Next-to-Leading Order Rates in Leptogenesis

Dissertation

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by

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Abstract

The lepton asymmetry L of the early Universe is generated through the out-of-equilibrium interactions of sterile neutrinos, which is described by linear kinetic equations. These equations are determined by dissipative coefficients (or rates). The focus of this work is the computation of next-to-leading order corrections to these rates. In order to avoid inconsistencies of Boltzmann equations we use Landau's theory of quasi-stationary fluctuations, which allows to compute the rates from Kubo-type relations. The rates are then determined by real-time correlation functions and susceptibilities of conserved charges and can be calculated at leading order in the sterile neutrino interactions and to any order in the SM interactions.

Firstly we compute the susceptibilities of conserved charges at order g^2 in the Standard Model couplings, which completes the order g^2 calculation of the $\Delta L = 1$ washout rate and provides a relation between the baryon number B and baryon number minus lepton number $B - L$ at order g^2.

Then we calculate the $\Delta L = 2$ washout rate in an effective theory for temperatures much smaller than the lightest sterile neutrino mass. In contrast to earlier calculations, we take full quantum statistics and so-called spectator processes into account. Furthermore, we consider next-to-leading order contributions from the thermal Higgs mass which are of order g in the Standard Model couplings.

Following that, we study the CP violating lepton asymmetry rate and derive a master formula which relates this rate to a three-point spectral function of Standard Model fields. We use this formula to compute the order g^2 corrections to the CP asymmetry at zero temperature.

Finally we show in a rather general framework that particle equilibration rates are simply related to particle production rates. This implies that the coefficient in the kinetic equations, which is identified as the sterile neutrino equilibration rate, is already known at order g^2 through the well-known sterile neutrino production rate.

Publications in this thesis

1. Chapter 4 of this thesis reproduces the calculations and results of the publication [1]:

   D. Bödeker and M. Sangel,
   Order g^2 susceptibilities in the symmetric phase of the Standard Model
   JCAP **1504** (2015) no.04, 040
   doi:10.1088/1475-7516/2015/04/040
   arXiv:1501.03151 [hep-ph].

2. The calculations and results of chapters 5 and appendix A of this thesis are planned to be published under the title [2]:

   M. Sangel and M. Wörmann,
   Low-temperature lepton washout rate and bounds on neutrino masses.

3. The calculations and results of chapters 6 and 7 and appendix B of this thesis are planned to be published under the title [3]:

   D. Bödeker and M. Sangel,
   NLO lepton asymmetry rate in the non-relativistic regime: Zero temperature contribution.

4. Chapter 8 of this thesis reproduces the calculations and results of the publication [4]:

   D. Bödeker, M. Sangel and M. Wörmann,
   Equilibration, particle production, and self-energy
   Phys. Rev. D **93** (2016) no.4, 045028
   doi:10.1103/PhysRevD.93.045028
   arXiv:1510.06742 [hep-ph].

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Chapter 1

Introduction

1.1 Baryogenesis

Despite the experimental and theoretical success of the Standard Model (SM) of particle physics and its confirmation by the experimental discovery of the Higgs boson at the LHC [5], the Universe is still full of mysteries. It is known from the cosmic microwave background (CMB) that only about 4.6% [6] of the energy density of the Universe is made of ordinary baryonic matter, the substance which we and every object in our daily life are made of. But even this small fraction of baryonic matter is not fully understood. In particular the fact that the Universe contains more baryons than anti-baryons cannot be explained within the SM. This *baryon asymmetry* is oftentimes quantified as the ratio of the baryon number[1] density n_B to the photon number density n_γ, the so-called *baryon to photon ratio*,

$$\eta_{\mathrm{B}} \equiv \frac{n_{\mathrm{B}}}{n_\gamma}, \tag{1.1.1}$$

and can be measured from CMB data. The newest data from the WMAP experiment yields the baryon to photon ratio [6]

$$\eta_{\mathrm{B}}^{\mathrm{CMB}} = (6.19 \pm 0.14) \times 10^{-10}. \tag{1.1.2}$$

The work of many theoretical physicists is motivated by the task to explain the observed value of η_{B}. If we assume that no baryon asymmetry was present at the beginning of the Universe, it must have been generated by a mechanism. Such a generation is called *baryogenesis* and, as stated by Sakharov in 1967 [7], requires at least three conditions:

1. The baryon number B must not be conserved.

2a. Charge conjugation (C) must not be a symmetry of the system. Otherwise a process which produces baryons, and its charge conjugate process which produces anti-baryons would occur with the same rate.

[1] The baryon number B is defined as the difference of the number of baryons and the number of anti-baryons.

2b. Charge-parity (CP) transformations must not be a symmetry of the system. Otherwise a process that produces left/right-handed baryons, and its conjugate process which produces right/left-handed anti-baryons would occur with the same rate.

3. The system must be out of equilibrium. Otherwise for every baryon generating process there would be an inverse process which reduces the number of baryons with the same rate.

The first condition is satisfied in the SM due to the chiral anomaly in the electroweak sector. Let $Q \in \{L_i, B/3\}$, where L_i is the lepton number with flavor i. Then the corresponding currents j_Q satisfy the relation[2] [8, 9, 10, 11]

$$\partial_\mu j^\mu_Q = \frac{1}{64\pi^2}\varepsilon^{\mu\nu\rho\sigma}\left(g_2^2 W^a_{\mu\nu}W^a_{\rho\sigma} - g_1^2 F_{\mu\nu}F_{\rho\sigma}\right), \tag{1.1.3}$$

where $F_{\mu\nu}$ and $W^a_{\mu\nu}$ are the field strength tensors of the $U(1)$ and $SU(2)$ gauge fields respectively. The corresponding gauge couplings are g_1 and g_2. Integrating over the space-time volume and assuming the spatial part of the currents to vanish at spatial infinity, (1.1.3) implies that the total charge

$$Q = \int d^3x j^0_Q \tag{1.1.4}$$

changes as

$$\Delta Q = Q(t) - Q(t_0) = \frac{1}{64\pi^2}\varepsilon^{\mu\nu\rho\sigma}\int_{t_0}^{t} dt' \int d^3x \left(g_2^2 W^a_{\mu\nu}W^a_{\rho\sigma} - g_1^2 F_{\mu\nu}F_{\rho\sigma}\right). \tag{1.1.5}$$

The solutions of the field equations, for which ΔQ is not zero, are related to the topology of the degenerate SM vacuum. As illustrated in figure 1.1, there exists a (discrete) set of infinitely many gauge field configurations for which the system is in the ground state [12]. These configurations are separated by an energy barrier with the maximal energy [12, 13]

$$E_{\text{sph}} = \frac{4\pi v}{g_2} f\left(\frac{\lambda}{g_2}\right) \sim \mathscr{O}(10)\ \text{TeV}, \tag{1.1.6}$$

where v is the Higgs vacuum expectation value, λ is the Higgs self-coupling and f is a function that varies from 2.40 to 3.56 when λ/g_2 is varied from 0 to ∞. The solutions for which $|\Delta Q| = 1$ are those for which the gauge fields evolve from one vacuum configuration to the next.

There are two possibilities for such a transition. At zero temperature *instanton* solutions [14] allow for the system to tunnel through the barrier, but these processes occur with an exponentially suppressed rate [15, 16, 17]

$$\Gamma_{\text{inst}} \sim e^{-2S_{\text{inst}}}, \tag{1.1.7}$$

[2]Note that the $SU(3)$ gauge fields do not contribute to the divergence of the total baryon number current because left and right-handed quarks are equally coupled to the $SU(3)$ gauge fields. Only the difference of left and right-handed baryons is violated due to the chiral anomaly in QCD.

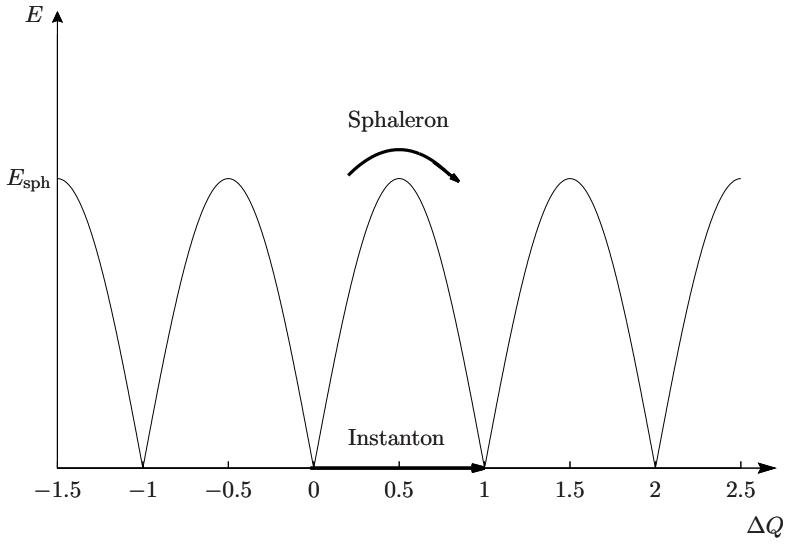


Figure 1.1: The vacuum structure of the electroweak theory. Each minimum corresponds to a vacuum configuration of the gauge fields. If the system changes from one minimum to the next, the charge changes by $\Delta Q = 1$.

where $S_{\rm inst} = 8\pi^2/g_2^2 \sim 186$. Therefore, this kind of transition is very inefficient and can be neglected as source of B violation. At high temperatures or energies the system can change between the vacua by evolving through field configurations which go over the top of energy barrier [13, 18, 19]. A field configuration on the top of the energy well is called *sphaleron* [13] and consequently, the transitions over the top of the energy well are called sphaleron transitions. While in the broken phase in the SM ($T < 160$ GeV), the sphaleron rate[3] is Boltzmann suppressed by the height of the energy barrier [20],

$$\Gamma_{\rm sph} \sim \exp(-E_{\rm sph}(T)/T), \tag{1.1.8}$$

the sphaleron transitions can be more frequent at high temperatures in the symmetric phase, where the rate has the form [23, 24]

$$\Gamma_{\rm sph} = \left(c_1 \ln \frac{m_{D,2}}{g_2^2 T} + c_2\right) \frac{g_2^2 T^2}{m_{D,2}^2} \left(\frac{g_2^2}{4\pi}\right)^5 T^4. \tag{1.1.9}$$

The coefficients $c_1 = 10.8 \pm 0.7$ [25] and $c_2 = 18 \pm 3$ [22] are numerical constants and $m_{D,2} = 11 g_2^2/6$ is the thermal Debye mass of the $SU(2)$ gauge fields.

In [26] it has been pointed out that the electroweak sphaleron transitions are in equilibrium for temperatures $T < 45/4\, T_{\rm sph}$, where $T_{\rm sph}$ is the temperature where the sphaleron rate

[3] A very detailed calculation of this rate can be found in [21] and recent lattice simulations can be found in [22].

(1.1.9) becomes equal to the Hubble rate [4], given as [27]

$$H \approx 1.66\sqrt{g^*}\frac{T^2}{m_{\rm pl}}. \tag{1.1.10}$$

Here $m_{\rm pl} = 1.22 \times 10^{19}$ GeV is the Planck mass and g^* is the number of effectively massless degrees of freedom, which for temperature $T \gtrsim 300$ GeV is equal to the Standard Model degrees of freedom $g_{\rm SM} = 106.75$ [27]. At high temperatures, g^* can in principle be larger than $g_{\rm SM}$ because heavier particles which are not part of the SM can be relativistic. We ignore this fact in order to give a rough estimate of the temperature, where the sphalerons are in equilibrium and find $T < 10^{12}$ GeV in accordance with [26].

We see that the first Sakharov condition of baryon violation is satisfied in the SM due to the electroweak sphaleron transitions. They violate the baryon number and the lepton numbers by

$$|\Delta B| = n_{\rm f}, \qquad |\Delta L_i| = 1, \tag{1.1.11}$$

whereas the differences of the charges

$$X_i \equiv L_i - B/n_{\rm f} \tag{1.1.12}$$

remain conserved in the SM.

The second Sakharov condition is satisfied in the SM as well. Charge conjugation C is intrinsically violated in the SM because only left-handed leptons couple to the $SU(2)$ gauge bosons. Here and throughout the whole thesis we define the charge conjugation of a spinor ψ as

$$\psi^c \equiv C\psi C^\dagger = \gamma_0 \mathscr{C}\psi^*, \tag{1.1.13}$$

where $\mathscr{C}$ is a unitary and antisymmetric matrix with the property [28]

$$\mathscr{C}\gamma_\mu\mathscr{C} = -\gamma_\mu^\top. \tag{1.1.14}$$

It is easy to see that left/right-handed currents $j^\mu_{L/R} = \overline{\psi}\gamma^\mu P_{\rm L/R}\psi$ with

$$P_L = \frac{1}{2}(1-\gamma_5), \qquad P_R = \frac{1}{2}(1+\gamma_5) \tag{1.1.15}$$

are not invariant under charge conjugation. They transform as

$$(j^\mu_L)^c = -j^\mu_R. \tag{1.1.16}$$

CP violation happens in the SM through Yukawa interactions

$$\mathscr{L}_{q,Y} = -(h_u)_{ij}\,\bar{q}_i\tilde{\varphi}u_{R,j} - (h_d)_{ij}\,\bar{q}_i\varphi d_{R,j} + \text{H.c} \tag{1.1.17}$$

of right-handed up and down quarks u_R and d_R and the left-handed quark doublet $q = (u_L, d_L)^\top$ with the Higgs field φ. The field $\tilde{\varphi} = i\sigma_2\varphi^*$ is the $SU(2)$ conjugate Higgs field and h_u and h_d are complex coupling matrices. In the broken phase, where the Higgs field obtains

[4] To be precise: $\Gamma_{\rm sph}/T^4|_{T=T_{\rm sph}} = H/T|_{T=T_{\rm sph}}$

a vacuum expectation value $\langle\varphi\rangle = (0, v)^\top$, these couplings define mass matrices $m_u = vh_u$ and $m_d = vh_d$ for up and down quarks respectively. In a basis where these matrices are diagonal, the left-handed quark fields are linear combinations, given by $d'_{L,i} = (U_d)_{ij} d_{L,j}$ and $u'_{L,i} = (U_u)_{ij} u_{L,j}$, where $d_{L,j}$ and $u_{L,j}$ are the original fields and U_u and U_d are transformation matrices. Then the weak quark current

$$j^{\mu+} \equiv \frac{1}{\sqrt{2}} \overline{u}_{L,i} \gamma^\mu d_{L,i} = \frac{1}{\sqrt{2}} \overline{u}'_{L,i} (U_u^\dagger)_{ik} (U_d)_{kj} \gamma^\mu d'_{L,j} \tag{1.1.18}$$

contains in the mass eigenbasis the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V = U_u^\dagger U_d$ [29]. For $n_\mathrm{f} = 3$ flavors the CKM matrix depends on three real parameters and one CP violating phase which cannot be absorbed by redefinition of the quark fields. The values of these parameters have quite extensively measured and a summary can be found in [30].

The third Sakharov condition about non-equilibrium is satisfied in the early Universe during the electroweak phase transition. However, in [20] is has been shown that the observed baryon to photon ratio (1.1.2) could only be generated within the SM during the electroweak phase transition, if it was a strongly first order one. The character of the phase transition depends on the magnitude of the Higgs mass. A first order phase transition is only possible for Higgs masses $m_H < 80$ GeV [31]. At the currently observed Higgs mass $m_H \approx 125$ GeV, the transition from the symmetric to the broken phase is rather a smooth crossover. Therefore, physics beyond the SM is needed for an explanation the observed baryon asymmetry of the Universe.

1.2 Leptogenesis

A very simple mechanism which can generate the baryon asymmetry in the Universe beyond the SM, has been suggested in 1986 by Fukugita and Yanagida and is called leptogenesis [32]. The idea behind this mechanism is that a baryon asymmetry can be produced through the generation of a lepton asymmetry L. If the sphaleron processes are in equilibrium during or after leptogenesis, they tend to reduce $B+L$ by partially converting the lepton asymmetry to a baryon asymmetry. After leptogenesis when the $B-L$ violating processes become inefficient, $B - L$ is a conserved charge and can be related to the baryon number through the relation

$$\langle B \rangle = \kappa \langle B - L \rangle, \tag{1.2.1}$$

where $\langle ... \rangle$ is the equilibrium average.

The coefficient κ was first determined in [33] at leading order in the SM couplings, in the symmetric phase and is given as

$$\kappa = \frac{28}{79}. \tag{1.2.2}$$

Later it has also been computed in the broken phase at leading order [34] and in the symmetric phase to order g^2 in the SM couplings [1]. The order g^2 calculation of [1] is part of this thesis in chapter 4.

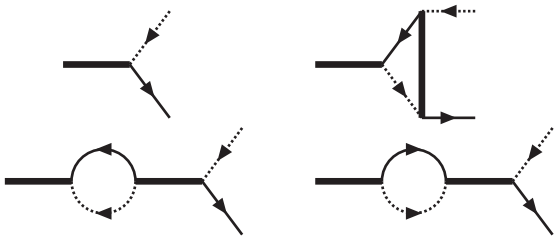


Figure 1.2: The tree-level and one-loop graphs which contribute to the sterile neutrino decay rate $\Gamma(N \to \ell\varphi)$. Thick lines are sterile neutrinos, dashed lines represent Higgs and solid lines with arrow are leptons.

Originally Yanagida and Fukugita suggested to extend the SM by right-handed neutrinos ν_R, which are singlets under the SM gauge groups. The most general gauge invariant and renormalizable Lagrangian for this theory allows a Majorana mass term

$$\mathscr{L}_M = -M_{IJ}\overline{\nu_{R,I}}\nu^c_{R,J} + \text{H.c}, \tag{1.2.3}$$

and a Yukawa interaction

$$\mathscr{L}_{\text{int}} = -(h_\nu)_{Ii}\overline{\nu_{R,I}}\tilde{\varphi}^\dagger\ell_i + \text{H.c} \tag{1.2.4}$$

with the left-handed lepton doublet $\ell_i = (\nu_{L,i}, e_{L,i})^\top$ and the neutrino coupling matrix $(h_\nu)_{Ii}$. Here i denotes the SM lepton flavors and I, J denotes the right-handed neutrino flavor[5].

In this work we find it convenient to express the Lagrangian in terms of Majorana fields

$$N = \nu_R + \nu^c_R. \tag{1.2.5}$$

Then, in a basis where the Majorana mass matrix is diagonal, $M_{IJ} = \delta_{IJ}M_I$, the complete Lagrangian of the system can be written as

$$\mathscr{L} = \mathscr{L}_{\text{SM}} + \frac{1}{2}\overline{N_I}(i\partial\!\!\!/ - M_I)N_I - \left[(h_\nu)_{Ii}\overline{N}_I\tilde{\varphi}^\dagger\ell_i + \text{H.c}\right]. \tag{1.2.6}$$

In the simplest picture a lepton asymmetry can be generated through the decay of the Majorana neutrinos. If the coupling h_ν in (1.2.4) is CP violating, the rate $\Gamma(N \to \ell\varphi)$ for the decay into particles differs from the rate $\Gamma(N \to \bar{\ell}\varphi^\dagger)$ for the decay into anti-particles. A measure of this difference is the so-called CP asymmetry ε_{Ii} which is defined as

$$\varepsilon_{Ii} \equiv \frac{\Gamma(N_I \to \ell_i\varphi) - \Gamma(N_I \to \bar{\ell}_i\varphi^\dagger)}{\Gamma(N_I \to \ell\varphi) + \Gamma(N_I \to \bar{\ell}\varphi^\dagger)}. \tag{1.2.7}$$

At leading order in h_ν both rates are equal and therefore the CP asymmetry vanishes. For example the zero temperature decay rates read [35]

$$\Gamma(N_I \to \ell\varphi) = \Gamma(N_I \to \bar{\ell}\varphi^\dagger) = \frac{(h_\nu h_\nu^\dagger)_{II}M_I^2}{16\pi E_{\mathbf{k}}}, \tag{1.2.8}$$

[5]In leptogenesis the the number of right-handed flavors has to be at least two for the coupling h_ν to be CP violating. But apart from that let us keep it arbitrary.

and their sum yields the total decay rate

$$\Gamma_{N_I} = \frac{(h_\nu h_\nu^\dagger)_{II} M_I^2}{8\pi E_{\mathbf{k}}}. \tag{1.2.9}$$

In order to generate non-vanishing CP asymmetries, one has to compute the difference of the rates to order h_ν^4, which takes into account loop effects. At leading order the CP violation comes from an interference of the tree level graph with the one-loop graphs in figure 1.2. The first complete[6] calculation of the CP asymmetries in leptogenesis at zero temperature has been done in [36] and yields the result[7]

$$\begin{aligned}\varepsilon_{Ii} =& \frac{1}{8\pi} \frac{1}{\left(h_\nu h_\nu^\dagger\right)_{II}} \sum_{J\neq I} \mathrm{Im}\left[(h_\nu)_{Ii}(h_\nu^\dagger)_{iJ}\left(h_\nu h_\nu^\dagger\right)_{IJ}\right] g(x_{JI}) \\ &+ \frac{1}{8\pi} \frac{1}{\left(h_\nu h_\nu^\dagger\right)_{II}} \sum_{J\neq I} \mathrm{Im}\left[(h_\nu)_{Ii}(h_\nu^\dagger)_{iJ}\left(h_\nu h_\nu^\dagger\right)_{JI}\right] \frac{1}{1-x_{JI}},\end{aligned} \tag{1.2.10}$$

where $x_{JI} = M_J^2/M_I^2$ and

$$g(x) = \sqrt{x}\left[\frac{1}{1-x} + 1 - (1+x)\ln\left(\frac{1+x}{x}\right)\right]. \tag{1.2.11}$$

In this work we are interested in the hierarchical limit, where the lightest Majorana neutrino with mass M_1 is much lighter than the other ones. In this case one may expand the CP asymmetries to the first order in M_1/M_J and one finds [36]

$$\varepsilon_{1i} = -\frac{3}{16\pi} \frac{1}{\left(h_\nu h_\nu^\dagger\right)_{11}} \sum_{J\neq 1} \frac{M_1}{M_J} \mathrm{Im}\left[(h_\nu)_{1i}(h_\nu^\dagger)_{iJ}\left(h_\nu h_\nu^\dagger\right)_{1J}\right]. \tag{1.2.12}$$

So far we have only considered the zero temperature decay of Majorana neutrinos. For a realistic description of a L generation in the Universe one has to consider the interactions of the Majorana neutrinos with the hot SM plasma in an expanding background. In leptogenesis the non-equilibrium system is characterized by a large separation of time-scales. There are *slow* quantities, which are changed by the Yukawa interaction (1.2.4) but conserved in the SM and *fast* quantities which are dominated by so-called *spectator processes* [37]. These processes are so fast (compared to the expansion of the Universe) that the quantities which are changed by them can be considered as being in equilibrium. We will discuss this separation of time-scales in more detail in chapter 3.

In leptogenesis such non-equilibrium systems are often times described by kinetic equations which determine the time evolution of the slow quantities. It depends on the temperature which quantities are considered as slow. Certainly during leptogenesis the Majorana neutrino phase-space densities $f_{I\mathbf{k}}$[8] and the charges $X_i = L_i - B/n_{\mathrm{f}}$ are slow. But for example, at temperatures above 10^{12} GeV, the electroweak sphalerons are out of equilibrium so

[6] Earlier calculations like in [35] did not take into account the self-energy graph.

[7] Note that the Yukawa couplings in [36] are defined as $\lambda_{iI} = (h_\nu)_{Ii}$

[8] We will from now on consider a finite volume $V = L^3$ with periodic boundary conditions. Then we have discrete momenta $k_i = 2\pi n_i/L$, which we consider as indices. In the end of our calculations we take the infinite volume limit, where $\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{d^3k}{(2\pi)^3}$.

that B is separately conserved. In this case the lepton numbers L_i are slow charges as well. Of course the lepton numbers L_i are not independent of the charges X_i, but we can choose a linearly independent set of charges out of the set of all linear combinations of X_i's and L_i's. At higher temperatures the Hubble rate becomes larger, so that further processes are out of equilibrium and more charges are conserved. Thus the set of slow charges becomes larger, the higher the temperature was. In the following we denote the elements of a set of linearly independent slow charges by Q_a.

We assume that the system is close to equilibrium during leptogenesis, so that the values of these charges are so small that their time evolution can be well described by linearized kinetic equations[9]

$$D_t f_{I\mathbf{k}} = -\left(\gamma_{\delta f \delta f}\right)_{I\mathbf{k}J\mathbf{k}'} \delta f_{J\mathbf{k}'} - \left(\gamma_{\delta f\, Q}\right)_{I\mathbf{k}a} Q_a, \tag{1.2.13}$$

$$D_t Q_a = -\left(\gamma_{Q\,\delta f}\right)_{aI\mathbf{k}} \delta f_{I\mathbf{k}} - \left(\gamma_{QQ}\right)_{ab} Q_b. \tag{1.2.14}$$

Here we have defined

$$\delta f_{I\mathbf{k}} \equiv f_{I\mathbf{k}} - f^{\mathrm{eq}}_{I\mathbf{k}} \tag{1.2.15}$$

as the deviation of the sterile neutrino phase-space density from its equilibrium value and D_t as the total time derivative which takes into account the effects of the expansion of the Universe.

For leptogenesis it is very common to derive kinetic equations such as (1.2.13) and (1.2.14) with Boltzmann equations. These equations describe the evolution of phase-space densities, such as $f_{I\mathbf{k}}$, through collision integrals $C_{I\mathbf{k}}[f]$ as

$$D_t f_{I\mathbf{k}} = \frac{1}{E_{I\mathbf{k}}} C_{I\mathbf{k}}[f], \tag{1.2.16}$$

where $C_{I\mathbf{k}}[f]$ depends on all processes which change the quantities of the non-equilibrium system. For a detailed definition of $C_{I\mathbf{k}}[f]$ see for example chapter 5 of [27]. The collision integrals can be linearized in $f_{I\mathbf{k}}$ and Q_a and determine in this way the *rates* γ. A rather detailed derivation of the leading order rates γ in (1.2.13) and (1.2.14), using Boltzmann equation and assuming kinetic equilibrium for the sterile neutrinos, can be found in [35, 38].

In this work we will often consider the non-relativistic limit, where the lightest sterile neutrino mass M_1 is much larger than the temperature T. In this limit, the rates can be expanded in powers of T/M_1 and $e^{-M_1/T}$. In this expansion, the leading contribution of the rates $\gamma_{\delta f \delta f}$ and $\gamma_{Q\,\delta f}$ is the zero temperature limit. In this limit, it has been found in [39] that the rates are determined through the decay rates (1.2.9) and the CP asymmetries (1.2.10) by

$$\left(\gamma_{\delta f \delta f}\right)_{I\mathbf{k}I'\mathbf{k}'}\big|_{T=0} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{II'}\Gamma_{N_I}(\mathbf{k}), \qquad \left(\gamma_{Q\,\delta f}\right)_{iI\mathbf{k}}\big|_{T=0} = \varepsilon_{Ii}\Gamma_{N_I}(\mathbf{k}). \tag{1.2.17}$$

Although the definition of the rates through Boltzmann equations is widely used in leptogenesis, this might lead to inconsistencies and difficulties even at leading order. For example the collision integrals depend on vacuum S-matrix elements. This might be inconsistent with the interactions in a thermal plasma. Some complications also arise in the derivation of the linear equation (1.2.14), using Boltzmann equations. The CP violating coefficient $\gamma_{Q\,\delta f}$ does

[9] Over repeated indices has to be summed

not only get contributions from $\Delta L = 1$ violating sterile neutrino decay, but also from the real intermediate state of s-channel $\Delta L = 2$ violating scattering processes with sterile neutrino exchange [35, 40] .

In this work we avoid these problems and extend the approach of Bödeker and Laine [41] which is based on Landau's theory of quasi-stationary fluctuations [42]. The large separation of time scales during leptogenesis guarantees that for small values of $\delta f_{I\mathbf{k}}$ and Q_a the non-equilibrium system is completely described by the linear kinetic equations (1.2.13) and (1.2.14). In contrast to Boltzmann equations, the rates γ can then computed from Kubo-type relations [43]. They relate the rates to real-time correlation functions and susceptibilities, which can be computed in thermal quantum field theory. Their calculation can be done at leading order in the Yukawa interaction (1.2.4) and to any order in the SM model couplings and thermal effects are naturally incorporated.

In this work we study radiative corrections to the rates in (1.2.13) and (1.2.14). At first, in chapter 4 we compute susceptibilities of the conserved charges Q_a at order g^2 in the SM couplings. This computation completes the order g^2 of the order h_ν^2 *washout rate* γ_{QQ} [41] and yields a relation between B and $B - L$ at order g^2.

In chapter 5 we compute the washout rate γ_{QQ} in an effective theory for $M_1 \gg T$, where the leading order is of order h_ν^4, determined by $\Delta L = 2$ processes. In earlier calculations [44] only classical statistics has been used for the thermal distribution functions of lepton and Higgs. Since the order h_ν^4 of this rate is an important ingredient for the determination of upper bounds on active neutrino masses [45], we compute this rate taking into account full quantum statistical effects, and study next-to-leading order corrections from thermal Higgs mass. These corrections are of order g in the SM couplings.

Up to this point the CP violating rate $\gamma_{Q\,\delta f}$ has only been completely known at leading order in the non-relativistic regime (see (1.2.17)). Furthermore, corrections from scatterings including gauge bosons have been computed in [46] and corrections including top quark scatterings have been computed in [47]. However, a complete consistent expansion in powers of SM couping has not been done yet. In chapter 7 we do the first step in this direction, computing the next-to-leading order zero-temperature corrections to the CP violating rate $\gamma_{Q\,\delta f}$ in the hierarchical limit. These corrections are of order g^2 in the SM couplings. Recently, also the first thermal corrections to the order g^2 in powers of T/M_1 have been computed in [48] in the hirachical limit.

Finally, in chapter 8 we show that the rate $\gamma_{\delta f\,\delta f}$ can be related to the *sterile neutrino production rate* by a simple relation which has first been suggested by Weldon [49]. The sterile neutrino production rate is known at leading order and next-to-leading order in several temperature regimes [50, 51, 52, 53, 54, 55, 56, 57, 58]. Therefore, no explicit next-to-leading order calculation for $\gamma_{\delta f \delta f}$ needs to be done.

The rate $\gamma_{\delta f\,Q}$ is oftentimes neglect in leptogenesis computations. We will do this here as well. The reason is that the rate $\gamma_{\delta f\,Q}$ is CP violating like $\gamma_{Q\,\delta f}$, and thus of order h_ν^4. In the kinetic equation (1.2.13) it is furthermore multiplied by the small asymmetry Q, which is assumed to be much smaller than δf. Therefore, the term containing this rate is the smallest one in the equations (1.2.13) and (1.2.14).

Chapter 2

Two-point functions at finite temperature

In chapter 3 we want to relate the coefficients γ to two-point functions and susceptibilities in finite temperature field theory. Let us therefore review some basic definitions and relations for such correlation functions.

2.1 Basic relations between two-point functions

Every real-time two-point correlator of elementary or composite operators A and B can be constructed from the Wightman correlators [59]

$$\Delta^{>}_{AB}(\omega) = \int dt e^{i\omega t}\langle A(t)B(0)\rangle, \qquad \Delta^{<}_{AB}(\omega) = \int dt e^{i\omega t}\sigma\langle B(0)A(t)\rangle, \tag{2.1.1}$$

where the average is defined as $\langle ...\rangle = \mathrm{Tr}\left[... \exp(-\beta H)\right]/Z$ with the inverse temperature $\beta = 1/T$ and $Z = \mathrm{Tr}\left[\exp(-\beta H)\right]$. Here $\sigma = 1$ if A and B are bosonic operators and $\sigma = -1$ if A and B are fermionic operators. The Wightman correlators can be used to define the so-called spectral function as[1] [59]

$$\rho_{AB}(\omega) \equiv \Delta^{>}_{AB}(\omega) - \Delta^{<}_{BA}(\omega) = \int dt e^{i\omega t}\left\langle [A(t), B(0)]\right\rangle, \tag{2.1.2}$$

with the (anti-)commutator

$$[A, B] \equiv AB - \sigma BA. \tag{2.1.3}$$

The cyclicity of the trace implies that both Wightman functions are related through the simple equation [59]

$$\Delta^{>}_{AB}(\omega) = \sigma e^{\beta\omega}\Delta^{<}_{AB}(\omega). \tag{2.1.4}$$

[1] Note that our definition of the spectral function differs from the one defined in [59] by a factor two.

Using this cyclicity property and the definition (2.1.2), one can also express each Wightman function in terms of the spectral functions as

$$\Delta^{>}_{AB}(\omega) = (1 + \sigma f_\sigma(E_{\mathbf{k}}))\, \rho_{AB}(\omega), \qquad \Delta^{<}_{AB}(\omega) = \sigma f_\sigma(E_{\mathbf{k}}) \rho_{AB}(\omega), \tag{2.1.5}$$

where f_σ is the Bose-Einstein distribution for $\sigma = 1$ and the Fermi-Dirac distribution for $\sigma = -1$.

In this work we are in particular interested in the imaginary time correlator which reads in frequency space

$$\Delta_{AB}(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \langle A(-i\tau) B(0) \rangle, \tag{2.1.6}$$

with Matsubara frequencies $\omega_n = \pi n T$, where n is even (odd) integer if A and B are bosonic (fermionic). This correlator can be written in the spectral representation [59]

$$\Delta_{AB}(i\omega_n) = \int \frac{d\omega}{2\pi} \frac{\rho_{AB}(\omega)}{\omega - i\omega_n}, \tag{2.1.7}$$

and can be analytically continued to the complex plane with frequencies ω off the real axis. In particular it is useful to define the retarded and advanced two-point functions as [59]

$$\Delta^{\mathrm{ret}}_{AB}(\omega) = \Delta_{AB}(\omega + i0^+), \qquad \Delta^{\mathrm{adv}}_{AB}(\omega) = \Delta_{AB}(\omega + i0^-), \tag{2.1.8}$$

with real ω. Using the spectral representation (2.1.7) in combination with the identity

$$\frac{1}{x - i0^+} = \mathrm{P.V.}\frac{1}{x} + i\pi\delta(x), \tag{2.1.9}$$

where P.V. denotes the principal value, one easily finds the inverse relation [59]

$$\rho_{AB}(\omega) = \frac{1}{i}\left(\Delta^{\mathrm{ret}}_{AB}(\omega) - \Delta^{\mathrm{adv}}_{AB}(\omega)\right) \equiv \frac{1}{i}\mathrm{Disc}\Delta_{AB}(\omega), \tag{2.1.10}$$

between the spectral function ρ_{AB} and the analytically continued two-point correlator Δ_{AB}.

Let A and B now be bosonic operators. An important quantity which appears in the computation of the coefficients γ is the so-called symmetric correlator

$$C_{AB}(t) = \frac{1}{2}\langle\{A(t), B(0)\}\rangle. \tag{2.1.11}$$

In Fourier space we have $C_{AB}(\omega) = \frac{1}{2}\left(\Delta^{>}_{AB}(\omega) + \Delta^{<}_{AB}(\omega)\right)$ and using the cyclicity property (2.1.4) one finds [59]

$$C_{AB}(\omega) = \left(\frac{1}{2} + f_{\mathrm{B}}(\omega)\right) \rho_{AB}(\omega). \tag{2.1.12}$$

2.2 Symmetries of two-point correlators

Let us in the following assume the operators A and B to be Hermitian, bosonic and either even or odd under CPT transformation, that is

$$CPTA(t)(CPT)^{-1} = \varepsilon_A A(-t) \tag{2.2.1}$$

with $\varepsilon_A = \pm 1$. This is for instance the case if A and B represent charges or phase-space densities. Then, invariance under CPT transformation [60, 61, 62] implies[2]

$$\rho_{AB}(\omega) = \varepsilon_A \varepsilon_B \rho_{AB}(\omega)^*. \tag{2.2.2}$$

Thus, if A and B have the same sign under CPT transformation, the spectral function is real and otherwise imaginary. On the other hand, hermiticity of A and B implies that

$$\rho_{AB}(\omega)^* = -\rho_{AB}(-\omega). \tag{2.2.3}$$

Therefore, a real spectral functions is odd and an imaginary spectral functions is even in ω. If the spectral function is real, the inverse relation (2.1.8) can be simplified to

$$\rho_{AB}(\omega) = 2\mathrm{Im}\Delta^{\mathrm{ret}}_{AB}(\omega). \tag{2.2.4}$$

In this work equal time two-point correlators play an important role as well. For hermitian operators A and B they are called susceptibilities and are denoted as

$$\chi_{AB} \equiv \langle AB \rangle. \tag{2.2.5}$$

Like for the spectral function the CPT theorem implies

$$\chi_{AB} = \varepsilon_A \varepsilon_B \chi^*_{AB}. \tag{2.2.6}$$

In combination with hermiticity of A and B we find

$$\chi_{AB} = \varepsilon_A \varepsilon_B \chi_{BA}. \tag{2.2.7}$$

If the quantities A and B are conserved, that is they commute with H, the susceptibilities have the symmetry

$$\chi_{AB} = \chi_{BA} \tag{2.2.8}$$

and therefore

$$\chi_{AB} = \varepsilon_A \varepsilon_B \chi_{AB}. \tag{2.2.9}$$

Consequently, susceptibilities of conserved quantities vanish, if the quantities have different signs under CPT transformation.

[2] A similar argument is given in [63] under the assumption of T invariance.

Chapter 3

Theory of quasi-stationary fluctuations

The aim of this chapter is to derive relations between the dissipative coefficient and correlation functions in thermal quantum field theory. For this purpose we review the concepts of Landau's theory of quasi-stationary fluctuations, closely following [41] and [42].

3.1 Linear kinetic equations

A thermodynamic system in equilibrium is completely determined by the temperature T and the values of all its conserved quantities like charges or number densities. We denote these quantities by y_i and define them in such a way that their equilibrium values $\langle y_i \rangle$ vanish. In thermal equilibrium the quantities y_i fluctuate around their equilibrium values with average fluctuations

$$\Delta y_i = \sqrt{\langle y_i^2 \rangle}. \tag{3.1.1}$$

Let us now assume that a subset of the quantities y_i has been prepared to have values $y_i \gg \Delta y_i$. As long as there are no processes which would drive the system further out of equilibrium, the quantities strive to reach their equilibrium values again. Some of these quantities might reach their equilibrium value very fast with a relaxation time t_{fast}, whereas other quantities might have a relaxation time $t_{\text{slow}} \gg t_{\text{fast}}$. Let us call these quantities *slow* and denote them by x_i. For example, in an expanding universe t_{slow} is of order $1/H$ with the Hubble rate H. If one considers the system on a time scale t with $t_{\text{fast}} \ll t \ll t_{\text{slow}}$, then the fast quantities have already reached their equilibrium values, whereas the slow quantities are still equilibrating.

Although the whole system is not in equilibrium, the subsystem of fast quantities can be considered as being in equilibrium for given values of the slow quantities. According to Landau the system is then in a state of *incomplete equilibrium*, which is entirely determined by the values of the slow quantities and the temperature (which determines the values of the fast quantities). The non-equilibrium system can then be described by effective classical equations of motion [42]

$$\dot{x}_i = \Gamma_i[x], \tag{3.1.2}$$

where Γ depends only on x_i and the temperature T. For very small values of x_i one can linearize the equations

$$\dot{x}_i = -\gamma_{x_i x_j} x_j, \tag{3.1.3}$$

with dissipative coefficients (or rates) $\gamma_{x_i x_j}$. For $x_i \in \{Q_a, \delta f_{I\mathbf{k}}\}$ we get the linear kinematic equations (1.2.13) and (1.2.14)[1].

The solution for x_i with the initial condition $x_i(0) = x_i$ reads[2] [41]

$$x_i(t) = x_j \left(e^{-\gamma t}\right)_{x_i x_j}. \tag{3.1.4}$$

If the size of the slow quantities becomes similar to the size of the average fluctuation, one additionally has to consider stochastic forces F_i, which causes the fluctuations of x_i. In this case the equations for the x_i read [42]

$$\dot{x}_i = -\gamma_{x_i x_j} x_j + F_i. \tag{3.1.5}$$

Up to this point everything has been entirely classical. Every information about quantum physics is hidden in the coefficients γ and in the next section we will show how to compute these coefficients from correlation functions in thermal quantum field theory.

3.2 Determination of the coefficients γ

At first we use the solution (3.1.4) to find a linear relation between the coefficients γ and the classical correlation function [42]

$$\mathscr{C}_{x_i x_j}(t) \equiv \langle x_i(t) x_j(0) \rangle. \tag{3.2.1}$$

Then we match the classical correlation function to its quantum mechanical equivalent for times t in the interval $t_{\text{fast}} \ll t \ll t_{\text{slow}}$. In order to allow for a quantum physical description one has to define reasonable operators for the quantities x_i. Then the quantum mechanical generalization of the correlation function (3.2.1) which does not depend on the order of the operators x_i is the symmetric correlator (2.1.11),

$$C_{x_i x_j}(t) = \frac{1}{2} \langle \{x_i(t), x_j(0)\} \rangle. \tag{3.2.2}$$

Let us start with the classical correlators (3.2.1). Landau found a relation for the rates γ, which does not depend on the stochastic forces F_i even if these forces cannot be neglected in the kinetic equations. The idea is to assume that the stochastic forces are not correlated at different times. This is a good approximation since the stochastic forces only control the fluctuations and are thus only correlated at times of the order t_{fast}. Since the quantities $x_i(t = 0)$ are only correlated with stochastic forces $F_i(t \leq 0)$ we have $\langle F_i(t) x_j(0) \rangle = 0$ for

[1] Note that in (1.2.13) and (1.2.14) we have, for the sake of notational simplicity, considered the rates as matrices in flavor and momentum space defined as $(\gamma_{Q\,\delta f})_{a,I\mathbf{k}} \equiv \gamma_{Q_a\,\delta f_{I\mathbf{k}}}$ and so on.

[2] Although the initial condition lies outside the time interval $[t_{\text{fast}}, t_{\text{slow}}]$ the solution is still correct for times within this interval.

$t > 0$. Thus, taking the time derivative of (3.2.1) and using eq. (3.1.5), the correlation function obeys for $t > 0$ the homogeneous equation [42]

$$\dot{\mathscr{C}}_{x_i x_j}(t) = -\gamma_{x_i x_k} \mathscr{C}_{x_k x_j}(t), \tag{3.2.3}$$

with the solution [41]

$$\mathscr{C}_{x_i x_j}(t) = \chi_{x_k x_j} \left(e^{-\gamma t}\right)_{x_i x_k}, \qquad t > 0, \tag{3.2.4}$$

where $\chi_{x_i x_j}$ are the susceptibilities of the quantities x_i.

A linear relation between γ and the correlator can be found if one expands the one-sided Fourier transform [41]

$$\mathscr{C}^+_{x_i x_j}(\omega) \equiv \int_{-\infty}^{\infty} dt \mathscr{C}_{x_i x_j}(t)\theta(t)e^{i\omega t} = -\left(i\omega - \gamma\right)^{-1}_{ik} \chi_{x_k x_j} \tag{3.2.5}$$

for frequencies ω in the range $\omega_{\mathrm{UV}} \gg \omega \gg |\gamma|$, where $|\gamma|$ is the absolute value of the largest eigenvalue of the matrix $\gamma_{x_i x_j}$ and $\omega_{\mathrm{UV}} \sim 1/t_{\mathrm{fast}}$. This yields [41]

$$\mathrm{Re}\mathscr{C}^+_{x_i x_j}(\omega) = \frac{1}{\omega^2}\gamma_{x_i x_k}\chi_{x_k x_j} + \mathscr{O}(\omega^{-4}). \tag{3.2.6}$$

In order to compute the one-sided Fourier transform of the quantum mechanical correlator we write the θ-function as

$$\theta(t) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i0^+} \tag{3.2.7}$$

and use the relation (2.1.12) between C and ρ. Then we find (see also [41])

$$C^+_{x_i x_j}(\omega) = \int \frac{d\omega'}{2\pi} \frac{i}{\omega - \omega' + i0^+} \left[\frac{1}{2} + f_{\mathrm{B}}(\omega')\right] \rho_{x_i x_j}(\omega'). \tag{3.2.8}$$

We only need the $1/\omega^2$ pole of this correlator as we can see from (3.2.6). For large frequencies ω' the integrand in (3.2.8) is of order ω^0 and does therefore not contribute. But for small frequencies $\omega' \ll T$ we can expand $(1/2 + f_{\mathrm{B}}(\omega)) = T/\omega + \mathscr{O}(\omega)$ and with (2.1.7) and (2.1.8) we find[3]

$$C^+_{x_i x_j}(\omega) = -i\frac{T}{\omega}\Delta^{\mathrm{ret}}_{x_i x_j}(\omega) + \mathscr{O}(\omega^{-1}). \tag{3.2.9}$$

Using equation (3.2.6) and matching the classical correlator with the quantum version (3.2.9), we find the Kubo-type relation

$$\gamma_{x_i x_j} = T \lim_{|\gamma| \ll \omega \ll \omega_{\mathrm{UV}}} \omega \mathrm{Im}\Delta^{\mathrm{ret}}_{x_i x_k}(\omega)(\chi^{-1})_{x_k x_j}. \tag{3.2.10}$$

The coefficients γ in (1.2.13) and (1.2.14) can now be computed with the formula (3.2.10), using proper operators for the quantities x_i. In order to simplify the problem we proceed as

[3] We used the partial fraction decomposition $\frac{1}{(\omega'-\omega)\omega'} = \left(\frac{1}{\omega'-\omega} - \frac{1}{\omega'}\right)\frac{1}{\omega}$

[41] and compute the correlation function of the time derivatives of x_i. Using integration by parts, this can easily related to the original correlator by

$$\Delta_{\dot{x}_i\dot{x}_j}(\omega) = \omega^2 \Delta_{x_i x_j}(\omega), \tag{3.2.11}$$

so that the formula (3.2.10) can be rewritten as

$$\gamma_{x_i x_j} = T \lim_{|\gamma| \ll \omega \ll \omega_{\mathrm{UV}}} \frac{\mathrm{Im}\Delta^{\mathrm{ret}}_{\dot{x}_i \dot{x}_k}(\omega)}{\omega} (\chi^{-1})_{x_k x_j}. \tag{3.2.12}$$

Since we consider quantities x_i which are conserved in the SM, their time derivatives are only determined by the interaction $\mathscr{L}_{\mathrm{int}}$, defined in (1.2.4), and only depend on fields which are changed by $\mathscr{L}_{\mathrm{int}}$. These are the left-handed SM leptons ℓ_i, the $SU(2)$ conjugate Higgs $\tilde{\varphi}$ and the Majorana neutrinos N_I. On the contrary, the quantities x_i themselves can depend on much more fields, which would only complicate the calculation of the correlator. Furthermore, we are interested in the leading order of the correlator in the Yukawa interaction (1.2.4). Since the time derivatives of x_i are already of order h_ν, the perturbative expansion can be abbreviated.

Keep in mind that $\Delta^{\mathrm{ret}}_{\dot{x}_i\dot{x}_j}(\omega)$ in (3.2.12) can be written in terms of the spectral function $\rho_{\dot{x}_i\dot{x}_j}(\omega)$ through the spectral representation (2.1.7). If the spectral function is real, the Kubo formula can be further simplified using (2.2.4) which yields [41]

$$\gamma_{x_i x_j} = \frac{T}{2} \lim_{|\gamma| \ll \omega \ll \omega_{\mathrm{UV}}} \frac{\rho_{\dot{x}_i \dot{x}_k}(\omega)}{\omega} (\chi^{-1})_{x_k x_j}. \tag{3.2.13}$$

We want to compute the rates γ_{QQ}, $\gamma_{Q\delta f}$ and $\gamma_{\delta f \delta f}$ in the kinetic equations (1.2.13) and (1.2.14) only at leading order in the sterile neutrino Yukawa interaction and to higher orders in the SM interactions. The retarded correlators in those equations already contains the full leading h_ν dependence and therefore the susceptibilities can be computed in a system, where the sterile neutrinos are free and the quantities $x_i \in \{Q, \delta f\}$ are conserved. Then the CPT properties from section (2.2) also imply[4]

$$\chi_{Q\delta f} = \chi_{\delta f Q} = 0. \tag{3.2.14}$$

The susceptibilities of the sterile neutrino phase-space densities $\chi_{\delta f \delta f}$ are completely determined by the free theory, but the computation of the susceptibilities χ_{QQ} is non-trivial. We have done this computation in [1] which is part of the next chapter.

[4]Here and in the following we consider the susceptibilities as matrices in flavor and momentum space, such that for example $(\chi_{Q\delta f})_{aI\mathbf{k}} \equiv \chi_{Q_a \delta f_{I\mathbf{k}}}$

Chapter 4

Susceptibilities of conserved charges

In this chapter we take a closer look at the susceptibilities χ_{QQ}[1] of conserved charges in the SM which are for example needed to complete the order g^2 calculation of the washout rate in [41] and for the relation between B and $B-L$ which we present in section 4.7. This chapter closely follows the author's publication [1].

4.1 Computation of the susceptibilities

This section is a modified version of chapter 3 of the author's publication [1].

The conserved charges in the SM can be divided into two categories, the almost conserved charges which are only conserved in the SM and the strictly conserved charges which are also conserved by the Yukawa interaction (1.2.4). Following [41], we denote the strictly conserved charges by $Q_{\bar{a}}$ and the almost conserved charges by Q_a. The SM plasma is then described by the partition function [64]

$$Z(T,\mu) \equiv \exp(-\Omega(T,\mu)/T) \equiv \mathrm{Tr}\exp\left[(\mu_A Q_A - H_{\mathrm{SM}})/T\right], \tag{4.1.1}$$

where Ω is the grand canonical thermodynamic potential, T is the temperature and μ_A are the chemical potentials of the charges Q_A with $A \in \{a, \bar{a}\}$.

The washout rate γ_{QQ} in the kinetic equation (1.2.14) is determined by the susceptibilities of the almost conserved charges Q_a in an ensemble, where the equilibrium value of the strictly conserved quantities is zero. This constrains the chemical potentials of the latter ones by the equations

$$\langle Q_{\bar{a}} \rangle = -\frac{\partial\Omega(T,\mu)}{\partial\mu_{\bar{a}}} = 0. \tag{4.1.2}$$

The susceptibilities of the almost conserved charges Q_a in the constrained system can then be computed as [41]

$$(\chi_{QQ})_{ab} = -T\frac{\partial^2}{\partial\mu_a\partial\mu_b}\left(\Omega(T,\mu)\Big|_{\partial\Omega/\partial\mu_{\bar{a}}=0}\right)\Big|_{\mu=0}. \tag{4.1.3}$$

[1] Mind that we consider the susceptibilities as a matrix with $(\chi_{QQ})_{ab} \equiv \chi_{Q_a Q_b}$

Explicit examples for these susceptibilities at different temperatures can be found in [41, 45].

The thermodynamic potential Ω defined through (4.1.1) can most conveniently be calculated with the imaginary time path-integral [34]

$$\exp(-\Omega/T) = \int \mathscr{D}\Phi_{\mathrm{SM}} \exp\left(\int_0^{1/T} d\tau \left[\mu_A Q_A(-i\tau) + \int d^3x \mathscr{L}_{\mathrm{SM}}\right]\right), \tag{4.1.4}$$

where Φ_{SM} represents all SM fields. For the computation of Ω we pay special attention to the hypercharge chemical potential μ_{Y}. It can be identified with the constant mode of the temporal component of the gauge fields [34, 65],

$$\mu_{\mathrm{Y}} \equiv g_1 \bar{B}_0. \tag{4.1.5}$$

Note that in the imaginary time formalism gauge invariance requires that the temporal components of the gauge fields are imaginary which implies that the gauge charge chemical potential is imaginary.

We proceed as in [34] and compute Ω in two steps by introducing an effective thermodynamic potential $\widetilde{\Omega}(T, \mu, \mu_{\mathrm{Y}})$ which can be obtained from the path integral (4.1.4) by integrating over all fields except the constant mode $\bar{B}_0$. Once $\widetilde{\Omega}$ has been computed, Ω is determined by the remaining integral over $\bar{B}_0$

$$\exp(-\Omega/T) = \int d\bar{B}_0 \exp\left(-\widetilde{\Omega}/T\right). \tag{4.1.6}$$

The integral over the constant mode ensures that Ω is independent of μ_{Y}, so that the total hypercharge Q_{Y} vanishes (Gauß' law). Since for the susceptibilities we only need the $\mathscr{O}(\mu^2)$ part of Ω, the integral over $\bar{B}_0$ can be computed in the saddle point approximation where μ_Y is determined through the condition [1]

$$\frac{\partial\widetilde{\Omega}}{\partial\bar{B}_0} = \frac{\partial\widetilde{\Omega}}{\partial\mu_{\mathrm{Y}}} = 0, \tag{4.1.7}$$

and yields

$$\Omega(T,\mu) = \widetilde{\Omega}(T,\mu,\mu_{\mathrm{Y}})|_{\frac{\partial\widetilde{\Omega}}{\partial\bar{B}_0}=0} + \mathrm{const} \times T. \tag{4.1.8}$$

We could have considered the chemical potentials of other gauge charges like the third component of the electroweak isospin as well. But in this work we consider only the symmetric phase where the Higgs vacuum expectation value is zero. In this case only the hypercharge chemical potential satisfies the saddle point condition (4.1.7) with a non-vanishing value.

In order to compute $\widetilde{\Omega}$ in perturbation theory it is convenient to introduce chemical potentials for each particle species $\alpha \in \{\varphi, \ell_i, e_i, q_i, u_i, d_i\}$ like in [34]. These are linearly related to the chemical potentials of the conserved charges Q_i and the hypercharge Q_{Y} by the relation

$$\mu_\alpha(\mu_{\mathrm{Y}}, \mu_{\mathrm{i}}) = y_\alpha \mu_{\mathrm{Y}} + \sum_i \mu_i T_{i,\alpha}, \tag{4.1.9}$$

where $T_{i,\alpha}$ is the generator of the symmetry transformation corresponding to the charge Q_i acting on the particle species α and y_α is the hypercharge of the species α which we normalize such that $y_\varphi = 1/2$. For example, the generator matrices of $B - L$ are proportional to the unit matrix, with $T_{B-L,q} = T_{B-L,u} = T_{B-L,d} = 1/3$ and $T_{B-L,\ell} = T_{B-L,e} = -1$.

4.2 Dimensional reduction

This section is a modified version of chapter 3 of the author's publication [1].

For a complete perturbative expansion of $\widetilde{\Omega}$ it is important to take special care of the effects of different momentum scales. At high temperature this can be conveniently done with the method of *dimensional reduction* [66, 67, 68, 69]. The contributions of the hard scale $p \sim T$, the soft scale $p \sim gT$ and the ultrasoft scale $p \sim g^2T$ are then separately computed within effective theories which can be obtained by integrating out one momentum scale after another. For this purpose consider the Fourier expansion of the fields

$$\Phi(x) = \sum\!\!\!\!\!\!\int_p e^{i(p_0 t - \mathbf{p}\cdot\mathbf{x})} \widetilde{\Phi}(ip_n, \mathbf{p}), \tag{4.2.1}$$

with imaginary time $t = -i\tau$, sum-integrals[2] $\sum\!\!\!\!\!\!\int_p = T\sum_{p_0}\int \frac{d^3p}{(2\pi)^3}$ and Matsubara frequencies $p_0 = i\pi nT$ with even (odd) integer n for bosons (fermions). In the bosonic case the expansion has a zero-mode $n = 0$ and the fields get contributions from soft and ultrasoft spatial momenta. Fermions do not have such a zero mode and thus all fermionic contributions are hard.

Starting from the path integral (4.1.4) one first integrates over all hard field modes which results in a potential $\widetilde{\Omega}_{\rm hard}$ and an effective Lagrangian $\mathscr{L}_{\rm soft}$ for spatial momenta $|\mathbf{p}| \ll T$ such that

$$\exp\left(-\widetilde{\Omega}/T\right) = \exp\left(-\widetilde{\Omega}_{\rm hard}/T\right) \int \mathscr{D}\Phi_{\rm ultrasoft}\mathscr{D}\Phi_{\rm soft} \exp\left(-\int d^3x \mathscr{L}_{\rm soft}\right). \tag{4.2.2}$$

Since in the first integration step all fields modes with $n \neq 0$ have been eliminated, the Lagrangian $\mathscr{L}_{\rm soft}$ does only contain effective three-dimensional fields.

In the second step one integrates over the remaining zero-modes with soft spatial momenta $|\mathbf{p}| \sim gT$ which leads to the potential $\Omega_{\rm soft}$ and an effective ultrasoft Lagrangian $\mathscr{L}_{\rm ultrasoft}$ for spatial momenta $|\mathbf{p}| \ll gT$. In the final step one integrates over the ultrasoft modes which leads to the potential $\widetilde{\Omega}_{\rm ultrasoft}$. The complete effective grand canonical potential can then be written as a sum of the three parts

$$\widetilde{\Omega} = \widetilde{\Omega}_{\rm hard} + \widetilde{\Omega}_{\rm soft} + \widetilde{\Omega}_{\rm ultrasoft}. \tag{4.2.3}$$

In practical calculations one works as follows [69]. Starting from the full four-dimensional Lagrangian one computes all diagrams which contribute to $\widetilde{\Omega}$ to the desired order in g within naive perturbation theory. But for the sum-integrals which appear in these diagrams one neglects the zero-modes. The result is then $\Omega_{\rm hard}$.

In order to get the effective three-dimensional Lagrangian $\mathscr{L}_{\rm soft}$ one has to write down all possible terms with three-dimensional fields which give contributions to momentum scales $p \ll T$. Of course there are no terms with fermionic fields in the effective Lagrangian because

[2] We assume the volume V to be so large that the sum over the spatial momenta can with a very good approximation be identified with an integral.

they are always hard. All terms in $\mathscr{L}_{\text{soft}}$ have to respect the symmetries of the original four-dimensional Lagrangian. Let us illustrate this by means of a simple example. The Lagrangian of the massless four-dimensional Yukawa theory with a real scalar field ϕ reads

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \bar{\psi}i\partial\!\!\!/\psi - g\bar{\psi}\psi\phi. \tag{4.2.4}$$

At finite temperature this Lagrangian is invariant under rotations and under $\phi \to -\phi$. The most general three-dimensional Lagrangian with the same symmetries, with hard modes integrated out, reads

$$\mathscr{L}_{\text{soft}} = \frac{1}{2}\partial_k\phi\partial^k\phi - \frac{1}{2}m^2\phi^2 - \lambda_1\phi^4 - \lambda_2\phi^6 + ..., \tag{4.2.5}$$

with three dimensional fields ϕ. The parameters m and λ_i in the effective Lagrangian are defined through the original four-dimensional theory and therefore depend only on g and T. They can be determined by matching correlation functions in the original theory at high temperatures $T \gg p$ with the corresponding correlation functions in the effective three-dimensional theory. In principle one has to write down infinitely many terms with different parameters λ_i. But since higher order vertices are of higher order in the original coupling constants, only those vertices have to be considered in the effective Lagrangian which contribute to the desired order in the perturbative expansion of Ω_{soft}.

In the same way one can proceed with the ultrasoft Lagrangian. One writes down all terms with fields which yields contributions to momenta $p \ll gT$. Analogously the parameters of $\mathscr{L}_{\text{ultrasoft}}$ are then determined by the parameter of $\mathscr{L}_{\text{soft}}$ and can be computed by matching correlation function of the soft theory with the corresponding correlation functions in the ultrasoft theory.

Note that in the following we separate the constant mode $\bar{B}_0$ from the gauge fields because it is associated with the hypercharge chemical potential, which is a free parameter of $\widetilde{\Omega}$. Then the constant mode is not part of the three-dimensional gauge fields in the effective theory neither. But it will then appear in the effective parameters of the dimensionally reduced theory.

4.3 Hard contributions

This section is a modified version of chapter 4 of the author's publication [1].

We are interested in contribution to $\widetilde{\Omega}$ to order μ_α^2 in the particle chemical potentials (4.1.9) up to order $g^2 \sim g_i^2 \sim \lambda \sim m_0^2/T^2 \sim |h_i|^2$ in the SM couplings, where g_3 is the strong $SU(3)$ gauge coupling, m_0 is the zero temperature Higgs mass and h_i stands for the Yukawa couplings h_e, h_u and h_d of the charged SM leptons and quarks. Since the fermionic contributions of $\widetilde{\Omega}$ have already been computed in [41], we only need to calculate the contributions of the Higgs chemical potential. The Higgs contributions to $\widetilde{\Omega}$ which contain Yukawa couplings are hard[3] and have also been computed in [41]. Therefore, we need only

[3]This is easy to see since the integrals for diagrams with fermions can be written as products of one-loop integrals.

the terms

$$\mathscr{L}_{\rm SM} = -\varphi^\dagger D_\mu D^\mu \varphi - m_0^2 \varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 \\ - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{2\xi_1}(\partial_\mu B^\mu)^2 - \frac{1}{2\xi_2}(\partial_\mu A^{a\mu})^2 + ..., \tag{4.3.1}$$

from the SM Lagrangian, where ξ_1 and ξ_2 are the gauge parameter of the $U(1)$ and $SU(2)$ gauge fields respectively. The terms which depend on the Higgs chemical potential $\mu_\varphi = y_\varphi \mu_{\rm Y}$ are

$$\delta\mathscr{L} = \mu_\varphi \left[\varphi^\dagger\left(\partial_\tau\varphi\right) - \left(\partial_\tau\varphi^\dagger\right)\varphi\right] + \mu_\varphi^2\varphi^\dagger\varphi + 2g_1\mu_\varphi B_0\varphi^\dagger\varphi + 2g_2\mu_\varphi\varphi^\dagger A_0\varphi. \tag{4.3.2}$$

For convenience we include the quadratic part of (4.3.2) into the Higgs propagator. The sum-integrals which appear in the diagrams do then depend on μ_φ through the Higgs propagator and can be expanded to order μ_φ^2. After this expansion we encounter the bosonic one-loop sum-integral class

$$K_a^b = \sum\!\!\!\!\!\!\int_P \frac{(-ip_0)^b}{(-p^2)^a}, \tag{4.3.3}$$

with the solution for $d = 3 - 2\varepsilon$ spatial dimensions [70]

$$K_a^b = \frac{2\pi^{3/2}T^4}{(2\pi T)^{2a-b}}\left(\frac{\mu^2}{\pi T^2}\right)^\varepsilon \frac{\Gamma(a-\frac{3}{2}+\varepsilon)}{\Gamma(a)}\zeta(2a-b-3+2\varepsilon). \tag{4.3.4}$$

As explained in the previous section, we need to neglect the zero modes in order to obtain the hard contributions. Since the zero modes of these sum-integrals are scaleless integrals, they vanish in dimensional regularization anyway. In the special case

$$K \equiv \sum\!\!\!\!\!\!\int_P \ln(-p^2), \tag{4.3.5}$$

we follow [70] and exploit the fact that $K \sim (T^2)^{(d/2+1/2)}$ and therefore

$$T^2\partial_{T^2}K = \frac{d+1}{2}K. \tag{4.3.6}$$

On the other hand, by acting with the derivative on the sum-integral explicitly one finds

$$T^2\partial_{T^2}K = \frac{1}{2}K + K_1^2. \tag{4.3.7}$$

Therefore, we have

$$K = \frac{2}{d}K_1^2. \tag{4.3.8}$$

All diagrams which contribute to $\Omega_{\rm hard}$ up to order g^2 can be reduced to the following μ_φ-dependent one-loop sum-integrals

$$J_0(\mu_\varphi) \equiv \sum\!\!\!\!\!\!\int_P \ln(-\tilde{p}^2), \qquad J_1(\mu_\varphi) \equiv \sum\!\!\!\!\!\!\int_P \frac{1}{-\tilde{p}^2}, \tag{4.3.9}$$

where $\tilde{p}$ is defined such that $\tilde{p}_0 = p_0 + \mu_\varphi$ and $\tilde{\mathbf{p}} = \mathbf{p}$, and the μ_φ-independent two-loop sum-integral

$$J_2 \equiv \sum\!\!\!\!\!\!\int_{p,q} \frac{1}{p^2 q^2 (p+q)^2} = 0. \tag{4.3.10}$$

We expand J_0 and J_1 to order μ_φ^2 and express the result in terms of the sum-integral class (4.3.3) and plug in the solution (4.3.4) which yields the finite results

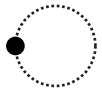
$$J_0(\mu_\varphi) = K + (2K_2^2 - K_1^0)\mu_\varphi^2 + O(\mu_\varphi^4) = -\frac{\pi^2 T^4}{45} - \mu_\varphi^2 \frac{T^2}{6} + O(\mu_\varphi^4), \tag{4.3.11}$$

$$J_1(\mu_\varphi) = K_1^0 + (K_2^0 - 4K_3^2)\mu_\varphi^2 + O(\mu_\varphi^4) = \frac{T^2}{12} - \frac{\mu_\varphi^2}{8\pi^2} + O(\mu_\varphi^4). \tag{4.3.12}$$

The two-loop sum-integral (4.3.10) is only needed for $\mu_\varphi = 0$ because it appears only in the diagram (4.3.18) which contains the 3-vertices in (4.3.2) and therefore a factor μ_φ^2. The fact that J_2 is zero has been found to order $O(\varepsilon)$ in [71, 72], and to all orders in [70].

We compute the diagrams which contribute to $-\widetilde{\Omega}_{\rm hard}/V$ up to order g^2. After performing the $SU(2)$ gauge group traces, we find the leading order and order g^2 Higgs contributions

$$= -2J_0(\mu_\varphi) = 2\left(\frac{\pi^2 T^4}{45} + \mu_\varphi^2 \frac{T^2}{6} + O(\mu_\varphi^4)\right), \tag{4.3.13}$$

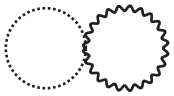


$$= -2J_1(\mu_\varphi) = -2m_0^2\left(\frac{T^2}{12} - \frac{\mu_\varphi^2}{8\pi^2} + O(\mu_\varphi^4)\right), \tag{4.3.14}$$

$\frac{1}{2}$ 

$$= -6\lambda J_1^2(\mu_\varphi) = -\lambda\frac{T^2}{2}\left(\frac{T^2}{12} - \frac{\mu_\varphi^2}{4\pi^2}\right) + O(\mu_\varphi^4). \tag{4.3.15}$$

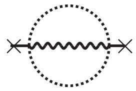
Here we have explicitly written the symmetry factor 1/2 in front of the two-loop integral. In the same way we find for the interactions of the Higgs with gauge bosons the contributions

$\frac{1}{2}$ 

$$\begin{aligned} &= -\frac{d+1}{2}\left(g_1^2 + 3g_2^2\right) J_1(\mu_\varphi) J_1(0) \\ &= -\frac{d+1}{2}\left(g_1^2 + 3g_2^2\right)\frac{T^2}{12}\left(\frac{T^2}{12} - \frac{\mu_\varphi^2}{8\pi^2} + O(\mu_\varphi^4)\right), \end{aligned} \tag{4.3.16}$$

$\frac{1}{2}$ 

$$\begin{aligned} &= \frac{1}{4}(g_1^2 + 3g_2^2) \sum\!\!\!\!\!\!\int_{p,q} \frac{(2p+q)^2}{p^2 q^2 (p+q)^2} \\ &= \frac{1}{4}(g_1^2 + 3g_2^2)\left[4J_1(\mu_\varphi)J_1(0) - J_1^2(\mu_\varphi) + O(\mu_\varphi^4)\right] \\ &= \frac{1}{4}(g_1^2 + 3g_2^2)\frac{T^2}{12}\left(3\frac{T^2}{12} - 2\frac{\mu_\varphi^2}{8\pi^2} + O(\mu_\varphi^4)\right). \end{aligned} \tag{4.3.17}$$

$$\frac{1}{2}$$  $$= -\frac{1}{4}\mu_\varphi^2(g_1^2 + 3g_2^2)J_2 + O(\mu_\varphi^4) = O(\mu_\varphi^4), \tag{4.3.18}$$

where we used the gauge boson propagators in Feynman gauge. But we have checked that the result is gauge parameter independent. The cross in the diagram (4.3.18) represents the 3-vertices in (4.3.2).

Now we combine the hard purely bosonic contributions above with the ones containing fermions from [41] and obtain

$$\begin{aligned}
-\frac{12}{VT^2}&\left[\widetilde{\Omega} - \widetilde{\Omega}(\mu = 0)\right]_{\text{hard}} \\
&= 6\left[1 - \frac{3}{8\pi^2}\left(\frac{g_1^2}{36} + \frac{3g_2^2}{4} + \frac{4g_3^2}{3}\right)\right]\text{tr}(\mu_q^2) \\
&+ 3\left[1 - \frac{3}{8\pi^2}\left(\frac{4g_1^2}{9} + \frac{4g_3^2}{3}\right)\right]\text{tr}(\mu_u^2) \\
&+ 3\left[1 - \frac{3}{8\pi^2}\left(\frac{g_1^2}{9} + \frac{4g_3^2}{3}\right)\right]\text{tr}(\mu_d^2) \\
&+ 2\left[1 - \frac{3}{8\pi^2}\left(\frac{g_1^2}{4} + \frac{3g_2^2}{4}\right)\right]\text{tr}(\mu_\ell^2) \\
&+ \quad \left[1 - \frac{3}{8\pi^2}g_1^2\right]\text{tr}(\mu_e^2) \\
&+ 4\left[1 + \frac{3}{4\pi^2}\left(\frac{1}{2}\lambda + \frac{g_1^2 + 3g_2^2}{8} + \frac{m_0^2}{T^2}\right)\right]\mu_\varphi^2 \\
&+ 3\left[\frac{1}{4\pi^2}\text{tr}(h_u h_u^\dagger)\mu_\varphi^2 - \frac{3}{8\pi^2}\text{tr}\Big(h_u^\dagger h_u \mu_q^2 + h_u h_u^\dagger \mu_u^2\Big)\right] \\
&+ 3\left[\frac{1}{4\pi^2}\text{tr}(h_d h_d^\dagger)\mu_\varphi^2 - \frac{3}{8\pi^2}\text{tr}\Big(h_d^\dagger h_d \mu_q^2 + h_d h_d^\dagger \mu_d^2\Big)\right] \\
&+ \left[\frac{1}{4\pi^2}\text{tr}(h_e h_e^\dagger)\mu_\varphi^2 - \frac{3}{8\pi^2}\text{tr}\Big(h_e^\dagger h_e \mu_\ell^2 + h_e h_e^\dagger \mu_e^2\Big)\right] + O(\mu^4)\ .
\end{aligned} \tag{4.3.19}$$

Here tr(...) denotes the trace in flavor space.

4.4 The dimensionally reduced theory

This section is a modified version of chapter 5 of the author's publication [1].

Now we consider the effective soft theory. As explained in section 4.2, one has to write down all terms with three-dimensional soft fields which respect the symmetries of the original theory. For the SM this has been done in [69] at zero μ and the terms of their result which

contribute to the order g^2 are [4]

$$\begin{aligned}
-\mathscr{L}_{\text{soft},\,\mu_\varphi=0} &= \frac{1}{4}F_{ij}F_{ij} + \frac{1}{4}W_{ij}W_{ij} + \frac{1}{2\xi_1}(\partial_i B_i)^2 + \frac{1}{2\xi_2}(\partial_i A_i^a)^2 \\
&+ \varphi^\dagger \mathbf{D}^2\varphi + m_3^2\varphi^\dagger\varphi + \lambda_3\left(\varphi^\dagger\varphi\right)^2 \\
&- \frac{1}{2}(\partial_i B_0)^2 - \frac{1}{2}m_{D,1}^2 B_0^2 - \frac{1}{2}(D_i A_0)^2 - \frac{1}{2}m_{D,2}^2\text{Tr}\left(A_0^2\right) \\
&- h_1\varphi^\dagger\varphi B_0^2 - h_2\varphi^\dagger\varphi\text{Tr}\left(A_0^2\right). \qquad (4.4.1)
\end{aligned}$$

In addition to (4.3.2) we get the following μ_φ dependent terms

$$-\delta\mathscr{L}_{\text{soft}} = -\mu_\varphi^2\varphi^\dagger\varphi - \rho_1\varphi^\dagger B_0\varphi - \rho_2\varphi^\dagger A_0\varphi. \qquad (4.4.2)$$

For the couplings in (4.4.1) we only need the leading order matching [69]

$$g_{i,3}^2 = g_i^2 T\ (i=1,2,3), \quad \lambda_3 = \lambda T, \quad h_1 = g_1^2 y_\varphi^2 T, \quad h_2 = \frac{1}{4}g_2^2 T. \qquad (4.4.3)$$

Furthermore, by matching three-point vertices of (4.3.2) with (4.4.2), we find the new parameters in $\delta\mathscr{L}_{\text{soft}}$,

$$\rho_1 = 2\mu_\varphi y_\varphi g_1, \quad \rho_2 = 2\mu_\varphi g_2. \qquad (4.4.4)$$

We need the thermal masses of the Higgs and the gauge bosons only at order g^2. Then the thermal Higgs mass is [67, 69]

$$m_3^2 = m_0^2 + T^2\left(\frac{1}{2}\lambda + \frac{3}{16}g_2^2 + \frac{1}{16}g_1^2 + \frac{1}{4}h_t^2\right), \qquad (4.4.5)$$

and the Debye masses for A_0, B_0 are [69],

$$\begin{aligned}
m_{D,1}^2 &= \left(\frac{N_s}{6} + \frac{5n_\text{f}}{9}\right)g_1^2T^2, \qquad (4.4.6)\\
m_{D,2}^2 &= \left(\frac{2}{3} + \frac{N_s}{6} + \frac{n_\text{f}}{3}\right)g_2^2T^2, \qquad (4.4.7)
\end{aligned}$$

where $N_s = 1$ is the number of Higgs doublets and $n_\text{f} = 3$ is the number of families. For convenience, we define a μ_φ-dependent Higgs mass as

$$m_{3,\mu_\varphi}^2 \equiv m_3^2 - \mu_\varphi^2. \qquad (4.4.8)$$

The loop integrals which contribute to $\widetilde{\Omega}_{\text{soft}}$ are, like in section 4.3, μ_φ - dependent through the Higgs propagator. In our calculation we encounter the standard one-loop integrals[5] [67]

$$I_0(m) = \int_{\mathbf{k}}\ln(k^2+m^2) = \frac{2m^d}{d}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{d/2}} = -\frac{m^3}{6\pi} + O(\varepsilon), \qquad (4.4.9)$$

$$I_1(m) = \int_{\mathbf{k}}\frac{1}{(k^2+m^2)} = m^{d-2}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{d/2}} = -\frac{m}{4\pi} + O(\varepsilon) \qquad . \qquad (4.4.10)$$

[4] The term $\varphi^\dagger A_0 B_0\varphi$ term does not contribute at $O(g^2)$.

[5] We use the notation $\int_{\mathbf{k}} = \int\frac{d^3k}{(2\pi)^3}$

In the case $m = m_{3,\mu_\varphi}$ we expand in powers of μ_φ^2,

$$I_0(m_{3,\mu_\varphi}) = -\frac{m_3^3}{6\pi} + \frac{\mu_\varphi^2 m_3}{4\pi} + O(\mu_\varphi^4), \tag{4.4.11}$$

$$I_1(m_{3,\mu_\varphi}) = -\frac{m_3}{4\pi} + \frac{\mu_\varphi^2}{8\pi m_3} + O(\mu_\varphi^4). \tag{4.4.12}$$

The only two-loop integral we need is [67, 71]

$$\begin{aligned} I(m_a, m_b, m_c) =& \int_{\mathbf{k}_1,\mathbf{k}_2} \frac{1}{\left(\mathbf{k}_1^2 + m_a^2\right)\left(\mathbf{k}_2^2 + m_b^2\right)\left[(\mathbf{k}_1+\mathbf{k}_2)^2 + m_c^2\right]} \\ =& \frac{1}{16\pi^2}\left[\frac{1}{4\varepsilon} + \ln\left(\frac{\bar{\mu}}{m_a+m_b+m_c}\right) + \frac{1}{2}\right] + O(\varepsilon). \end{aligned} \tag{4.4.13}$$

where $\bar{\mu}$ is the $\overline{\text{MS}}$ scale parameter. In the special case $m_a = m_{3,\mu_\varphi}$, $m_b = m \in \{0, m_{D,1}, m_{D,2}\}$ and $m_c = m_{3,\mu_\varphi}$ it is useful to expand in μ_φ^2,

$$\begin{aligned} I(m_{3,\mu_\varphi}, m, m_{3,\mu_\varphi}) =& \frac{1}{16\pi^2}\left[\frac{1}{4\varepsilon} + \ln\left(\frac{\bar{\mu}}{2m_{3,\mu_\varphi}+m}\right) + \frac{1}{2}\right] \\ =& \frac{1}{16\pi^2}\left[\frac{1}{4\varepsilon} + \ln\left(\frac{\bar{\mu}}{2m_3+m}\right) + \frac{1}{2} + \frac{\mu_\varphi^2}{m_3(2m_3+m)}\right] + O(\mu_\varphi^4). \end{aligned} \tag{4.4.14}$$

4.5 Soft contributions for soft Higgs mass

This section is a modified version of chapter 6 of the author's publication [1].

Let us take a closer look at the thermal Higgs mass (4.4.5). The zero temperature Higgs mass parameter m_0^2 is negative with $\sqrt{-m_0^2} \approx 125$ GeV and can cancel the g^2T contributions at temperatures close to the electroweak scale ($T \sim 160$ GeV). Then the thermal Higgs mass is ultrasoft. This case will be discussed in section 4.6. In this section we consider temperatures high enough so that m_3^2 is of order $(gT)^2$ and positive.

In the following we compute the diagrams which contribute to $\widetilde{\Omega}_{\text{soft}}$ up to order g^2. Performing traces in $SU(2)$ gauge group space, we find the leading order and order g^2 Higgs contributions

$$= -2TI_0(m_{3,\mu\varphi}) = 2T\left(\frac{m_3^3}{6\pi} - \frac{\mu_\varphi^2 m_3}{4\pi} + O(\mu_\varphi^4)\right), \tag{4.5.1}$$

$\frac{1}{2}$ 

$$= -6\lambda T^2\left[I_1(m_{3,\mu_\varphi})\right]^2 = -\frac{3\lambda T^2}{8\pi^2}\left(m_3^2 - \mu_\varphi^2\right) + O(\mu_\varphi^4). \tag{4.5.2}$$

Note that the μ_φ^2-term has the same parametric form as the one in (4.3.15). The sum of (4.5.2) and (4.3.15) yields the $O(\lambda)$ correction, that has been computed in [41] by a Higgs mass resummation. For the interaction between Higgs and the gauge fields we present the

calculation again only in Feynman gauge. But we have also checked that the result is gauge parameter independent. This yields

$\frac{1}{2}$ 

$$
\begin{aligned}
=&\frac{T^2}{4}(g_1^2+3g_2^2)\int_{\mathbf{k}_1,\mathbf{k}_2}\frac{(2\mathbf{k}_1+\mathbf{k}_2)^2}{(\mathbf{k}_1^2+m_{3,\mu_\varphi}^2)\mathbf{k}_2^2[(\mathbf{k}_1+\mathbf{k}_2)^2+m_{3,\mu_\varphi}^2]}\\
=&-\frac{T^2}{4}(g_1^2+3g_2^2)\left\{\left[I_1(m_{3,\mu_\varphi})\right]^2+4m_{3,\mu_\varphi}^2 I(m_{3,\mu_\varphi},0,m_{3,\mu_\varphi})\right\}\\
=&\frac{\mu_\varphi^2T^2}{32\pi^2}\,(g_1^2+3g_2^2)\left[\frac{1}{2\varepsilon}+\frac{1}{2}+2\ln\left(\frac{\bar\mu}{2m_3}\right)\right]+\cdots\,,
\end{aligned}
\tag{4.5.3}
$$

$\frac{1}{2}$ 

$$
\begin{aligned}
=&-\mu_\varphi^2T^2\left[g_1^2I(m_{3,\mu_\varphi},m_{D,1},m_{3,\mu_\varphi})+3g_2^2I(m_{3,\mu_\varphi},m_{D,2},m_{3,\mu_\varphi})\right]\\
=&-\frac{\mu_\varphi^2T^2}{32\pi^2}\left\{g_1^2\left[\frac{1}{2\varepsilon}+1+2\ln\left(\frac{\bar\mu}{2m_3+m_{D,1}}\right)\right]\right.\\
&\left.+3g_2^2\left[\frac{1}{2\varepsilon}+1+2\ln\left(\frac{\bar\mu}{2m_3+m_{D,2}}\right)\right]\right\}+\cdots,
\end{aligned}
\tag{4.5.4}
$$

$\frac{1}{2}$ 

$$
\begin{aligned}
=&-\frac{1}{2}g_1^2T^2I_1(m_{3,\mu_\varphi})I_1(m_{D,1})-\frac{3}{2}g_2^2T^2I_1(m_{3,\mu_\varphi})I_1(m_{D,2})\\
=&-\frac{T^2}{32\pi^2}\left(g_1^2m_{3,\mu_\varphi}m_{D,1}+3g_2^2m_{3,\mu_\varphi}m_{D,2}\right)\\
=&\frac{\mu_\varphi^2T^2}{32\pi^2}\frac{1}{2m_3}\left(g_1^2m_{D,1}+3g_2^2m_{D,2}\right)+\cdots\,,
\end{aligned}
\tag{4.5.5}
$$

where we omitted terms of orders other than μ_φ^2. The solid line in the above diagrams represent the soft temporal components A_0 and B_0. Adding up all contributions we obtain the finite result

$$
\begin{aligned}
&-\frac{12}{VT^2}\left[\widetilde\Omega(\mu)-\widetilde\Omega(0)\right]_{\rm soft}\\
&\qquad=2\mu_\varphi^2\left\{-\frac{3m_3}{\pi T}+\frac{9\lambda}{4\pi^2}+\frac{3}{32\pi^2}\left[g_1^2C_1+3g_2^2C_2\right]\right\}+O(\mu_\varphi^4),
\end{aligned}
\tag{4.5.6}
$$

with

$$
C_i\equiv\frac{m_{D,i}}{m_3}-1-4\ln\left(\frac{2m_3}{2m_3+m_{D,i}}\right).
\tag{4.5.7}
$$

After integrating out the soft fields, we are left with an effective theory for the ultrasoft ones. For soft m_3 the ultrasoft theory contains only the spatial gauge fields and at order g^2 and μ_φ^2 the effective Lagrangian is independent of μ_φ so that this sector does not contribute to the susceptibilities. This implies $\widetilde\Omega_{\rm ultrasoft}=0$.

4.6 Ultrasoft Higgs mass

This section is a modified version of chapter 7 of the author's publication [1].

The perturbative expansion within the effective theory in section 4.5 works well, as long as the Higgs mass is not much smaller than gT. From our result (4.5.6) we can estimate how soft m_3 can be for the perturbative expansion in section 4.5 to be valid. The diagram (4.5.5) contains a self-energy contribution to the Higgs field from soft gauge fields. If $m_3^2 \sim g^3T^2$, this contributions is of the same size as m_3 itself. The two-loop contribution (4.5.5) becomes therefore as large as the one-loop contribution (4.5.1) and the perturbative expansion breaks down. In this case the Higgs field has to be included in an effective theory for momenta $p \ll gT$. For such small Higgs masses m_3 does not contribute to Ω_{soft}. Therefore, we have to set $m_3 = 0$ in all diagrams in section 4.5. Then all diagrams but (4.5.4) vanish in dimensional regularization and Ω_{soft} is divergent and reads

$$-\frac{12}{VT^2}\left[\widetilde{\Omega}(\mu)-\widetilde{\Omega}(0)\right]_{\text{soft}} = -\frac{3\mu_\varphi^2}{8\pi^2}\left\{g_1^2\left[\frac{1}{2\varepsilon}+1+2\ln\left(\frac{\bar{\mu}}{m_{D,1}}\right)\right] +3g_2^2\left[\frac{1}{2\varepsilon}+1+2\ln\left(\frac{\bar{\mu}}{m_{D,2}}\right)\right]\right\}. \tag{4.6.1}$$

We obtain the contributions for momenta $p \ll gT$ from the effective ultrasoft Lagrangian

$$-\mathscr{L}_{\text{ultrasoft}} = \frac{1}{4}F_{ij}F_{ij} + \frac{1}{4}W_{ij}W_{ij} - \varphi^\dagger \mathbf{D}^2\varphi + \overline{m}_{3,\mu_\varphi}^2\varphi^\dagger\varphi + \bar{\lambda}_3\left(\varphi^\dagger\varphi\right)^2, \tag{4.6.2}$$

with the parameters [69]

$$\overline{m}_{3,\mu_\varphi}^2 = m_{3,\mu_\varphi}^2 - \frac{1}{4\pi}\left(3h_2 m_{D,2} + y_\varphi h_1 m_{D,1}\right), \tag{4.6.3}$$

$$\bar{\lambda}_3 = \lambda_3. \tag{4.6.4}$$

The effective Higgs mass squared $\overline{m}_3^2$ does now contain a negative contribution of the order g^3T^2, corresponding to the self-energy correction from the soft temporal gauge fields in (4.5.5). The perturbative expansion in this theory works well as long as the dimensionless expansion parameter $g^2T/\overline{m}_3 \ll 1$, which is true as long as $\overline{m}_3$ is much larger than g^2T. Let us consider the case $0 < \overline{m}_3^2 \sim g^3T$. Then the expansion parameter is $g^{1/2}$ and we are still in the symmetric phase. The diagrams which contribute to $\widetilde{\Omega}_{\text{ultrasoft}}$ are (4.5.1), (4.5.2) and (4.5.3) with m_3 replaced by $\overline{m}_3$. Then we obtain

$$-\frac{12}{VT^2}\left[\widetilde{\Omega}(\mu)-\widetilde{\Omega}(0)\right]_{\text{ultrasoft}} = \mu_\varphi^2\left\{-\frac{6\overline{m}_3}{\pi T}+\frac{9\lambda}{2\pi^2} + 3\frac{g_1^2+3g_2^2}{8\pi^2}\left[\frac{1}{2\varepsilon}+\frac{1}{2}+2\ln\left(\frac{\bar{\mu}}{2\overline{m}_3}\right)\right]\right\}. \tag{4.6.5}$$

Combining the soft contributions (4.6.1) with the ultra soft contributions (4.6.6), the diver-

gences cancel and we find the finite result

$$-\frac{12}{VT^2}\left[\widetilde{\Omega}(\mu)-\widetilde{\Omega}(0)\right]_{\text{soft+ultrasoft}} = 2\mu_\varphi^2\left[-\frac{3\bar{m}_3}{\pi T}+\frac{9\lambda}{4\pi^2}+\frac{3}{32\pi^2}\left(g_1^2\bar{C}_1+3g_2^2\bar{C}_2\right)\right], \tag{4.6.6}$$

with

$$\bar{C}_i \equiv -1-4\ln\left(\frac{2\overline{m}_3}{m_{D,i}}\right). \tag{4.6.7}$$

Since we consider $\overline{m}_3 \sim g^3T$, the leading order is $\mathscr{O}(g^{3/2})$ and the next-to-leading order is $\mathscr{O}(g^2)$. Furthermore, we have terms in this expression which are parametrically $\ln(m_{Di}/\overline{m}_3) \sim \ln(1/g)$.

Now let us consider the case that $\overline{m}_3 \sim g^2T$. In this case the one-loop contribution is of order g^2 such as the two-loop contributions and perturbation theory again breaks down. The reason is that the only mass scale in the ultrasoft theory is then the magnetic screening scale g^2T and the dimensionless expansion parameter is of order one. Thus an infinite number of diagrams contribute to the order g^2, which is the so-called Linde problem [73].

However, one can derive an expression for (4.6.6) which can also be evaluated with non-perturbative methods. Writing the ultrasoft Lagrangian as

$$\mathscr{L}_{\text{ultrasoft}} = (\mathscr{L}_{\text{ultrasoft}})_{\mu_\varphi=0} + \mu_\varphi^2\varphi^\dagger\varphi, \tag{4.6.8}$$

one can easily expand the path integral

$$\exp(-\widetilde{\Omega}_{\text{ultrasoft}}/T) = \int \mathscr{D}\Phi_{\text{ultrasoft}} \exp\left\{\int d^3x \mathscr{L}_{\text{ultrasoft}}\right\} \tag{4.6.9}$$

to second order in μ_φ which yields

$$\left[\widetilde{\Omega}(\mu)-\widetilde{\Omega}(0)\right]_{\text{ultrasoft}} = -VT\mu_\varphi^2\left\langle\varphi^\dagger\varphi\right\rangle + O(\mu_\varphi^4). \tag{4.6.10}$$

The expectation value of $\varphi^\dagger\varphi$ can be extracted from effective potential and can be perturbatively expanded as long as $\overline{m}_3 \gg g^2T$. The two two-loop expansion is [53, 67, 69]

$$\langle\varphi^\dagger\varphi\rangle_{2-\text{loop}} = -\frac{\overline{m}_3T}{2\pi}+\frac{T^2}{16\pi^2}\left\{6\lambda+\left(g_1^2+3g_2^2\right)\left[\frac{1}{4\varepsilon}+\ln\left(\frac{\bar{\mu}}{2\overline{m}_3}\right)+\frac{1}{4}\right]\right\}, \tag{4.6.11}$$

which again leads to our result (4.6.6).

When the Higgs mass becomes as small as the magnetic screening scale, $\overline{m}_3 \sim g^2T$, the expression (4.6.10) is still valid, but its perturbative expansion in terms of loop diagrams breaks down. From the perturbative expansion (4.6.11) we have seen that the ultrasoft one-loop and two-loop contributions are of order g^2 (plus order $g^2\ln(g)$) if $\overline{m}_3 \sim g^2T$. With a simple argument one can show that the complete contribution from ultrasoft fields is of this order when $\overline{m}_3 \sim g^2T$. Since the three-dimensional fields have mass dimension 1/2, and since

the only mass scale in the ultrasoft theory is g^2T, we have $\langle\varphi^\dagger\varphi\rangle \sim g^2T$ plus possible order $g^2\ln(g)$ contributions.

A reliable calculation of $\langle\varphi^\dagger\varphi\rangle$ near the electroweak crossover can be done with lattice simulations of the three-dimensional theory (4.6.2). Such simulations have been done in [74]for a $SU(2)$+Higgs theory for several Higgs masses and in [22] for $m_H = (125-126)$ GeV. A study which also takes into account the $U(1)$ gauge fields can be found in [75]. Near the electroweak crossover $\langle\varphi^\dagger\varphi\rangle$ turned out to be a rather smooth function of the temperature.

We do not consider the case where $\overline{m}_3^2$ becomes negative. In this case the system would be in the broken phase, where the Higgs field develops an expectation value. Furthermore, in the presence of chemical potentials of the broken charges, the $SU(2)$ gauge fields would have a non-vanishing expectation value as well which can be identified with the chemical potential of the third component of the electroweak isospin. This case has been studied at leading order in [34].

4.7 Relation between B and $B-L$

This section is a modified version of chapter 8 of the author's publication [1].

In this section we use our results for $\widetilde{\Omega}$ to compute the relation between the baryon number B and $B-L$ at order g^2 in the symmetric phase. For the computation of $\langle B\rangle$ it is convenient to introduce a chemical potential μ_B in the partition function as

$$\exp(-\Omega'/T) \equiv \mathrm{Tr}\exp\left[(\mu_A Q_A + \mu_B B - H_{\mathrm{SM}})/T\right]. \tag{4.7.1}$$

The artificial chemical potential μ_B is assumed to be independent of the chemical potentials μ_i of the conserved charges. Then, the expectation value of B can be computed as

$$\langle B\rangle = -\left.\frac{\partial\Omega'}{\partial\mu_B}\right|_{\mu_B=0}. \tag{4.7.2}$$

The introduction μ_{B} is only valid as long as the partition function is expanded to the linear order in μ_{B}. At this order the cyclicity of the trace ensures that the ordering of the operators B and H does not matter.

The results (4.3.19), (4.5.6) and (4.6.6) for $\widetilde{\Omega}'$ depend on the particle chemical potentials which have to be related to the chemical potential of the conserved charges $X_i = B/n_{\mathrm{f}} - L_i$ through the relation (4.1.9). Then, the chemical potentials of the particles are related to μ_{X_i} and μ_B by

$$\begin{aligned}
\mu_{q_i} &= \frac{\mu_Y}{6} + \frac{\mu_X+\mu_B}{3},\\
\mu_{u_i} &= \frac{2\mu_Y}{3} + \frac{\mu_X+\mu_B}{3},\\
\mu_{d_i} &= -\frac{\mu_Y}{3} + \frac{\mu_X+\mu_B}{3},\\
\mu_{l_i} &= -\frac{\mu_Y}{2} - \mu_{X_i},\\
\mu_{e_i} &= -\mu_Y - \mu_{X_i},
\end{aligned}$$

$$\mu_\varphi = \frac{\mu_Y}{2}, \tag{4.7.3}$$

where $\mu_X \equiv \frac{1}{n_\mathrm{f}} \sum_{i=1}^{n_\mathrm{f}} \mu_{X_i}$. Using these relations, we express $\widetilde{\Omega}'$ in terms of μ_Y, μ_{X_i} and μ_B and use the saddle point condition (4.1.7) to eliminate μ_Y and to obtain Ω'. Then, using (4.7.2), yields $\langle B \rangle$ as a linear function of μ_X. Furthermore, we obtain a linear relation between X_i and μ_{X_i} from

$$X_i = -\frac{\partial \Omega}{\partial \mu_{X_i}}, \tag{4.7.4}$$

which can be used to express the μ_X dependence of $\langle B \rangle$ in terms of $\langle B - L \rangle$, such that

$$\langle B \rangle = \kappa \langle B - L \rangle. \tag{4.7.5}$$

For m_3 of order gT we obtain

$$\begin{aligned}
\kappa =& \frac{4(2n_\mathrm{f} + N_s)}{22n_\mathrm{f} + 13N_s} + \frac{m_3}{\pi T} \frac{24 n_\mathrm{f} N_s}{(22n_\mathrm{f} + 13N_s)^2} \\
&+ \frac{g_1^2}{16\pi^2} \frac{236 n_\mathrm{f}^2 - (12C_1 - 212) n_\mathrm{f} N_s + 75 N_s^2}{(22n_\mathrm{f} + 13N_s)^2} \\
&+ \frac{g_2^2}{16\pi^2} \frac{9(12 n_\mathrm{f}^2 - 4(C_2 - 1) n_\mathrm{f} N_s + 3N_s^2)}{(22n_\mathrm{f} + 13N_s)^2} \\
&- \frac{g_3^2}{16\pi^2} \frac{96(8 n_\mathrm{f}^2 + 11 n_\mathrm{f} N_s + 3N_s^2)}{(22n_\mathrm{f} + 13N_s)^2} \\
&+ \frac{h_t^2}{16\pi^2} \frac{6(6 n_\mathrm{f}^2 - 41 n_\mathrm{f} N_s - 18N_s^2)}{(22n_\mathrm{f} + 13N_s)^2} \\
&- \frac{\lambda}{16\pi^2} \frac{384 n_\mathrm{f} N_s}{(22n_\mathrm{f} + 13N_s)^2} \\
&- \frac{m_0^2}{(\pi T)^2} \frac{12 n_\mathrm{f} N_s}{(22n_\mathrm{f} + 13N_s)^2},
\end{aligned} \tag{4.7.6}$$

with the same definitions as in (4.4.7) and (4.5.7). When $\bar{m}_3^2 \sim g^3 T^2$, the result for κ can be obtained from (4.7.6) by replacing m_3 by $\overline{m}_3$ and C_i by $\bar{C}_i$ defined in (4.6.3) and (4.6.7).

In figure 4.1 it can be seen that the next-to-leading order (NLO) correction to κ which comes from the one-loop Higgs correction is smaller than 1%. At next-to-next-to-leading order (NNLO) the strong coupling g_3 enters the perturbative expansion. Therefore, the NNLO corrections are much larger than the NLO Higgs correction, but still smaller than 5%. However, as one can see in figure 4.1, the pure electroweak NNLO corrections are much smaller than the NLO correction which indicates that perturbation theory works well.

Let us now consider temperatures close to the electroweak crossover. As long as $\overline{m}_3$ is not much smaller than $g^{3/2}T$, we can still identify the one-loop Higgs corrections with the NLO and the two-loop corrections with the NNLO. In figure 4.2 have plotted the NLO, the NNLO with the Higgs mass m_3 treated as soft and and the NNLO with ultrasoft Higgs mass $\overline{m}_3$. For such small temperatures the thermal Higgs mass m_3 becomes very small and therefore the NLO contribution tends to zero close to the electroweak crossover. If the Higgs mass

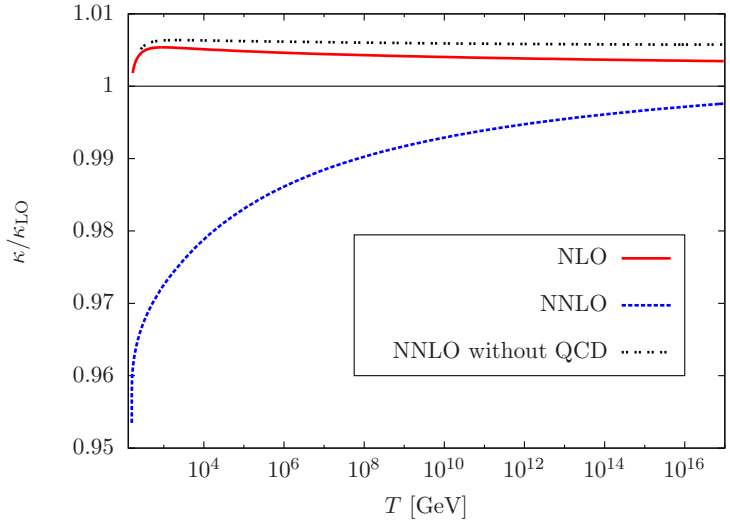


Figure 4.1: The radiative corrections of κ compared to the leading order for $m_H = 126$ GeV, $n_{\rm f} = 3$ and $N_s = 1$. The NLO corrections are smaller than 1% and the electroweak NNLO correction are even smaller. At NNLO the QCD corrections dominate the result, but they are still smaller than 5%. This plot has been been published in [1]

m_3 is treated as soft, the NNLO diverges like $\sim 1/m_3$ at low temperatures. This divergence disappears if the Higgs is treated in the ultrasoft theory with a mass parameter $\overline{m}_3$. However, there is still a logarithmic divergence at $T \gtrsim 165$ GeV where $\overline{m}_3 \ll g^{3/2}T$. At this point the perturbative expansion breaks down and lattice simulations for the expression (4.6.10) have to be considered.

We have also compared the size of the contributions from the Higgs chemical potential with the size of the contributions from fermionic chemical potentials. Figure 4.3 indicates that the NNLO contributions from the Higgs chemical potential is nearly as large as the electroweak NNLO contributions from fermionic chemical potentials.

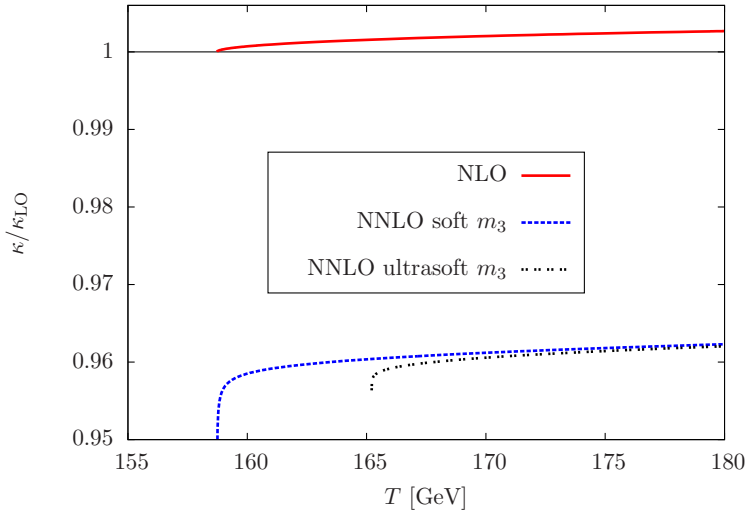


Figure 4.2: The radiative corrections of κ at low temperatures with $m_H = 126$ GeV, $n_{\rm f} = 3$ and $N_s = 1$. If the Higgs mass is treated as soft, the result diverges for $T < 160$ GeV, and if the the Higgs mass is treated as ultrasoft, it diverges even at $T \sim 165$ GeV. This is the region where perturbation theory breaks down. This plot has been been published in [1]

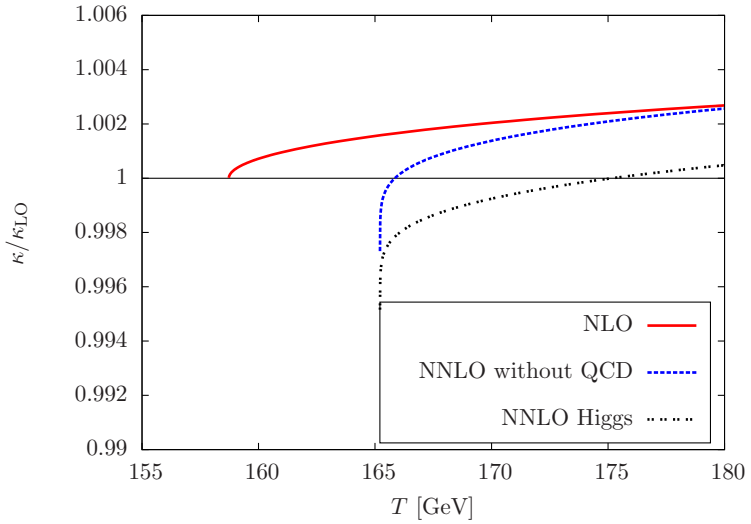


Figure 4.3: The contribution of the Higgs chemical potential is similar large as the contributions from fermionic chemical potentials. At $T > 175$ GeV the Higgs correction almost cancels the fermionic NNLO corrections, so that the electroweak NNLO is almost as large as the NLO.

Chapter 5

Lepton washout rate at low temperatures

The susceptibilities which we considered in the previous chapter are in particular important for the computation of the washout rate γ_{QQ}. This rate has been computed at order h_ν^2 in [41]. In this chapter we consider the washout rate at low temperatures $T \ll M_1$. In this limit the rate is of order h_ν^4 and is dominated by $\Delta L = 2$ sterile neutrino mediated scattering processes. This rate is an important ingredient for finding upper bounds on sterile neutrino masses, as pointed out in [45]. The calculations and results of this chapter are planned to be published in [2].

5.1 Neutrino masses and the washout rate

The discovery of solar [76] and atmospheric [77] neutrino oscillations proves that at least two of the three known (active) neutrinos are massive. These masses cannot be explained in the SM, but the Yukawa interaction (1.2.4) which couples left-handed neutrinos with right-handed neutrinos can be a source for neutrino masses. In the broken phase the Higgs field $\tilde{\varphi}$ obtains a vacuum expectation value $\langle\tilde{\varphi}\rangle = (v, 0)^\top$ so that the interaction (1.2.4) leads to a Dirac mass term

$$\mathscr{L}_{\text{Dirac}} = -(m_{\text{D}})_{Ii}\overline{\nu_{R,I}}\nu_{L,i} + \text{H.c}, \tag{5.1.1}$$

with the Dirac mass matrix $(m_{\text{D}})_{Ii} = v(h_\nu)_{Ii}$. In combination with the Majorana mass term (1.2.3) for the right-handed neutrinos one can write the complete neutrino mass term as [78]

$$\mathscr{L}_{\text{mass}} = -\frac{1}{2}\left(\overline{\nu_L}, \overline{\nu_R^c}\right)\mathscr{M}\begin{pmatrix}\nu_L^c \\ \nu_R\end{pmatrix} + \text{H.c}, \tag{5.1.2}$$

with the mass matrix

$$\mathscr{M} = \begin{pmatrix}0 & m_{\text{D}} \\ m_{\text{D}}^\top & M\end{pmatrix}. \tag{5.1.3}$$

The mass matrix is symmetric and can therefore be diagonalized. In general, for an arbitrary number of flavors, it is rather difficult to do this diagonalization. However, if we

assume the Majorana masses to be much larger than the Dirac masses, one can approximate the block-diagonalized version of $\mathscr{M}$ as [78, 79]

$$D_{\mathscr{M}} \approx \begin{pmatrix} m_\nu & 0 \\ 0 & M \end{pmatrix}, \tag{5.1.4}$$

with the 3×3 block of light neutrino masses

$$m_\nu \approx -m_{\mathrm{D}}^\top M^{-1} m_{\mathrm{D}}. \tag{5.1.5}$$

This formula has an intuitive interpretation which is known as the seesaw mechanism. The light neutrino masses are light because they are inverse proportional to the heavy Majorana masses.

The light neutrino masses play an important role in leptogenesis because they are directly related to the Yukawa coupling h_ν which is responsible for the $B - L$ violating interactions. In particular, at temperatures much lower than the lightest sterile neutrino mass the mass matrix m_ν becomes important for the washout rate γ_{QQ}. At temperatures $T \sim M_1$, where M_1 is the lightest sterile neutrino mass, the washout rate γ_{QQ} is dominated by $\Delta L = 1$ processes which are of order h_ν^2. In [41] an expression for this rate has been derived using the formula (3.2.13) for $x_i = Q_a$. This expression is valid to order h_ν^2 and to all orders in the SM couplings. However, at low temperatures ($T \ll M_1$) the $\Delta L = 1$ processes are exponentially suppressed and the $\Delta L = 2$ processes

$$\ell + \ell \leftrightarrow \tilde{\varphi} + \tilde{\varphi}, \tag{5.1.6}$$

$$\bar{\ell} + \bar{\ell} \leftrightarrow \tilde{\varphi}^\dagger + \tilde{\varphi}^\dagger, \tag{5.1.7}$$

$$\ell + \tilde{\varphi}^\dagger \leftrightarrow \bar{\ell} + \tilde{\varphi}, \tag{5.1.8}$$

dominate γ_{QQ} [44]. These processes are mediated by virtual sterile neutrinos and are of order h_ν^4.

At very low temperatures $T \ll M_1$ it has been shown in [44] in the single-flavor approximation (see also section (5.5)), neglecting thermal effects and spectator processes, that the the leading contribution to the washout rate is

$$\gamma_{QQ} = \frac{cT^3}{v^4} \bar{m}^2, \tag{5.1.9}$$

where $\bar{m}^2 = \mathrm{tr}(m_\nu^\dagger m_\nu)$ is the sum of all squared light neutrino masses, and c is a numerical constant. Since in this limit γ_{QQ} is proportional to $\bar{m}^2$, this rate is an important ingredient for finding upper bounds on the parameter $\bar{m}$. In [44] the value of c has been obtained neglecting quantum-statistical effects in the collision integrals of Boltzmann equations. We expect these effects on the constant c to be important because the sterile neutrino mediated processes are, unlike the $\Delta L = 1$ processes, not Boltzmann suppressed. Therefore, in this work we compute γ_{QQ} in the low-temperature regime, including full quantum-statistical effects. Furthermore, our approach naturally allows to include the effects of spectator processes through the susceptibilities which we considered in the previous chapter. In this case c depends on the temperature.

For the calculation of the rate γ_{QQ} we proceed as [41] for the $\Delta L = 1$ washout rate, using the Kubo-type formula (3.2.13). We work within an effective low-temperature theory which approximates the sterile neutrino exchange as point-like interactions, described by a dimension-5 operator (see (5.2.6)).

Another analysis of the $\Delta L = 2$ washout rate including quantum statistical effects and thermal masses has been done in [80]. However no explicit analytical expression in the low-temperature regime is given there.

A full leading order calculation of the $\Delta L = 2$ washout rate within a low-temperature effective theory might also be interesting for a recently proposed minimal leptogenesis scenario [81] where a dimension-5 operator is used which is very similar to (5.2.6).

5.2 Kubo relations for the washout rate in an effective theory

In the following we consider the scattering processes (5.1.6)-(5.1.8) which are mediated by sterile neutrino exchange. We assume these processes to occur at temperatures which are much smaller than the lightest sterile neutrino mass M_1. Then there are no sterile neutrinos present in the hot plasma. Furthermore, virtual sterile neutrinos typically have momenta much smaller than their mass. The interactions (5.1.6)-(5.1.8) can then be described by an effective dimension-5 operator, which can be obtained from (1.2.4) by integrating out the sterile neutrinos in the path integral. This has for example been done in [82]. Let us shortly explain how it works. For the Majorana fields we write $\bar{N} = N^\top \mathscr{C}^{-1}$. The full action of the theory can then be written as

$$S = \int d^4x \left(\frac{1}{2} N_I^\top D_I N_I - \bar{\mathscr{I}}_I N_I + \mathscr{L}_{\rm SM} \right), \tag{5.2.1}$$

where $D_I \equiv \mathscr{C}^{-1}(i\partial\!\!\!/ - M_I)$ and $\mathscr{I} = J + J^c$ with $J = \tilde{\varphi} h_\nu \ell$. Integrating over the neutrinos in the path integral, we obtain the effective action

$$S_{\rm int}^{\rm eff} = -\frac{1}{2} \int d^4x \int d^4y \overline{\mathscr{I}}_I(x) G_I(x-y) \mathscr{I}_I(y), \tag{5.2.2}$$

with

$$G_I(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{k\!\!\!/ + M_I}{k^2 - M_I^2 + i0^+}. \tag{5.2.3}$$

If $k^2 \ll M_I^2$, we can approximate

$$G_I(x-y) \approx -\delta^{(4)}(x-y) \frac{1}{M_I} \tag{5.2.4}$$

and then the effective interaction leads to the Lagrangian

$$\mathscr{L}_{\rm int}^{\rm eff} = \frac{1}{2M_I} \int d^3x \overline{\mathscr{I}}_I \mathscr{I}_I, \tag{5.2.5}$$

which in terms of the Standard Model fields reads

$$\mathscr{L}_{\text{int}}^{\text{eff}} = \frac{1}{2M_I}\int d^3x\Big\{\left(\tilde{\varphi}^\dagger \ell_i\right)^\top \mathscr{C}^{-1}\left(\tilde{\varphi}^\dagger \ell_j\right)(h_\nu)_{Ij}(h_\nu^\top)_{iI} + \text{H.c.}\Big\}. \tag{5.2.6}$$

Here we have used $\overline{\ell}\ell = 0$ due to the left-handed nature of the leptons. The dimension-5 operator in (5.2.6) has also been found by [83] as a source of $\Delta L = 2$ violating processes.

Let us now use the approach from section 3.2 to compute the washout rate within this effective theory. First of all we use the fact that the equal time correlations of the charges Q_a with the phase-space density $f_{I\mathbf{k}}$ vanish (see 3.2.14). Then, for the real-time correlator we need only the correlations of quantities with the same sign under CPT transformation, namely the charges Q_a. Then the washout rate can be computed with the Kubo-type relation (3.2.13) which explicitly reads [41]

$$(\gamma_{QQ})_{ab} = \frac{1}{2V}\lim_{\gamma\ll\omega\ll\omega_{\text{UV}}}\frac{\rho_{\dot{Q}_a\dot{Q}_c}(\omega)}{\omega}(\Xi^{-1})_{cb}, \tag{5.2.7}$$

where we have defined the volume independent susceptibilities, according to [41], as

$$\Xi_{ab} \equiv \frac{1}{TV}(\chi_{QQ})_{ab}. \tag{5.2.8}$$

Since we consider temperatures much lower than the lightest sterile neutrino mass, the time derivative of the almost conserved charges is determined by the effective interaction (5.2.6) and can be computed in the Heisenberg picture as

$$\dot{Q}_a = i[H_{\text{int}}^{\text{eff}}, Q_a], \tag{5.2.9}$$

where $H_{\text{int}}^{\text{eff}} = -\mathscr{L}_{\text{int}}^{\text{eff}}$. Due to the fact that (5.2.6) changes only the number of left-handed leptons, it is sufficient for the computation of the time derivatives to consider the dependence of Q_a on these fields only. Let T_a^ℓ be the generator of the symmetry transformation corresponding to the charge Q_a acting on the left-handed lepton fields. Then we can write

$$Q_a = \int d^3x\Big(\bar{\ell}_i\gamma^0(T_a^\ell)_{ij}\ell_j + \text{contributions from other fields}\Big), \tag{5.2.10}$$

and a straightforward calculation, using (5.2.9) and (5.2.6) yields the time derivatives

$$\dot{Q}_a = \frac{i}{M_I}\int d^3x\Big\{\overline{J}_{aI}\mathscr{C}\overline{J}_I^\top + J_I^\top\mathscr{C}^\dagger J_{aI}\Big\}, \tag{5.2.11}$$

where $J_{aI} = (h_\nu)_{Ii}(T_a^\ell)_{ij}\tilde{\varphi}^\dagger\ell_j$. Since the $\dot{Q}_a$ are of order h_ν^2 the spectral function in (5.2.7) is already of the order h_ν^4 which we are interested in. Therefore, only SM interactions have to be taken into account for the calculation of the spectral function and the susceptibilities.

For perturbative calculations of the spectral function in (5.2.7) it is most convenient to compute first the imaginary time correlator

$$\Delta_{\dot{Q}_a\dot{Q}_b}(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau}\left\langle\dot{Q}_a(-i\tau)\dot{Q}_b(0)\right\rangle \tag{5.2.12}$$

and then to analytically continue the result to complex frequencies ω via $i\omega_n \to \omega$ and to use the inverse relation (2.1.10) to obtain the spectral function.

5.3 Leading order

First we calculate the imaginary time correlator (5.2.12) at leading order, using the operator (5.2.11). Applying Wick's theorem, contracting all gauge and flavor indices and performing Dirac traces yields

$$\Delta^{\rm LO}_{\dot{Q}_a\dot{Q}_b}(i\omega_n) = -V\frac{d(r)(d(r)+1)}{M_I M_J}\Big\{2\left(h_\nu T^\ell_a h^\dagger_\nu\right)_{IJ}\left(h_\nu T^\ell_b h^\dagger_\nu\right)_{IJ} + \left(h_\nu h^\dagger_\nu\right)_{IJ}\left(h_\nu\{T^\ell_a,T^\ell_b\}h^\dagger_\nu\right)_{IJ}\Big\}\Delta_0(i\omega_n), \tag{5.3.1}$$

where $d(r)=2$ is the dimension of the representation of the gauge group $SU(2)$ and

$$\Delta_0(i\omega_n) = \sum\!\!\!\!\!\!\int_{\{k_1\},\{k_2\},k_3} \frac{2k_1\cdot k_2}{k_1^2k_2^2k_3^2(k_1+k_2+k_3+q)^2} \tag{5.3.2}$$

is a three-loop sum-integral corresponding to the diagram in figure 5.1. Here $\{k_i\}$ denotes fermionic Matsubara frequencies and $q=(i\omega_n,\mathbf{0})$. The analytical continuation of (5.3.2)

Figure 5.1: Diagrammatic representation of the sum-integral entering the leading order of $\Delta_{\dot{Q}_a\dot{Q}_b}(i\omega_n)$. Dotted lines are bosons and solid lines are fermions.

can only be done after performing the Matsubara sums. Then we obtain an expression consisting of three-dimensional integrals containing ratios of the form $\Delta/(\omega_n^2+\Delta^2)$, where $\Delta = E_{\rm in}-E_{\rm out}$ can be identified with the difference of the energies of incoming and outgoing particles in $0\leftrightarrow 4$, $1\leftrightarrow 3$ and $2\leftrightarrow 2$ processes. Only the $2\leftrightarrow 2$ processes (5.1.6)-(5.1.8) are kinematically allowed and contribute to the rate. These terms are

$$\begin{aligned}\Delta_0(i\omega_n) = -4\int d\Pi_1\int d\Pi_2\int d\Pi_3\int d\Pi_4(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,(E_1E_2-\mathbf{k}_1\mathbf{k}_2)\\ \times\Big\{\frac{E_1+E_2-E_3-E_4}{\omega_n^2+(E_1+E_2-E_3-E_4)^2}\\ \times\Big(f_{{\rm F},1}f_{{\rm F},2}(1+f_{{\rm B},3}+f_{{\rm B},4})-f_{{\rm B},3}f_{{\rm B},4}(1-f_{{\rm F},1}-f_{{\rm F},2})\Big)\\ -2\frac{E_1+E_3-E_2-E_4}{\omega_n^2+(E_1+E_3-E_2-E_4)^2}\\ \times\Big(f_{{\rm F},1}f_{{\rm B},3}(1+f_{{\rm B},2}-f_{{\rm F},4})-f_{{\rm F},2}f_{{\rm B},4}(1-f_{F,1}+f_{{\rm B},3})\Big)\\ +\text{similar terms which do not contribute to the rate}\Big\},\end{aligned} \tag{5.3.3}$$

where $d\Pi_i = d^3k_i/(2E_i)$, $E_i = |\mathbf{k}_i|$ and $f_{\mathrm{F,i}} = f_{\mathrm{F}}(E_i)$ and $f_{\mathrm{B,i}} = f_{\mathrm{B}}(E_i)$ are the Fermi-Dirac and Bose-Einstein distributions, respectively. We calculate the discontinuity of (5.3.3), making use of

$$\mathrm{Disc}\frac{\Delta}{\omega_n^2+\Delta^2}\bigg|_{i\omega_n\to\omega} = i\pi\left(\delta(\omega-\Delta)-\delta(\omega+\Delta)\right) \tag{5.3.4}$$

and expand the result to order ω. Then, using (5.2.7) we obtain

$$\begin{aligned}(\gamma_{QQ})_{ab} =& \frac{12}{TM_IM_J}\left(I_{12}^{1100}+2I_{13}^{1010}\right)\left(\Xi^{-1}\right)_{cb}\\ &\times\left(2\left(h_\nu T_a^\ell h_\nu^\dagger\right)_{IJ}\left(h_\nu T_c^\ell h_\nu^\dagger\right)_{IJ}+\left(h_\nu h_\nu^\dagger\right)_{IJ}\left(h_\nu\{T_a^\ell,T_c^\ell\}h_\nu^\dagger\right)_{IJ}\right).\end{aligned} \tag{5.3.5}$$

Here we have defined the integral class

$$\begin{aligned}I_{ij}^{\sigma_1\sigma_2\sigma_3\sigma_4} =& \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 (2\pi)^4\delta^{(4)}(k_1+k_2-k_3-k_4)\\ &\times(k_i\cdot k_j)f_{\sigma_1}^{eq}f_{\sigma_2}^{eq}(1+\sigma_3 f_{\sigma_3}^{eq})(1+\sigma_4 f_{\sigma_4}^{eq})\Big|_{k_i^0=E_i},\end{aligned} \tag{5.3.6}$$

where σ_i is 1 for bosons and -1 fermions, $f_1{=}f_{\mathrm{B}}$ and $f_{-1}{=}f_{\mathrm{F}}$. We calculate the integrals I_{12}^{1100} and I_{13}^{1010} in appendix A, using full quantum statistics and their results are

$$I_{12}^{1100} = 1.14\times10^{-4}T^6, \tag{5.3.7}$$

$$I_{13}^{1010} = 5.91\times10^{-5}T^6. \tag{5.3.8}$$

Our result (5.3.5) gives the leading order washout rate for the charges Q_a in the symmetric phase for temperatures $T \ll M_1$.

5.4 Beyond leading order

As we have seen in chapter 4, the NLO susceptibilities are of order g due to soft thermal Higgs mass (4.4.5). Therefore, the next-to-leading order corrections to the washout rate should be of order g as well. Keep in mind that the susceptibilities of almost conserved charges are defined trough the relation (4.1.3), that is, in an ensemble where the strictly conserved charges vanish. It depends on the temperature which charges are conserved . A table for the order g susceptibilities in the single flavor approximation for different temperature regimes can be found in [45].

If one naively expands the spectral functions in (5.2.7) to order g^2 one finds infrared divergent diagrams which correspond to thermal Higgs mass corrections to the Higgs propagator. These divergences can be cured by thermal Higgs mass resummation. The idea works as follows. We write the Lagrangian as

$$\mathscr{L} = \mathscr{L} - m_3^2\varphi^\dagger\varphi + m_3^2\varphi^\dagger\varphi, \tag{5.4.1}$$

with m_3 defined in (4.4.5), and treat the first mass term as a part of the Higgs propagator and the second one as a two-vertex in which we expand in perturbation theory. The infrared

divergences are then cured due to the massive Higgs propagator. On the other hand, the contributions of order g^2 diagrams which correspond to thermal Higgs mass corrections are canceled by diagrams with the two-vertex insertion.

The parametric dependence of the leading order diagram on the Higgs mass reflects the nature of the infrared divergence and thus yields a contribution which is parametrically larger than order g^2. In appendix A we investigate the leading contribution of the Higgs mass, computing the leading order diagram in figure 5.1 with massive Higgs propagator and expand the result for small values of m_3/T. Then we find that the integrals (5.3.6) have the logarithmic mass dependence

$$I_{12}^{1100}(m_3) = \left(1.14 + 0.333\frac{m_3^2}{T^2}\ln(\frac{m_3^2}{T^2}) + \mathscr{O}(g^2)\right) \times 10^{-4} T^6 \tag{5.4.2}$$

$$I_{13}^{1010}(m_3) = \left(5.91 + 2.54\frac{m_3^2}{T^2}\ln(\frac{m_3^2}{T^2}) + \mathscr{O}(g^2)\right) \times 10^{-5} T^6. \tag{5.4.3}$$

Unlike for the susceptibilities, the Higgs mass resummation does not lead to an order g correction of the spectral function, but rather to an order $g^2\ln(g)$ correction. Therefore, the leading Higgs mass effect is due to the susceptibilities. We have not computed the effects of thermal lepton and gauge boson masses on the spectral function. A complete calculation of these effects would be a substantial amount of work, but we expect them, like the thermal Higgs mass corrections to the spectral function, to be of order $g^2\ln(g^2)$ or g^2.

5.5 Size of the quantum statistical effects and order g Higgs mass contribution

In the following we define the left-handed lepton which couples to the lightest sterile neutrino N_1 as ℓ_{N_1} and we define the lepton asymmetry in this flavor direction as L_{N_1}. We assume that during the generation of the lepton asymmetry only N_1 is present in the plasma so that only an asymmetry in ℓ_{N_1} is generated through the sterile neutrino Yukawa interaction. This is a valid approximation in the hierarchical limit $M_1 \ll M_{I\neq 1}$. If the charged lepton Yukawa interactions are in equilibrium, the asymmetry in the left-handed lepton ℓ_{N_1} is partially converted into an asymmetry in the right-handed sector. Since the charged lepton Yukawa interactions are also flavor violating, the asymmetry L_{N_1} is also partially converted into an asymmetry in other flavors. In the following we use the single-flavor approximation, assuming that the flavor violation can be neglected. Then asymmetries in directions other than L_{N_1} are zero and we can choose the charge Q_a such that it contains the lepton number summed over all flavors. This corresponds to setting $T_a^\ell = 1$ in (5.3.5) and leads to the rate

$$\gamma_{QQ} = 1.12 \times 10^{-2}\frac{T^5\bar{m}^2}{v^4\Xi}. \tag{5.5.1}$$

Let us investigate how strongly the quantum-statistical effects and spectator processes affect the washout rate by comparing our result (5.5.1) to the one from [44], which translates in our notation to

$$\gamma_{QQ}\Big|_{\text{classical}} = 2.68 \times 10^{-2}\frac{T^3\bar{m}^2}{v^4}. \tag{5.5.2}$$

There the rate has been computed with classical statistics and spectator processes have been neglected. If we also neglect spectator processes, this corresponds to $T^2\Xi^{-2} = 3$ [45]. Then our result is 24.8% larger than (5.5.2) due to quantum-statistical effects.

In [45] our result for the $\Delta L = 2$ washout rate has been used for finding upper bounds on the sum of the squared neutrino masses $\bar{m}^2$. There it turned out that full quantum statistics yields a 5% tighter mass bound than classical statistics which yields the upper bound $\bar{m}^{\max} < 0.2$ eV.

In the same reference it has been shown that the $\Delta L = 2$ washout rate becomes relevant for sterile neutrino masses $M_1 \gtrsim 10^{14}$ GeV. For such sterile neutrino masses let us consider two examples for the susceptibilities which have been computed in [45] with the formula (4.1.3). At temperatures $T \gtrsim 10^{13}$ GeV the strong sphalerons become active and the susceptibilities read [45]

$$T^2\Xi^{-1} = \frac{90}{23}\left(1 + \frac{49}{230}\frac{m_3}{\pi T}\right). \tag{5.5.3}$$

In this case the washout rate is 30.4% larger compared to case where the spectator processes have been neglected.

For lower temperatures between 10^{12} GeV $< T < 10^{13}$ GeV the τ-Yukawa interaction becomes active. In this case one has to distinguish the two cases whether L_{N_1} is equal to the τ flavor asymmetry L_τ or not. Then the susceptibilities are [45]

$$T^2\Xi^{-1} = \begin{cases} \frac{57}{16}\left(1 + \frac{27}{304}\frac{m_3}{\pi T}\right), & L_{N_1} \neq L_\tau \\ 3\left(1 + \frac{3}{16}\frac{m_3}{\pi T}\right), & L_{N_1} = L_\tau \end{cases} \tag{5.5.4}$$

In the first case the washout rate is 18.8% larger compared to the case there the spectator processes have been neglected. In the second case ($L_{N_1} = L_\tau$) the washout rate differs from the case where spectator processes are neglected only due to the Higgs mass correction.

For temperatures between 10^{12} GeV $< T < 10^{14}$ GeV we have $0.13 < \frac{m_3}{\pi T} < 0.18$. Then we find, if $L_{N_1} \neq L_\tau$, that the order g (or NLO) correction to the $\Delta L = 2$ washout rate from the thermal Higgs mass are smaller than 2% in this temperature regime. If $L_{N_1} \neq L_\tau$, they are smaller than 3% which has also been pointed out in [45].

Chapter 6

Three-point functions at finite temperature

Before we continue with the computation of the CP violating lepton asymmetry rate $\gamma_{Q\delta f}$, we have to introduce some mathematical tools. In chapter (7) we show that this rate can be related to three-point spectral functions at finite temperature. Therefore, the aim of this chapter is to derive relations for three-point functions analogous to the relation (2.1.7), (2.1.2) and (2.1.10) which hold for two-point functions. Furthermore, we investigate the properties of three-point functions if the system is symmetric under CP and T transformations. The calculations and results of this chapter are planned to be published in [3].

6.1 The three-point spectral representation

Three-point functions at finite temperature and their spectral representation have been studied in several references in the real-time formalism. For example, in 1990 Kobes and Evans showed in [84, 85] that a spectral representation exists for three-point functions at finite temperature. But their notion of a spectral representation differs from ours. In particular, they do not give an integral representation analogous to (2.1.7).

Spectral representations similar to (2.1.7) have been derived in [86, 87] for retarded and advanced real-time three-point correlators, but not for the imaginary time correlator. Furthermore, they use different representations for each real-time correlator, but we only want a single spectral representation for the imaginary time correlator.

An integral representation which relates the imaginary time correlator to real-time correlators has been derived in [88]. However, this integral representation differs somewhat from the simple structure of (2.1.7) which we are interested in. For example, their integral representation still depends on thermal distributions and does not give a simple relation to three-point spectral functions which are defined through (anti-)commutators similar to (2.1.2). In this section we closely follow [88] and show that a simple spectral representation can be obtained which is very similar to (2.1.7). In [88] it has also been shown that the imaginary time correlator can be related to advanced and retarded correlators via analytical continuation similar to (2.1.8). Using this fact, we can furthermore show that inverse relations similar to (2.1.10)

exist.

We consider three operators A, B, and C, which can be fermionic or bosonic, elementary or composite. They define six three-point correlation functions, one for each permutation of ABC, of which one is given as

$$\Gamma_{ABC}(t_A, t_B) \equiv \langle A(t_A)B(t_B)C(0)\rangle \,. \tag{6.1.1}$$

Here we have already used translational invariance in time which allows to set $t_C = 0$. The correlator is well defined for complex times with

$$0 \geq \mathrm{Im}\, t_B \geq \mathrm{Im}\, t_A \geq -\beta \tag{6.1.2}$$

and in this region its Fourier representation

$$\Gamma_{ABC}(t_A, t_B) = \int \frac{d\omega_A}{2\pi} \int \frac{d\omega_B}{2\pi} e^{-i(\omega_A t_A + \omega_B t_B)} \gamma_{ABC}(\omega_A, \omega_B) \tag{6.1.3}$$

exists. Due to the cyclicity of the trace the Fourier transform satisfies the relation

$$\gamma_{ABC}(\omega_A, \omega_B) = e^{-\omega_C/T}\, \gamma_{CAB}(\omega_A, \omega_B), \tag{6.1.4}$$

with $\omega_C = -\omega_A - \omega_B$.

For the asymmetry rate in the next chapter we need the time ordered imaginary time three-point correlator in frequency space, which we define as

$$\Gamma_{ABC}(i\omega_n, i\omega_{n'}) \equiv \int_0^\beta d\tau \int_0^\beta d\tau' \, \exp(i\omega_n \tau + i\omega_{n'}\tau') \left\langle \mathrm{T}\left\{A(-i\tau)B(-i\tau')C(0)\right\}\right\rangle , \tag{6.1.5}$$

with Matsubara frequencies $\omega_n = n\pi T$, with even (odd) integers n for bosonic (fermionic) operators. Writing the time ordering in (6.1.5) explicitly, we have

$$\begin{aligned}\Gamma_{ABC}(i\omega_n, i\omega_{n'}) = \int_0^\beta d\tau \int_0^\beta d\tau' \, \exp(i\omega_n \tau + i\omega_{n'}\tau') \Big[&\theta(\tau - \tau') \big\langle A(-i\tau)B(-i\tau')C(0)\big\rangle \\ &+ (-1)^{\deg A \deg B} \theta(\tau' - \tau) \big\langle B(-i\tau')A(-i\tau)C(0)\big\rangle \Big],\end{aligned} \tag{6.1.6}$$

where deg stands for the *degree* defined as

$$\deg A = \begin{cases} 0 & \text{if } A \text{ bosonic} \\ 1 & \text{if } A \text{ fermionic.} \end{cases} \tag{6.1.7}$$

Like in [88], we plug in the Fourier representation (6.1.3) of the correlators on the right-hand

side and perform the τ and τ' integral. This yields

$$\begin{aligned}\Gamma_{ABC}(i\omega_n, i\omega_{n'}) = &\int \frac{d\omega}{2\pi}\int \frac{d\omega'}{2\pi}\frac{\left((-1)^{\deg A+\deg B}e^{-\beta(\omega+\omega')}-1\right)\gamma_{ABC}(\omega,\omega')}{(i\omega_n+i\omega'_n-\omega-\omega')(i\omega_n-\omega)}\\ &+\frac{(-1)^{\mathrm{degA}\deg B}\left((-1)^{\deg A+\deg B}e^{-\beta(\omega+\omega')}-1\right)\gamma_{BAC}(\omega',\omega)}{(i\omega_n+i\omega'_n-\omega-\omega')(i\omega'_n-\omega')}\\ &+\frac{\left(1-(-1)^{\deg A}e^{-\beta\omega}\right)\gamma_{ABC}(\omega,\omega')}{(i\omega'_n-\omega')(i\omega_n-\omega)}\\ &+\frac{(-1)^{\mathrm{degA}\deg B}\left(1-(-1)^{\deg B}e^{-\beta w'}\right)\gamma_{BAC}(\omega',\omega)}{(i\omega'_n-\omega')(i\omega_n-\omega)}.\end{aligned} \tag{6.1.8}$$

In contrast to [88], we now use the cyclicity property (6.1.4) to eliminate all exponential functions and do the partial fraction decomposition

$$\frac{1}{(i\omega_n-\omega)(i\omega_{n'}-\omega')}=\frac{1}{i\omega_n+i\omega'_n-\omega-\omega'}\left(\frac{1}{i\omega_n-\omega}+\frac{1}{i\omega'_n-\omega'}\right). \tag{6.1.9}$$

Furthermore, we assume that ABC is a bosonic operator, so that $\deg C = \deg(AB)$. Then we obtain the spectral representation

$$\begin{aligned}\Gamma_{ABC}(i\omega_n, i\omega_{n'}) = &\int \frac{d\omega}{2\pi}\int \frac{d\omega'}{2\pi}\frac{1}{i\omega_n+i\omega_{n'}-\omega-\omega'}\\ &\left[\frac{\rho_{ABC}(\omega,\omega')}{i\omega_n-\omega_1}+(-1)^{\deg A\deg B}\frac{\rho_{BAC}(\omega',\omega)}{i\omega_{n'}-\omega'}\right],\end{aligned} \tag{6.1.10}$$

which contains the spectral functions

$$\rho_{ABC}(\omega,\omega')\equiv\int dt\int dt'\,\exp(i\omega t+i\omega' t')\left\langle\left[A(t),\left[B(t'),C(0)\right]\right]\right\rangle, \tag{6.1.11}$$

with the graded commutator

$$[A,B]\equiv AB-(-1)^{\deg A\deg B}BA. \tag{6.1.12}$$

According to [88], we can obtain all retarded correlators via analytical continuation of (6.1.5). In the notation of [88] the six different retarded functions are

$$R_1(\omega_A,\omega_B)=\Gamma_{ABC}(\omega_A+2i\varepsilon,\omega_B-i\varepsilon), \tag{6.1.13}$$

$$R_2(\omega_A,\omega_B)=\Gamma_{ABC}(\omega_A-i\varepsilon,\omega_B+2i\varepsilon), \tag{6.1.14}$$

$$R_3(\omega_A,\omega_B)=\Gamma_{ABC}(\omega_A-i\varepsilon,\omega_B-i\varepsilon), \tag{6.1.15}$$

$$\overline{R}_i(\omega_A,\omega_B)=R_i(\omega_A,\omega_B)\Big|_{\epsilon\to-\varepsilon}\qquad(i=1,\ldots,3). \tag{6.1.16}$$

Writing these retarded and advanced correlators in terms of the spectral representation (6.1.10) and using (2.1.9), we find two combinations where all principal values drop out and

only delta functions remain in the spectral representation. This allows to write the spectral functions explicitly in terms of advanced and retarded correlators as

$$\rho_{ABC} = R_2 + \overline{R}_2 - R_3 - \overline{R}_3, \tag{6.1.17}$$

$$\rho_{BAC} = (R_1 + \overline{R}_1 - R_3 - \overline{R}_3)(-1)^{\deg A \deg B}. \tag{6.1.18}$$

These are the inverse relations analogous to (2.1.10). If the spectral functions are real, it can easily be seen from the spectral representation (6.1.10) that these relations can be further simplified to

$$\rho_{ABC} = 2\mathrm{Re}(R_2 - R_3) \tag{6.1.19}$$

$$\rho_{BAC} = 2\mathrm{Re}(R_1 - R_3)(-1)^{\deg A \deg B}. \tag{6.1.20}$$

In the next section we give conditions for the spectral functions to be real.

6.2 Implications of CP and T invariance

Like for the two-point spectral functions in section 2.2, we can find conditions for the three-point spectral functions to be real valued. Here we assume that the system is CP and consequently, T invariant. Let us for a moment suppress a possible dependence of the operators A, B and C on spatial vectors. Then we assume that the operators in the spectral function (6.1.11) transform under T and CP transformations such that

$$T\left[A(t_A), [B(t_B), C(0)]\right] T^{-1} = \varepsilon_T \left[A(-t_A), [B(-t_B), C(0)]\right] \tag{6.2.1}$$

$$CP\left[A(t_A), [B(t_B), C(0)]\right] (CP)^{-1} = \varepsilon_{CP} \left[A(t_A), [B(t_B), C(0)]\right]^*, \tag{6.2.2}$$

where ε_T and ε_{CP} are ± 1. Then it is easy to see that the spectral functions obey the relations

$$\rho_{ABC}(\omega_A, \omega_B) = \varepsilon_T \rho^*_{ABC}(\omega_A, \omega_B), \tag{6.2.3}$$

$$\rho_{ABC}(\omega_A, \omega_B) = \varepsilon_{CP} \rho^*_{ABC}(-\omega_A, -\omega_B), \tag{6.2.4}$$

where for the first relation we have used the anti-unitarity of T. Both relations can be combined to

$$\rho_{ABC}(\omega_A, \omega_B) \quad = \quad \varepsilon_T \varepsilon_{CP} \rho_{ABC}(-\omega_A, -\omega_B). \tag{6.2.5}$$

In the next chapter we are explicitly confronted with the case

$$A(x) = J_i^\alpha(x), \qquad B(y) = (\mathscr{C}^\dagger J_j)^\alpha(y), \qquad C(0) = \overline{J_k} \mathscr{C} \overline{J_l}^\top(0), \tag{6.2.6}$$

with the fields $J_i = \tilde{\varphi}^\dagger \ell_i$. Here α is a spinor index which is summed over in the product AB. Let us investigate the CP and T properties of these operators. Like in [89], we choose a representation where $\mathscr{C} = i\gamma_2\gamma_0$ and

$$PJ(t, \mathbf{x})P^{-1} = \eta \gamma_0 J(t, -\mathbf{x}), \qquad TJ(t, \mathbf{x})T^{-1} = \gamma_1\gamma_3 J(-t, \mathbf{x}), \tag{6.2.7}$$

where η is a complex number with $|\eta| = 1$. Then we find

$$(CP)A(x_0, \mathbf{x})B(y_0, \mathbf{y})(CP)^{-1} = (B(y_0, -\mathbf{y})A(x_0, -\mathbf{x}))^\dagger\,, \tag{6.2.8}$$

$$(CP)C(0)(CP)^{-1} = -C^\dagger(0), \tag{6.2.9}$$

and

$$TA(x_0, \mathbf{x})B(y_0, \mathbf{y})T^{-1} = A(-x_0, \mathbf{x})B(-y_0, \mathbf{y}), \tag{6.2.10}$$

$$TC(0)T^{-1} = C(0). \tag{6.2.11}$$

For the spectral functions we can use rotational invariance. This implies that the spectral functions only depend on scalar combinations $\mathbf{k}_i \cdot \mathbf{k}_j$ of the spatial momenta and the frequencies. This implies the symmetry

$$\rho_{ABC}(\omega_1, \mathbf{k}_1, \omega_2, \mathbf{k}_2) = \rho_{ABC}(\omega_1, -\mathbf{k}_1, \omega_2, -\mathbf{k}_2). \tag{6.2.12}$$

Now, using the CP properties (6.2.8) and (6.2.9) in combination with rotational invariance, we find

$$\rho_{ABC}(\omega_1, \mathbf{k}_1, \omega_2, \mathbf{k}_2) = \rho_{ABC}(-\omega_1, -\mathbf{k}_1, -\omega_2, -\mathbf{k}_2)^*. \tag{6.2.13}$$

The properties under T transformation (6.2.10) and (6.2.11) imply

$$\rho_{ABC}(\omega_1, \mathbf{k}_1, \omega_2, \mathbf{k}_2) = \rho_{ABC}(\omega_1, \mathbf{k}_1, \omega_2, \mathbf{k}_2)^*, \tag{6.2.14}$$

that is, $\varepsilon_T = 1$ and $\varepsilon_{CP} = 1$. The spectral representation (6.1.10) implies that these symmetries hold for the corresponding Euclidean correlator (7.2.14) as well,

$$\Gamma^*(k_1, k_2) = \Gamma(k_1, k_2) \qquad\qquad \Gamma(-k_1, -k_2) = \Gamma(k_1, k_2). \tag{6.2.15}$$

Chapter 7

Lepton asymmetry rate

In this chapter we compute the CP violating coefficient $(\gamma_{Q\,\delta f})_{aI\mathbf{k}}$ which describes the rate of the generation of the asymmetries Q_a due the deviation of the sterile neutrino phase-space densities $f_{I\mathbf{k}}$ from equilibrium. We first derive a master formula for the rate in the hierarchical limit. This formula relates the rate to a three-point spectral function of SM fields. Then as an application, we compute the leading order and the next-to-leading order rate at zero temperature. The calculations and results of this chapter are planned to be published in [3].

7.1 Kubo relations for the asymmetry rate

The CP violating rate can, according to (3.2.12) be computed from the relation

$$(\gamma_{Q\,\delta f})_{aI\mathbf{k}} = \frac{T}{\omega}\mathrm{Im}\,\Delta^{\mathrm{ret}}_{\dot{Q}_a \dot{f}_{J\mathbf{k}'}}(\omega)(\chi^{-1}_{\delta f\delta f})_{J\mathbf{k}'I\mathbf{k}} \qquad (\gamma \ll \omega \ll \omega_{\mathrm{UV}}), \tag{7.1.1}$$

where we have used that the equal-time correlators (4.1.3) of X_a with $\delta f_{I\mathbf{k}}$ vanish due to CPT invariance (see also section 2.2).

The charges Q_a are described by the same operators (5.2.10) as for the washout rate, but we compute their time derivatives now in the full theory (1.2.4) which yields

$$\dot{Q}_a(t) = i\int d^3x \left[K_a(x) - K_a^\dagger(x)\right], \tag{7.1.2}$$

where $K_a = \overline{N_I} J_{Ia}$ with

$$J_{Ia} = (h_\nu)_{Ii}(T_a)_{ij}\tilde{\varphi}^\dagger \ell_i. \tag{7.1.3}$$

In order to define operators for the sterile neutrino phase-space densities, we consider the sterile neutrino fields in the interaction picture. We define the Hamiltonian as

$$H = H_0 + H_{\mathrm{int}}, \tag{7.1.4}$$

where H_0 contains the full SM and the free sterile neutrinos and $H_{\mathrm{int}} = -\mathscr{L}_{\mathrm{int}}$ represents the Yukawa interaction (1.2.4). Then in the interaction picture, the sterile neutrinos can be

written as solutions of the free field equations in terms of annihilation and creation operators $c^\dagger_{I\mathbf{k}s}$ and $c_{I\mathbf{k}s}$ as[1]

$$[N_I(x)]_{\rm int} = \sum_{\mathbf{k},s} \frac{1}{\sqrt{2E_{I\mathbf{k}}V}} \left[e^{-ikx}\, u_{I\mathbf{k}s}\, c_{I\mathbf{k}s} + e^{ikx}\, v_{I\mathbf{k}s}\, c^\dagger_{I\mathbf{k}s} \right]_{k^0 \to E_{I\mathbf{k}}}, \tag{7.1.5}$$

where $E_{I\mathbf{k}} = (\mathbf{k}^2 + M_I^2)^{1/2}$, and V is the volume. We normalized the annihilation and creation operators such that

$$\{c_{I\mathbf{k}s}\,,\, c^\dagger_{I'\mathbf{k}'s'}\} = \delta_{II'}\delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}. \tag{7.1.6}$$

We define the phase-space densities (or occupation number) in the interaction picture as the spin averages operator

$$[f_{I\mathbf{k}}]_{\rm int} \equiv \frac{1}{2}\sum_s c^\dagger_{I\mathbf{k}s} c_{I\mathbf{k}s}. \tag{7.1.7}$$

Since the occupation number is conserved in the free theory it commutes with H_0. Therefore, we have $f_{I\mathbf{k}}(t) = e^{iHt}[f_{I\mathbf{k}}]_{\rm int} e^{-iHt}$ in the Heisenberg picture.

Now we use the orthogonality of the spinors

$$u^\dagger_{I\mathbf{k}s} u_{I'\mathbf{k}s'} = v^\dagger_{I\mathbf{k}s} v_{I\mathbf{k}s'} = 2E_{I\mathbf{k}}\delta_{ss'}\delta_{II'}, \tag{7.1.8}$$

$$v^\dagger_{I\mathbf{k}s} u_{I'-\mathbf{k}'s'} = u^\dagger_{I\mathbf{k}sI} v_{I'-\mathbf{k}'s'} = 0, \tag{7.1.9}$$

to write the creation and annihilation operators in terms of N as

$$c_{I\mathbf{k}s} = \frac{1}{\sqrt{2E_{I\mathbf{k}}V}}\, u^\dagger_{I\mathbf{k}s} N_I(0,\mathbf{k})\,, \quad c^\dagger_{I\mathbf{k}s} = \frac{1}{\sqrt{2E_{I\mathbf{k}}V}}\, v^\dagger_{I\mathbf{k}s} N_I(0,-\mathbf{k}), \tag{7.1.10}$$

where $N(t,\mathbf{k}) \equiv \int d^3x e^{-i\mathbf{k}\mathbf{x}} N(t,\mathbf{x})$ denotes the spatial Fourier transform of N. Then, with the completeness relations

$$\sum_s u_{I\mathbf{k}s}\bar{u}_{I\mathbf{k}s} = \not{k} + M_I, \tag{7.1.11}$$

$$\sum_s v_{I\mathbf{k}s}\bar{v}_{I\mathbf{k}s} = \not{k} - M_I, \tag{7.1.12}$$

we can easily write the phase-space density in the Heisenberg picture as

$$f_{I\mathbf{k}}(t) = \frac{1}{2VE_{I\mathbf{k}}} \int d^3x d^3x' e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \overline{N}_I(t,\mathbf{x})\gamma^0 \left(\not{k} + M_I\right)\gamma^0 N_I(t,\mathbf{x}') \Big|_{k_0 = E_{I\mathbf{k}}}. \tag{7.1.13}$$

Using the Heisenberg equations of motion, the time derivative reads

$$\dot{f}_{I\mathbf{k}}(t) = \frac{-i}{4VE_{I\mathbf{k}}} \Big\{ \Big[R_{I\mathbf{k}}(t) - R^\dagger_{I\mathbf{k}}(t) \Big] + (k \to -k) \Big\}, \tag{7.1.14}$$

[1]Keep in mind that we consider a finite volume V with periodic boundary conditions, where the momenta $\mathbf{k}$ are discrete.

with

$$R_{I\mathbf{k}}(t) \equiv \int d^3x d^3x' e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \overline{N}_I(t,\mathbf{x}) \gamma^0 \big(\not{k} + M_I\big) J_I(t,\mathbf{x}') \Big|_{k_0=E_{I\mathbf{k}}} \tag{7.1.15}$$

and

$$J_I = \widetilde{\varphi}(h_\nu)_{Ii} \ell_i. \tag{7.1.16}$$

The retarded correlator in (7.1.1) is of order h_ν^4 and therefore, the susceptibilities $\chi_{\delta f \delta f}$ are determined by the free theory. In the free theory we have

$$\langle c^\dagger_{I\mathbf{k}s} c_{J\mathbf{k}'s'} \rangle = \delta_{IJ} \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'} f_{\mathrm{F}}(E_{I\mathbf{k}}), \tag{7.1.17}$$

which implies for the susceptibilities, using Wick's theorem,

$$(\chi_{\delta f, \delta f})_{I\mathbf{k}, J\mathbf{k}'} = \delta_{IJ} \delta_{\mathbf{k}\mathbf{k}'} \chi_{I\mathbf{k}}, \tag{7.1.18}$$

with

$$\chi_{I\mathbf{k}} = f_{\mathrm{F}}(E_{I\mathbf{k}}) \left[1 - f_{\mathrm{F}}(E_{I\mathbf{k}})\right] = -T f'_{\mathrm{F}}(E_{I\mathbf{k}}). \tag{7.1.19}$$

Let us without loss of generality choose $I = 1$. Then (7.1.1) simplifies to

$$(\gamma_{Q\,\delta f})_{a1\mathbf{k}} = \frac{1}{\chi_{1\mathbf{k}}} \frac{T}{\omega} \mathrm{Im}\, \Delta^{\mathrm{ret}}_{\dot{Q}_a \dot{f}_{1\mathbf{k}}}(\omega) \qquad (\gamma \ll \omega \ll \omega_{\mathrm{UV}}). \tag{7.1.20}$$

7.2 Relation to three-point functions in the hierarchical limit

In this section we show that the retarded correlator in (7.1.20) can at order h_ν^4 be related to a single three-point function of SM fields, if the hierarchical limit $M_1 \ll M_{I\neq 1}$ is considered. We start from the imaginary time correlator

$$\Delta_{\dot{Q}_a \dot{f}_{1\mathbf{k}}}(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \left\langle \dot{Q}_a(-i\tau) \dot{f}_{1\mathbf{k}}(0) \right\rangle, \tag{7.2.1}$$

and insert the results for the time derivatives (7.1.2) and (7.1.14) which yields

$$\Delta_{\dot{Q}_a \dot{f}_{1\mathbf{k}}}(i\omega_n) = \frac{1}{4VE_{1\mathbf{k}}} \int \big(d^4x\big)_{\mathrm{E}}\, e^{i\omega_n \tau} \left\langle \Big[K_a(x) - K_a^\dagger(x)\Big] \Big[R_{1\mathbf{k}}(0) - R^\dagger_{1\mathbf{k}}(0)\Big] \right\rangle, \tag{7.2.2}$$

where $(d^4x)_{\mathrm{E}} \equiv d\tau d^3x$. The cyclicity of the trace allows to write

$$\left\langle K_a(-i\tau, \mathbf{x}) R^\dagger_{1\mathbf{k}}(0) \right\rangle = \left\langle R^\dagger_{1\mathbf{k}}(0) K_a(i\beta - i\tau, \mathbf{x}) \right\rangle, \tag{7.2.3}$$

and using the fact that in imaginary time $K(-i\tau, \mathbf{x})^\dagger = K^\dagger(i\tau, \mathbf{x})$, we find

$$\left\langle K_a(-i\tau, \mathbf{x}) R^\dagger_{1\mathbf{k}}(0) \right\rangle = \left\langle K^\dagger_a(-i\beta + i\tau, \mathbf{x}) R_{1\mathbf{k}}(0) \right\rangle^*. \tag{7.2.4}$$

The same relations hold for $K_a \leftrightarrow K_a^\dagger$. Applying these relations to (7.2.2) and using periodicity in imaginary time, we find

$$\Delta_{\dot{Q}_a \dot{f}_{1\mathbf{k}}}(i\omega_n) = \frac{1}{2E_{1\mathbf{k}}V}\mathrm{Re}\int \left(d^4x\right)_{\mathrm{E}} e^{i\omega_n\tau}\Big\langle \Big[K_a(x) - K_a^\dagger(x)\Big] R_{1\mathbf{k}}(0) + (k \to -k)\Big\rangle_{k^0\to E_{1\mathbf{k}}}. \quad (7.2.5)$$

We want to compute the rate which describes the generation of asymmetries Q_a due to the CP violating Yukawa interaction (1.2.4). This interaction does not violate CP at order h_ν^2, as we have seen in section 1.2. Therefore, we have to expand the Euclidean correlator (7.2.5) to order h_ν^4. After the perturbative expansion the sterile neutrinos can be considered as free particles and we can use Wick's theorem for them. Then (7.2.5) contains, for instance, correlation functions like $\langle N_I(x)N_J(0)\rangle$ and $\langle \overline{N}_I(x)\overline{N}_J(0)\rangle$. Using the fact that N is Majorana, that is, $N_I(x) = -\mathscr{C}\overline{N}_I^\top(x)$, we write these correlation functions in momentum space as the sterile neutrino propagator

$$S_I(p) \equiv \int \left(d^4x\right)_{\mathrm{E}} e^{ipx}\langle N_I(x)\overline{N}_I(0)\rangle. \quad (7.2.6)$$

We also assume the sterile neutrinos to be hierarchical, that is $M_1 \ll M_I$ for $I \neq 1$ and that the temperature is not much larger than M_1. Then we have $T \ll M_{I\neq 1}$ and as for the effective theory in section 5.2 we can approximate the propagator for $I \neq 1$ as

$$S_I(p) \simeq \frac{1}{M_I} \text{ for } I \neq 1. \quad (7.2.7)$$

If we also use that the operators J_{aI} and J_I are left-handed, then terms containing the product $\overline{J}\,J$ vanish. Finally, we end up with the contributions[2]

$$\int \left(d^4x\right)_{\mathrm{E}} \langle K_a(x)R_{1\mathbf{k}}\rangle\, e^{i\omega_n\tau} = V\int \left(d^4x\right)_{\mathrm{E}}\left(d^4x'\right)_{\mathrm{E}} T\sum_{\{\omega_{n'}\}} e^{i(\omega_n-\omega_{n'})\tau + i\mathbf{kx}}\sum_I \frac{1}{M_I}$$
$$\frac{1}{2}e^{i\mathbf{kx}}\Big\langle \mathrm{T}\Big\{J_{1a}^\top(x)\mathscr{C}S_1(i\omega_{n'},\mathbf{k})\gamma^0(\not{k}+M_1)J_1(0)\overline{J_I}\mathscr{C}\overline{J_I}^\top(x')\Big\}\Big\rangle\Big|_{k_0=E_{1\mathbf{k}}} \quad (7.2.8)$$

and

$$\int \left(d^4x\right)_{\mathrm{E}} \Big\langle K_a^\dagger(x)R_{1\mathbf{k}}\Big\rangle\, e^{i\omega_n\tau} = V\int \left(d^4x\right)_{\mathrm{E}}\left(d^4x'\right)_{\mathrm{E}} T\sum_{\{\omega_{n'}\}} e^{i(\omega_n-\omega_{n'})\tau}\sum_I \frac{1}{M_I}$$
$$e^{i\mathbf{kx}'}\Big\langle \mathrm{T}\Big\{\overline{J_{1a}}\mathscr{C}\overline{J_1}^\top(x)J_1^\top(x')\mathscr{C}S_1(i\omega_{n'},\mathbf{k})\gamma^0(\not{k}+M_1)J_1(0)\Big\}\Big\rangle\,|_{k_0=E_{1\mathbf{k}}}. \quad (7.2.9)$$

Let us write the sterile neutrino propagator as

$$S_1(k) = (\not{k}+M_1)\Delta_1(k) \quad (7.2.10)$$

[2]Keep in mind that $\{\omega_{n'}\}$ denotes fermionic Matsubara frequencies.

with $\Delta_1 \equiv (-k^2 + M_I^2)^{-1}$. The left-handed nature of the Standard Model fermions can be used to simplify the Dirac structure in (7.2.8) and (7.2.9). For example, we have

$$P_{\mathrm{L}} S_1 \left(i\omega_{n'}, \mathbf{k}\right) \gamma^0 \left(\not{k} + M_1\right) P_{\mathrm{L}} = M_1 (i\omega_{n'} + k^0) \Delta_1 \left(i\omega_{n'}, \mathbf{k}\right) P_{\mathrm{L}}. \tag{7.2.11}$$

Furthermore, we cancel the numerator with the denominator

$$(i\omega_{n'} + k^0) \Delta_I \left(i\omega_{n'}, \mathbf{k}\right) = \frac{1}{k^0 - i\omega_{n'}} \qquad (k^0 = E_{I\mathbf{k}}) \tag{7.2.12}$$

and we pull out the Yukawa coupling by writing J_I and J_{Ia} defined in (7.1.16) and (7.1.3) in terms of

$$J_i = \widetilde{\varphi} \ell_i. \tag{7.2.13}$$

The Euclidean correlator (7.2.5) can then be written in terms of the three-point function

$$\Gamma_{ijlm}(k_1, k_2) \equiv \int \left(d^4 x_1\right)_{\mathrm{E}} \left(d^4 x_2\right)_{\mathrm{E}} e^{i(k_1 x_1 + k_2 x_2)} \left\langle \mathrm{T} \left\{ J_i^\top(x_1) \mathscr{C}^\dagger J_j(x_2) \overline{J}_l \mathscr{C} \overline{J_m}^\top(0) \right\} \right\rangle \tag{7.2.14}$$

as

$$\begin{aligned} \Delta_{\dot{Q}_a \dot{f}_{1\mathbf{k}}} \left(i\omega_n\right) =& \frac{1}{2E_{1\mathbf{k}}} \sum_I \frac{M_1}{M_I} \mathrm{Re} \Bigg\{ T \sum_{\{k^0\}} \frac{1}{k^0 - E_{1\mathbf{k}}} \qquad (7.2.15) \\ & \Bigg(\Bigg[\frac{1}{2} (h_\nu)^*_{Il} (h_\nu)^*_{Im} (h_\nu T_a)_{1i} (h_\nu)_{1j} \Gamma_{ijlm}(k - q, q - k) \\ & - (T_a h_\nu)^*_{Il} (h_\nu)^*_{Im} (h_\nu)_{1i} (h_\nu)_{1j} \Gamma_{ijlm}(k - q, -k) \Bigg] - \left[k \to -k\right] \Bigg) \Bigg\}, \qquad (7.2.16) \end{aligned}$$

where we have renamed $\omega_{n'}$ to k_0 and $q = (i\omega_n, \mathbf{0})$. Keeping in mind that J is fermionic and that the matrix $\mathscr{C}$ is antisymmetric, we have

$$\mathrm{T} \left\{ J_i^\top(x) \mathscr{C}^\dagger J_j(y) \right\} = \mathrm{T} \left\{ J_j^\top(y) \mathscr{C}^\dagger J_i(x) \right\}, \tag{7.2.17}$$

$$\mathrm{T} \left\{ \overline{J}_k(x) \mathscr{C} \overline{J}_l^\top(y) \right\} = \mathrm{T} \left\{ \overline{J}_l(y) \mathscr{C} \overline{J}_k^\top(x) \right\}, \tag{7.2.18}$$

and consequently, the three-point correlator (7.2.14) has the symmetries

$$\Gamma_{ijlm}(k_1, k_2) = \Gamma_{jilm}(k_2, k_1), \tag{7.2.19}$$

$$\Gamma_{ijlm}(k_1, k_2) = \Gamma_{ijml}(k_1, k_2). \tag{7.2.20}$$

7.3 CP, T and $SU(n_{\mathrm{f}})$-flavor symmetry

In this work we only study the effects of CP violation due to the sterile neutrino Yukawa interactions. Therefore, we neglect the CP violation of the Standard Model in the following.

In this case we can use the symmetries for the three-point function from section 6.2. First of all, we use that Γ is real. Then we can simplify the Euclidean correlator to

$$\begin{aligned}\Delta_{\dot{Q}_a\dot{f}_{1\mathbf{k}}}(i\omega_n) = \frac{i}{2E_{1\mathbf{k}}}\sum_{I>1}\frac{M_1}{M_I}T\sum_{\{k^0\}}\frac{1}{k^0-E_{1\mathbf{k}}}\\ \Big\{-\frac{1}{2}\,\mathrm{Im}\big[(h_\nu)_{1i}(h_\nu T_a)_{1j}(h_\nu)^*_{Il}(h_\nu)^*_{Im}\big]\Gamma_{ijlm}(-k-q,k+q)\\ +\mathrm{Im}\big[(h_\nu)_{1i}(h_\nu)_{1j}(T_a h_\nu)^*_{Il}(h_\nu)^*_{Im}\big]\Gamma_{ijlm}(-k-q,k)-(k\to-k)\Big\}.\end{aligned} \tag{7.3.1}$$

Then we use the fact that $\Gamma(k_1,k_2)=\Gamma(-k_1,-k_2)$. This allows to write the correlator as

$$\begin{aligned}\Delta_{\dot{Q}_a\dot{f}_{1\mathbf{k}}}(i\omega_n) = \frac{i}{2E_{1\mathbf{k}}}\sum_{I>1}\frac{M_1}{M_I}T\sum_{\{k^0\}}\frac{1}{k^0-E_{1\mathbf{k}}}\\ \Big\{-\frac{1}{2}\,\mathrm{Im}\big[(h_\nu)_{1i}(h_\nu T_a)_{1j}(h_\nu)^*_{Il}(h_\nu)^*_{Im}\big]\Gamma_{ijlm}(-k-q,k+q)\\ +\mathrm{Im}\big[(h_\nu)_{1i}(h_\nu)_{1j}(T_a h_\nu)^*_{Il}(h_\nu)^*_{Im}\big]\Gamma_{ijlm}(-k-q,k)\Big\}-(\omega_n\to-\omega_n).\end{aligned} \tag{7.3.2}$$

The three-point correlator (7.2.14) is in general a complicated tensor in flavors space which depends on the leptonic SM Yukawa interactions. We can tremendously simplify the problem if we neglect the Yukawa interactions of SM leptons. This is a good approximation since these interactions are very weak during leptogenesis. Then the remaining interactions are invariant under leptonic $SU(n_\mathrm{f})$ flavor transformations. In combination with the symmetry (7.2.20) this implies that the correlation function has the flavor structure

$$\Gamma_{ijlm}=\frac{1}{2}(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\Gamma, \tag{7.3.3}$$

where

$$\Gamma=\delta_{ij}\delta_{lm}\Gamma_{ijlm}/n_\mathrm{f}. \tag{7.3.4}$$

Using this, we obtain

$$\Delta_{\dot{Q}_a\dot{f}_{1\mathbf{k}}}(i\omega_n)=\left(\sum_{I>1}\frac{M_1}{M_I}\mathrm{Im}\left[\left(h_\nu h_\nu^\dagger\right)_{1I}\left(h_\nu T_a h_\nu^\dagger\right)_{1I}\right]\right)\mathscr{M}_{\mathbf{k}}(i\omega_n), \tag{7.3.5}$$

where we defined the function

$$\mathscr{M}_{\mathbf{k}}(i\omega_n)=\frac{iT}{4E_{\mathbf{k}}}\sum_{\{k^0\}}\frac{-\Gamma(-k-q,k+q)+2\Gamma(-k-q,k)}{k^0-E_{\mathbf{k}}}-(\omega_n\to-\omega_n)\,. \tag{7.3.6}$$

Note that (7.3.6) only depends on Standard Model parameters. Therefore, from here on we drop the subscript 1 on $E_{\mathbf{k}}$.

7.4 Analytic continuation and ω-expansion

Before we analytically continue the retarded correlator (7.3.5) to complex frequencies, we have to perform the Matsubara sum over the heavy neutrino frequency in (7.3.6). We do this by using the three-point spectral representation (6.1.10). The symmetries (7.2.19) and (7.2.20) imply that the two spectral functions ρ_{ABC} and ρ_{BAC} in (6.1.10) are related through

$$\rho_{BAC}(\omega_2, \mathbf{k}_2, \omega_1, \mathbf{k}_1) = -\rho_{ABC}(\omega_2, \mathbf{k}_2, \omega_1, \mathbf{k}_1), \tag{7.4.1}$$

so that the spectral representation consists of a single spectral function which we denote as $\rho \equiv \rho_{ABC}$. Using this and the fact that the spectral function is even in (k_1, k_2), we can write the spectral representation (6.1.11) of the three-point correlator (7.3.4) as as

$$\begin{aligned}\Gamma(k_1, k_2) = \int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{1}{k_1^0 + k_2^0 - \omega_1 - \omega_2} \\ \times \left[\frac{\rho(\omega_1, \mathbf{k}_1, \omega_2, \mathbf{k}_2)}{k_1^0 - \omega_1} + \frac{\rho(-\omega_2, -\mathbf{k}_2, -\omega_1, -\mathbf{k}_1)}{k_2^0 - \omega_2}\right].\end{aligned} \tag{7.4.2}$$

We insert (7.4.2) in (7.3.6) and obtain

$$\begin{aligned}\mathscr{M}_{\mathbf{k}}(i\omega_n) =& \frac{i}{4E_{\mathbf{k}}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}T\sum_{\{k_0\}}\frac{1}{k^0 - E_{\mathbf{k}}} \\ &\times\Bigg\{\rho(\omega_1, -\mathbf{k}, \omega_2, \mathbf{k})\left[-\frac{1}{\omega_1+\omega_2}\frac{1}{k^0+\omega_1+i\omega_n} + \frac{2}{\omega_1+\omega_2+i\omega_n}\frac{1}{k^0+\omega_1+\omega}\right] \\ &+\rho(-\omega_2, -\mathbf{k}, -\omega_1, -\mathbf{k})\left[\frac{1}{\omega_1+\omega_2}\frac{1}{k^0-\omega_2+i\omega_n} - \frac{2}{\omega_1+\omega_2+i\omega_n}\frac{1}{k^0-\omega_2}\right]\Bigg\} \\ &-(\omega_n \to -\omega_n).\end{aligned} \tag{7.4.3}$$

The apparent singularity in the terms with $1/(\omega_1+\omega_2)$ cancels because the numerator vanishes for $\omega_1 = -\omega_2$. We can therefore replace $1/(\omega_1+\omega_2)$ by its principal value. Then we substitute $\omega_1 \leftrightarrow \omega_2$ in the third line and rewrite the terms such that

$$\begin{aligned}\mathscr{M}_{\mathbf{k}}(i\omega_n) = \frac{i}{2E_{\mathbf{k}}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\rho(\omega_1, -\mathbf{k}, \omega_2, \mathbf{k})T\sum_{\{k_0\}}\frac{1}{k^0 - E_{\mathbf{k}}} \\ \times\left(\frac{1}{k^0+\omega_1+i\omega_n} - \frac{1}{k^0+\omega_1}\right)\left(\frac{1}{\omega_1+\omega_2+i\omega_n} - \text{P.V.}\frac{1}{\omega_1+\omega_2}\right) - (\omega_n \to -\omega_n).\end{aligned} \tag{7.4.4}$$

Using this expression, we perform the Matsubara sum which gives

$$\begin{aligned}\mathscr{M}_{\mathbf{k}}(i\omega_n) = \frac{i}{2E_{\mathbf{k}}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\rho(\omega_1, -\mathbf{k}, \omega_2, \mathbf{k})\left[f_{\text{F}}(-\omega_1) - f_{\text{F}}(E_{\mathbf{k}})\right] \\ \times\frac{-i\omega_n}{(E_{\mathbf{k}}+\omega_1+i\omega_n)(E_{\mathbf{k}}+\omega_1)}\left(\frac{1}{\omega_1+\omega_2+i\omega_n} - \text{P.V.}\frac{1}{\omega_1+\omega_2}\right) - (\omega_n \to -\omega_n).\end{aligned} \tag{7.4.5}$$

Now we can analytically continue $i\omega_n \to \omega + i0+$ with real ω. Then we can use

$$\frac{1}{x - i0^+} = \text{P.V.}\frac{1}{x} + i\pi\delta(x), \tag{7.4.6}$$

and expand (7.4.5) to the leading order in ω. Then the principal values cancel inside the bracket and only the δ-function remains. This gives

$$\mathscr{M}_{\mathbf{k}}(\omega + i0^+) = \frac{i}{2E_{\mathbf{k}}} \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \rho(\omega_1, -\mathbf{k}, \omega_2, \mathbf{k}) \left[f_{\mathrm{F}}(-\omega_1) - f_{\mathrm{F}}(E_{\mathbf{k}}) \right] \\ \times \frac{-\omega}{(E_{\mathbf{k}} + \omega_1 + i0^+)(E_{\mathbf{k}} + \omega_1)} (-i\pi) \delta(\omega_1 + \omega_2) - (\omega \to -\omega) + O(\omega^3). \tag{7.4.7}$$

For the rate we need the imaginary part of this expression. Since ρ is real, as we have shown in section 6.2, we get a second δ-function, so that the ω_1 and ω_2 integrals are trivial. This yields

$$\mathrm{Im}\mathscr{M}_{\mathbf{k}}(\omega + i0^+) = -\frac{\omega}{4E_{\mathbf{k}}} \rho(-E_{\mathbf{k}}, -\mathbf{k}, E_{\mathbf{k}}, \mathbf{k}) f'_{\mathrm{F}}(E_{\mathbf{k}}) + O(\omega^3). \tag{7.4.8}$$

Using the formula (7.1.20) for the asymmetry rate in combination with (7.3.5), (7.4.8) and with $\chi_{1\mathbf{k}} = -T f'_{\mathrm{F}}(E_{\mathbf{k}})$, we obtain the *master formula*

$$(\gamma_{Q\,\delta f})_{a1\mathbf{k}} = \frac{\rho(-E_{\mathbf{k}}, -\mathbf{k}, E_{\mathbf{k}}, \mathbf{k})}{4E_{\mathbf{k}}} \left(\sum_{I>1} \frac{M_1}{M_I} \mathrm{Im} \left[\left(h_\nu h_\nu^\dagger \right)_{1I} \left(h_\nu T_a h_\nu^\dagger \right)_{1I} \right] \right). \tag{7.4.9}$$

for the lepton asymmetry rate. This formula is valid to order h_ν^4 and only in the hierarchical limit, where the lightest sterile neutrino mass is much smaller than the heavier sterile neutrino. It can be expanded to all orders in the SM couplings, except in the Yukawa interactions of SM leptons. The formula does also not take into account effects of SM CP-violation.

7.5 Leading order at zero temperature

As a first application of the master formula (7.4.9) we compute the zero-temperature contribution to the asymmetry rate at leading order in the symmetric phase. We start from the Euclidean three-point correlator (7.2.14), which at leading order corresponds to the diagram

$\Gamma^{(0)}(k_1, k_2) =$  , (7.5.1)

where solid thick lines are heavy neutrinos carrying the ingoing momenta k_1 an k_2 respectively. The solid lines with arrows are SM leptons and the dotted lines are Higgs. The dashed line represents the outgoing momentum k_1+k_2 which will be set zero in the corresponding spectral function. Using Wick's theorem, contracting all gauge indices, using the properties of the $\mathscr{C}$ matrix (1.1.14) and computing the Dirac trace, we find[3]

$$\Gamma^{(0)}(k_1, k_2) = 4d(r)(d(r)+1) \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{p_1 \cdot p_2}{p_1^2 (p_1 - k_1)^2 (p_2 + k_2)^2 p_2^2}, \tag{7.5.2}$$

[3]We use $d^d p = d\bar{p}_0 d^{d-1} p$ and $p_0 = i\bar{p}_0$

where $d(r) = 2$ is the dimension of the representation of the gauge group $SU(2)$. This two-loop integral can be interpreted as a product of two one-loop tensor integrals of the form

$$I_1^\mu(k) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu}{p^2(p-k)^2}. \tag{7.5.3}$$

Now, substituting $p \to -p + k$, we find

$$I_1^\mu(k) = -I_1^\mu(k) + k^\mu I_1(k), \tag{7.5.4}$$

where

$$I_1(k) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2(p-k)^2}. \tag{7.5.5}$$

Therefore, we can write

$$I_1^\mu(k) = \frac{k^\mu}{2} I_1(k) \tag{7.5.6}$$

which simplifies the three-point functions to

$$\Gamma^{(0)}(k_1, k_2) = -6k_1 \cdot k_2 I_1(k_1) I_1(-k_2). \tag{7.5.7}$$

Now we use the inverse relation (B.3.2) in combination with (B.3.7). Then we set $k_1 = -k$ and $k_2 = k$ with $k^2 = M_1^2$ and obtain for the leading order spectral function in (7.4.9)

$$\rho^{(0)}(-k, k) = -\frac{24M_1^2}{(16\pi)^2}. \tag{7.5.8}$$

The master formula (7.4.9) then yields

$$(\gamma^{(0)}_{Q\,\delta f})_{a1\mathbf{k}} = -\frac{6M_1^2}{(16\pi)^2 E_\mathbf{k}} \sum_{I>1} \frac{1}{M_I} \mathrm{Im}\left[\left(h_\nu T_a h_\nu^\dagger\right)_{1I} \left(h_\nu h_\nu^\dagger\right)_{1I}\right]. \tag{7.5.9}$$

This result can be written as

$$(\gamma^{(0)}_{Q\,\delta f})_{a1\mathbf{k}} = \varepsilon^{(0)}_{1a} (\gamma^{(0)}_{\delta f \delta f})_{1\mathbf{k}}, \tag{7.5.10}$$

where

$$(\gamma^{(0)}_{\delta f\,\delta f})_{1\mathbf{k}} = \frac{M_1^2 (h_\nu h_\nu^\dagger)_{11}}{8\pi E_\mathbf{k}} \tag{7.5.11}$$

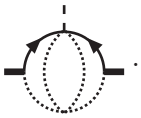
is the leading order neutrino decay rate (c.f (1.2.9)) and

$$\varepsilon^{(0)}_{1a} = -\frac{3}{16\pi} \sum_{I>1} \frac{M_1}{M_I} \frac{\mathrm{Im}\left[\left(h_\nu T_a h_\nu^\dagger\right)_{1I} \left(h_\nu h_\nu^\dagger\right)_{1I}\right]}{(h_\nu h_\nu^\dagger)_{11}} \tag{7.5.12}$$

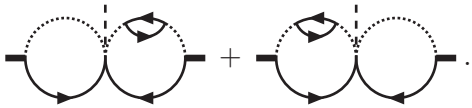
is the CP asymmetry which agrees with (1.2.12) for $(T_a)_{ij} = \delta_{ai}\delta_{aj}$. We have therefore reproduced the well-known leading order relation from section (1.2).

7.6 Next-to-leading order at zero temperature

Now we compute the next-to-leading order SM corrections to the asymmetry rate at zero temperature. For the next-to-leading order three-point correlator we consider only contributions from the Higgs self-interaction λ, the top Yukawa coupling h_t and the $\mathrm{U}(1) \times \mathrm{SU}(2)$ gauge couplings g_1 and g_2. The only Higgs contribution at zero-temperature at order g^2 is

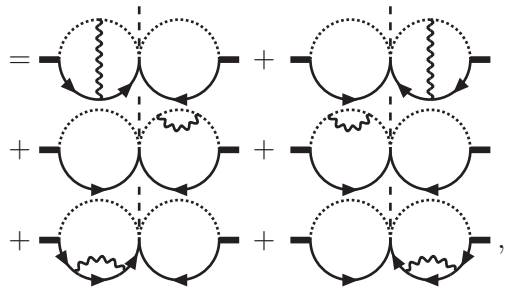
$\Gamma_\lambda(k_1, k_2) \equiv$  . (7.6.1)

The top quark contribution comes from the diagrams

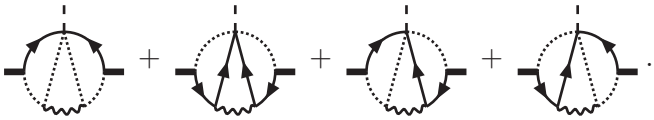
$\Gamma_t(k_1, k_2) \equiv$  . (7.6.2)

Here the fermionic lines in the closed lepton loop represent the top-quarks.

The diagrams contributing to g_1 and g_2 decompose into two gauge invariant sets, the factorisable diagrams

$\Gamma_{g,\mathrm{fac}}(k_1, k_2) =$  , (7.6.3)

and the non-factorisable diagrams

$\Gamma_{g,\mathrm{nfac}}(k_1, k_2) =$  . (7.6.4)

Here the wiggled lines are the gauge bosons. The complete next leading order correlator is then given by

$$\Gamma^{(2)}(k_1, k_2) = \Gamma_{g,\mathrm{nfac}}(k_1, k_2) + \Gamma_{g,\mathrm{fac}}(k_1, k_2) + \Gamma_\lambda(k_1, k_2) + \Gamma_t(k_1, k_2). \tag{7.6.5}$$

We compute the spectral functions of the three-point functions $\Gamma_{g,\mathrm{nfac}}$, $\Gamma_{g,\mathrm{fac}}$, Γ_λ and Γ_t in appendix B. At first we study the Dirac traces in B.1 and show that all terms containing γ_5 drop out. After performing the Dirac traces the three-point functions can be expressed in terms of scalar Feynman integrals. We reduce these integrals to a minimal set of master integrals in B.2. Then, in B.3 - B.5 we compute the master spectral functions of the master integrals. Combining all results we find the spectral functions

$$\rho_{g,\mathrm{nfac}}(-k,k) = \frac{3(g_1^2+g_2^2)M_1^2}{(16\pi)^2 8\pi^2}\left(\frac{3}{\varepsilon}+\frac{23}{2}+8\ln(2)+9\ln\frac{\bar{\mu}^2}{M_1^2}\right), \tag{7.6.6}$$

$$\rho_{g,\mathrm{fac}}(-k,k) = -\frac{3(g_1^2+3g_2^2)M_1^2}{(16\pi)^2 4\pi^2}\left(\frac{3}{\varepsilon}+\frac{53}{2}+9\ln\frac{\bar{\mu}^2}{M_1^2}\right), \tag{7.6.7}$$

$$\rho_{\lambda}(-k,k) = \frac{3\lambda M_1^2}{(16\pi)^2\pi^2}\left(\frac{1}{\varepsilon}+\frac{13}{2}+3\ln\frac{\bar{\mu}^2}{M_1^2}\right), \tag{7.6.8}$$

$$\rho_{t}(-k,k) = \frac{3|h_t|^2 M_1^2}{(16\pi)^2\pi^2}\left(\frac{3}{\varepsilon}+\frac{45}{2}+9\ln\frac{\bar{\mu}^2}{M_1^2}\right). \tag{7.6.9}$$

For the renormalization it turns out to be convenient to describe the exchange of virtual heavy neutrinos N_I with $I \neq 1$ by an effective theory with an dimension-5 operator. Following section 5.2 we find

$$\mathscr{L}_{\mathrm{int}}^{\mathrm{eff}} = -\bar{N}_1(h_\nu)_{1i}J_i + \frac{1}{2}(g_\nu)_{ij}J_i^\top \mathscr{C}^{-1}J_j + \mathrm{H.c}, \tag{7.6.10}$$

where $(g_\nu)_{ij} = \sum_{I\neq 1}(h_\nu)_{Ii}(h_\nu)_{Ij}/M_I$. We renormalize the fields by

$$\varphi = \varphi_{\mathrm{R}} Z_\varphi^{1/2}, \qquad \ell = \ell_{\mathrm{R}} Z_\ell^{1/2} \tag{7.6.11}$$

and the couplings by

$$(h_\nu)_{1i} = (h_{\nu\mathrm{R}})_{1i} Z_h \qquad (g_\nu)_{ij} = (g_{\nu\mathrm{R}})_{ij} Z_g. \tag{7.6.12}$$

In appendix B.6 we summarize the calculation of Z_h and Z_g and find

$$Z_h = 1 + \frac{1}{(4\pi)^2\varepsilon}\left(-\frac{3}{8}(g_1^2+3g_2^2)+\frac{N_c}{2}|h_t|^2\right), \tag{7.6.13}$$

$$Z_g = 1 + \frac{1}{(4\pi)^2\varepsilon}\left(-\frac{3}{4}(g_1^2+3g_2^2)+\frac{3}{4}(g_1^2+g_2^2)+2\lambda+N_c|h_t|^2\right), \tag{7.6.14}$$

where in Z_g we have explicitly distinguished the coupling structures $(g_1^2+g_2^2)$ and $(g_1^2+3g_2^2)$. The reason is that the first one cancels the divergences of the non-factorisable and the second one in combination with Z_h the factorisable gauge field corrections. Note that our result for Z_h is consistent with the one in [53]. Using the master formula (7.4.9) in combination with the spectral functions (7.6.6)-(7.6.9) and expressing the result in terms of the renormalized couplings (B.6.3), we find the finite expression

$$\begin{aligned}(\gamma^{(2)}_{Q\,\delta f})_{a1\mathbf{k}} = (\gamma^{(0)}_{Q\,\delta f})_{a1\mathbf{k}}\Bigg\{&1+\frac{g_1^2+3g_2^2}{(8\pi)^2}\left(29+6\ln\frac{\bar{\mu}^2}{M_1^2}\right)\\ &+\frac{g_1^2+g_2^2}{(8\pi)^2}\left(\frac{1}{2}-8\ln(2)-3\ln\frac{\bar{\mu}^2}{M_1^2}\right)\\ &-\frac{|h_t|^2}{(8\pi)^2}\left(84+24\ln\frac{\bar{\mu}^2}{M_1^2}\right)-\frac{\lambda}{(8\pi)^2}\left(20+8\ln\frac{\bar{\mu}^2}{M_1^2}\right)\Bigg\},\end{aligned} \tag{7.6.15}$$

where $(\gamma^{(0)}_{Q\,\delta f})_{a1\mathbf{k}}$ is the leading order rate (7.5.9). Note that the factorisable corrections are exactly twice the order g^2 corrections to the zero-temperature sterile neutrino productions rate in [53]. We choose the renormalization scale $\bar{\mu} = T$ and define $z = M_1/T$. The size of the corrections is given in table 7.1 and they are found to be very small. Even in the very non-relativistic limit $z = 10$, where the contributions of the logarithms become relevant, the corrections are smaller than 3%.

M_1/GeV	z	$\gamma^{(2)}_{aI\mathbf{k}}/\gamma^{(0)}_{aI\mathbf{k}}$
10^9	1	0.9998
10^9	5	1.0181
10^9	10	1.0277
10^{13}	1	1.0096
10^{13}	5	1.0176
10^{13}	10	1.0219

Table 7.1: Size of the corrections to the asymmetry rate for different values of M_1 and z.

Using the zero-temperature limit of the NLO neutrino decay rate of [53], we find for the NLO CP asymmetry

$$\begin{aligned}\varepsilon^{(2)}_{1a} = \varepsilon^{(0)}_{1a}\Bigg\{1 &+ \frac{g_1^2+3g_2^2}{(8\pi)^2}\left(\frac{29}{2}+3\ln\frac{\bar{\mu}^2}{M_1^2}\right)\\ &+\frac{g_1^2+g_2^2}{(8\pi)^2}\left(\frac{1}{2}-8\ln(2)-3\ln\frac{\bar{\mu}^2}{M_1^2}\right)\\ &-\frac{|h_t|^2}{(8\pi)^2}\left(42+12\ln\frac{\bar{\mu}^2}{M_1^2}\right)-\frac{\lambda}{(8\pi)^2}\left(20+8\ln\frac{\bar{\mu}^2}{M_1^2}\right)\Bigg\}. \qquad (7.6.16)\end{aligned}$$

Chapter 8

Sterile neutrino equilibration rate

In this chapter we study the coefficient $\gamma_{\delta f \delta f}$ in the kinetic equation (1.2.13) which describes the evolution of the sterile neutrino phase-space densities $f_{I\mathbf{k}}$. This chapter is based on the author's publication [4].

8.1 Relation between production and equilibration rates

For the derivation of the linear kinetic equations (1.2.13) and (1.2.14) we assumed that the phase-space densities $f_{I\mathbf{k}}$ and the charges Q_a are close to their equilibrium values. One would therefore naturally think that the equations are wrong if $f_{I\mathbf{k}} \ll f^{\rm eq}_{I\mathbf{k}}$ and consequently, $\delta f_{I\mathbf{k}} \approx -f^{\rm eq}_{I\mathbf{k}}$, which can be of order one. In order to get the correct kinetic equation for small $f_{I\mathbf{k}}$, one would not expand the kinetic equations (3.1.2) in $\delta f_{I\mathbf{k}}$ but in $f_{I\mathbf{k}}$. The leading contribution to this equation does then read

$$(\partial_t - H\mathbf{k}\partial_{\mathbf{k}})\, f_{I\mathbf{k}} = \Gamma^{\rm pro}_{I\mathbf{k}} + ..., \tag{8.1.1}$$

where "..." contains terms of order $f_{I\mathbf{k}}$ and Q_a. The coefficient $\Gamma^{\rm pro}$ is the sterile neutrino production rate and has been widely studied in several temperature regimes at leading order and next-to-leading order in the SM couplings [50, 51, 52, 53, 54, 55, 56, 57, 58]. Let us for a moment assume that the linear equation (1.2.13) is true for small values of $f_{I\mathbf{k}}$ as well. Then, for $f_{I\mathbf{k}} = 0$ and $Q_a = 0$ (1.2.13) implies that

$$\Gamma^{\rm pro}_{I\mathbf{k}} = -(\gamma_{\delta f \delta f})_{I\mathbf{k}I'\mathbf{k}'} f^{\rm eq}_{I\mathbf{k}}. \tag{8.1.2}$$

We will see that at leading order in h_ν

$$(\gamma_{\delta f \delta f})_{I\mathbf{k}I'\mathbf{k}'} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{II'}\Gamma^{\rm eq}_{I\mathbf{k}}, \tag{8.1.3}$$

which defines the sterile neutrino equilibration rate $\Gamma^{\rm eq}_{I\mathbf{k}}$. This implies the simple relation

$$\Gamma^{\rm pro}_{I\mathbf{k}} = -\Gamma^{\rm eq}_{I\mathbf{k}} f^{\rm eq}_{I\mathbf{k}}, \tag{8.1.4}$$

between the sterile neutrino production rate and the equilibration rate. The great achievement of this relation is that one can simply use everything which is known about the so well

studied production rate also for the equilibration rate $\Gamma^{\rm eq}_{I\mathbf{k}}$. In 1983 Weldon [49] showed with Boltzmann equations that this relation is indeed correct at leading order in perturbation theory. However, if one wants to take radiative corrections like in [39] into account, (8.1.4) has to be proved beyond leading order. Shortly before we did the proof in [4], it has been proved implicitly in [90]. In this reference, a kinetic equation for $f_{I\mathbf{k}}$ has been derived, which is valid at h_ν^2, for any value of $f_{I\mathbf{k}}$ and to all orders in the SM couplings. In the absence of chemical potentials of SM charges the equation agrees with the linear equation (1.2.13) at order h_ν^2.

Here we prove the relation (8.1.4)in a rather general framework and consider the production and equilibration rate of a particle with any spin or helicity, bosonic or fermionic, charged or uncharged, which is described by a field Φ. We assume the particle to be weakly and linearly coupled to a plasma such that its interactions can be considered as slow. Then the non-equilibrium system is completely determined by all slow quantities such as the phase-space density $f_{\mathbf{k}\lambda}$ of the particle Φ. Here λ labels possible spins or helicities. If the phase-space density is close to its equilibrium value, its time evolution is described by the linear equation

$$(\partial_t - H\mathbf{k}\partial_{\mathbf{k}})f_{\mathbf{k}\lambda} = -(\gamma_{\delta f\delta f})_{\mathbf{k}\lambda\mathbf{k}'\lambda'}\delta f_{\mathbf{k}'\lambda'} + ..., \tag{8.1.5}$$

where "..." denotes all other slow quantities in the system. The coefficient $\gamma_{\delta f\delta f}$ can then be computed according to (3.2.13) as

$$(\gamma_{\delta f\delta f})_{\mathbf{k}\lambda\mathbf{k}'\lambda'} = \frac{T}{2}\lim_{\gamma\ll\omega\ll\omega_{\rm UV}}\frac{\rho_{\dot f_{\mathbf{k}\lambda}\dot f_{\mathbf{k}'\lambda'}}(\omega)}{\omega}(\chi^{-1}_{\delta f\delta f})_{\mathbf{k}''\lambda''\mathbf{k}'\lambda'}. \tag{8.1.6}$$

We assume that the system is described by the Hamiltonian

$$H = H_0 + U, \tag{8.1.7}$$

where H_0 is the Hamiltonian which describes the plasma and the free fields Φ and U describes the weak interaction of the fields Φ with the plasma. In the following, we distinguish two cases. At first, we consider a charged particle species Φ and derive a master formula for the coefficient $\gamma_{\delta f\delta f}$ which relates the coefficient to the self-energy of the particle Φ. This master formula is valid to the leading order in U and to all orders in the plasma interactions. Then we show that similar relations hold for uncharged particles.

In the following sections 8.2 and 8.3 we closely follow the publication [4] which has been written by the author of this thesis in collaboration with D. Bödeker and M. Wörmann. A similar discussion of the following derivations can also be found in [45].

8.2 Charged particle species

At first we assume the particle Φ to be charged. The interaction can be written as

$$U = \int d^3x \left(\overline{J}\Phi - \overline{\Phi}J\right), \tag{8.2.1}$$

where J is any elementary or composite operator which does not contain the fields Φ. We define $\overline{\Phi} \equiv \Phi^\dagger$ for bosons and $\overline{\Phi} \equiv \Phi^\dagger\gamma_0$ for spin 1/2.

Like in section (7.1), we define the phase-space density in the interaction picture with respect to U. The fields Φ can in the interaction picture be written as free on-shell fields

$$[\Phi(x)]_{\rm int} = \sum_{\mathbf{k},\lambda} \frac{1}{\sqrt{2E_{\mathbf{k}}V}} \left[e^{-ikx}\, u_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda} + e^{ikx}\, v_{\mathbf{k}\lambda} d^\dagger_{\mathbf{k}\lambda} \right]_{k^0=E_{\mathbf{k}}}, \tag{8.2.2}$$

where the annihilation and creation operators are normalized such that

$$[c_{\mathbf{k}\lambda}, c^\dagger_{\mathbf{k}'\lambda'}] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}, \tag{8.2.3}$$

with the (anti-)commutator $[A,B] = AB - \sigma BA$ like in (2.1.3) with $\sigma = 1$ for bosons and $\sigma = -1$ for fermions. We define the phase-space density operator as

$$[f_{\mathbf{k}\lambda}]_{\rm int} = c^\dagger_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda}. \tag{8.2.4}$$

Since $[f_{\mathbf{k}\lambda}]_{\rm int}$ commutes with H_0 the Heisenberg operator is given as

$$f_{\mathbf{k}\lambda}(t) = e^{iHt}[f_{\mathbf{k}\lambda}]_{\rm int} e^{-iHt} \tag{8.2.5}$$

and thus its time derivative is determined by the Heisenberg equation of motion

$$\dot{f}_{\mathbf{k}\lambda}(t) = i[H, f_{\mathbf{k}\lambda}(t)]. \tag{8.2.6}$$

The commutator can be most easily computed in the interaction picture, where

$$\dot{f}_{\mathbf{k}\lambda}(t) = ie^{iHt}e^{-iH_0t}[U_{\rm int}, [f_{\mathbf{k}\lambda}]_{\rm int}]e^{iH_0t}e^{-iHt}. \tag{8.2.7}$$

Since we need the time derivative only at leading order in U, we can approximate

$$\dot{f}_{\mathbf{k}\lambda}(t) = i[U_{\rm int}, [f_{\mathbf{k}\lambda}]_{\rm int}] + \mathscr{O}(U^2). \tag{8.2.8}$$

Then we use (8.2.2) and (8.2.3) which yields

$$[[\Phi]_{\rm int}, [f_{\mathbf{k}\lambda}]_{\rm int}] = \frac{e^{-ikx}}{\sqrt{2E_{\mathbf{k}}V}} u_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda}, \tag{8.2.9}$$

and thus in combination with (8.2.1), the time derivative at leading order in U reads

$$\dot{f}_{\mathbf{k}\lambda}(t) = \frac{i}{\sqrt{2E_kV}} \int d^3x \left[\overline{J}(x) e^{-ikx} u_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda} - \text{ H.c.} \right]. \tag{8.2.10}$$

Let us now use the formula (8.1.6) to compute the equilibration rate. In contrast to the washout rate in chapter 5 it turns out to be useful here to compute the spectral function

$$\rho_{\dot{f}_{\mathbf{k}\lambda}\dot{f}_{\mathbf{k}'\lambda'}}(\omega) = \int dt\; e^{i\omega t} \left\langle \left[\dot{f}_{\mathbf{k}\lambda}(t), \dot{f}_{\mathbf{k}'\lambda'}(0) \right] \right\rangle, \tag{8.2.11}$$

directly instead of obtaining it as the imaginary part of the retarded correlator. Since we only need the leading order in U, the average in (8.2.11) can be computed in an ensemble with free fields Φ. Thus after plugging (8.2.10) into (8.2.11) we may use

$$\langle c^\dagger_{\mathbf{k}\lambda} c_{\mathbf{k}'\lambda'}\rangle = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'} f_\sigma(E_\mathbf{k}), \tag{8.2.12}$$

$$\langle c_{\mathbf{k}\lambda} c^\dagger_{\mathbf{k}'\lambda'}\rangle = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}(1+\sigma f_\sigma(E_\mathbf{k})), \tag{8.2.13}$$

where f_σ is equal to the Bose-Einstein or Fermi-Dirac distribution for $\sigma = 1$ and $\sigma = -1$ receptively. The spectral function can then be expressed in terms of the Wightman functions in (2.1.1) with $A = \overline{u}_{\mathbf{k}\lambda} J$ and $B = A^\dagger$ as

$$\rho_{\dot f_{\mathbf{k}\lambda}\dot f_{\mathbf{k}'\lambda'}}(\omega) = \frac{\delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}}{2E_\mathbf{k}}\Big[f_\sigma(E_\mathbf{k})\Delta^{>}_{\overline{u}J\,\overline{J}u}(E_\mathbf{k}+\omega,\mathbf{k}) \tag{8.2.14}$$

$$-\sigma(1+\sigma f_\sigma(E_\mathbf{k}))\Delta^{<}_{\overline{u}J\,\overline{J}u}(E_\mathbf{k}+\omega,\mathbf{k}) - (\omega\to-\omega)\Big]. \tag{8.2.15}$$

Now we use the identities (2.1.5) to express the Wightman functions in terms of the spectral functions of the operators $A = \overline{u}_{\mathbf{k}\lambda} J$ and $B = A^\dagger$. This yields

$$\rho_{\dot f_{\mathbf{k}\lambda}\dot f_{\mathbf{k}'\lambda'}}(\omega) = \frac{\delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}}{2E_\mathbf{k}}\Big[f_\sigma(E_\mathbf{k}) - f_\sigma(E_\mathbf{k}+\omega)\Big]\rho_{\overline{u}J\,\overline{J}u}(E_\mathbf{k}+\omega,\mathbf{k}) - (\omega\to-\omega), \tag{8.2.16}$$

and expanding this to the first order in ω leads to

$$\rho_{\dot f_{\mathbf{k}\lambda}\dot f_{\mathbf{k}'\lambda'}}(\omega) = -\omega\frac{\delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}}{E_\mathbf{k}} f'_\sigma(E_\mathbf{k})\rho_{\overline{u}J\,\overline{J}u}(E_\mathbf{k},\mathbf{k}) + \mathscr{O}(\omega^3). \tag{8.2.17}$$

The susceptibilities $\chi_{\delta f\delta f}$ are determined by the free theory. Using Wick's theorem and the averages (8.2.12) and (8.2.13), we find

$$(\chi_{\delta f\delta f})_{\mathbf{k}\lambda\mathbf{k}'\lambda'} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'} f_\sigma(E_\mathbf{k})(1+\sigma f_\sigma(E_\mathbf{k})) = -T f'_\sigma(E_\mathbf{k}). \tag{8.2.18}$$

Now we can use the Kubo-type relation (8.1.6) and obtain the result

$$(\gamma_{\delta f\delta f})_{\mathbf{k}\lambda\mathbf{k}'\lambda'} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}\Gamma^{\mathrm{eq}}_{\mathbf{k}\lambda}, \tag{8.2.19}$$

with the equilibration rate

$$\Gamma^{\mathrm{eq}}_{\mathbf{k}\lambda} = \frac{1}{2E_\mathbf{k}}\rho_{\overline{u}J\,\overline{J}u}(E_\mathbf{k},\mathbf{k}). \tag{8.2.20}$$

The Φ self-energy at leading order in the interaction U is equal to

$$\Sigma(i\omega_n,\mathbf{k}) = \Delta_{J\overline{J}}(i\omega_n,\mathbf{k}). \tag{8.2.21}$$

Therefore, using the relation (2.1.10) between the spectral function and the two-point correlator, we find

$$\Gamma^{\mathrm{eq}}_{\mathbf{k}\lambda} = \frac{1}{2iE_\mathbf{k}}\overline{u}_{\mathbf{k}\lambda}\mathrm{Disc}\Sigma(E_\mathbf{k},\mathbf{k})u_{\mathbf{k}\lambda}. \tag{8.2.22}$$

This is one of the relations which Weldon proved at leading order [49]. Our derivation of this equations is valid at leading order in the Φ interaction U and to all orders in H_0.

A similar relation for the production rate and the self-energy of the particle Φ, which is valid at leading order in U and to all orders in H_0 has for example been found in [91, 92, 93] and reads

$$\Gamma^{\rm pro}_{\mathbf{k}\lambda} = \frac{\sigma}{2E_{\mathbf{k}}} \Delta^{<}_{\overline{u}J\overline{J}u}(E_{\mathbf{k}}, \mathbf{k}), \tag{8.2.23}$$

For a review see for example [94]. Using the relation (2.1.5) between the Wightman functions and the spectral function one easily finds

$$\Gamma^{\rm pro}_{\mathbf{k}\lambda} = -\Gamma^{\rm eq}_{\mathbf{k}\lambda} f_\sigma(E_{\mathbf{k}}). \tag{8.2.24}$$

This is the other relation which has been found by Weldon [49] at leading order. Our proof of this relation is valid at leading order in U and to all orders in H_0.

8.3 Uncharged particle

The annihilation operators of uncharged particles satisfy $c_{\mathbf{k}\lambda} = d_{\mathbf{k}\lambda}$. Therefore, the fields Φ read in the interaction picture

$$[\Phi(x)]_{\rm int} = \sum_{\mathbf{k},\lambda} \frac{1}{\sqrt{2E_{\mathbf{k}}V}} \left[e^{-ikx}\, u_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda} + e^{ikx}\, v_{\mathbf{k}\lambda} c^\dagger_{\mathbf{k}\lambda} \right]_{k^0 = E_{\mathbf{k}}}. \tag{8.3.1}$$

Let us assume that the interaction can be written as

$$U = \overline{I}\Phi = \overline{\Phi} I, \tag{8.3.2}$$

where I is an elementary or composed operator which does not depend on Φ. This is clearly true if Φ is a real scalar field or a gauge field. We will later justify that this is also true for fermions and show that an interaction like (8.2.1) can be written as (8.3.2), if Φ is uncharged. We use the same definition for the phase-space density as for the charged field (8.2.4) and compute the time derivate according to (8.2.8). For the uncharged field operator (8.3.1) we do then find the commutator

$$[[\Phi]_{\rm int}, [f_{\mathbf{k}\lambda}]_{\rm int}] = \frac{1}{\sqrt{2E_{\mathbf{k}}V}} \left(e^{-ikx} u_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda} - e^{ikx} v_{\mathbf{k}\lambda} c^\dagger_{\mathbf{k}\lambda} \right) \tag{8.3.3}$$

and therefore, the time-derivative at leading order in U reads

$$\dot{f}_{\mathbf{k}\lambda}(t) = \frac{i}{\sqrt{2E_{\mathbf{k}}V}} \int d^3x \overline{I}(x) \left(e^{-ikx} u_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda} - e^{ikx} v_{\mathbf{k}\lambda} c^\dagger_{\mathbf{k}\lambda} \right). \tag{8.3.4}$$

The fact that $\dot{f}_{\mathbf{k}\lambda}(t)$ is real requires $\overline{I}(x) v_{\mathbf{k}\lambda} = (\overline{I}(x) u_{\mathbf{k}\lambda})^\dagger$ and therefore the time derivative of $f_{\mathbf{k}\lambda}$ for an uncharged particle is the same as for the charged particle with J replaced by I,

$$\dot{f}_{\mathbf{k}\lambda}(t) = \frac{i}{\sqrt{2E_{\mathbf{k}}V}} \int d^3x \left(\overline{I}(x) e^{-ikx} u_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda} - \text{H.c} \right). \tag{8.3.5}$$

For this reason, the steps for the computation of the spectral function are the same as for the charged particle and we find

$$\Gamma^{\rm eq}_{\mathbf{k}\lambda} = \frac{1}{2E_{\mathbf{k}}}\rho_{\overline{u}I\,u\overline{I}}(E_{\mathbf{k}},\mathbf{k}). \tag{8.3.6}$$

The self-energy of the uncharged particle is at leading order in U the same as for the charged particle with J replaced by I. Therefore, we find the same relation (8.2.22) between the equilibration rate and the self energy for an uncharged particle. For the production rate one also finds the relation (8.2.23) to the self-energy, with J replaced by I. Therefore, the relation of Weldon (8.2.24) is also true for uncharged particles.

Let us now have a closer look at the operator I in (8.3.2). The fact that Φ is uncharged means that it is invariant under charge conjugation

$$C\Phi C^\dagger \equiv \Phi^c = \Phi \tag{8.3.7}$$

and we assume that fields transform under charge conjugation as

$$\Phi^c \equiv S\overline{\Phi}^\top, \tag{8.3.8}$$

where S is a matrix with appropriate properties. For example, for spin $\frac{1}{2}$ fermions we would have $S = -\mathscr{C}$. In general we see from terms like

$$\overline{\Phi}\Phi = \overline{\Phi}S\overline{\Phi}^\top = \sigma\overline{\Phi}S^\top\overline{\Phi}^\top, \tag{8.3.9}$$

that S has to satisfy the condition

$$S^\top = \sigma S. \tag{8.3.10}$$

Let us now consider the interaction (8.2.1) which we used for the charged particle and write Φ as its charge conjugated. This yields

$$U = \overline{J}S\overline{\Phi}^\top + \overline{\Phi}J. \tag{8.3.11}$$

Then, in combination with (8.3.10) we have

$$U = \overline{\Phi}(J + J^c). \tag{8.3.12}$$

If we define $I = (J + J^c)$, we end up with the interaction (8.3.2) which we used for the uncharged particles. Therefore, if Φ is uncharged, we can write the interaction (8.2.1) as (8.3.2).

Let us now go back to the sterile neutrino Yukawa interaction (1.2.4). Then $\Phi(x)$ is equal to the Majorana neutrinos $N_I(x)$ and the interaction with the SM plasma is given by $J_I = (h_\nu)_{Ii}\tilde{\varphi}\ell_i$. At first we rewrite the terms $\overline{J^c_I}u_{I\mathbf{k}s} = \overline{v}_{I\mathbf{k}s}J_I$ and $\overline{u}_{I\mathbf{k}s}J^c_I = \overline{J}_I v_{I\mathbf{k}s}$ and then we use the fact that the expectation values $\langle J_I(x)J_I(0)\rangle$ and $\langle\overline{J}_I(x)\overline{J}_I(0)\rangle$ vanish due to $B-L$ conservation in the SM. Then the equilibration rate for the Majorana neutrinos can be written as

$$\Gamma^{\rm eq}_{I\mathbf{k}s} = \frac{1}{2E_{I\mathbf{k}}}\left[\rho_{\overline{u}J\,u\overline{J}}(E_{\mathbf{k}},\mathbf{k}) + \rho_{\overline{v}J\,v\overline{J}}(-E_{\mathbf{k}},-\mathbf{k})\right]. \tag{8.3.13}$$

A similar expression has been derived in [50] for the production rate of spin-averaged sterile neutrinos. In order to compare their result with ours, let us compute the spin averaged rate as

$$\Gamma^{\text{eq}}_{I\mathbf{k}} = \frac{1}{2}\sum_{s}\Gamma^{\text{eq}}_{I\mathbf{k}s}. \tag{8.3.14}$$

Using the completeness relations (7.1.11) and (7.1.12) and using the fact that the fields J are left-handed, so that the masses M_I drop out, we end up with

$$\Gamma^{\text{eq}}_{I\mathbf{k}} = \frac{1}{4E_{I\mathbf{k}}}\text{Tr}\left(\not{k}\left[\rho_{J\overline{J}}(E_{\mathbf{k}},\mathbf{k}) + \rho_{J\overline{J}}(-E_{\mathbf{k}},-\mathbf{k})\right]\right). \tag{8.3.15}$$

Combining this result with the relation (8.1.4) we reproduce the the formula for sterile neutrino production rate which has first been derived in [50]. We conclude that the sterile neutrino equilibration rate can be obtained from the known results for the sterile neutrino production rate. In particular this justifies the usage of the NLO production rate of [53] in [39] within the linear kinetic equation (1.2.13) .

Chapter 9

Summary and Outlook

We used the Kubo-type relations (3.2.12) to compute dissipation rates of the linear kinetic equations (1.2.13) and (1.2.14) which describe the evolution of the sterile neutrino phase-space densities and charges which are broken in the presence of the sterile neutrino Yukawa interaction. These Kubo-type formulas relate the dissipation rates to real-time correlation functions and susceptibilities which can be calculated in thermal quantum field theory.

In chapter 4 we computed the susceptibilities of conserved charges in the Standard Model by calculating the grand canonical potential to order g^2 in the Standard Model couplings and to order μ^2 in the particle chemical potentials. The computation of the order g^2 susceptibilities completes the order g^2 calculation of the $\Delta L = 1$ washout rate [41]. We also used them to compute the relation between B and $B-L$ at order g^2. The susceptibilities receive contributions from different momentum scales, which we calculated in effective theories within the framework of dimensional reduction. The NLO corrections are only due to the Higgs and are smaller than 1% for the ratio $\kappa = B/(B-L)$. The NNLO corrections are much larger due to QCD corrections, but still smaller than 5%. At low temperatures close to the electroweak scale where the Higgs mass becomes ultrasoft ($m_3 \sim g^2 T$), we find that the contributions to the Higgs chemical potential are determined by the non-perturbative electroweak magnetic screening scale $g^2 T$, where the loop expansion breaks down. For a reliable calculation of these contributions lattice simulations are needed.

In chapter 5 we computed the $\Delta L = 2$ washout rate in the low temperature limit ($T \ll M_1$), where the sterile neutrino exchange can be approximated as a point-interaction which can be described by a dimension-5 operator. Using the Kubo-type formula we computed the leading order $\Delta L = 2$ washout rate, using full quantum statistics. The next-to-leading order of this rate is of order g in the SM couplings because the next-to-leading order of the susceptibilities is of order g due to the thermal Higgs mass. We find that the spectral function obtains a order $g^2 \ln g$ correction due to Higgs mass resummation which is beyond next-to-leading order. Numerically we find that quantum statistics gives a 24.6% larger rate than classical statistics and the order g corrections from the thermal Higgs mass are smaller than 3%.

In chapter 6 we derived a spectral representation for imaginary time three-point functions at finite temperature. We found inverse relations between the spectral functions and the

retarded correlators, which are very similar to the well-known relations for two-point spectral functions summarized in section 2.1.

In chapter 7 we used the findings of chapter 6 to compute the CP violating asymmetry rate. We derived from the Kubo-relation (3.2.12) a master formula for this rate in the hierarchical limit. This formula relates the asymmetry rate to a single three-point spectral function of Standard Model fields, which can be computed to any order in the Standard Model couplings, except the Yukawa interactions of SM leptons. As a first application we used this formula to compute next-to-leading order corrections to the asymmetry rate at zero temperature. In the non-relativistic limit $T \ll M_1$ this is the leading contribution of the expansion in powers of T/M_1 and $e^{-M_1/T}$. We find the corrections to be smaller than 2%. An interesting future project could be to compute higher orders in T/M_1 with the master formula (7.4.9) and to compare them with the recently published results of [48]. Another important project would be the computation of the leading order in the ultra-relativistic regime $M_1 \sim gT$. For this purpose one possibly has to generalize the master formula to the non hierarchical limit. At high temperatures the three-point spectral function should receive large contributions from infinitely many soft gauge bosons like in the case for the sterile neutrino production rate [51].

Finally, in chapter 8 we showed that the sterile neutrino equilibration rate and the sterile neutrino production rate are related by the simple equation (8.1.4). This relation holds to leading order in the Yukawa interaction (1.2.4) and to all orders in the Standard Model interaction. Therefore, all results of the next-to-leading order analysis of the well studied production rate can be simply related to the equilibration rate. We proved this formula in a rather general framework, considering a charged or uncharged particle Φ, which can be a fermion or a boson with any spin and which is weakly and linearly coupled to a plasma. It would be interesting to study if a similar relation also holds for the production rate of the lepton asymmetry. A first step in this direction has been made in [95] where a formula for the lepton asymmetry with $f_{I\mathbf{k}} = 0$ has been derived. In principle it should be possible to use their approach to find a relation between the asymmetry rate and three-point spectral functions similar to our master formula (7.4.9).

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Appendix A

Washout rate: Computation of the phase-space integrals

The calculations and results of this appendix are planned to be published in [2].

A.1 Definitions

In this appendix we calculate the integrals I_{12}^{1100} and I_{13}^{1010} belonging to the integral class

$$I_{ij}^{\sigma_1\sigma_2\sigma_3\sigma_4} = \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 (2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_3 - k_4) \times (k_i \cdot k_j) f_{\sigma_1}^{eq}(E_1) f_{\sigma_2}^{eq}(E_2)(1 + \sigma_3 f_{\sigma_3}^{eq}(E_3))(1 + \sigma_4 f_{\sigma_4}^{eq}(E_4))\Big|_{k_i^0 = E_i}, \tag{A.1.1}$$

where σ_i is 1 for bosons and -1 fermions, $f_1{=}f_{\rm B}$ and $f_{-1}{=}f_{\rm F}$, $d\Pi_i = d^3k_i/(2E_i)$, $E_i = \sqrt{m_i^2 + \mathbf{k}_i^2}$ with $m_i \in \{0, m_3\}$ and m_3 is defined in (4.4.5). Thereby we expand the integrals to the next-to-leading order in m_3. For the calculation we will use the relation

$$f_{\sigma_i}(E_i) f_{\sigma_j}(E_j) = f_{\sigma_{ij}}(E_i + E_j)\left(1 + \sigma_i f_{\sigma_i}(E_i) + \sigma_j f_{\sigma_j}(E_j)\right), \tag{A.1.2}$$

where $\sigma_{ij} = \sigma_i\sigma_j$. In addition we will use in the following the notation

$$f_{\rm F}(x) = \frac{1}{\exp(x/T) + 1}, \qquad f_{\rm B}(x) = \frac{1}{\exp(x/T) - 1} \tag{A.1.3}$$

$$\tilde{f}_{\rm F}(x) = \frac{1}{\exp(x) + 1}, \qquad \tilde{f}_{\rm B}(x) = \frac{1}{\exp(x) - 1}. \tag{A.1.4}$$

A.2 Calculation of I_{12}^{1100}

At first we bring the integral I_{12}^{1100} to a simpler form using (A.1.2) for $f_{\rm B}(E_3)$ and $f_{\rm B}(E_4)$ and find

$$I_{12}^{1100}(m_3) = \int d\Pi_1 d\Pi_2 (k_1 \cdot k_2) f_{\rm F}(E_1) f_{\rm F}(E_2)(1 + f_{\rm B}(E_1 + E_2)) F_{m_3}(k_1 + k_2), \tag{A.2.1}$$

where

$$F_{m_3}(k) = \int d\Pi_3 d\Pi_4 (2\pi)^4 \delta^{(4)}(k - k_3 - k_4)(1 + f_{\rm B}(E_3) + f_{\rm B}(E_4)), \tag{A.2.2}$$

with $k = (E_1 + E_2, \mathbf{k}_1 + \mathbf{k}_2)$, $E_{1,2} = |\mathbf{k}_{1,2}|$ and $E_{3,4} = \sqrt{\mathbf{k}_{3,4}^2 + m_3^2}$. We calculate (A.2.2) analytically with finite Higgs mass. For that purpose we choose $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ as z-axis, perform the k_4 integral and the integrals over the angles and end up with

$$\begin{aligned} F_{m_3}(k) \equiv \frac{1}{8\pi|\mathbf{k}|} \int_{m_3}^{\infty} dE_3 (1 + 2 f_{\rm B}(E_3)) \theta(k_0 - E_3) \theta(2|\mathbf{k}|\sqrt{E_3^2 - m_3^2} + k^2 - 2k_0 E_3) \\ \times \theta(2|\mathbf{k}|\sqrt{E_3^2 - m_3^2} - k^2 + 2k_0 E_3). \end{aligned} \tag{A.2.3}$$

The remaining integral over E_3 can be solved exactly and the solution reads

$$F_{m_3}(k) = -\frac{1}{8\pi|\mathbf{k}|} \left\{ |\mathbf{k}| \left(\frac{k^2 - 4m_3^2}{k^2} \right)^{1/2} + 2T \ln \left(\frac{f_{\rm B}\left(E_3^+\right)}{f_{\rm B}\left(E_3^-\right)} \right) \right\} \theta(k^2 - 4m_3^2), \tag{A.2.4}$$

where $E_3^{\pm} = k_0/2 \pm |\mathbf{k}| \left(1 - 4m_3^2/k^2\right)^{1/2}/2$ are the zeros of the $\theta-$functions. For $m_3 = 0$ we find

$$F_0(k) = -\frac{1}{8\pi|\mathbf{k}|} \left\{ |\mathbf{k}| + 2T \ln \left(\frac{f_{\rm B}\left(\frac{k_0 + |\mathbf{k}|}{2}\right)}{f_{\rm B}\left(\frac{k_0 - |\mathbf{k}|}{2}\right)} \right) \right\}. \tag{A.2.5}$$

Substituting $E_1 = (x+y)T/2$, $E_2 = (x-y)T/2$ and $|\mathbf{k}_1 + \mathbf{k}_2| = zT$, the integral $I_{12}^{1100}(m_3 = 0)$ can be written as

$$\begin{aligned} I_{12}^{1100}(m_3 = 0) = -\frac{T^6}{16(2\pi)^5} \int_0^{\infty} dx \int_0^{\infty} dy \int_0^{\infty} dz \left(x^2 - z^2\right) \left(z + 2 \ln \left(\frac{\tilde{f}_{\rm B}\left(\frac{z+x}{2}\right)}{\tilde{f}_{\rm B}(\frac{x-z}{2})} \right) \right) \\ \times \tilde{f}_{\rm F}\left(\frac{x+y}{2}\right) \tilde{f}_{\rm F}\left(\frac{x-y}{2}\right) (1 + \tilde{f}_{\rm B}(x)) \theta(x - z) \theta(z - y). \end{aligned} \tag{A.2.6}$$

The y-integral can be solved exactly and reads

$$\int_0^{\infty} dy \tilde{f}_{\rm F}\left(\frac{x+y}{2}\right) \tilde{f}_{\rm F}\left(\frac{x-y}{2}\right) \theta(z - y) = \left(z + 2 \ln \left(\frac{\tilde{f}_{\rm F}\left(\frac{z+x}{2}\right)}{\tilde{f}_{\rm F}(\frac{x-z}{2})} \right) \right) \tilde{f}_{\rm B}(x). \tag{A.2.7}$$

Thus we are left with the 2-dimensional integral

$$\begin{aligned} I_{12}^{1100}(m_3 = 0) = -\frac{T^6}{16(2\pi)^5} \int_0^{\infty} dx \int_0^{\infty} dz \left(x^2 - z^2\right) \tilde{f}_{\rm B}(x)(1 + \tilde{f}_{\rm B}(x)) \theta(x - z) \\ \times \left(z + 2 \ln \left(\frac{\tilde{f}_{\rm B}\left(\frac{z+x}{2}\right)}{\tilde{f}_{\rm B}(\frac{x-z}{2})} \right) \right) \left(z + 2 \ln \left(\frac{\tilde{f}_{\rm F}\left(\frac{z+x}{2}\right)}{\tilde{f}_{\rm F}(\frac{x-z}{2})} \right) \right), \end{aligned} \tag{A.2.8}$$

which can be further simplified by doing the substitution $(x+z)/2 = s$ and $(x-z)/2 = t$, leading to

$$I_{12}^{1100}(m_3=0) = \frac{T^6}{4(2\pi)^5}\int_0^\infty ds \int_0^\infty dt \cdot s \cdot t \tilde{f}_{\rm B}(s+t)(1+\tilde{f}_{\rm B}(s+t)) \times \left(s-t+2\ln\left(\frac{\tilde{f}_{\rm B}(s)}{\tilde{f}_{\rm B}(t)}\right)\right)\left(s-t+2\ln\left(\frac{\tilde{f}_{\rm F}(s)}{\tilde{f}_{\rm F}(t)}\right)\right). \tag{A.2.9}$$

The integral can now conveniently solved numerically and the solution is

$$I_{12}^{1100}(m_3=0) = 1.14\times 10^{-4} T^6. \tag{A.2.10}$$

The next step is to find the leading m_3 order of $I_{12}^{1100}(m_3)$. This can be obtained by restricting to the infrared sensitive part of (A.2.2), denoted by $F_{m_3}^{IR}$. It corresponds to the small arguments in the Bose distribution, where $f_{\rm B}(E) \approx T/E$. In this region we find

$$F_{m_3}^{IR}(k) = -\frac{T}{8\pi|\mathbf{k}|} 2\ln\left(\frac{E_3^-}{E_3^+}\right)\theta(k^2-4m_3^2). \tag{A.2.11}$$

It turns out that the mass derivative of $F_{m_3}^{IR}$ can be more easily integrated. Therefore, taking the derivative and again restricting to the infrared sensitive part, yields

$$-\frac{1}{2}\frac{d}{dm_3^2}F^{IR}(k) = \frac{T}{4\pi}\frac{k_0}{k^4+4|\mathbf{k}|^2m_3^2}\theta(k^2-4m_3^2). \tag{A.2.12}$$

We substitute $E_1 = (x+y)T/2$, $E_2 = (x-y)T/2$ and $z_{12} = \mathbf{k}_1\cdot\mathbf{k}_2/(E_1E_2)$ and solve the integral over z_{12} which yields

$$\frac{Tk_0}{4\pi}\int_{-1}^1 dz_{12}\frac{k^2\theta(k^2-4m_3^2)}{k^4+4|\mathbf{k}|^2m_3^2} = -\frac{1}{4\pi}\frac{x}{x^2-y^2}\left(\ln\left(\frac{x^2}{x^2-y^2}\right)+\ln(\frac{m_3^2}{T^2})+\mathscr{O}(m_3)\right). \tag{A.2.13}$$

The remaining 2-dimensional integral is

$$-\frac{1}{2}\frac{d}{dm_3^2}I_{ll\to\varphi\varphi}^{IR} = -\frac{T^6}{8}\left(\frac{1}{2\pi}\right)^5\left(\ln(\frac{m_3^2}{T^2})+\mathscr{O}(m_3^0)\right)\int_0^\infty dx\tilde{f}_{\rm F}\left(\frac{x+y}{2}\right) \times\int_0^\infty dy\tilde{f}_{\rm F}\left(\frac{x-y}{2}\right)x(1+\tilde{f}_{\rm B}(x)), \tag{A.2.14}$$

and can be solved exactly. The result is

$$\begin{aligned}-\frac{1}{2}\frac{d}{dm_3^2}I_{ll\to\varphi\varphi}^{IR} =& \frac{1}{8(2\pi)^5}\left(\frac{\pi^2}{12}+\ln(2)^2\right)\ln(\frac{m_3^2}{T^2})T^4+\mathscr{O}(m_3^0)\\ =& 1.66\times 10^{-5}\ln(\frac{m_3^2}{T^2})T^4+\mathscr{O}(m_3^0).\end{aligned} \tag{A.2.15}$$

Integrating over m_3 and using the condition that $I_{12}^{1100}(m_3=0)$ is equal to our result (A.2.10), we find

$$I_{12}^{1100}(m_3) = \left(1.14+0.333\frac{m_3^2}{T^2}\ln(\frac{m_3^2}{T^2})\right)\times 10^{-4}T^6+\mathscr{O}(m_3^2). \tag{A.2.16}$$

A.3 Calculation of I_{13}^{1010}

At first we rename $\mathbf{k}_2 \leftrightarrow \mathbf{k}_3$ and then use (A.1.2) for $f_{\mathrm{B}}(E_3)$ and $f_{\mathrm{B}}(E_4)$. Then we can write

$$I_{13}^{1010} = \int d\Pi_1 d\Pi_2 (k_1 \cdot k_2) f_{\mathrm{F}}(E_1)(1 - f_{\mathrm{F}}(E_2)) f_{\mathrm{B}}(E_2 + E_1) G_{m_3}(k_1 - k_2), \tag{A.3.1}$$

where

$$G_{m_3}(k) = \int d\Pi_3 d\Pi_4 (2\pi)^4 \delta^{(4)}(k - k_3 - k_4) \left(f_{\mathrm{B}}(E_3) - f_{\mathrm{B}}(E_4)\right). \tag{A.3.2}$$

In this case $k_0 = E_1 - E_2$ and $\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2$. Again we first perform the k_4 integral and the integrals over the angles and obtain

$$\begin{aligned} G_{m_3}(k) = \frac{1}{8\pi|\mathbf{k}|} \int_{m_3}^{\infty} dE_3 f_{\mathrm{B}}(E_3) \Big\{ & \theta(2|\mathbf{k}|\sqrt{E_3^2 - m_3^2} + k^2 - 2k_0 E_3) \\ & \theta(2|\mathbf{k}|\sqrt{E_3^2 - m_3^2} - k^2 + 2k_0 E_3)\theta(E_3 + k_0) \\ & - \theta(2|\mathbf{k}|\sqrt{E_3^2 - m_3^2} + k^2 + 2k_0 E_3) \\ & \theta(2|\mathbf{k}|\sqrt{E_3^2 - m_3^2} - k^2 - 2k_0 E_3)\theta(E_3 - k_0) \Big\}. \end{aligned} \tag{A.3.3}$$

The remaining integral over E_3 has the simple solution

$$G_{m_3}(k) = \frac{1}{8\pi|\mathbf{k}|} \left(\frac{k_0}{T} + \ln\left(\frac{f_{\mathrm{B}}(E_3^+)}{f_{\mathrm{B}}(-E_3^-)} \right) \right). \tag{A.3.4}$$

Again, we first consider the massless case and we substitute $E_1 = xT$, $E_1 - E_2 = yT$ and $|\mathbf{k}_1 - \mathbf{k}_2| = zT$. Then we obtain

$$\begin{aligned} I_{13}^{1010}(m_3 = 0) = & - \frac{T^6}{8(2\pi)^5} \int_0^{\infty} dx \int_0^{\infty} dy \int_0^{\infty} dz \tilde{f}_{\mathrm{F}}(x) \tilde{f}_{\mathrm{B}}(-y)(1 - \tilde{f}_{\mathrm{F}}(x - y))(y^2 - z^2) \\ & \times \left(y + \ln\left(\frac{\tilde{f}_{\mathrm{B}}(\frac{y-z}{2})}{\tilde{f}_{\mathrm{B}}(\frac{z-y}{2})} \right) \right) \theta(2x - y - z)\theta(z - y). \end{aligned} \tag{A.3.5}$$

The x-integral can be solved exactly and yields

$$\int_0^{\infty} dx (1 - \tilde{f}_{\mathrm{F}}(x - y)) \tilde{f}_{\mathrm{F}}(x) \theta(2x - y - z) = y + \ln\left(\frac{\tilde{f}_{\mathrm{F}}(\frac{y+z}{2})}{\tilde{f}_{\mathrm{F}}(\frac{y-z}{2})} \right). \tag{A.3.6}$$

Finally we end up with the 2-dimensional integral

$$\begin{aligned} I_{13}^{1010}(m_3 = 0) = & \frac{T^6}{8(2\pi)^5} \int_0^{\infty} dy \int_0^{\infty} dz \tilde{f}_{\mathrm{B}}(y)(1 + \tilde{f}_{\mathrm{B}}(y)) \theta(z - y)(y^2 - z^2) \\ & \times \left(y + \ln\left(\frac{\tilde{f}_{\mathrm{B}}(\frac{y-z}{2})}{\tilde{f}_{\mathrm{B}}(\frac{z-y}{2})} \right) \right) \left(y + \ln\left(\frac{\tilde{f}_{\mathrm{F}}(\frac{y-z}{2})}{\tilde{f}_{\mathrm{F}}(\frac{z-y}{2})} \right) \right), \end{aligned} \tag{A.3.7}$$

which can be calculated numerically. The result is

$$I_{13}^{1010}(m_3 = 0) = 5.91 \times 10^{-5} T^6. \tag{A.3.8}$$

Now for the small mass behavior we consider again the mass derivative of the infrared sensitive part and obtain

$$-\frac{1}{2}\frac{d}{dm_3^2} G_{m_3}^{IR}(k) = -\frac{1}{4\pi}\frac{Tk_0}{k^4 + 4|\mathbf{k}|^2 m_3^2}. \tag{A.3.9}$$

Doing the substitutions $E_1 = xT$, $E_1 - E_2 = yT$ and $z_{12} = \mathbf{k}_1 \cdot \mathbf{k}_2/(E_1 E_2)$ and integrating over z_{12} yields

$$\frac{Tk_0}{4\pi}\int_{-1}^{1} dz_{12} \frac{k^2}{k^4 + 4|\mathbf{k}|^2 m_3^2} = \frac{y}{16\pi x(x-y)}\left(\ln\left(\frac{y^2}{4x^2(x-y)^2}\right) + \ln\left(\frac{m_3^2}{T^2}\right)\right) + \mathscr{O}(m_3). \tag{A.3.10}$$

Then we get the 2-dimensional integral

$$\begin{aligned} -\frac{1}{2}\frac{d}{dm_3^2} I_{13}^{1010}(m_3) =& \frac{T^6}{8}\left(\frac{1}{2\pi}\right)^5 \left(\ln(\frac{m_3^2}{T^2}) + \mathscr{O}(m_3^0)\right) \\ & \times \int_0^\infty dx \int_0^\infty dy \left(1 - \tilde{f}_{\mathrm{F}}(x)\right) \tilde{f}_{\mathrm{F}}(x-y) y \tilde{f}_{\mathrm{B}}(y) \theta(x-y), \end{aligned} \tag{A.3.11}$$

whose solution reads

$$\begin{aligned} -\frac{1}{2}\frac{d}{dm_3^2} I_{13}^{1010}(m_3) =& \frac{T^4}{64(2\pi)^5}\left(\pi^2 - 4\ln(2)^2\right)\left(\ln(\frac{m_3^2}{T^2}) + \mathscr{O}(m_3^0)\right) \\ =& 1.27 \times 10^{-5} T^4 \left(\ln(\frac{m_3^2}{T^2}) + \mathscr{O}(m_3^0)\right), \end{aligned} \tag{A.3.12}$$

and therefore

$$I_{13}^{1010}(m_3) = T^6\left(5.91 + 2.54\ln(\frac{m_3^2}{T^2})\right) \times 10^{-5} + \mathscr{O}(m_3^0). \tag{A.3.13}$$

Appendix B

Asymmetry rate: Computation of the spectral functions

In this appendix we present the computation of the spectral functions of the three-point correlator (7.6.1), (7.6.2), (7.6.3) and (7.6.4). The calculations and results of this appendix are planned to be published in [3].

B.1 Dirac traces

At first we use the Symbolic Manipulation System FORM [96] to generate the diagrams with Wick's theorem, compute traces in the gauge group space and use the properties (1.1.14) of the matrix $\mathscr{C}$. Since the resulting loop integrals are UV divergent, we use dimensional regularization. We do not yet perform the traces over Dirac matrices because in dimensional regularization in calculations beyond leading order we have to be careful with γ_5, appearing in the chiral projectors $P_{L/R}$.

Then we find the following expressions[1] for the NLO diagrams[2]

$$= 2d(r)(d(r)+1)(y_{\tilde{\varphi}}^2 g_1^2 + C_2(r) g_2^2)\mathrm{Tr}\left(\gamma_{\mu_1} P_\mathrm{L} \gamma_{\mu_2} P_\mathrm{R}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1} p_3^{\mu_2} (p_1+p_2-2k_1)^2}{p_1^2 p_3^2 (p_1-k_1)^4 (p_2-k_1)^2 (p_3+k_2)^2 (p_1-p_2)^2}, \tag{B.1.1}$$

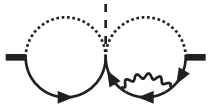
$$= 2d(r)(d(r)+1)(y_{\tilde{\varphi}}^2 g_1^2 + C_2(r) g_2^2)\mathrm{Tr}\left(\gamma_{\mu_1} P_\mathrm{L} \gamma_{\mu_2} P_\mathrm{R}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} (p_3+p_2+2k_2)^2}{p_1^2 p_2^2 (p_1-k_1)^2 (p_2+k_2)^4 (p_3+k_2)^2 (p_3-p_2)^2}, \tag{B.1.2}$$

$$= 2d(r)(d(r)+1)(y_\ell^2 g_1^2 + C_2(r) g_2^2)$$

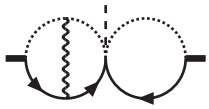
[1] We use $\int_p = \int \frac{d^d p}{(2\pi)^d}$ with $d^d p = d\bar{p}_0 d^{d-1} p$ and $p_0 = i\bar{p}_0$.

[2] Due to lack of space we only show the diagrams in Feynman gauge $\xi_1 = \xi_2 = 1$

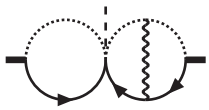
$$\times \mathrm{Tr}\left(\gamma_{\mu_1} P_{\mathrm{L}} \gamma_{\mu_5} P_{\mathrm{R}} \gamma_{\mu_2} P_{\mathrm{L}} \gamma^{\mu_5} P_{\mathrm{R}} \gamma_{\mu_3} P_{\mathrm{L}} \gamma_{\mu_4} P_{\mathrm{R}}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} p_1^{\mu_3} p_3^{\mu_4}}{p_1^4 p_2^2 p_3^2 (p_1-k_1)^2 (p_3+k_2)^2 (p_1-p_2)^2}, \tag{B.1.3}$$



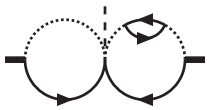
$$= 2d(r)(d(r)+1)(y_\ell^2 g_1^2 + C_2(r) g_2^2)$$
$$\times \mathrm{Tr}\left(\gamma_{\mu_1} P_{\mathrm{L}} \gamma_{\mu_2} P_{\mathrm{R}} \gamma^{\mu_5} P_{\mathrm{L}} \gamma_{\mu_3} P_{\mathrm{R}} \gamma_{\mu_5} P_{\mathrm{L}} \gamma_{\mu_4} P_{\mathrm{R}}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_2^{\mu_4}}{p_1^2 p_2^4 p_3^2 (p_1-k_1)^2 (p_2+k_2)^2 (p_2-p_3)^2}, \tag{B.1.4}$$



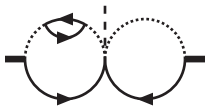
$$= 2d(r)(d(r)+1)(y_{\tilde{\varphi}} y_\ell g_1^2 + C_2(r) g_2^2) \mathrm{Tr}\left(\gamma_{\mu_1} P_{\mathrm{L}} \gamma_{\mu_5} P_{\mathrm{R}} \gamma_{\mu_2} P_{\mathrm{L}} \gamma_{\mu_3} P_{\mathrm{R}}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} (p_1+p_2-2k_1)^{\mu_5}}{p_1^2 p_2^2 p_3^2 (p_1-k_1)^2 (p_2-k_1)^2 (p_3+k_2)^2 (p_1-p_2)^2}, \tag{B.1.5}$$



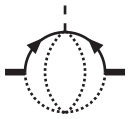
$$= 2d(r)(d(r)+1)(y_{\tilde{\varphi}} y_\ell g_1^2 + C_2(r) g_2^2) \mathrm{Tr}\left(\gamma_{\mu_1} P_{\mathrm{L}} \gamma_{\mu_2} P_{\mathrm{R}} \gamma_{\mu_5} P_{\mathrm{L}} \gamma_{\mu_3} P_{\mathrm{R}}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} (p_2+p_3+2k_2)^{\mu_5}}{p_1^2 p_2^2 p_3^2 (p_1-k_1)^2 (p_2+k_2)^2 (p_3+k_2)^2 (p_2-p_3)^2}, \tag{B.1.6}$$



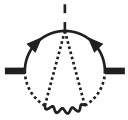
$$= -2d(r)(d(r)+1) N_c |h_t|^2 \mathrm{Tr}\left(\gamma_{\mu_1} P_{\mathrm{L}} \gamma_{\mu_2} P_{\mathrm{R}}\right) \mathrm{Tr}\left(\gamma_{\mu_3} P_{\mathrm{L}} \gamma_{\mu_4} P_{\mathrm{R}}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} (p_3+k_2)^{\mu_3} (p_2-p_3)^{\mu_4}}{p_1^2 p_2^2 (p_1-k_1)^2 (p_2+k_2)^4 (p_3+k_2)^2 (p_3-p_2)^2}, \tag{B.1.7}$$



$$= -2d(r)(d(r)+1) N_c |h_t|^2 \mathrm{Tr}\left(\gamma_{\mu_1} P_{\mathrm{L}} \gamma_{\mu_2} P_{\mathrm{R}}\right) \mathrm{Tr}\left(\gamma_{\mu_3} P_{\mathrm{L}} \gamma_{\mu_4} P_{\mathrm{R}}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1} p_3^{\mu_2} (k_1-p_2)^{\mu_3} (p_2-p_1)^{\mu_4}}{p_1^2 p_3^2 (p_1-k_1)^4 (p_2-k_1)^2 (p_3+k_2)^2 (p_1-p_2)^2}, \tag{B.1.8}$$

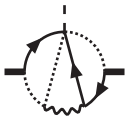


$$= 4d(r)(d(r)+1)\lambda \mathrm{Tr}\left(\gamma_{\mu_1} P_{\mathrm{L}} \gamma_{\mu_2} P_{\mathrm{R}}\right)$$
$$\times \int_{p_1 p_2 p_3} \frac{(k_1-p_1)^{\mu_1} (p_2+k_2)^{\mu_2}}{p_1^2 p_2^2 (p_1-k_1)^2 (p_2+k_2)^2 (p_1-p_3)^2 (p_2-p_3)^2}, \tag{B.1.9}$$

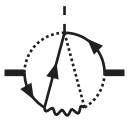


$$= 2d(r)(y_{\tilde{\varphi}}^2 (d(r)+1) g_1^2 + C_2(r) g_2^2) \mathrm{Tr}\left(\gamma_{\mu_1} P_{\mathrm{L}} \gamma_{\mu_2} P_{\mathrm{R}}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{(k_1-p_1)^{\mu_1} (p_2+k_2)^{\mu_2} (p_3-2p_2)^{\mu_5} (p_3-2p_1)_{\mu_5}}{p_1^2 p_2^2 p_3^2 (p_1-k_1)^2 (p_2+k_2)^2 (p_1-p_3)^2 (p_2-p_3)^2}, \tag{B.1.10}$$

$$= -2d(r)(y_\ell^2(d(r)+1)g_1^2 + C_2(r)g_2^2)$$
$$\times \mathrm{Tr}\left(\gamma_{\mu_1} P_\mathrm{L}\gamma_{\mu_5} P_\mathrm{R}\gamma_{\mu_2} P_\mathrm{L}\gamma_{\mu_3} P_\mathrm{R}\gamma^{\mu_5} P_\mathrm{L}\gamma_{\mu_4} P_\mathrm{R}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1}(p_1-p_3)^{\mu_2}(p_2-p_3)^{\mu_3}p_2^{\mu_4}}{p_1^2p_2^2p_3^2(p_1-k_1)^2(p_2+k_2)^2(p_1-p_3)^2(p_2-p_3)^2}, \tag{B.1.11}$$



$$= 2d(r)(y_{\tilde\varphi}y_\ell(d(r)+1)g_1^2 + C_2(r)g_2^2)\mathrm{Tr}\left(\gamma_{\mu_1} P_\mathrm{L}\gamma_{\mu_2} P_\mathrm{R}\gamma_{\mu_5} P_\mathrm{L}\gamma_{\mu_3} P_\mathrm{R}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{(k_1-p_1)^{\mu_1}(p_3-p_1)^{\mu_2}p_2^{\mu_3}(p_3-2p_1)^{\mu_5}}{p_1^2p_2^2p_3^2(p_1-k_1)^2(p_2+k_2)^2(p_1-p_3)^2(p_2-p_3)^2}, \tag{B.1.12}$$



$$= 2d(r)(y_{\tilde\varphi}y_\ell(d(r)+1)g_1^2 + C_2(r)g_2^2)\mathrm{Tr}\left(\gamma_{\mu_1} P_\mathrm{L}\gamma_{\mu_5} P_\mathrm{R}\gamma_{\mu_2} P_\mathrm{L}\gamma_{\mu_3} P_\mathrm{R}\right)$$
$$\times \int_{p_1,p_2,p_3} \frac{p_1^{\mu_1}(p_1-p_3)^{\mu_2}(p_2+q)^{\mu_3}(p_3-2p_2)^{\mu_5}}{p_1^2p_2^2p_3^2(p_1-k_1)^2(p_2+k_2)^2(p_1-p_3)^2(p_2-p_3)^2}. \tag{B.1.13}$$

Here we use $y_\varphi = -y_{\tilde\varphi} = -y_\ell = 1/2$, $d(r) = 2$, $C_2(r) = 3/4$ and $N_c = 3$.

For the treatment of γ_5 we proceed similar to [53] and use the definition

$$\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{B.1.14}$$

of 't Hooft and Veltman [97] and apply the prescription of [98] which allows a naively commuting γ_5 with $\gamma_5^2 = 1$ in traces with more than one γ_5, except in closed fermion loops. Then only traces with one or no γ_5 remain. Let us for example consider the trace in the diagram (B.1.9). Using the prescription of [98] we can write

$$\mathrm{Tr}\left(\gamma^\mu \mathrm{P_L}\gamma^\nu \mathrm{P_R}\right) = \mathrm{Tr}\left(\gamma^\mu\gamma^\nu - \gamma^\mu[\gamma_5,\gamma^\nu]\right). \tag{B.1.15}$$

Furthermore, with the definition of γ_5 through (B.1.14) we obtain (c.f. [53])

$$[\gamma_5,\gamma^\nu] = \frac{i}{3!}\varepsilon^{\nu\mu\rho\sigma}(\gamma_\mu\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho\gamma_\mu). \tag{B.1.16}$$

Then the γ_5 term in the trace drops out due to the total anti-symmetry of the ε tensor and the cyclicity of the trace. We can do exactly the same in the diagram (B.1.10).

The factorisable diagrams (B.1.1)-(B.1.8) are very similar to the diagrams which contribute to the sterile neutrino productions rate in the non-relativistic regime [53]. In fact, the diagrams are proportional to the NLO sterile neutrino self-energy for which it has been shown in [53] that the γ_5 contributions cancel.

For the diagrams (B.1.11)-(B.1.13) such a cancellation does not happen and we need a further argument. First of all we use a naively anti-commuting γ_5 and $\gamma_5^2 = 1$ in traces with more than one γ_5. Then only traces with no or one γ_5 remain. Using the definition (B.1.14) the traces containing one γ_5 can be written as

$$\mathrm{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_5\right) = \frac{i}{4!}\epsilon^{\nu_1\nu_2\nu_3\nu_4}\mathrm{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}\gamma_{\nu_4}\right). \tag{B.1.17}$$

Due to the total anti-symmetry of the ε tensor we can drop all combinations of

$$\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4}$$

which are anti-symmetric in the indices $\mu_1, ..., \mu_4$. Therefore, we have

$$\mathrm{Tr}\,(\not{k}_1\not{k}_2\not{k}_3\not{k}_4\gamma_5) = 4i\epsilon_{\mu\nu\rho\sigma}k_1^{[\mu}k_2^{\nu}k_3^{\rho}k_4^{\sigma]} \tag{B.1.18}$$

and consequently, in the diagrams (B.1.11) - (B.1.13) the tensor integrals which come from traces with one γ_5 have the generic form

$$I_T^{\mu\nu\rho\sigma}(k_1,k_2) = \int_{p_1,p_2,p_3} \frac{T^{\mu\nu\rho\sigma}(\{p_i\},k_1,k_2)}{p_1^2p_2^2p_3^2(p_1-k_1)^2(p_2+k_2)^2(p_1-p_3)^2(p_2-p_3)^2}, \tag{B.1.19}$$

where $T^{\mu_1\mu_2\mu_3\mu_4}(\{p_i\},k_1,k_2)$ is one of the total anti-symmetric rank-4-tensors of the set

$$\{p_1^{[\mu}p_2^{\nu}p_3^{\rho}k_1^{\sigma]}, p_1^{[\mu}p_2^{\nu}p_3^{\rho}k_2^{\sigma]}, p_i^{[\mu}p_j^{\nu}k_1^{\rho}k_2^{\sigma]}\}. \tag{B.1.20}$$

Lorentz symmetry guarantees that a sub-integral such as

$$J^{\mu}(p_1,p_2) = \int_{p_3} \frac{p_3^{\mu}}{p_3^2(p_3-p_1)^2(p_3-p_2)^2} \tag{B.1.21}$$

can be written in terms of scalar functions f_1 and f_2 as

$$J^{\mu}(p_1,p_2) = p_1^{\mu}f_1(p_1,p_2) + p_2^{\mu}f_2(p_1,p_2). \tag{B.1.22}$$

Thus all tensor integrals can be expressed in terms of integrals containing the tensor

$$T^{\mu\nu\rho\sigma}(p_1,p_2,k_1,k_2) = p_1^{[\mu}p_2^{\nu}k_1^{\rho}k_2^{\sigma]} \tag{B.1.23}$$

Lorentz symmetry guarantees that we can compute the three-point correlator for $k_1 = (k_1^0, \mathbf{0})$. Since the spectral function function in (7.4.9) is evaluated at $k_1 = -k_2 = -k$ we can also set $k_2 = (k_2^0, \mathbf{0})$. Since $T^{\mu\nu 00}(p_1,p_2,k_1,k_2) = 0$, we see that traces with one γ_5 do not contribute. Note that this argument does not hold at finite temperature since Lorentz symmetry is broken there. For the finite temperature computation one would need to compute the tensor sum-integrals in terms with γ_5 explicitly.

B.2 Reduction to master integrals and ε-expansion

We compute the diagrams (B.1.1) - (B.1.13) with arbitrary gauge parameters ξ_1 and ξ_2. We perform the Dirac traces in FORM [96] which yields scalar products in the numerators of the Feynman integrals, which can be expressed in terms of inverse scalar propagators through the relations

$$p_i \cdot p_j = \frac{1}{2}\left(p_i^2 + p_j^2 - (p_i - p_j)^2\right), \tag{B.2.1}$$

$$p_i \cdot k_1 = \frac{1}{2}\left(p_i^2 + k_1^2 - (p_i - k_1)^2\right), \tag{B.2.2}$$

$$p_i \cdot k_2 = \frac{1}{2}\left((p_i + k_2)^2 - p_i^2 - k_2^2\right). \tag{B.2.3}$$

Then all three-point Feynman-integrals appearing in the computation of the NLO spectral functions have the generic form

$$
\begin{aligned}
I_{a_1\ldots a_{12}}(k_1,k_2) = \int \frac{d^dp_1}{(2\pi)^d}\frac{d^dp_2}{(2\pi)^d}\frac{d^dp_3}{(2\pi)^d}\frac{1}{p_1^{2a_1}p_2^{2a_2}p_3^{2a_3}} \\
\times \frac{1}{(p_1-k_1)^{2a_4}(p_2-k_1)^{2a_5}(p_3-k_1)^{2a_6}} \\
\times \frac{1}{(p_1+k_2)^{2a_7}(p_2+k_2)^{2a_8}(p_3+k_2)^{2a_9}} \\
\times \frac{1}{(p_1-p_2)^{2a_{10}}(p_1-p_3)^{2a_{11}}(p_2-p_3)^{2a_{12}}},
\end{aligned}
\tag{B.2.4}
$$

with integers a_i. They can be mapped to a minimal set of master integrals, using the method of integration by parts (IBP) [99]. With help of the program Reduze [100], which uses the Laporta algorithm [101] for IBP, we get the following gauge-parameter independent expressions for the three-point correlators in terms of master integrals:

$$
\begin{aligned}
\Gamma_{g,\mathrm{nfac}}(k_1,k_2) = -2d(r)(y_{\tilde{\varphi}}y_\ell(d(r)+1)g_1^2 + C_2(r)g_2^2) \\
\times\Bigg(\frac{-(d-2)(2d-5)(-20+79d-48d^2+8d^3)}{(d-3)^2(3d-10)(3d-8)k_2^2}I_{001000010110} \\
+\frac{4(d-2)(2d-5)(2d-3)}{(d-4)(3d-8)k_2^2}I_{000001010110} \\
+\frac{(d-2)(2(9-9d+2d^2)(k_1+k_2)^2+(-25+23d-5d^2)k_2^2)}{(d-3)(3d-8)k_2^2}I_{001001010110} \\
+\frac{(42560-78192d+58256d^2-22318d^3+4561d^4-456d^5+16d^6)}{(d-4)^2(d-3)^2(3d-10)(3d-8)k_1^2} \\
\times(2d-5)I_{010001000110} \\
+\frac{-2320+2900d-1168d^2+93d^3+41d^4-7d^5}{(d-4)(d-3)(3d-10)(3d-8)}I_{010001010110} \\
+\frac{2(-60+55d-15d^2+d^3)k_1^2-2(d-2)^2(4d-13)k_1\cdot k_2}{(3d-8)^2}I_{011001010110} \\
+\frac{(d-2)^2}{(d-3)}I_{011001100100} \\
+\frac{8(-60+55d-15d^2+d^3)k_1^2k_2^2}{(3d-10)(3d-8)^2}I_{021001010110} \\
+\frac{-4(d-2)}{d-4}I_{100001010110} \\
+\frac{(d-2)k_1^2}{(d-3)}I_{101001010110} \\
+\frac{(d-2)k_2^2}{(d-3)}I_{110001010110}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(d-2)(2d-3)k_1^2(k_1+k_2)^2}{(d-3)(3d-8)k_2^2}I_{001002010110}\\
&+\frac{(12-5d)(k_1+k_2)^2}{(d-3)(3d-8)}I_{010002010110}\\
&+\frac{(d-2)^2}{(d-3)}I_{101010001100}\Bigg), \qquad \text{(B.2.5)}\\
\Gamma_{g,\mathrm{fac}}(k_1,k_2) =&2d(r)(d(r)+1)(y_{\tilde{\varphi}}^2 g_1^2+C_2(r)g_2^2)\\
&\times\Bigg(\frac{(d-2)(-4-d+d^2)k_2\cdot k_1}{(-4+d)^2k_2^2}I_{011001100100}\\
&+\frac{(d-2)(-4-d+d^2)k_2\cdot k_1}{(-4+d)^2k_1^2}I_{101010001100}\\
&-\frac{(d-2)k_2\cdot k_1}{(d-4)}I_{111011100000}\\
&-\frac{(d-2)k_2\cdot k_1}{(d-4)}I_{111001110000}\Bigg), \qquad \text{(B.2.6)}\\
\Gamma_\lambda(k_1,k_2) =&-4d(r)(d(r)+1)\lambda\\
&\times\Bigg(\frac{(-2+d)(-5+2d)(-20+7d)}{(-3+d)(-10+3d)(-8+3d)k_2^2}I_{001000010110}\\
&+\frac{-4(-2+d)(-5+2d)}{(-4+d)(-8+3d)k_2^2}I_{000001010110}\\
&+\frac{(-2+d)}{(-8+3d)}I_{001001010110}\\
&+\frac{-(-5+2d)(-1040+1064d-362d^2+41d^3)}{(-4+d)(-3+d)(-10+3d)(-8+3d)k_1^2}I_{010001000110}\\
&+\frac{-(100-72d+13d^2)}{(-10+3d)(-8+3d)}I_{010001010110}\\
&+\frac{-2((-4+d)(-5+2d)k_1^2+(-2+d)^2k_1\cdot k_2)}{(-8+3d)^2}I_{011001010110}\\
&+\frac{-8(-4+d)(-5+2d)k_1^2k_2^2}{(-10+3d)(-8+3d)^2}I_{021001010110}\\
&+\frac{-(-2+d)k_1^2(k_1+k_2)^2}{(-3+d)(-8+3d)k_2^2}I_{001002010110}\\
&+\frac{-(-4+d)(k_1+k_2)^2}{(-3+d)(-8+3d)}I_{010002010110}\Bigg), \qquad \text{(B.2.7)}\\
\Gamma_t(k_1,k_2) =& 2d(r)(d(r)+1)N_c|h_t|^2\\
&\times\Bigg(\frac{(d-2)k_2\cdot k_1}{(d-4)k_1^2}I_{101010001100}
\end{aligned}
$$

$$+\frac{(d-2)k_2\cdot k_1}{(d-4)k_2^2}I_{011001100100}\Big). \tag{B.2.8}$$

Applying the inverse relation (6.1.19) to the above reduction, we can express the spectral functions corresponding to the Γ's in terms of master spectral functions. The master spectral functions are obtained from the master integrals in the reductions above from the inverse relation (6.1.19) as well. We compute these master spectral functions in appendix B.3 - B.5.

Let us discuss the master integrals which appear in the above reductions in more detail. The factorisable contributions Γ_t and $\Gamma_{g,\text{fac}}$ contain only the factorisable master integrals

$$I_{\text{fac3L}}(k_1,k_2) \equiv I_{101010001100}(k_1,k_2), \tag{B.2.9}$$
$$I_{\text{fac3R}}(k_1,k_2) \equiv I_{011001100100}(k_1,k_2), \tag{B.2.10}$$
$$I_{\text{fac4L}}(k_1,k_2) \equiv I_{111011100000}(k_1,k_2), \tag{B.2.11}$$
$$I_{\text{fac4R}}(k_1,k_2) \equiv I_{111001110000}(k_1,k_2). \tag{B.2.12}$$

We compute the $\varepsilon=(4-d)/2$ expansion of the spectral functions of these integrals in (B.3) and find

$$\rho_{\text{fac3L}}(-k,k) = \frac{k^2}{(16\pi)^2 8\pi^2}\left(1+\varepsilon\left(\frac{17}{2}+3\ln\left(\frac{\bar\mu^2}{k^2}\right)\right)\right), \tag{B.2.13}$$
$$\rho_{\text{fac3R}}(-k,k) = \frac{k^2}{(16\pi)^2 8\pi^2}\left(1+\varepsilon\left(\frac{17}{2}+3\ln\left(\frac{\bar\mu^2}{k^2}\right)\right)\right), \tag{B.2.14}$$
$$\rho_{\text{fac4L}}(-k,k) = -\frac{1}{(16\pi)^2 2\pi^2}\left(\frac{1}{\varepsilon}+6+3\ln\left(\frac{\bar\mu^2}{k^2}\right)\right), \tag{B.2.15}$$
$$\rho_{\text{fac4R}}(-k,k) = -\frac{1}{(16\pi)^2 2\pi^2}\left(\frac{1}{\varepsilon}+6+3\ln\left(\frac{\bar\mu^2}{k^2}\right)\right), \tag{B.2.16}$$

where we introduced the $\overline{\text{MS}}$ - scale parameter $\bar\mu^2 = 4\pi\mu^2 e^{-\gamma_E}$.

In the reduction of the non-factorisable contributions $\Gamma_{g,\text{nfac}}$ and Γ_λ appear some master integrals which trivially lead to vanishing spectral functions because they depend only on a single variable. These integrals are

$$I_0(k_2) \equiv I_{001000010110}(k_1,k_2), \tag{B.2.17}$$
$$I_0(k_1) \equiv I_{000001010110}(k_1,k_2), \tag{B.2.18}$$
$$I_0(k_1+k_2) \equiv I_{010001000110}(k_1,k_2). \tag{B.2.19}$$

The remaining non-trivial non-factorisable integrals are

$$I_{\text{SSR}}(k_1,k_2) \equiv I_{001001010110}(k_1,k_2), \tag{B.2.20}$$
$$I_{\text{SSL}}(k_1,k_2) \equiv I_{010001010110}(k_1,k_2), \tag{B.2.21}$$
$$I_{\text{SSRdot}}(k_1,k_2) \equiv I_{001002010110}(k_1,k_2), \tag{B.2.22}$$
$$I_{\text{SSLdot}}(k_1,k_2) \equiv I_{010002010110}(k_1,k_2), \tag{B.2.23}$$
$$I_{\text{BB}}(k_1,k_2) \equiv I_{011001010110}(k_1,k_2), \tag{B.2.24}$$

$$I_{\mathrm{BBdot}}(k_1,k_2) \equiv I_{021001010110}(k_1,k_2), \tag{B.2.25}$$
$$I_{\mathrm{LR}}(k_1,k_2) \equiv I_{100001010110}(k_1,k_2), \tag{B.2.26}$$
$$I_{\mathrm{2L}}(k_1,k_2) \equiv I_{110001010110}(k_1,k_2), \tag{B.2.27}$$
$$I_{\mathrm{2R}}(k_1,k_2) \equiv I_{101001010110}(k_1,k_2). \tag{B.2.28}$$
$$\tag{B.2.29}$$

The integrals $I_{\mathrm{SSRdot}}(k_1,k_2)$ and $I_{\mathrm{SSLdot}}(k_1,k_2)$ are multiplied by $(k_1+k_2)^2$ in (B.2.5). Therefore, we only need to check whether their spectral functions have a pole for $k_1 = -k_2$. In appendix B.5.2 we find that this is not the case. We compute the other spectral functions in sections B.4 and B.5 and find the ε-expansion

$$\rho_{\mathrm{SSL}}(-k,k) = 0, \tag{B.2.30}$$
$$\rho_{\mathrm{SSR}}(-k,k) = 0, \tag{B.2.31}$$
$$\rho_{\mathrm{BB}}(-k,k) = -\frac{1}{(16\pi)^2 4\pi^2}\left(\frac{1}{\varepsilon}+7+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right), \tag{B.2.32}$$
$$\rho_{\mathrm{BBdot}}(-k,k) = \frac{1}{(16\pi)^2 4\pi^2 k^2}\left(\frac{1}{\varepsilon}+4+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right), \tag{B.2.33}$$
$$\rho_{\mathrm{LR}}(-k,k) = \frac{k^2}{(16\pi)^2 8\pi^2}\left(1+\varepsilon\left(10+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right)\right), \tag{B.2.34}$$
$$\rho_{2L}(-k,k) = -\frac{1+\ln(2)}{(16\pi)^2 4\pi^2}, \tag{B.2.35}$$
$$\rho_{2R}(-k,k) = -\frac{1+\ln(2)}{(16\pi)^2 4\pi^2}, \tag{B.2.36}$$
$$\rho_{\mathrm{fac3L}}(-k,k) = \frac{k^2}{(16\pi)^2 8\pi^2}\left(1+\varepsilon\left(\frac{17}{2}+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right)\right), \tag{B.2.37}$$
$$\rho_{\mathrm{fac3R}}(-k,k) = \frac{k^2}{(16\pi)^2 8\pi^2}\left(1+\varepsilon\left(\frac{17}{2}+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right)\right), \tag{B.2.38}$$
$$\rho_{\mathrm{fac4L}}(-k,k) = -\frac{1}{(16\pi)^2 2\pi^2}\left(\frac{1}{\varepsilon}+6+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right), \tag{B.2.39}$$
$$\rho_{\mathrm{fac4R}}(-k,k) = -\frac{1}{(16\pi)^2 2\pi^2}\left(\frac{1}{\varepsilon}+6+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right). \tag{B.2.40}$$

B.3 Factorisable three-point spectral functions

Some of three-point master integrals of the NLO correlator factorize into a product of two real two-point integrals $I(k_1)$ and $J(-k_2)$,

$$\Gamma(k_1,k_2) = I(k_1)J(-k_2). \tag{B.3.1}$$

In this case one can further simplify the inverse relation (6.1.19) to

$$\rho(k_1,k_2) = -4\mathrm{Im}(I(k_1+i0^+))\mathrm{Im}(J(-k_2+i0^+)). \tag{B.3.2}$$

Thus the three-point spectral-functions also factorizes into a product of two two-point spectral functions.

B.3.1 Computation of ρ_{fac4R} and ρ_{fac4L}

We compute the spectral functions of the integrals

$$I_{\text{fac4L}}(k_1,k_2)=\int_{p_1,p_2,p_3}\frac{1}{p_1^2p_2^2p_3^2(p_2-k_1)^2(p_3-k_1)^2(p_1+k_2)^2}, \tag{B.3.3}$$

and $I_{\text{fac4R}}(k_1,k_2)=I_{\text{fac4L}}(k_2,k_1)$ which can be written as a product of the two integrals

$$I_1(k)=\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2(p-k)^2} \tag{B.3.4}$$

and

$$I_2(k)=I_1(k)^2 \tag{B.3.5}$$

as

$$I_{\text{fac4L}}(k_1,k_2)=I_2(k_1)I_1(-k_2). \tag{B.3.6}$$

Using the inverse relation (B.3.2) and [53]

$$\text{Im}I_1(k_0+i0^+,\mathbf{k})=\frac{\text{sgn}(k_0)}{16\pi}\left[1+\varepsilon\left(\ln\frac{\bar{\mu}^2}{k^2}+2\right)\right]+\mathscr{O}(\varepsilon^2), \tag{B.3.7}$$

$$\text{Im}I_2(k_0+i0^+,\mathbf{k})=\frac{\text{sgn}(k_0)}{2(4\pi)^3}\left(\frac{1}{\varepsilon}+2\ln\frac{\bar{\mu}^2}{k^2}+4\right)+\mathscr{O}(\varepsilon), \tag{B.3.8}$$

we find

$$\rho_{\text{fac4L}}(-k,k)=\rho_{\text{fac4R}}(-k,k)=-\frac{1}{(16\pi)^2 2\pi^2}\left(\frac{1}{\varepsilon}+6+3\ln\frac{\bar{\mu}^2}{k^2}\right)+\mathscr{O}(\varepsilon), \tag{B.3.9}$$

where $\overline{\mu}^2=4\pi\mu^2e^{-\gamma_E}$ is the $\overline{\text{MS}}$ renormalization scale parameter.

B.3.2 Computation of ρ_{fac3R} and ρ_{fac3L}

We compute the spectral functions of the integrals

$$I_{\text{fac3L}}(k_1,k_2)=\int_{p_1,p_2,p_3}\frac{1}{p_1^2p_3^2(p_2-k_1)^2(p_3+k_2)^2(p_1-p_2)^2} \tag{B.3.10}$$

and $I_{\text{fac3R}}(k_1,k_2)=I_{\text{fac3L}}(k_2,k_1)$, which can be written as

$$I_{\text{fac3L}}(k_1,k_2)=I_3(k_1)I_1(-k_2), \tag{B.3.11}$$

with

$$I_3(k)\equiv\int_{p_1p_2}\frac{1}{p_1^2(p_1-p_2)^2(p_2-k_1)^2}. \tag{B.3.12}$$

Using the inverse relation (B.3.2) and [53]

$$\mathrm{Im}I_3(k_0+i0^+,\mathbf{k}) = -\frac{\mathrm{sgn}(k_0)k^2}{8(4\pi)^3} + \mathscr{O}(\varepsilon), \tag{B.3.13}$$

we find

$$\rho_{\mathrm{fac3L}}(-k,k) = \rho_{\mathrm{fac3R}}(-k,k) = \frac{k^2}{(16\pi)^2 8\pi^2}\left(1+\varepsilon\left(\frac{17}{2}+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right)\right). \tag{B.3.14}$$

B.4 Non-factorisable spectral functions without squared propagators

B.4.1 Computation of ρ_{BB}

We compute the spectral function to the integral

$$I_{\mathrm{BB}}(k_1,k_2) = \int_{p_1,p_2,p_3} \frac{1}{p_1^2(p_1-k)^2 p_2^2(p_2+q)^2(p_3-p_1)^2(p_3-p_2)^2}. \tag{B.4.1}$$

At first we use a FORM [96] program to compute the integrals over the temporal components $\bar{p}_l^0 = -ip_l^0$ for $l=1,2$. In order to simplify this task, we write the one-loop sub-integral over p_3 as $I_1(p_1-p_2)$ defined in (B.3.4). This yields

$$I_{\mathrm{BB}}(k_1,k_2) = \int_{p_1,p_2,p_3} \frac{1}{p_1^2(p_1-k)^2 p_2^2(p_2+q)^2} I_1(p_1-p_2). \tag{B.4.2}$$

Now we use the spectral representation[3]

$$I_1(p_1-p_2) = 2\int_0^\infty \frac{ds}{2\pi}\frac{s\,\rho_{I_1}(s,|\mathbf{p}_1-\mathbf{p}_2|)}{s^2+(\bar{p}_1^0-\bar{p}_2^0)}. \tag{B.4.3}$$

For simplicity we set $k_1=(k_1^0,\mathbf{0})$ and $k_2=(k_2^0,\mathbf{0})$. After the integration over $\bar{p}_1^0$ and $\bar{p}_2^0$ we find

$$\begin{aligned} I_{\mathrm{BB}}(k_1,k_2)|_{\mathbf{k}_1=0,\mathbf{k}_2=0} &= \frac{1}{8}\int_{\mathbf{p}_1,\mathbf{p}_2}\int_0^\infty \frac{ds}{2\pi}\frac{s\,\rho_{I_1}(s,|\mathbf{p}_1-\mathbf{p}_2|)}{s^2-(|\mathbf{p}_1|-|\mathbf{p}_2|)^2} \\ &\quad\times \frac{1}{|\mathbf{p}_1|^2|\mathbf{p}_2|^2|}\frac{1}{(k_1^0-2|\mathbf{p}_1|)(-k_2^0-|\mathbf{p}_2|)} \\ &\quad + \text{many terms which do not contribute to } \rho(-k,k). \end{aligned} \tag{B.4.4}$$

In this representation one can conveniently use the inverse relation (6.1.19) in combination with (2.1.9). Furthermore, we use the spectral representation of the integral I backwards. Then we find

$$\begin{aligned} \rho_{\mathrm{BB}}(-k,k) &= -\frac{4\pi^2}{k_0^4}\int_{\mathbf{p}_1,\mathbf{p}_2} I_1(0,|\mathbf{p}_1-\mathbf{p}_2|)\delta((k_0-2|\mathbf{p}_1|)\delta(k_0-2|\mathbf{p}_2|) \\ &\quad + \text{many terms which do not contribute to } \rho(-k,k). \end{aligned} \tag{B.4.5}$$

[3]Here we made use of the fact that the spectral functions is odd in s.

Working in $(d-1)$-dimensional spherical coordinates with

$$\int_{\mathbf{p}_i} = \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int d|\mathbf{p}|_i |\mathbf{p}_i|^{d-2} \tag{B.4.6}$$

and $\Omega_d \equiv 2\pi^{d/2}/\Gamma(d/2)$, we evaluate the radial integrals over $|\mathbf{p}_i|$ and obtain

$$\begin{aligned}\rho_{\mathrm{BB}}(-k,k) = -\left(\frac{k_0}{2}\right)^{2d-6} \frac{\pi^{d+1}}{(2\pi)^{2d-2}\Gamma(\frac{d-1}{2})\Gamma(\frac{d-2}{2})} \int_{-1}^{1} dz_{12}(1-z_{12}^2)^{\frac{d-4}{2}} \\ \times I_1(0,|\mathbf{p}_1-\mathbf{p}_2|)_{|\mathbf{p}_1|=|\mathbf{p}_1|=\frac{k_0}{2}},\end{aligned} \tag{B.4.7}$$

where

$$z_{12} \equiv \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{p}_1||\mathbf{p}_2|}. \tag{B.4.8}$$

For the one-loop sub-integral we use the solution

$$I_1(k) = \frac{\Gamma(\frac{d}{2}-1)^2\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(d-2)} \left(k^2\right)^{\frac{d}{2}-2}. \tag{B.4.9}$$

Now the z_{12} integral is straightforward and we expand the resulting expression for the spectral function in $\varepsilon = (4-d)/2$. This yields

$$\rho_{\mathrm{BB}}(-k,k) = -\frac{1}{(16\pi)^2 4\pi^2}\left(\frac{1}{\varepsilon} + \frac{7}{2} + 3\ln\frac{\bar{\mu}^2}{k^2}\right) + \mathscr{O}(\varepsilon), \tag{B.4.10}$$

where we have used Lorentz symmetry to replace k_0^2 by k^2.

B.4.2 Computation of ρ_{LR}

We compute the spectral function of the integral

$$I_{\mathrm{LR}}(k_1,k_2) \equiv \int_{p_1,p_2,p_3} \frac{1}{p_1^2(p_3-k_1)^2(p_2+k_2)^2(p_1-p_2)^2(p_1-p_3)^2}. \tag{B.4.11}$$

We proceed as for I_{BB} and write the integral in terms of the sub-integral I_1 which yields

$$I_{\mathrm{LR}}(k_1,k_2) = \int_{p_1} \frac{1}{p_1^2} I_1(p_1+k_2) I_1(p_1-k_1). \tag{B.4.12}$$

Then we use the spectral representation (B.4.3) to obtain

$$I_{\mathrm{LR}}(k_1,k_2) = \int_{p_1}\int_0^\infty \frac{dt}{2\pi}\int_0^\infty \frac{ds}{2\pi}\frac{4st}{p_1^2}\frac{\rho_{I_1}(s,|\mathbf{p}_1+\mathbf{k}_2|)\rho_{I_1}(t,|\mathbf{p}_1-\mathbf{k}_1|)}{(s^2+(\bar{p}_1^0+ik_1^0)^2)(t^2+(\bar{p}_1^0-ik_2^0)^2)}. \tag{B.4.13}$$

Now we solve the $\bar{p}_1^0$ integral with a FORM program and use the inverse relation (6.1.19) to obtain δ-functions. Then it is easy to solve the s- and t-integral and we find

$$\rho_{\mathrm{LR}}(-k,k) = \int_{\mathbf{p}_1} \frac{1}{2|\mathbf{p}_1|}\rho_{I_1}(E_{\mathbf{k}}-|\mathbf{p}_1|,|\mathbf{p}_1|)^2\theta(E_{\mathbf{k}}-|\mathbf{p}_1|). \tag{B.4.14}$$

We use $\rho_{I_1}(k) = 2\mathrm{Im}I_1(k_0+i0^+,\mathbf{k})$ together with (B.3.7) and solve the $|p_1|$ with spherical coordinates in $(d-1)$ dimensions. Then, expanding in $\varepsilon = (d-4)/2$, we finally get

$$\rho_{\mathrm{LR}}(-k,k) = \frac{k^2}{(16\pi)^2 8\pi^2}\left(1+\varepsilon\left(10+3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right)\right). \tag{B.4.15}$$

B.4.3 Computation of $\rho_{\rm SSR}$ and $\rho_{\rm SSL}$

We compute the spectral functions of the integrals

$$I_{\rm SSL}(k_1,k_2)=\int_{p_1,p_2,p_3}\frac{1}{p_2^2(p_3-k_1)^2(p_2+k_2)^2(p_1-p_2)^2(p_1-p_3)^2} \tag{B.4.16}$$

and $I_{\rm SSR}(k_1,k_2)=I_{\rm SSL}(k_2,k_1)$. It can be written in terms of the sub-integral I_3 as

$$I_{\rm SSL}(k_1,k_2)=\int_{p_1}\frac{1}{p_2^2(p_2+k_2)}I_3(p_2-k_1). \tag{B.4.17}$$

Again, we write the sub-integral in the spectral representation and obtain

$$I_{\rm SSL}(k_1,k_2)=\int_{p_2}\int_0^\infty\frac{ds}{2\pi}\frac{1}{p_2^2(p_2+k_2)}\frac{2s\,\rho_{I_3}(s,|\mathbf{p}_2-\mathbf{k}_1|)}{s^2+(\bar{p}_2^0+ik_1^0)^2}. \tag{B.4.18}$$

Then we solve the $\bar{p}_2^0$-integral with a FORM program and use the inverse relation (6.1.19) to obtain delta functions. Then we solve the s-integral and find

$$\rho_{\rm SSL}(-k,k)=\int_{\mathbf{p}_2}\frac{1}{4\mathbf{p}_2^2}\pi\delta(k_0-2|\mathbf{p}_2|)\rho_{I_3}(k_0-|\mathbf{p}_2|,|\mathbf{p}_2|). \tag{B.4.19}$$

The δ-function yields $\rho_{I_3}(k_0/2,k_0/2)$ which is zero as we can see from (B.3.13). Therefore, we find

$$\rho_{\rm SSL}(-k,k)=\rho_{\rm SSR}(-k,k)=0. \tag{B.4.20}$$

B.4.4 Computation of $\rho_{\rm 2L}$ and $\rho_{\rm 2R}$

We compute the spectral functions of the integrals

$$I_{\rm 2L}(k_1,k_2)=\int_{p_1,p_2,p_3}\frac{1}{p_1^2p_2^2(p_3-k_1)^2(p_2+k_2)^2(p_1-p_2)^2(p_1-p_3)^2} \tag{B.4.21}$$

and $I_{\rm 2R}(k_1,k_2)=I_{\rm 2L}(k_2,k_1)$. The integral can be written as

$$I_{\rm 2L}(k_1,k_2)=\int_{p_1,p_2}\frac{1}{p_1^2p_2^2(p_2+k_2)^2(p_1-p_2)^2}I_1(p_1-k_1). \tag{B.4.22}$$

We find it also convenient to use the spectral representation for the propagator

$$\frac{1}{(p_1-p_2)^2}=\int_0^\infty\frac{dt}{2\pi}\frac{2t\,\rho_\delta(t,|\mathbf{p}_1-\mathbf{p}_2|)}{s^2+(\bar{p}_1^0-\bar{p}_2^0)^2}, \tag{B.4.23}$$

where

$$\rho_\delta(k_0,\mathbf{k})=-\frac{\pi}{|\mathbf{k}|}\left(\delta(k_0-|\mathbf{k}|)-\delta(k_0+|\mathbf{k}|)\right). \tag{B.4.24}$$

Then, computing the integrals over $\bar{p}_1^0$ and $\bar{p}_2^0$ with a FORM program and using the inverse relation (6.1.19), we find

$$\begin{aligned}\rho_{2L}(-k,k) =&\pi \int_{\mathbf{p}_1\mathbf{p}_2} \int_0^\infty \frac{dt}{2\pi} \rho_\delta(t, |\mathbf{p}_1 - \mathbf{p}_2|) \\ &\times \Big[8\delta(2|\mathbf{p}_2| - k_0) f_1(|\mathbf{p}_1|, |\mathbf{p}_2|, t) \\ &+8\delta(2|\mathbf{p}_2| - k_0) f_2(|\mathbf{p}_1|, |\mathbf{p}_2|, t) \\ &-\delta(-k_0 + t + |\mathbf{p}_1| + |\mathbf{p}_2|) f_3(|\mathbf{p}_1|, |\mathbf{p}_2|, t) \\ &+\delta(-k_0 - t - |\mathbf{p}_2| + |\mathbf{p}_1|) f_4(|\mathbf{p}_1|, |\mathbf{p}_2|, t) \Big], \end{aligned} \tag{B.4.25}$$

where

$$f_1(|\mathbf{p}_1|, |\mathbf{p}_2|, t) = \frac{\rho_{I_1}(k_0/2 - t, |\mathbf{p}_1|)\theta(k_0/2 - t)}{k_0^2(k_0 - 2|\mathbf{p}_1| + 2t)(k_0 + 2|\mathbf{p}_1| + 2t)}, \tag{B.4.26}$$

$$f_2(|\mathbf{p}_1|, |\mathbf{p}_2|, t) = \frac{t\,\rho_{I_1}(k_0 - |\mathbf{p}_1|, |\mathbf{p}_1|)\theta(k_0 - |\mathbf{p}_1|)\theta(k_0 + |\mathbf{p}_1| - |\mathbf{p}_2|)}{k_0^2|\mathbf{p}_1|(k_0 - 2|\mathbf{p}_1| - 2t)(k_0 - 2|\mathbf{p}_1| + 2t)}, \tag{B.4.27}$$

$$f_3(|\mathbf{p}_1|, |\mathbf{p}_2|, t) = \frac{\rho_{I_1}(k_0 - |\mathbf{p}_1|, |\mathbf{p}_1|)\theta(k_0 - |\mathbf{p}_1|)\theta(k_0 - |\mathbf{p}_1| - t)}{2k_0|\mathbf{p}_1|(k_0 - 2|\mathbf{p}_1| - 2t)(k_0 - |\mathbf{p}_1| - t)}, \tag{B.4.28}$$

$$f_4(|\mathbf{p}_1|, |\mathbf{p}_2|, t) = \frac{\rho_{I_1}(k_0 - |\mathbf{p}_1|, |\mathbf{p}_1|)\theta(k_0 - |\mathbf{p}_1|)\theta(|\mathbf{p}_1| - t - k_0)}{2k_0|\mathbf{p}_1|(k_0 - |\mathbf{p}_1| + t)(k_0 - 2|\mathbf{p}_1| + 2t)}. \tag{B.4.29}$$

Like in the case for ρ_{BB} we first solve the integral over $z_{12} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{p}_1||\mathbf{p}_2|}$ in $(d-1)$-dimensional spherical coordinates. For this purpose it turns out to be convenient to substitute $x \equiv |\mathbf{p}_1 + \mathbf{p}_2|$ with

$$dz_{12} = -\frac{x dx}{|\mathbf{p}_1||\mathbf{p}_2|}. \tag{B.4.30}$$

Then the integral over x is trivial due to the delta function in $\rho_\delta(t, x)$ and yields

$$\int_{||\mathbf{p}_1| - |\mathbf{p}_2||}^{|\mathbf{p}_1| + |\mathbf{p}_2|} dx x \rho_\delta(t, x) = -\pi\theta(|\mathbf{p}_1| + |\mathbf{p}_2| - t)\theta(t - ||\mathbf{p}_1| + |\mathbf{p}_2||). \tag{B.4.31}$$

Similarly we solve the integral over $|\mathbf{p}_2|$, making use of the other δ-functions. Finally we solve the remaining integrals over t and $|\mathbf{p}_1|$ with Mathematica [102] and obtain the finite result

$$\rho_{2L}(-k,k) = \rho_{2R}(-k,k) = -\frac{1 + \ln(2)}{(16\pi)^2 4\pi^2}. \tag{B.4.32}$$

B.5 Non-factorisable spectral functions with one squared propagator

B.5.1 Computation of ρ_{BBdot}

The situation is more complicated if propagators in the master integrals are squared. For example in the reduction (B.2.7) and (B.2.5) appears the integral

$I_{\mathrm{BBdot}}(k_1,k_2) \equiv I_{021001010110}(k_1,k_2)$, which explicitly reads

$$I_{\mathrm{BBdot}}(k_1,k_2) = \int_{p_1,p_2,p_3} \frac{1}{p_1^2 p_2^4 (p_1-k)^2 (p_2+q)^2 (p_1-p_2)^2 (p_2-p_3)^2}. \tag{B.5.1}$$

We can compute this integral with the same techniques as in section (B.4.1) if we introduce an artificial mass in the integral (B.4.1) which defines the integral

$$I_{\mathrm{BB}}(k_1,k_2,m) \equiv \int_{p_1,p_2,p_3} \frac{1}{p_1^2\left(p_2^2+m^2\right)(p_1-k)^2 (p_2+q)^2 (p_1-p_2)^2 (p_2-p_3)^2}. \tag{B.5.2}$$

The spectral function of I_{BBdot} is then determined by the mass derivative

$$\rho_{\mathrm{BBdot}}(k_1,k_2) = -\left[\frac{d^2}{dm^2}\rho_{\mathrm{BB}}(k_1,k_2,m)\right]_{m=0}. \tag{B.5.3}$$

We compute the spectral function of the massive integral $I_{\mathrm{BB}}(k_1,k_2,m)$, applying the same steps as in (B.4.1), but with much more terms during the calculation. After a long calculation and with help of Mathematica [102], we find

$$\rho_{\mathrm{BBdot}}(-k,k) = \frac{1}{(16\pi)^2 4\pi^2 k^2}\left(\frac{1}{\varepsilon} + 4 + 3\ln\left(\frac{\bar{\mu}^2}{k^2}\right)\right), \tag{B.5.4}$$

after expanding in $\varepsilon = (d-4)/2$ and taking the mass derivative (B.5.3)

B.5.2 Computation of ρ_{SSRdot} and ρ_{SSLdot}

We do not need to compute ρ_{SSRdot} and ρ_{SSLdot} explicitly because in the reduction (B.2.6) the corresponding master integrals are multiplied with $(k_1+k_2)^2$ which vanishes for $k_1 = -k_2 = -k$. Therefore, we only have to check if the spectral functions have a pole in $k_1 = -k_2$. Let us start with

$$I_{\mathrm{SSLdot}} = \int_{p_1,p_2,p_3} \frac{1}{p_2^2 (p_3-k_1)^4 (p_2+k_2)^2 (p_1-p_2)^2 (p_1-p_3)^2}, \tag{B.5.5}$$

which we can write as

$$I_{\mathrm{SSL}}(k_1,k_2) = \int_{p_1} \frac{1}{p_2^2 (p_2+k_2)} I_{3\mathrm{dot}}(p_2-k_1), \tag{B.5.6}$$

where

$$I_{3\mathrm{dot}}(k) = \int_{p_2,p_3} \frac{1}{p_3^2 (p_3-k)^4 (p_3-p_2)^2}. \tag{B.5.7}$$

For simplicity we set $\mathbf{k}_1 = \mathbf{k}_2 = 0$ and assume $k_1^0 < 0$ and $k_2^0 > 0$. Then, analogously to section B.4.3, we find

$$\rho_{\mathrm{SSLdot}}(k_1,k_2)\Big|_{\mathbf{k}_1=\mathbf{k}_2=0} = \int_{\mathbf{p}_2} \frac{1}{4\mathbf{p}_2^2}\pi\delta(k_2^0 - 2|\mathbf{p}_2|)\rho_{I_{3\mathrm{dot}}}(-k_1^0-|\mathbf{p}_2|,|\mathbf{p}_2|)\theta(-k_1^0-|\mathbf{p}_2|). \tag{B.5.8}$$

In order to compute $\rho_{I_{3\text{dot}}}$, we proceed as follows. At first we take the derivative

$$k^\mu \frac{\partial}{\partial k^\mu} I_3(k) = \int_{p_2,p_3} \frac{2k\cdot(p_3-k)}{p_3^2(p_3-k)^4(p_3-p_2)^2}, \tag{B.5.9}$$

and cancel the scalar products in the numerator which yields

$$k^\mu \frac{\partial}{\partial k^\mu} I_3(k) = -k^2 I_{3\text{dot}}(k) - I_3(k). \tag{B.5.10}$$

On the other hand, we know that $I_3(k) \sim (k^2)^{(d-3)}$ such that

$$k^\mu \frac{\partial}{\partial k^\mu} I_3(k) = 2(d-3) I_3(k). \tag{B.5.11}$$

Therefore, we have

$$I_3(k) = \frac{k^2}{2d-5} I_{3\text{dot}}(k) \tag{B.5.12}$$

and consequently, for the spectral function

$$\rho_{I_{3\text{dot}}}(k) = \frac{3}{k^2}\rho_{I_3}(k) = -\frac{3\text{sgn}(k_0)}{4(4\pi)^4}. \tag{B.5.13}$$

Plugging this result into (B.5.15) and solving the integral $\mathbf{p}_2$-integral, it is easy to see that the result is well defined for $k_1 = -k_2$. Therefore ρ_{SSLdot} does not contribute to the asymmetry rate.

For the computation of $\rho_{\text{SSRdot}}(k_1,k_2)$ we introduce an artificial mass for the squared propagator. Then $\rho_{\text{SSRdot}}(k_1,k_2)$ can be computed as the mass derivate of the spectral function of

$$I_{\text{SSR}}(k_1,k_2,m) \equiv \int_{p_3} \frac{1}{p_3^2((p_3-k_1)^2+m^2)} I_3(p_3+k_2). \tag{B.5.14}$$

We write the massive propagator and the integral I_3 in terms of their spectral functions, perform the $\bar{p}_3^0$ integral and use the inverse relations. For simplicity we set $\mathbf{k}_1 = \mathbf{k}_2 = 0$ and we assume $k_1^0 < 0$ and $k_2^0 > 0$. This yields

$$\begin{aligned}\rho_{\text{SSR}}(k_1,k_2,m)\Big|_{\mathbf{k}_1=\mathbf{k}_2=0} &= \int \frac{d^3p_3}{(2\pi)^3}\frac{1}{2|\mathbf{p}_3|}\rho_\delta(-k_1^0-|\mathbf{p}_3|,\sqrt{|\mathbf{p}_3|^2+m^2}) \\ &\times\Big\{\rho_{I_3}(k_2^0-|\mathbf{p}_3|,|\mathbf{p}_3|)\theta(-k_1^0-|\mathbf{p}_3|)\theta(k_2^0-|\mathbf{p}_3|) \\ &-\rho_{I_3}(-k_2^0+|\mathbf{p}_3|,|\mathbf{p}_3|)\theta(-k_1^0-|\mathbf{p}_3|)\theta(-k_2^0+|\mathbf{p}_3|)\Big\},\end{aligned} \tag{B.5.15}$$

where ρ_δ and ρ_{I_3} are given in (B.4.24) and (B.3.13) respectively. We assume that $(k_1^0)^2 > m^2$, which allows to write

$$\rho_\delta(-k_1^0-|\mathbf{p}_3|,\sqrt{|\mathbf{p}_3|^2+m^2}) = -\frac{\pi((k_1^0)^2+m^2)}{2(k_1^0)^2\sqrt{\mathbf{p}_3^2+m^2}}\delta\left(|\mathbf{p}_3| - \frac{m^2-(k_1^0)^2}{2k_1^0}\right). \tag{B.5.16}$$

The $\mathbf{p_3}$ integral can now easily be solved which leads to

$$\rho_{\rm SSR}(k_1,k_2,m)\Big|_{\mathbf{k_1}=\mathbf{k_2}=0} = -\frac{k_2^0((k_1^0)^2-m^2)((k_1^0)^2+k_1^0k_2^0-m^2)}{2048\pi^4(k_1^0)^3}. \tag{B.5.17}$$

Taking the derivative with respect to m^2 it is easy to see that no pole for $k_2^0=-k_1^0$ appears. Therefore, $\rho_{\rm SSRdot}$ does not contribute to the asymmetry.

B.6 Counter terms

In this section we give some more details to the calculations of the counter terms. We consider the interaction

$$\mathscr{L}_{\rm int}^{\rm eff} = -\overline{N}_1(h_\nu)_{1i}J_i + \frac{1}{2}(g_\nu)_{ij}J_i^\top\mathscr{C}^{-1}J_j + \text{H.c}, \tag{B.6.1}$$

where $(g_\nu)_{ij}=\sum_{I\neq 1}(h_\nu)_{Ii}(h_\nu)_{Ij}/M_I$ and $J_i=\tilde{\varphi}^\dagger\ell_i$. We renormalize the fields by

$$\varphi = \varphi_{\rm R}Z_\varphi^{1/2}, \qquad \ell = \ell_{\rm R}Z_\ell^{1/2} \tag{B.6.2}$$

and the couplings by

$$(h_\nu)_{1i} = (h_{\nu{\rm R}})_{1i}Z_h \qquad (g_\nu)_{ij} = (g_{\nu{\rm R}})_{ij}Z_g. \tag{B.6.3}$$

In the following we write $Z_i=1+\delta Z_i$ and consider the leading SM contributions to δZ_i. In this case the counter-term Lagrangian can be written as

$$\begin{aligned}\mathscr{L}_{\rm CT} =&\delta Z_\ell\bar{\ell}i\partial\!\!\!/\ell + \delta Z_\varphi(\partial_\mu\varphi)^\dagger\partial^\mu\varphi && \text{(B.6.4)}\\ &-\frac{1}{2}\left(\delta Z_\varphi+\delta Z_\ell+2\delta Z_h\right)\left(\overline{N}_1(h_\nu)_{1i}J_i+\text{H.c}\right) && \text{(B.6.5)}\\ &+\frac{1}{2}\left(\delta Z_\varphi+\delta Z_\ell+\delta Z_g\right)\left((g_\nu)_{ij}J_i^\top\mathscr{C}^{-1}J_j+\text{H.c}\right). && \text{(B.6.6)}\end{aligned}$$

B.6.1 Determination of δZ_ℓ and δZ_φ

Let $\Sigma_\ell=\Sigma_\ell^{\rm 1L}+\Sigma_\ell^{\rm CT}$ and $\Sigma_\varphi=\Sigma_\varphi^{\rm 1L}+\Sigma_\varphi^{\rm CT}$ be the lepton and Higgs self-energy respectively. They consists of the one-loop parts $\Sigma_\ell^{\rm 1L}$ and $\Sigma_\varphi^{\rm 1L}$ and the counter-term parts

$$\begin{aligned}(\Sigma_\ell^{\rm CT})_{ab} =&\delta Z_\ell p\!\!\!/P_{\rm L}\delta_{ab}, && \text{(B.6.7)}\\ (\Sigma_\varphi^{\rm CT})_{ab} =&\delta Z_\varphi p^2\delta_{ab}, && \text{(B.6.8)}\end{aligned}$$

where a and b denote indices in electroweak $SU(2)$ space. We compute SM contributions to the one-loop parts, given by the diagrams in figure B.1, with arbitrary gauge parameter ξ_1 and ξ_2. In order to simplify the calculation, we take the trace in $SU(2)$ space, by multiplying the self-energies with δ_{ab}. After applying standard Feynman rules, simplify the Dirac structure and canceling scalar products in numerators, we find

$$\begin{aligned}\frac{1}{2}\delta_{ab}(\Sigma_\ell^{\rm 1L}(p))_{ab} =&\frac{d-2}{8}p\!\!\!/P_{\rm L}I_1(p)\left(g_1^2\xi_1+3g_2^2\xi_2\right), && \text{(B.6.9)}\\ \frac{1}{2}\delta_{ab}(\Sigma_\varphi^{\rm 1L}(p))_{ab} =&-\frac{1}{4}p^2I_1(p)\left(g_1^2(3-\xi_1)+3g_2^2(3-\xi_2)-4N_c|h_t|^2\right). && \text{(B.6.10)}\end{aligned}$$

The integral $I_1(p)$ is the standard one-loop integral defined in (B.3.4). In $d = 4-2\varepsilon$ dimensions it has the ε expansion

$$I_1(p) = \frac{1}{(4\pi)^2\varepsilon} + \mathscr{O}(\varepsilon^0). \tag{B.6.11}$$

Enforcing the complete self-energies Σ_ℓ and Σ_φ to be finite leads to

$$\delta Z_\ell = -\frac{1}{(4\pi)^2\varepsilon}\left(\frac{1}{4}\xi_1 g_1^2 + \frac{1}{4}\xi_2 g_2^2\right), \tag{B.6.12}$$

$$\delta Z_\varphi = \frac{1}{(4\pi)^2\varepsilon}\left(\frac{1}{4}g_1^2(3-\xi_1) + \frac{3}{4}g_2^2(3-\xi_2) - 4N_c|h_t|^2\right). \tag{B.6.13}$$

Figure B.1: One-loop Diagrams contributing to the Higgs and lepton self-energies $\Sigma_\varphi^{1\mathrm{L}}$ and $\Sigma_\ell^{1\mathrm{L}}$. The lepton self-energy (left) gets only corrections from gauge bosons. Higgs Higgs self-energy gets also a correction from the top quark (right).

B.6.2 Determination of δZ_h

For the computation of δZ_h we have to consider the one-loop SM correction to the three-vertex $(V_{1i}^{1\mathrm{L}})_{ab}$ in figure B.2. Again, after using standard Feynman rules, we take the trace in $SU(2)$ space. In order to get rid off the Dirac structure we also take the trace over the Dirac matrices. Then, after canceling all scalar products in the numerator of the Feynman integrals, we can express the the one-loop vertex function in terms of integrals of the class

$$I_{abc}(p_1,p_2) \equiv \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^{2a}(k-p_1)^{2b}(k+p_2)^{2c}}. \tag{B.6.14}$$

Here p_1 defines the external momentum of the Higgs. We use the program Reduze [100] to write the result in terms of master integrals and find

$$\begin{aligned}\mathrm{Tr}\left(\delta_{ab}(V_{1i}^{1\mathrm{L}})_{ab}\right) =& -i(h_{\nu R})_{1i}g_1^2\left[I_{011} - (1-\xi_1)\left(\frac{1}{2}I_{011} + \frac{d-3}{2}I_{101}\right)\right] \\ & -i(h_{\nu R})_{1i}3g_2^2\left[I_{011} - (1-\xi_2)\left(\frac{1}{2}I_{011} + \frac{d-3}{2}I_{101}\right)\right],\end{aligned} \tag{B.6.15}$$

where Tr denotes the trace in spinor space. All master integrals can be written in terms of the integral I_1 and therefore yield the same infinite contribution (B.6.11).This implies

$$\mathrm{Tr}\left(\delta_{ab}(V_{1i}^{1\mathrm{L}})_{ab}\right) = -i(h_{\nu R})_{1i}\left[g_1^2\xi_1 + 3g_2^2\xi_2\right]\frac{1}{(4\pi)^2\varepsilon} + \mathscr{O}(\varepsilon^0). \tag{B.6.16}$$

From (B.6.6) we can read off the counter-term three-vertex

$$(V_{1i}^{\mathrm{CT}})_{ab} = -i(h_{\nu R})_{1i}\delta_{ab}\left(\frac{1}{2}\delta Z_\varphi + \frac{1}{2}\delta Z_\ell + \delta Z_h\right)P_{\mathrm{L}}. \tag{B.6.17}$$

Then, enforcing the complete three-vertex to be finite, that is,

$$\delta_{ab}\mathrm{Tr}\left((V_{1i}^{1\mathrm{L}})_{ab}+(V_{1i}^{\mathrm{CT}})_{ab}\right)=\mathscr{O}(\varepsilon^0), \tag{B.6.18}$$

we find in the gauge invariant result

$$\delta Z_h=-\left[\frac{3}{8}(g_1^2+3g_2^2)-\frac{N_c}{2}\right]\frac{1}{(4\pi)^2\varepsilon} \tag{B.6.19}$$

for the three-vertex coupling renormalization. Our result is consistent with the one in [53].

Figure B.2: One-loop correction to the three-vertex $(V_{1i}^{1\mathrm{L}})_{ab}$

B.6.3 Determination of δZ_g

For the computation of δZ_g we proceed as for δZ_h. We compute the one-loop SM corrections to the four-vertex $(V_{ij}^{1\mathrm{L}})_{abcd}$ in figure B.3. Here a,b and c,d are the lepton and Higgs elec-

Figure B.3: One-loop SM corrections to the four-vertex $(V_{ij}^{1\mathrm{L}})_{abcd}$. There are also contributions from the first diagram with permutations of external lines.

troweak $SU(2)$ indices respectively. First we use a FORM [96] code to generate all diagrams and multiply them with the projector

$$P_{abcd}=\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}, \tag{B.6.20}$$

in order to get rid of the $SU(2)$ structure. Furthermore, we multiply the diagrams with $\mathscr{C}$ in order to eliminate $\mathscr{C}^{-1}$. Then we take the Dirac trace and cancel all scalar products in the numerator. The remaining integrals can be expressed in terms of the one-loop four-point integrals

$$I_{abcd}^{(1)}(p_1,p_2,p_3)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^{2a}(k+p_1)^{2b}(k+p_3)^{2c}(k+p_1+p_2)^{2d}}, \tag{B.6.21}$$

$$I_{abcd}^{(2)}(p_1,p_2,p_3)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^{2a}(k+p_2)^{2b}(k+p_3)^{2c}(k+p_1+p_2)^{2d}}. \tag{B.6.22}$$

Here p_1 and p_2 define the external momenta of the SM leptons. We use the program Reduze [100] to reduce these integrals to master integrals. Then, the terms which contribute to the

infinite part are

$$\begin{aligned}\frac{1}{24}P_{abcd}\mathrm{Tr}\left[(V_{ij}^{1\mathrm{L}})_{abcd}\mathscr{C}\right] &= -i(g_{\nu R})_{ij}g_1^2\Big[(6-3d)(I^{(1)}_{0101}+I^{(1)}_{1100})+(6-3\xi_1)(I^{(2)}_{0101}+I^{(2)}_{1100}) \\ &\quad +(6+\xi_1(6d-18))(I^{(1)}_{1001}-I^{(1)}_{0110}-I^{(2)}_{0110})\Big] \\ &\quad +-i(g_{\nu R})_{ij}g_2^2\Big[6(I^{(1)}_{0011}+I^{(1)}_{1010})+(12-3d-3\xi_2)(I^{(1)}_{0101}+I^{(1)}_{1100}) \qquad \text{(B.6.23)} \\ &\quad +(6+\xi_2(6d-18))(I^{(1)}_{1001}-2I^{(1)}_{0110}-2I^{(2)}_{0110}) \qquad \text{(B.6.24)} \\ &\quad +(12-6\xi_2)(I^{(2)}_{0101}+I^{(2)}_{1100})\Big] \\ &\quad +i(g_{\nu R})_{ij}2\lambda I^{(1)}_{1001} \\ &\quad +\text{terms which do not contribute to the infinite part.} \qquad \text{(B.6.25)}\end{aligned}$$

Here we can again express all master integrals in terms of the integral I_1 so that each integral yields the same infinite part (B.6.11). Then the leading order in the ε expansion is

$$\frac{1}{24}P_{abcd}\mathrm{Tr}\left[(V_{ij}^{1\mathrm{L}})_{abcd}\mathscr{C}\right] = \frac{i(g_{\nu R})_{ij}}{(4\pi)^2\varepsilon}\left[\frac{3(g_1^2+g_2^2)}{4}-\frac{g_1^2\xi_1+3g_2^2\xi_2}{2}+2\lambda\right]. \qquad \text{(B.6.26)}$$

From (B.6.6) we read off the counter-term four-vertex

$$(V_{ij}^{\mathrm{CT}})_{abcd} = \frac{i}{2}(g_{\nu R})_{ij}(\delta Z_\varphi+\delta Z_\ell+\delta Z_g)\left(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}\right)\mathscr{C}^{-1}P_{\mathrm{L}}. \qquad \text{(B.6.27)}$$

Again, we enforce the complete corrections to be finite, that is,

$$P_{abcd}\mathrm{Tr}\left(\left[(V_{ij}^{1\mathrm{L}})_{abcd}+(V_{ij}^{\mathrm{CT}})_{abcd}\right]\mathscr{C}\right) = \mathrm{O}(\varepsilon^0) \qquad \text{(B.6.28)}$$

which yields the gauge invariant result

$$\delta Z_g = \frac{1}{(4\pi)^2\varepsilon}\left[\frac{3}{4}(g_1^2+g_2^2)-\frac{3}{4}(g_1^2+3g_2^2)+2\lambda+N_c|h_t|^2\right] \qquad \text{(B.6.29)}$$

for the coupling renormalization of the four-vertex.

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