq \lambda$ if $\sum_{i=1}^{l} \mu_i \leq \sum_{i=1}^{l} \lambda_i$ for all $1 \leq l \leq n$. This is a *partial ordering.*
- The *lexicographic order* is given in the following way. We have $\mu \leq \lambda$ if for some integer s, $1 \leq s \leq n$, $\mu_i = \lambda_i$ for all $i < s$ and $\mu_s \leq \lambda_s$. This is a *total ordering.*

The dominance order implies the lexicographic order, i.e. if $\mu \trianglelefteq \lambda$, then $\mu \leq \lambda$.

We will need to consider the following set of matrices defined by two compositions. For $\lambda \in \Lambda(m, d)$ and $\mu \in \Lambda(n, d)$ define

$$A_\mu^\lambda = \{A = (a_{ij}) \in \mathbf{M}_{m\times n}(\mathbb{N}_0) |\ \lambda_i = \sum_j a_{ij},\ \mu_j = \sum_i a_{ij}\}. \tag{1.1}$$

Notice that every matrix in A_μ^λ can be seen as a composition of d in mn parts.

Example 1.1. Let $\lambda = (3,1) \in \Lambda(2,4)$ and $\mu = (2,1,1) \in \Lambda(3,4)$. Then A^λ_μ consists of the following matrices:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

1.1.1 Orbits of multi-indeces

Multi-indeces are useful to decribe bases of some modules, such as tensor powers of a k-module. A permutation of the factors of a tensor product can be seen as an action of the symmetric group on the indeces of the elements of a basis. Compositions will help to describe the orbits of this action.

For positive integers n, d let

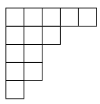
$$I(n,d) := \{\underline{i} = (i_1 \dots i_d) \mid 1 \leq i_l \leq n\}$$

be the set of d-tuples of positive integers smaller equal than n. The symmetric group $\mathfrak{S}_d$ acts naturally on the right on $I(n,d)$ by $\underline{i}\pi = (i_{\pi(1)}, \dots, i_{\pi(d)})$. For any d-tuple $\underline{i} \in I(n,d)$ let the *content* of $\underline{i}$ be the vector $\alpha = (\alpha_1, \dots, \alpha_n)$ such that, for each $1 \leq l \leq n$, α_l equals the number of the entries of $\underline{i}$ that are equal to l. This can be seen as a composition of d in n parts. If $\lambda \in \Lambda(n,d)$ is the content of $\underline{i}$, we say that $\underline{i}$ belongs to λ (and write $\underline{i} \in \lambda$). Clearly, the content of a d-tuple is invariant under permutation of the entries. Hence, $\Lambda(n,d)$ can be seen as the set of all $\mathfrak{S}_d$-orbits of $I(n,d)$. We write $\underline{i} \sim \underline{j}$ to indicate that the elements $\underline{i}$ and $\underline{j}$ are in the same orbit. We will also consider the action of $\mathfrak{S}_d$ on $I(n,d) \times I(n,d)$ and write $(\underline{i}, \underline{j}) \sim (\underline{i}', \underline{j}')$ iff $\underline{i} \sim \underline{i}'$ and $\underline{j} \sim \underline{j}'$.

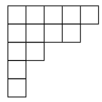
1.2 Young diagrams and Tableaux

It is useful to associate to a partition $\lambda \in \Lambda^+(n,d)$ a *diagram*. This is the set of ordered pairs $(i,j) \in \mathbb{N}^2$ with $i \geq 1$ and $i \leq j \leq \lambda_i$. Graphically, we can represent a diagram by drawing a square for each pair.

For example, the diagram of $\lambda = (5,3,2,2,1)$ will look like



Using diagrams, one can see that if λ is a partition, then $\tilde{\lambda}$ is the partition whose jth term is the number of squares in the jth column of the diagram of λ, where columns are counted from left to right. It is therefore clear that $|\lambda| = |\tilde{\lambda}|$. In the example above, the diagram of $\tilde{\lambda}$ is



The representation of a partition λ by using boxes is usually called the *Young diagram* of λ and denoted by Δ_λ.

If S is a totally ordered set and $\lambda \in \Lambda^+(n, d)$ a partition, a *tableau* of shape λ with values in the set S is a function from Δ_λ to S, i.e. a *filling* of the Young diagram of λ. Denote by $\mathrm{Tab}_\lambda(S)$ the set of all tableaux of shape λ and values in S. If $S = \{1, \dots, n\} = [n]$, we will write $\mathrm{Tab}_\lambda(n)$ instead of $\mathrm{Tab}_\lambda([n])$.

A tableau $T \in \mathrm{Tab}_\lambda(S)$ is called *row-standard* if its entries are strictly increasing along each row and *column-standard* if the columns are weakly increasing. We denote by $\mathrm{Tab}_\lambda^{rs}(S)$ and by $\mathrm{Tab}_\lambda^{cs}(S)$ the sets of row-standard and column-standard tableaux with entries in S, respectively. A *standard* tableau is both row- and column-standard. Write $\mathrm{Tab}_\lambda^{s}(S)$ for the set of standard tableaux.

Dually, T is called *row-costandard* if its entries are weakly increasing along each row and *column-costandard* if the columns are strictly increasing. A *costandard* tableau is both row- and column-costandard. As before, denote by $\mathrm{Tab}_\lambda^{rc}(S)$, $\mathrm{Tab}_\lambda^{cc}(S)$ the sets of row-costandard and column-costandard tableaux and by $\mathrm{Tab}_\lambda^{c}(S)$ the set of costandard tableaux. We may want to consider all tableaux of a given weight d, that is tableaux of shape λ for all partitions λ of d. In this case, we will write $\mathrm{Tab}_d(S)$ and use superscripts as above if we want to restrict to (row-/column-) standard or costandard tableaux.

Example 1.2. The tableau

1	2	3
1	3	

is standard but not costandard.

For a tableau $T \in \mathrm{Tab}_\lambda(n)$ with $\lambda \in \Lambda^+(r, d)$, we call the composition $\mu \in \Lambda(n, d)$, where μ_i equals the number of times the element i occurs in T, the *content* of T and write $c(T) = \mu$.

Example 1.3. The content of the tableau in the example above is given by $\mu = (2, 1, 2)$.

We have the following easy result.

Lemma 1.1. *Let* λ *and* μ *be partitions of weight* d*. There exist a costandard tableau of shape* λ *and content* μ *if and only if* $\mu \trianglelefteq \lambda$*. There exist a standard tableau of shape* λ *and content* μ *if and only if* $\tilde{\mu} \trianglelefteq \tilde{\lambda}$*. Here we consider the dominance order.*

Proof. Let $\mu \trianglelefteq \lambda$ and consider the tableau S of shape λ defined as follows. If we count the boxes from the right to the left, from the top to the bottom, the entries of the first μ_1 boxes of S are equal to 1, the following μ_2 boxes have entries equal to 2 and so on. This is a costandard tableau because of the condition $\mu \trianglelefteq \lambda$. Conversely, assume that there exists a costandard tableau T of shape λ and content μ. Since the entries of T are strictly increasing along columns, we can only have entries equal to 1 in the first row of T. Hence, $\lambda_1 \geq \mu_1$. For the same reason, entries equal to 2 can only occur in the first two rows, thus $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$ and so on. It follows $\mu \trianglelefteq \lambda$. The proof for standard tableaux is analogue. □

2 Strict polynomial functors

In this section, following [21], we define strict polynomial functors as the category of k-linear representations of the category of divided powers, for an arbitrary commutative ring k. In particular, we recall the definition the internal tensor product for strict polynomial functors given in [21, (2.5)]. We recall some properties and results.

We fix a commutative ring k. Let P_k denote the category of finitely generated projective k-modules For each $V, W \in \mathsf{P}_k$, denote by $V \otimes W$ their tensor product over k and by $\mathrm{Hom}(V, W)$ the group of k-linear maps $V \to W$. This provides two bifunctors

$$- \otimes - : \mathsf{P}_k \times \mathsf{P}_k \longrightarrow \mathsf{P}_k$$

$$\mathrm{Hom}(-,-) : (\mathsf{P}_k)^{op} \times \mathsf{P}_k \longrightarrow \mathsf{P}_k$$

with a natural isomorphism

$$\mathrm{Hom}_{\mathsf{P}_k}(U \otimes V, W) \cong \mathrm{Hom}_{\mathsf{P}_k}(U, \mathrm{Hom}(V, W)).$$

Moreover we have the a duality

$$(-)^* : (\mathsf{P}_k)^{op} \longrightarrow \mathsf{P}_k$$

defined by the functor sending V to $V^* = \mathrm{Hom}(V, k)$.

For U, V, W, V', W' in P_k one has natural isomorphisms

$$\begin{aligned} V^* \otimes W &\cong \mathrm{Hom}(V, W) \\ \mathrm{Hom}(U, V) \otimes W &\cong \mathrm{Hom}(U, V \otimes W) \\ \mathrm{Hom}(V, W) \otimes \mathrm{Hom}(V', W') &\cong \mathrm{Hom}(V \otimes V', W \otimes W'). \end{aligned} \tag{2.1}$$

2.1 The category of divided powers

For a non negative integer d, denote by $\mathfrak{S}_d$ the symmetric group permuting d elements. For any $V \in \mathsf{P}_k$, $\mathfrak{S}_d$ acts on the right on the tensor power $V^{\otimes d}$ by permuting the factors of the tensor product, i.e.

$$\text{for} \quad v_i \in V, \quad \sigma \in \mathfrak{S}_d, \quad (v_1 \otimes \ldots \otimes v_d)\sigma = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}.$$

The elements of $V^{\otimes d}$ which are invariant under this action form a submodule $(V^{\otimes d})^{\mathfrak{S}_d} \subset V^{\otimes d}$, that we denote $\Gamma^d(V)$ and call the module of *divided powers of degree d*. Set $\Gamma^0 V \cong k$.

To check that $\Gamma^d V$ is in P_k it suffices to observe that $\Gamma^d V$ is a finitely generated free k-module, provided that V is free finitely generated. Thus, we have the following (cf. [12, Proposition 4.2])

Proposition 2.1. *For $V, W \in \mathsf{P}_k$ and for any non negative integer d there is a natural map*

$$\psi^d : \Gamma^d V \otimes \Gamma^d W \to \Gamma^d(V \otimes W)$$

given by the restriction of the isomorphism

$$V^{\otimes d} \otimes W^{\otimes d} \xrightarrow{\sim} (V \otimes W)^{\otimes d}.$$

Also, we have a natural isomorphism $\Gamma^d k \cong k$. This morphisms endow Γ^d with a structure of a symmetric monoidal functor $\Gamma^d : \mathsf{P}_k \to \mathsf{P}_k$. □

Definition 2.2. The *category of degree d divided powers* $\Gamma^d\mathsf{P}_k$ is given as follows. The objects of $\Gamma^d\mathsf{P}_k$ are same objects as of P_k. Moreover, the morphisms between two objects V and W are given by

$$\mathrm{Hom}_{\Gamma^d\mathsf{P}_k}(V, W) := \Gamma^d \,\mathrm{Hom}(V, W) = (\mathrm{Hom}(V, W)^{\otimes d})^{\mathfrak{S}_d}.$$

The composition is induced by the symmetric monoidal structure on Γ^d from Proposition 2.1.

Note that $\mathrm{Hom}_{\Gamma^d\mathsf{P}_k}(V, W)$ can be identified with $\mathrm{Hom}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$ where for $\sigma \in \mathfrak{S}_d$, $f \in \mathrm{Hom}(V^{\otimes d}, W^{\otimes d})$ and $v_i \in V$ the action is given by

$$f\sigma(v_1 \otimes \cdots \otimes v_d) := f((v_1 \otimes \cdots \otimes v_d)\sigma^{-1})\sigma = f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)})\sigma.$$

In other words, the set of morphisms $\mathrm{Hom}_{\Gamma^d\mathsf{P}_k}(V, W)$ is isomorphic to the set of $\mathfrak{S}_d$-equivariant morphisms from $V^{\otimes d}$ to $W^{\otimes d}$.

2.2 The category of strict polynomial functors

Definition 2.3. The *category of strict polynomial functors of degree d* over k is the category of k-linear representations of $\Gamma^d\mathsf{P}_k$

$$\mathsf{Rep}\,\Gamma^d_k = \mathrm{Fun}_k(\Gamma^d\mathsf{P}_k, \mathsf{M}_k),$$

where M_k denotes the category of k-modules. The morphisms between two strict polynomial functors X, Y are denoted by $\mathrm{Hom}_{\Gamma^d_k}(X, Y)$.

A strict polynomial functor X is called *finite* if $X(V) \in \mathsf{P}_k$ for any $V \in \mathsf{P}_k$. The full subcategory of finite strict polynomial functors is denoted by $\mathsf{rep}\,\Gamma^d_k = \mathrm{Fun}_k(\Gamma^d\mathsf{P}_k, \mathsf{P}_k)$.

The category $\mathsf{Rep}\,\Gamma^d_k$ is abelian with infinite exact direct sums, further $\mathsf{rep}\,\Gamma^d_k \subset \mathsf{Rep}\,\Gamma^d_k$ is an exact subcategory.

Classical examples of strict polynomial functors of degree d, for d any non negative integer, are

- The divided power functor Γ^d, as already observed;
- The symmetric power functor S^d, defined by

$$S^d V := V^{\otimes d}/\langle v\otimes w - w\otimes v,\ v,w\in V\rangle;$$

  Namely, for each $d\geq 0$, the k-module S^dV is free provided that V is free. Thus S^dV belongs to P_k for all $V\in\mathsf{P}_k$ and this gives a functor $\Gamma^d\mathsf{P}_k\to\mathsf{P}_k$, since the ideal generated by elements of the form $v\otimes w - w\otimes v$ is invariant under the action of $\mathfrak{S}_d$ on $V^{\otimes d}$.
- The exterior power functor Λ^d, defined by

$$\Lambda^d V := V^{\otimes d}/\langle v\otimes v,\ v\in V\rangle.$$

  For each $d\geq 0$, the k-module Λ^dV is free provided that V is free. Thus Λ^dV belongs to P_k for all $V\in\mathsf{P}_k$ and this gives a functor $\Gamma^d\mathsf{P}_k\to\mathsf{P}_k$, since the ideal generated by elements of the form $v\otimes v$ is invariant under the action of $\mathfrak{S}_d$ on $V^{\otimes d}$.

All functors listed above are finite: $\Gamma^d, S^d, \Lambda^d\in \mathsf{rep}\,\Gamma^d_k$.

2.2.1 Representable functors

For any module $V\in\Gamma^d\mathsf{P}_k$, denote by $\Gamma^{d,V}$ the functor represented by V, i.e.

$$\Gamma^{d,V} := \mathrm{Hom}_{\Gamma^d\mathsf{P}_k}(V,-).$$

In other words,

$$\Gamma^{d,V}(W) = \Gamma^d\,\mathrm{Hom}(V,W).$$

By the Yoneda lemma, for any $F\in\mathsf{Rep}\,\Gamma^d_k$, we have

$$\mathrm{Hom}_{\Gamma^d_k}(\Gamma^{d,V},F) = F(V),$$

hence $\Gamma^{d,V}$ is a projective object of $\mathsf{Rep}\,\Gamma^d_k$ and $\mathsf{rep}\,\Gamma^d_k$.

Via the Yoneda embedding

$$\begin{aligned}(\Gamma^d\mathsf{P}_k)^{\mathrm{op}} &\to \mathsf{Rep}\,\Gamma^d_k\\ V &\mapsto \Gamma^{d,V}\end{aligned}$$

the category $(\Gamma^d\mathsf{P}_k)^{\mathrm{op}}$ can be identified with the full subcategory of $\mathsf{Rep}\,\Gamma^d_k$ consisting of all representable functors.

Example 2.1. For any non negative integer d, the divided power functor of degree d is the representable functor represented by k. In fact, for every $W\in\Gamma^d\mathsf{P}_k$ we have

$$\Gamma^d W\cong\Gamma^d\,\mathrm{Hom}(k,W) = \Gamma^{d,k}(W).$$

2.2.2 Colimits of representable functors

Representable functors are particularly important in this discussion, because every object in $\mathsf{Rep}\,\Gamma^d_k$ can be obtained as a colimit of them. This is an analogue of a free presentation of a module over a ring, see [26, III.7] and will make possible to give some definitions or to show some results with the help of representable functors and to extend them to arbitrary objects, by using colimits.

In our situation this can be done as follows. Let X be an object in $\mathsf{Rep}\,\Gamma^d_k$ and $V \in \Gamma^d\mathsf{P}_k$. By the Yoneda lemma, every element $v \in X(V)$ corresponds to a natural transformation $F_v : \Gamma^{d,V} \to X$. Let

$$\mathcal{C}_X = \{F_v : \Gamma^{d,V} \to X \mid V \in \Gamma^d\mathsf{P}_k,\ v \in X(V)\}$$

be the category whose objects are natural transformations F_v from representable functors $\Gamma^{d,V}$ to X, where V runs through all elements in $\Gamma^d\mathsf{P}_k$, and where a morphism between F_v and F_w, with $v \in X(V),\ w \in X(W)$, is given by a natural transformation $\phi_{v,w} : \Gamma^{d,V} \to \Gamma^{d,W}$ such that $F_v = F_w \circ \phi_{v,w}$. Define $\mathcal{F}_X : \mathcal{C}_X \to \mathsf{Rep}\,\Gamma^d_k$ to be the functor sending a natural transformation F_v to its domain, the representable functor $\Gamma^{d,V}$. Then $X = \operatorname{colim} \mathcal{F}_X$.

2.2.3 Duality

Given a representation $X \in \mathsf{Rep}\,\Gamma^d_k$, its *dual* X° is defined by

$$X^\circ(V) = X(V^*)^*.$$

For all $X, Y \in \mathsf{Rep}\,\Gamma^d_k$ we have a natural isomrphism

$$\mathrm{Hom}_{\Gamma^d_k}(X, Y^\circ) \cong \mathrm{Hom}_{\Gamma^d_k}(Y, X^\circ).$$

The evaluation morphism $X \to X^{\circ\circ}$ is an isomorphism when X takes values in P_k.

Example 2.2.
- The divided power functor Γ^d and the symmetric power functor S^d are dual to each other. It follows easily from the definitions that $S^d \cong (\Gamma^d)^\circ$;
- There is a natural isomorphism $\Lambda^d(V^*) \cong (\Lambda^d V)^*$, induced by

$$(f_1 \wedge \ldots \wedge f_d)(v_1 \wedge \ldots \wedge v_d) = \det(f_i(v_j)),$$

and therefore $(\Lambda^d)^\circ \cong \Lambda^d$.

2.2.4 External tensor product

Definition 2.4. For non-negative integers d, e and functors $X \in \mathsf{Rep}\,\Gamma_k^d$ and $Y \in \mathsf{Rep}\,\Gamma_k^e$ we define the *external tensor product*

$$- \otimes - : \mathsf{Rep}\,\Gamma_k^d \otimes \mathsf{Rep}\,\Gamma_k^e \longrightarrow \mathsf{Rep}\,\Gamma_k^{d+e}$$

as follows. On objects it is given by

$$(X \otimes Y)(V) = X(V) \otimes_k Y(V),$$

where $\otimes_k$ denotes the usual tensor product of k-modules, and on morphisms via the natural inclusion

$$\Gamma^{d+e}\operatorname{Hom}(V, W) \xrightarrow{\sim} (\operatorname{Hom}(V, W)^{(d+e)})^{\mathfrak{S}_{d+e}} \hookrightarrow (\operatorname{Hom}(V, W)^{(d+e)})^{\mathfrak{S}_d \times \mathfrak{S}_e}$$

$$\xrightarrow{\sim} \Gamma^d \operatorname{Hom}(V, W) \otimes \Gamma^e \operatorname{Hom}(V, W)$$

Of course, we can consider the tensor product of several functors. In particular, for positive integers n, d and a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of d in n parts, we can take

$$\Gamma^\lambda = \Gamma^{\lambda_1} \otimes \cdots \otimes \Gamma^{\lambda_n} \in \mathsf{Rep}\,\Gamma_k^d$$

where $\Gamma^{\lambda_1,k} \in \mathsf{Rep}\,\Gamma_k^{\lambda_1}, \ldots, \Gamma^{\lambda_n,k} \in \mathsf{Rep}\,\Gamma_k^{\lambda_n}$. Analogously, we can define

$$S^\lambda = S^{\lambda_1} \otimes \ldots \otimes S^{\lambda_n} \quad \text{and} \quad \Lambda^\lambda = \Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_n}.$$

Example 2.3. If we denote by (d) the partition of d having only one entry, we clearly have $\Gamma^{(d)} \cong \Gamma^d$.

2.2.5 The tensor product of strict polynomial functors

For V, W in P_k, the usual tensor product $V \otimes_k W$ of k-modules induces a tensor product on $\Gamma^d\mathsf{P}_k$, the category of divided powers. It coincides on objects with the one for P_k and on morphisms it is given via the following composite:

$$\Gamma^d \operatorname{Hom}(V, V') \times \Gamma^d \operatorname{Hom}(W, W') \to \Gamma^d(\operatorname{Hom}(V, V') \otimes \operatorname{Hom}(W, W'))$$

$$\xrightarrow{\sim} \Gamma^d \operatorname{Hom}(V \otimes W, V' \otimes W').$$

This last tensor product induces a tensor product for strict polynomial functors via the Yoneda embedding, which as been defined in [21].

Definition 2.5. The *internal tensor product* on $\mathsf{Rep}\,\Gamma^d_k$, is given for representable functors $\Gamma^{d,V}$ and $\Gamma^{d,W}$ in $\mathsf{Rep}\,\Gamma^d_k$ as follows

$$\Gamma^{d,V} \otimes_{\Gamma^d_k} \Gamma^{d,W} := \Gamma^{d,V\otimes W}.$$

For arbitrary objects X and Y in $\mathsf{Rep}\,\Gamma^d_k$ define

$$\Gamma^{d,V} \otimes_{\Gamma^d_k} X := \operatorname{colim}(\Gamma^{d,V} \otimes_{\Gamma^d_k} \mathcal{F}_X),$$
$$X \otimes_{\Gamma^d_k} Y := \operatorname{colim}(\mathcal{F}_X \otimes_{\Gamma^d_k} Y),$$

where $\Gamma^{d,V} \otimes_{\Gamma^d_k} \mathcal{F}_X$ resp. $\mathcal{F}_X \otimes_{\Gamma^d_k} Y$ is the functor sending the natural transformation F_v to $\Gamma^{d,V} \otimes_{\Gamma^d_k} \mathcal{F}_X(F_v)$ resp. $\mathcal{F}_X(F_v) \otimes_{\Gamma^d_k} Y$.

In other words, $-\otimes_{\Gamma^d_k} -$ is the unique bifunctor $\mathsf{Rep}\,\Gamma^d_k \times \mathsf{Rep}\,\Gamma^d_k \to \mathsf{Rep}\,\Gamma^d_k$ wich commutes with colimits in both arguments and extends the bifunctor $-\otimes - : \Gamma^d\mathsf{P}_k \times \Gamma^d\mathsf{P}_k \to \Gamma^d\mathsf{P}_k$ via the Yoneda embedding.

The tensor unit is given by

$$I_{\Gamma^d_k} := \Gamma^{d,k} \cong \Gamma^{(d)}.$$

2.2.6 Graded representations and exponential functors

It is sometimes convenient to consider the category

$$\prod_{d\geq 0} \mathsf{Rep}\,\Gamma^d_k$$

consisting of graded representations $X^* = (X^0, X^1, X^2, \ldots)$. We say that X^* is *finite* if $X^i \in \mathsf{rep}\,\Gamma^i_k$ for all $i \geq 0$. The tensor product $X \otimes Y$ of two graded representations X, Y is defined in degree d by

$$(X \otimes Y)^d = \sum_{i+j=d} X^i \otimes Y^j.$$

The classical polynomial functors we considered so far also yield examples of graded functors

$$\Gamma^* = (\Gamma^1, \Gamma^2, \Gamma^3, \ldots),$$
$$S^* = (S^1, S^2, S^3, \ldots),$$
$$\Lambda^* = (\Lambda^1, \Lambda^2, \Lambda^3, \ldots).$$

Moreover, they satisfy another interesting property (cf. [13] or [31])

Definition 2.6. An *exponential functor* is a finite graded functor

$$X^* = (X^0, X^1, X^2, \dots, X^n, \dots)$$

together with natural isomorphisms

$$X^0(V) \cong k, \quad X^n(V \oplus W) \cong \bigoplus_{m=o}^{n} X^m(V) \otimes_k X^{n-m}(W),\ n > 0.$$

The functors Γ^*, Λ^* and S^* are often called *classical exponential functors.*

From the exponential property for the graded divided power functor, it follows that, for each positive integer n, one obtains in degree d a decomposition

$$\Gamma^{d,k^n} = \sum_{i-0}^{d} (\Gamma^{d-i,k^{n-1}} \otimes \Gamma^i)$$

and using induction a *canonical decomposition* [21, (2.8)]

$$\Gamma^{d,k^n} = \bigoplus_{\lambda \in \Lambda(n,d)} \Gamma^\lambda, \tag{2.2}$$

where $\Lambda(n,d)$ denotes the set of compositions of d in n parts.

The decomposition of divided powers has the following consequence

Proposition 2.7. *The category of finitely generated projective objects in* $\mathsf{Rep}\,\Gamma^d_k$ *is equivalent to the category* $\mathsf{add}\,\Gamma$ *of direct summands of finite direct sums of functors* Γ^λ*, where* $\lambda \in \Lambda(n,d)$ *is any composition and* n *a non negative integer.*

Proof. As we already observed it follows from Yoneda lemma that representable functors are projective. For any $V \in \Gamma^d\mathsf{P}_k$ the functor $\Gamma^{d,V}$ is a direct summand of a direct sum of copies of Γ^{d,k^n} for some n. Hence every representable functor admits a decomposition of the form (2.2) and it follows easily that every element in $\mathsf{add}\,\Gamma$ is finitely generated projective. □

2.2.7 Representations of Schur algebras

Schur algebras are finite dimensional algebras named after Issai Schur by J.A. Green (and extensively treated in [18]), that are very important in the representation theory of general linear groups. As showed by Friedlander and Suslin in [14], they are closely related to strict polynomial functors.

Definition 2.8. For n, d positive integers, define the Schur algebra as

$$S_k(n,d) = \mathrm{End}_{k\mathfrak{S}_d}((k^n)^{\otimes d}) = \mathrm{End}_{\Gamma^d_k}(\Gamma^{d,k^n})^{\mathrm{op}}.$$

We want to describe a basis for $S_k(n,d)$ explicitly. To do this, observe that, if $E = k^n$, the set $I = I(n,d) = \{\underline{i} = (i_1, \dots, i_d) \mid 1 \leq i_l \leq n\}$ naturally indexes a basis of the dth tensor power $E^{\otimes d}$. Namely, for a fixed basis $\{e_1, \dots, e_n\}$ of E, write

$$e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_d} \text{ for } \underline{i} = (i_1, \dots, i_d) \in I.$$

An element $\theta \in \mathrm{End}_k(E^{\otimes d})$ has a matrix, say $(T_{\underline{i}\underline{j}})$, relative to the basis $\{e_{\underline{i}} \mid \underline{i} \in I\}$ given above. Here, $T_{\underline{i}\underline{j}} \in k$ and $\underline{i}$ and $\underline{j}$ run independently over the set I. The action of $\mathfrak{S}_d$ on I induces an action on $E^{\otimes d}$ by

$$e_{\underline{i}}\pi = e_{\underline{i}\pi}, \quad \text{for all } \ \underline{i} \in I, \ \pi \in \mathfrak{S}_d.$$

It follows that θ lies in $\mathrm{End}_{k\mathfrak{S}_d}((k^n)^{\otimes d})$ if and only if

$$(T_{\underline{i}\pi\underline{j}\pi}) = (T_{\underline{i}\underline{j}}) \ \text{ for all } \ \underline{i}, \underline{j} \in I, \ \pi \in \mathfrak{S}_d. \tag{2.3}$$

Consequently, $\mathrm{End}_{k\mathfrak{S}_d}(E^{\otimes d})$ has a k-basis in one-to-one correspondence with the set Ω of all $\mathfrak{S}_d$-orbits on $I \times I$, with respect to the action given by $(\underline{i}, \underline{j})\pi = (\underline{i}\pi, \underline{j}\pi)$, for any $(\underline{i}, \underline{j}) \in I \times I$ and $\pi \in \mathfrak{S}_d$. Namely, if ω is such an orbit, define the corresponding basis element θ_ω to be that $\theta \in \mathrm{End}_k(E^{\otimes d})$ whose matrix $(T_{\underline{i}\underline{j}})$ has $T_{\underline{i}\underline{j}} = 0$ or 1 according as $(\underline{i}, \underline{j}) \in \omega$ or not. For $\omega \in \Omega$ and $(\underline{i}, \underline{j}) \in \omega$ any representative, denote by $\zeta_{\underline{i},\underline{j}}$ the corresponding basis element of $\mathrm{End}_{k\mathfrak{S}_d}(E^{\otimes d})$. As a k-space the Schur algebra $S_k(n,d)$ has basis

$$\{\zeta_{\underline{i},\underline{j}} \mid \underline{i}, \underline{j} \in I\}, \text{ where } \zeta_{\underline{i},\underline{j}} = \zeta_{\underline{p},\underline{q}} \ \text{ if and only if } \ (\underline{i}, \underline{j}) \sim (\underline{p}, \underline{q}). \tag{2.4}$$

It is a well known result (cf. [18], p.67) that, for every integer $N \geq n \geq d$ the k-algebras $S_k(n,d)$ and $S_k(N,d)$ are *Morita equivalent*, i.e. we have an equivalence of categories $S_k(n,d)\,\mathsf{Mod} \cong S_k(N,d)\,\mathsf{Mod}$. Moreover, we have the following

Proposition 2.9. *([14, Theorem 3.2]) If $n \geq d$ there is an equivalence of categories $\mathsf{Rep}\,\Gamma_k^d \cong \mathsf{Mod}\,\mathrm{End}_{\Gamma_k^d}(\Gamma^{d,k^n}) = S_k(n,d)\,\mathsf{Mod}$.*

Proof. If $P \in \mathsf{Rep}\,\Gamma_k^d$ is a small projective generator, that is P is projective and the functor $\mathrm{Hom}_{\Gamma_k^d}(P, -)$ is faithful and preserves set-indexed direct sums, then

$$\mathrm{Hom}_{\Gamma_k^d}(P, -) : \mathsf{Rep}\,\Gamma_k^d \to \mathsf{Mod}\,\mathrm{End}_{\Gamma_k^d}(P)$$

is an equivalence of categories. The representable functors $\Gamma^{d,V}$ with $V \in \mathsf{P}_k$ form a family of small projective generators by Yoneda lemma. If $n \geq d$, the functors Γ^λ with $\lambda \in \Lambda(n,d)$ also form a family of small projective generators because of the decomposition (2.2). Thus $P = \Gamma^{d,k^n}$ is a small projective generator. Now the assertion follows by observing that $\mathrm{End}_{\Gamma^d_k}(P) = \mathrm{End}_{\mathfrak{S}_d}((k^n)^{\otimes d}) = S_k(n,d)$. $\square$

2.2.8 Weight spaces

Let V be a free k-module with basis $\{v_1,\ldots,v_n\}$ and $X \in \mathsf{Rep}\,\Gamma^d_k$ any functor. The canonical decomposition (2.2) yields a decomposition of $X(V)$ into *weight spaces.*

Lemma 2.10. *Let $\mu \in \Lambda(n,d)$ and set $V = k^n$. For $X \in \mathsf{Rep}\,\Gamma^d_k$ there are natural isomorphisms*

$$\mathrm{Hom}_{\Gamma^d_k}(\Gamma^\mu, X) \xrightarrow{\sim} \mathrm{Hom}_{S_k(n,d)}(\Gamma^\mu(V), X(V)) \xrightarrow{\sim} X(V)_\mu$$

which induce a decomposition

$$X(V) = \bigoplus_{\mu\in\Lambda(n,d)} X(V)_\mu \quad \textit{where} \quad \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\mu, X) \xrightarrow{\sim} X(V)_\mu.$$

Proof. Consider, for each $\mu \in \Lambda(n,d)$, the composition of

$$\begin{aligned} \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\mu, X) &\xrightarrow{\sim} \mathrm{Hom}_{S_k(n,d)}(\Gamma^\mu(V), X(V)) \\ \Phi &\mapsto \Phi_V \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \mathrm{Hom}_{S_k(n,d)}(\Gamma^\mu(V), X(V)) &\xrightarrow{\sim} X(V)_\mu \\ \psi &\mapsto \psi(v_1^{\otimes\mu_1} \otimes \ldots \otimes v_n^{\mu_n}). \end{aligned} \tag{2.6}$$

The first map is an isomorphisms because of the equivalence given in Proposition 2.9. For the second isomorphism we identify $\mathrm{End}_{\Gamma^d_k}(V) \cong S_k(n,d)$ and note that $v_1^{\otimes\mu_1} \otimes \ldots \otimes v_n^{\mu_n}$ generates $\Gamma^\mu(V)$ as an $S_k(n,d)$-module.

Now, the canonical decomposition (2.2) induces, via the Yoneda isomorphism

$$\mathrm{Hom}_{\Gamma^d_k}(\Gamma^{d,V}, X) \xrightarrow{\sim} X(V),$$

the decomposition

$$X(V) = \bigoplus_{\mu\in\Lambda(n,d)} X(V)_\mu \quad \text{with} \quad \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\mu, X) \xrightarrow{\sim} X(V)_\mu.$$

$\square$

We observe that the duality preserves the weight space decomposition.

Lemma 2.11. *Let $\mu \in \Lambda(n,d)$ and set $V = k^n$. For $X \in \mathsf{Rep}\,\Gamma^d_k$ there is a natural isomorphism*

$$X^\circ(V)_\mu \cong X(V^*)^*_\mu.$$

defined in the previous section

Proof. We have

$$\mathrm{Hom}_{\Gamma^d_k}(\Gamma^{d,V}, X^\circ) \cong X^\circ(V) = X(V^*)^* \cong \mathrm{Hom}_{\Gamma^d_k}(\Gamma^{d,V^*}, X)^*.$$

Now use Lemma 2.10 and the canonical decomposition

$$\Gamma^{d,k^n} = \bigoplus_{\mu \in \Lambda(n,d)} \Gamma^\mu \cong \Gamma^{d,V^*}.$$

□

2.3 The algebra of divided powers

Given $V \in \mathsf{P}_k$, set $\Gamma V = \sum_{d \geq 0} \Gamma^d V$. For any $V \in \mathsf{P}_k$ and any pair of non negative integers d, e the inclusion $\mathfrak{S}_d \times \mathfrak{S}_e \subseteq \mathfrak{S}_{d+e}$ induces two natural maps.

The first one $\Gamma^{d+e} \to \Gamma^d V \otimes \Gamma^e V$ is given by

$$(V^{\otimes d+e})^{\mathfrak{S}_{d+e}} \subseteq (V^{\otimes d+e})^{\mathfrak{S}_d \times \mathfrak{S}_e} \cong (V^{\otimes d})^{\mathfrak{S}_d} \otimes (V^{\otimes e})^{\mathfrak{S}_e} \tag{2.7}$$

and the second map $\Gamma^d V \otimes \Gamma^e V \to \Gamma^{d+e}$ sends $x \otimes y \in \Gamma^d V \otimes \Gamma^e V$ to the multiplication

$$xy = \sum_{g \in \mathfrak{S}_{d+e}/\mathfrak{S}_d \times \mathfrak{S}_e} g(x \otimes y). \tag{2.8}$$

This multiplication gives ΓV the structure of a commutative k-algebra. Suppose that V is a free k-module with basis $\{v_1, \ldots, v_n\}$. Then the elements

$$v_\lambda = \prod_{i=1}^{n} v_i^{\otimes \lambda_i} \tag{2.9}$$

with $\lambda \in \Lambda(n,d)$ form a k-basis of $\Gamma^d V$.

Remark 2.12. Observe that, if we multiply two tensor powers of the same element $v \in V$, we obtain an integer multiple of their tensor product

$$v^{\otimes d} v^{\otimes e} = \sum_{g \in \mathfrak{S}_{d+e}/\mathfrak{S}_d \times \mathfrak{S}_e} g(v \otimes v) = c\; v^{\otimes d+e},$$

where $c = |\mathfrak{S}_{d+e}/\mathfrak{S}_d \times \mathfrak{S}_e|$.

2.4 Young tableaux and bases

Let $V \in \mathsf{P}_k$ be a free module and fix a basis $\{v_1, \ldots, v_n\}$. With the help of *Young tableaux*, defined in the first chapter, we want to describe k-bases of the free modules $\Gamma^\lambda V$ and $\Lambda^\lambda V$ for any $\lambda \in \Lambda(r, d)$ and any non negative integer r. Notice that, since (the isomorphism classes of) $\Gamma^\lambda V$ and $\Lambda^\lambda V$ do not depend on the order of the parts of λ, and since every composition $\nu \in \Lambda(r, d)$, after reordering of parts, gives rise to a unique partition λ, it is enough to consider the case $\lambda \in \Lambda^+(r, d)$. Consider the set $\mathrm{Tab}_\lambda(n)$ of tableaux of shape λ and entries $\{1, \ldots, n\}$.

Given $T \in \mathrm{Tab}_\lambda(n)$, let $T(i, j)$ denote the entry of the box (i, j) of the diagram of λ and define a composition $\alpha^i \in \Lambda(n, \lambda_i)$ by setting

$$\alpha^i_j = \mathsf{card}\{1 \le t \le \lambda_i \mid T(i, t) = j\}, \quad 1 \le i \le r,\ 1 \le j \le n.$$

$$\text{Set} \quad v_T = v_{\alpha^1} \otimes \ldots \otimes v_{\alpha^r},$$

where the factors v_{α^i} are defined as in (2.9). Observe that α^i only depends on the entries of the ith row of T and not on their order, thus we only consider row-costandard tableaux. Moreover, as T runs through all tableaux in $\mathrm{Tab}^{rc}_\lambda(n)$, the corresponding compositions α^i, for $i = 1, \ldots, r$, run through all compositions of λ_i in n parts, hence the elements v_{α^i} form a basis of $\Gamma^{\lambda_i} V$ (as in (2.9)). It follows now easily

Lemma 2.13. *Let $V \in \mathsf{P}_k$ be a free module and fix a basis $\{v_1, \ldots, v_n\}$. The set $\{v_T \mid T \in \mathrm{Tab}^{rc}_\lambda(n)\}$ defines a k-basis of $\Gamma^\lambda V$.* □

For a strictly increasing subset $I = i_1 < i_2 < \ldots < i_l$ of $\{1, \ldots, n\}$, denote by v_I the element $v_{i_1} \wedge \ldots \wedge v_{i_l} \in \Lambda^l V$. Let $T \in \mathrm{Tab}^{cc}_\lambda(n)$ and denote by

$$I^j = T(1, j) < T(2, j) < \ldots < T(\tilde{\lambda}_j, j),\ 1 \le j \le \lambda_1.$$

$$\text{Set} \quad \hat{v}_T = v_{I^1} \otimes \ldots \otimes v_{I^{\lambda_1}} \in \Lambda^{\tilde{\lambda}} V.$$

We have

Lemma 2.14. *Let $V \in \mathsf{P}_k$ be a free module and fix a basis $\{v_1, \ldots, v_n\}$. Then $\{\hat{v}_T \mid T \in \mathrm{Tab}^{cc}_\lambda(n)\}$ defines a k-basis of $\Lambda^{\tilde{\lambda}} V$.* □

Example 2.4. Let $\lambda = (5, 3, 3, 2) \in \Lambda^+(4, 13)$ and V a 6-dimensional free k-module with basis $v_1, \ldots, v_6$. Let $T \in \mathrm{Tab}_\lambda(6)$ be the following tableau

1	2	2	3	3
2	3	5		
4	4	6		
5	6			

Then we have

$$v_T = (v_1(v_2 \otimes v_2)(v_3 \otimes v_3)) \otimes (v_2 v_3 v_5) \otimes ((v_4 \otimes v_4)v_6) \otimes (v_5 v_6)$$

$$\hat{v}_T = (v_1 \wedge v_2 \wedge v_4 \wedge v_5) \otimes (v_2 \wedge v_3 \wedge v_4 \wedge v_6) \otimes (v_2 \wedge v_5 \wedge v_6) \otimes v_3 \otimes v_3.$$

2.5 Standard morphisms

In this section, we want to compute the weight spaces for Γ^λ and S^λ. As showed by Totaro in [30], it is possible to describe bases for them combinatorially, with the help of the matrices in A^λ_μ, defined in (1.1).

Fix a non negative integer d and let $\lambda, \mu \in \Lambda(d)$ be two compositions. For a matrix $A = (a_{ij}) \in A^\lambda_\mu$, define the *standard morphisms* $\gamma_A : \Gamma^\mu \to \Gamma^\lambda$ and $\sigma_A : \Gamma^\mu \to S^\lambda$ as follows.

$$\gamma_A : \Gamma^\mu = \bigotimes_j \Gamma^{\mu_j} \to \bigotimes_j (\bigotimes_i \Gamma^{a_{ij}}) = \bigotimes_i (\bigotimes_j \Gamma^{a_{ij}}) \to \bigotimes_i \Gamma^{\lambda_i} = \Gamma^\lambda \quad (2.10)$$

where the first morphism is the tensor product of the natural inclusions $\Gamma^{\mu_j} \to \bigotimes_i \Gamma^{a_{ij}}$ given in (2.7) and the second morphism is the tensor product of the natural product maps $\bigotimes_j \Gamma^{a_{ij}} \to \Gamma^{\lambda_i}$ given in (2.8).

Analogously,

$$\sigma_A : \Gamma^\mu = \bigotimes_j \Gamma^{\mu_j} \to \bigotimes_j (\bigotimes_i T^{a_{ij}}) = \bigotimes_i (\bigotimes_j T^{a_{ij}}) \to \bigotimes_i S^{\lambda_i} = S^\lambda \quad (2.11)$$

where T^r denotes the functor $(-)^{\otimes r}$ for any non negative integer r, the first morphism is the tensor product of the natural inclusions $\Gamma^{\mu_j} \to \bigotimes_i T^{a_{ij}}$, and the second morphism is the tensor product of the natural projection maps $\bigotimes_j T^{a_{ij}} \to S^{\lambda_i}$.

The following lemma is given in [[30], p.8] (cf. also [22, Lemma 5.3]).

Lemma 2.15. *Let $\lambda,\ \mu \in \Lambda(d)$, be compositions of d.*

1. *The morphisms γ_A with $A \in A^\lambda_\mu$ form a k-basis of* $\mathrm{Hom}_{\Gamma^d_k}(\Gamma^\mu, \Gamma^\lambda)$.

2. *The morphisms σ_A with $A \in A^\lambda_\mu$ form a k-basis of* $\mathrm{Hom}_{\Gamma^d_k}(\Gamma^\mu, S^\lambda)$.

Proof. Let $\lambda \in \Lambda(r, d),\ \ \mu \in \Lambda(s, d)$. Let n be a non negative integer such that $n \geq d$ and $n \geq r, s$. By extending λ and μ by zeros, if necessary, we can assume $\lambda, \mu \in \Lambda(n, d)$. Moreover, assume λ is a partition (as we already observed, $\Gamma^\nu \cong \Gamma^\lambda$ for any composition ν which, up to the order, coincides with λ). Fix a free k-module V with basis $\{v_1, \dots, v_n\}$. From Proposition 2.9 and Lemma 2.10, we have isomorphisms

$$\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\mu, \Gamma^\lambda) \xrightarrow{\sim} \mathrm{Hom}_{S_k(n,d)}(\Gamma^\mu V, \Gamma^\lambda V) \xrightarrow{\sim} (\Gamma^\lambda V)_\mu.$$

A standard morphism γ_A, evaluated at V, sends

$$v_1^{\otimes \mu_1} \otimes \dots \otimes v_n^{\otimes \mu_n} \mapsto v_A = v_{\alpha^1} \otimes \dots \otimes v_{\alpha^n}$$

where $\alpha^i \in \Lambda(n, \lambda_i)$ is the composition defined by $\alpha^i_j = a_{ij}$. Note that, if $T \in \mathrm{Tab}^{rc}_\lambda(n)$ is the tableau having a_{ij} entries equal to j in the ith row, for any $i = 1, \dots, n$, then $v_A = v_T$.

Moreover, for any tableau $T \in \mathrm{Tab}^{rc}_\lambda(n)$, the element v_T belongs to $(\Gamma^\lambda V)_\mu$, where μ equals the content of T. The standard morphism $\gamma_A : \Gamma^\mu \to \Gamma^\lambda$ given by $A = (a_{ij})$, where $a_{ij} = \mathsf{card}\{t | T(i,t) = j\}$, and evaluated at V sends $v_1^{\otimes \mu_1} \otimes \dots \otimes v_n^{\otimes \mu_n}$ to v_T. If $\mu = \lambda$, then T is the unique (row) costandard tableau such that all boxes of the ith row have entry i

Now the first assertion follows from the fact that the elements v_A form a basis of $\Gamma^\lambda V$ as μ runs through $\Lambda(n,d)$. The proof of the second assertion is analogous. □

Example 2.5. Let $\lambda = (5,3,3,2)$ and $\mu = (1,3,3,2,2,2)$. For

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

the morphism γ_A at $V = k^6$ takes $v_1^{\otimes \mu_1} \otimes \dots \otimes v_6^{\mu_6}$ to

$$(v_1(v_2 \otimes v_2)(v_3 \otimes v_3)) \otimes (v_2 v_3 v_5) \otimes ((v_4 \otimes v_4) v_6) \otimes (v_5 v_6).$$

Example 2.6. The special case $\lambda = (1, \dots, 1) = \mu$ yields the isomorphism

$$\mathrm{End}_{\Gamma_k^d}(\Gamma^{(1,\dots,1)}) \cong k\mathfrak{S}_d.$$

2.6 Schur and Weyl modules and functors

Fix a partition $\lambda \in \Lambda(r,d)$, denote by $\tilde{\lambda} \in \Lambda(s,d)$ the conjugate partition. Each integer $t \in \{1, \dots, d\}$ can be written uniquely as a sum

$$t = \lambda_1 + \dots + \lambda_{i-1} + j, \quad \text{with } \ 1 \le j \le \lambda_i.$$

The pair (i,j) describes the position (ith row, jth column) of the tth box of the *Young diagram* of λ. The partition λ determines a permutation $\sigma_\lambda \in \mathfrak{S}_d$ by $\sigma_\lambda(t) = \tilde{\lambda}_1 + \dots = \tilde{\lambda}_{j-1} + i$, where $1 \le i \le \lambda_j$. Note that $\sigma_\lambda^{-1} = \sigma_{\tilde{\lambda}}$.

Example 2.7.

$$\lambda = (2,3) \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \qquad \tilde{\lambda} = (2,2,1) \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \sigma_\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

Fix a k-module $V \in \mathsf{P}_k$ and consider the following composition, which we denote by $\delta_\lambda V$

$$\Lambda^{\tilde{\lambda}_1} V \otimes \ldots \otimes \Lambda^{\tilde{\lambda}_s} V \xrightarrow{\Delta \otimes \ldots \otimes \Delta} V^{\otimes d} \xrightarrow{\quad s_\lambda \quad} V^{\otimes d} \xrightarrow{\nabla \otimes \ldots \otimes \nabla} S^{\lambda_1} V \otimes \ldots \otimes S^{\lambda_r} V \tag{2.12}$$

Here, for an integer t, we denote by $\Delta : \Lambda^t V \to V^{\otimes t}$ the comultiplication given by

$$\Delta(v_1 \wedge \ldots \wedge v_t) = \sum_{\sigma \in \mathfrak{S}_t} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots v_{\sigma(t)}.$$

We denote by $\nabla : V^{\otimes d} \to S^d V$ the multiplication and by $s_\lambda : V^{\otimes d} \to V^{\otimes d}$ the map given by

$$s_\lambda(v_1 \otimes \ldots \otimes v_d) = v_{\sigma_\lambda(1)} \otimes \ldots \otimes v_{\sigma_\lambda(d)}.$$

Definition 2.16. The *Schur module* $\mathbb{S}_\lambda V$ is the image of $\delta_\lambda V$.

Dually, consider the composition

$$\Gamma^{\lambda_1} V \otimes \ldots \otimes \Gamma^{\lambda_r} V \xrightarrow{\Delta \otimes \ldots \otimes \Delta} V^{\otimes d} \xrightarrow{\quad s_{\tilde{\lambda}} \quad} V^{\otimes d} \xrightarrow{\nabla \otimes \ldots \otimes \nabla} \Lambda^{\tilde{\lambda}_1} V \otimes \ldots \otimes \Lambda^{\tilde{\lambda}_s} V \tag{2.13}$$

where Δ denotes the natural inclusion $\Delta : \Gamma^{\lambda_i} V \to V^{\otimes \lambda_i}$ and ∇ is the natural projection $\nabla : V^{\otimes \tilde{\lambda}_j} \to \Lambda^{\tilde{\lambda}_j}$. Denote it by $\delta'_\lambda V$.

Definition 2.17. The *Weyl module* $\mathbb{W}_\lambda V$ is the image of $\delta'_\lambda V$.

It is well known (cf. [16], Lecture 6) that, if k is a field of characteristic 0, then for any partition λ and any k-module V

$$\mathbb{S}_\lambda V \cong \mathbb{W}_\lambda V$$

is a simple $\mathrm{GL}(V)$-module and the Schur modules $\mathbb{S}_\lambda V$, as λ runs over all partitions with no more that $n = \dim V$ parts, give a complete set of irreducible $\mathrm{GL}(V)$ representations.

The definitions above give rise to functors $\mathbb{S}_\lambda$ and $\mathbb{W}_\lambda$ in $\mathsf{Rep}\,\Gamma^d_k$ for each partition λ and each weight d. Note that $\mathbb{S}^\circ_\lambda \cong \mathbb{W}_\lambda$ and $\mathbb{W}^\circ_\lambda \cong \mathbb{S}_\lambda$.

Example 2.8. Symmetric and exterior powers are particular cases of Schur functors. In fact, $\mathbb{S}_{(1,\ldots,1)} = \Lambda^d$ and $\mathbb{S}_{(d)} = S^d$.

Observe that, if V is a free k-module, for k any commutative ring, then

$$V = k \otimes_{\mathbb{Z}} V_0$$

for a suitable free $\mathbb{Z}$-module V_0 of the same rank of V. We will need some functors that commute with this *base change*.

2.7 Universal functors and universal transformations

The following Definition is given in [19, Definition I.3.10].

Definition 2.18. Let $T_k : \mathsf{P}_k \times \ldots \times \mathsf{P}_k \to \mathbf{M}_k$ be a k-linear functor, which is defined for all commutative rings k and for all n-tuples of free k-modules $V_1, \ldots, V_n$. The functor T_k is called *universal* if, for any $\phi : k \to \ell$ ring homomorphism, we have a natural equivalence

$$T_\ell(\ell \otimes -, \ldots, \ell \otimes -) \cong \ell \otimes T_k(V_1, \ldots, V_n).$$

Moreover, T_k is called *universally free* if it is universal and $T_k(V_1, \ldots, V_n)$ is a free k-module for any n-tuple $V_1, \ldots, V_n$.

Let $\Psi_k : T_k \to T'_k$ be a natural transformation of universal functors, defined for all commutative rings k. We say that Ψ_k is *universal* if for any ring homomorphism $\phi : \ell \to k$ and any n-tuple of free k-modules $V_1, \ldots, V_n$, the diagram

$$\begin{array}{ccc} \ell \otimes T_k(V_1, \ldots, V_n) & \xrightarrow{\ell \otimes \Psi_k(V_1, \ldots, V_n)} & \ell \otimes T'_k(V_1, \ldots, V_n) \\ \downarrow \cong & & \downarrow \cong \\ T_\ell(\ell \otimes V_1, \ldots, \ell \otimes V_n) & \xrightarrow{\Psi_\ell(\ell \otimes V_1, \ldots, \ell \otimes V_n)} & T'_\ell(\ell \otimes V_1, \ldots, \ell \otimes V_n) \end{array} \qquad (2.14)$$

is commutative.

It follows immediately

Lemma 2.19. *Let $\Psi_k : T_k \to T'_k$ be a universal natural transformation. Assume that, for some ring k, Ψ_k is a natural isomorphism, that is, for any n-tuple $V_1, \ldots, V_n$ of free k-modules*

$$\Psi_k(V_1, \ldots, V_n) : T_k(V_1, \ldots, V_n) \to T'_k(V_1, \ldots, V_n)$$

is an isomorphism. Then, for any ring homomorphism $\phi : k \to \ell$, Ψ_ℓ is a natural isomorphism.

Remark 2.20. If Ψ_k is a universal natural transformation as above, then the functor $\operatorname{Im}\Psi_k$ is universal. Indeed, it follows from the commutativity of the diagram (2.14) that, for any ring homomorphism $\phi : \ell \to k$, we have

$$\ell \otimes \operatorname{Im}\Psi_k(V_1, \dots, V_n) \cong \operatorname{Im}\Psi_\ell(\ell \otimes V_1, \dots, \ell \otimes V_n),$$

for any n-tuple $V_1, \dots, V_n$.

Moreover, since $\ell \otimes -$ is a right exact functor, it preserves cokernels. Hence, $\operatorname{Coker}\Psi_k$ is a universal functor, for any universal natural transformation Ψ_k.

The functors Λ^d, S^d and Γ^d have been definded for an arbitrary commutative ring and it follows immediately from the definitions that are universally free. Moreover, tensor products and direct sums of universally free functors are universally free. It follows that, for any composition λ, Λ^λ, S^λ and Γ^λ are universally free, such as the graded versions $\Lambda = \bigoplus_{d\geq 0} \Lambda^d$, $S = \bigoplus_{d\geq 0} S^d$ and $\Gamma = \bigoplus_{d\geq 0} \Gamma^d$.

Although it is less obvious, Schur and Weyl functors are also universally free and it is possible to give bases for them that are indexed by *(co)standard tableaux* of shape λ. This is stated in the following result, proved in [1, Theorems II.2.16 and II.3.16].

Theorem 2.21 (Standard basis theorem for Schur and Weyl functors)**.** *Let λ be a partition and V a free k-module with basis $\{v_1, \dots, v_n\}$. Then*

1. $\{\delta'_\lambda V(v_T) |\ T \in \operatorname{Tab}_\lambda(n)$ *is costandard* $\}$ *is a free basis for* $\mathbb{W}_\lambda V$.
2. $\{\delta_\lambda V(\hat{v}_T) |\ T \in \operatorname{Tab}_\lambda(n)$ *is standard* $\}$ *is a free basis for* $\mathbb{S}_\lambda V$.

Moreover, the functors $\mathbb{S}_\lambda$ and $\mathbb{W}_\lambda$ are universally free. □

Sketch of the proof. We briefly describe the idea of the proof for the first statement, the proof of the second one is analogous and can be found in [1, Theorem II.2.16], in the more general context of skew-partitions.

Let $\lambda \in \Lambda^+(r, d)$. For any integers i, t such that $1 \leq i < r$ and $1 \leq t \leq \lambda_{i+1}$, let

$$\lambda(i, t) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + t, \lambda_{i+1} - t, \lambda_{i+2}, \dots, \lambda_r)$$

and consider the *standard morphism*

$$\gamma_{A(i,t)} : \Gamma^{\lambda(i,t)} \to \Gamma^\lambda$$

given by the matrix $A(i, t) = \operatorname{diag}(\lambda_1, \dots, \lambda_r) + tE_{i+1,i} - tE_{i+1,i+1}$.

By direct computation, one sees that $\delta'_\lambda \circ \gamma_{A(i,t)} = 0$, for any i, t as above. It follows that, if α is the sum of all standard morphisms of this form

$$\bigoplus_{i\geq 1} \bigoplus_{t=1}^{\lambda_{i+1}} \Gamma^{\lambda(i,t)} \xrightarrow{\alpha} \Gamma^\lambda \tag{2.15}$$

one has $\operatorname{Im}\alpha \subseteq \ker(\delta'_\lambda)$. Moreover, it follows easily from the Definition (2.10) of standard morphisms that they are universal. Hence α is a universal natural transformation.

We already observed that row-costandard tableaux of shape λ and filling in $\{1, \dots, n\}$, where $n = \dim V$, index a k-basis of $\Gamma^\lambda V$. By defining a pseudo order (i.e. a relation which is reflexive and transitive, but not antysymmetric) on those tableaux, it is possible to show that, if T is a row-costandard tableau, which is not costandard, then there exist costandard tableaux T_i such that

$$v_T - \sum \pm v_{T_i} \in \operatorname{Im}\alpha.$$

This is done by constructing a suitable tableau $\tilde{T}$ of shape $\mu > \lambda$, such that $\alpha(v_{\tilde{T}}) = \sum v_{T_i}$, where one summand equals v_T and for all other summands we have $T_i < T$. Since, with respect to this (pseudo) order, costandard tableaux are smaller than other tableaux with the same entries, by repeating this process, we obtain $v_T - \sum \pm v_{T_i} \in \operatorname{Im}\alpha$, with all T_i costandard. It follows that the images of costandard tableaux generate $\mathbb{W}_\lambda V$.

By observing that the pseudo ordering given above is a total ordering on the set of costandard tableaux, one shows that they are linearly independent, thus $\{\delta'_\lambda V(v_T) |\ T \in \mathrm{Tab}_\lambda(n) \text{ is costandard}\}$ is a free basis for $\mathbb{W}_\lambda V$. But the same elements also form a basis for $\operatorname{Coker}\alpha_V$ thus, $\mathbb{W}_\lambda V \cong \operatorname{Coker}\alpha_V$ and we have a presentation

$$\bigoplus_{i\geq 1} \bigoplus_{t=1}^{\lambda_{i+1}} \Gamma^{\lambda(i,t)} \xrightarrow{\alpha} \Gamma^\lambda \longrightarrow \mathbb{W}_\lambda \longrightarrow 0 \tag{2.16}$$

Now, since it is the cokernel of a universal natural transformation, the Weyl functor $\mathbb{W}_\lambda$ is a universally free functor. $\square$

The next proposition describes the weight spaces for Schur and Weyl functors.

Proposition 2.22. *Let λ and μ be partitions of weight d.*

1. $\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\mu, \mathbb{W}_\lambda) \neq 0$ *if and only if* $\mu \trianglelefteq \lambda$. *Moreover,* $\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\lambda, \mathbb{W}_\lambda) \cong k$.

2. $\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\mu, \mathbb{S}_\lambda) \neq 0$ *if and only if* $\mu \trianglelefteq \lambda$. *Moreover,* $\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\lambda, \mathbb{S}_\lambda) \cong k$.

Proof. Let $V \in \mathsf{P}_k$ be a free module of dimension n. Apply Lemma 2.10. As observed in the proof of Lemma 2.15, a basis element v_T of $\Gamma^\lambda V$ is contained in the weight space $(\Gamma^\lambda V)_\mu$, where μ is the content of T. The assertion for $\mathbb{W}_\lambda V$ follows now from Lemma 1.1 by using the basis for the Weyl module given in Theorem 2.21.

Since $\mathbb{S}_\lambda \cong \mathbb{W}_\lambda^\circ$, the assertion for Schur functors follows from the first one, by using Lemma 2.11 □

3 Representations of symmetric groups

In this chapter we want to investigate the monoidal structure on the category of representations of the symmetric group $\mathfrak{S}_d$ and to explain its connection with the monoidal structure on $\mathsf{Rep}\,\Gamma_k^d$ given by the tensor product from Definition 2.5.

3.1 Permutation modules

Let E be a free k-module and fix a basis $\{e_1, \ldots, e_n\}$. Consider the right action of the symmetric group $\mathfrak{S}_d$ on $E^{\otimes d}$ by place permutation

$$(v_1 \otimes \cdots \otimes v_d)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \quad \text{for } \sigma \in \mathfrak{S}_d,\ v_1 \otimes \cdots \otimes v_d \in E^{\otimes d}.$$

By linear extension of the action, $E^{\otimes d}$ becomes a $k\mathfrak{S}_d$-module.

Consider the basis of the tensor power $E^{\otimes d}$ given by $\{e_{\underline{i}} \mid \underline{i} \in I(n,d)\}$, as in Section 2.2.7. If two basis elements are in the same orbit with respect to the action of the symmetric group, that is, if $e_{\underline{j}} = e_{\underline{i}}\sigma$, for some $\sigma \in \mathfrak{S}_d$, then one has $\underline{i}, \underline{j} \in \lambda$, for some composition λ, where $\Lambda(n,d)$ is considered as the set of all $\mathfrak{S}_d$-orbits on $I(n,d)$, as in Section 1.1.1. Thus, we can give the following

Definition 3.1. Let $\lambda \in \Lambda(n,d)$ be a composition. The *transitive permutation module* M^λ corresponding to λ is the k-span of the set $\{e_{\underline{i}} \mid \underline{i} \text{ belongs to } \lambda\}$.

We have the following decomposition of $E^{\otimes d}$ as a $k\mathfrak{S}_d$-module.

$$E^{\otimes d} = \bigoplus_{\lambda \in \Lambda(n,d)} M^\lambda \tag{3.1}$$

Note that, if we denote by $\underline{i}_\lambda = (1 \ldots 1\, 2 \ldots 2 \ldots n \ldots n)$ the d-tuple having λ_l entries equal to l, we have

$$\{e_{\underline{i}} \mid \underline{i} \text{ belongs to } \lambda\} = \{e_{\underline{i}_\lambda}\sigma \mid \sigma \in \mathfrak{S}_d/\mathfrak{S}_\lambda\} \tag{3.2}$$

where $\mathfrak{S}_\lambda$ denotes the Young subgroup $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n} \subset \mathfrak{S}_d$. So we have a one to one correspondence between the elements of a basis of M^λ and the elements of the group $\mathfrak{S}_d/\mathfrak{S}_\lambda$.

Example 3.1. If $\lambda = (d)$, the corresponding permutation module $M^{(d)}$ is the trivial $k\mathfrak{S}_d$ representation of dimension d. If $\omega = (1, \ldots, 1) \in \Lambda(d,d)$, then $M^\omega \cong k\mathfrak{S}_d$.

We give now an equivalent description of permutation modules and of the action of the symmetric group on them, that will be useful in what follows.

To a d-tuple $\underline{i} \in I(n,d)$ we can associate a dissection $d_{\underline{i}}$ of the set $\{1,\ldots,d\}$ as follows

$$d_{\underline{i}} := \{d_{\underline{i}}^1, \ldots, d_{\underline{i}}^n\} \text{ with } d_{\underline{i}}^l := \{j \mid i_j = l\}.$$

That is, $d_{\underline{i}}^l$ is the list of the indices of all entries of $\underline{i}$ which are equal to l. Note that $d_{\underline{i}}^l \cap d_{\underline{i}}^h = \emptyset$ for $l \neq h$. If $\underline{i} \in \lambda$, then we have $\mathsf{card}(d_{\underline{i}}^l) = \lambda_l$, for any $l = 1, \ldots, n$.

Observe that, $e_{\underline{i}}\sigma = e_{i_{\sigma(1)}} \otimes \ldots \otimes e_{i_{\sigma(1)}} = e_{\underline{j}}$ if and only if $\underline{i} = \sigma^{-1}\underline{j}$, for $\sigma \in \mathfrak{S}_d$. Thus, the action of the symmetric group on $d_{\underline{i}}^l$ is given by

$$\sigma^{-1}j \quad \text{for every} \quad j \in d_{\underline{i}}^l, \sigma \in \mathfrak{S}_d.$$

A permutation module M^λ can be identified with k-span d_λ of all dissections $d_{\underline{i}}$ with $\underline{i} \in \lambda$.

Remark 3.2. In the literature the permutation modules are often described as the k-span of tabloids. Recall that a tabloid $\{T\}$ is an equivalence class of tableaux of shape λ with filling $\{1,\ldots,d\}$, with no repeats allowed, two being equivalent if corresponding rows contain the same entries. In other words, only the content of each row matters and not the order of its entries. The permutation module M^λ can be given as the k-free module with basis all tabloids of shape λ. It is easy to see that this is equivalent to the description given above. Indeed, we can see a dissection $d_{\underline{i}}$, $\underline{i} \in \lambda$, as the tabloid of shape λ having rows $d_{\underline{i}}^1, \ldots, d_{\underline{i}}^n$. Conversely, the rows of a tabloid always give a dissection of this form.

3.1.1 Left modules

If we denote by $\mathsf{mod}\, k\mathfrak{S}_d$ the category of right $k\mathfrak{S}_d$-modules that are finitely generated projective over k, we have an equivalence of categories

$$\begin{aligned} \mathrm{Hom}_{k\mathfrak{S}_d}(-, k\mathfrak{S}_d) \colon \mathsf{mod}\, k\mathfrak{S}_d^{\mathrm{op}} &\to k\mathfrak{S}_d\, \mathsf{mod} \\ M &\mapsto \mathrm{Hom}_{k\mathfrak{S}_d}(M, k\mathfrak{S}_d) \end{aligned}$$

where the left action of the symmetric group on $\mathrm{Hom}_{k\mathfrak{S}_d}(M, k\mathfrak{S}_d)$ is given by $(\pi f) \colon m \mapsto \pi \cdot f(m)$ for $\pi \in \mathfrak{S}_d$, $f \in \mathrm{Hom}_{k\mathfrak{S}_d}(M, k\mathfrak{S}_d)$ and $m \in M$. Note that we can identify $\mathrm{Hom}_{k\mathfrak{S}_d}(E^{\otimes d}, k\mathfrak{S}_d)$ with the left module $E^{\otimes d}$ where the action for $\pi \in \mathfrak{S}_d$ and $v_1 \otimes \cdots \otimes v_d \in E^{\otimes d}$ is given by

$$\pi(v_1 \otimes \cdots \otimes v_d) := v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(d)}.$$

Namely, if we denote by ${}^{\lambda}M \subseteq E^{\otimes d}$ the left permutation module corresponding to λ, i.e. the module with basis $\{\pi e_{\underline{i}_\lambda} : \pi \in \mathfrak{S}_d/\mathfrak{S}_\lambda\}$, we get the following isomorphism of left modules:

$$\begin{aligned} {}^{\lambda}M &\to \operatorname{Hom}_{k\mathfrak{S}_d}(M^\lambda, k\mathfrak{S}_d) \\ \pi e_{\underline{i}_\lambda} &\mapsto (f_\pi : e_{\underline{i}_\lambda} \mapsto \pi) \end{aligned}$$

We will identify ${}^{\lambda}M$ and $\operatorname{Hom}_{k\mathfrak{S}_d}(M^\lambda, k\mathfrak{S}_d)$ via this isomorphism.

3.2 The tensor product of representations of symmetric groups

The $k\mathfrak{S}_d$-module structure on the tensor product of two representations of $\mathfrak{S}_d$ is given via the Hopf algebra structure of the group algebra $k\mathfrak{S}_d$, i.e. for $M,\ N \in \mathsf{Mod}\, k\mathfrak{S}_d$, let $M \otimes_k N$ be the usual tensor product over k together with the following diagonal action of $\mathfrak{S}_d$

$$(M \otimes_k N) \times k\mathfrak{S}_d \to (M \otimes_k N) \tag{3.3}$$

$$((m \otimes n), \pi) \mapsto (m\pi \otimes n\pi) \tag{3.4}$$

The tensor unit is given by $M^{(d)} \cong k$, the trivial $k\mathfrak{S}_d$-module.

3.2.1 Tensoring permutation modules

For a field k of characteristic 0, James and Kerber showed in [20] how to decompose the tensor product of two permutation modules in terms of their characters. In the following, we want to generalize their description for k an arbitrary commutative ring.

The tensor product of two permutation modules $M^\lambda \otimes_k M^\mu$, $\lambda, \mu \in \Lambda(d)$, can be described with the help of the set of matrices A^λ_μ defined in (1.1).

Lemma 3.3. *The tensor product of two permutation modules M^λ and M^μ, for $\lambda \in \Lambda(m,d)$ and $\mu \in \Lambda(n,d)$, can be decomposed into permutation modules as follows*

$$M^\lambda \otimes_k M^\mu \cong \bigoplus_{A \in A^\lambda_\mu} M^A,$$

where A is regarded as the composition $(a_{11}, a_{12}, \ldots, a_{21}, a_{22}, \ldots, a_{mn})$.

Proof. The idea of the proof is taken from [20]. For any composition ν of d, consider the basis of the permutation module $M^\nu = d_\nu$ given by the set of all dissections $d_{\underline{i}}$ with $\underline{i} \in \nu$. Then a basis of $M^\lambda \otimes_k M^\mu = d_\lambda \otimes d_\mu$ is given by all products $d_{\underline{i}} \otimes d_{\underline{j}}$ with $\underline{i} \in \lambda$ and $\underline{j} \in \mu$.

Consider now the orbits of $d_\lambda \otimes d_\mu$ under the action of $\mathfrak{S}_d$. This is equivalent to considering the $\mathfrak{S}_d$-orbits on $I_m \times I_n$ with respect to the action $(\underline{i}, \underline{j})\pi = (\underline{i}\pi, \underline{j}\pi)$, for any $(\underline{i}, \underline{j}) \in I_m \times I_n$, $\pi \in \mathfrak{S}_d$, where I_m denotes the set $I(m,d)$ and $I_n = I(n,d)$. Denote by $O(\underline{i}, \underline{j})$ the orbit of the pair $(\underline{i}, \underline{j})$. Set

$$A^{\underline{i}}_{\underline{j}} := (|d^s_{\underline{i}} \cap d^t_{\underline{j}}|)_{st} \in \mathbf{M}_{m\times n}(\mathbb{N}) \quad \text{for all} \quad \underline{i} \in \lambda, \underline{j} \in \mu.$$

Note that $A^{\underline{i}}_{\underline{j}} \in A^\lambda_\mu$. The set A^λ_μ is in one to one correspondence to the orbits of the pairs $(\underline{i}, \underline{j})$ with $\underline{i} \in \lambda$, $\underline{j} \in \mu$. Namely, we have the following bijections

$$\{O(\underline{i}, \underline{j}) \mid \underline{i} \in \lambda,\ \underline{j} \in \mu\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} A^\lambda_\mu \tag{3.5}$$

which can be given as follows. Set $\phi(O(\underline{i}, \underline{j})) = A^{\underline{i}}_{\underline{j}}$. Observe that this is well defined because $A^{\underline{i}}_{\underline{j}} = A^{\underline{i}\pi}_{\underline{j}\pi}$ for all $\pi \in \mathfrak{S}_d$. Assume $n \geq m$. For a matrix $A \in A^\lambda_\mu$, set $\psi(A) = O(\underline{i}_A, \underline{j}_A)$, where $\underline{i}_A = \underline{i}_\lambda = (1, ...1, 2, ..., 2, ..., m, ..., m)$ and

$$\underline{j}_A = (\underbrace{1,\dots,1}_{a_{11}}, \dots, \underbrace{m,\dots,m}_{a_{1m}}, \dots, \underbrace{n,\dots,n}_{a_{1n}}, \underbrace{1,\dots,1}_{a_{21}}, \dots, \underbrace{n,\dots,n}_{a_{2n}}, \dots, \underbrace{1,\dots,1}_{a_{n1}}, \dots, \underbrace{n,\dots,n}_{a_{mn}}).$$

It is not difficult to see that $\phi(O(\underline{i}_A, \underline{j}_A)) = A$.

It follows that we can decompose the $k\mathfrak{S}_d$-module $d_\lambda \otimes d_\mu$ into a direct sum of submodules

$$d_\lambda \otimes d_\mu = \bigoplus_{A \in A^\lambda_\mu} (d_\lambda \otimes d_\mu)_A,$$

where $(d_\lambda \otimes d_\mu)_A \subseteq d_\lambda \otimes d_\mu$ is spanned by all $d_{\underline{i}} \otimes d_{\underline{j}}$ such that $A = A^{\underline{i}}_{\underline{j}}$. Now, we have an isomorphism $(d_\lambda \otimes d_\mu)_A \cong d_A$ as $k\mathfrak{S}_d$-modules, given by

$$\{d^1_{\underline{i}}, \dots, d^m_{\underline{i}}\} \otimes \{d^1_{\underline{j}}, \dots, d^n_{\underline{j}}\} \mapsto \{d^1_{\underline{i}} \cap d^1_{\underline{j}}, d^1_{\underline{i}} \cap d^2_{\underline{j}}, \dots, d^2_{\underline{i}} \cap d^1_{\underline{j}}, \dots, d^m_{\underline{i}} \cap d^n_{\underline{j}}\},$$

that completes the proof.

□

Remark 3.4. In the same way, it is possible to tensor left permutation modules. Namely for ${}^\lambda M = \mathrm{Hom}_{k\mathfrak{S}_d}(M^\lambda, k\mathfrak{S}_d)$ and ${}^\mu M = \mathrm{Hom}_{k\mathfrak{S}_d}(M^\mu, k\mathfrak{S}_d)$, we get

$${}^\lambda M \otimes_k {}^\mu M \cong \bigoplus_{A \in A^\lambda_\mu} {}^A M.$$

For left modules, the tensor unit is given by $I_{k\mathfrak{S}_d} := {}^{(d)}M$.

Example 3.2. Let $\lambda = (3,1) \in \Lambda(2,4)$ and $\mu = (2,1,1) \in \Lambda(3,4)$. Then A^λ_μ consist of the following matrices:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let now $\underline{i} = (1112) \in \lambda$ and $\underline{j} = (1312) \in \mu$. Then $d_{\underline{i}} = \{\{123\},\{4\}\}$ and $d_{\underline{j}} = \{\{13\},\{4\},\{2\}\}$. The orbit of $d_{\underline{i}} \otimes d_{\underline{j}}$ consists of the elements

$$\begin{aligned}\{(d_{\underline{i}}\pi \otimes d_{\underline{j}}\pi) \mid \pi \in \mathfrak{S}_4\} = &\{\{123\},\{4\}\} \otimes \{\{13\},\{4\},\{2\}\}, \\ &\{\{213\},\{4\}\} \otimes \{\{23\},\{4\},\{1\}\}, \\ &\{\{132\},\{4\}\} \otimes \{\{12\},\{4\},\{3\}\}, \\ &\{\{231\},\{4\}\} \otimes \{\{12\},\{4\},\{3\}\}, \\ &\{\{312\},\{4\}\} \otimes \{\{23\},\{4\},\{1\}\}, \dots\end{aligned}$$

and $A^{\underline{i}}_{\underline{j}} = \begin{pmatrix} |\{123\} \cap \{23\}| & |\{123\} \cap \{4\}| & |\{123\} \cap \{1\}| \\ |\{4\} \cap \{23\}| & |\{4\} \cap \{4\}| & |\{4\} \cap \{1\}| \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.
Recall that $A^{\underline{i}}_{\underline{j}} = A^{\underline{i}\pi}_{\underline{j}\pi}$ for all $\pi \in \mathfrak{S}_d$ and that $(d_\lambda \otimes d_\mu)_{A^{\underline{i}}_{\underline{j}}} \cong d_{A^{\underline{i}}_{\underline{j}}}$. So we get

$$(d_\lambda \otimes d_\mu)_{A^{\underline{i}}_{\underline{j}}} \cong M^{A^{\underline{i}}_{\underline{j}}} = M^{(2,0,1,0,1,0)} \cong M^{(2,1,1)}.$$

There are two more orbits that can be obtained by taking, for example, the elements

$$\underline{i}' = (1112) \text{ and } \underline{j}' = (1213) \quad \text{resp.} \quad \underline{i}'' = (1112) \text{ and } \underline{j}'' = (1321),$$

which span the submodules $M^{(2,1,0,0,0,1)} \cong M^{(2,1,1)}$ resp. $M^{(1,1,1,1,0,0)} \cong M^{(1,1,1,1)}$. All in all we get

$$M^{(3,1)} \otimes M^{(2,1,1)} \cong M^{(2,1,1)} \oplus M^{(2,1,1)} \oplus M^{(1,1,1,1)} = 2 \cdot M^{(2,1,1)} \oplus M^{(1,1,1,1)}.$$

3.3 From strict polynomial functors to representations of the symmetric group

Set $E = k^n$. Using 2.2 and 3.1 one obtains the following decompositions

$$\begin{aligned}\mathrm{End}_{\Gamma^d_k}(\Gamma^{d,k^n}) &= \bigoplus_{\lambda,\mu \in \Lambda(n,d)} \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\lambda, \Gamma^\mu) \\ S_k(n,d) = \mathrm{End}_{k\mathfrak{S}_d}(E^{\otimes d}) &= \bigoplus_{\lambda,\mu \in \Lambda(n,d)} \mathrm{Hom}_{k\mathfrak{S}_d}(M^\mu, M^\lambda)\end{aligned}$$

and using the description of a basis of the Schur algebra given in (2.4), one gets even more

Lemma 3.5. *Let* $\lambda,\ \mu \in \Lambda(n,d)$ *be compositions of* d*. Then*

$$\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\lambda, \Gamma^\mu) \cong \mathrm{Hom}_{k\mathfrak{S}_d}(M^\mu, M^\lambda).$$

Proof. Consider the basis $\{\zeta_{\underline{i},\underline{j}} \mid \underline{i},\underline{j} \in I(n,d)\}$ of the $S_k(n,d)$. It follows immediately from the definition that, if $\underline{i} \in \lambda, \underline{j} \in \mu$,

$$\zeta_{\underline{i},\underline{j}}(e_{\underline{l}}) = \begin{cases} e_{\underline{i}\pi} & \text{if } \underline{l} = \underline{j}\pi \text{ for some } \pi \in \mathfrak{S}_d \\ 0 & \text{else} \end{cases}$$

That is, elements from M^μ are sent to elements of M^λ and every other element is sent to zero. It follows that a basis for $\mathrm{Hom}_{k\mathfrak{S}_d}(M^\mu, M^\lambda)$ is given by $\{\zeta_{\underline{i},\underline{j}} \mid \underline{i} \in \lambda,\ \underline{j} \in \mu\}$, which is in one-to-one correspondence with A_μ^λ by (3.5). We know from Lemma 2.15 that A_μ^λ indexes a basis of $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\lambda, \Gamma^\mu)$, which completes the proof. □

The above discussion yields a functor $\mathcal{F}$ from the category of strict polynomial functors to the category of representations of the symmetric group, as follows.

Consider the composition $\omega = (1, \dots, 1) \in \Lambda(d,d)$ consisting of d times 1. For any $X \in \mathsf{Rep}\,\Gamma_k^d$, the set $\mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, X)$ has a natural structure of a right $\mathrm{End}_{\Gamma_k^d}(\Gamma^\omega)$-module. Namely, for $\varphi \in \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, X)$ and $f \in \mathrm{End}_{\Gamma_k^d}(\Gamma^\omega)$, set $\varphi f = \varphi \circ f$. Moreover, by Lemma 3.5, we have

$$\mathrm{End}_{\Gamma_k^d}(\Gamma^\omega) \cong \mathrm{End}_{k\mathfrak{S}_d}(M^\omega)^{\mathrm{op}} \cong \mathrm{End}_{k\mathfrak{S}_d}(k\mathfrak{S}_d)^{\mathrm{op}} \cong k\mathfrak{S}_d^{\mathrm{op}},$$

where we use the identification $M^\omega \cong k\mathfrak{S}_d$.

If we identify $\mathsf{Mod}\, k\mathfrak{S}_d^{\mathrm{op}} \cong k\mathfrak{S}_d\,\mathsf{Mod}$, we get a functor

$$\mathcal{F} = \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, -)\colon\ \mathsf{Rep}\,\Gamma_k^d \to k\mathfrak{S}_d\,\mathsf{Mod}\,.$$

The projective objects of $\mathsf{Rep}\,\Gamma_k^d$, that is the functors of the form Γ^λ, are mapped under $\mathcal{F}$ to the permutation modules

$$\mathcal{F}(\Gamma^\lambda) = \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda) \cong \mathrm{Hom}_{k\mathfrak{S}_d}(M^\lambda, M^\omega) \cong \mathrm{Hom}_{k\mathfrak{S}_d}(M^\lambda, k\mathfrak{S}_d) = {}^\lambda M.$$

Note that, in particular, the representable functor $\Gamma^{d,k^n} = \bigoplus_{\lambda \in \Lambda(n,d)} \Gamma^\lambda$ is mapped to $\mathrm{Hom}_{k\mathfrak{S}_d}(E^{\otimes d}, k\mathfrak{S}_d) = \bigoplus_{\lambda \in \Lambda(n,d)} {}^\lambda M$.

3.3.1 An equivalence of categories

Let $\Gamma = \{\Gamma^\lambda\}_{\lambda \in \Lambda(n,d)}$ and $M = \{{}^\lambda M\}_{\lambda \in \Lambda(n,d)}$. Denote by $\mathsf{add}\,\Gamma$ the full subcategory of $\mathsf{Rep}\,\Gamma_k^d$ whose objects are direct summands of finite direct sums of elements of Γ. Define $\mathsf{add}\,M$ similarly as a subcategory of $k\mathfrak{S}_d\,\mathsf{Mod}$.

Lemma 3.6. *The functor $\mathcal{F} = \operatorname{Hom}_{\Gamma_k^d}(\Gamma^\omega, -)$ induces an equivalence of categories between $\mathsf{add}\,\Gamma$ and $\mathsf{add}\,M$.*

Proof. Since $\Gamma^{d,k^n} = \bigoplus_{\lambda \in \Lambda(n,d)} \Gamma^\lambda$ we have $\mathsf{add}\,\Gamma = \mathsf{add}\,\Gamma^{d,k^n}$. Similarly one can see that $\mathsf{add}\,M = \mathsf{add}\operatorname{Hom}_{k\mathfrak{S}_d}(E^{\otimes d}, k\mathfrak{S}_d)$. Thus we get the following commutative diagram:

$$\begin{array}{ccc}
\mathsf{Rep}\,\Gamma_k^d & \xrightarrow{\ \mathcal{F}\ } & k\mathfrak{S}_d\,\mathsf{Mod} \\
\uparrow & & \uparrow \\
\mathsf{add}\,\Gamma & \xrightarrow{\ \mathcal{F}|_{\mathsf{add}\,\Gamma}\ } & \mathsf{add}\,M
\end{array}$$

The object Γ^{d,k^n} is mapped under $\mathcal{F}$ to $\operatorname{Hom}_{k\mathfrak{S}_d}(E^{\otimes d}, k\mathfrak{S}_d)$.

For the morphisms $\mathcal{F}$ induces the following isomorphism:

$$\begin{aligned}
\operatorname{Hom}_{\Gamma_k^d}(\Gamma^\lambda, \Gamma^\mu) &\to \operatorname{Hom}_{k\mathfrak{S}_d}(\operatorname{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\lambda), \operatorname{Hom}_{\Gamma_k^d}(\Gamma^\omega, \Gamma^\mu)) \\
&\cong \operatorname{Hom}_{k\mathfrak{S}_d}(\operatorname{Hom}_{k\mathfrak{S}_d}(M^\lambda, k\mathfrak{S}_d), \operatorname{Hom}_{k\mathfrak{S}_d}(M^\mu, k\mathfrak{S}_d)) \\
&= \operatorname{Hom}_{k\mathfrak{S}_d}({}^\lambda M, {}^\mu M)
\end{aligned}$$

□

Remark 3.7. Assume k is a field of characteristic 0. Since the categories $\mathsf{Rep}\,\Gamma_k^d$ and $k\mathfrak{S}_d\,\mathsf{Mod}$ are semisimple, we have

$$\mathsf{add}\,\Gamma = \mathsf{Rep}\,\Gamma_k^d \quad \text{and} \quad \mathsf{add}\,M = k\mathfrak{S}_d\,\mathsf{Mod}\,.$$

Thus the functor $\mathcal{F}$ is an equivalence of categories.This has already be proved by Schur.

Remark 3.8. If we do not restrict to the subcategories $\mathsf{add}\,\Gamma$ and $\mathsf{add}\,M$ the functor $\mathcal{F}$ is not an equivalence in general. For example, if k is a field of positive characteristic p, the categories $\mathsf{Rep}\,\Gamma_k^p$ and $k\mathfrak{S}_p\,\mathsf{Mod}$ are not equivalent (cf. [18, 6.4]).

3.3.2 The monoidal structure

The following result comes from a joint work with Rebecca Reischuk, [2] and shows that the monoidal structure given on the category $\mathsf{Rep}\,\Gamma_k^d$ corresponds through the functor $\mathcal{F}$ to the tensor product of $k\mathfrak{S}_d$ modules given in (3.3).

Independently of any assumption on the commutative ring k we have the following

Theorem 3.9. *The functor*

$$\mathcal{F} = \operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, -)\colon \mathsf{Rep}\,\Gamma^d_k \to k\mathfrak{S}_d\,\mathsf{Mod}$$

preserves the monoidal structure defined on strict polynomial functors, i.e.

$$\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, X \otimes_{\Gamma^d_k} Y) \cong \operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, X) \otimes_k \operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, Y) \tag{3.6}$$

for all X and Y in $\mathsf{Rep}\,\Gamma^d_k$ and

$$\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, I_{\Gamma^d_k}) = I_{k\mathfrak{S}_d}.$$

Proof. As observed in the first section, every functor X in $\mathsf{Rep}\,\Gamma^d_k$ is a colimit of representable functors. One can show that we obtain the same if we only take the colimit with respect to those functors that are represented by finitely generated free modules.

Moreover the functor $\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, -)$ preserves colimits, since it has a right adjoint. Thus it is enough to show the isomorphism (3.6) for functors represented by free modules. Let $V = k^n$ and $W = k^m$ for some non-negative integers n and m. Using the definition of the internal tensor product and the canonical decomposition (2.2) we get

$$\Gamma^{d,k^n} \otimes_{\Gamma^d_k} \Gamma^{d,k^m} = \Gamma^{d,k^n \otimes k^m} \cong \Gamma^{d,k^{n\cdot m}} = \bigoplus_{\nu \in \Lambda(n\cdot m,d)} \Gamma^\nu.$$

Writing down the entries of $\nu \in \Lambda(n\cdot m, d)$ in an $n \times m$ matrix, we obtain a bijection between the set $\Lambda(n \cdot m, d)$ and the set of all $n \times m$ matrices with entries in $\mathbb{N}$ such that the sum of all entries is d. By setting $\lambda_i = \sum_j a_{ij}$ and $\mu_j = \sum_i a_{ij}$, every such matrix $A = (a_{ij})$ defines a couple (λ, μ), with $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$, so that $A \in A^\lambda_\mu$. We get bijections of sets

$$\Lambda(n\cdot m, d) \longleftrightarrow \{A \in \mathbf{M}_{n\times m}(\mathbb{N}) \mid \sum_{st} a_{st} = d\} \longleftrightarrow \bigcup_{\substack{\lambda\in\Lambda(n,d)\\ \mu\in\Lambda(m,d)}} A^\lambda_\mu$$

and thus the following decomposition

$$\Gamma^{d,k^n} \otimes_{\Gamma^d_k} \Gamma^{d,k^m} \cong \bigoplus_{\nu\in\Lambda(n\cdot m,d)} \Gamma^\nu = \bigoplus_{\substack{\lambda\in\Lambda(n,d)\\ \mu\in\Lambda(m,d)}} \bigoplus_{A\in A^\lambda_\mu} \Gamma^A,$$

where the matrix $A = (a_{ij})$ is seen as the composition

$$(a_{11}, a_{12}, \ldots, a_{21}, a_{22}, \ldots, a_{mn}) \in \Lambda(mn, d).$$

Finally this yields

$$\begin{aligned}
\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, \Gamma^{d,k^n} \otimes_{\Gamma^d_k} \Gamma^{d,k^m}) &\cong \operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, \bigoplus_{\substack{\lambda\in\Lambda(n,d)\\ \mu\in\Lambda(m,d)}} \bigoplus_{A\in A^\lambda_\mu} \Gamma^A) \\
&\cong \bigoplus_{\substack{\lambda\in\Lambda(n,d)\\ \mu\in\Lambda(m,d)}} \bigoplus_{A\in A^\lambda_\mu} \operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, \Gamma^A) \\
&\cong \bigoplus_{\substack{\lambda\in\Lambda(n,d)\\ \mu\in\Lambda(m,d)}} \bigoplus_{A\in A^\lambda_\mu} {}^A M \\
&\overset{(*)}{\cong} \bigoplus_{\substack{\lambda\in\Lambda(n,d)\\ \mu\in\Lambda(m,d)}} {}^\lambda M \otimes_k {}^\mu M \\
&\cong \left(\bigoplus_{\lambda\in\Lambda(n,d)} {}^\lambda M\right) \otimes_k \left(\bigoplus_{\mu\in\Lambda(m,d)} {}^\mu M\right) \\
&\cong \left(\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, \Gamma^{d,k^n})\right) \otimes_k \left(\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, \Gamma^{d,k^m})\right),
\end{aligned}$$

where $(*)$ is due to Lemma 3.3.

For the respective tensor units we get:

$$\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, I_{\Gamma^d_k}) = \operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, \Gamma^d) \cong \operatorname{Hom}_{k\mathfrak{S}_d}(M^{(d)}, k\mathfrak{S}_d) = {}^{(d)}M = I_{k\mathfrak{S}_d}.$$

The naturality of the coherence maps is obtained using that $\mathcal{F}(\Gamma^\lambda) = \operatorname{Hom}_{k\mathfrak{S}_d}(M^\lambda, k\mathfrak{S}_d)$ and the naturality of $\operatorname{Hom}_{k\mathfrak{S}_d}(-, k\mathfrak{S}_d)$.

□

Now, some relations given for $k\mathfrak{S}_d$-modules easily translate for strict polynomial functors. In particular,

Corollary 3.10. *The tensor product of Γ^λ and Γ^μ can be decomposed by the same rule as the tensor product of M^λ and M^μ, namely*

$$\Gamma^\lambda \otimes_{\Gamma^d_k} \Gamma^\mu \cong \bigoplus_{A\in A^\lambda_\mu} \Gamma^A$$

Proof. We have that

$$\begin{aligned}
\bigoplus_{\nu\in\Lambda(n\cdot m,d)} \Gamma^{\nu} &= \Gamma^{d,k^{n\cdot m}} \\
&\cong \Gamma^{d,k^{n}} \otimes_{\Gamma_k^d} \Gamma^{d,k^{m}} \\
&= \left(\bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^{\lambda}\right) \otimes_{\Gamma_k^d} \left(\bigoplus_{\mu\in\Lambda(m,d)} \Gamma^{\mu}\right) \\
&\cong \bigoplus_{\substack{\lambda\in\Lambda(n,d)\\ \mu\in\Lambda(m,d)}} \Gamma^{\lambda} \otimes_{\Gamma_k^d} \Gamma^{\mu}
\end{aligned}$$

and thus $\Gamma^{\lambda}\otimes_{\Gamma_k^d}\Gamma^{\mu}$ belongs to $\mathsf{add}\,\Gamma$. The equivalence in Lemma 3.6 yields the stated decomposition. □

4 Relation to symmetric functions in characteristic 0

In this section we will assume that k is a field of characteristic 0. In this case, the categories $\mathsf{Rep}\,\Gamma_k^d$ and $k\mathfrak{S}_d\,\mathsf{Mod}$ are semisimple and the functor $\mathcal{F} = \mathrm{Hom}_{\Gamma_k^d}(\Gamma^\omega, -)$ induces an equivalence. Since $k\mathfrak{S}_d\,\mathsf{Mod}$ is semisimple we can identify every module with its character. As Macdonald explains in [25] there is an isometric isomorphism from the ring of irreducible characters of $k\mathfrak{S}_d$, for all $d \geq 0$, to the ring of symmetric functions. In this section we will give the most important definitions and results about symmetric functions, following [25], and explain the relations to strict polynomial functors and representations of the symmetric group.

4.1 Ring of symmetric functions

Consider the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in n indipendent variables with integer coefficients. The symmetric group $\mathfrak{S}_n$ acts on this ring by permuting the variables

$$\sigma(f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

for $f \in \mathbb{Z}[x_1, \ldots, x_n]$, $\sigma \in \mathfrak{S}_n$. A polynomial is *symmetric* if it is invariant under this action. The symmetric polynomials form a subring

$$\mathbf{\Lambda}_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n},$$

which is a graded ring. We have

$$\mathbf{\Lambda}_n = \bigoplus_{d \geq 0} \mathbf{\Lambda}_n^d$$

where $\mathbf{\Lambda}_n^d$ consists of the homogeneous symmetric polynomials of degree d, together with the zero polynomial.

For any sequence of natural numbers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by x^α the monomial

$$x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

The polynomial

$$m_\alpha(x_1, \ldots, x_n) := \sum_{\beta \sim \alpha} x^\beta = \sum_{\beta \sim \alpha} x_1^{\beta_1} \cdots x_n^{\beta^n},$$

where the sum is taken over all different permutations β of α, is clearly symmetric. It is easy to see that the *monomial symmetric functions* m_λ form a $\mathbb{Z}$-basis of $\mathbf{\Lambda}_n$, as λ runs over all partitions of length $\leq n$. Hence the

m_λ such that $l(\lambda) \leq n$ and $|\lambda| = d$ form a $\mathbb{Z}$-basis of $\mathbf{\Lambda}_n^d$. In particular, as soon as $n \geq d$, the m_λ such that $|\lambda| = d$ form a $\mathbb{Z}$-basis of $\mathbf{\Lambda}_n^d$.

It is often more convenient to work with symmetric function in infinitely many variables. To make this precise, let $n \geq m$ and consider the homomorphism

$$\mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}[x_1, \dots, x_m]$$

which sends $x_{m+1}, \dots, x_n$ to zero and the other variables to themselves. If we restrict to $\mathbf{\Lambda}_n$, this gives an homomorphism

$$\rho_{n,m} : \mathbf{\Lambda}_n \to \mathbf{\Lambda}_m$$

which has the following effect on the monomial symmetric function: it sends $m_\lambda(x_1, \dots, x_n)$ to $m_\lambda(x_1, \dots, x_m)$ if $l(\lambda) \leq m$ and to 0 if $l(\lambda) > m$. It follows that $\rho_{n,m}$ is surjective. On restriction to $\mathbf{\Lambda}_n^d$, we have homomorphisms

$$\rho_{n,m}^d : \mathbf{\Lambda}_n^d \to \mathbf{\Lambda}_m^d$$

for all $d \geq 0$ and $n \geq m$, which are always surjective, and bijective for $n \geq m \geq d$. We now form the inverse limit

$$\mathbf{\Lambda}^d = \varprojlim_n \mathbf{\Lambda}_n^d$$

of the $\mathbb{Z}$-module $\mathbf{\Lambda}_n^d$ relative to the homomorphisms $\rho_{n,m}^d$. An element of $\mathbf{\Lambda}^d$ is a sequence $f = (f_n)_{n \geq 0}$, where each $f_n = f_n(x_1, \dots, x_n)$ is a homogeneous symmetric polynomial of degree d in $x_1, \dots, x_n$ and $f_n(x_1, \dots, x_m, 0, \dots, 0) = f_m(x_1, \dots, x_m)$ whenever $n \geq m$. Since $\rho_{n,m}^d$ is an isomorphism for $n \geq m \geq d$, the projection

$$\rho_n^d : \mathbf{\Lambda}^d \to \mathbf{\Lambda}_n^d,$$

which sends f to f_n, is an isomorphism for all $n \geq d$. Thus, it follows that $\mathbf{\Lambda}^d$ has a $\mathbb{Z}$-basis consisting of all monomial symmetric functions m_λ, for all partitions λ of d, defined by

$$\rho_n^d(m_\lambda) = m_\lambda(x_1, \dots, x_n)$$

for all $n \geq d$. Hence $\mathbf{\Lambda}^d$ is a free $\mathbb{Z}$-module, whose rank equals the number of partitions of d. Now let

$$\mathbf{\Lambda} = \bigoplus_{d \geq 0} \mathbf{\Lambda}^d,$$

so that $\mathbf{\Lambda}$ is the free $\mathbb{Z}$-module generated by m_λ, for all partitions λ. We have surjective homomorphisms

$$\rho_n = \bigoplus_{d \geq 0} \rho_n^d : \mathbf{\Lambda} \to \mathbf{\Lambda}_n$$

for each $n \geq 0$, ρ_n is an isomorphism in degrees $d \leq n$. It is clear that $\mathbf{\Lambda}$ has a structure of graded ring, such that all ρ_n are ring homomorphisms. The graded ring $\mathbf{\Lambda}$ is called *ring of symmetric functions* in countably many indipendent variables.

4.1.1 Bases and relations

In this section we collect some important symmetric functions, which also give bases for the ring $\mathbf{\Lambda}$, and describe some relations between them. A more complete discussion and proofs can be found in [25].

Elementary symmetric functions For any natural number n define the n-th elementary symmetric function

$$e_n := m_{(1^n)}$$

For any set of variables $\underline{x}$ this is the sum of all products of n distinct variables, i.e.

$$e_n(\underline{x}) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} \ldots x_{i_n}.$$

If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is any sequence of natural numbers, set

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_n}.$$

From direct computation, one has

$$e_{\tilde{\lambda}} = \sum_{\mu \trianglelefteq \lambda} a_{\lambda\mu} m_\mu,$$

where $a_{\lambda\mu}$ are non negative integers.

Hence, the set $\{e_\lambda \mid \lambda \text{ partition of } d\}$ is a basis for the symmetric functions of degree d. If we consider all partitions of all non-negative integers d we obtain a $\mathbb{Z}$-basis of the ring $\mathbf{\Lambda}$.

Complete symmetric functions For any natural number n define the n-th complete symmetric function

$$h_n := \sum_{|\alpha| = n} m_\alpha.$$

For any set of variables this is the sum of all monomials of total degree n, i.e.

$$h_n(\underline{x}) = \sum_{|\alpha| = n} \sum_{\beta \sim \alpha} x_1^{\beta_1} \cdots x_n^{\beta^n},$$

where the second sum is taken over all distinct permutations of α. With the help of *generating functions*, one can show that elementary and complete symmetric functions are related by the formula

$$\sum_{r=0}^{d} e_r h_{d-r} = 0,$$

for all $d \geq 0$. Since the e_r are algebrically independent, we may define a homomorphism of graded rings $\omega : \mathbf{\Lambda} \to \mathbf{\Lambda}$, by $\omega(e_r) = h_r$ for all $r \geq 0$. Because of the symmetry of the above relation, ω is an involution, i.e. ω^2 is the identity map. Hence, $h_1, h_2, \dots$ are algebraically independent over $\mathbb{Z}$ and it follows that the set

$$\{h_\lambda := h_{\lambda_1} \cdots h_{\lambda_n} \mid \lambda \text{ a partition}\},$$

forms a $\mathbb{Z}$-basis of the ring $\mathbf{\Lambda}$.

Power sum For any natural number n define the n-th power sum

$$p_n := m_{(n)}.$$

Again with the help of generating functions, one can see that

$$dh_d = \sum_{r=1}^{d} p_r h_{d-r}.$$

It follows thar $\mathbb{Q}[p_1, \dots, p_n] = \mathbb{Q}[h_1, \dots, h_n]$, for any n and the $p_1, p_2, \dots$ are algebraically independent over $\mathbb{Q}$. Then

$$\{p_\lambda := p_{\lambda_1} \cdots p_{\lambda_n} \mid \lambda \text{ a partition}\}$$

is a $\mathbb{Q}$-basis of $\mathbf{\Lambda}_\mathbb{Q} = \mathbf{\Lambda} \otimes_\mathbb{Z} \mathbb{Q}$.

If we want to express elementary and complete symmetric functions in terms of power sums, we have

$$h_d = \sum_{|\lambda|=d} z_\lambda^{-1} p_\lambda \quad \text{and} \quad e_d = \sum_{|\lambda|=d} \varepsilon_\lambda z_\lambda^{-1} p_\lambda, \tag{4.1}$$

where $\varepsilon_\lambda = (-1)^{|\lambda|-l(\lambda)}$ and $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$, with $m_i = m_i(\lambda)$ the number of parts of λ that are equal to i.

Schur functions Suppose to begin with that the number of variables is finite, say $x_1, \dots, x_n$. Let $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ be a monomial, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$, as above. Consider the polynomial a_α obtained by antisymmetrizing x^α, that is

$$a_\alpha = a_\alpha(x_1, \dots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \cdot \sigma(x^\alpha),$$

where $\epsilon(\sigma)$ is the sign of the permutation $\sigma \in \mathfrak{S}_n$. This polynomial is skew-symmetric, that is, we have

$$\sigma(a_\alpha) = \epsilon(\sigma) a_\alpha$$

for any $\sigma \in \mathfrak{S}_n$; in particular, a_α vanishes unless $\alpha_1, \alpha_2, \dots, \alpha_n$ are all distinct. Hence, we may assume $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$, and therefore we can write $\alpha = \lambda + \delta$ where λ is a partition of length $\leq n$, and $\delta = (n-1, n-2, \dots, 1, 0)$. Then

$$a_\alpha = a_{\lambda+\delta} = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \cdot \sigma(x^{\lambda+\delta}).$$

This can be written as a determinant

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j+n-j})_{1 \leq i,j \leq n}.$$

This determinant is divisible in $\mathbb{Z}[x_1, \dots, x_n]$ by each of the differences $x_i - x_j$ with $1 \leq i < j \leq n$, and hence by their product, which is the *Vandermonde determinant*

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j}) = a_\delta.$$

So $a_{\lambda+\delta}$ is divisible by a_δ in $\mathbb{Z}[x_1, \dots, x_n]$ and the quotient

$$s_\lambda = s_\lambda(x_1, \dots, x_n) = a_{\lambda+\delta}/a_\delta \tag{4.2}$$

is *symmetric*, i.e. $s_\lambda \in \mathbf{\Lambda}_n$. It is called the *Schur function* in the variables $x_1, \dots, x_n$, corresponding to the partition λ, where $l(\lambda) \leq n$, and it is homogeneous of degree $|\lambda|$. The polynomials $a_{\lambda+\delta}$, where λ runs through all partitions of length $\leq n$, form a basis of the $\mathbb{Z}$-module A_n of skew-symmetric polynomials in $x_1, \dots, x_n$. Multiplication by a_δ is an isomorphism of $\mathbf{\Lambda}_n$ onto A_n. Therefore the Schur functions $s_\lambda(x_1, \dots, x_n)$, where $l(\lambda) \leq n$, form a $\mathbb{Z}$-basis of $\mathbf{\Lambda}_n$.

Now we want to increase the number of variables. If $l(\lambda) \leq n$, it is clear that $a_\alpha(x_1, \ldots, x_n, 0) = a_\alpha(x_1, \ldots, x_n)$. Hence,

$$\rho_{n+1,n}(s_\lambda(x_1, \ldots, x_{n+1})) = s_\lambda(x_1, \ldots, x_n).$$

It follows that for each partition λ the polynomials $s_\lambda(x_1, \ldots, x_n)$, as $n \to \infty$, define a unique element $s_\lambda \in \mathbf{\Lambda}$, homogeneous of degree $|\lambda|$.

Therefore, the set of Schur functions corresponding to all partitions is a $\mathbb{Z}$-basis of the ring $\mathbf{\Lambda}$ and for each $d \geq 0$, the s_λ such that $|\lambda| = d$ form a $\mathbb{Z}$-basis of $\mathbf{\Lambda}^d$. Clearly, each Schur function can be expressed as a polynomial in the elementary symmetric functions, and as a polynomial in the complete symmetric functions. With $\tilde{\lambda} \in \Lambda(\tilde{n}, d)$ the conjugate partition of $\lambda \in \Lambda(n, d)$, one has

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n} = \det(e_{\tilde{\lambda}_i - i + j})_{1 \leq i,j \leq \tilde{n}}. \tag{4.3}$$

On the other hand, to express complete symmetric functions in terms of Schur functions, we need the so called *Kostka numbers*. For λ, μ any partitions, $K_{\lambda\mu}$ is the number of costandard tableaux of shape λ and content μ. They are non negative integers and are computed combinatorially (cf. [25, 6. Table I]). By Lemma 1.1, one has $K_{\lambda\mu} = 0$ unless $\lambda \geq \mu$, and $K_{\lambda\lambda} = 1$. We have

$$h_\lambda = \sum_{\mu \in \Lambda(n,d)} K_{\mu\lambda} s_\mu \quad \text{and} \quad e_{\tilde{\lambda}} = \sum_{\mu \in \Lambda(n,d)} K_{\tilde{\mu}\tilde{\lambda}} s_\mu. \tag{4.4}$$

In particular, $h_{(d)} = h_d = s_{(d)}$ and $e_{(d)} = e_d = s_{(1^d)}$.

4.1.2 Scalar product

One can define a scalar product on $\mathbf{\Lambda}$ by requiring that the bases $\{h_\lambda\}$ and $\{m_\lambda\}$ should be dual to each other, i.e.

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

Where $\delta_{\lambda\mu}$ is the Kronecker delta. This implies that

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \tag{4.5}$$

so that the p_λ form an *orthogonal* basis of $\mathbf{\Lambda}_\mathbb{Q}$. Moreover,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} \tag{4.6}$$

so that the s_λ form an orthonormal basis of $\mathbf{\Lambda}$.

4.2 The characters of the symmetric groups

In this section we want to explain how symmetric functions describe the characters of symmetric groups representations and how this translates in terms of strict polynomial functors. We start by briefly recalling some standard facts about the representation theory of symmetric groups.

4.2.1 Representations of symmetric groups

Assume k is a field of characteristic 0, in particular the category $k\mathfrak{S}_d\,\mathsf{Mod}$ is semisimple. We want to describe the simple $k\mathfrak{S}_d$-modules; we know from the classical representation theory of finite groups, that the number of simple modules equals the number of the conjugacy classes of $\mathfrak{S}_d$.

The number of conjugacy classes of the symmetric groups on d elements is the number of partitions of d. Indeed, the conjugacy classes of $\mathfrak{S}_d$ are indexed by partitions as follows.

Each permutation $\sigma \in \mathfrak{S}_d$ factorizes uniquely as a product of disjoint cycles. If the orders of these cycles are $\rho_1, \rho_2, \ldots$, where $\rho_1 \geq \rho_2 \geq \ldots$, then

$$\rho(\sigma) = (\rho_1, \rho_2, \ldots)$$

is a partition of d, that we call the *cycle type* of σ. It is well known that this determines σ up to conjugacy in $\mathfrak{S}_d$.

The simple $k\mathfrak{S}_d$-module corresponding to a partition of d is the submodule $\mathrm{Sp}(\lambda)$ of the permutation module M^λ defined as follows. Recall that M^λ can be seen as the k-vector space with basis the tabloids $\{T\}$ of shape λ (cf. Remark 3.2). For each tableau T of shape λ and filling $\{1, \ldots, d\}$, that is, every number from 1 to d occurs once, we can define an element of M^λ by

$$v_T = \sum_{\sigma \in \mathfrak{S}_{\tilde{\lambda}}} \mathrm{sgn}(\sigma)\{\sigma T\},$$

where σ runs over all permutation of the Young subgroup of $\mathfrak{S}_d$ corresponding to the partition $\tilde{\lambda}$ dual to λ, that is the subgroup given by permutations preserving the columns of T. The *Specht module* $\mathrm{Sp}(\lambda)$ is the subspace of M^λ spanned by the elements v_T, as T runs over all tableaux of shape λ and filling $\{1, \ldots, d\}$. It is easy to see that this is a submodule and that, in fact, we have

$$\mathrm{Sp}(\lambda) = k\mathfrak{S}_d\, v_T,$$

for any such T.

By [15, 7.2 Proposition 2], a k-basis of $\mathrm{Sp}(\lambda)$ is given by those v_T coming from a standard tableau T. Note that, since every entry of T occurs once, such a standard tableau is also costandard.

Equivalently, by using the description of M^λ as the k-span of the set $\{e_{\underline{i}} \mid \underline{i} \text{ belongs to } \lambda\}$, given in Section 3.1, we can describe $\mathrm{Sp}(\lambda)$ by

$$\mathrm{Sp}(\lambda) = k\mathfrak{S}_d\, v_\lambda,$$

where $v_\lambda := \sum_{\sigma\in\mathfrak{S}_{\lambda'}} \mathrm{sgn}(\sigma) e_{\underline{i}_\lambda}\sigma$ and $\underline{i}_\lambda$ is the d-tuple where 1 occurs λ_1 times, 2 occurs λ_2 times and so on, i.e. $\underline{i}_\lambda = (1\dots 12\dots 2\dots n\dots n) \in \lambda$.

Since $k\mathfrak{S}_d$ is semisimple, each module can be decomposed into simple ones, thus as a direct sum of Specht modules. The decomposition of the permutation modules is given via Kostka numbers.

$$^\mu M = \bigoplus_{\lambda\in\Lambda(n,d)} K_{\lambda\mu}\mathrm{Sp}(\lambda). \tag{4.7}$$

The tensor product of two Specht modules is given by

$$\mathrm{Sp}(\lambda)\otimes_k \mathrm{Sp}(\mu) \cong \bigoplus_{\nu\in\Lambda(n,d)} g^\nu_{\lambda\mu}\mathrm{Sp}(\nu), \tag{4.8}$$

where $g^\nu_{\lambda\mu}$ are called *Kronecker coefficients* (cf. [20, 2.8.13]).

4.2.2 From characters of the symmetric group to symmetric functions

Recall that, if G is a finite group and f, g are functions on G with values in a commutative $\mathbb{Q}$-algebra, the *scalar product* of f and g is defined by

$$\langle f, g\rangle_G = \frac{1}{|G|}\sum_{x\in G} f(x)g(x^{-1}).$$

If H is a subgroup of G and f is a character of H, the induced character on G will be denoted by $\mathrm{ind}_H^G(f)$. If g is a character of G, the restriction to H will be denoted by $\mathrm{res}_G^H(g)$.

We define a mapping $\varphi : \mathfrak{S}_d \to \mathbf{\Lambda}^d$ by

$$\varphi(\sigma) = p_{\rho(\sigma)},$$

where $\rho(\sigma)$ is the cycle type of σ. If d, e are positive integers, we may embed $\mathfrak{S}_e\times\mathfrak{S}_d$ in $\mathfrak{S}_{d+e}$ as usual, by making $\mathfrak{S}_e$ and $\mathfrak{S}_d$ act on complementary subsets of $\{1,\dots,d+e\}$. There are many different ways of doing this, but the resulting subgroups of $\mathfrak{S}_{d+e}$ are all conjugate. Hence $\sigma\times\tau$, for $\sigma\in\mathfrak{S}_d$ and $\tau\in\mathfrak{S}_e$, is well defined up to conjugacy in $\mathfrak{S}_{d+e}$ and its cycle type is given by $\rho(\sigma\times\tau) = \rho(\sigma)\cup\rho(\tau)$, so that

$$\varphi(\sigma\times\tau) = \varphi(\sigma)\varphi(\tau). \tag{4.9}$$

Let R^d be the $\mathbb{Z}$-module generated by the irreducible characters of $\mathfrak{S}_d$, and let

$$R = \bigoplus_{d \geq 0} R^d,$$

with $\mathfrak{S}_0 = 1$, so that $R^0 = \mathbb{Z}$. The $\mathbb{Z}$-module R has a ring structure, defined as follows. Let $f \in R^d$ and $g \in R^e$ and embed $\mathfrak{S}_e \times \mathfrak{S}_d$ in $\mathfrak{S}_{d+e}$. Then $f \times g$ is a character of $\mathfrak{S}_e \times \mathfrak{S}_d$, and we define

$$f \cdot g = \operatorname{ind}_{\mathfrak{S}_e \times \mathfrak{S}_d}^{\mathfrak{S}_{d+e}} (f \times g), \tag{4.10}$$

which is a character of $\mathfrak{S}_{d+e}$, i.e. an element of R^{d+e}. Thus, we have defined a bilinear multiplication $R^d \times R^e \to R^{d+e}$, and it is not difficult to verify that with this multiplication R is a commutative, associative, graded ring with identity.

Moreover, R carries a scalar product: if $f, g \in R$, say $f = \sum f_d$, $g = \sum g_d$, with $f_d, g_d \in R^d$, we define

$$\langle f, g \rangle = \sum_{d \geq 0} \langle f_d, g_d \rangle_{\mathfrak{S}_d}.$$

Next we define a $\mathbb{Z}$-linear mapping

$$\operatorname{ch} : R \to \mathbf{\Lambda}_{\mathbb{C}} = \mathbf{\Lambda} \otimes_{\mathbb{Z}} \mathbb{C}$$

as follows : if $f \in R^d$, then

$$\operatorname{ch}(f) = \langle f, \varphi \rangle_{\mathfrak{S}_d} = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} f(\sigma) \varphi(\sigma), \tag{4.11}$$

since $\varphi(\sigma) = \varphi(\sigma^{-1})$.

If f_ρ is the value of f at elements of cycle type ρ, we have

$$\operatorname{ch}(f) = \sum_{|\rho| = d} z_\rho^{-1} f_\rho p_\rho. \tag{4.12}$$

We call $\operatorname{ch}(f)$ the *characteristic* of f and ch is the *characteristic map.*

Proposition 4.1. *[25, I, (7.3)] The characteristic map is an isometric isomorphism of R onto Λ.*

Proof. From (4.12) and (4.5), it follows that, for $f, g \in R^d$,

$$\langle \operatorname{ch}(f), \operatorname{ch}(g) \rangle = \sum_{|\rho| = d} z_\rho^{-1} f_\rho g_\rho = \langle f, g \rangle_{\mathfrak{S}_d}$$

and hence ch is an isometry.

To verify that ch is a ring homomorphism, let $f \in R^d$, $g \in R^e$. We have

$$\begin{aligned}\mathrm{ch}(f \cdot g) &= \left\langle \mathrm{ind}_{\mathfrak{S}_d \times \mathfrak{S}_e}^{\mathfrak{S}_{d+e}}(f \times g), \varphi \right\rangle_{\mathfrak{S}_{d+e}} \\ &= \left\langle f \times g, \mathrm{res}_{\mathfrak{S}_{d+e}}^{\mathfrak{S}_d \times \mathfrak{S}_e}(\varphi) \right\rangle_{\mathfrak{S}_d \times \mathfrak{S}_e}\end{aligned}$$

by *Frobenius reciprocity*,

$$= \langle f, \varphi \rangle_{\mathfrak{S}_d} \langle g, \varphi \rangle_{\mathfrak{S}_e} = \mathrm{ch}(f) \cdot \mathrm{ch}(g)$$

by (4.9).

Now, let η_d be the identity character on $\mathfrak{S}_d$, that is the character of the trivial permutation module ${}^{(d)}M$. Then

$$\mathrm{ch}(\eta_d) = \sum_{|\rho|=d} z_\rho^{-1} p_\rho = h_d$$

by (4.12) and (4.1). If $\lambda \in \Lambda^+(r, d)$ is any partition, then $\eta_\lambda = \eta_{\lambda_1}\eta_{\lambda_2} \ldots \eta_{\lambda_r}$ is the character of $\mathfrak{S}_d$ induced by the identity character of $\mathfrak{S}_\lambda$, and we have $\mathrm{ch}(\eta_\lambda) = h_\lambda$.

For each partition λ, define

$$\chi^\lambda = \det(\eta_{\lambda_i - i + j})_{1 \le i,j \le d} \in R^d, \tag{4.13}$$

i.e. χ^λ is a (possibly virtual) character of $\mathfrak{S}_d$ and by (4.3) we have

$$\mathrm{ch}(\chi^\lambda) = s_\lambda. \tag{4.14}$$

Since ch is an isometry, it follows from (4.6) that $\left\langle \chi^\lambda, \chi^\mu \right\rangle = \delta_{\lambda\mu}$ for any two partitions λ, μ. Hence, up to the sign, χ^λ are irreducible characters of $\mathfrak{S}_d$. Since the number of coniugacy classes of $\mathfrak{S}_d$, hence the number of irreducible characters, equals the number of partition of d, these characters exhaust all the irreducible characters of $\mathfrak{S}_d$. Hence χ^λ, for $\lambda \in \Lambda^+(d)$, form a basis of R^d and ch is an isomorphism of R^d onto $\mathbf{\Lambda}^d$, thus of R onto $\mathbf{\Lambda}$. □

Remark 4.2. If we denote by $[V]$ the character of the $k\mathfrak{S}_d$-module V, one has $\chi^\lambda = [\mathrm{Sp}(\lambda)]$ hence

$$\mathrm{ch}([\mathrm{Sp}(\lambda)]) = s_\lambda,$$

in particular $\mathrm{ch}([\mathrm{Sp}((n))]) = h_n$.

From (3.2), it follows that the permutation module ${}^\lambda M$ is isomorphic to the induced representation of the trivial representation from $\mathfrak{S}_\lambda$ to $\mathfrak{S}_d$ Hence, for its character we get

$$\mathrm{ch}([{}^\lambda M]) = \mathrm{ch}(\eta_\lambda) = h_\lambda.$$

4.2.3 Kronecker product

Using the characteristic map, one can define an internal product, sometimes called Kronecker product, via the internal tensor product of modules over the symmetric group. Let $f, g \in \mathbf{\Lambda}^d$ and let $f = \mathrm{ch}(\phi)$ and $g = \mathrm{ch}(\psi)$ where ϕ, ψ are class functions on $\mathfrak{S}_d$. The *internal product* of f and g is defined to be

$$f * g = \mathrm{ch}(\phi \cdot \psi),$$

where $\phi \cdot \psi$ is the function $\sigma \mapsto \phi(\sigma)\psi(\sigma)$ on $\mathfrak{S}_d$.

With respect to this product $\mathbf{\Lambda}^d$ becomes a commutative and associative ring with identity element h_d.

From (4.11) and (4.14) we obtain an explicit formula for the internal product of Schur functions. If λ, μ, ν are partitions of d, we have

$$s_\lambda * s_\mu = \sum_\nu g^\nu_{\lambda\mu} s_\nu, \tag{4.15}$$

where

$$g^\nu_{\lambda\mu} = \langle \chi^\nu, \chi^\lambda \chi^\mu \rangle_{\mathfrak{S}_d} = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma).$$

This also allows, by (4.4), to compute internal products of elementary and complete symmetric functions.

Notice that the coefficients $g^\nu_{\lambda\mu}$ coincide with the Kronecker coefficients given in (4.8). We will see that this is a consequence of a more general correspondence (cf. Corollary 4.4).

Since we know how to decompose the tensor product of two permutation modules, we can compute the Kronecker product of two complete symmetric functions:

$$h_\lambda * h_\mu = \mathrm{ch}([{}^\lambda M] \cdot [{}^\mu M]) = \mathrm{ch}([{}^\lambda M \otimes_k {}^\mu M]) = \mathrm{ch}([\bigoplus_{A \in A^\lambda_\mu} {}^A M]) = \sum_{A \in A^\lambda_\mu} h_A.$$

4.3 Connection to strict polynomial functors

Recall that the functor

$$\mathcal{F} = \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\omega, -) \colon \mathsf{Rep}\, \Gamma^d_k \to k\mathfrak{S}_d \,\mathsf{Mod},$$

defined in (3.3) is an equivance of categories if k is a field of characteristic 0. The simple objects of $\mathsf{Rep}\,\Gamma^d_k$ are given by the Schur functors $\mathbb{S}_\lambda$ defined in Section 2.6. We want to show that $\mathcal{F}$ sends the Schur functor $\mathbb{S}_\lambda$ to the Specht module $\mathrm{Sp}(\lambda)$ (cf. also [18, 6]).

Proposition 4.3. *For any partition $\lambda \in \Lambda^+(d)$, we have an isomorphism*

$$\mathcal{F}(\mathbb{S}_\lambda) = \mathrm{Hom}_{\Gamma^d_k}(\Gamma^\omega, \mathbb{S}_\lambda) \cong \mathrm{Sp}(\lambda).$$

Proof. From Lemma 2.10, we know that $\mathrm{Hom}_{\Gamma^d_k}(\Gamma^\omega, \mathbb{S}_\lambda)$ is isomorphic to the weight space $\mathbb{S}_\lambda(V)_\omega$, with $V \cong k^n$, for any $n \geq d$. By Theorem 2.21, a basis of $\mathbb{S}_\lambda$ is indexed by standard tableaux of shape λ and filling $\{1, \ldots, n\}$. Thus, as in Lemma 2.15, a basis of $\mathbb{S}_\lambda(V)_\omega$ is given by standard tableaux of shape λ and content ω, that is, by standard tableaux of shape λ with entry $1, \ldots, d$, where each entries occurs once. But this is also a k-basis of $\mathrm{Sp}(\lambda)$. □

We are now able to translate in terms of strict polynomial functors some relation given for $k\mathfrak{S}_d$-modules.

Recall that $\Gamma^d = \mathbb{S}_{(d)}$. In particular this means that we have

$$\Gamma^\lambda = \mathbb{S}_{(\lambda_1)} \otimes \cdots \otimes \mathbb{S}_{(\lambda_n)}.$$

In terms of strict polynomial functors, the decomposition (4.7) becomes

$$\Gamma^\lambda = \bigoplus_{\mu \in \Lambda(n,d)} K_{\mu\lambda} \mathbb{S}_\mu.$$

Moreover, by 4.8 and Theorem 3.9, the tensor product of Schur functors is again given by the Kronecker coefficients:

$$\mathbb{S}_\lambda \otimes_{\Gamma^d_k} \mathbb{S}_\mu \cong \bigoplus_{\nu \in \Lambda(n,d)} g^\nu_{\lambda\mu} \mathbb{S}_\nu$$

By identifying, once again, representations of the symmetric group with their characters $k\mathfrak{S}_d\,\mathsf{Mod} \cong R^d$ and considering the composition $\mathrm{ch} \circ \mathcal{F}$ we obtain a map

$$\mathsf{Rep}\,\Gamma^d_k \to \mathbf{\Lambda}^d.$$

But there is an alternative approach that endows strict polynomial functors with a ring structure, making possible to define a map going directly from strict polynomial functors to symmetric functions, which is a ring homomorphism.

4.3.1 From strict polynomial functors directly to symmetric functions

Let $X \in \mathsf{Rep}\,\Gamma^d_k$ and $Y \in \mathsf{Rep}\,\Gamma^e_k$ and consider their *external tensor product*

$$(X \otimes Y)(V) = X(V) \otimes_k Y(V), \quad \text{for any} \quad V \in \mathsf{P}_k$$

as in definition 2.4.

For a non negative integer d, denote by $\mathcal{F}_d = \mathrm{Hom}_{\Gamma^k_d}(\Gamma^\omega_d, -)$ the functor $\mathcal{F}$ from Definition 3.3 in degree d, where $\omega_d = (1, \dots, 1)$ is the partition consisting of d parts equal to 1.

Consider the $k[\mathfrak{S}_d \times \mathfrak{S}_e]$-module $\mathcal{F}_d(X) \otimes \mathcal{F}_e(Y)$, with diagonal action. We want to look at it as a $k\mathfrak{S}_{d+e}$ representation, that is, we consider

$$\begin{aligned} &\mathrm{ind}^{\mathfrak{S}_{d+e}}_{\mathfrak{S}_d \times \mathfrak{S}_e}(\mathcal{F}_d(X) \otimes \mathcal{F}_e(Y)) \\ &= \mathrm{ind}^{\mathfrak{S}_{d+e}}_{\mathfrak{S}_d \times \mathfrak{S}_e}\Big(\mathrm{Hom}_{\Gamma^d_k}(\Gamma^{\omega_d}, X) \otimes \mathrm{Hom}_{\Gamma^e_k}(\Gamma^{\omega_e}, Y)\Big) \\ &\cong \mathrm{Hom}_{\Gamma^{d+e}_k}(\Gamma^{\omega_{d+e}}, X \otimes Y) = \mathcal{F}_{d+e}(X \otimes Y). \end{aligned}$$

The character of this module is given by

$$[\mathrm{ind}^{\mathfrak{S}_{d+e}}_{\mathfrak{S}_d \times \mathfrak{S}_e}(\mathcal{F}_d(X) \otimes \mathcal{F}_e(Y))] = [\mathcal{F}_d(X)] \cdot [\mathcal{F}_e(Y)],$$

the product of characters given in (4.10). Thus, the external tensor product of two strict polynomial funtors of degree d and e, respectively, corresponds through the equivalence $\mathcal{F}_{d+e}$, to the *induction product*

$$\mathcal{F}_d(X) \cdot \mathcal{F}_e(Y) = \mathrm{ind}^{\mathfrak{S}_{d+e}}_{\mathfrak{S}_d \times \mathfrak{S}_e}(\mathcal{F}_d(X) \otimes \mathcal{F}_e(Y))$$

of the corresponding $k\mathfrak{S}_d$ resp. $k\mathfrak{S}_e$-modules. The external tensor product of polynomial functors defines a product on the Grothendieck group $K(\mathfrak{F})$, where

$$\mathfrak{F} = \bigoplus_{d \geq 0} \mathsf{Rep}\, \Gamma^d_k$$

is the category of strict polynomial functors of bounded degree, which gives it the structure of a commutative, associative, graded ring with identity (cf. [25, Appendix A]).

We want to define a homomorphism of graded rings

$$\chi\colon K(\mathfrak{F}) \to \mathbf{\Lambda}.$$

For $a = (a_1, \dots, a_n) \in k^n$, denote by $\mathrm{diag}(a)$ the diagonal endomorphism of k^n with eigenvalues $(a_1, \dots, a_n)$. If X is a polynomial functor, the trace of $X(\mathrm{diag}(a))$ is a polynomial function of $(a_1, \dots, a_n)$, which is symmetric because

$$\mathrm{diag}(\sigma(a_1, \dots, a_n)) = \sigma\, \mathrm{diag}(a) \sigma^{-1}, \quad \text{for any} \quad \sigma \in \mathfrak{S}_n,$$

where in the second term we denote by σ the *permutation matrix* corresponding to σ. Hence, we have

$$\mathrm{trace}\, X(\mathrm{diag}(\sigma a)) = \mathrm{trace}(X(\sigma) X(\mathrm{diag}(a)) X(\sigma^{-1})) = \mathrm{trace}\, X(\mathrm{diag}(a)).$$

Setting

$$\chi(X)(a_1, \dots, a_n) = \operatorname{trace} X(\operatorname{diag}(a))$$

yields the desired homomorphism.

If we observe that

$$K(\mathsf{Rep}\,\Gamma_k^d) \cong K(\mathsf{Mod}\,k\mathfrak{S}_d) \cong R^d$$

we may identify $K(\mathfrak{F})$ with R. Under this identification, the map χ coincides with ch. To see this, it is enough to observe that $\chi(\Gamma^\lambda) = h_\lambda$.

Indeed, we have

$$\Gamma^d(\operatorname{diag}(a)) = \operatorname{diag}(a) \otimes \cdots \otimes \operatorname{diag}(a) = \operatorname{diag}(a)^{\otimes d}, \quad \text{hence}$$

$$\chi(\Gamma^d)(a_1, \dots, a_n) = \operatorname{trace}(\operatorname{diag}(a)^{\otimes d}) = (\operatorname{trace}(\operatorname{diag}(a)))^d = (a_1 + \cdots + a_n)^d.$$

Recall that

$$(a_1 + \cdots + a_n)^d = \sum \binom{d}{m_1, \dots, m_n} a_1^{m_1} \dots a_n^{m_n}$$

where the sum is taken over all compositions $(m_1, \dots, m_n)$ of d and the coefficient of $a_1^{m_1} \dots a_n^{m_n}$ equals $\frac{d!}{m_1! \dots m_n!}$. If we observe that this coefficient gives the number of permutations that fix the partition $(m_1, \dots, m_n)$, we can rewrite the sum as

$$\sum_{|\lambda|=d} \sum_{\beta\sim\lambda} a_1^{\beta_1} \cdots a_n^{\beta^n} = \sum_{|\lambda|=d} m_\lambda(a_1, \dots, a_n) = h_d(a_1, \dots, a_n).$$

It follows that $\chi(\Gamma^d) = h_d$, thus $\chi(\Gamma^\lambda) = h_\lambda$.

As a consequence, we get in particular, that the characters of the Schur functor corresponding to a partition λ is the Schur function corresponding to the same partition

$$\chi(\mathbb{S}_\lambda) = s_\lambda.$$

From Proposition 4.1 and Theorem 3.9, it follows that χ also preserves *internal products*

Corollary 4.4. *The characteristic map χ sends the internal tensor product of strict polynomial functors to the Kronecker product of symmetric functions, i.e.*

$$\chi(X \otimes_{\Gamma_k^d} Y) = \chi(X) * \chi(Y).$$

□

5 Cauchy filtration

5.1 History and development

The Cauchy Decomposition first appeared in characteristic zero (cf. [24, (1.5.1)]), with the name *Cauchy Formula*, in the theory of symmetric functions as the expansion

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(\underline{x}) s_\lambda(\underline{y}) \tag{5.1}$$

where λ runs over all partitions and s_λ is the Schur function corresponding to λ. It is considered due to Cauchy, although he does not state the formula explicitly. However, it follows easily from Cauchy's work [4], as we describe in more detail later.

We have seen that symmetric functions give the *formal characters* of strict polynomial functors and that polynomial functors are strongly related to Schur algebras and representations of general linear groups. Thus, (5.1) can be generalized in various ways, from which the expansion above can be reobtained by computation of characters in characteristic 0 and by recalling some standard connections between symmetric functions (cf. Section 5.3).

A common generalization of (5.1) is the decomposition of the symmetric algebra of a tensor product of two k-modules, where k is an arbitrary commutative ring. Namely, it is possible to give a natural filtration of $S(V \otimes W)$, whose *associated graded object* is $\bigoplus_\lambda \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W)$ (cf. [1, Theorem III.1.4]), which gives rise to a direct sum decomposition if k is a field of characteristic 0, since the products $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W)$ are irreducible $\mathrm{GL}(V) \times \mathrm{GL}(W)$-modules.

Moreover, if we recall that $(\Gamma^d)^\circ \cong S^d$, for any k-module V, we can rewrite the *Cauchy filtration* in terms of divided powers, where the terms of the associated graded object are given by Weyl functors (cf. [22]).

The symmetric algebra $S(V)$ of an n-dimensional k-module V is isomorphic to the polynomial ring $k[x_1, \ldots, x_n]$. Thus if V and W are k-modules of dimension m and n respectively, $S(V \otimes W)$ can be seen as the polynomial ring $R = k[X_{ij}]$ in mn indeterminates, where we can regard at X_{ij} as a matrix. Hence, (5.1) can be also be generalized as a decomposition of R.

As a decomposition of R in characteristic 0, the Cauchy decomposition can already be followed from Schur's thesis (cf. Introduction of [7]), where a complete classification of the irreducible $\mathrm{GL}(n, k)$-modules is given and their characters are computed. However, the decomposition has not been stated by Schur.

5.1.1 Cauchy filtration for regular function on matrices

Doubilet, Rota and Stein [11] prove in 1974 a first characteristic-free decomposition of $R = k[X_{ij}]$, which they call *straightening formula*. They show that the double standard tableaux form a k-free basis for the polynomial ring R, where they define double tableaux as products of minors of matrices.

More precisely, let $(S|T)$ be a double tableau, with $S, T \in \mathrm{Tab}_\lambda(n)$. If $X = (X_{ij})$ is an $n \times m$ matrix with $n \leq m$ and if S and T have at most n columns, then we associate to $(S|T)$ the product of minors of X whose i-th factor is the minor involving the rows $S(i,1), S(i,2), \ldots, S(i,\lambda_i)$ and the columns $T(i,1), \ldots T(i,\lambda_i)$. Thus, the i-th factor is a minor of order λ_i.

If S, S' are tableaux, of possibly different form, we write $S \leq S'$ if and only if for all p, q the first p rows of S contain fewer occurrences of integers $\leq q$ than the corresponding rows of S', i.e.

$$\mathsf{card}\{(i,j)|i \leq p \text{ and } S(i,j) \leq q\} \leq \mathsf{card}\{(i,j)|i \leq p \text{ and } S'(i,j) \leq q\}.$$

The last tableau in this order C_λ has entries $1, \ldots, \lambda_i$ in the ith row, for any i, and is called *canonical tableau* of shape λ.

We can partially order the double tableaux of a fixed shape λ by setting $(S|T) \leq (S'|T')$ if $S \leq S'$ and $T \leq T'$. The content of a double tableau $c(S|T)$ is the pair $(c(S), c(T))$. If we now see a double tableau as a product of minors of a matrix, as described above, the Straightening formula is, for each non-standard double tableau $\mathcal{M} \in R$ an expression of the form $\mathcal{M} = \sum n_i \mathcal{M}_i$ with $n_i \in \mathbb{Z}$ and $c(\mathcal{M}) = c(\mathcal{M}_i)$. Since there are only finitely many double tableaux of a given content, it follows by induction that every double tableau is a linear combination of standard double tableaux. Moreover,

Theorem 5.1 (Doubilet-Rota-Stein)**.** *The double standard tableaux form a k-free basis for the polynomial ring $R = k[X_{ij}]$.*

Some years later, in 1980, De Concini, Eisenbud and Procesi [7] exploit the methods of [11] to give a characteristic free decomposition of R, which holds up to filtration, into $G = \mathrm{GL}(n,k) \times \mathrm{GL}(m,k)$ modules

$$R \cong \sum L_\sigma \otimes_k {}_\sigma L,$$

where the action is given by $A^{-1}X_{ij}B$, for $(A,B) \in G$ and the modules L_σ are a characteristic-free version of the distinct irreducible polynomial representations of general linear groups, indexed by partitions and given by Schur in his Thesis for k a field of characteristic zero.

Denote by A_λ the k-linear span of the double tableaux of shape $\geq \lambda$. Since one clearly has $A_{(1,\ldots,1)} = R$, this yields a filtration

$$A_{(d)} \subseteq A_{(d-1,1)} \subseteq \ldots \subseteq A_{(1,\ldots,1)} = R$$

where we consider the lexicographic order on partitions and write λ^+ for the successor of a given partition λ. The double standard tableaux of shape $\geq \lambda$ form a basis for A_λ ([7, Corollary 2.3]). We say that a double tableau is right (resp. left) *semicanonical* if it has the form $(A|C_\lambda)$ (respectively $(C_\lambda|B)$). Let L_λ and ${}_\lambda L$ be the spaces spanned respectively by all right and left semicanonical tableaux of shape λ. One can show that L_λ is a $\mathrm{GL}(n,k)$ submodule of R, respectively ${}_\lambda L$ is a $\mathrm{GL}(m,k)$ submodule of R. We have

Theorem 5.2 (De Concini-Eisenbud-Procesi). *L_λ (resp. ${}_\lambda L$) has a k-basis consisting of all right (resp. left) semicanonical standard tableaux. Moreover, there is a G-isomorphism*

$$L_\lambda \otimes_k {}_\lambda L \to A_\lambda / A_{\lambda^+}.$$

Thus the associated graded object of the filtration of R by the ideals A_λ is $\sum_\lambda L_\lambda \otimes_k {}_\lambda L$. If k is a field of characteristic 0, for any partition λ, the G-module $L_\lambda \otimes_k {}_\lambda L$ is irreducible and the above filtration gives a direct sum decomposition of the G-module R into irreducible submodules.

5.1.2 Cauchy filtration for the symmetric algebra

Almost simultaneously, Akin, Buchsbaum and Weyman [1] prove the same result by using a different language. They prove a decomposition of the symmetric algebra in terms of Schur functors which holds up to filtration in positive characteristic. To do this, they define a natural pairing

$$\langle -, - \rangle : \Lambda^\lambda V \otimes \Lambda^\lambda W \to S^d(V \otimes W),$$

where λ is a partition of d, which gives a natural filtration

$$0 \subseteq M_{(d)} \subseteq M_{(d-1,1)} \subseteq \ldots \subseteq M_{(1,\ldots,1)} = S_d(V \otimes W),$$

where $M_\lambda = \sum_{\mu \geq \lambda} \langle \Lambda^\mu V, \Lambda^\mu W \rangle$. Again, we consider the lexicographic order on partitions. The following result is proved in [1, III.1.4].

Theorem 5.3 (Akin-Buchsbaum-Weyman). *For any partition λ, there is an isomorphism*

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\lambda W \to M_\lambda / M_{\lambda^+},$$

hence the associated graded object of the filtration $\{M_\lambda\}$ is

$$\bigoplus_{|\lambda|=d} \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W).$$

It seems that the Cauchy filtration in terms of divided powers has been first stated by Hashimoto and Kurano in 1992 in [19], where it is obtained as a corollary of a version of the decomposition of symmetric algebras given in [1] extendeded to chain complexes [19, Theorem III.2.7, Corollary III.2.9].

5.1.3 Cauchy filtration as a good filtration for Schur algebras

The Cauchy filtration, in terms of strict polynomial functors, is one of the properties that make $\mathsf{Rep}\,\Gamma_k^d$ an *highest weight category*. The module category of an algebra A is an highest weight category if and only if A is a *quasi-hereditary algebra* [5]. Thus by Proposition 2.9, to prove the highest weight structure for strict polynomial functors one can show that the Schur algebra $S_k(n,d)$ is quasi hereditary. This has been first proved by S. Donkin [9], [8] some years before the name was coined. He works over an arbitrary commutative ring k and defines a k-algebra $S_k(\pi)$ for each finite saturated set π of dominant weights of a semisimple, complex, finite dimensional Lie algebra $\mathfrak{g}$, which is free of finite rank over k. For some particular choices of π and $\mathfrak{g}$ one obtains the Schur algebra. These algebras are all quasi-hereditary. The filtration of $S^d(V\otimes W)$ is the equivalent in this context to show that the injective $S_k(\pi)$-modules have a *good filtration*. This was already proved in [8, Theorem 2.6 and Remark (2), p.7] in the category of rational modules for a a semisimple, simply connected affine algebraic group G over an algebraically closed field of prime characteristic p, and is showed more explicitly for generalized Schur algebras in [9, 2.2h], for k an arbitrary ring. There are several other proofs of the Schur algebra being quasi-hereditary. R. Parshall [28, Section 41] proves that $S_k(n,d)$ is quasi-hereditary for k an algebraically closed field (1989), and in [6, Theorem 3.7.21] the same is done for an arbitrary noetherian commutative ring R (1990). The same result is proved by Green [17, Theorem 7.1] in 1992 with combinatorial methods.

In this section, we will discuss the Cauchy filtration for $\Gamma(V\otimes W)$, as described in [22], and prove it by methods similar to [1]. By computing characters, we will then find the Cauchy Formula (5.1). In the last section we discuss how the Cauchy Formula can be obtained from Cauchy's work.

5.2 Cauchy filtration for divided powers

Let $V,W\in \mathsf{P}_k$. For any non negative integer d there is a unique map

$$\psi^d_{V,W}:\Gamma^dV\otimes\Gamma^dW\to\Gamma^d(V\otimes W)$$

making the following square commutative (recall Proposition 2.1),

$$\begin{array}{ccc}
\Gamma^dV\otimes\Gamma^dW & \xrightarrow{\psi^d_{V,W}} & \Gamma^d(V\otimes W)\\
\downarrow & & \downarrow\\
V^{\otimes d}\otimes W^{\otimes d} & \xrightarrow{\sim} & (V\otimes W)^{\otimes d}
\end{array}$$

hence, we have a natural transformation of functors

$$\psi^d : \Gamma^d(-) \otimes \Gamma^d(-) \to \Gamma^d(- \otimes -).$$

Assume that V, W are free and fix bases $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$, respectively. Consider two compositions $\mu \in \Lambda(m, d)$, $\nu \in \Lambda(n, d)$, so that (cf. (2.9))

$$v_\mu \otimes w_\nu \in \Gamma^d V \otimes \Gamma^d W.$$

The tensor product $v_\mu \otimes w_\nu = \prod_{i=1}^m v^{\otimes \mu_i} \otimes \prod_{j=1}^n w^{\otimes \nu_i}$ is the sum of all possible different expressions of the form $(v_{i_1} \otimes \dots \otimes v_{i_d}) \otimes (w_{j_1} \otimes \dots \otimes w_{j_d})$, in which v_i occurs μ_i times and w_j occurs ν_j times, for any $i = 1, \dots, m$ and $j = 1, \dots n$. This is *rearranged* by $\psi^d_{V,W}$ to give $(v_{i_1} \otimes w_{j_1}) \otimes \dots \otimes (v_{i_d} \otimes w_{j_d})$. Let a_{ij} be the number of times $v_i \otimes w_j$ appears in the last product. The matrix $A = (a_{ij})$ formed in this way is an element of A^μ_ν and all distinct permutations of $(v_{i_1} \otimes w_{j_1}) \otimes \dots \otimes (v_{i_d} \otimes w_{i_d})$ that give rise to the same matrix occur as a summand of $\psi^d_{V,W}(v_\mu \otimes w_\nu)$. Thus it is not difficult to see that

$$\psi^d_{V,W}(v_\mu \otimes w_\nu) = \sum_{(a_{ij}) \in A^\nu_\mu} \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (v_i \otimes w_j)^{\otimes a_{ij}}. \tag{5.2}$$

For a partition $\lambda \in \Lambda^+(r, d)$ we can extend $\psi^d_{V,W}$ to a map

$$\psi^\lambda_{V,W} : \ \Gamma^\lambda V \otimes \Gamma^\lambda W \to \Gamma^d(V \otimes W)$$

given by the composite

$$\Gamma^\lambda V \otimes \Gamma^\lambda W \xrightarrow{\sim} (\Gamma^{\lambda_1} V \otimes \Gamma^{\lambda_1} W) \otimes \cdots \otimes (\Gamma^{\lambda_r} V \otimes \Gamma^{\lambda_r} W)$$

$$\xrightarrow{\psi^{\lambda_1}_{V,W} \otimes \dots \otimes \psi^{\lambda_r}_{V,W}} \Gamma^{\lambda_1}(V \otimes W) \otimes \cdots \otimes \Gamma^{\lambda_r}(V \otimes W) \to \Gamma^d(V \otimes W)$$

where the last map is given by the multiplication.

We write ψ^λ for the corresponding natural transformation

$$\psi^\lambda : \Gamma^\lambda(-) \otimes \Gamma^\lambda(-) \to \Gamma^d(- \otimes -).$$

Recall that, if V, W are free $\Gamma^\lambda V$ and $\Gamma^\lambda W$ are also free and have a k-basis indexed by $\mathrm{Tab}^{rc}_\lambda(m)$ and $\mathrm{Tab}^{rc}_\lambda(n)$ respectively. An element of a basis of $\Gamma^\lambda V \otimes \Gamma^\lambda W$ is given by $v_S \otimes w_T$, where $v_S \in \Gamma^\lambda V$ and $w_T \in \Gamma^\lambda W$ are basis elements, $S \in \mathrm{Tab}^{rc}_\lambda(m)$, $T \in \mathrm{Tab}^{rc}_\lambda(n)$.

We will also say that the double tableaux $(S|T)$, with $S \in \mathrm{Tab}^{rc}_\lambda(m)$ and $T \in \mathrm{Tab}^{rc}_\lambda(n)$, are a basis for $\Gamma^\lambda V \otimes \Gamma^\lambda W$. We will say that a double tableau

$(S|T)$ has some property (e.g. is costandard), if both S and T satisfy it (e.g. they are both costandard). We can give the following description of $\psi^{\lambda}_{V,W}$ with respect to these bases. Let

$$v_S = v_{\alpha^1} \otimes \ldots \otimes v_{\alpha^r} \quad \text{and} \quad w_T = w_{\beta^1} \otimes \ldots \otimes w_{\beta^r}.$$

Then $v_S \otimes w_T$ is sent by the first map of the above composition to

$$(v_{\alpha^1} \otimes w_{\beta^1}) \otimes \ldots \otimes (v_{\alpha^r} \otimes w_{\beta^r}).$$

By expressing the image through ψ^{λ_i} of every tensor factor $(v_{\alpha^i} \otimes w_{\beta^i})$ as in (5.2) and multiplying, one has

$$\psi^{\lambda}_{V,W}(v_S \otimes w_T) = \prod_{l=1}^{r} \left(\sum_{(a_{ij}) \in A^{\alpha^l}_{\beta^l}} \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (v_i \otimes w_j)^{\otimes a_{ij}} \right). \tag{5.3}$$

Remark 5.4. For every partition λ, the map $\psi^{\lambda}_{V,W}$ is a k-linear map and the number of times a basis element v_i (resp. w_i) occurs in every summand of $\psi^{\lambda}_{V,W}(v_S \otimes w_T)$ equals the number of entries of S (resp. T) which are equal to i. We will say that $\psi^{\lambda}_{V,W}$ *preserves the content* of the double tableau $(S|T)$.

Consider the *lexicographic order* on the set of partitions of weight d. For a partition λ, denote by λ^+ its immediate successor and by λ^- its immediate precedessor. Set $(1, \ldots, 1)^- = -\infty$ and $(d)^+ = +\infty$. Let

$$F_{\lambda}(V, W) = \sum_{\mu \ge \lambda} \operatorname{Im} \psi^{\mu}_{V,W},$$

so that we have a functor

$$F_{\lambda} : \Gamma^d \mathsf{P}_k \otimes \Gamma^d \mathsf{P}_k \to \mathbf{M}_k.$$

Lemma 5.5. *For any partition λ, the functor F_{λ} is* universal*.*

Proof. The natural transformation ψ^{λ} is universal. Indeed, $\Gamma^{\lambda}(-) \otimes \Gamma^{\lambda}(-)$ and $\Gamma^d(- \otimes -)$ are universal functors, hence they are defined over any ring k and so is ψ^{λ}. Write $\Gamma^{\lambda}_k \otimes \Gamma^{\lambda}_k$, $\Gamma^d_k(- \otimes -)$ and $\psi^{\lambda,k}$ for the functors and the natural transformation defined over a ring k. Moreover, let $\phi : k \to \ell$ be a ring homomorphism and, if V and W are k-modules, write V_{ℓ} for $\ell \otimes V$, resp. W_{ℓ} for $\ell \otimes W$.

By the definition of ψ^λ, the diagram

$$\begin{array}{ccc} \ell \otimes (\Gamma^\lambda_k V \otimes \Gamma^\lambda_k W) & \xrightarrow{\ell\otimes\psi^{\lambda,k}_{V_k,W_k}} & \ell \otimes \Gamma^d_k(V \otimes W) \\ \downarrow \cong & & \downarrow \cong \\ \Gamma^\lambda_\ell V_\ell \otimes \Gamma^\lambda_\ell W_\ell & \xrightarrow{\psi^{\lambda,\ell}_{V_\ell,W_\ell}} & \Gamma^d_k(V_\ell \otimes W_\ell) \end{array}$$

is commutative.

It follows now from Remark 2.20, that $F^k_\lambda = \sum_{\mu\geq\lambda} \operatorname{Im}\psi^{\lambda,k}$ is a universal functor. □

Corollary 5.6. *For any partition λ, the functor F_λ/F_{λ^+} is universal.*

Proof. Consider the following map

$$\Psi^\lambda = \sum_{\mu>\lambda} \psi^\mu : \bigoplus_{\mu>\lambda} \Gamma^\mu(-) \otimes \Gamma^\mu(-) \to F_\lambda.$$

By Lemma 5.5, Ψ^λ is universal. since we have $F_\lambda/F_{\lambda^+} \cong \operatorname{Coker}\Psi^\lambda$, the statement follows from Remark 2.20. □

Definition 5.7. The *Cauchy filtration* is the chain

$$0 = F_\infty \subset F_{(d)} \subset F_{(d-1,1)} \subset \cdots \subset F_{(2,1,\ldots,1)} \subset F_{(1,\ldots,1)} = \Gamma^d(- \otimes -)$$

Our next goal is to show that the *associated graded object* of the Cauchy filtration is given by

$$\bigoplus_{|\lambda|=d} \mathbb{W}_\lambda \otimes \mathbb{W}_\lambda, \tag{5.4}$$

that is, for any λ partition of d and any $V, W \in \mathsf{P}_k$, one has

$$F_\lambda/F_{\lambda^+}(V,W) \cong \mathbb{W}_\lambda V \otimes_k \mathbb{W}_\lambda W.$$

To do this, we will need some technical results.

Lemma 5.8. *For any $V, W \in \mathsf{P}_k$ and any $\lambda \in \Lambda^+(r,d)$, $\mathbb{W}_\lambda V \otimes \mathbb{W}_\lambda W \neq 0$ implies $F_\lambda/F_{\lambda^+}(V,W) \neq 0$.*

Proof. Let V, W be free and let $\{v_1,\ldots,v_m\}$ and $\{w_1,\ldots,w_n\}$ be ordered bases of V and W, respectively. By Theorem 2.21, from $\mathbb{W}_\lambda V \otimes \mathbb{W}_\lambda W \neq 0$ it follows $r \leq \min\{m,n\}$, so that we can consider $v^{\otimes\lambda} \otimes w^{\otimes\lambda} \in \Gamma^\lambda V \otimes \Gamma^\lambda W$. We have

$$\psi^\lambda_{V,W}(v^{\otimes\lambda} \otimes w^{\otimes\lambda}) = (v_1 \otimes w_1)^{\otimes\lambda_1} \ldots (v_r \otimes w_r)^{\otimes\lambda_r}$$

which is clearly a nonzero element of $F_\lambda(V,W)$. We want to show that this element is not contained in $F_{\lambda^+}(V,W)$.

Suppose $(v_1\otimes w_1)^{\otimes\lambda_1}\dots(v_r\otimes w_r)^{\otimes\lambda_r}\in F_{\lambda^+}(V,W)$, then it is contained in $\operatorname{Im}\psi^{\mu}_{V,W}$, for some $\mu>\lambda$, $\mu\in\Lambda^+(s,d)$. In particular, by Remark 5.4, one can find tableaux S and T of content λ, so that we have $S,T\in\operatorname{Tab}^{rc}_{\mu}(r)$, such that

$$\psi^{\mu}_{V,W}(v_S\otimes w_T)=(v_1\otimes w_1)^{\otimes\lambda_1}\dots(v_r\otimes w_r)^{\otimes\lambda_r}.$$

If $v_S=v_{\alpha^1}\otimes\dots\otimes v_{\alpha^s}$ and $w_T=w_{\beta^1}\otimes\dots\otimes w_{\beta^s}$, this becomes by (5.3)

$$\prod_{l=1}^{s}\left(\sum_{(a_{ij})\in A^{\alpha^l}_{\beta^l}}\prod_{\substack{1\le i\le m\\1\le j\le n}}(v_i\otimes w_j)^{\otimes a_{ij}}\right)=(v_1\otimes w_1)^{\otimes\lambda_1}\dots(v_r\otimes w_r)^{\otimes\lambda_r}.$$

By comparing the two expression, one has $a_{ij}=0$ if $i\neq j$, for every $(a_{ij})\in A^{\alpha^l}_{\beta^l}$, and any $l=1,\dots,s$, that is, all matrices occurring in the sum are diagonal. Since the sums of rows and columns are fixed, we must have for every l, $\alpha^l=\beta^l$ and $A^{\alpha^l}_{\beta^l}=\{\operatorname{diag}(\alpha^l_1,\dots,\alpha^l_r)\}$, so that $T=S$. This is only the case when $\alpha^l=\beta^l$ has only one non-zero entry. Indeed, if $\alpha^l_i,\alpha^l_j\geq 1$, for some $i<j$, one has

$$\operatorname{diag}(\alpha^l_1,\dots,\alpha^l_i-1,\dots\alpha^l_j-1,\dots,\alpha^l_r)+E_{ij}+E_{ji}\in A^{\alpha^l}_{\beta^l},$$

which is not diagonal.

It follows that $\alpha^l=\beta^l$ has one non-zero entry, equal to μ_l. This means that, for any $1\le l\le s$, all entries of the l-th row of $S=T$ are equal, which is not possible, because $\mu\geq\lambda$ and λ equals the content of T.

This shows that $(v_1\otimes w_1)^{\otimes\lambda_1}\cdot\ldots\cdot(v_r\otimes w_r)^{\otimes\lambda_r}$ is contained in $F_\lambda(V,W)$ but not in $F_{\lambda^+}(V,W)$, therefore $F_\lambda/F_{\lambda^+}(V,W)\neq 0$. □

Lemma 5.9. *Let $V,W\in\mathsf{P}_k$. Let $\lambda\in\Lambda^+(r,d)$ be a fixed partition. For $1\le i\le r$ and $1\le t\le\lambda_{i+1}$, the composition*

$$\Gamma^{\lambda(i,t)}V\otimes\Gamma^{\lambda(i,t)}W\xrightarrow{\gamma_{A(i,t)}V\otimes\gamma_{A(i,t)}W}\Gamma^{\lambda}V\otimes\Gamma^{\lambda}W\xrightarrow{\psi^{\lambda}_{V,W}}\Gamma^d(V\otimes W)$$

where $\lambda(i,t)$ is the partition

$$\lambda(i,t)=(\lambda_1,\dots,\lambda_{i-1},\lambda_i+t,\lambda_{i+1}-t,\lambda_{i+2},\dots,\lambda_r)$$

and $\gamma_{A(i,t)}:\Gamma^{\lambda(i,t)}\to\Gamma^{\lambda}$ *is the standard morphism given by the matrix*

$$A(i,t)=\operatorname{diag}(\lambda_1,\dots,\lambda_r)+tE_{i+1,i}-tE_{i+1,i+1},$$

equals a multiple of $\psi^{\lambda(i,t)}_{V,W}$.

Proof. Recall the presentation (2.16)

$$\bigoplus_{i\geq 1}\bigoplus_{t=1}^{\lambda_{i+1}}\Gamma^{\lambda(i,t)}\xrightarrow{\ \alpha\ }\Gamma^{\lambda}\longrightarrow\mathbb{W}_{\lambda}\longrightarrow 0,$$

of $\mathbb{W}_{\lambda}$.

Let V, W be free and fix $\{v_1,\ldots,v_m\}$ and $\{w_1,\ldots,w_n\}$ ordered bases of V and W, respectively. Assume $r\leq\min\{m,n\}$. Write $v^{\otimes\lambda}$ for the element

$$v^{\otimes\lambda}=v_1^{\otimes\lambda_1}\otimes\ldots\otimes v_r^{\otimes\lambda_r}.$$

Define $w^{\otimes\lambda}$ analogously. The basis element

$$v^{\otimes\lambda(i,t)}\otimes w^{\otimes\lambda(i,t)}\in\Gamma^{\lambda(i,t)}V\otimes\Gamma^{\lambda(i,t)}W$$

is sent by $\gamma_{A(i,t)}V\otimes\gamma_{A(i,t)}W$ to

$$(v_1^{\otimes\lambda_1}\otimes\cdots\otimes v_{i-1}^{\otimes\lambda_{i-1}}\otimes v_i^{\otimes\lambda_i}\otimes v_i^{\otimes t}v_{i+1}^{\otimes\lambda_{i+1}-t}\otimes v_{i+2}^{\lambda_{i+2}}\otimes\cdots\otimes v_r^{\otimes\lambda_r})\otimes$$

$$(w_1^{\otimes\lambda_1}\otimes\cdots\otimes w_{i-1}^{\otimes\lambda_{i-1}}\otimes w_i^{\otimes\lambda_i}\otimes w_i^{\otimes t}w_{i+1}^{\otimes\lambda_{i+1}-t}\otimes w_{i+2}^{\lambda_{i+2}}\otimes\cdots\otimes w_r^{\otimes\lambda_r}).$$

The image of this under $\psi^{\lambda}_{V,W}$ is

$$\prod_{l=1}^{i-1}(v_l\otimes w_l)^{\otimes\lambda_l}(v_i\otimes w_i)^{\otimes\lambda_i}(v_i\otimes w_i)^{\otimes t}(v_{i+1}\otimes w_{i+1})^{\otimes\lambda_{i+1}-t}\prod_{l=i+1}^{r}(v_l\otimes w_l)^{\otimes\lambda_l}.$$

We want to compare this expression with $\psi^{\lambda(i,t)}(v^{\lambda(i,t)}\otimes w^{\lambda(i,t)})=$

$$\prod_{l=1}^{i-1}(v_l\otimes w_l)^{\otimes\lambda_l}(v_i\otimes w_i)^{\otimes\lambda_i+t}(v_{i+1}\otimes w_{i+1})^{\otimes\lambda_{i+1}-t}\prod_{l=i+1}^{r}(v_l\otimes w_l)^{\otimes\lambda_l}.$$

We notice that all but one factor of the both expressions agree. Namely, the product $(v_i\otimes w_i)^{\otimes\lambda_i}(v_i\otimes w_i)^{\otimes t}$ in the first expression, is replaced by the factor $(v_i\otimes w_i)^{\otimes\lambda_i+t}$ in the second one. Now, from the Remark (2.12) it follows that the composition $\psi^{\lambda}_{V,W}\circ(\gamma_{A(i,t)}V\otimes\gamma_{A(i,t)}W)$ is an integer multiple of $\psi^{\lambda(i,t)}_{V,W}$. □

We give a last general results, that can be found in [33], for the case $k=\mathbb{C}$.

Lemma 5.10. *Let k be a field and let G and H be groups, not necessarily finite. Let X be a finite dimensional kG-module and Y a finite dimensional kH-module. Then the tensor product $X\otimes_k Y$ is an irreducible $G\times H$-module if and only if X and Y are irreducible.*

Proof. It is clear that, if X or Y is reducible, then so is $X \otimes Y$, so we have to prove only the "only if" part. As a G-module, the tensor product $X \otimes Y$ is isomorphic to $X^{\oplus n}$, where $n = \dim Y$. Since X is irreducible, we have $\mathrm{End}_G(X) \cong k$, $X^{\oplus n}$ is a completely reducible G-module and its submodules are of the form $X^m \cong X \otimes k^m$, with $m \leq n$, that is G-submodules of $X \otimes Y$ are of the form $X \otimes Y'$, where Y' is a k-subspace of Y. If $X \otimes Y'$ also has a H-module structure, we must have $X \otimes Y' = X \otimes Y$, since by the irreducibility of Y we have $H \cdot Y' = Y$. □

Theorem 5.11. *Fix a commutative ring k and let $V, W \in \mathsf{P}_k$. Let $\lambda \in \Lambda^+(r,d)$ be a fixed partition. Then the morphism*

$$\psi^\lambda : \Gamma^\lambda V \otimes \Gamma^\lambda W \to F_\lambda(V,W)$$

induces an isomorphism

$$\mathbb{W}_\lambda V \otimes \mathbb{W}_\lambda W \xrightarrow{\sim} F_\lambda / F_{\lambda^+}(V,W), \tag{5.5}$$

which is functorial in V and W with respect to morphisms in $\Gamma^d \mathsf{P}_k$.

Proof. Assume V, W free of dimension m and n respectively. As a first step, we want to show that there is a morphism $\bar{\psi}^\lambda_{V,W}$ making the following square commutative.

$$\begin{array}{ccc} \Gamma^\lambda V \otimes \Gamma^\lambda W & \xrightarrow{\psi^\lambda_{V,W}} & F_\lambda(V,W) \\ \downarrow p & & \downarrow q \\ \mathbb{W}_\lambda V \otimes \mathbb{W}_\lambda W & \xrightarrow{\bar{\psi}^\lambda_{V,W}} & F_\lambda/F_{\lambda^+}(V,W) \end{array} \tag{5.6}$$

From the presentation (2.16), we have

$$\mathrm{Im}(\gamma_{A(i,t)} V \otimes \gamma_{A(i,t)} W) \subseteq \ker(p).$$

By Lemma 5.9, it holds

$$\mathrm{Im}(\psi^\lambda_{V,W} \circ (\gamma_{A(i,t)} V \otimes \gamma_{A(i,t)} W)) \subseteq \mathrm{Im}\, \psi^{\lambda(i,t)}_{V,W} \subseteq F_{\lambda^+}(V,W),$$

where the last inclusion follows from $\lambda(i,t) > \lambda$, for any i, t.

Hence $\psi^\lambda(\ker(p)) \subseteq \ker(q)$. This yields $\bar{\psi}^\lambda_{V,W}$.

Now we want to prove that $\bar{\psi}^\lambda_{V,W}$ is an isomorphism. From the definition of $F_\lambda(V,W)$ it follows that

$$F_\lambda/F_{\lambda^+}(V,W) \cong \mathrm{Im}\, \psi^\lambda_{V,W} / (F_{\lambda^+}(V,W) \cap \mathrm{Im}\, \psi^\lambda_{V,W}),$$

thus the composition

$$\Gamma^\lambda V \otimes \Gamma^\lambda W \xrightarrow{\psi^\lambda_{V,W}} F_\lambda(V,W) \xrightarrow{q} F_\lambda/F_{\lambda^+}(V,W)$$

is surjective and, by the commutativity of the diagram (5.6), so is $\bar{\psi}^\lambda_{V,W}$. Of course we can define $\bar{\psi}^{\lambda,k}$ for any ring k, because ψ^λ is universal. Since the above discussion is independent of the choice of the ring k, $\bar{\psi}^{\lambda,k}$ is an epimorphism for any ring k.

We want to show that $\bar{\psi}^\lambda$ is a universal natural transformation. Indeed, from Theorem 2.21 and Corollary 5.6, we know that both $\mathbb{W}_\lambda \otimes \mathbb{W}_\lambda$ and F_λ/F_{λ^+} are universal functors. Moreover, from the proof of Theorem 2.21, we know that the p and $\alpha\otimes\alpha$ are universal natural tranformations, where α is the map appearing in the presentation (2.16).

We need to check the commutativity of the diagram (2.14), that is

$$\bar{\psi}^{\lambda,\ell}_{V_\ell,W_\ell} \cong \ell \otimes \bar{\psi}^{\lambda,k}_{V,W}$$

for any ring homomorphism $\phi : k \to \ell$ and any k-modules V,W. Write p_ℓ and q_ℓ for the projections given in the diagram (5.6), defined for a ring ℓ. Since, from Lemma 5.5, we have

$$\ell \otimes \ker q_k = \ell \otimes F^k_{\lambda^+}(V,W) = F^\ell_{\lambda^+}(V_\ell,W_\ell) = \ker q_\ell,$$

we only have to show that

$$\ell \otimes \psi^{\lambda,k}_{V,W}(\ker p_k) = \ker p_\ell.$$

We have $\ell\otimes\psi^{\lambda,k}_{V,W}(\ker p_k) = \ell\otimes\psi^{\lambda,k}_{V,W}(\operatorname{Im}\alpha^k_V\otimes\alpha^k_W) = \ell\otimes\operatorname{Im}(\psi^{\lambda,k}_{V,W}\circ\alpha^k_V\otimes\alpha^k_W)$ $= \operatorname{Im}(\psi^{\lambda,\ell}_{V_\ell,W_\ell}\circ\alpha^\ell_V\otimes\alpha^\ell_W)$ where the last equality follows from Remark 2.20. Since $\operatorname{Im}(\psi^{\lambda,\ell}_{V_\ell,W_\ell}\circ\alpha^\ell_V\otimes\alpha^\ell_W) = \ker p_\ell$, this proves the universality of $\bar{\psi}^\lambda$.

Suppose $\mathbb{W}_\lambda V\otimes\mathbb{W}_\lambda W \neq 0$, by Lemma 5.8 we have $F_\lambda/F_{\lambda^+}(V,W)\neq 0$. If we take $k=\mathbb{Q}$, then $\mathbb{W}_\lambda V\otimes\mathbb{W}_\lambda W$ is an irreducible $\mathrm{GL}(V)\times\mathrm{GL}(W)$-module for any partition λ, by Lemma 5.10. Hence $\bar{\psi}^{\lambda,\mathbb{Q}}$ is forced to be an isomorphism, because we already know that it is an epimorphism. Because of the universality, we have $0=\ker\bar{\psi}^{\lambda,\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}}\ker\bar{\psi}^{\lambda,\mathbb{Z}}$ and, since a submodule of a free $\mathbb{Z}$-module is free (cf. [23], Appendix 2), this implies $\ker\bar{\psi}^{\lambda,\mathbb{Z}} = 0$. It follows that $\bar{\psi}^{\lambda,\mathbb{Z}}$ is a monomorphism and hence an isomorphism. In particular, it follows that F_λ/F_{λ^+} is universally free. Then $\bar{\psi}^{\lambda,\mathbb{Z}}$ is a natural isomorphism between universally free functors, hence by Lemma 2.19, $\bar{\psi}^{\lambda,k}$ is a natural isomorphism for any k and this concludes the proof.

□

5.3 Computing characters

Let k be a field of characteristic 0 and consider the Cauchy decomposition discussed in the last section

$$\Gamma(V \otimes W) = \bigoplus_{\lambda} \mathbb{W}_\lambda(V) \otimes \mathbb{W}_\lambda(W) \tag{5.7}$$

into irreducible $\mathrm{GL}(V) \times \mathrm{GL}(W)-$modules, where λ runs over all partitions and $\mathbb{W}_\lambda$ denotes the Weyl functor corresponding to λ.

We want to compute the characters of both sides of this formula, that is, we want to compute their image through the homomorphism χ, defined in (4.3.1).

Recall that, for a polynomial functor $X \in K(\mathfrak{F})$,

$$\chi(X)(a_1, \ldots, a_n) = \operatorname{trace} X((a))$$

where for $a = (a_1, \ldots, a_n) \in k^n$, $(a) = \operatorname{diag}(a)$ denotes the diagonal endomorphism of k^n with eigenvalues $(a_1, \ldots, a_n)$. It is a symmetric function of $(a_1, \ldots, a_n)$.

Consider the functor $X = \Gamma(- \otimes -)$ and let V and W be free k-modules with bases $\{v_1, \ldots v_m\}$ and $\{w_1, \ldots, w_n\}$, respectively. A diagonal endomorphism $(a) \otimes (b)$ of the free k-module $V \otimes W$ is given by a couple of diagonal endomorphisms $(a) = ((a_1, \ldots, a_m))$ and $(b) = ((b_1, \ldots, b_n))$ of V and W.

If we take, as a basis for $V \otimes W$,

$$\{z_{11}, z_{12}, \ldots, z_{1n}, z_{21}, z_{22}, \ldots, z_{mn}\},$$

where $z_{ij} = v_i \otimes w_j$, we may rewrite $(a) \otimes (b)$ as the diagonal endomorphism of $V \otimes W \cong k^{mn}$ given by the $mn-$tuple $(a_1b_1, a_1b_2, \ldots a_2b_1, a_2b_2, \ldots, a_mb_n)$.

Thus we have

$$\chi(\Gamma)(a_1b_1, \ldots, a_mb_n) = \operatorname{trace} \Gamma((a) \otimes (b)).$$

Recall that $\chi(\Gamma^d) = h_d$, where h_d denotes the d-th complete symmetric function. Now we have

$$\chi(\Gamma)(a_1b_1, \ldots, a_mb_n) = \sum_{d \geq 0} \chi(\Gamma^d)(a_1b_1, \ldots, a_mb_n) = \sum_{d \geq 0} h_d(a_1b_1, \ldots, a_mb_n).$$

Denote by $h_d(a \cdot b)$ the polynomial $h_d(a_1b_1, \ldots, a_mb_n)$.

Recall that, in characteristic 0 one has $\mathbb{W}_\lambda \cong \mathbb{S}_\lambda$, thus $\chi(\mathbb{W}_\lambda) = s_\lambda$. If we now consider both diagonal morphisms (a) and (b) on V and W separately again, we obtain

$$\chi(\bigoplus_\lambda \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W))(a \cdot b) = \sum_\lambda \chi(\mathbb{S}_\lambda)(a) \cdot \chi(\mathbb{S}_\lambda)(b) = \sum_\lambda s_\lambda(a) s_\lambda(b).$$

Thus, by passing to the characters, the formula (5.7) translates as

$$\sum_{d \geq 0} h_d(a \cdot b) = \sum_\lambda s_\lambda(a) s_\lambda(b), \tag{5.8}$$

where λ runs over all partitions of d.

Observe that

$$(1 - a_i b_j)^{-1} = \sum_{d \geq 0} (a_i b_j)^d. \tag{5.9}$$

If $a = \{a_1, \ldots a_m\}$ and $b = \{b_1, \ldots, b_n\}$ are finite sets of variables, then from the last equality it follows

$$\prod_{i,j} (1 - a_i b_j)^{-1} = \prod_{i,j} \Big(\sum_{d \geq 0} (a_i b_j)^d \Big) = \Big(\sum_{d \geq 0} (a_1 b_1)^d \Big) \cdots \Big(\sum_{d \geq 0} (a_m b_n)^d \Big).$$

A degree d term of this function will be of the form

$$(a_1 b_1)^{l_{1,1}} \cdot (a_1 b_2)^{l_{1,2}} \cdots (a_m b_n)^{l_{m,n}},$$

where $l_{i,j}$ are integers such that $\sum l_{i,j} = d$. It is easy to see that a monomial of the form $(ab)^\lambda = (a_1 b_1)^{\lambda_1} \cdots (a_m b_n)^{\lambda_{mn}}$ is a term of our function, for every sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_{mn})$ such that $\sum_{i=1}^{mn} = d$. It follows that the degree d part of $\prod_{i,j} (1 - a_i b_j)^{-1}$ is given by the sum of all monomials in the variables $a_1 b_1, \ldots, a_m b_n$ of total degree d. This is, by definition, the d-th complete symmetric function $h_d(a_1 b_1, \ldots, a_m b_n)$. Thus,

$$\sum_d h_d(a \cdot b) = \prod_{i,j} (1 - a_i b_j)^{-1}.$$

By inserting the last equality in (5.8), we find

$$\prod_{i,j} (1 - a_i b_j)^{-1} = \sum_\lambda s_\lambda(a) s_\lambda(b) \tag{5.10}$$

the classical *Cauchy Formula* for symmetric functions. Of course the formula can be proved directly in the theory of symmetric functions, as showed for example in [24, (1.5.1)].

5.4 Some history: from Chauchy's work to Cauchy Formula

The expansion of $\prod_{i,j}(1-a_ib_j)^{-1}$ in terms of Schur functions is universally attributed to Cauchy and is therefore called the *Cauchy Formula*, but in the work of Cauchy there is not a clear statement of this identity.

However, (5.10) follows easily from a work of Cauchy on alternating functions [4] and the *Cauchy-Binet Formula*, a result of linear algebra that expresses the determinant of a product of matrices in terms of the determinants of their minors. We recall them in the next sections.

5.4.1 The Cauchy-Binet formula

Let A and B be matrices of size $m \times n$ and $n \times m$ respectively, with $n \geq m$. Let $[n] = \{1, \dots, n\}$ and denote by $\binom{[n]}{m}$ the set of m-combinations of elements of $[n]$, that is subsets of $[n]$ with m elements. If $S \in \binom{[n]}{m}$, denote by $A_{[m],S}$ the $m \times m$ minor of A, given by the columns of A indexed by S. Similarly, let $B_{S,[m]}$ the minor of B given by rows indexed by S. Then we have

$$\det(AB) = \sum_{S \in \binom{[n]}{m}} = \det(A_{[m],S}) \det(B_{S,[m]}) \tag{5.11}$$

Cauchy and Binet came to the same formula by using different methods. The *Mémoirs* of the both authors where presented for publication separately, but on the same day. Although there are no documents conferming this, there is the presumption that they, knowing beforehand to have reached similar results, arrangenged in a friendly way for simultaneous publicity (See [27] for interesting historical discussions).

5.4.2 Cauchy's work - Mémoire sur le fonctions alternées et sur les sommes alternées

This is a brief summary of Cauchy's work in more modern language.

Consider a sequence of variables $\underline{x} = x_1, x_2, x_3, \dots, x_n$ and form the product $P_{\underline{x}} = \prod_{1 \leq i < j \leq n}(x_i - x_j)$. It is clear that $P_{\underline{x}}$ is an *alternating function* of the variables $x_1, x_2, \dots x_n$, i.e. it changes its sign every time we interchange two of the variables. It follows directly from the definition that an alternating function F of $\underline{x}$ vanishes if we put $x_i = x_j$ for any two variables in the sequence $\underline{x}$. Hence, if F is an *integer function*, i.e. a function that only involves integer powers of the variables, it must be algebraically divisible by each of the differences $(x_i - x_j)$, with $j > i$. It follows that $P_{\underline{x}}$ divides F.

A rational function of $\underline{x}$, whose denominator is a symmetric function and whose numerator is an alternating function, is clearly again an alternating function of $\underline{x}$. Let now $f(x_1, x_2, \dots, x_n)$ be an arbitary function. Consider the following sum

$$\mathbf{s}_f(x_1, \dots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \dots x_{\sigma(n)}).$$

We call it the *alternating sum* of f. It is easy to see that $\mathbf{s}_f$ is an alternating function of $x_1, \dots, x_n$. If f is an integer function, $\mathbf{s}_f$ will be integer too and, as before, it will be divisible by the product $P_{\underline{x}}$. If f is rational, so will be its alternating sum and one can write $\mathbf{s}_f = U/V$, where U and V are interger functions of $x_1, x_2, x_3, \dots, x_n$. If V is a symmetric function and it is divisible by the product of all denominators that appear in the sum $\mathbf{s}_f$, U must necessarily be an alternating function of $\underline{x}$. Therefore $P_{\underline{x}}$ divides U and we can write $U = P_{\underline{x}} W$ and $\mathbf{s} = P_{\underline{x}} \frac{W}{V}$, where W is an integer symmetric function of $x_1, x_2, x_3, \dots, x_n$. It follows that the alternating sum is a product of $P_{\underline{x}}$ and a rational symmetric function W/V.

As an application of this, consider

$$f(\underline{x}) = \prod_{i \geq 1}^{n} (x_i - y_i)^{-1}.$$

Clearly, if we consider the alternating sum $\mathbf{s}_f$ as above (note that we only sum over the permutations of the variables $x_1, \dots, x_n$), we can set $V = \prod_{i,j \geq 1}^{n} (x_i - y_j)$. Then U will be an integer function of $\underline{x} = x_1, \dots, x_n$ and $\underline{y} = y_1, \dots, y_n$. Therefore, U will be divisible by $P_{\underline{x}}$ and $P_{\underline{y}}$. Hence we have

$$U = c P_{\underline{x}} P_{\underline{y}} \quad \text{and} \quad \mathbf{s}_f(\underline{x}, \underline{y}) = \frac{c P_{\underline{x}} P_{\underline{y}}}{V},$$

where c is a constant or a symmetric function of $\underline{x}, \underline{y}$. Every product of the form $\prod_{j \geq 1} (x_i - y_j)$, for a fixed i, will have degree n, as a function of $\underline{x}$ and $\underline{y}$. It follows $n = \deg(V) - \deg(U)$. From $\deg(V) = n^2$ it follows now $\deg(U) = n^2 - n$. On the other hand, $\deg(P_{\underline{x}}) = \deg(P_{\underline{y}}) = \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$, thus the degree of $c = \frac{U}{P_{\underline{x}} P_{\underline{y}}}$ has to be zero and c is a constant.

To determine c one can put $x_i = y_i$ for $i = 1, \dots, n$, in $\mathbf{s}_f(\underline{x}, \underline{y}) = \frac{c P_{\underline{x}} P_{\underline{y}}}{V}$ reduced to the form $c P_{\underline{x}} P_{\underline{y}} = \mathbf{s}_g$, where $g = \frac{V}{\prod_i (x_i - y_i)}$. In this way one finds $c P_{\underline{y}}^2 = \frac{V}{\prod_i (x_i - y_i)}$ or, equivalently $c P_{\underline{y}}^2 = \prod_{i \neq j} (y_i - y_j) = (-1)^{\frac{n(n-1)}{2}} P_{\underline{y}}^2$. It follows $c = (-1)^{\frac{n(n-1)}{2}}$. We can now rewrite

$$\mathbf{s}_f = (-1)^{\frac{n(n-1)}{2}} \frac{P_{\underline{x}} P_{\underline{y}}}{V}. \tag{5.12}$$

5.4.3 How to deduce Cauchy Formula from this?

In nowadays terms, Cauchy computes a determinant, that can be seen as a double version of a Vandermonde's determinant. Namely, consider two sets of n indipendent variables $\underline{x} = x_1, x_2, \ldots, x_n$ and $\underline{y} = y_1, y_2, \ldots, y_n$. Writing the function $\mathbf{s}_f(\underline{x})$ from (5.12) explicitly yields

$$\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \ldots x_{\sigma(n)}) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \frac{1}{(x_{\sigma(i)} - y_i)}.$$

This is nothing else than the *Leibniz Formula* for calculating the $n \times n$ determinant of

$$W = \begin{pmatrix} 1/(x_1-y_1) & 1/(x_1-y_2) & 1/(x_1-y_3) & \ldots & 1/(x_1-y_n) \\ 1/(x_2-y_1) & 1/(x_2-y_2) & 1/(x_2-y_3) & \ldots & 1/(x_2-y_n) \\ 1/(x_3-y_1) & 1/(x_3-y_2) & 1/(x_3-y_3) & \ldots & 1/(x_3-y_n) \\ \vdots & \vdots & \vdots & & \vdots \\ \\ 1/(x_n-y_1) & 1/(x_1-y_2) & 1/(x_n-y_3) & \ldots & 1/(x_n-y_n) \end{pmatrix}$$

Observe now that the product $P_{\underline{x}} = \prod_{1\le i<j\le n}(x_i - x_j)$ is, up to a sign $(-1)^{\frac{n(n-1)}{2}}$, the determinant of the Vandermonde's matrix $V(x_1, \ldots, x_n)$. Analogously $P_{\underline{y}} = (-1)^{\frac{n(n-1)}{2}} \det(V(y_1, \ldots y_n))$. Thus (5.12) states the following identity, that is known from linear algebra

$$\det(W) = (-1)^{\frac{n(n-1)}{2}} \cdot \frac{\det(V(x_1, \ldots, x_n)) \cdot \det(V(y_1, \ldots, y_n))}{\prod_{1\le i,j\le n}(x_i - y_j)}. \tag{5.13}$$

If we now multiply both sides of (5.13) by $\frac{(x_1 \cdot x_2 \cdots x_n)^n}{(\det(V(x_1,\ldots,x_n)) \cdot \det(V(y_1,\ldots,y_n)))}$ We obtain

$$\prod_{1\le i,j\le n} (1 - z_i y_j)^{-1} = \frac{\det(W')}{\det(V(z_1, \ldots, z_n)) \det(V(y_1, \ldots, y_n))}, \tag{5.14}$$

where $z_i = (x_i)^{-1}$ and

$$W' = \begin{pmatrix} 1/(1-z_1y_1) & 1/(1-z_1y_2) & 1/(1-z_1y_3) & \ldots & 1/(1-z_1y_n) \\ 1/(1-z_2y_1) & 1/(1-z_2y_2) & 1/(1-z_2y_3) & \ldots & 1/(1-z_2y_n) \\ 1/(1-z_3y_1) & 1/(1-z_3y_2) & 1/(1-z_3y_3) & \ldots & 1/(1-z_3y_n) \\ \vdots & \vdots & \vdots & & \vdots \\ \\ 1/(1-z_ny_1) & 1/(1-z_1y_2) & 1/(1-z_ny_3) & \ldots & 1/(1-z_ny_n) \end{pmatrix}.$$

Note that the sign disappears.

Now it remains to expand the left-hand side as a sum of products of Schur functions. To do this, observe that, as in (5.9)), W' can be seen as the product of an $(n \times \infty)$-matrix Z and an $(\infty \times n)$-matrix Y as follows

$$\begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 & \dots \\ 1 & z_2 & z_2^2 & z_2^3 & \dots \\ 1 & z_3 & z_3^2 & z_3^3 & \dots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_n & z_n^2 & z_n^3 & \dots \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ y_1 & y_2 & y_3 & \dots & y_n \\ y_1^2 & y_2^2 & y_3^2 & \dots & y_n^2 \\ y_1^3 & y_2^3 & y_3^3 & \dots & y_n^3 \\ \vdots & \vdots & \vdots & & \end{pmatrix}.$$

Using *Cauchy-Binet Formula* (5.11) it follows

$$\det(W') = \sum_{S \in \binom{\mathbb{N}^*}{n}} \det(Z_{[n],S}) \det(Y_{S,[n]}).$$

The order we consider on the composition S, hence the order we write columns and rows of the two minors respectively, does not change the product of their determinants. Namely, if we choose a different order, both determinants may change sign, but their product is preserved. This means that we can choose an order and write $S = \{s_1, \dots, s_n\}$ such that we have $s_1 > s_2 > \dots > s_n$. Observe also that one has $s_1 \geq n-1$. Thus we can see S as a partition and write it as a sum of two partitions $S = \lambda + \delta$, where $\delta = (n-1, n-2, \dots, 1, 0)$ and λ is a partition of length $\leq n$. In this way, every different composition S corresponds to a different partition λ of length at most n. The determinant of a minor of Z can be now written as follows

$$\det(Z_{[n],S}) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} z_{\sigma(i)}^{s_i} = \det(z_i^{\lambda_j+n-j})_{ij}.$$

Using the same notation, one has $\det(z_i^{n-j})_{ij} = \det(V(z_1, \dots, z_n))$.

The quotient $\det(z_i^{\lambda_j+n-j}) / \det(z_i^{n-j})$ is by definition the Schur function $s_\lambda(z_1, \dots, z_n)$ (cf. (4.2)).

Analogously, $\det(Y_{S,[n]}) = s_\lambda(y_1, \dots, y_n)$. It follows

$$\frac{\det(W')}{\det(V(z_1, \dots, z_n)) \det(V(y_1, \dots, y_n))} = \sum_{\lambda} s_\lambda(\underline{z}) s_\lambda(\underline{y}).$$

By inserting this in (5.14), we find the Cauchy Formula (5.10), as desired.

References

[1] K. Akin, D. A. Buchsbaum and J. Weyman,*Schur functors and Schur complexes*, Adv. in Math. **44** (1982), no. 3, 207–278.

[2] C. Aquilino, R. Reischuk, *The monoidal structure on strict polynomial functors*, arXiv:1503.05108v1

[3] M. Chałupnik, *Koszul duality and extensions of exponential functors*, Adv. Math. **218** (2008), no. 3,969–982.

[4] A.L. Cauchy, *Mémoire sur le fonctions alternées et sur les sommes alternées*, Exercices d'analyse et de phys. math., ii (1841), 151–159;

[5] E. Cline, B. Parshall and L. L. Scott, *Finite dimensional algebras and highest weight categories*, J. reine angew. Math. **391** (1988), 85-99

[6] E. Cline. B. Parshall and L. Scott, *Integral and graded quasi-hereditary algebras*, I. J. of Algebra **131** (1990) 126-160.

[7] C. De Concini, D. Eisenbud and C. Procesi,*Young diagrams and determinantal varieties* , Invent. Math. 56 (1980),129-165

[8] S. Donkin, *A filtration for rational modules*, Math. Zeit. **177** (1981), 1-8

[9] S. Donkin, *On Schur algebras and related algebras I*, J. Algebra **104** (1986), 310-328

[10] S. Donkin, *On Schur algebras and related algebras II*, J. Algebra 111 (1987),354-364

[11] P. Doubilet, G.-C. Rota and J. Stein, *On the foundations of combinatorial theory: IX, Combinatorial methods in invariant theory*, Stud. Appl. Math. **53** (3) (1974).

[12] A. I. Efimov, *Derived categories of grassmannians over integers and modular Representation Theory*, arXiv:1410.7462v1, 28 Oct 2014

[13] V.Franjou, E. M. Friedlander, A. Scorichenko, A. Suslin, *General linear and functor cohomology over finite fields*, Annals of Mathematics, **150** (1999), 663–728

[14] E. M. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), no.2, 209-270.

[15] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, **35**, Cambridge Univ. Press, Cambridge, 1997.

[16] W. Fulton, J. Harris, *Representation Theory - a first course*, Graduate Texts in Mathematics, Springer, New York (1991).

[17] J.A. Green *Combinatorics and the Schur algebra*, Journal of Pure and Applied Algebra **88** (1993) 89-106.

[18] J.A. Green, *Polynomial Representations of GLn*, Second corrected and augmented edition. With an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, Green and M. Schocker. Lecture Notes in Mathematics, **830**. Springer, Berlin, 2007.

[19] M Hashimoto, K. Kurano, *Resolutions of determinantal ideals, n-Minors of (n+2) square matrices*, Advances in Mathematics **94** (1992)

[20] G.D. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications **16**, Addison-Wesley, Reading, Mass., 1981.

[21] H. Krause, *Koszul, Ringel and Serre duality for strict polynomial functors*, Compos. Math. **149** (2013), no. 6, 996–1018.

[22] H. Krause, *Highest weight categories and strict polynomial functors*, arXiv:1405.1691v3.

[23] S. Lang, *Algebra*, Graduate Texts in Mathematics 211 (Revised third ed.), Springer-Verlag, New York (2002)

[24] A. Lascoux, *Symmetric functions and combinatorial operators on polynomials*, CBMS Regional Conference Series in Mathematics n.99, Raleigh (2001)

[25] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Second Edition, Oxford Mathematical Monographs, Oxford University Press, New York, 1995.

[26] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5** (1997).

[27] T. Muir, *The theory of determinants in the historical order of development*, Dover Publications, New York (1860)

[28] B. Parshall, *Finite dimensional algebras and algebraic groups*, Contemp. Math. X2 (1989) 97-l 14.

[29] L. Scott, *Simulating algebraic geometry with algebra, I: the algebraic theory of derived categories*, Proc. Sym. Pure Math. **47** (1987).

[30] B. Totaro, *Projective resolutions of representations of GL(n)*, J. Reine Angew. Math. **482** (1997) 1-13.

[31] A Touzé,*Bar complexes and extensions of classical exponential functors*, arXiv:1012.2724v4, 9 Sep 2013

[32] A. Touzé, *Koszul duality and derivatives of non-additive functors*, arXiv:1103.4580v1

[33] Fiona Murnaghan, *Introduction to Representation Theory*, http://www.math.toronto.edu/murnaghan/courses/mat445/ch1.pdf

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