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Essays on Two-player Games with Asymmetric Information

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To my beloved daughter and son, Shujun and Yide

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Chapter 1

General Introduction

Strategic feedback and uncertainty are the main ingredients in the decision-making process. As of this writing, in China, the annual National College Entrance Examination (NCEE) finished last week. Immediately after completing the examination, all students have to rank their three most-preferred colleges while accounting for their expectation of the results of their exams. Not all of the students can attend their preferred colleges because there are hundreds of millions of students competing for places. Therefore, students have to make decisions under uncertainty before they know their actual scores and the admission scores of different colleges. They may over-estimate or under-estimate their scores, and they also need to take into account the decisions of other students because the admission score could be higher or lower than the reference value depending on how many students chose this college.

Such examples are even more widespread in economics. When a firm is considering whether to enter a market, relative to incumbents, it may have less information regarding market demand, costs or other factors that could affect its future profits; moreover, it has to take into account the reactions of the incumbents. By contrast, while an incumbent may better know the market, it must be aware of the possible threat from potential entrants. The focus of game theory research is on how groups of people interact. In particular, information asymmetries are omnipresent in games because various types of uncertainty can affect the players, including the examples mentioned above. Further examples include asset trading between institutional investors and individual investors in financial markets; the relationships between banks and borrowers in capital markets; and contract setting between agents and principals in labor markets.

The present thesis introduces asymmetric information into two classes of games that are widely analysed and utilized in a variety of economic problems: zero-sum games[1] and signalling games. In chapter 2, we seek to analyse the price dynamics of a risky asset in a financial market through a zero-sum repeated trading game involving a more informed sector and a less informed sector. Here, the information asymmetry involved in the trading between the two sectors is referred to as incomplete information on one-and-a-half sides.

In chapter 3, we study the belief updating and equilibrium refinement problem in classical signalling games, in which an informed player moves first and conveys private information to an uninformed player through messages; the uninformed player attempts to make inferences about hidden information and takes an action that can influence both players' payoffs. Here, measurable uncertainty exists only on the second-mover side, while the first mover knows the true state of nature.

In chapter 4, the information asymmetries are extended in a broader sense, whereby one of the players has multi-priors on the state of nature. This concept of uncertainty is called *Knightian Uncertainty* or *Ambiguity* in the literature. We introduce ambiguity into entry deterrence games and discuss the impact of ambiguity in two cases of asymmetric information structures. In both cases, the entrant faces ambiguity regarding the state of the market, but the incumbent is either fully informed or faces classical *measurable uncertainty.*

1.1 Price dynamics in financial markets with asymmetric information

In financial markets, one of the most relevant problems is to accurately identify stock price dynamics, which not only influence trading in stock markets but also determine the price formulae for derivatives. Price dynamics are often exogenously modelled in terms of Brownian motion in the financial analysis literature, for example, using Bachelier dynamics, Black and Scholes dynamics, diffusion models, stochastic volatility models, and GARCH models. Information asymmetries are well known in financial markets. The institutional sector have better access to

[1]See Sorin (2002), Aumann and Maschler (1995), Milgrom and Stokey(1982), De Meyer and Marino (2005), Von Neumann and Morgenstern (2007),etc. for example

information than does the individual sector. The daily, repeated trades between the two sets of actors influence the price of stocks. For information on the series of models based on repeated exchange games between an informed sector and an uninformed sector to endogenously determine price dynamics, see De Meyer and Saley (2003), De Meyer (2010), and De Meyer and Fournier (2015).

In De Meyer and Saley (2003), a zero-sum repeated game between two market makers with incomplete information on one side is analysed. In a risk-neutral environment, under a particular trading mechanism, this paper proves that the price process converges to a continuous martingale involving a Brownian motion. De Meyer (2010) generalizes this idea to a broader setting and proves that the price dynamics must be a so-called "Continuous martingale of maximal variational" (CMMV); see the explanation in section 2.1. This property does not rely on the trading mechanism. De Meyer and Fournier (2015) generalize this analysis to the case of a risk-averse market. They show that the price process is still a CMMV under a martingale equivalent measure.

In all of these three models, the informed player is fully informed of the state (the value of the risky asset) and has full knowledge of the beliefs of the uninformed player. In the first chapter of this thesis, we attempt to characterize a financial model out of this scenario. In this setting, the more informed player has private information about the value of the risky asset, but the less informed player is informed of some private message associated with the value. Therefore, the more informed player is uncertain about the beliefs of his trading partner. An N-period repeated exchange of this risky asset using the numéraire between these two players under this information structure is modelled by a repeated zero-sum game with incomplete information on one-and-a-half sides, as introduced by Sorin and Zamir (1985). We show that completely different price dynamics from those in De Meyer's (2010) results can be obtained by simply slightly disturbing the information of the uninformed player.

1.2 Belief updating and equilibrium refinements in signalling games

In a dynamic decision-making process, people update their beliefs constantly as new information arrives. Bayes' Rule is a common assumption on belief updating in learning theory. However, it has two limitations: first, it does not predict how agents should react to information to which they assigned probability zero; second, a series of psychological experiments suggest that people's behaviour may deviate from Bayes' rule[2]. In addition, the Bayesian approach requires that we be able to quantify this uncertainty using a single prior, which is also an imperfect method. For example, imagine that we are considering spending a one-week holiday in some beachfront city abroad this summer. If we make the decision earlier, we can spend less on the hotel and flights but we are more uncertain about the weather. Can we really assign one number to the probability of good weather conditional on the information that we know? These findings spurred increased interest in research on non-Bayesian learning[3].

Signalling games are a widely utilized class of games in economics, as reviewed in Riley (2001) and Sobel (2007). A signalling game typically admits multiple sequential Nash equilibria because the second mover's belief is not well defined by Bayes' rule when zero-probability messages sent by the first mover are observed. Multi-equilibria cannot provide a precise prediction, and therefore, refinements were developed incrementally. Some refinements rely on *ad hoc* criteria, while some studies attempt to define a new concept of equilibrium; see the literature review in section 2.1, for example.

We are interested in nesting an alternative updating rule of the Hypothesis Testing model axiomatically characterized by Ortoleva (2012) into a class of general signalling games and thereby providing a new equilibrium refinement method in signalling games. Here, we present an example to briefly illustrate how the non-Bayesian updating rule proceeds. Imagine that our agent is uncertain about a

[2] See the psychological experiments in Kahneman and Tversky (1974), surveys by Camerer (1995) and Rabin (1998), and the arguments in Epstein et al. (2010) and Ortoleva (2012), among others.

[3] For example, Gul and Pesendorfer (2001, 2004), Epstein (2006, 2008, 2010), Golub and Jackson (2010), Jadbabaie et al., (2012), Gilboa et al, (2008, 2009, 2012),Teng (2014), and Ortoleva (2012).

state of nature; the state space is finite $\Omega = \{\omega_1, ..., \omega_N\}$. The set of all of the subsets of Ω is $\Sigma = 2^{\Omega}$. She does not have the confidence to assign one unique probability distribution on the state space; let us say that she has two priors: π_1 and π_2. Based on her current information and knowledge, she believes that π_1 is more likely than π_2. In brief, assume that she has a prior ρ over the set of priors, with $\rho(\pi_1) > \rho(\pi_2) > 0$. She has π_1 as her original prior before any new information arrives. She has a subjective threshold for deciding whether an event is a low-probability event, for example $\epsilon = 5\%$. After some event $A \in \Sigma$ is observed (new information arrives), she computes the probability of A as $\pi_1(A)$. She retains π_1 if $\pi_1(A) > \epsilon$ and proceeds with Bayesian updating on π_1 using A; otherwise, if $\pi_1(A) \leq \epsilon$, she rejects π_1, and compares the posterior probability of π_1 and π_2 conditional on A:

$$\frac{\pi_i(A)\rho(\pi_i)}{\pi_1(A)\rho(\pi_1) + \pi_2(A)\rho(\pi_2)}, \quad i \in \{1, 2\}.$$

She selects the prior from $\{\pi_1, \pi_2\}$ that maximizes the posterior probability above. For example, assume that π_2 is selected. Then, she proceeds with Bayesian updating on π_2 using A. She makes her decision as subjective expectation maximizer using the updated probability distribution. We can see that this updating rule is non-Bayesian if $\epsilon > 0$, and dynamic consistency is violated but only up to ϵ. If $\epsilon = 0$, then it is dynamic consistent, and the update rule is also well defined after zero-probability events.

In the main chapter of this thesis, we formulate signalling games nested by the updating rule of Ortoleva (2012) and define a new equilibrium concept, Hypothesis Testing Equilibrium (HTE). When $\epsilon = 0$, an HTE is a refinement of sequential Nash equilibria. In general signalling games, HTE survives the Intuitive Criterion. When $\epsilon > 0$, this is a game with non-Bayesian players. In signalling games, this implies the uninformed player is a non-Bayesian player, the informed player knows that the uninformed player is non-Baysian, and so forth. In this situation, an HTE can differ from a sequential Nash equilibrium. For a broad class of signalling games, we provide the existence and uniqueness theorems.

1.3 Risk, ambiguity, and limit pricing

As in the example above regarding an attempt to anticipate the weather in a city in another country, there are many situations for which we cannot assign a unique prior to the uncertainty. Consider further examples. We are unable to know the precise probability of an employee's performance in a job for which she has no experience before she starts to work in this position. We are also unable to assign a unique probability to the future returns or volatilities of an IPO. However, there also exist the situations in which we are able to precisely quantify the uncertainty in a situation by a probability; for example, when tossing a fair coin once, we believe that "heads" will appear with probability $\frac{1}{2}$; a pregnant woman will give birth to a girl with probability $\frac{1}{2}$; and there are other examples for which our subjective beliefs are determined by objective events.

Knight (1921) distinguished two types of uncertainty. Situations in which the uncertainty can be governed by a unique probability measure are called "*measurable uncertainty*" or "*risk*". In contrast, we use "*Knightian uncertainty*" or "*ambiguity*" to refer to situations in which individuals cannot or do not assign subjective probabilities to uncertain events. The Ellsberg Paradox (Ellsberg, 1963) shows that this distinction is behaviourally meaningful since people treat ambiguous bets differently from risky bets. Importantly, the lack of confidence reflected by choices in the Ellsberg Paradox cannot be rationalized by any probabilistic belief.

Many theoretical models of individuals' preferences in decisions under ambiguity have been proposed, including Maxmin Expected Utility (MEU) (Gilboa and Schmeidler, 1989), smooth ambiguity preference (Klibanoff et al., 2005), and variational representation of preferences (Maccheroni, et al., 2006). All of these utilities can capture ambiguity aversion, but they are rarely related to one another and are often expressed in drastically different formal languages (Epstein and Schneider, 2010). There are still difficulties in applying these theories to dynamic decision theory because most of the models do not satisfy the dynamic consistency property; see the arguments in Epstein et al, (2007), Eichberger et al, (2009) and Hanany, Klibanoff, and Mukerji (2015).

The issue of whether low prices can, in theory, deter entry is critical in competition policy. A vast literature studies the theory of limit pricing and can be dated back to Bain (1949). The seminal work by Milgrom and Roberts (1982) studies limit pricing theory in a two-period entry deterrence game with asymmetric information in which both the incumbent and the entrant have private information on their own costs but are uncertain about their opponent's costs. Other studies of the limit pricing theory for Oligopoly see Bagwell and Ramey (1991), limit pricing theory for Bertrand equilibrium by Chowdhury (2002), and signalling and learning in limit pricing game by Cooper et. al (1997). In the final part of this thesis, we introduce ambiguity into an asymmetric information framework using a simplified version of Milgrom and Roberts' limit pricing model. We discuss the impact of ambiguity in two cases of information asymmetry: in the first case, the incumbent is fully informed of the true state of the market; in contrast, the potential entrant is ambiguous about the state. In the second case, both players are uncertain about the state, but the incumbent is deciding under risk while the entrant is deciding under ambiguity. Because of information asymmetries, liit pricing appears under some conditions. Under ambiguity, the entrant behaves more cautiously than in the case under risk. Therefore, ambiguity decreases the probability of entry under certain conditions.

Chapter 2

Two-player Trading Games in A Stock Market with Incomplete Information on One-and-a-half Sides[1]

Abstract

Information asymmetries are well known in the financial markets. In this chapter, we formulate two-player trading games with incomplete information on one-and-a-half sides. Under this information structure, player 1 is informed of the true state of the nature (the value of the risky asset) but is uncertain about player 2's belief about the state because player 2 is privately informed through a message related to the state. In the N-stage repeated zero-sum game in a 2×2 framework, player 1 does not benefit from his informational advantage unless the message M known by player 2 and the true value L are independent. Therefore, the price dynamics are completely different from those in De Meyer's (2010) result by simply slightly disturbing the information of the uninformed player. In a non-zero-sum game in which player 2 is risk averse, player 1 can benefit from his informational advantage under more relaxed conditions on the joint distribution of L and M.

[1]This chapter is joint work with Bernard De Meyer.

2.1 Introduction

In financial markets, it is well known that the institutional investors have better access to market-relevant information than do private investors: the former are better skilled at analysing the flow of information and in some cases are even members of the board of directors of the firms of which they are trading shares. De Meyer (2010) analysed the effects of information asymmetries on price dynamics.

Specifically, the market is represented by a repeated exchange game between an informed sector (player 1) and an uninformed sector (player 2). In his model, both players are exchanging one risky asset (R) using the numéraire (N). An exogenous random event determines the liquidation value L of the risky asset. L is thus a random variable, and the players are assumed to have a common prior distribution μ on L. Player 1 is assumed to be initially informed about L while player 2 is not. During n consecutive periods, both players will exchange R for N using a "trading mechanism" T: at stage q, the two players select an action, i_q and j_q, respectively, and the resulting trade T_{i_q,j_q} is performed. The aim of player 1 is to maximize the expected liquidation value of his final portfolio. To represent a real-life exchange, the trading mechanism has to satisfy certain axioms. In this case, the mechanism is called a "natural exchange mechanism". If the mechanism is natural, then, asymptotically, as N goes to ∞, the price process will follow particular dynamics: It will be a Continuous Martingale of Maximal Variation, (i.e., a martingale P_t that can be written as $P_t = f(B_t, t)$, where B is a Brownian motion, f is increasing in B at a fixed t, and $t = \frac{q}{N}$). The asymptotic price process is independent of the "natural trading mechanism" used in each round.

De Meyer and Saley (2003) and De Meyer and Fournier (2015) are two additional models with ideas similar to that of De Meyer (2010). One criticism of these previous models is the information structure, whereby the informed player has full knowledge of the beliefs of the uninformed player. This chapter represents the first attempt to characterize a financial model out of this scenario. In our setting, player 1 is informed of the value L of the risky asset, and player 2 is informed of some private message M associated with L. Therefore, player 1 is uncertain about player 2's belief regarding the state of L. We have a game with incomplete information on one-and-a-half sides, as introduced by Sorin and Zamir (1985).

In section 2.2, we introduce a 2×2 benchmark model in which both L and M

take two possible values. If both players are risk neutral, the N-stage repeated zero-sum trading game between the two players proceeds as follows: In each stage, player 1 decides to sell or buy one share of the risky asset at a price that is simultaneously proposed by player 2. The choices are simultaneous, which at first glance, seems surprising. Indeed, one usually assumes that the trader will buy or sell after observing the market maker's price. However, following the argument in De Meyer and Fournier (2015), a sequential model in which player 1 reacts to the price posted by player 2 is equivalent to our model. In the zero-sum in the simultaneous game considered here, player 1's payoff is linear in player 2's choice of price, therefore, the equilibrium strategy of player 2 is a pure strategy. Player 2's move p_q thus can be completely anticipated by player 1 in period q. Player 1 would obtain no benefit from observing p_q before selecting u_q, namely deciding whether to buy or sell. In the non zero-sum game where player 2 is risk averse, due to Jensen's inequality, the equilibrium strategy of player 2 is also a pure strategy.

In section 2.3, we analyse two cases, and we obtain all of the results for the 2×2 benchmark model. The first result is quite intuitive: if the message M and the value L are independent, then player 2 clearly cannot induce any information on the state L; therefore, player 1 can ignore the message of player 2. We are back to the game with incomplete information on one side analysed by De Meyer (2010). We know that in this case the value v_N of the game is strictly positive except when L is deterministic. Player 1 can benefit from his private information on the value of the risky asset.

In the second case, M and L are not independent. Here, we surprisingly prove that the value v_N of the game is zero and there is no revelation by player 1. As a function of a probability vector defined on the unit simplex Δ^3, the value of the game is not continuous. It is zero everywhere except on the manifold where L and M are independent. For the N-stage repeated game, there is no optimal strategy for player 2, but we can identify the $\epsilon-$optimal strategy of player 2. In fact, in each stage, player 2 plays optimally, but just introducing $\epsilon-$perturbation such that the posteriors of L and M are not independent. Since ϵ is small enough, player 2 can guarantee a payoff of zero. Therefore, in the non-independent case, the value of the game v_N is zero, which means that player 1 does not benefit from his private information. We can see that by just slightly disturbing the information of the uninformed player, the posterior belief of player 2 is not known to player 1, the price dynamics are nearly constant. This result is completely different from De Meyer's (2010).

In the $N-$stage repeated zero-sum game, in each stage, player 2 makes a lottery involving two prices such that the conditional expectation of the price equals the value L. Playing in this way entails a negative price, which is not a natural interpretation in economics. However, we cannot impose a restriction requiring a positive price because this would violate the invariance axiom of the natural trading mechanism. That is, the value of the game must remain unchanged if one shifts the liquidation value L by a constant amount. This result might be improved in the event that player 2 is risk averse. Therefore, in section 2.4, we discuss a non-zero-sum one-shot game in which player 2 is risk averse. In this setting, we show that the value of the game is positive under more relaxed conditions on the joint distribution of M and L. We conjecture that in a repeated game, player 2 cannot guarantee the value of the game to be zero by slightly modifying his optimal strategy in each stage. Given the complexity of analysing the repeated game and characterizing the price dynamics in this setting, we leave such work to further research.

This chapter proceeds as follows. Section 2.2 introduces a 2×2 benchmark model, in which the value L of the risky asset and the message M only take two possible values. In section 2.3, we discuss the value of the one-stage game and the $N-$stage repeated zero-sum game. Section 2.4 discusses one-stage games with more general random variables representing M and L and a one-stage 2×2 game in which player 2 is risk averse. Section 2.5 provides the conclusion.

2.2 Description of the 2×2 zero-sum game

In the game we consider, player 1 initially receives some private information about the risky asset and player 2 receives some message M associated with this information. This information will be publicly disclosed at a future date. At that date, the value L of the risky asset will depend solely on the information initially received by player 1. In the 2×2 framework, $L \in \{0, 1\}$, and $M \in \{a, b\}$. We denote $\text{Prob}(L = l \cap M = m) = \mu_{lm}$, $\text{Prob}(M = m) = \mu_m$, for $m \in \{a, b\}$, $\text{Prob}(L = l) = \mu_l$, for $l \in \{0, 1\}$, and $\text{Prob}(L = l|M = m) = \mu_{l|m}$, $\mu = (\mu_{0|a}, \mu_{0|b}, \mu_a)$. We use the method of Sorin and Zamir (1985) to define $G_N(\mu), N \geq 1$ as the $N-$stage repeated zero-sum game in which the two players exchange the risky asset for a numéraire in each stage. For each $\mu_{0|a}$, $\mu_{0|b}$, $\mu_a \in [0,1]^3$, the game is proceeded as follows:

Step 00. Nature selects $m \in \{a, b\}$ with $\text{Prob}(m = a) = \mu_a$ and this choice is told to player 2 only.

Step 0. Nature selects $l \in \{0, 1\}$ with $\text{Prob}(l = 0) = \mu_{0|m}$ and this choice is told to player 1 only.

Step 1. Player 1 decides to buy ($u_1 = 1$) or sell ($u_1 = -1$) one unit of the risky asset. Then the action set of player 1 is $u_q \in \{1, -1\}$. Simultaneously, player 2 selects a price $p_1 \in \mathbb{R}$ for the transaction. Then this choice (u_1, p_1) is announced to both players and the trade performed with this price.

Step $q \in \{2, ..., N\}$. In addition to their private information, both players knowing the history of actions prior to round q, that is $h_q = (u_1, p_1; ...; u_q, p_q)$, select some move and this pair (u_q, p_q) is announced to both players.

In this game, observing m, player 2 has a prior that $L = 0$ with probability $\mu_{0|m}$. However, because player 1 can not observe m, player 1 does not observe the prior of player 2 on the state L and is just aware of the probability distribution of player 2's prior. Let h_q denote the history of actions prior to round q, that is $h_q = (u_1, p_1; ...; u_q, p_q)$ with $h_0 = \emptyset$, and $\mathbb{H}_q$ denote the set of all possible histories until round q. A behavioural strategy of player 1 in this game is $\sigma = (\sigma_1, ...\sigma_N)$ with $\sigma_q : (L, h_{q-1}) \to \Delta(\{1, -1\})$. A behavioural strategy of player 2 is $\tau = (\tau_1, ...\tau_N)$ with $\tau_q : (M, h_{q-1}) \to \Delta(\mathbb{R})$. Then, a triple (μ, σ, τ) induces a unique probability distribution on $(L, M, \mathbb{H}_N)$. When X is a random variable, we denote $E_{\mu,\sigma,\tau}[X]$ as the expectation of X with respect to this probability. We assume that both players are risk neutral; then, the payoff of player 1 in this game is the expected value of his final portfolio:

$$g_N(\mu, \sigma, \tau) = E_{\mu,\sigma,\tau}\left[\sum_{q=1}^{N} u_q(L - p_q)\right]. \tag{2.1}$$

The payoff of player 2 is $-g_N(\mu, \sigma.\tau)$. To simplify notation, we denote the expectation $E_{\mu,\sigma,\tau}$ by E.

2.3 Main Results

2.3.1 One-shot game

To characterize the value v_N of the $N-$stage repeated zero-sum game, we first analyse the one-shot game. In the one-shot game, $q = 1$, the payoff function (2.1) is simply the following:

$$\begin{aligned} g_1(\mu, \sigma.\tau) &= E[u(L-p)] \\ &= E_\mu[(2\sigma(L)-1)(L - E_{\tau_M}(p))], \end{aligned}$$

where $\sigma(l)$ is the probability with which player 1 chooses $u = 1$ if $L = l$, for all $l \in \{0, 1\}$. This equation shows that the pure strategy of player 2, $p = (p_a, p_b)$, where $p_m = E_{\tau_m}(p)$, $\forall m \in \{a, b\}$, yields the same payoff as the mixed strategy τ given player 1's strategy σ. Player 1's payoff function is linear in player 2's choice of price, and therefore player 1 cares about any random price choice by player 2 only through the mean price. Therefore, player 2 does not need to play mixed strategies. Player 2 plays p_a (resp. p_b) with probability 1 if he is type $M = a$ (resp. $M = b$). Then we can rewrite the payoff function as follows:

$$\begin{aligned} g_1(\mu, \sigma.\tau) &= E_\mu[(2\sigma(L)-1)(L - E_{\tau_M}(p))] \\ &= EE[(2\sigma(L)-1)(L - p_M)|L] \\ &= E[(2\sigma(L)-1)(L - E[p_M|L])]. \end{aligned}$$

This payoff function involves the expectation of p_M conditional on L, and thus, we can analyse the equilibrium in two cases:

Case (i): *L and M are independent.*

If L and M are independent, then

$$\begin{aligned} g_1(\mu, \sigma.\tau) &= E[(2\sigma(L)-1)(L - E[p_M|L])] \\ &= E[(2\sigma(L)-1)(L - E[p_M]) \\ &= \mu_0 P[1 - 2\sigma(0)] + \mu_1(P-1)[1 - 2\sigma(1)], \end{aligned}$$

where $P = \mu_a p_a + \mu_b p_b$. The best response, $\sigma^* = (\sigma(0)^*, \sigma(1)^*)$, of player 1 to a given strategy P^* of player 2 is

$$(\sigma(0)^*, \sigma(1)^*) = \begin{cases} (1,1) & \text{if } P^* < 0, \\ (0,1) & \text{if } 0 \leq P^* < 1, \\ (0,0) & \text{if } P^* \geq 1. \end{cases}$$

That is,

$$\sigma(l)^* = \mathbb{1}_{\{P^* < l\}} + s\mathbb{1}_{\{P^* = l\}} \quad \forall s \in [0,1], \quad \forall l \in \{0,1\}. \tag{2.2}$$

Given the best response of player 1, the payoff function becomes

$$\begin{aligned} g_1(\mu, \sigma^*, P) &= \sum_{l \in \{0,1\}} \mu_l (P - l)[1 - 2(\mathbb{1}_{\{P^* < l\}} + s\mathbb{1}_{\{P^* = l\}})] \\ &= \sum_{l \in \{0,1\}} \mu_l [1 - 2(\mathbb{1}_{\{P^* < l\}} + s\mathbb{1}_{\{P^* = l\}})] P - \sum_{l \in \{0,1\}} L\mu_l [1 - 2(\mathbb{1}_{\{P^* < l\}} + s\mathbb{1}_{\{P^* = l\}})] \\ &= \sum_{l \in \{0,1\}} \mu_l [1 - 2(\mathbb{1}_{\{P^* < l\}} + s\mathbb{1}_{\{P^* = l\}})] P + \mu_1 [2(\mathbb{1}_{\{P^* < 1\}} + s\mathbb{1}_{\{P^* = 1\}}) - 1]. \end{aligned}$$

In equilibrium, we have the following condition:

$$\begin{aligned} &\sum_{l \in \{0,1\}} \mu_l [1 - 2(\mathbb{1}_{\{P^* < l\}} + s\mathbb{1}_{\{P^* = l\}})] = 0, \\ \Rightarrow &E[\mathbb{1}_{\{P^* < L\}}] + sE[\mathbb{1}_{\{P^* = L\}})] = \frac{1}{2}. \end{aligned} \tag{2.3}$$

Equation (2.3) implies that the equilibrium price $P^* \leq median(L)$. If L is not degenerated to 1, then $P^* < 1$. Otherwise, if $P^* = 1$, we can always choose $s \in (\frac{1}{2}, 1]$ such that $2s\mathbb{1}_{\{P^*=1\}} - 1 > 0$. Immediately, we can deduce that the value of the game is positive, that is:

$$g_1(\sigma^*, P^*) = \mu_1 [2(\mathbb{1}_{\{P^* < 1\}} + s\mathbb{1}_{\{P^* = 1\}}) - 1] > 0. \tag{2.4}$$

Now, we can summarize this result in the following proposition:

Proposition 2.1. *In the one-shot zero-sum game in the* 2×2 *framework, if* L *and* M *are independent, then in equilibrium, the optimal strategy of player 1 is given in equation* (2.2)*, the optimal strategy of player 2 is given in equation* (2.3)*, and the value of the game is positive and given in* (2.4)*.*

Case (ii): *L and M are not independent.*

We can easily show that player 1 can guarantee a payoff of zero by choosing $\sigma(l) = \frac{1}{2}$, for all $l \in \{0, 1\}$. However, as

$$\begin{aligned} g_1(\mu, \sigma.\tau) &= E[(2\sigma(L) - 1)(L - E[p_M|L])] \\ &\leq E[|L - E[p_M|L]|], \end{aligned}$$

if player 2 can choose $p^* = (p_a^*, p_b^*)$, such that

$$E[p_M^*|L] = L,$$

then player 2 also can guarantee a payoff of zero. In fact, if L and M are not independent, then $\det(\mu) \neq 0$, and there exists an equilibrium price $p^* = (p_a^*, p_b^*)$ satisfying the following conditions:

$$\begin{aligned} \mu_{0a} p_a^* + \mu_{0b} p_b^* &= 0 \\ \mu_{1a} p_a^* + \mu_{1b} p_b^* &= \mu_1. \end{aligned}$$

Solving these equations, we obtain

$$p_a^* = -\frac{\mu_1 \mu_{0b}}{det(\mu)}, \quad \text{and} \quad p_b^* = \frac{\mu_1 \mu_{0a}}{det(\mu)}. \tag{2.5}$$

In this case, the value of the game is zero. The informed player 1 does not benefit from his private information on the value of the asset.

Proposition 2.2. *In the one-shot zero-sum game in the 2×2 framework, if L and M are not independent, then given a joint distribution μ, there exists an equilibrium strategy $p_\mu = (p_\mu^*(a), p_\mu^*(b))$ of player 2 given by equation* (2.5)*, and the value of the game is zero.*

2.3.2 $N-$stage repeated zero-sum game

With the preliminary results from the one-shot game, we proceed to analyse the $N-$stage repeated game. In the one-shot game, the value of the game can be explicitly written in the form

$$v_1 = f(\mu_{0a}, \mu_{0b}, \mu_{1a}, \mu_{1b}),$$

where

$$\sum_{l\in\{0,1\},m\in\{a,b\}} \mu_{lm} = 1, \quad \mu = (\mu_{0a}, \mu_{0b}, \mu_{1a}, \mu_{1b});$$

then, as a function of a probability vector μ defined in a unit simplex Δ^3, the value $v_1(\mu)$ of the game is not continuous. It is zero everywhere except on the manifold where $\det(\mu) = 0$. To obtain this result for the $N-$stage repeated zero-sum game, we again consider two cases: (i). L and M are independent, and (ii). L and M are not independent.

Case (i). *L and M are independent.*

We assume that $\mu^1(h_0) = \mu$; then, at each stage $q \in \{1, ..., N-1\}$, a triple (μ, σ, τ) induces a unique posterior joint distribution of (L, M). That is, at the end of stage q, a history $h_q = (u_1, p_1; ...; u_q, p_q)$ is observed; then, in stage $q+1$, the joint posterior probability distribution of $(L = l, M = m)$ given this history can be computed recursively as follows:

$$\begin{aligned}
\mu_{lm}^{q+1}(h_q) &= \mathbb{P}(L = l, M = m | u_q, p_q, h_{q-1}) \\
&= \frac{\mathbb{P}(L = l, M = m, u_q, p_q | h_{q-1})\mathbb{P}(h_{q-1})}{\mathbb{P}(h_q)} \\
&= \frac{\mathbb{P}(h_{q-1})}{\mathbb{P}(h_q)} \mathbb{P}(u_q | L = l, h_{q-1})\mathbb{P}(p_q | M = m, h_{q-1})\mathbb{P}(L = l, M = m | h_{q-1}) \\
&= \frac{\mathbb{P}(h_{q-1})}{\mathbb{P}(h_q)} \sigma_q(l, h_{q-1})(u_q) \cdot \tau_q(m, h_{q-1})(p_q) \cdot \mu_{lm}^q(h_{q-1}),
\end{aligned}$$

for all $l \in \{0, 1\}, m \in \{a, b\}$. The third equality holds because u_q and p_q are selected simultaneously by player 1 and player 2 in stage q. In the last equation, the notations denote that, in stage q, conditional on the history path h_{q-1}, type l of player 1 chooses u_q with probability $\sigma_q(l, h_{q-1})(u_q)$, and type m of player 2 chooses price p_q with probability $\tau_q(m, h_{q-1})(p_q)$. Then, we have the following property:

Proposition 2.3. *Let $\mu^1(h_0) = \mu$; at the end of stage q, where $q \in \{1, ...N-1\}$, a history $h_q = (u_1, p_1; ...; u_q, p_q)$ is observed. Then the posterior of the joint* *distribution $\mu^{q+1}(h_q)$ can be computed recursively as follows:*

$$\mu^{q+1}(h_q) = \frac{\mathbb{P}(h_{q-1})}{\mathbb{P}(h_q)} \begin{pmatrix} \sigma^q(0,h_{q-1})(u_q) & 0 \\ 0 & \sigma^q(1,h_{q-1})(u_q) \end{pmatrix} \mu^q \begin{pmatrix} \tau^q(a,h_{q-1})(p_q) & 0 \\ 0 & \tau^q(b,h_{q-1})(p_q) \end{pmatrix} \tag{2.6}$$

Proposition 2.4. *In this 2×2 framework, if L and M are independent, then for all $q \in \{1, ...N\}$, the posteriors are also independent.*

Proof. From proposition 2.3, we know that for all $q \in \{1, ..., N-1\}$,

$$\det(\mu^{q+1}) = \frac{[\mathbb{P}(h_{q-1})]^2 \det(\mu^q)}{[\mathbb{P}(h_q)]^2} \prod_{l \in \{0,1\}} \sigma^q(l, h_{q-1})(u_q) \prod_{m \in \{a,b\}} \tau^q(m, h_{q-1})(p_q).$$

Since both L and M only take two values, the independence assumption between L and M is equivalent to $\det(\mu^1) = 0$. The equation above implies that $\det(\mu^q) = 0$, for all $q \in \{1, ..., N\}$. That is, all the posteriors of L and M are independent. ■

In this case, the N-stage repeated zero-sum game with incomplete information on one-and-a-half sides coincides with the repeated zero-sum game with incomplete information on one side discussed in De Meyer (2010).

Case (ii). *L and M are not independent.*

First, we claim that player 1 can guarantee a zero payoff.

Lemma 2.1. *In the N-stage repeated zero-sum game in the 2×2 framework, if L and M are not independent, then $\underline{v}_N \geq 0$.*

Proof. Consider the following strategy of player 1:

$$\tilde{\sigma} = (\tilde{\sigma}_1, ..., \tilde{\sigma}_N), \quad \text{where} \quad \tilde{\sigma}_q(l) = \frac{1}{2}, \quad \forall l \in \{0,1\}, \quad \forall q \in \{1, ..., N\}.$$

Using $\tilde{\sigma}$, the payoff $g_1(\mu, \tilde{\sigma}, \tau)$ of player 1 is zero regardless of the strategy of player 2. Therefore,

$$\underline{v}_N = \max_{\sigma} \min_{\tau} g_1(\sigma, \tau) \geq \min_{\tau} g_1(\tilde{\sigma}, \tau) = 0.$$

■

Second, let us prove that player 2 can guarantee a payoff of zero, that is $\overline{v}_N \leq 0$. To obtain this result, we need the following preliminary knowledge. From the posterior joint distribution of (L, M) given by equation (2.6), at stage $q+1$, $q \in \{1, ..., N-1\}$, we compute the conditional distribution of M given L and the past path h_q as follows:

$$\begin{aligned}
\mathbb{P}(M = m|L = l, h_q) &= \frac{\mathbb{P}(M = m, L = l, h_q)}{\sum\limits_{m' \in \{a,b\}} \mathbb{P}(M = m', L = l, h_q)} \\
&= \frac{\mathbb{P}(h_q|M = m, L = l)\mu_{lm}}{\sum\limits_{m' \in \{a,b\}} \mathbb{P}(h_q|M = m', L = l)\mu_{lm'}} \\
&= \frac{\prod\limits_{k=1}^{q} \sigma^k(l, h_{k-1})(u_k) \prod\limits_{k=1}^{q} \tau^k(m, h_{k-1})(p_k)\mu_{lm}}{\sum\limits_{m' \in \{a,b\}} \prod\limits_{k=1}^{q} \sigma^k(l, h_{k-1})(u_k) \prod\limits_{k=1}^{q} \tau^k(m', h_{k-1})(p_k)\mu_{lm'}} \\
&= \frac{\prod\limits_{k=1}^{q} \tau^k(m, h_{k-1})(p_k)\mu_{lm}}{\sum\limits_{m' \in \{a,b\}} \prod\limits_{k=1}^{q} \tau^k(m' h_{k-1})(p_k)\mu_{lm'}}.
\end{aligned} \tag{2.7}$$

Again, the third equality holds because, in each stage, player 1 and player 2 make moves simultaneously. This equation means that the posterior of M conditional on (L, h_q) does not depend on the behavioural strategy σ of player 1. This implies that in each stage, player 1's past strategies do not influence his current strategy.

As in the one shot-game, we analogously rewrite the payoff of player 1 as follows:

$$\begin{aligned}
g_N(\mu, \sigma, \tau) &= E_{\mu,\sigma,\tau}\left[\sum_{q=1}^{N} u_q(L - p_q)\right] \\
&= \sum_{q=1}^{N} E\left[E\left[u_q(L - p_q)|L, h_{q-1}\right]\right] \\
&= \sum_{q=1}^{N} E[E[u_q|L, h_{q-1}](L - E[p_q|L, h_{q-1}])] \\
&= \sum_{q=1}^{N} E[(2\sigma_q(L, h_{q-1})(1) - 1)(L - E[p_q|L, h_{q-1}])] \\
&\leq \sum_{q=1}^{N} E|L - E[p_q|L, h_{q-1}]|.
\end{aligned}$$

If there exists price $(p_q^*(a), p_q^*(b))$ for each $q \in \{1, ..., N\}$, such that

$$E[p_q^*|L, h_{q-1}] = L,$$

that is, for each q, $(p_q^*(a), p_q^*(b))$ solves the equations

$$\begin{aligned} p_q^*(a)\mathbb{P}(M = a|L = 0, h_{q-1}) + p_q^*(b)\mathbb{P}(M = b|L = 0, h_{q-1}) = 0, \\ p_q^*(a)\mathbb{P}(M = a|L = 1, h_{q-1}) + p_q^*(b)\mathbb{P}(M = b|L = 1, h_{q-1}) = 1, \end{aligned} \tag{2.8}$$

then the payoff of player 1 is equal to or less than zero. Since we have shown that the posterior of M conditional on (L, h_{q-1}) does not depend on the behavioural strategy σ of player 1, the solution $p_q^*(a)$ and $p_q^*(b)$ of (2.8) does not depend on σ. Now we are prepared to prove the following lemma:

Lemma 2.2. *In the* 2×2 *N-stage repeated zero-sum game, if* L *and* M *are not independent, then* $\overline{v}_N \leq 0$.

Proof. To prove this lemma, recall that in the one-shot 2×2 zero-sum game, player 2 plays a pure strategy p_μ^* as given in proposition 2.2. For a given μ, let us define a perturbed strategy $\gamma^{\mu,\eta} = (\gamma_a^{\mu,\eta}, \gamma_b^{\mu,\eta})$ of player 2 as follows: For a given $\eta > 0$, for some ϵ strictly positive to be defined later, this strategy consists in playing $p_\mu^*(a)$ with probability $1 - \epsilon$ and $p_\mu^*(b)$ with probability ϵ when he is type a; otherwise, it consists in playing $p_\mu^*(b)$ with probability $1 - \epsilon$ and $p_\mu^*(a)$ with probability ϵ when he is type b, that is

$$\begin{aligned} \gamma_a^{\mu,\eta}(p_\mu^*(a)) &= 1 - \epsilon, \qquad \gamma_a^{\mu,\eta}(p_\mu^*(b)) = \epsilon, \\ \gamma_b^{\mu,\eta}(p_\mu^*(b)) &= 1 - \epsilon, \qquad \gamma_b^{\mu,\eta}(p_\mu^*(a)) = \epsilon. \end{aligned}$$

In the one-shot game, this strategy guarantees that player 1's payoff is very small, regardless of his strategy σ, and we obtain

$$\begin{aligned} g_1(\mu, \sigma, \gamma^{\mu,\eta}) &= g_1(\sigma, p_\mu^*) + \epsilon K(\mu), \\ &= \epsilon K(\mu), \end{aligned}$$

where

$$K(\mu) = \frac{\mu_0 \mu_1}{\det(\mu)} \sum_{l \in \{0,1\}} [1 - 2\sigma(l)][\mu_{la} - \mu_{lb}].$$

Note that $K(\mu) \leq \left|\frac{2\mu_0\mu_1}{\det(\mu)}\right|$. We then define $\epsilon = \frac{\eta}{\left|\frac{2\mu_0\mu_1}{\det(\mu)}\right|}$. Therefore, this perturbed strategy $\gamma^{\mu,\eta}$ of player 2 can guarantee that $g_1(\mu, \sigma, \gamma^{\mu,\eta}) \leq \eta$ regardless of the strategy of player 1.

With this definition of strategy $\gamma^{\mu,\eta}$ in mind, we are prepared to prove this lemma. Since L and M are not independent, $\det(\mu) \neq 0$. Let us consider the following strategy τ of player 2: In stage 1, he plays $\gamma^{\mu,\eta}$, and for the following stages, $q \in \{2, ..., N\}$, he first computes the conditional posterior $\mu^q(M|L, h_{q-1})$ of M conditional on (L, h_{q-1}) as given in (2.7), which does not depend on the strategy of player 1, as we discussed above. When $\det(\mu^q) \neq 0$, he plays $\gamma^{\mu^q,\eta}$. However, if $\det(\mu^q) = 0$, let us consider the first-stage $q_0 \geq 2$ such that at this stage,

$$\det(\mu^{q_0}) = \frac{[\mathbb{P}(h_{q_0-2})]^2 \det(\mu^{q_0-1})}{[\mathbb{P}(h_{q_0-1})]^2} \prod_{l\in\{0,1\}} \sigma^{q_0}(l, h_{q_0-1})(u_{q_0}) \prod_{m\in\{a,b\}} \tau^{q_0}(m, h_{q_0-1})(p_{q_0}) = 0.$$

Since in stage q_0, player 2 plays $\tau_{q_0} \neq 0$, the only possibility is that

$$\sigma^q(l, h_{q_0-1})(u_{q_0}) = 0, \quad \text{for some} \quad l \in \{0, 1\}.$$

In other words, in this stage, observing u_{q_0}, player 2 can deduce the state of L. Since he knows the value of L, he plays L until the end of the game. Clearly, this strategy yields the following payoff to player 1:

$$g_N(\mu, \sigma, \gamma^\eta) = \sum_{q=1}^{N} g_1(\mu^q, \sigma^q, \tau) \leq N\eta$$

Since η is arbitrary small, there is no strategy of player 1 that can guarantee a strictly positive payoff. In each stage $q \geq 2$, player 2 simply introduces an $\epsilon-$perturbation of his optimal strategy $p_q^*(a)$ and $p_q^*(b)$, which is the solution of Eq. (2.8), such that M and L are not independent, and the choices of $p_q^*(a)$ and $p_q^*(b)$ for all $q \in \{2, ..., N\}$ do not depend on the strategy σ of player 1, this strategy of player 2 can guarantee a payoff of zero. ■

Combining Lemma 2.1 and Lemma 2.2 implies the following theorem:

Theorem 2.1. *In the 2×2 N-stage repeated zero-sum game, if L and M are not independent, then the value v_N of the game is zero.*

2.4 Discussion

2.4.1 A general one-shot zero-sum game

In the previous sections, we analysed a simple 2×2 benchmark model, and we showed that, in the N-period repeated zero-sum trading game, the more informed player cannot exploit his private information unless the message M and the value L are independent. A natural extension of this model is to examine the case in which both L and M can take more possible values on $\mathbb{R}$. With the same argument as for the one-shot game in the 2×2 framework, when L and M are independent, the value v_1 of the game is still positive in the general framework. However, for the case in which L and M are not independent, the condition,

$$L = E[p^*(M)|L)] \qquad a.s. \tag{2.9}$$

to guarantee a zero payoff for player 2 may not always hold. If there exists $p^*(M)$ such that condition (2.9) holds true, then the value v_1 of the game is zero. Otherwise, v_1 is strictly positive. Therefore condition (2.9) is critical to guarantee a zero payoff.

Example 1. Assume that M is a mean-preserving spread of L, i.e., $M = L + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2)$, ϵ and L are independent. It is easy to show that the price $p^*(M) = M$ can satisfy condition (2.9).

Example 2. Assume the joint density function of (L, M) is

$$f(l, m) = \begin{cases} 2, & \text{if } 0 \leq l \leq 1, l \leq m \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

By solving the condition (2.9) as

$$\begin{aligned} &\int_l^1 \frac{p(m)}{1-l} dm = l, \\ \Rightarrow &\int_l^1 p(m) dm = l(1-l). \end{aligned} \tag{2.10}$$

we can obtain the equilibrium price $p^*(M) = 2M - 1$. In these two examples, L and M are not independent, and the value of the game is zero in both examples. However, condition (2.9) does not hold true under the following conditions:

(i). If both L and M are continuous variables, then the value of M is known whenever L is known but not vice versa, for example $M = L^2$.
(ii). If both L and M are discrete variables with a joint distribution μ, then $\text{rank}(\mu) < \#(L)$, where $\#(L)$ is the cardinality of the set of alternatives of L.
Example 3. Let us consider the following example in which the joint distribution μ of L and M is as given in table 2.2. In this example, there clearly is no $p = (p_a, p_b)$

Table 2.1: Joint distribution μ of (L, M)

	M=a	M= b
$L = 0$	$\frac{1}{6}$	$\frac{1}{6}$
$L = 1$	$\frac{1}{6}$	$\frac{1}{6}$
$L = 2$	0	$\frac{1}{3}$

such that

$$E(p(M|L=0)) = \frac{p_a + p_b}{2} = 0,$$
$$E(p(M|L=1)) = \frac{p_a + p_b}{2} = 1,$$
$$E(p(M|L=2)) = 2 - p_b = 2.$$

We can compute that the optimal strategy of player 1 is

$$\sigma^* = (\sigma^*(0), \sigma^*(1), \sigma^*(2)) = (0, 1, \frac{1}{2}),$$

and the optimal strategy of player 2 is

$$p^* = (p_a^*, p_b^*) = (-2, 2).$$

The value of the game is $v = \frac{1}{3} > 0$.

To summarize the argument above, we conclude that in the general one-shot zero-sum game, if L and M are independent, then the value of the game $v_1 > 0$. If L and M are not independent and there exists p^* satisfying condition (2.9), then the value of the game $v_1 = 0$; otherwise, $v_1 > 0$.

2.4.2 One-shot non zero-sum game

The value of the game is discontinuous in the models discussed in the 2×2 framework under the assumption that player 2 is risk neutral. Under this assumption, player 2 is indifferent between zero and a lottery with zero expectation. Therefore, he can bear very high risk to drag the value down to zero. In the $N-$stage repeated zero-sum game, in each stage, player 2 constructs a lottery between two prices such that the conditional expectation of the price equals the value of L. Playing in this way allows a negative price, which is not a natural interpretation in economics. However, we cannot impose the restriction of a positive price because doing so would violate the invariance axiom of the natural trading mechanism. That is, the value of the game must remain unchanged if one shifts the liquidation value L by a constant amount. This result might be improved upon if player 2 is risk averse. To obtain an intuitive explanation, we consider a non-zero-sum one shot game in which player 2 is risk averse.

In this setting, player 2's risk-aversion preference is represented by a von Neumann-Morgenstern utility function, $E[H(x)]$, where H is concave and increasing. We denote the expected payoff of player i as $g^i(\mu, \sigma, \tau)$, $i \in \{1, 2\}$. Since player 1 is risk neutral, $g^1(\mu, \sigma, \tau)$ has the same expression as we discussed in the previous sections. The expected payoff of player 2, $g^2(\mu, \sigma, \tau)$, becomes

$$\begin{aligned} g^2(\mu, \sigma, \tau) &= E[H(u(L-p))] \\ &= E[\sigma(L)E_{\tau_m}[H(p-L)] + (1-\sigma(L))E_{\tau_m}[H(L-p)]]. \end{aligned}$$

Since H is a concave function, Jensen's inequality implies that

$$g^2(\mu, \sigma, \tau) \leq E[\sigma(L)H(E_{\tau_m}[p] - L) + (1-\sigma(L))H(L - E_{\tau_m}[p])].$$

Therefore, the pure strategy $p = (p_a, p_b)$ yields a payoff for player 2 that is at least as the same as a mixed strategy, where $p_a = E_{\tau_a}(p)$ and $p_b = E_{\tau_b}(p)$. Therefore, the optimal strategies of player 2 are pure strategies, and the payoff given a pure strategy $p = (p_a, p_b)$ is

$$g^2_\mu(\sigma, p) = E_\mu[\sigma(L)H(E[p(M)] - L) + (1-\sigma(L))H(L - E[p(M)])]. \quad (2.11)$$

To analyse the equilibrium of the game, we assume that

$$H(x) = x - e^{-x}, \quad x \in \mathbb{R}; \tag{2.12}$$

then,

$$\begin{aligned} H'(x) &= 1 + e^{-x} > 0, \quad \lim_{x \to +\infty} H(x) = +\infty, \\ H''(x) &= -e^{-x} < 0, \quad \lim_{x \to -\infty} H(x) = -\infty. \end{aligned}$$

Without loss of generality, we assume that $\mu_0 = \mu_1 = \frac{1}{2}$ in the $2\times$ 2 framework. First, we attempt to identify the conditions such that, in equilibrium, player 1 completely reveals his information; that is, he sells the risky asset when $L = 0$ and buys when $L = 1$. In such an equilibrium, the value v_1 of the game is positive.

Definition 2.1. *A Nash equilibrium* (σ^*, p^*) *is called a revealing equilibrium if* $(\sigma(0)^*, \sigma(1)^*) = (0, 1)$.

In this case, the expected payoff of player 2 is

$$g^2(\mu, (0,1), (p_a, p_b)) = \mu_{0a} H(-p_a) + \mu_{0b} H(-p_b) + \mu_{1a} H(p_a - 1) + \mu_{1b} H(p_b - 1).$$

The strategy of player 2, (p_a^*, p_b^*), maximizing $g^2(\mu, (0,1), (p_a, p_b))$ solves the following first-order conditions:

$$\begin{aligned} \frac{\partial g_\mu^2}{\partial p_a} &= -\mu_{0a} H'(-p_a) + \mu_{1a} H'(p_a - 1) = 0, \\ \frac{\partial g_\mu^2}{\partial p_b} &= -\mu_{0b} H'(-p_b) + \mu_{1b} H'(p_b - 1) = 0. \end{aligned} \tag{2.13}$$

By substituting $H'(x) = 1 + e^{-x}$ into Eq. (2.13), we obtain $p_a^* = p_a^*(\pi_0, \pi_1), p_b^* = p_b^*(\pi_0, \pi_1)$, where $\pi_0 = \mathbb{P}(M = a | L = 0)$, and $\pi_1 = \mathbb{P}(M = a | L = 1)$. In a revealing equilibrium, $(\sigma(0)^*, \sigma(1)^*) = (0, 1)$ is also the best reply by player 1 to the strategy (p_a^*, p_b^*) of player 2. The payoff function of player 1 does not change; therefore, if $(\sigma(0)^*, \sigma(1)^*) = (0, 1)$ is the best response of player 1 to a given strategy $p^* = (p_a^*, p_b^*)$ of player 2, and the following conditions hold:

$$\begin{aligned} E[p^*(M) | L = 0] &= \pi_0 p_a^* + (1 - \pi_0) p_b^* > 0, \\ E[p^*(M) | L = 1] &= \pi_1 p_a^* + (1 - \pi_1) p_b^* < 1. \end{aligned} \tag{2.14}$$

Jointly solving inequalities (2.14) and (2.13),we find the area on the space of (π_0, π_1) where the revealing equilibrium $((\sigma(0)^*, \sigma(1)^*), (p_a^*, p_b^*)) = ((0,1), (p_a^*(\pi_0, \pi_1), p_b^*(\pi_0, \pi_1)))$ exists (the middle-area in fig 1. (a)). It is easy to verify that the expected payoff of player 1 is strictly positive in a revealing equilibrium.

Definition 2.2. *A Nash equilibrium (σ^*, p^*) is called an equalizing equilibrium if $\sigma(l)^* = s \in [0,1]$ for $l \in \{0,1\}$.*

In an equalizing equilibrium, both types of player 1 are indifferent between revealing and not revealing information. In this case, the optimal strategy of player 1 is any value between 0 and 1. An equalizing equilibrium, $((\sigma(0)^*, \sigma(1)^*), (p_a^*, p_b^*))$, satisfies the following conditions:

$$
\begin{aligned}
&g_\mu^2((\sigma(0)^*, \sigma(1)^*), (p_a^*, p_b^*)) \geq g_\mu^2((\sigma(0)^*, \sigma(1)^*), (p_a, p_b)), \quad \forall p_a, p_b \in \mathbb{R}, \\
&E[p^*(M)|L] = L.
\end{aligned}
$$

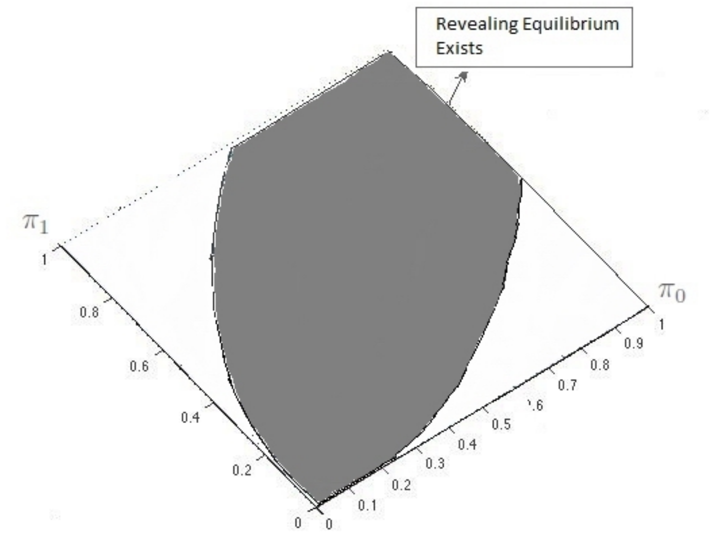
Then, $(\sigma(0)^*, \sigma(1)^*, (p_a^*, p_b^*))$ solves the following system of equations:

$$
\begin{aligned}
&\pi_0\sigma(0)^*(1+e^{-p_a^*}) - \pi_0(1-\sigma(0)^*)(1+e^{p_a^*}) + \pi_1\sigma(1)^*(1+e^{1-p_a^*}) - \\
&\qquad \pi_1(1-\sigma(1)^*)(1+e^{p_a^*-1}) = 0, \\
&(1-\pi_0)\sigma(0)^*(1+e^{-p_b^*}) - (1-\pi_0)(1-\sigma(0)^*)(1+e^{p_b^*}) + \\
&\qquad (1-\pi_1)\sigma(1)^*(1+e^{1-p_b^*}) - (1-\pi_1)(1-\sigma(1)^*)(1+e^{p_b^*-1}) = 0, \\
&\pi_0 p_a^* + (1-\pi_0)p_b^* = 0, \\
&\pi_1 p_a^* + (1-\pi_1)p_b^* = 1.
\end{aligned}
\tag{2.15}
$$

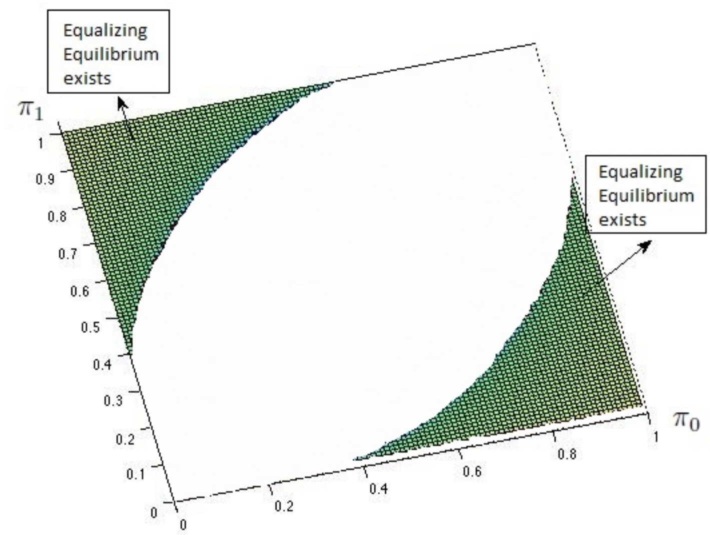
The first two equations are the first-order conditions for maximizing player 2's expected payoff. The last two equations satisfy condition (2.9). Solving this system with constraints $\sigma(0)^* \in [0,1]$, and $\sigma(1)^* \in [0,1]$, we can find the area on the space of (π_0, π_1) where the equalizing equilibrium exists (the two edge areas in fig 1. (b)). The expected payoff of player 1 is zero in the equalizing equilibrium. From the picture, we can see that player 1 can exploit his private information under a more relaxed condition on the joint distribution relative to the risk-neutral environment. In that case, the value of the game is positive only if $\pi_0 = \pi_1$. Player 2 cannot guarantee a zero payoff to player 1 by simply slightly perturbing his optimal choice, as is true in risk-neutral case. Therefore, we cannot use the same argument to characterize the price dynamics in an $N-$stage repeated game. The price dynamics may be different.

2.5 Conclusion

In this chapter, we present two-player trading games with incomplete information on one-and-a-half sides. For risk-neutral players, the more informed player can only exploit his informational advantage when the value of the risky asset and the message known by his trading partner are independent. By contrast, he can benefit from his informational advantage under more relaxed conditions on the joint distribution when his trading partner is risk averse. However, we obtain these conclusions for finite-stage trading games based on a special trading mechanism; that is, both players are forced to trade by buying or selling one share of the risky asset. Further studies of trading behaviour and asymptotic price dynamics in a general natural trading mechanism are encouraged.


Revealing Equilibrium
Exists
π_1
π_0
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.7
0.8
0.9
1

(a)


Equalizing
Equilibrium
exists
Equalizing
Equilibrium
exists
π_1
π_0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1

(b)

Figure 2.1

Chapter 3

Hypothesis Testing Equilibrium in Signalling Games[1]

Abstract

In this chapter, we propose a definition of Hypothesis Testing Equilibrium (HTE) for general signalling games with non-Bayesian players nested by an updating rule according to the Hypothesis Testing model characterized by Ortoleva (2012). An HTE may differ from a sequential Nash equilibrium because of dynamic inconsistency. However, in the case in which player 2 only treats a zero-probability message as an unexpected news, an HTE is a refinement of sequential Nash equilibrium and survives the Intuitive Criterion in general signalling games but not vice versa. We provide an existence theorem covering a broad class of signalling games often studied in economics, and the completely separating *constrained HTE* is unique in such signalling games. We also present the results according to Milgrom-Roberts' (1982) model of limit pricing and obtain a unique HTE for each interesting case.

Keywords: Signalling Games, Hypothesis Testing Equilibrium, Equilibrium Refinement.

[1]Part of this chapter was published in IMW working papers series

3.1 Introduction

Ortoleva (2012) models an agent who does not update according to Bayes' rule but instead "rationally" chooses a new prior among a set of priors when her original prior assigns a small probability to a realized event. He provides axiomatic foundations for his model in the form of a Hypothesis Testing representation theorem for suitably defined preferences. Both the testing threshold and the set of priors are subjective; therefore, an agent who follows this updating rule is aware of and can anticipate her updating behaviour when formulating plans.

Specifically, we consider the preferences of an agent over acts $\mathcal{F}$ that are functions from state space Ω to a set of consequences X. If the preference relation is characterized by Dynamic Coherence in conjunction with other standard postulates, then the agent's behaviour can be represented by a Hypothesis Testing model (u, ρ, ϵ). According to this representation, the agent has a utility function u over consequences; a prior over priors ρ; and a threshold $\epsilon \in [0, 1)$. She then acts as follows: Before any information arrives, she has a set of priors Π with probability assessment ρ over Π. She chooses π_Ω as her original prior, which is assigned the highest probability by ρ among all $\pi \in supp(\rho)$. Then, she forms her preference as the standard expected utility maximizer. As new information (an event) A is revealed, the agent evaluates the probability of the occurrence of the event as $\pi_\Omega(A)$. She retains her original prior π_Ω and proceeds with Bayesian updating on π_Ω using A if the event A is anticipated, i.e., $\pi_\Omega(A) > \epsilon$. However, if $\pi_\Omega(A) \leq \epsilon$, she rejects her original prior π_Ω and searches for a new prior π^* among $supp(\rho)$ such that π^* is the most likely one conditional on event A, that is, $\pi^* = \underset{\pi \in supp(\rho)}{\operatorname{argmax}} \mathbb{P}(\pi|A)$, where

$$\begin{aligned} \mathbb{P}(\pi|A) &= \frac{\mathbb{P}(A|\pi)\mathbb{P}(\pi)}{\int_{\pi' \in supp(\rho)} \mathbb{P}(A|\pi')\mathbb{P}(\pi')d\pi'} \\ &= \frac{\pi(A)\rho(\pi)}{\int_{\pi' \in supp(\rho)} \pi'(A)\rho(\pi')d\pi'}. \end{aligned} \tag{3.1}$$

Using this π^*, she proceeds with Bayesian updating and forms her preference by maximizing expected utility.

Ortoleva (2012) applied his model in the "Beer-Quiche" game and defined a Hypothesis Testing Equilibrium (HTE) when $\epsilon = 0$ for this specific game. In this game, there exists a unique HTE that coincides with the selection of the Intuitive Criterion of Cho and Kreps (1987). This chapter develops the idea of nesting this updating model in general signalling games with finite states and proposes a general concept of HTE. In the general definition of HTE, we allow the testing threshold $\epsilon \geq 0$. If player 2 has a testing threshold $\epsilon > 0$, then she changes her original belief when a small (but non-zero) probability event occurs. This dynamic inconsistency leads to the result that an HTE may deviate from sequential Nash equilibrium. However, we show that when $\epsilon = 0$, an HTE is a refinement of sequential Nash equilibria. In this case, player 2 only considers the zero-probability event as an unexpected event. To compare our approach with other refinement criteria, we focus primarily on the properties of this class of HTE. We have three main findings: (a). As a method of refinement, an HTE survives the Intuitive Criterion in general signalling games, but not vice versa. (b). A general HTE exists in a broad class of signalling games that is widely utilized in economics and satisfies the Single Crossing Property together with other standard assumptions. (c). We propose a concept of *constrained HTE* in which the set of alternative beliefs of player 2 is restricted to be around her original belief. We show that the *constrained HTE* is unique if only completely separating equilibria exist. As an example, we present these results in Milgrom-Roberts' limit pricing model and obtain a unique HTE for each interesting case.

This chapter focuses on signalling games, a class of games in which an informed player (player 1) conveys private information to an uninformed player (player 2) through messages, and player 2 attempts to make inferences about hidden information and takes an action that can influence both players' payoffs. There is an enormous literature that analyses and utilizes signalling games in a wide range of economic problems, as reviewed in Riley (2001) and Sobel (2007); see also Spence's model of the labor market (Spence, 1974), Milgrom-Roberts' model of limit pricing (Milgrom and Roberts, 1982), bargaining models (Fudenberg and Tirole, 1983), and models in finance (Brealey, et. al, 1977), for example. Typically, a signalling game gives rise to many sequential Nash equilibria (Kreps and Wilson, 1982) because, under the assumption of Bayesian updating, in equilibrium, there are no other restrictions on the message m that is sent with zero probability by player 1 except that player 2's responses to m can be rationalized by *some* belief held by player 2. Therefore, the natural approach to refining sequential Nash equilibria is to impose additional restrictions on the out-of-equilibrium beliefs, as we can see in the literature reviewed in Govindan and Wilson (2008, 2009), Hillas and Kohlberg (2002), Kohlberg (1990), and van Damme (2002).

One branch of the refinement criteria, which has been widely applied in signalling games, is motivated by the concept of strategic stability for finite games addressed by Kohlberg and Mertens (1986). The Intuitive Criterion, D1 and D2 Criteria (Cho and Kreps, 1987), and Divinity (Banks and Sobel, 1987), for example, are all weaker versions of strategic stability that are defined more easily for signalling games. These refinements interpret the meaning of the out-of-equilibrium messages depending on the current equilibrium, meaning that, in a reasonable equilibrium, sending an out-of-equilibrium message is costly and unattractive to player 1. There is also a branch of refinements that are intended to define a new concept of equilibrium, for example, the perfect sequential equilibria proposed by Grossman and Perry (1986), different versions of perfect Bayesian equilibrium (PBE) discussed by Fudenburg and Tirole (1991), and forward induction equilibrium defined by Govindan and Wilsons (2009) and modified by Man (2012), the consistent forward induction equilibrium path proposed by Umbauer (1991), the undefeated equilibrium of Mailath, et. al, (1993), and some methods of equilibrium selection in cheap talk game by Matthews et. al (1991) and de Groot Ruiz and Offerman (2012).There is no consensus in the literature that one refinement is better than another. One refinement can be favourable in certain settings but unfavourable in other settings.

All of the refinements mentioned above concern signalling games with Bayesian players. However, behaviour deviating from Bayesian updating has been observed by psychologists,[2] and these experiments have motivated increasing interest in studies on non-Bayesian updating; see, for example, the model of temptation and self-control proposed by Gul and Pesendorfer (2001, 2004) characterized axiomatically by Epstein (2006) and extended by Epstein et al., (2008, 2010), models of learning in social networks developed by Golub and Jackson (2010) and Jadbabaie et al., (2012), the arguments from rational beliefs advanced by Gilboa et. al, (2008, 2009, 2012) and Teng (2014), and the Hypothesis Testing model of Ortoleva (2012). In signalling games with non-Bayesian players, this chapter proposes a concept of HTE and provides a refinement based on the idea of

[2]For example, see Tversky and Kahneman (1974), Camerer (1995), Rabin (1998, 2002), and Mullainathan (2000).

non-Bayesian reactions to low-probability messages according to the Hypothesis Testing model.

The non-Bayesian updating rule is nested in signalling games as follows: Before player 1 moves, player 2 has a prior over a (finite) set of strategies that player 1 may use, and she determines the strategy that she believes player 1 will use most likely, which induces her original belief. After she observes a message sent by player 1, she evaluates the probability of the observed message using her original belief. She retains her original belief and uses it to proceed with Bayesian updating if the probability of the message she observed is greater than her testing threshold. However, if the probability is less than or equal to her threshold, she discards her original belief (she believes that player 1 may use a strategy other than her original conjecture). She then searches for a new belief that can be induced by another "rational" strategy by player 1 such that it is the most likely strategy conditional on the observed message. In a Hypothesis Testing equilibrium, the strategy of player 1 that induces player 2's original belief coincides with the strategy that player 1 actually uses. The difficulty concerns how to construct the set of player 2's beliefs and how to assign a prior over the set of possible strategies. We first follow the idea of Ortoleva (2012) and allow all beliefs that can be "rationalized" by at least one strategy of player 2, which is a weak restriction on the beliefs available to player 2. Then, we propose the *constrained HTE* in which only those beliefs around her original belief are under consideration. When player 2 observes a zero-probability message under her original belief, she looks for the most likely types who are willing to send this message, and she modifies her belief only by revising her original belief according to this message.

This chapter is organized as follows. In the next section, we briefly recall the basic concepts and definitions from Ortoleva (2012) on the updating rule of the Hypothesis Testing model and the framework of general signalling games. Section 3.3 defines the general HTE and discusses its main properties. Section 3.4 proves the existence and uniqueness theorems. Section 3.5 compares the refinements of the *constrained* HTE and Intuitive Criterion. Section 3.5 analyses the HTE of Milgrom-Roberts' limit pricing model in a finite framework, and section 3.7 provides the conclusion and some remarks.

3.2 Formulations and preliminaries

3.2.1 The updating rule of Hypothesis Testing Model

First, we recall the basic concepts, definitions and main results of the Hypothesis Testing model in general decision theory. Adopting the notations in Ortoleva (2012), consider a probability space $(\Omega, \Sigma, \Delta(\Omega))$, where Ω is a finite (nonempty) state space, Σ is set of all subsets of Ω, and $\Delta(\Omega)$ is the set of all probability measures (beliefs) on Ω. Write $\Delta(\Delta(\Omega))$ as the set of all beliefs over beliefs. Let

$$BU(\pi, A)(B) = \frac{\pi(A \cap B)}{\pi(A)}$$

denote the Bayesian update of $\pi \in \Delta(\Omega)$ using $A \in \Sigma$ if $\pi(A) > 0$. As discussed in the introduction, equation (3.1) provides the Bayesian update of the second-order prior $\rho \in \Delta(\Delta(\Omega))$ using $A \in \Sigma$ if $\pi(A) > 0$ for some $\pi \in \text{supp}(\rho)$. We denote it as follows:

$$BU(\rho, A)(\pi) := \frac{\pi(A)\rho(\pi)}{\int\limits_{\Delta(\Omega)} \pi'(A)\rho(\pi')d\pi'}.$$

Let us consider the preferences of an agent over acts $\mathcal{F}$, which are functions from state space Ω to a set of consequences X. For example, X could be a set of possible prizes that depend on the realizations of the state.

Definition 3.1. *(Ortoleva, 2012) A class of preference relations $\{\succeq_A\}_{A\in\Sigma}$ admits a Hypothesis Testing representation if there exists a nonconstant affine function $u : X \to \mathbb{R}$, a prior over priors $\rho \in \Delta(\Delta(\Omega))$ with finite support, and $\epsilon \in [0, 1)$ such that, for any $A \in \Sigma$, there exists $\pi_A \in \Delta(\Omega)$ such that*

(i) for any $f, g \in \mathcal{F}$

$$f \succeq_A g \Leftrightarrow \sum_{\omega\in\Omega} \pi_A(\omega)u(f(\omega)) \geq \sum_{\omega\in\Omega} \pi_A(\omega)u(g(\omega))$$

(ii) $\{\pi_\Omega\} = \underset{\pi\in\Delta(\Omega)}{\text{argmax}}\, \rho(\pi)$

(iii)

$$\pi_A = \begin{cases} BU(\pi_\Omega, A) & \pi_\Omega(A) > \epsilon \\ BU(\pi^*_A, A) & \textit{otherwise}, \end{cases}$$

$$\textit{where } \{\pi^*_A\} = \operatorname*{argmax}_{\pi \in \Delta(\Omega)} BU(\rho, A)(\pi).$$

Under this definition, if a decision maker's preference is represented by the updating rule according to the Hypothesis Testing model, then she proceeds with updating according to the following procedure:

Step 0. The agent is uncertain about some important state of the nature. Instead of a single subjective probability distribution over the alternative possibilities, she has a set of probability distributions (priors) Π and a probability distribution (second-order prior) ρ on Π, and $\text{supp}(\rho) \neq \emptyset$. The agent has a subjective threshold ϵ for hypothesis testing.

Step 1. Before any new information is revealed, the agent chooses a prior $\pi_\Omega \in \text{supp}(\rho)$ that is the most likely prior according to her belief ρ. In this hypothesis test, π_Ω serves as a *null hypothesis* and all the other priors $\pi \in \text{supp}(\rho)$ serve as *alternative hypotheses.*

Step 2. As new information (an event) A is revealed, the agent evaluates the probability of the occurrence of A as $\pi_\Omega(A)$. The null hypothesis will not be rejected if $\pi_\Omega(A) > \epsilon$, and the agent can proceed to apply Bayes' rule to the prior π_Ω. However, the null hypothesis will be rejected if $\pi_\Omega(A) \leq \epsilon$. The agent doubts her original prior π_Ω because an unexpected event occurred. The agent will choose an alternative prior $\pi^* \in \text{supp}(\rho)$ that is the most likely prior conditional on the event A. Then she proceeds with Bayes' rule on the prior π^*. When $\epsilon > 0$, we have non-Bayesian updating and dynamic consistency is violated up to ϵ. When $\epsilon = 0$, the dynamic consistency condition holds and the posteriors are also well defined after zero-probability events.

The aim of this chapter is to nest the non-Bayesian updating rule according to the Hypothesis Testing model in signalling games; therefore, we now briefly introduce the general framework of signalling games with Bayesian players.

3.2.2 Signalling games

Nature selects the type of player 1 according to some probability distribution μ over a finite set T with $\text{supp}(\mu) \neq \emptyset$ (for simplicity, we take $T = \text{supp}(\mu)$). Player 1 is informed of his type $t \in T$, but player 2 is not. After player 1 has learnt his type, he chooses to send a message m from a finite set M. Observing the message m, player 2 updates his belief on the types of player 1 and selects a response

r in a finite action set R. The game ends with this response, and payoffs are made to the two players. The payoff to player i, $i = 1, 2$, is given by a function $u_i : T \times M \times R \to \mathbb{R}$. The distribution μ and the description of the game are common knowledge.

A behavioural strategy of player 1 is a function $\sigma : T \to \Delta(M)$ such that $\sum_{m \in M} \sigma(m; t) = 1$ for all $t \in T$. Type t of player 1 chooses to send message m with probability $\sigma(m; t)$ for all $t \in T$. A behavioural strategy of player 2 is a function $\tau : M \to \Delta(R)$ such that $\sum_{r \in R} \tau(r; m) = 1$ for all $m \in M$. Player 2 plays a response r to the message m with probability $\tau(r; m)$. We adopt the notations in Cho and Kreps (1978) and write $\mathrm{BR}(m, \mu)$ for the set of best responses of player 2 after observing m if she has posterior belief $\mu(\cdot|m)$.

$$\mathrm{BR}(m, \mu) = \underset{r \in R}{\operatorname{argmax}} \sum_{t \in T} u_2(t, m, r)\mu(t|m).$$

If $T' \subseteq T$, let $\mathrm{BR}(T', m)$ denote the set of best responses of player 2 to posteriors concentrated on the set T'. That is,

$$\mathrm{BR}(T', m) = \bigcup_{\{\mu : \mu(T'|m) = 1\}} \mathrm{BR}(m, \mu).$$

Let $\mathrm{BR}(T', m, \mu)$ be the set of best responses of player 2 to the observed message m if she has posterior belief $\mu(\cdot|m)$ concentrated on the subset T', and let MBR denote the set of mixed best responses of player 2. Since we concentrate on the finite sets of T, M, and R, the sequential Nash equilibrium can be straightforwardly defined.

Proposition 3.1. *A profile of players' behavioural strategies (σ^*, τ^*) forms a sequential Nash equilibrium (SNE) in a finite signalling game if it satisfies the following conditions:*

(i) Given player 2's strategy τ^*, each type t evaluates the expected utility from sending message m as $\sum_{r \in R} u_1(t, m, r)\tau^*(r; m)$ and $\sigma^*(\cdot; t)$ assigns a weight to m only if it is among the maximizing ms in this expected utility.

(ii) Given player 1's strategy σ^*, for all m that are sent by some type t with positive probability $\mu(t|m) > 0$, every response $r \in R$ such that $\tau^*(r; m) > 0$ must

be a best response to m given belief $\mu(t|m)$; that is,

$$\tau^*(\cdot; m) \in MBR(m, \mu(\cdot|m)), \tag{3.2}$$

where $\mu(t|m) = \frac{\sigma^*(m;t)\mu(t)}{\sum_{t' \in T} \sigma^*(m;t')\mu(t')}$.

(iii) *For every message m that is sent with zero probability by player 1 (for all m such that $\sum_t \sigma^*(m;t)\mu(t) = 0$), there must be some probability distribution $\mu(\cdot|m)$ over types T such that* (3.2) *holds.*

In an SNE, given the strategy of player 1, player 2 proceeds in three steps: she computes the probability of an observed message m as $\mathbb{P}(m) = \sum_{t \in T} \sigma^*(m;t)\mu(t)$. If $\mathbb{P}(m) > 0$, that is, there exists some $t \in T$, such that $\sigma^*(m;t) > 0$, then she uses Bayes' rule to compute the posterior assessment $\mu(\cdot|m)$, and she then chooses her best response to m compatible with her belief $\mu(\cdot|m)$. If $\mathbb{P}(m) = 0$, then the only requirement is that there exists some belief $\mu(\cdot|m)$ such that her response to the out-of-equilibrium message is rational.

What happens if player 2's reactions to a low-probability events deviate from Bayes' rule? In the next section, our aim is to define an alternative equilibrium in such signalling games when player 2 uses the updating rule according to the Hypothesis Testing model.

3.3 Hypothesis Testing Equilibrium

3.3.1 Definition of HTE in general signalling games

In a signalling game, player 2 cannot observe player 1's strategies but can observe the messages sent by player 1; therefore, it is helpful to understand an SNE in the following way: imagine that player 2 has a conjecture, $\hat{\sigma}(\cdot; t), \forall t \in T$, about player 1's behaviour before player 1 moves. She attempts to formulate a best response using her conjecture. Similarly, player 1 also has a conjecture, $\hat{\tau}$, about player 2's behaviour. In equilibrium, the conjecture profile $(\hat{\sigma}, \hat{\tau})$ coincides with the strategy profile (σ^*, τ^*) that players actually use. Each conjecture $\hat{\sigma}$ available to player 2 about player 1's behaviour induces a prior (belief) π on the state space $\Omega = T \times M$ if the marginal distribution $\sum_{m \in M} \pi(t, m)$ coincides with the initial distribution μ. For every realization (t, m), $\pi(t, m)$ is a joint probability of type t and message m, which is

$$\pi(t,m) = \hat{\sigma}(m;t)\mu(t), \quad \text{and} \quad \sum_{t\in T}\sum_{m\in M}\pi(t,m) = \sum_{t}\mu(t) = 1.$$

If player 2 follows an updating rule according to the Hypothesis Testing model, then player 2 has a set of "rational" conjectures (priors), and she has a probability distribution ρ on the set of conjectures (priors). First, we address some requirements for the second-order prior ρ.

Definition 3.2. *Consider a finite signalling game $\Gamma(\mu)$ in which player 2's preference admits a Hypothesis Testing model (ρ,ϵ). ρ is consistent if it satisfies the following requirements:*

(i). $\forall \pi \in supp(\rho)$, π is compatible with the initial information of the game, that is, $\sum_{m\in M}\pi(t,m) = \mu(t)$.

(ii). $\forall \pi \in supp(\rho)$, π can be rationalized by at least one possible strategy of player 2. That is, there exists some strategy $\tau : M \to \Delta(R)$ of player 2 such that $\pi(t,m) = 0, \forall t \in T, \forall m \in M$, if the type-message pair (t,m) is not a best response to τ.

As addressed in Ortoleva (2012), this requirement for rationality is a weak condition in the sense that player 2 can take any conjecture into consideration as long as it is compatible with player 1's best response to some possible strategy τ of player 2. Now, we are prepared to define HTE in signalling games.

Definition 3.3. *In a finite signalling game $\Gamma(\mu)$, a profile of behavioural strategies (σ^*,τ^*) is an HTE based on a Hypothesis Testing model (ρ,ϵ) if*

(i). ρ is consistent.

(ii). The support of ρ contains π_Ω induced by σ^*, and

$$\pi_\Omega = \operatorname*{argmax}_{\pi\in supp(\rho)} \rho(\pi).$$

Let

$$M^E = \{m \in M : \sum_{t\in T}\pi_\Omega(m|t)\mu(t) > \epsilon\},$$

then for any $m \in M \backslash M^E$, there exists some $\pi_m \in supp(\rho)$, such that

$$\pi_m = \underset{\pi \in supp(\rho)}{\text{argmax}}\, BU(\rho, m)(\pi).$$

(iii).For all $t \in T$, $\sigma^*(m;t) > 0$ implies that m maximizes the expected utility of player 1, and

$$\tau^*(\cdot; m) \in MBR(m, \mu(\cdot|m)),$$

where

$$\mu(t|m) = \begin{cases} \pi_\Omega(t|m) = \frac{\sigma^*(m;t)\mu(t)}{\sum_{t'} \sigma^*(m;t')\mu(t')} & \text{if} \quad m \in M^E \\ \pi_m(t|m) = \frac{\pi_m(m|t)\mu(t)}{\sum_{t'} \pi_m(m|t')\mu(t')}, & \text{otherwise.} \end{cases}$$

The idea behind this definition is similar to that of Nash equilibrium except that we allow non-Bayesian reactions for out-of-equilibrium messages when $\epsilon > 0$. Therefore, dynamic consistency is violated but only up to ϵ. However, dynamic consistency holds when $\epsilon = 0$, and the updating rule is also well defined after zero-probability messages. According to the definition, for a given ϵ, to prove that a profile of strategies is an HTE based on (ρ, ϵ), we simply need to find a proper ρ to support the equilibrium.

Example 1. As an illustration, we apply this definition to the simple game depicted in Figure 1.

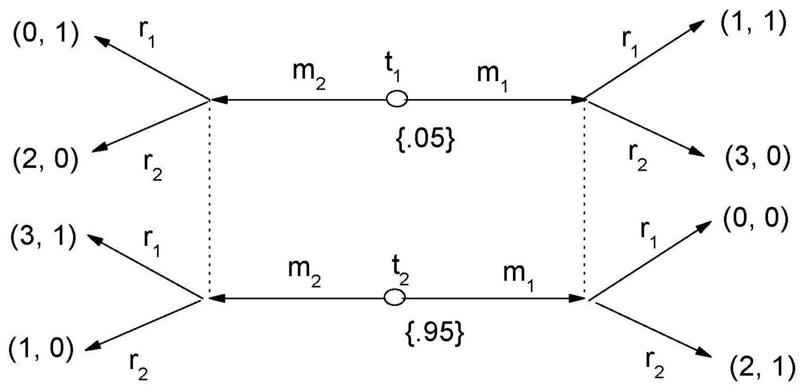


Figure 1

In this game, it is very easy to verify that there is one separating SNE: type t_1 of player 1 chooses message m_1, and type t_2 chooses message m_2; player 2, regardless of which message is observed, chooses r_1. If player 2 has a threshold $\epsilon = 5\%$, then this equilibrium is an HTE supported by a Hypothesis Testing model (ρ, ϵ), where $\text{supp}(\rho)$ only contains one element π induced by the strategy of player 1. That is, π satisfies the following conditions:

$$\begin{aligned}
&\pi(t_1, m_1) + \pi(t_1, m_2) = 0.05, \quad \pi(t_2, m_1) + \pi(t_2, m_2) = 0.95;\\
&\pi(m_1|t_1) = 1, \quad \pi(m_2|t_2) = 1.
\end{aligned}$$

This prior (belief) π can be rationalized by a strategy of player 2, which is choosing r_1 regardless of which message is observed.

In this example, there also exists an HTE that is not an SNE:

$$\begin{aligned}
&\sigma^*(m_1; t_1) = 1, \quad \sigma^*(m_2; t_2) = 1;\\
&\tau^*(r_2; m_1) = 1, \quad \tau^*(r_1; m_2) = 1.
\end{aligned}$$

The support of ρ contains two elements π and $\hat{\pi}$ such that $0 < \rho(\hat{\pi}) < \rho(\pi) < 0.9524$, and

$$\begin{aligned}
&\pi(t_1) = 0.05, \quad \pi(t_2) = 0.95, \quad \pi(m_1|t_1) = 1, \quad \pi(m_2|t_2) = 1;\\
&\hat{\pi}(t_1) = 0.05, \quad \hat{\pi}(t_2) = 0.95, \quad \hat{\pi}(m_1|t_1) = 1, \quad \hat{\pi}(m_1|t_2) = 1.
\end{aligned}$$

π can be rationalized by the strategy τ^*, and $\hat{\pi}$ can be rationalized by choosing r_2 regardless of the message observed. Now let us verify that (σ^*, τ^*) is an HTE supported by (ρ, ϵ). Given τ^*,

$$\begin{aligned}
&u_1(t_1, m_1, \tau^*) = 3 > u_1(t_1, m_2, \tau^*) = 0;\\
&u_1(t_2, m_2, \tau^*) = 3 > u_1(t_2, m_1, \tau^*) = 2;
\end{aligned}$$

therefore, σ^* maximizes player 1's expected payoff for both types. Given σ^*, player 2 starts with belief π since $\rho(\pi) > \rho(\hat{\pi})$ and retains π if she observes m_2 since

$$\pi(m_2) = \pi(m_2|t_1)\pi(t_1) + \pi(m_2|t_2)\pi(t_2) = 0.95 > \epsilon.$$

With the posterior belief $\pi(\cdot|m_2)$, observing m_2, player 2's best response is r_1. She switches to belief $\hat{\pi}$ if she observes m_1 since

$$\pi(m_1) = \pi(m_1|t_1)\pi(t_1) + \pi(m_1|t_2)\pi(t_2) = 0.05 \leq \epsilon,$$

and

$$BU(\rho, m_1)(\hat{\pi}) > BU(\rho, m_1)(\pi).$$

Therefore, observing m_1, player 2 has the posterior assessment of $\hat{\pi}(t_1|m_1) = 0.05$ and $\hat{\pi}(t_2|m_1) = 0.95$. With this posterior belief, player 2 computes her expected payoffs as follows:

$$u_2(r_1; m_1, \sigma^*) = \hat{\pi}(t_1|m_1) \times 1 + \hat{\pi}(t_2|m_1) \times 0 = 0.05;$$
$$u_2(r_2; m_1, \sigma^*) = \hat{\pi}(t_1|m_1) \times 0 + \hat{\pi}(t_2|m_1) \times 1 = 0.95,$$

which implies that r_2 is the best response to m_1. Therefore, (σ^*, τ^*) is an HTE but not a Nash Equilibrium, as r_2 is not a best response of player 2 to the message m_1 if she only has one belief π.

3.3.2 Properties of Hypothesis Testing Equilibrium

From the previous example, we can immediately obtain the following property:

Proposition 3.2. *In a finite signalling game $\Gamma(\mu)$, if a profile of behavioural strategies (σ, τ) is an HTE supported by a Hypothesis Testing model $(\rho, 0)$, then it is also an HTE supported by a Hypothesis Testing model $(\rho_\epsilon, \epsilon)$, for all $\epsilon > 0$.*

Proof. Let M_0^E and M_ϵ^E denote the sets of messages that are sent by player 1 with probability zero or a probability less than or equal to ϵ, respectively. Then, $M_\epsilon^E \subseteq M_0^E$. We can simply take

$$\rho_\epsilon = \rho = \{\pi_\Omega, \pi_m, m \in M \backslash M^E\}.$$

Player 2 starts with $\pi_\Omega = \operatorname{argmax}_{\pi \in \text{supp}(\rho)} \rho(\pi)$, she retains π_Ω and proceeds with Bayesian updating if she observes $m \in M_\epsilon^E$. If $m \in M_0^E \backslash M_\epsilon^E$, there exists $\pi_m = \pi_\Omega$ such that σ and τ are sequentially rational. If $m \notin M_0^E$, there exists π'_m that is identical to π_m such that σ and τ are sequentially rational. ■

As we can see from the previous example, if $\epsilon > 0$, then any message sent with probability less than or equal to ϵ is an out-of-equilibrium message. Because of dynamic inconsistency, an HTE may deviate from an SNE. However, when $\epsilon = 0$, only the messages sent with zero probability are off-the-equilibrium path; therefore, it is not surprising that there is a close relationship between this special class of HTE and SNE.

Proposition 3.3. *In a finite signalling game $\Gamma(\mu)$, an HTE supported by a Hypothesis Testing model $(\rho, 0)$ is a refinement of SNE.*

This requires no proof, as it relies solely on the definitions of HTE and SNE. If a profile of strategies (σ^*, τ^*) is an HTE supported by $(\rho, 0)$, then for any message sent by player 1 such that $\sigma^*(m; t) > 0$ for some $t \in T$, player 2's posterior belief derived by Bayesian updating using σ^*. For any message sent with zero probability, that is, $\sigma^*(m; t) = 0$ for all $t \in T$, there exists some belief on the side of player 2 to rationalize her behaviour. In addition, (σ^*, τ^*) is sequentially rational. Therefore, (σ^*, τ^*) is an SNE. Moreover, according to definition 3.2, we require that any belief in the support of ρ must can be "rationalized" by at least one strategy of player 2, which means, in addition to the requirement of equation (3.2), that we actually impose a further restriction on the out-of-equilibrium beliefs of player 2. Therefore, it is not surprising that an SNE may not be an HTE supported by some $(\rho, 0)$, as in the example of the "Quiche-Beer" game in Ortoleva (2012). Here, we also provide a simple example[3] to demonstrate this property.

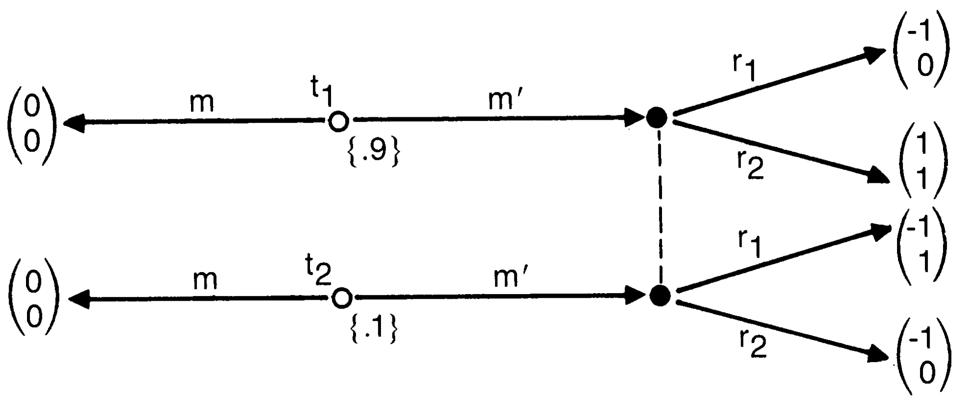


Figure 2

In the game depicted in Figure. 2, $(t_1, m), (t_2, m)$ is an SNE supported by a belief of player 2 such that $\mu(t_1|m) = 0.9$ and $\mu(t_2|m') \geq 0.5$. However, this is not an HTE supported by $(\rho, 0)$ because there does not exist a ρ such that supp(ρ)

[3]This example is from Cho and Kreps (1987)

contains a "rationalized" belief $\pi_{m'}$ regarding the out-of-equilibrium message m'. By contradiction, assume that there exists $\pi_{m'}$ such that $\pi_{m'}(t_2|m') \geq 0.5$ and $\pi_{m'}$ can be rationalized by some strategy of player 2. If $\pi_{m'}(t_2|m') \geq 0.5$, then $\pi_{m'}(t_2, m') > 0$, which implies that for any strategy τ that is rationalized $\pi_{m'}$, m' must be a best response of t_2 to strategy τ. However, for type t_2, m' is strictly dominated by m, which implies that there is no such strategy τ such that m' is a best response to t_2. This is contrary to condition (ii) in consistency definition 3.2.

3.4 Existence of HTE

3.4.1 Definition, notations, and assumptions

Proposition 3.1 indicates that, in a signalling game, an HTE supported by (ρ, ϵ) exists if an HTE supported by $(\rho, 0)$ exists. Moreover, an HTE supported by $(\rho, 0)$ is a refinement of SNE, which interests us because it allows us to compare our definition of HTE with other refinement criteria. Therefore, in the analysis to follow, we simply need to restrict our attention to this class of HTE by imposing $\epsilon = 0$. Since mixed strategies are not needed to prove existence, we only consider the Pure Sequential Nash Equilibrium (PSNE). A pure strategy of player 1 is a mapping $s_1 : T \to M$, and a pure strategy of player 2 is a response function $s_2; M \to R$.

Definition 3.4. *In a finite signalling game $\Gamma(\mu)$, a profile of strategies (s_1^*, s_2^*) forms a PSNE if there exists $\beta_m \in \Delta(T)$, $\forall\ m \notin M^E$, such that:*

(i). Given s_2^*,

$$u_1(t, s_1^*(t), s_2^*(s_1^*(t))) \geq u_1(t, m, s_2^*(m)), \quad \forall m \in M, \quad \forall t \in T.$$

(ii). Given s_1^*, for any $m \in M$, $s_2^*(m) \in BR(m, \mu_2(\cdot|m))$, where

$$\mu_2(t|m) = \begin{cases} \beta(t|m) & \text{if} \quad m \in M^E \\ \beta_m(t|m) & \text{otherwise,} \end{cases}$$

and

$$\beta(t|m) = \begin{cases} \frac{\mu(t)}{\sum_{t' \in T_{m,s_1^*}} \mu(t')} & \textit{if} \quad t \in T_{m,s_1^*} \\ 0, & \textit{otherwise}, \end{cases}$$

where $T_{m,s_1^*} = \{t \in T : s_1^*(t) = m\}$.

Before we prove the existence theorem, we provide some notations that we may use in the statements.

T_{m,s_1}: the subset of types of player 1 who send message m under strategy s_1, that is,

$$T_{m,s_1} = \{t \in T : s_1(t) = m\}.$$

$M^E_{(s_1,s_2)}$: the set of on-the-equilibrium messages if $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$, that is,

$$M^E_{(s_1,s_2)} = \{m \in M : \exists t \in T, \quad s.\ t. \quad s_1(t) = m\}.$$

$u_1(t; s_1, s_2)$: the payoff of type t under strategy (s_1, s_2), that is,

$$u_1(t; s_1, s_2) = u_1(t, s_1(t), s_2(s_1(t))).$$

$\beta_{T_m} \in \Delta(T)$: the probability assessment concentrating on types $t \in T_m$, that is,

$$\beta_{T_m}(t) = \begin{cases} \frac{\mu(t)}{\sum_{t' \in T_m} \mu(t')} & \text{if} \quad t \in T_m \\ 0 & \text{otherwise.} \end{cases}$$

$\beta_t \in \Delta(T)$: the probability assessment concentrating on type t, that is,

$$\beta_t(t') = \begin{cases} 1 & \text{if} \quad t' = t \\ 0 & \text{otherwise.} \end{cases}$$

We cannot be confident that there exists an HTE for a game that is randomly selected from the space of signalling games with finite states; instead, we prove the existence theorem in a class of signalling games that satisfy the following assumptions.

Assumption 3.1. *T, M, and R are finite. The type of player 1 has a probability distribution, $\mu \in \Delta(T)$, with full support. Further, $u_i(t, s_1, s_2)$, $i \in \{1, 2\}$, exists* *and is finite for all* $t \in T$ *and all nondecreasing functions* $s_1 : T \to M$ *and* $s_2 : M \to R$.

Assumption 3.2. *For any* $t \in T$*, for any fixed* $r \in R$*, if we connect the points* $\{u_1(t, m, r) : m \in M\}$ *in order by a smooth line, then* $u_1(t)$ *is strictly concave in* m*, and* u_2 *is strictly concave in* r.

Assumption 3.3. *First-order stochastic dominance:* $\forall t \in T$, $\forall m \in M$, $\forall \beta, \beta' \in \Delta(T)$*, whenever* β *stochastically dominants* β'*, that is,*

$$\sum_{t' \leq t} \beta'(t') \geq \sum_{t' \leq t} \beta(t'), \quad \forall t \in T,$$

and strict inequality holds for some $t \in T$*, then*

$$u_1(t, m, BR(m, \beta)) > u_1(t, m, BR(m, \beta')).$$

Assumption 3.4. *Single Crossing Property:*
(i). *For all* $m > m'$*, and all* $t' > t$*,*$\forall r, r' \in R$*,*

$$\begin{aligned} &u_1(t, m, r) \geq (>) u_1(t, m', r'), \quad \textit{implies} \\ &u_1(t', m, r) \geq (>) u_1(t', m', r'). \end{aligned}$$

(ii). *For all* $\hat{r} > r$*, and all* $\hat{m} > m$*,* $\forall\ t \in T$*,*

$$\begin{aligned} &u_2(t, m, \hat{r}) \geq (>) u_2(t, m, r), \quad \textit{implies} \\ &u_2(t, \hat{m}, \hat{r}) \geq (>) u_2(t, \hat{m}, r). \end{aligned}$$

Assumption 3.1 is primarily a technical assumption to fit our definition of HTE. Assumption 3.2 insures that only pure strategies are considered by both players. Assumption 3.3 states that all types of player 1 prefer the best response of player 2 when player 2 believes that player 1 is more likely to be of a higher type. The fourth assumption is the Milgrom-Shannon Single Crossing Property (SCP) (Milgrom and Shannon, 1994) for both players, which is a widely used assumption in signalling games to model many economic problems. It states that if type t prefers a higher message-response pair (m, r) to a lower message-response pair (m', r'), then any higher type $t' > t$ also prefers the higher message-response pair (m, r). This captures the idea that higher messages are more easily sent by a higher type. The utility of player 2 also satisfies the SCP: If a higher response $\hat{r}$ is a better response to a message m sent by type t than r, then it is also a better response to a higher message $\hat{m}$ sent by t than r. Athey (2001) characterizes the single crossing condition in several classes of incomplete information games, such as all types of games with supermodular and log-supermodular payoffs, limit pricing games, and auctions.

Before we proceed to the existence theorem, let us review the concept of lexicographical dominance introduced by Mailath et al., (1993).

Definition 3.5. *In a signalling game $\Gamma(\mu)$, a strategy profile $(s_1^*, s_2^*) \in PSNE(\Gamma(\mu))$ lexicographically dominates (l-dominates) another strategy profile $(s_1, s_2) \in PSNE(\Gamma(\mu))$ if there exists $j \in T$, such that*

$$\begin{aligned} u_1(t; s_1^*, s_2^*) &> u_1(t; s_1, s_2) \quad if \quad t = j \\ u_1(t; s_1^*, s_2^*) &\geq u_1(t; s_1, s_2) \quad if \quad t \geq j + 1. \end{aligned}$$

A strategy profile $(s_1^*, s_2^*) \in PSNE(\Gamma(\mu))$ is a lexicographically maximum sequential equilibrium (LMSE) if there is no $(s_1, s_2) \in PSNE(\Gamma(\mu))$ l-dominates (s_1^*, s_2^*).

If we restrict player 1's types to a subset of T, we can define a truncated game from G. Formally, for any $j \in T$, let

$$T^j = \{1, ..., j\}, \quad \mu^j(t) = \beta_{T^j}.$$

A truncated game G^j is defined by substituting T^j for T and the T^j-conditional prior μ^j for the prior μ in original game. Then, we can obtain the following properties:

Proposition 3.4. *Assume that $(s_1, s_2) \in PSNE(\Gamma(\mu))$, $\forall\ j \in T$, if $s_1(t) \neq s_1(j)$, $\forall t > j$, then $(s_1^j, s_2^j) \in PSNE(\Gamma^j(\mu^j))$, where $s_1^j(t) = s_1(t)$, $\forall\ t \leq j$, and $s_2^j(m) = s_2(m)\ \forall\ m \in M$.*

The following lemma derived in Mailath et al. (1993) is very important for our proof. The reader is urged to read their paper to obtain a detailed analysis of this result.

Proposition 3.5. *Mailath et al. (1993): Under A3.1-A3.4, suppose that $(s_1, s_2) \in$*

PSNE$(\Gamma(\mu))$, $(\hat{s}_1, \hat{s}_2) \in$ *PSNE*$(\Gamma^j(\mu^j))$, *and for some* $j \in T$,

$$u_1(j; \hat{s}_1, \hat{s}_2) > u_1(j; s_1, s_2),$$

then there exists $(s_1^*, s_2^*) \in$ *PSNE*$(\Gamma(\mu))$, *such that*

$$\begin{aligned} u_1(t; s_1^*, s_2^*) &\geq u_1(t; \hat{s}_1, \hat{s}_2) \quad \textit{for all} \quad t \leq j \quad \textit{and} \\ u_1(t; s_1^*, s_2^*) &\geq u_1(t; s_1, s_2) \quad \textit{for all} \quad t > j. \end{aligned}$$

That is, (s_1^*, s_2^*) *l-dominates* (s_1, s_2).

3.4.2 Existence of HTE supported by a Hypothesis Testing model $(\rho, 0)$

Theorem 3.1. *Under* $A3.1 - A3.4$, *an LMSE is an HTE.*

To prove the theorem, we need the following critical results:

Lemma 3.1. *(Athey, 2001): Under A3.1 and A3.4, there exists a PSNE in* $\Gamma(\mu)$.

Therefore, an *LMSE* exists. Moreover, both players play nondecreasing strategies:

Lemma 3.2. *Under A3.1 and A3.4,* $\forall$ $(s_1^*, s_2^*) \in$ *PSNE*$(\Gamma(\mu))$, $s_1^*(t) \leq s_1^*(t')$ *if* $t < t'$.

Proof. In equilibrium (s_1^*, s_2^*) , $\forall t, t' \in T$,

$$u_1(t; s_1^*, s_2^*) \geq u_1(t, s_1^*(t'), s_2^*(s_1^*(t'))).$$

Suppose that $s_1^*(t) > s_1^*(t')$, and $t' > t$, by assumption of the SCP,

$$u_1(t'; s_1^*, s_2^*) > u_1(t'; s_1^*(t'), s_2^*(s_1^*(t'))) = u_1^*(t'; s_1^*, s_2^*),$$

which upsets the equilibrium. ■

Lemma 3.3. *Under A3.1 and A3.4,* $\forall$ $(s_1^*, s_2^*) \in$ *PSNE*$(\Gamma(\mu))$, $s_2^*(m) \leq s_2^*(m')$ *if* $m < m'$.

Proof. In equilibrium (s_1^*, s_2^*) , $\forall m' > m \in M$, since $s_2^*(m)$ is a best response to message m for any t,

$$u_2(t', m, s_2^*(m)) \geq u_2(t', m, s_2^*(m')).$$

Suppose that $s_2^*(m) > s_2^*(m')$, from assumption A3.4

$$u_2(t', m', s_2^*(m)) \geq u_2(t', m', s_2^*(m')),$$

which is contrary to the fact that $s_2^*(m')$ is a best response to m'. ■

Lemma 3.4. *For an* $(s_1^*, s_2^*) \in PSNE(\Gamma(\mu))$*, let*

$$T(r) = \{t \in T : u_1(t, m, r) > u_1(t; s_1^*, s_2^*)\}, \tag{3.3}$$

then under A3.1 and A3.4, $T(r)$ *is convex.*

Proof. For all t', $t'' \in T(r)$, $t' < t''$, suppose that there $\exists\, t \in [t', t'']$, such that

$$u_1(t, m, r) \leq u_1(t; s_1^*, s_2^*).$$

If $m > s_2^*(t)$, as $t' < t$, by A4, we obtain

$$u_1(t', m, r) \leq u_1(t', s_1^*(t), s_2^*(s_1^*(t))).$$

In equilibrium,

$$u_1(t', s_1^*(t), s_2^*(s_1^*(t))) \leq u_1(t'; s_1^*, s_2^*);$$

therefore,

$$u_1(t', m, r) \leq u_1(t'; s_1^*, s_2^*),$$

which is contrary to the assumption that $t' \in T(r)$. We can analogously prove the other case in which $m < s_2^*(t)$ to obtain a contradiction with $t'' \in T(r)$. Therefore, $\forall\, t \in [t', t'']$, $t \in T(r)$, which implies that $T(r)$ is convex. ■

Given a response r of player 2 to message m, $T(r)$ is the set of types who are willing to deviate from the equilibrium strategy.

Lemma 3.5. *If* $r < r'$*, then* $T(r) \subseteq T(r')$*.*

Proof. $\forall\, t \in T(r)$,

$$u_1(t, m, r) < u_1(t, m, r')$$
$$u_1(t, m, r) > u_1(t; s_1^*, s_2^*).$$

The first inequality holds because of A3.3. Therefore,

$$u_1(t; s_1^*, s_2^*) < u_1(t, m, r'),$$

which implies that $t \in T(r')$. ■

This lemma implies that a higher response to message m induces more types of player 1 to deviate from the equilibrium strategy to m.

For each message $m \in M$, we can form a set of responses of player 2 under which at least one type of player 1 is willing to deviate from the equilibrium strategy to m. Let

$$R_m = \{r \in R : \exists t \in T, \quad s.\ t. \quad u_1(t, m, r) > u_1(t; s_1, s_2)\}. \tag{3.4}$$

Case (i). $R_m = \emptyset$. In this case, no type would deviate to m from his equilibrium strategy given any response by player 2, that is, $\beta(t|m) = 0$, $\forall\, t \in T$.
Case (ii). If $R_m \neq \emptyset$, then let $r_m = \min R_m$. Then, $T(r_m)$ is the set of types who are willing to deviate to m driven by the smallest trigger response by player 2.

The main idea of the proof of the existence theorem as follows: For each out-of-equilibrium message m, there is a posterior belief supporting the *LMSE* that can be rationalized by one strategy of player 2. We find such a strategy of player 2: She plays a response r_m to m and retains her equilibrium responses to any other message. Now, let us prove Theorem 3.1.

Proof of Theorem 3.1. Let (s_1, s_2) be an *LMSE* in $\Gamma(\mu)$. To prove that (s_1, s_2) is an HTE, we simply need to prove that for any out-of-equilibrium message m, there exists a belief $\beta(\cdot|m) \in \Delta(T)$, such that $s_2(m) = \mathrm{BR}(m, \beta(\cdot|m))$, can be rationalized by some strategy $\tilde{s}_{2,m} : M \to R$. Then, we can construct the Hypothesis Testing model $(\rho, 0)$ in which the support of ρ contains priors derived from beliefs of on-the-equilibrium messages and out-of-equilibrium messages. Suppose that m is an out-of-equilibrium message. Consider an out-of-equilibrium message m; if $R_m = \emptyset$, then any $\beta(\cdot|m) \in \Delta(T)$, such that $s_2(m) = \mathrm{BR}(m, \beta(\cdot|m))$, can be rationalized by s_2. If $R_m \neq \emptyset$, we aim to find a strategy for player 2 to rationalize $\beta(\cdot|m)$. According to Lemma 3.4, we can denote $T(r_m) = [i, j]$ (in our finite setting, in fact, $T(r_m)$ is the set of types located between i and j.), where

$$\begin{aligned} i &= \min\{t \in T : u_1(t, m, r_m) > u_1(t; s_1, s_2)\} \\ j &= \max\{t \in T : u_1(t, m, r_m) > u_1(t; s_1, s_2)\}. \end{aligned}$$

We denote $m_j = s_1(j)$ and $k = \max\{t \in T : s_1(t) = s_1(j)\}$. If $m > m_j$, then $k = j$ by assumption A4. If $m < m_j$, then $u(t, m_j, r_m) > u(t, m, r_m)$, for all $t \in [j, k]$, because of the concavity assumption A2. Now let us consider the k-truncated game $\Gamma^k(\mu^k)$. We claim that there is no profile of strategies $(\hat{s}_1^k, \hat{s}_2^k) \in \text{PSNE}(\Gamma^k(\mu^k))$ such that $\hat{s}_1^k(t) = m$ and $\hat{s}_2^k(m) = r_m$ for any $t \in [i, j]$. Suppose, contrary to the assertion, that there exists such an equilibrium $(\hat{s}_1^k, \hat{s}_2^k)$, and in equilibrium, $\exists j_0 \in [i, j]$,

$$\begin{aligned} \hat{s}_1^k(j_0) &= m, \quad \text{and} \\ \hat{s}_2^k(m) &= r_m. \end{aligned}$$

We denote

$$h = \max\{t \in [i, k], \hat{s}_1^k(t) = \hat{s}_1^k(j_0)\};$$

then, $h < j$ because either $j = k$ or $\hat{s}_1^k(t) > m, \forall t \in [j, k]$. Proposition. 3.4 implies that $(\hat{s}_1^k, \hat{s}_2^k)$ is a PSNE in the h-truncated game $\Gamma^h(\mu^h)$ by simply dropping the strategies of types higher than h, and

$$u_1(h; \hat{s}_1^k, \hat{s}_2^k) > u_1(h; s_1, s_2).$$

Therefore, Prop. 3.5 implies that there exists (s_1^*, s_2^*) that l-dominates (s_1, s_2), which is contrary to the assumption that (s_1, s_2) is an *LMSE*. This analysis means that for any $(\hat{s}_1^k, \hat{s}_2^k) \in \text{PSNE}(\Gamma^k(\mu^k))$, $\text{BR}(m, \beta_{[i,j]}) < r_m$. In particular, (s_1, s_2) is a PSNE of $\Gamma^k(\mu^k)$ by simply deleting the strategies of the types higher than k; therefore, $s_2(m) < r_m$. To summarize the argument above, for any out-of-equilibrium message m, there exists a belief $\beta(\cdot|m) \in \Delta(T)$, such that

$s_2(m) = \mathrm{BR}(m, \beta(\cdot|m))$, can be rationalized by the strategy $\tilde{s}_2$ of player 2:

$$\begin{aligned}
&\tilde{s}_2(m) = r_m, \\
&\tilde{s}_2(m') = s_2(m'), \quad \forall m' \neq m.
\end{aligned}$$

For all $t \in T(r_m)$,

$$\begin{aligned}
u_1(t, m, \tilde{s}_2(m)) &= u_1(t, m, r_m) \\
&> u_1(t; s_1, s_2) \\
&\geq u_1(t, m', s_2(m')) \quad \forall m' \in M \\
&= u_1(t, m', \tilde{s}_2(m')) \quad \forall m' \in M,
\end{aligned}$$

and for all $t \notin T(r_m)$, $\exists\ s_1(t) \neq m$, such that $u_1(t, m, \tilde{s}_2(m)) \leq u_1(t; s_1, s_2)$. Therefore, given $\tilde{s}$,

$$\begin{aligned}
&\beta(m|t) = 1, \quad \forall t \in T(r_m), \\
&\beta(m|t) = 0, \quad \forall t \notin T(r_m).
\end{aligned}$$

We can construct a Hypothesis Testing model $(\rho, 0)$ in which

$$\mathrm{supp}(\rho) = \{\pi_\Omega, \{\pi_m : \forall m \notin M^E_{(s_1,s_2)}\}\},$$

where

$$\begin{aligned}
&\pi_\Omega(\cdot|m) = \beta_{T_{m,s_1}}, \quad \forall m \in M^E_{(s_1,s_2)} \\
&\pi_m(t|m) = \beta(t|m) = \beta_{T(r_m)}, \quad \forall t \in T, \quad \forall m \notin M^E_{(s_1,s_2)},
\end{aligned}$$

with

$$\begin{aligned}
&\rho(\pi_m) = 0, \quad \text{if} \quad R_m = \emptyset, \\
&\text{otherwise,} \quad 0 < \rho(\pi_m) = \rho(\pi'_m) < \rho(\pi_\Omega) \quad \text{if} \quad m, m' \notin M^E_{(s_1,s_2)}, \quad R_m, R'_m \neq \emptyset, \\
&\text{and} \quad \sum_{m \in M \setminus M^E_{(s_1,s_2)}} \rho(\pi_m) + \rho(\pi_\Omega) = 1.
\end{aligned}$$

By construction, (s_1, s_2) is an HTE supported by $(\rho, 0)$. *Q.E.D*

3.4.3 Uniqueness of *constrained HTE*

As noted above, under condition (*ii*) on the requirements for the consistency of ρ, player 2 is allowed to consider any strategy of player 1 as long as the strategy is compatible with a best response to a strategy of player 2, which enlarges the set of alternative beliefs of player 2. However, it is natural for us to restrict the strategies of player 1 such that player 2's alternative beliefs are around her original belief.

Definition 3.6. *An HTE $(s_1, s_2) \in PSNE(\Gamma(\mu))$ is a constrained HTE if, for any out-of-equilibrium message m, there exists a posterior belief conditional on m supporting the equilibrium that can be rationalized by*

$$\begin{aligned} \tilde{s}_2(m) &= r_m \\ \tilde{s}_2(m') &= s_2(m') \quad \forall m' \neq m. \end{aligned} \tag{3.5}$$

Remark. In a *constrained HTE*, any belief of an out-of-equilibrium message supporting the equilibrium can be rationalized by a strategy of player 2 that is not far from her equilibrium strategy. This idea is quite intuitive; when player 2 observes a message that deviates from her original belief, she searches for the most likely types who have a potential incentive to send this message and forms her new belief by simply perturbing her original belief regarding this message.

Remark. In an HTE, it is required that there exists $\pi_m \in supp(\rho)$, such that π_m is compatible with $\tilde{s}_2$. If π_m can be deduced by player 1's strategy $\tilde{s}_1$, then $\tilde{s}_2$ may not be a best response to $\tilde{s}_1$; only $\tilde{s}_1$ is required to be a best response to $\tilde{s}_2$. If $(\tilde{s}_1, \tilde{s}_2)$ forms an equilibrium, then in this equilibrium, $u_1(t, m, \tilde{s}_2) \leq u_1(t; s_1, s_2)$, $\forall\, t \in T$. This case coincides with the *Undefeated Equilibrium* proposed by Mailath et al. (1993).

Look through the proof of the existence theorem, we can immediately obtain the following proposition.

Proposition 3.6. *Under A3.1- A3.4, an LMSE is a constrained HTE.*

Now, we seek to prove the uniqueness of a *constrained HTE* if an *LMSE* is unique.

Proposition 3.7. *Under A3.1-A3.4, an LMSE is unique.*

Proof. Suppose that both (s_1^*, s_2^*) and (s_1, s_2) are *LMSE*, as (s_1^*, s_2^*) is not l-dominating (s_1, s_2), for any $t_0 \in T$, such that:

$$u_1(t_0; s_1^*, s_2^*) > u_1(t_0; s_1, s_2),$$

$\exists t_1 > t_0, \quad$ such that

$$u_1(t_1; s_1^*, s_2^*) < u_1(t_1; s_1, s_2).$$

We have the same expression for (s_1, s_2), and T is finite; therefore, (s_1^*, s_2^*) and (s_1, s_2) are identical. ■

Clearly, all other strategy profiles $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$ must be l-dominated by the unique *LMSE*. Let us denote the unique *LMSE* as (s_1^{LM}, s_2^{LM}).

Theorem 3.2. *Under A3.1-A3.4, if the unique LMSE is completely separating, and M is rich enough, then the outcome of a constrained HTE supported by $(\rho, 0)$ is unique.*

Lemma 3.6. *Under A3.1-A3.4, if $(s_1, s_2) \in PSNE(\Gamma(\mu))$ is a completely separating equilibrium, let $j = \min\{t \in T : u_1(t; s_1^{LM}, s_2^{LM}) > u_1(t; s_1, s_2)\}$, and $\forall\ t \in T$, $s_1(t) \neq s_1^{LM}(j)$, then (s_1, s_2) is not a constrained HTE.*

Proof. If $j = \min\{t \in T : u_1(t; s_1^{LM}, s_2^{LM}) > u_1(t; s_1, s_2)\}$, we denote

$$\begin{aligned} &s_1^{LM}(j) = m, \quad s_2^{LM}(m) = r_j^*, \\ &s_1(j) = m_j, \quad s_2(m_j) = r_j. \end{aligned}$$

We assume that $\beta(\cdot|m) \in \Delta(T)$ is a posterior belief supporting the equilibrium (s_1, s_2); then, $R(m) \neq \emptyset$ since $u_1(j, m, r_j^*) > u_1(j; s_1, s_2)$. Let $r_m = \min R_m$; then, $r_j^* \geq r_m$. Further, we can show that $r_m = r_j^*$. This is true because of the following argument. By assumption, we have the following conditions:

$$\begin{aligned} &u_1(j, m, r_j^*) > u_1(j, m_j, r_j) \quad \text{and} \\ &u_1(j, m_j, r_j) > u_1(j, m, s_2(m)), \\ \Rightarrow &u_1(j, m, r_j^*) > u_1(j, m, s_2(m)) \\ \Rightarrow &s_2(m) < r_j^* \\ \Rightarrow &\text{BR}(m, \beta(\cdot|m)) < \text{BR}(m, \beta_j), \end{aligned}$$

the last condition holds because (s_1^{LM}, s_2^{LM}) is a completely separating equilibrium. Therefore, by the SCP of the utility function of player 2, we have

$$\sum_{t \in T} \beta(t|m) < \mu(j), \quad \text{and} \quad \exists t_0 < j, \quad \beta(t_0|m) > 0.$$

Let

$$T(r_m) = \{t \in T : u_1(t, m, r_m) > u_1(t, s_1, s_2)\}.$$

By contradiction, suppose that (s_1, s_2) is a *constrained HTE*; then $\beta(\cdot|m)$ can be rationalized by $\tilde{s}_2$ given in Equ. (3.5). Moreover $\beta(t_0|m) > 0$ implies that m is a best response of t_0 to $\tilde{s}_2$, that is, $u_1(t_0, m, \tilde{s}_2(m)) > u_1(t_0, m', \tilde{s}_2(m'))$, $\forall\ m' \in M$. Therefore, $t_0 \in T(r_m)$. However,

$$u_1(t; s_1^{LM}, s_2^{LM}) \leq u_1(t; s_1, s_2) \quad \forall t < j,$$

and together with the following condition

$$u_1(t; s_1^{LM}, s_2^{LM}) \geq u_1(t; m, r_j^*),$$

implies that

$$u_1(t, m, r_j^*) < u_1(t; s_1, s_2) \quad \forall t < j.$$

Therefore, $\forall\ r < r_j^*$,

$$u_1(t_0, m, r) \leq u_1(t_0; m, r_j^*) < u_1(t_0; s_1, s_2).$$

Therefore, $r_m \geq r_j^*$, which induces $r_m = r_j^*$. However, with this strategy $\tilde{s}_2$,

$$u_1(j, m, \tilde{s}_2) = u_1(j, m, r_j^*) > u_1(j; s_1, s_2) \geq u_1(j, m', \tilde{s}_2(m')), \quad \forall m' \in M,$$

which implies that $j \in T(r_m)$. Therefore, $\beta(j|m) = \frac{\mu(j)}{\sum_{t \in T(r_m)} \mu(t)} \geq \mu(j)$. We have a contradiction with $\sum_{t \in T} \beta(t|m) < \mu(j)$. ■

Now let us prove Theorem 3.2.

Proof of Theorem 3.2. We simply need to show that any $(s_1, s_2) \in \text{PSNE}(\Gamma(\mu))$ that is l-dominated by (s_1^{LM}, s_2^{LM}) is not a *constrained HTE*. For any $(s_1, s_2) \in$

PSNE($\Gamma(\mu)$), let

$$j = \min\{t \in T : u_1(t; s_1^{LM}, s_2^{LM}) > u_1(t; s_1, s_2)\},$$

and $s_1^{LM}(j) = m_j$. If m_j is an out-of-equilibrium message under (s_1, s_2), by Lemma 3.6, we can immediately obtain the conclusion. However, if $m_j \in M^E_{(s_1,s_2)}$, according to our assumption, if M is rich enough, we can select an m slightly greater than m_j but less than $s_1^{LM}(j+1)$ such that $m \notin M^E_{(s_1,s_2)}$ and form a new PSNE (s_1^*, s_2^*) by simply perturbing (s_1^{LM}, s_2^{LM}) at j. Then, (s_1^*, s_2^*) still l-dominates (s_1, s_2). This can be done because (s_1^{LM}, s_2^{LM}) is a completely separating equilibrium. Again, Lemma 3.6 implies that (s_1, s_2) is not a *constrained HTE*. *Q.E.D*

Corollary 3.1. *Under A3.1- A3.4, if there only exist completely separating equilibria, then a "Pareto-dominant equilibrium" or "Riley equilibrium" is the unique constrained HTE.*

Proof. A "Riley equilibrium" maximizes the payoffs of all types in the set of completely separating equilibria, which means that it is the unique *LMSE*; therefore, it is the unique *constrained HTE*. ■

This proposition ensures that our HTE concept can capture the well-known "Pareto-dominant separating equilibrium" or "Riley outcome" that is often selected in economic applications.

3.5 Comparison with Intuitive Criterion

Intuitive Criterion: (Cho and Kreps, 1987) Fix a sequential equilibrium outcome and let $u_1^*(t)$ be the payoff of a type t of player 1 in this equilibrium. For each out-of-equilibrium message m, form the set

$$S(m) = \{t \in T : \quad u_1^*(t) > \max_{r \in \mathrm{BR}(T(m),m)} u_1(t, m, r)\}. \tag{3.6}$$

If, for some out-of-equilibrium message m there exists a type $t' \in T$ such that

$$u_1^*(t') < \min_{r \in \mathrm{BR}(T \backslash S(m),m)} u_1(t', m, r), \tag{3.7}$$

then the equilibrium outcome fails the Intuitive Criterion. Here $T(m)$ denotes the set of types of player 1 who have options to send the message m.

Proposition 3.8. *In a finite signalling game $\Gamma(\mu)$, if $(\sigma_1, \sigma_2) \in SNE(\Gamma(\mu))$ fails the Intuitive Criterion, then it is also not a Constrained HTE supported by a Hypothesis Testing model $(\rho, 0)$.*

Proof. If $(\sigma_1, \sigma_2) \in \mathrm{SNE}(\Gamma(\mu))$ fails the Intuitive Criterion, then for some $m \notin M^E_{(\sigma_1,\sigma_2)}$, $\exists\, t' \in T$, such that condition (3.7) holds. We prove that for this m, any belief $\pi_m \in \Delta(T \times M)$, such that

$$u_1(t; s_1, s_2) \geq u_1(t, m, \mathrm{BR}(m, \pi_m(\cdot|m))), \quad \forall t \in T, \tag{3.8}$$

cannot be rationalized by the strategy of player 2 given in (3.5). By contradiction, suppose that there exists $\tilde{s}_2 : M \to R$ that rationalizes $\pi_m(\cdot|m)$. T and M are finite; for this $m \in M$, there exists a posterior distribution $\mu_2(\cdot|m) \in \Delta(T)$ such that $\tilde{s}_2(m) = \mathrm{BR}(T(m), m, \mu_2(\cdot|m))$. By the definition of $S(m)$,

$$u_1^*(t) > \max_{r \in \mathrm{BR}(T(m),m)} u_1(t, m, r) > u_1(t, m, \tilde{s}_2(m)),$$

which implies that the types in $S(m)$ would never deviate from their equilibrium strategy to m. Therefore,

$$\pi_m(t, m) = 0, \quad \forall t \in S(m),$$

which implies that $\mathrm{BR}(\pi_m, T\backslash S, m) = \mathrm{BR}(\pi_m, T, m)$. Since there exists $t' \in T$,

$$\begin{aligned} u_1^*(t') &< \min_{r \in \mathrm{BR}(T(m)\backslash S(m),m)} u_1(t', m, r) \\ &\leq u_1(t', m, r), \quad r \in \mathrm{BR}(\pi_m, T(m)\backslash S(m), m) \\ &\leq u_1(t', m, r), \quad r \in \mathrm{BR}(\pi_m, T(m), m). \end{aligned}$$

Therefore, type t' could achieve a payoff strictly higher than his expected equilibrium payoff by sending the message m. This is contrary to condition (3.8).

■

Remark. We do not need to impose assumptions to obtain this property. In addition, we do not need to restrict to pure strategies in this property. This result is illustrated in the following simple example.

Example 2. The simple game depicted in Figure 3 provides an example to illustrate the result in Prop. 3.8. There are two PSNE in this game. In the first,

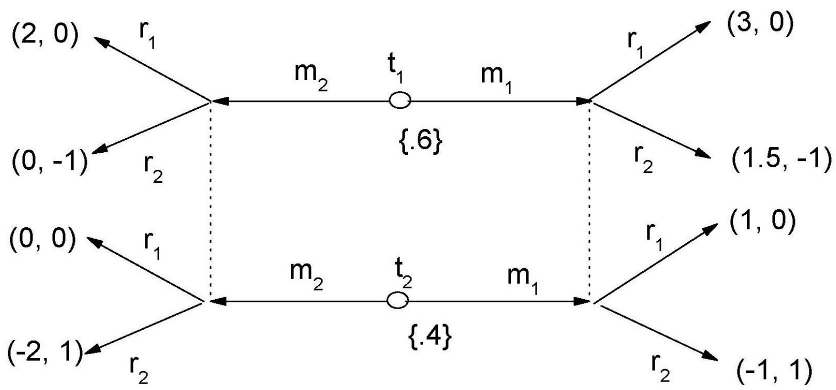


Figure 3

the strategies of player 1 and player 2 are given by

$$\begin{aligned}&\sigma^*(m_1;t_1)=1, \quad \sigma^*(m_1;t_2)=1;\\ &\tau^*(r_1;m_1)=1,\end{aligned} \tag{3.9}$$

This is a PSNE supported by any belief of player 2 regarding out-of-equilibrium message m_2. We can easily verify that this equilibrium can satisfy the Intuitive Criterion and is also an HTE. The second equilibrium is as follows:

$$\begin{aligned}&\sigma^*(m_2;t_1)=1, \quad \sigma^*(m_2;t_2)=1;\\ &\tau^*(r_1;m_2)=1.\end{aligned} \tag{3.10}$$

This is supported by a belief such that $\mu_2(t_1|m_1) < 0.5$. First, we verify that this PSNE can survive the Intuitive Criterion. For the out-of-equilibrium message m_1, we form the set

$$S(m_1) = \{t \in T : u_1^*(t) > \max_{r \in \mathrm{BR}(T,m_1)} u_1(t,m_1,r)\}.$$

Since

$$u_1^*(t_1) = 2 < \max_{r \in \mathrm{BR}(T,m_1)} u_1(t_1,m_1,r) = u_1(t_1,m_1,r_1) = 3,$$

and

$$u_1^*(t_2) = 0 < \max_{r \in \mathrm{BR}(T, m_1)} u_1(t_2, m_1, r) = u_1(t_1, m_1, r_1) = 1,$$

we have $S(m_1) = \emptyset$. Therefore, there is no type t' such that

$$u_1^*(t') < \min_{r \in \mathrm{BR}(T \setminus S(m_1), m_1)} u_1(t', m_1, r).$$

Second, we can show that this PSNE is not an HTE. By contradiction, if it is an HTE, then for the out-of-equilibrium message m_1, there must exist a belief π' such that $\pi'(t_1|m_1) < 0.5$ can be rationalized by a strategy of player 2. $\pi'(t_1|m_1) < 0.5$ implies that $\pi'(t_2, m_1) > 0$. Then, according to the condition (ii) of consistency definition 3.2 , any strategy of player 2 that rationalizes π' must be such that m_1 is a best response for type t_2. If we write such a strategy of player 2 as $\phi(r_1; m_1) = x \in [0, 1]$ and $\phi(r_1; m_2) = y \in [0, 1]$, then x and y must satisfy the following condition:

$$\begin{aligned} & u_1(t_2, m_1, \phi) \geq u_1(t_2, m_2, \phi) \\ \Rightarrow\ & 1 \cdot x + (-1) \cdot (1 - x) \geq 0 \cdot y + (-2) \cdot (1 - y) \\ \Rightarrow\ & 2x - 2y + 1 \geq 0. \end{aligned}$$

Given such a strategy, the payoffs of type t_1 are

$$\begin{aligned} & u_1(t_1, m_1, \phi) = 3 \cdot x + 1.5 \cdot (1 - x) = 2.5x + 1.5, \\ & u_1(t_1, m_2, \phi) = 2 \cdot y + 0 \cdot (1 - y) = 2y. \end{aligned}$$

Since

$$\begin{cases} 2x - 2y + 1 \geq 0 \\ x \in [0, 1], y \in [0, 1] \end{cases} \Rightarrow 2.5x + 1.5 > 2y,$$

which means that m_1 is the unique best response for type t_1. That is, $\pi'(m_1|t_1) = 1$. Therefore,

$$\pi'(t_1|m_1) \geq \pi'(m_1|t_1)\pi'(t_1) = 0.6,$$

which is contrary to $\pi'(t_1|m_1) < 0.5$.

3.6 HTE of the Milgrom-Roberts model

In this section, we present an example of simplified version of the Milgrom and Roberts (1982) limit pricing entry model to illustrate how to find an HTE. In this example, there are two periods and two firms. Firm 1, the incumbent, has private information concerning his unit cost. Firm 1 chooses a first-period quantity Q_1. In the second period, firm 2, the entrant, observes the quantity and decides whether to enter the market. It pays a fixed cost $K > 0$ if it enters. Then, the private information is revealed and last stage of the game is played, either by both firms competing in a Cournot duopoly or by only the incumbent acting as a monopolist[4]. Suppose that there are two possible costs for firm 1, (c_L, c_H). The common prior is $\mu(c_1 = c_L) = 1 - \mu(c_1 = c_H) = x$. Firm 2's cost $c_2 > 0$ is common knowledge.

We assume that the inverse market demand is given by $P(Q) = a - bQ, a, b > 0$. Let $\Pi_1^t(Q_1)$ denote a firm 1 of type t's monopoly profit in the first period if its production quantity is Q_1, that is

$$\Pi_1^t(Q_1) = (a - bQ_1 - c_t)Q_1 \quad t \in \{L, H\},$$

then it is easily to see that $\Pi_1^t(Q_1)$ is strictly concave in Q_1. Let Q_M^L and Q_M^H denote firm 1's monopoly quantities that maximize its short-run profit when its cost is low or high, respectively. With our linear demand function, we obtain

$$Q_M^L = \frac{a - c_L}{2b} \quad \text{and} \quad Q_M^H = \frac{a - c_H}{2b}.$$

Let Π_M^L and Π_M^H denote the monopoly profits under low and high cost, respectively, that is

$$\Pi_M^L = \Pi_1^L(Q_M^L) = \frac{(a - c_L)^2}{4b} \quad \text{and} \quad \Pi_M^H = \Pi_1^H(Q_M^H) = \frac{(a - c_H)^2}{4b}.$$

Since we assume (as do Milgrom and Roberts, 1982) that firm 2 learns firm 1's cost immediately if firm 2 decides to enter the market; we can explicitly compute the two firms' Cournot duopoly profits with complete information.

$$\Pi_{1C}^t = \frac{(a - 2c_t + c_2)^2}{9b}, \quad \text{and} \quad \Pi_{2C}^t = \frac{(a - 2c_2 + c_t)^2}{9b} - K,$$

[4] J. Tirole (1988) analysed a similar model in which the two firms compete via a Bertrand competition game if entry occurs

where $t \in \{L, H\}$. We assume the discount factors $\delta_1 = \delta_2 = \delta$. To make things interesting, we assume that $\Pi_{2C}^H > 0 > \Pi_{2C}^L$, which implies that, under complete information, firm 2 enters the market if and only if firm 1 is of the high-cost type. Furthermore, $E_x(\Pi_{2C}) < 0$, which implies that firm 2 will not enter the market if it only has the prior information.

To fit our HTE concept, we assume that firm 1 only has four choices of Q_1, that is, $Q_1 \in \{Q_M^L, Q_M^H, Q_1^L, Q_1^H\}$, where $Q_1^t > Q_M^t$, $\forall\ t \in \{L, H\}$ that satisfy the following conditions:

$$\Pi_1^L(Q_1^t) + \delta\Pi_M^t > \Pi_M^t + \delta\Pi_{1C}^t, \quad \forall t \in \{L, H\}.$$

This condition implies that the low-cost (high-cost) type of firm 1 prefers to deter the entrant by choosing a higher quantity Q_1^L (Q_1^H) if the monopoly quantity Q_M^L (Q_M^H) induces entry. Let $\tilde{Q}_1^t$ for $t \in \{L, H\}$ denote the quantity at which level a type t of firm 1 is indifferent between deterring and not deterring entry. That is,

$$\Pi_1^t(\tilde{Q}_1^t) + \delta\Pi_M^t = \Pi_M^t + \delta\Pi_{1C}^t, \quad \text{for} \quad t \in \{L, H\}.$$

Due to the strict concavity of $\Pi_1^t(Q_1^t)$, we have $\tilde{Q}_1^t < Q_1^t$. Now, there are only three interesting cases left to us to analyse the equilibria:

$$\begin{aligned}
&(i). \quad \tilde{Q}_1^L > Q_1^L > Q_M^L > \tilde{Q}_1^H > Q_1^H > Q_M^H,\\
&(ii). \quad \tilde{Q}_1^L > Q_1^L > \tilde{Q}_1^H > Q_1^H > Q_M^L > Q_M^H,\\
&(iii). \quad \tilde{Q}_1^L > Q_1^L > \tilde{Q}_1^H > Q_M^L > Q_1^H > Q_M^H.
\end{aligned}$$

In this example, we only consider the pure strategies of the firms; let

$$s : \{L, H\} \to \{Q_M^L, Q_M^H, Q_1^L, Q_1^H\}$$

and

$$t : \{Q_M^L, Q_M^H, Q_1^L, Q_1^H\} \to \{0, 1\}$$

be the strategies of firm 1 and firm 2, respectively. We can verify that, in this game, assumptions A3.1-A3.4 are satisfied, and therefore, an HTE exists.

Proposition 3.9. *In case (i), there exist two separating PSNE:*

$$s(L) = Q_M^L, \quad \textit{and} \quad s(H) = Q_M^H$$
$$t(Q_1) = \begin{cases} 0 & \textit{if} \quad Q_1 \geq Q_M^L \\ 1 & \textit{otherwise} \end{cases} \qquad (PSNE \quad 3.9.1)$$

and

$$s(L) = Q_1^L, \quad \textit{and} \quad s(H) = Q_M^H$$
$$t(Q_1) = \begin{cases} 0 & \textit{if} \quad Q_1 \geq Q_1^L \\ 1 & \textit{otherwise} \end{cases} \qquad (PSNE \quad 3.9.2)$$

Proposition 3.10. *In case (i), both of the equilibria can survive the Intuitive Criteria, but only the efficient separating equilibrium (PSNE 3.9.1) is an HTE.*

The proof is in appendix A.

Proposition 3.11. *In case (ii), there exist two pooling PSNE and one separating PSNE:*

$$s(L) = s(H) = Q_M^L$$
$$t(Q_1) = \begin{cases} 0 & \textit{if} \quad Q_1 \geq Q_M^L \\ 1 & \textit{otherwise,} \end{cases} \qquad (PSNE \quad 3.11.1)$$

$$s(L) = s(H) = Q_1^H$$
$$t(Q_1) = \begin{cases} 0 & \textit{if} \quad Q_1 \geq Q_1^H \\ 1 & \textit{otherwise,} \end{cases} \qquad (PSNE \quad 3.11.2)$$

and

$$s(L) = Q_1^L, \quad \textit{and} \quad s(H) = Q_M^H$$
$$t(Q_1) = \begin{cases} 0 & \textit{if} \quad Q_1 \geq Q_1^L \\ 1 & \textit{otherwise} \end{cases} \qquad (PSNE \quad 3.11.3)$$

Proposition 3.12. *In case (ii), all three PSNE can survive the Intuitive Criterion,* *but only the efficient equilibrium* ($PSNE \quad 3.11.1$) *is an HTE.*

The proof is given in Appendix B.

Proposition 3.13. *In case* (iii)*, there exist one separating PSNE and one pooling PSNE:*

$$s(L) = s(H) = Q_M^L$$
$$t(Q_1) = \begin{cases} 0 & if \quad Q_1 \geq Q_M^L \\ 1 & otherwise \end{cases} \qquad (PSNE \quad 3.13.1)$$

and

$$s(L) = Q_1^L, \quad and \quad s(H) = Q_M^H$$
$$t(Q_1) = \begin{cases} 0 & if \quad Q_1 \geq Q_1^L \\ 1 & otherwise \end{cases} \qquad (PSNE \quad 3.13.2)$$

Proposition 3.14. *In case* (iii)*, both of the PSNE can survive the Intuitive Criterion, but only the efficient pooling equilibrium* ($PSNE \quad 3.13.1$) *is an HTE.*

We can prove this proposition using almost the same argument as in the proof of Prop. 3.12. Thus, we omit the proof here.

To summarize this analysis of the limit pricing entry deterrence model, in each case, there exists a unique HTE. In the HTE, the high-cost type of firm 1 either chooses its monopoly quantity and allows entry or engages in pooling with the low-cost type and deters entry depending on the cost of pooling. The low-cost type choose its monopoly quantity and entry does not occur. By contrast, in each case, there exist multi-equilibria that can survive the Intuitive Criterion. In separating equilibria such as (*PSNE* 3.9.2), (*PSNE* 3.11.3), and (*PSNE* 3.12.2), although it is costly for the high-cost type to pool with the low-cost type, the low-cost type of firm 1 still needs to sacrifice its short-run profit and engages in limit pricing to distinguish himself from the high-cost type. We argue that this type of equilibrium cannot be an HTE because there is no a rational belief system to support the equilibrium when an out-of-equilibrium message Q_M^L is observed.

3.7 Conclusions

In this chapter, we propose a general definition of Hypothesis Testing Equilibrium (HTE) in a framework of general signalling games with non-Bayesian players. We focus on the analysis of a special class of HTE with the threshold $\epsilon = 0$ as a means of equilibrium refinement that can survive the Intuitive Criterion. For a broad class of signalling games, a Lexicographically Maximum Nash sequential equilibrium (*LMSE*) is an HTE, and if the *LMSE* is completely separating, then the *constrained HTE* is unique. In the example of an entry deterrence game, we show that there exists a unique HTE in each interesting case. However, there are aspects of HTE that we have not considered, in particular, the existence and uniqueness in more general signalling games. Natural extensions are to apply the concept of HTE in signalling games in which the state space is infinite or to consider general dynamic games. When $\epsilon > 0$, the dynamic consistency condition is violated, and there will be some difficulties in making these extensions. Nevertheless, when $\epsilon = 0$, dynamic consistency holds, and in that case further research to apply the idea of HTE to dynamic games, even more general games, for equilibrium refinement is encouraged.

Chapter 4

Entry Deterrence Games under Ambiguity

Abstract

In this chapter, we introduce ambiguity into an entry deterrence game in which the potential entrant has multiple priors on the true state of aggregate demand. We formulate two models with respect to the information asymmetries between the established firm and the potential entrant. In the first model, the established firm is fully informed of the true state, and in the second, the established firm is also uncertain about the state but is informed of the distribution of the state. In both models, we characterize the conditions under which limit pricing emerges in equilibria, and thus ambiguity decreases the probability of entry. Welfare analysis shows that limit pricing is more harmful in a market with higher expected demand than in a market with lower expected demand.

Keywords: Entry Deterrence Game, Asymmetric Information, Ambiguity, Message-Monotone Equilibrium.

4.1 Introduction

Games with incomplete information typically involve the situation in which the players are uncertain about some important parameters of the game, such as the payoff functions, the strategies available to various players, or the information that other players have about the game. The normal form of such games is well studied (Harsanyi, 1967, 1968a, 1968b). Extensive-form games with incomplete information in which one informed player, who possesses private information, sends a signal to a second party, who thereupon takes an action have been also widely considered as signalling games (Spence, 1972, Cho & Kreps, 1987, Fudenberg & Tirole, 1991, etc.). However, in most of the literature, it is assumed that the probability distributions entertained by the different players are mutually ”consistent”, in the sense that they can be regarded as conditional probability distributions derived from a certain “basic probability distribution” or “common prior” over the parameters unknown to the various players.

In this chapter, we introduce a different concept of the asymmetric information structure in an entry deterrence game between one more informed player and one less informed player. The established firm is more informed since it has already made investment commitment. Either it has knowledge of the true state of the market (the aggregate demand, costs, for example), or at least, it has confidence in the probability distribution of the state. While the potential entrant has little information and hence little confidence regarding to the true state of the market, it may have a set of probability measures over the state space. These two different situations with respect to uncertainty are distinguished by Knight (1921). The situation in which the uncertainty can be governed by a unique probability measure is called “*measurable uncertainty*” or “*risk*”. In contrast, we use “*Knightian uncertainty*” or“*ambiguity*” to refer to situations in which individuals cannot or do not assign subjective probabilities to uncertain events. The Ellsberg Paradox (Ellsberg, 1963) has shown that this distinction is behaviourally meaningful since people treat ambiguous bets differently from risky bets. Importantly, the lack of confidence reflected by choices in the Ellsberg Paradox cannot be rationalized by any probabilistic belief; see Ellsberg (1963). In the literature, theoretical models of individual preferences for decisions under ambiguity include Maxmin Expected Utility (MEU) (Gilboa and Schmeidler, 1989), smooth ambiguity preference (Klibanoff et al., 2005), and variational representation of preferences (Maccheroni, et al., 2006).

In a seminal study of limiting pricing and entry deterrence models with incomplete information, Milgrom and Roberts (1982) formulated a two-period two-firm model in which the firms know the realizations of their own unit costs but those of their opponent. However, the joint distribution of the costs is common knowledge. In their setting, both firms face "*measurable uncertainty*" regarding their opponent's costs. They studied the impact of the information asymmetries on the entry probability compared with the complete information case. We introduce "*Knightian uncertainty*" in a simplified version of the Milgrom-Roberts model and discuss two cases of information asymmetry between the established firm and the potential entrant.

We formulate a two-period game in which the payoffs of the two players depend on an aggregate demand function with two possible alternatives. In the first model, the established firm (firm 1) is fully informed of the true state, while the potential entrant (firm 2) lacks sufficiently precise information to assign a unique probability measure governing the uncertainty and instead has a set of priors on the state. We identify the conditions under which limit pricing emerges in equilibrium and discuss the impact of ambiguity on the probability of entry. We compare our results with other two situations: the situation in which firm 2 is also informed of the true state and the situation in which firm 2 faces measurable uncertainty.

However, the established firm may not know the realization of demand but only expected demand before it begins to produce and enters the market. Therefore, we formulate a second model in which both of the firms face uncertainty, but firm 1 is under risk, while firm 2 is under ambiguity. Since we employ a linear demand function, choosing quantity as firm 1's strategy in the first period is equivalent to choosing price. In this model, firm 1 chooses a price to charge in the first period, and it will learn the realization of the uncertain demand in the first period. Firm 2 updates its set of priors after observing the price charged by firm 1 in the first period and decides whether to enter the market. If firm 2 enters, it also makes an investment commitment and learns the real prior immediately; then the two firms engage in Cournot competition. Otherwise, firm 1 remains its monopoly. From the information structure, we can see that firm 1 has an informational advantage in both pre- and post-entry markets.

We focus on the second model and characterize a message-monotone equilibrium in which firm 1's strategy (price charged in the first period) is weakly increasing in the probability of high aggregate demand. One of the main findings is that, in this equilibrium, the limit pricing strategy emerges when the probability of high demand is between two thresholds and the entrant is deterred by the limit pricing. The lower threshold is the point at which firm 2 is indifferent between entering the market and staying out the market. The upper threshold is the point at which firm 1 is indifferent between engaging in limit pricing to successfully deter the entrant and charging the monopoly price to induce entry. We use Maxmin Expected Utility (MEU) to represent firm 2's preferences with respect to ambiguity aversion. Compared with the symmetric information case in which firm 2 also knows the distribution of the unknown parameter, the entrant behaves more cautiously when there is ambiguity. The probability that entry actually occurs in such an equilibrium is equal to or less than that in a symmetric information case. However, the entry probability decreases only when the degree of ambiguity is large enough that the indifference point of firm 2 is located in the set of its priors. A welfare analysis reveals that the changes in expected consumer surplus due to limit pricing are increasing in the expected aggregate demand under certain standard conditions. However, limit pricing decreases the expected industry surplus and total welfare. Further, it is more harmful in a market with higher expected demand than in a market with lower expected demand.

The remainder of this chapter is organized as follows. Section 4.2 briefly discusses a model in which the established firm is completely informed of the demand function but the entrant is not informed and instead has multi-priors on the unknown parameter in the demand function. Section 4.3 is the main part of this chapter. In this section, we describe a model in which both firms are uncertain about the demand function, but there is asymmetric information with respect to uncertainty. The established firm is under risk, but the entrant is under ambiguity. In this section, we characterize one message-monotone equilibrium and discuss the impact of ambiguity. Section 4.4 discusses the impact of information asymmetry on welfare. Finally, section 4.5 presents the conclusions.

4.2 Entry deterrence game under uncertainty on one side

4.2.1 Description of the game

Consider the market for a homogeneous good in which there are an established firm, denoted firm 1, and a potential entrant, denoted firm 2. The inverse demand function for the industry output is $P = a - bQ$. The parameter a is assumed to be one of the two possible values a^H and a^L, with $0 < a^L < a^H$. Thus $a = a^H$ reflects a higher aggregate demand state than $a = a^L$. A two-period model of entry deterrence and entry under incomplete information proceeds as follows: At stage 0, nature selects the value of a according to a distribution $\mathbb{P}(a = a^H) = 1 - \mathbb{P}(a = a^L) = x \in [0, 1]$ and fixes a through the next two periods. Firm 1 is informed the value of a. But firm 2 only has knowledge that $x \in [\underline{x}, \overline{x}] \subseteq [0, 1]$.

At stage 1, firm 1 has to choose a price to charge for the production. Observing the price charged by firm 1, firm 2 decides to enter the market or not.

At stage 2, if firm 2 decides to enter the market and makes commitment on the investment with a fixed cost K, it also can learn the value of a and the two firms proceed Cournot competition in this stage. Otherwise, if firm 2 doesn't enter, firm 1 will enjoy the monopoly profit.

We assume that the firms are risk neutral and ambiguity averse. The unit costs, c_i, $i = 1, 2$, which are constants, and the description of the game are common knowledge.

4.2.2 Strategies and payoffs

The price selected by firm 1 in the first period influences the choice of firm 2, which will affect the total profit of firm 1 in the whole two periods. Therefore, if we restrict attention to the pure strategy, a strategy of firm 1 is a mapping:

$$s : \{a^L, a^H\} \to \mathbb{R}_+,$$

where $s(a)$ is the price chosen by firm 1 in the first period conditional on a, $a \in \{a^L, a^H\}$. A pure strategy of firm 2 is a mapping:

$$t : \mathbb{R}_+ \to \{0, 1\}.$$

Then $t(s) = 0$ means that firm 2 decides to stay out of the market observing price s, and $t(s) = 1$ implies that firm 2 decides to enter the market observing price s. In order to derive the total payoffs of the two firms, let us analyze the post-entry Cournot duopoly market first. If firm 2 decides to enter the market, it also learns the value of a, this is a typical one shot game with complete information. In this sub-game, the profit of firm i, $i, j = 1, 2$ conditional on a is given by

$$\Pi_C^i = \frac{(a - 2c_i + c_j)^2}{9b}.$$

Given the strategy t of firm 2, a strategy s gives a total payoff to firm 1:

$$\begin{aligned}\Pi^1 &= \Pi_1^1 + \delta_1 \Pi_2^1 \\ &= \frac{(s - c_1)(a - s)}{b} + \delta[t(s)\Pi_C^1 + (1 - t(s))\Pi_M^1],\end{aligned}$$

where Π_M^1 is the monopoly profit if firm 2 doesn't enter the market, and

$$\Pi_M^1 = \frac{(a - c_1)^2}{4b}.$$

Therefore, the total profit of firm 1 is

$$\Pi^1 = \frac{(s - c_1)(a - s)}{b} + \delta\left[t(s)\frac{(a - 2c_1 + c_2)^2}{9b} + (1 - t(s))\frac{(a - c_1)^2}{4b}\right]. \tag{4.1}$$

To derive the expected profit of firm 2 is a bit complicate since firm 2 thinks in terms of a set of probability laws of the unknown parameter a, that is, it assigns an interval $[\underline{x}, \overline{x}]$ to the probability of $a = a^H$. Here we adopt the Maxmin Expected Utility (MEU) in Gilboa & Schmeidler(1989) to represent firm 2's ambiguity aversion. Formally and generally, the multiple-prior model postulates the following utility function on the set of AA acts:

$$U^{MP}(h) = \min_{p \in C} \int_{\Omega} u(h) dp.$$

Here, $u : \Delta(C) \to \mathbb{R}$ is a vNM functional on lotteries that is affine. Conditional on any observed $s \in \mathbb{R}_+$, firm 2 assigns probabilities to s having been the choice of each type of firm 1, then firm 2 updates its set of priors first. Denote the updated set of priors as $[\underline{x}', \overline{x}']$. For each s, the expected payoff of firm 2 with a strategy $t(s)$ is:

$$\Pi^2 = \min_{x \in [\underline{x}', \overline{x}']} E[t(s)(\Pi_C^2(a) - K)]. \tag{4.2}$$

4.2.3 Equilibrium Analysis

A pair of strategies profile (s^*, t^*) forms a Nash equilibrium if, s^* maximizes firm 1's total payoff (4.1) given t^*, and t^* is a best response for any observed s given s^*, that is, t^* maximizes (4.2) for each s. Since there are only two possible alternatives of a, firm 1 has only two types. Therefore, in equilibrium, the only values of s which could be observed are $s^*(a^L)$ and $s^*(a^H)$. Now in this set-up there only exist either pooling equilibria, that is, $s^*(a^L) = s^*(a^H)$, or separating equilibria, that is, $s^*(a^L) \neq s^*(a^H)$. We will discuss the conditions of the Nash equilibrium under the following assumptions:

Assumption 4.1.

$$\begin{aligned} &(i). \quad \Pi_C^2(a^L) - K < 0 \quad \textit{and} \quad \Pi_C^2(a^H) - K > 0. \\ &(ii). \quad a^L - 2c_2 + c_1 > 0. \end{aligned}$$

Assumption 4.2. *Firm 2's optimal choice is staying out of the market under it's original information.*

Assumption 4.1(i) just simply means that firm 2 won't enter the market if it is informed that $a = a^L$ and it will enter the market if $a = a^H$. Assumption 4.1(2) implies that if firm 2 enters the market, then it produces positive quantity in the Cournot market even at the low state of demand. Under Maxmin Expected utility, Assumption 4.2 can be explicitly expressed as:

$$\begin{aligned}
&\min_{x\in[\underline{x},\overline{x}]} E[(\Pi_C^2(a)-K)] \\
&=\min_{x\in[\underline{x},\overline{x}]} \frac{(a^e(x)-2c_2+c_1)^2}{9b}-K \\
&=\frac{(a^e(\underline{x})-2c_2+c_1)^2}{9b}-K<0,
\end{aligned}$$

where $a^e(x) = xa^H + (1-x)a^L$ is the expectation of a under the law of a, $\mathbb{P}(a^H) = x$. The last equality holds because of A4.2(ii). Let $f(x) = \frac{(a^e(x)-2c_2+c_1)^2}{9b}$, then $f'(x) > 0$, $\forall\ x \in [0,1]$, and $f'(0) > 0$.

In a pooling equilibrium (s^*, t^*), $s^*(a^H) = s^*(a^L) = s$. Given the strategy of firm 1, firm 2 can't learn any information observing s. Therefore, firm 2 still has the set of multiple priors $[\underline{x}, \overline{x}]$. Assumption 4.2 implies that the best response of firm 2 conditional on s is $t^*(s) = 0$. If firm 1 uses separating strategy, firm 2 can learn the true value of a, therefore, the ambiguity is resolved. Then we are back to the model with complete information. Clearly, there exist multiple equilibria, however, we're interested in the "efficient equilibrium" where the competition between types of firm 1 won't be unnecessarily wasteful. The following proposition provides the conditions under which a pooling equilibrium or a separating equilibrium exists.

Proposition 4.1. *A pooling equilibrium,s*

$$s^*(a^H) = s^*(a^L) = s^M(a^H)$$

$$t^*(s) = \begin{cases} 0 & \textit{if} \quad s \le s^M(a^H) \\ 1 & \textit{otherwise} \end{cases}$$

exists if

$$\Pi_1^1(s^M(a^H)) + \delta\Pi_M^1(a^L) > \Pi_M^1(a^L) + \delta\Pi_C^1(a^L). \tag{4.3}$$

Otherwise, if (4.3) *doesn't hold, there exists a separating equilibrium:*

$$s^*(a^H) = s^M(a^H); \quad s^*(a^L) = s^M(a^L)$$
$$t^*(s) = \begin{cases} 0 & \text{if} \quad s < s^M(a^H) \\ 1 & \text{if} \quad s \geq s^M(a^H). \end{cases}$$

Where $s^M(a^t)$ *is the monopoly price of type* t, *and* $s^M(a^t) = \frac{a+c^t}{2}$, *for* $t \in \{L, H\}$.

As long as we clarify the updated set of priors of firm 2, the proof is quite straightforward, we omit the proof here. Since $s^M(a^L) < s^M(a^H)$, the limit pricing emerges when condition (4.3) holds.

4.2.4 Discussion of the impact of ambiguity

We can discuss the impact of the ambiguity by comparing with other two situations: firm 2 is also informed the value of a, and firm 2 is informed the true distribution of a.

Situation 1: firm 2 is also completely informed. In this case, there doesn't exist pooling equilibrium, that is, limit pricing doesn't emerge. Therefore, ambiguity decreases the probability of entry comparing with regime with complete information if condition (4.3) holds. Otherwise, if (4.3) doesn't hold, ambiguity has no influence on the probability of entry.

Situation 2: firm 2 is informed of the true distribution of a. In this case, firm 2 is under "measurable uncertainty" or "risk". We denote the true distribution as $\mathbb{P}(a = a^H) = 1 - \mathbb{P}(a = a^L) = x_0$. In order to discuss the impact of ambiguity, we need to address different cases combining the condition (4.3) and the following one:

$$\frac{(a^e(x_0) - 2c_2 + c_1)^2}{9b} - K < 0. \tag{4.4}$$

If both (4.3) and (4.4) hold, limit pricing emerges just as the ambiguity situation. In this case, ambiguity doesn't influence the probability of entry. If (4.4) doesn't hold, limit pricing doesn't emerge. Then ambiguity decreases the probability of entry if (4.3) also holds.

4.3 Entry deterrence game with asymmetric information on uncertainty

4.3.1 Description of the asymmetric information structure

In the previous model, the established firm has knowledge of the true state of the demand before it makes move in the first period, and firm 2 is even unaware of the distribution of the state. However, even though firm 1 has committed the investment (for example, set up the plants, did some market research, etc.), it is still difficult to know the true state before it sets a price and sells the production in the market. In this section, we modify the structure of the information asymmetries in the entry deterrence game as both the firms are uncertain on the true state of the demand. Since the established firm has committed the investment, it has collected more precise information of the market. In this setting, firm 1 knows the true distribution of a, that is, it knows that $x = x_0$. Firm 2 is in the same situation as the previous model, it has multiple priors on a, $x \in [\underline{x}, \overline{x}] \subseteq [0, 1]$. We assume that $x_0 \in [\underline{x}, \overline{x}]$ for consistency. The asymmetric information situation is common knowledge but not the information itself. The rest of the description of the game is the same as the model in section 4.2. The production unit cost c_i of firm i ($i = 1, 2$) is constant and known to both firms. Firms are risk-neutral and ambiguity averse, and firms 2's ambiguity aversion is represented by Maxmin Expected Utility.

4.3.2 Strategies and payoffs

In order to simplify the the payoff functions, we take the method given by Milgrom and Roberts (1982) to normalize the profit to firm 1 in the second period to be zero if entry occurs. Otherwise firm 1 gets a reward which equals to the difference between the monopoly profit and the Cournot profit if entry doesn't occur. We assume that firm 2 can learn the distribution of a immediately as soon as it decides to enter the market and makes investment commitment with a fixed cost K. Firm 1 learnt the true state of a by observing its demand in the first period. Under this framework, firm 1 has information advantage both in the pre-entry market and in the post-entry market.

We analyze the post-entry Cournot duopoly market first, which is a typical one shot game with incomplete information on one side. In this sub-game, firm 1's pure strategy is to choose an output level conditional on a, $Q_1(\cdot) : \{a^H, a^L\} \to \mathbb{R}_+$, to maximize it's profit:

$$\max_{Q_1}[a - b(Q_1 + Q_2) - c_1]Q_1.$$

Given the quantity of production of firm 2, Q_2, the reaction curve of firm 1 is:

$$Q_1(Q_2; a) = \frac{a - bQ_2 - c_1}{2b}. \tag{4.5}$$

Firm 2's pure strategy is to choose an output level Q_2 conditional on x, $Q_2(\cdot) : [0, 1] \to \mathbb{R}_+$, to maximize its expected profit:

$$\begin{aligned} &\max_{Q_2} E_a[a - b(Q_1(a) + Q_2) - c_2]Q_2 \\ =&\max_{Q_2}[a^e(x) - b(Q_1^e(x) + Q_2) - c_2]Q_2, \end{aligned}$$

where $a^e(x) = xa^H + (1 - x)a^L$ and $Q_1^e(x) = xQ_1(a^H) + (1 - x)Q_1(a^L)$. Then the reaction curve of firm 2 is:

$$Q_2(Q_1; x) = \frac{a^e(x) - bQ_1^e(x) - c_2}{2b}. \tag{4.6}$$

We can get the Equilibrium Cournot quantities from equation (4.5) and (4.6):

$$\begin{aligned} Q_2(x) &= \frac{a^e(x) + c_1 - 2c_2}{3b}, \\ Q_1(a) &= \frac{3a - a^e(x) - 4c_1 + 2c_2}{6b}, \qquad \text{for} \quad a \in \{a^H, a^L\}. \end{aligned}$$

According to the aggregate demand function, we can obtain the *ex post* equilibrium price which firm 1 can anticipate *ex ante*:

$$P(a) = \frac{3a - a^e(x) + 2c_1 + 2c_2}{6}, \qquad \text{for} \quad a \in \{a^H, a^L\}.$$

Now we can compute the Cournot profit of firm 1 conditional on a:

$$\begin{aligned}\Pi_{1C}(a,x) &= [P(a)-c_1]Q_1(a) \\ &= \frac{(3a-a^e(x)-4c_1+2c_2)^2}{36b},\end{aligned}$$

and the expected Cournot profit of firm 2:

$$\begin{aligned}\Pi_{2C}^e(x) &= E_a[P(a)-c_2]Q_2(x) \\ &= \frac{(a^e(x)+c_1-2c_2)^2}{9b}.\end{aligned}$$

As long as we pin down the Cournot competition game in the second period, we can analyze the strategies and payoffs of the two firms in the two periods. As we illustrated in the game description, firm 1's strategy in the first period influences its total payoff, and it also serves as a signal transferred to firm 2 to influence firm 2's decision in the second period. We restrict attention to pure strategy equilibria. Denote the pure strategy of firm 1 as:

$$s : [0,1] \to \mathbb{R}_+,$$

then $s(x)$ is the price chosen by firm 1 in the first period conditional on the probability distribution of a. Denote the pure strategy of firm 2 as:

$$t : \mathbb{R}_+ \to \{0,1\}.$$

If firm 2 decides to stay out of the market observing s, then $t(s)=0$. Otherwise, if firm 2 decides to enter the market, then $t(s)=1$. Given these strategies, we can write down the total expected payoffs of the two firms. Let $\Pi_1^e(s;x,t)$ and $\Pi_2^e(t;x,s)$ be the total expected profits of firm 1 and firm 2 respectively. Then

$$\begin{aligned}\Pi_1^e(s;x,t) &= \Pi_1^{0e}(s;x)+\delta_1 R^e(x)(1-t) \\ &= \Pi_1^{0e}(s;x)+\delta_1 E_x[\Pi_M(a)-\Pi_{1C}(a,x)](1-t),\end{aligned}$$

and

$$\Pi_2^e(t;x,s) = \delta_2 t(s)[\Pi_{2C}^e(x)-K],$$

where δ_i is the discount factor of firm $i, i = 1, 2$, $\Pi_1^{0e}(s; x)$ is firm 1's expected profit in the first period if it chooses price s, and $R^e(x)$ is firm 1's expected reward profit if it succeeds in deterring firm 2 which is equal to the expected monopoly profit $E_a[\Pi_M(a)]$ minus the expected Cournot profit $E_a[\Pi_{1C}(a, x)]$. If firm 1 decides to reveal itself in the first period, then knowing the probability distribution of a, firm 1 chooses a monopoly price to maximize its expected profit:

$$\max_s E\left[\frac{(s - c_1)(a - s)}{b}\right].$$

Solving this optimization problem,

$$s_M^e(x) = \frac{a^e(x) + c_1}{2},$$

then we can get the expected monopoly profit.

$$E_a[\Pi_M(a)] = \Pi_{1M}^e(x) = \frac{[a^e(x) - c_1]^2}{4b}.$$

To make things interesting, we assume that firm 1's expected monopoly profit is greater than its expected Cournot profit, that is, $\Pi_{1M}^e(x) > \Pi_{1C}^e(x)$.

4.3.3 Equilibrium analysis

Again, we adopt the Maxmin Expected Utility to represent the ambiguity aversion of firm 2 with multiple priors . Assume that firm 1 plays some strategy $\overline{s} : [0, 1] \to \mathbb{R}_+$, then any P (price) in the range of $\overline{s}$, firm 2 updates its set of priors by $x \in \overline{s}^{-1}(P)$. Firm 2's best response is "enter" if and only if the minimum expected value of the post-entry profit, $\inf_{x\in\overline{s}^{-1}(P)}(\Pi_{2C}^e(x) - K)$, is positive. Formally, we give the definition of Nash equilibrium as follows:

Definition 4.1. *A strategy profile (s^*, t^*) is a Nash equilibrium of the entry deterrence game if it satisfies the following conditions:*

(i) for any $x \in [0,1]$ and any $s : [0,1] \to \mathbb{R}_+$

$$\Pi_1^{0e}(s^*;x)+\delta_1 R^e(x)(1-t^*(s^*)) \geq \Pi_1^{0e}(s;x)+\delta_1 R^e(x)(1-t^*(s)).$$

(ii) for any $t : \mathbb{R}_+ \to \{0,1\}$, for all $P \in \mathbb{R}_+$, such that $\exists x \in [0,1]$, $s^*(x) = P$,

$$\inf_{x\in\{x:s^*(x)=P\}} t^*(P)(\Pi_{2C}^e(x)-K) \geq \inf_{x\in\{x:s^*(x)=P\}} t(P)(\Pi_{2C}^e(x)-K).$$

For simplicity, we analyze the equilibrium under an additional assumption:

Assumption 4.3. *$[\underline{x},\overline{x}] = [0,1]$, $c_2 \geq c_1$ and $\delta_1 = 1$.*

In the equilibrium analysis, we focus on message-monotone equilibrium (Y. Chen, 2011).

Definition 4.2. *In a message-monotone equilibrium in the entry deterrence game, the strategy of firm 1, $s(x)$, is weakly increasing in x.*

The continuity of the duopoly Cournot payoff function together with assumption 4.1 imply that there exists a belief $\hat{x} \in (0,1)$ such that firm 2 is indifferent between "enter" and "not enter". With a simple calculation, we can get the indifferent point which is:

$$\hat{x} = \frac{\sqrt{9bK}-(a^L+c_1-2c_2)}{a^H-a^L}.$$

Since we assume that $\Pi_M^e(x) > \Pi_{1C}^e(a,x)$ for all $x \in [0,1]$, which implies that firm 1 has incentive to deter the entrant by sacrificing the profit in the first period due to charging a lower price. And it's getting more and more costly to deter the entrant as x increases. Under assumption $\delta_1 = 1$, the indifference point of firm 1, $\tilde{x}$, satisfies the following condition (see Appendix A for details):

$$\frac{(a^e(\hat{x})-c_1)^2}{4} = \frac{(a^e(\tilde{x})-2c_1+c_2)^2}{9}.$$

Given these two critical points $\hat{x}$ and $\tilde{x}$ together with assumptions A4.1-A4.3, we can get the following property of a Nash equilibrium:

Proposition 4.2. *In this entry deterrence model, under A4.1-A4.4, a Nash equilibrium (s^*, t^*) is a message-monotone equilibrium.*

The proof is given in Appendix B.

An increasing x indicates a higher expected aggregate demand. With message-monotone strategies, firm 1 sends a more attractive message to firm 2 by choosing a non-decreasing price when the expected aggregate demand increases. On the other hand, observing a higher price, firm 2 can induce a higher expected demand. Therefore, firm 2's strategy is also non-decreasing with respect to the price it observed. The following theorem characterizes such a message-monotone equilibrium.

Theorem 4.1. *Under the assumptions A4.1-A4.3, there exists a message-monotone equilibrium (s^*, t^*) in the entry deterrence game in two different cases:*

(i) if $\tilde{x} \leq \hat{x}$, then there exists a separating equilibrium (s^*, t^*):

$$s^*(x) = s^e_M(x) \quad \forall x \in [0, 1],$$

$$t^*(s) = \begin{cases} 0, & \textit{if} \quad s \leq s^e_M(\hat{x}) \\ 1, & \textit{otherwise} \end{cases}$$

(ii) if $\tilde{x} \geq \hat{x}$, then there exists a semi-separating equilibrium (s^*, t^*):

$$s^*(x) = \begin{cases} s^e_M(x) & \textit{if} \quad x \in [0, \hat{x}) \cup (\min\{\tilde{x}, 1\}, 1] \\ s^e_M(\hat{x}) & \textit{if} \quad x \in [\hat{x}, \min\{\tilde{x}, 1\}], \end{cases}$$

$$t^*(s) = \begin{cases} 0, & \textit{if} \quad s \leq s^e_M(\hat{x}) \\ 1, & \textit{otherwise} \end{cases}$$

Proof (see the appendix C)

Remark In the first case, for any level of expected demand, it is so costly to deter the entrant that the reward is not large enough to compensate the lost if firm 1 conducts limit pricing in the first period, therefore, limit pricing doesn't emerge in this situation. In the second case, in contrast, when the expected demand is low enough such that the entrant is not interested in entering the market, firm 1 doesn't need to conduct limit pricing. Or the expected demand is high enough that the signal sent by firm 1 is also high such that the entrant will enter the market observing the signal, then it is unnecessary for firm 1 to conduct limit pricing. But when the expected demand is between the two thresholds, limit pricing emerges.

4.4 Welfare Analysis

4.4.1 The impact of ambiguity on expected welfares

In this sector, we discuss the impact of ambiguity by comparing our results with the situation where firm 2 is also informed the probability distribution of a. If there doesn't exist information asymmetries between firm 1 and firm 2, then firm 1 cannot deter firm 2 by conducting limit pricing because firm 2 doesn't need to induce any information from the behavior of firm 1 in the first period. Firm 2 will choose to enter the market when $x > \hat{x}$ and stay out the market otherwise. Therefore, limit pricing doesn't emerge and firm 1 just charges the expected monopoly price in the first period for any $x \in [0, 1]$. In order to discuss the impact of ambiguity on expected welfare, we just need to analyze the impact of limit pricing on the expected welfares.

To focus on the impact of limit pricing, we assume that $\hat{x} < \tilde{x} \leq 1$.

Consumer's surplus: In the previous section, we have shown that the price decreases from the expected monopoly price $P_M(x) = s^e_M(x)$ to the limit price $P_M(\hat{x}) = s^e_M(\hat{x})$ when $x \in [\hat{x}, \tilde{x}]$ in the first period, and

$$P_M(\hat{x}) - P_M(x) = \frac{a^e(\hat{x}) - a^e(x)}{2} < 0.$$

Due to limit pricing, the entrant is deterred in the second period, the monopoly price is greater than the Cournot price:

$$\begin{aligned} P_M(a) - P_C(a) &= \frac{a + c_1}{2} - \frac{3a - a^e(x) + 2c_1 + 2c_2}{6} \\ &= \frac{a^e(x) + c_1 - 2c_2}{6} > 0. \end{aligned}$$

Therefore, the consumer's surplus increases in the first period and decreases in the second period. The change of total consumer's surplus is ambiguous. However, we still can analyze the changes of consumer's surplus with respect to the state of the market. We can clearly see that the price decreases in the first period faster than it increases in the second period as x increases, i.e.

$$\left|\frac{\partial(P_M(\hat{x}) - P_M(x))}{\partial x}\right| > \frac{\partial(P_M(a) - P_C(a))}{\partial x}.$$

Therefore, we expect that the consumer's surplus increases in x. Formally, The net consumer's surplus from without limit pricing to limit pricing is:

$$\begin{aligned}
\Delta S^C &= -\int_{P_M(x)}^{P_M(\hat{x})} \frac{a-p}{b} dp - \int_{P_C(a)}^{P_M(a)} \frac{a-p}{b} dp \\
\frac{\partial \Delta E(S^C)}{\partial x} &= \frac{a^H - a^L}{2}\left[E\left[\frac{a - P_M(x)}{b}\right] - \frac{1}{3}E\left[\frac{a - P_C(a)}{b}\right]\right] \\
&= \frac{a^H - a^L}{2}(E_x[Q_M] - \frac{1}{3}E_x[Q_C]) \\
&= \frac{a^H - a^L}{2}(Q_M^e(x) - \frac{1}{3}Q_C^e(x))
\end{aligned}$$

We can see that the expected consumer's surplus is increasing in x as long as the expected aggregate monopoly production $E_x[Q_M]$ at x is greater than one third of the expected aggregate Cournot production $E_x[Q_C]$.

Firm's surplus: Firm's surplus is equal to firm's profit. The net firm's surplus from without limit pricing to limit pricing is:

$$\begin{aligned}
E[\Delta S^F] &= \Pi_{1M}^e(\hat{x}) + \Pi_{1M}^e(a) - (\Pi_{1M}^e(x) + \Pi_{1C}(a) + \Pi_{2C}^e(x)) \\
&= \{(P_M(\hat{x}) - c_1)Q_M(\hat{x})) + (P_M(a) - c_1)Q_M(a)\} \\
&\quad - \{(P_M(x) - c_1)Q_M(x) + (P_C(a) - c_1)Q_{1C} + (P_C(a) - c_2)Q_{2C}\} \\
&= \frac{(P_M(\hat{x}) - c_1)^2}{4b} + \frac{(P_M(a) - c_1)^2}{4b} \\
&\quad - \left\{\frac{(P_M(x) - c_1)^2}{4b} + \frac{(P_C(a) - c_1)^2}{b} + \frac{(P_C(a) - c_2)(a^e(x) + c_1 - 2c_2)}{3b}\right\} \\
&= \frac{(a^e(\hat{x}) - c_1)^2}{4b} - \left(\frac{(a^e(x) - 2c_1 + c_2)^2}{9b} + \frac{(a^e(x) - 2c_2 + c_1)^2}{9b}\right).
\end{aligned}$$

The net industry' surplus equals to the difference between the expected monopoly profit at $\hat{x}$ and the total expected Cournot profits at x. Since $x > \hat{x}$, the net industry's surplus is negative if c_2 is not very large. We conclude that deterring an moderate cost entrant by limit pricing harms the industry. Moreover,

$$\begin{aligned}\frac{\partial E[\Delta S^F]}{\partial x} &= -\frac{2(a^H - a^L)(a^e(x) - c_1 - c_2)}{9b} \\ &= -\frac{2(a^H - a^L)Q^e_C(x)}{3} < 0,\end{aligned}$$

where $Q^e_C(x)$ is the expected total Cournot production at x if firm 2 enters the market in the second period. This result shows that the limit pricing harms the industry more in a market with higher expected demand than a lower one.

Total welfare: The total welfare is the sum of the consumer's surplus and the firm's surplus. Therefore,

$$\Delta TW = \Delta S^C + \Delta S^F,$$

and

$$\begin{aligned}\frac{\partial E[\Delta TW]}{\partial x} &= -\frac{(a^H - a^L)}{6}(3Q^e_M(x) - 5Q^e_C(x)) \\ &\begin{cases} < 0 & \text{if} \quad 3Q^e_M(x) - 5Q^e_C(x) < 0 \\ > 0 & \text{if} \quad 3Q^e_M(x) - 5Q^e_C(x) > 0. \end{cases}\end{aligned}$$

Limit pricing decreases the total social welfare more in a higher expected demand market than in a lower one when the expected monopoly output is less than 60% of the expected Cournot output.

In a general framework, if firm 2 has a set of priors $x \in [\underline{x}, \overline{x}]$, the impact of ambiguity on expected welfare occurs only when an increase of ambiguity changes the strategy of firm 1 from non-limit pricing to limit pricing, or vice versa . As we can see from the equilibrium analysis, limit pricing doesn't occur as long as the realization of x is in the region $[\underline{x}, \hat{x}] \cup [\tilde{x}, \overline{x}]$. Therefore, the ambiguity doesn't influence the strategy of firm 1 or the welfare when $x \in [\underline{x}, \hat{x}] \cup [\tilde{x}, \overline{x}]$. Ambiguity decreases the total expected welfare when $x \in [\hat{x}, \tilde{x}]$ comparing with symmetric information situation. However, the change of the degree of ambiguity influences the welfare only when firm 2's set of priors changes such that $\underline{x}$ moves from one side of $\hat{x}$ to the other. We present an example in the next section to illustrate the results.

4.4.2 An example

In this section, we provide a numerical example to compute the equilibrium strategies in the entry deterrence game with asymmetric information on uncertainty. Suppose that $a^H = 10$, $a^L = 5$, $b = 1$, $\delta_1 = \delta_2 = 1$, $c_1 = c_2 = 1$, $K = 3.5$. The probability distribution of a is $\mathbb{P}(a = a^H) = 1 - \mathbb{P}(a = a^L) = x$. The set of priors of firm 2 on the distribution of a is $[0, 1]$. One can compute that the indifferent point of firm 2 is $\hat{x} = 0.322$, and the indifferent point of firm 1 is $\tilde{x} = 0.884$. The expected monopoly price charged by firm 1 at $x = \hat{x}$ is: $s^e_M(\hat{x}) = \frac{a^e(\hat{x})+c_1}{2} = 3.805$. At equilibrium, the message-monotone strategy of firm 1 is:

$$s^*(x) = \begin{cases} 3.805 & \text{if} \quad x \in [0.322, 0.884) \\ \frac{5x+6}{2} & \text{if} \quad x \in [0, 0.322) \cup [0.884, 1], \end{cases}$$

and the strategy of firm 2 is

$$t^*(s) = \begin{cases} 0 & \text{if} \quad s \leq 3.805 \\ 1 & \text{if} \quad s > 3.805. \end{cases}$$

In this example firm 1 can take the advantage of the information asymmetries by using pooling strategy to deter the entrant if the probability of the higher aggregate demand is not too low or too high. In this example, we can see (Figure 2) that both the expected consumer's surplus and the expected industry's surplus decrease due to the limit pricing, and limit pricing also decreases the expected total welfare. The expected net consumer's surplus is increasing in x but the expected total welfare is decreasing in x, which means that the established firm has to sacrifice more profits in the first period in order to deter the entrant in a market with higher expected demand. This result suggests that an antitrust policy is more important in a market with higher expected demand.

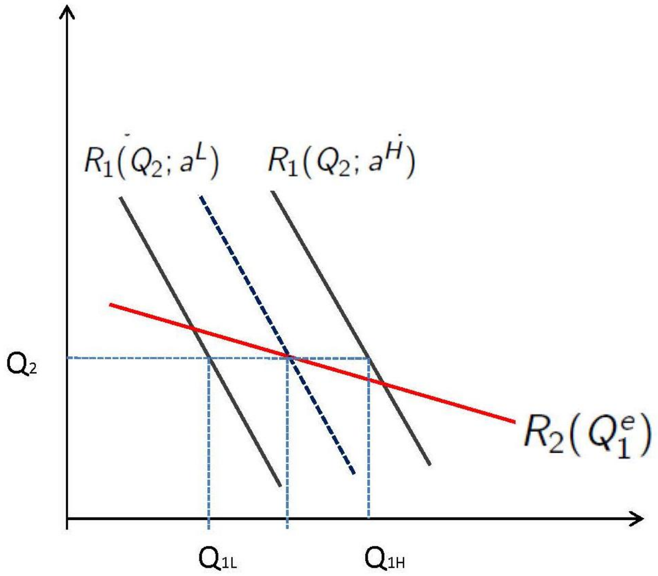


Figure 4.1: The response functions in the Cournot competion game

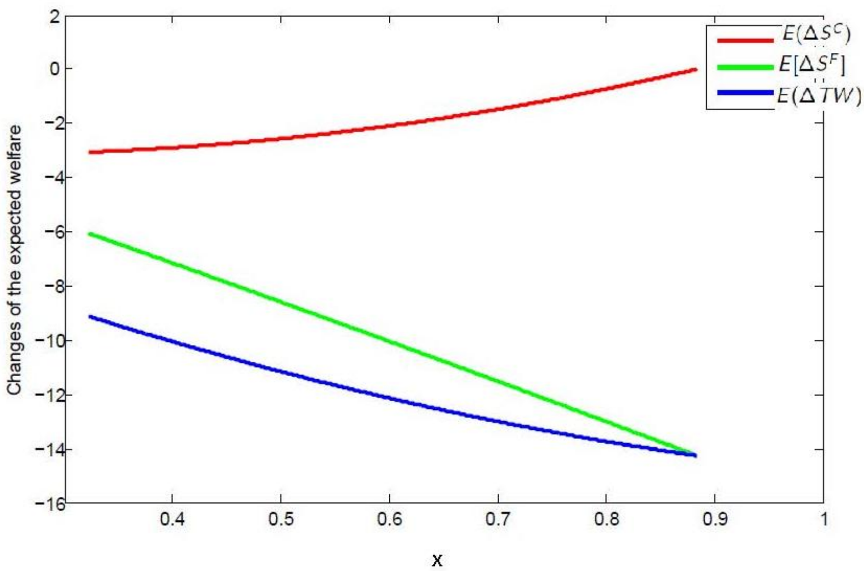


Figure 4.2: The changes of welfare due to limit pricing

4.5 Conclusions

In this chapter we re-examine the entry deterrence game in which two firms have asymmetric information (Milgrom & Roberts, 1982) but under different information structures. We formulate two models with asymmetric information. In both models, the entrant faces ambiguity about the true state, in the sense that it has multiple priors on the true state. In the first model, the established firm is fully informed of the true state, and in the second model, the established firm is only informed of the distribution of the true state. The informational advantage encourages the established firm to send an unattractive signal to the entrant by engaging in limit pricing in the first period and deters the entrant under certain circumstances. Compared with symmetric information, ambiguity decreases the probability of entry under certain conditions, but it is also possible for it to have no influence on the probability of entry. A numerical example reveals that both the expected consumer surplus and the expected industry surplus are decreased due to limit pricing, and thus total welfare is decreased. Deterring a moderate-cost entrant harms social welfare to a greater extent in a market with higher expected demand than in one with lower expected demand. However, in our analysis, we assume, for simplicity, that firm 1 is also informed of firm 2's set of priors. An extension of the analysis in which firm 2 also has private information on its set of priors is not trivial and the results would be different.

APPENDIX

A. The indifferent point $\tilde{x}$ of firm 1

If firm 1 knows that $\mathbb{P}(a = a^H) = x > \hat{x}$, it can choose to reveal this information to firm 2 by separating strategy or to conceal the information by pooling strategy. Assume that at $\tilde{x}$, firm 1 is indifferent between separating and pooling. If firm 1 reveals $\tilde{x}$ to firm 2, it chooses the monopoly price in the first period $s_m(\tilde{x}) = \frac{a^e(\tilde{x})+c_1}{2}$ and it allows firm 2 to enter the market. The total expected profit of firm 1 is the expected monopoly profit in the first period:

$$\Pi_{1M}^e(\tilde{x}) = \frac{(a^e(\tilde{x}) - c_1)^2}{4b}.$$

But if firm 1 chooses to conceal the information and successfully deters the entrant, the optimal price it can choose is

$$s(\tilde{x}) = s_m(\hat{x}) = \frac{a^e(\hat{x}) + c_1}{2}$$

and it will get a total expected profit:

$$\begin{aligned}
\Pi_1^e(\hat{x}, \tilde{x}) &= \Pi_{1M}^e(\hat{x}) + \delta_1 R^e(\tilde{x}) \\
&= \frac{(a^e(\hat{x}) - c_1)^2}{4b} + \delta_1 E_a[\frac{(a - c_1)^2}{4b} - \frac{(3a - a^e(\tilde{x}) - 4c_1 + 2c_2)^2}{36b}] \\
&= \frac{(a^e(\hat{x}) - c_1)^2}{4b} + \delta_1[\frac{Var(a) + (a^e(\tilde{x}) - c_1)^2}{4b} - \frac{9Var(a) + (2a^e(\tilde{x}) - 4c_1 + 2c_2)^2}{36b}] \\
&= \frac{(a^e(\hat{x}) - c_1)^2}{4b} + \delta_1[\frac{(a^e(\tilde{x}) - c_1)^2}{4b} - \frac{(a^e(\tilde{x}) - 2c_1 + c_2)^2}{9b}]
\end{aligned}$$

Let $\Pi_{1M}^e(\tilde{x}) = \Pi_1^e(\hat{x}, \tilde{x})$, we can get that the indifferent point $\tilde{x}$ satisfies:

$$\frac{(a^e(\hat{x}) - c_1)^2}{4} = (1 - \delta_1)\frac{(a^e(\tilde{x}) - c_1)^2}{4} + \delta_1 \frac{(a^e(\tilde{x}) - 2c_1 + c_2)^2}{9}.$$

If we take $\delta_1 = 1$, then we have

$$\frac{(a^e(\hat{x}) - c_1)^2}{4} = \frac{(a^e(\tilde{x}) - 2c_1 + c_2)^2}{9}.$$

B. Proof of Proposition 4.2

Proof. In this proposition, we intent to prove that at equilibrium, for $\forall x_1, x_2 \in [0,1]$, if $x_1 < x_2$, then $s_1^* = s^*(x_1) \leq s^*(x_2) = s_2^*$. For the case that $\hat{x} < \tilde{x}$, there are four possibilities that can occur at equilibrium:

(i). $t^*(s_1^*) = t^*(s_2^*) = 0$;

(ii). $t^*(s_1^*) = 0$, and $t^*(s_2^*) = 1$;

(iii). $t^*(s_1^*) = 1$, and $t^*(s_2^*) = 1$.

(iv). $t^*(s_1^*) = 1$, and $t^*(s_2^*) = 0$;

In each case, we can show that a violation of the monotonicity leads to a deviation from the equilibrium. Let $A_1 = \{x : s^*(x) = s_1^*\}$ and $A_2 = \{x : s^*(x) = s_2^*\}$ be the strategies of firm 1 at the particular equilibrium. By contradiction, we assume that $s_1^* > s_2^*$.

For case (i), let $x_1^* = \inf\{x : s^*(x) = s_1^*\}$ and $x_2^* = \inf\{x : s^*(x) = s_2^*\}$. Due to the strictly increasing of $\Pi_{2C}^e(x)$ in x, firm 2 takes x_1^* and x_2^* as the worst case by MaxMin utility. If firm 2 doesn't enter the market at x_1^* and x_2^*, then the optimal response of firm 1 will be

$$s_1^* = \frac{a^e(x_1^*) + c_1}{2b}, \quad \text{and} \quad s_2^* = \frac{a^e(x_2^*) + c_1}{2b}.$$

If $s_1^* > s_2^*$, then $x_2^* < x_1^* \leq x_1 < x_2$. Then the type x_2 of firm 1 would deviate to s_1^*. By deviation, $x_2 \in A_1$, and still we have $x_1^* = \inf A_1$, which doesn't influence the decision of firm 2. The x_2 type of firm 1 can get higher profit in the first period and the same monopoly profit in the second period.

For case (ii), we have similar argument. If $s_1^* > s_2^*$, and firm 2 enters the market observing s_2^*, then type x_2 of firm 1 would deviate to get higher profit in the first period and deter the entry by choosing s_1^*.

For case (iii), since both s_1^* and s_2^* can't deter the entry, type x_2 of firm 1 would deviate to get higher profit in the first period by choosing s_1^*.

For case (iv), Firstly, we show $x_1^* = x_1$. If $x_1^* < x_1$, then $s_1^* < \frac{a^e(x_1)+c_1}{2b}$. Since s_1^* induces entry, x_1 type of firm 1's profit is $\Pi^e_{1M}(x_1^*)$. However, the worst case by choosing s_1 is inducing entry and the profit is $\Pi^e_{1M}(x_1)$, and $\Pi^e_{1M}(x_1^*) < \Pi^e_{1M}(x_1)$. The x_1 type of firm 1 would deviate to $s_1 = \frac{a^e(x_1)+c_1}{2b}$. So $x_1^* = x_1$. Now let's show that if $s_1^* > s_2^*$, one type of firm 1 would deviate. At equilibrium, $x_2^* < x_1 < x_2$ and the profits of firm 1 given by:

$$\Pi^e_1(s_1^*; x_1) = \Pi^e_M(x_1)$$
$$\Pi^e_1(s_2^*; x_2) = \Pi^e_M(x_2^*) + R^e(x_2)$$

If $\Pi^e_1(s_1^*; x_1) > \Pi^e_1(s_2^*; x_2)$, then x_2 type of firm 1 would deviate. Choosing a higher price s_1^* induces entry but it can get higher profit.
If $\Pi^e_1(s_1^*; x_1) < \Pi^e_1(s_2^*; x_2)$, Equation (3) implies that

$$\Pi^e_M(x_2^*) + R^e(x_2) - \Pi^e_1(x_2) < \Pi^e_M(x_2^*) + R^e(x_1) - \Pi^e_1(x_1). \tag{4.7}$$

Since s_2^* is the optimal choice for x_2 type of firm 1,

$$\Pi^e_M(x_2^*) + R^e(x_2) - \Pi^e_1(x_2) \geq 0. \tag{4.8}$$

From Equation (4) and (5), we obtain

$$\Pi^e_M(x_2^*) + R^e(x_1) - \Pi^e_1(x_1) > 0,$$

which implies that x_1 type of firm 1 would deviate to get higher payoff by choosing a lower price s_2^* . ■

C. Proof of Theorem 4.1

Proof. To prove this theorem, we just follow the definition of Nash Equilibrium (Def. 2.1). Given the strategy of firm 1, observing $s < \frac{(a^e(\hat{x})-c_1)}{2}$, firm 2 induces that $x < \hat{x}$. Firm 2 chooses to stay out of the market with maxmin preference. So t^* is an optimal response to s^*. On the other hand, Given the strategy of firm 2, t^*, firm 1's optimal strategy is to maximize its total expected profit. Let $\delta_1 = 1$, then the total expected profits of firm 1 is:

$$\Pi_1^e(s;x) = \begin{cases} \Pi_1^{0e}(s;x) + R^e(x) & \text{if} \quad s \leq \frac{a^e(\hat{x})+c_1}{2b} \\ \Pi_1^{0e}(s;x) & \text{if} \quad s > \frac{a^e(\hat{x})+c_1}{2b}, \end{cases}$$

We can show that the difference of the profits of firm 1 between deterring and not deterring the entrant is decreasing in x if $\delta_1 = 1$:

$$\frac{d(\Pi_1^e(\hat{x},x) - \Pi_{1M}^e(x))}{dx} = -\frac{2(a^H - a^L)(a^e(x) - 2c_1 + c_2)}{9b} < 0. \tag{4.9}$$

And we have shown that firm 1 is indifferent at $x = \tilde{x}$. So firm 1 prefers deterring to accommodating the entrant when $x < \tilde{x}$. When $\tilde{x} < \hat{x}$, for any $x \in [0,1]$, firm 1 will choose the monopoly price in the first period because it is too costly to deter the entrant. But when $\tilde{x} > \hat{x}$, firm 1 has incentive to deter the entrant by pooling strategies for $x \in [\hat{x}, \tilde{x}]$ and what it can do the best is to choose the monoply price at $x = \hat{x}$. Given the strategy of firm 2, for any $x \in [0,1]$, firm 1 doesn't want to deviate. ■

References

Athey, S. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica 69*, 4 (2001), 861–889.

Aumann, R. J., Maschler, M., and Stearns, R. E. *Repeated games with incomplete information*. MIT press, 1995.

Bagwell, K., and Ramey, G. Oligopoly limit pricing. *The Rand Journal of Economics* (1991), 155–172.

Bain, J. S. A note on pricing in monopoly and oligopoly. *The American Economic Review* (1949), 448–464.

Banks, J. S., and Sobel, J. Equilibrium selection in signaling games. *Econometrica: Journal of the Econometric Society* (1987), 647–661.

Brealey, R., Leland, H. E., and Pyle, D. H. Informational asymmetries, financial structure, and financial intermediation. *The journal of Finance 32*, 2 (1977), 371–387.

Camerer, C. *Individual decision making*. Princeton University Press, 1995.

Chen, Y. Perturbed communication games with honest senders and naive receivers. *Journal of Economic Theory 146*, 2 (2011), 401–424.

Cho, I.-K. A refinement of sequential equilibrium. *Econometrica: Journal of the Econometric Society* (1987), 1367–1389.

Cho, I.-K., and Kreps, D. M. Signaling games and stable equilibria. *The Quarterly Journal of Economics* (1987), 179–221.

Chowdhury, P. R. Limit-pricing as bertrand equilibrium. *Economic Theory 19*, 4 (2002), 811–822.

Cooper, D. J., Garvin, S., and Kagel, J. H. Signalling and adaptive learning in an entry limit pricing game. *The RAND Journal of Economics* (1997), 662–683.

de Groot Ruiz, A., Offerman, T. J. S., and Onderstal, S. Equilibrium selection in cheap talk games: Acdc rocks when other criteria remain silent. *Tinbergen Institute Discussion Paper 11-037/1* (2012).

De Meyer, B. Price dynamics on a stock market with asymmetric information. *Games and economic behavior 69*, 1 (2010), 42–71.

De Meyer, B., and Fournier, G. Price dynamics on a risk averse market with asymmetric information, June 2015. Documents de travail du Centre d'Economie de la Sorbonne 2015.54 - ISSN : 1955-611X.

De Meyer, B., Lehrer, E., and Rosenberg, D. Evaluating information in zero-sum games with incomplete information on both sides. *Mathematics of Operations Research 35*, 4 (2010), 851–863.

De Meyer, B., and Marino, A. Duality and optimal strategies in the finitely repeated zero-sum games with incomplete information on both sides, Mar. 2005. Cahiers de la Maison des Sciences Economiques 2005.27 - ISSN : 1624-0340.

De Meyer, B., and Saley, H. M. On the strategic origin of brownian motion in finance. *International Journal of Game Theory 31*, 2 (2003), 285–319.

Dixit, A. The role of investment in entry-deterrence. *The economic journal 90*, 357 (1980), 95–106.

Eguia, J. X., Llorente-Saguer, A., Morton, R., and Nicolò, A. Equilibrium selection in sequential games with imperfect information. *MPI Collective Goods Preprint*, 2014/4 (2014).

Eichberger, J., Kelsey, D., et al. Optimism and pessimism in games. *University of Heidelberg: Unpublished manuscript* (2009).

Ellsberg, D. Risk, ambiguity, and the savage axioms. *The quarterly journal of economics* (1961), 643–669.

Epstein, L. G. An axiomatic model of non-bayesian updating. *The Review of Economic Studies 73*, 2 (2006), 413–436.

Epstein, L. G., Noor, J., and Sandroni, A. Non-bayesian learning. *The BE Journal of Theoretical Economics 10*, 1 (2010).

Epstein, L. G., and Schneider, M. Learning under ambiguity. *The Review of Economic Studies 74*, 4 (2007), 1275–1303.

Epstein, L. G., and Schneider, M. Ambiguity and asset markets. Tech. rep., National Bureau of Economic Research, 2010.

Foster, D. P., and Young, H. P. Learning, hypothesis testing, and nash equilibrium. *Games and Economic Behavior 45*, 1 (2003), 73–96.

Fudenberg, D., and Tirole, J. Sequential bargaining with incomplete information. *The Review of Economic Studies 50*, 2 (1983), 221–247.

Fudenberg, D., and Tirole, J. Perfect bayesian equilibrium and sequential equilibrium. *journal of Economic Theory 53*, 2 (1991), 236–260.

G, L., Noor, J., and Sandroni, A. Non-bayesian updating: a theoretical framework. *Theoretical Economics 3*, 2 (2008), 193–229.

Gaskins, D. W. Dynamic limit pricing: Optimal pricing under threat of entry. *Journal of Economic Theory 3*, 3 (1971), 306–322.

Geroski, P. A. What do we know about entry? *International Journal of Industrial Organization 13*, 4 (1995), 421–440.

Gilboa, I., Postlewaite, A., and Schmeidler, D. Is it always rational to satisfy savage's axioms? *Economics and Philosophy 25*, 03 (2009), 285–296.

Gilboa, I., Postlewaite, A., and Schmeidler, D. Rationality of belief or: why savages axioms are neither necessary nor sufficient for rationality. *Synthese 187*, 1 (2012), 11–31.

GILBOA, I., POSTLEWAITE, A. W., AND SCHMEIDLER, D. Probability and uncertainty in economic modeling. *The Journal of Economic Perspectives 22*, 3 (2008), 173–188.

GILBOA, I., AND SCHMEIDLER, D. Maxmin expected utility with non-unique prior. *Journal of mathematical economics 18*, 2 (1989), 141–153.

GOLUB, B., AND JACKSON, M. O. Naive learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics* (2010), 112–149.

GOVINDAN, S., AND WILSON, R. Global newton method for stochastic games. *Journal of Economic Theory 144*, 1 (2009), 414–421.

GOVINDAN, S., AND WILSON, R. A decomposition algorithm for n-player games. *Economic theory 42*, 1 (2010), 97–117.

GOVINDAN, S., AND WILSON, R. Axiomatic equilibrium selection for generic two-player games. *Econometrica 80*, 4 (2012), 1639–1699.

HANANY, E., KLIBANOFF, P., AND MUKERJI, S. Incomplete information games with smooth ambiguity preferences. Tech. rep., mimeo, 2015.

HARSANYI, J. C. Games with incomplete information played by bayesian players, i–iii: part i. the basic model&. *Management science 50*, 12_supplement (2004), 1804–1817.

HILLAS, J., AND KOHLBERG, E. Foundations of strategic equilibrium. *Handbook of Game Theory with Economic Applications 3* (2002), 1597–1663.

KLIBANOFF, P., MARINACCI, M., AND MUKERJI, S. A smooth model of decision making under ambiguity. *Econometrica 73*, 6 (2005), 1849–1892.

KNIGHT, F. H. *Risk, uncertainty and profit.* Courier Corporation, 2012.

KOHLBERG, E., AND MERTENS, J.-F. On the strategic stability of equilibria. *Econometrica: Journal of the Econometric Society* (1986), 1003–1037.

KREPS, D. M., AND WILSON, R. Sequential equilibria. *Econometrica: Journal of the Econometric Society* (1982), 863–894.

Maccheroni, F., Marinacci, M., and Rustichini, A. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica 74*, 6 (2006), 1447–1498.

Mailath, G. J., Okuno-Fujiwara, M., and Postlewaite, A. Belief-based refinements in signalling games. *Journal of Economic Theory 60*, 2 (1993), 241–276.

Man, P. T. Forward induction equilibrium. *Games and Economic Behavior 75*, 1 (2012), 265–276.

Matthews, S. A., Okuno-Fujiwara, M., and Postlewaite, A. Refining cheap-talk equilibria. *Journal of Economic Theory 55*, 2 (1991), 247–273.

Milgrom, P., and Roberts, J. Limit pricing and entry under incomplete information: An equilibrium analysis. *Econometrica: Journal of the Econometric Society* (1982), 443–459.

Milgrom, P., and Shannon, C. Monotone comparative statics. *Econometrica: Journal of the Econometric Society* (1994), 157–180.

Milgrom, P., and Stokey, N. Information, trade and common knowledge. *Journal of Economic Theory 26*, 1 (1982), 17–27.

Mitzkewitz, M. Equilibrium selection in signaling games. Tech. rep., Mimeo, Bonn, 1990.

Mullainathan, S. Thinking through categories. *NBER working paper* (2002).

Munoz-Garcia, F., and Espinola-Arredondo, A. The intuitive and divinity criterion: Interpretation and step-by-step examples. *Journal of Industrial Organization Education 5*, 1 (2011), 1–20.

Ortoleva, P. Modeling the change of paradigm: Non-bayesian reactions to unexpected news. *The American Economic Review 102*, 6 (2012), 2410–2436.

Pecorino, P., and Van Boening, M. Costly voluntary disclosure in a screening game. *International Review of Law and Economics 44* (2015), 16–28.

RABIN, M., ET AL. *Inference by believers in the law of small numbers.* Institute of Business and Economic Research, 2000.

RILEY, J. G. Silver signals: Twenty-five years of screening and signaling. *Journal of Economic literature 39*, 2 (2001), 432–478.

SALOP, S. C. Strategic entry deterrence. *The American Economic Review 69*, 2 (1979), 335–338.

SEAMANS, R. C. Threat of entry, asymmetric information, and pricing. *Strategic Management Journal 34*, 4 (2013), 426–444.

SMILEY, R. Empirical evidence on strategic entry deterrence. *International Journal of Industrial Organization 6*, 2 (1988), 167–180.

SOBEL, J. Signaling games. *Encyclopedia of Complexity and System Science, Springer, forthcoming* (2007).

SORIN, S. *A first course on zero-sum repeated games.* Springer Science & Business Media, 2002.

SORIN, S., AND ZAMIR, S. A 2-person game with lack of information on 11/2 sides. *Mathematics of operations research 10*, 1 (1985), 17–23.

SPENCE, M. Job market signaling. *The quarterly journal of Economics* (1973), 355–374.

TENG, J. *A Bayesian Theory of Games: Iterative Conjectures and Determination of Equilibrium.* Chartridge Books Oxford, 2013.

TIROLE, J. *The theory of industrial organization.* MIT press, 1988.

TVERSKY, A., AND KAHNEMAN, D. Judgment under uncertainty: Heuristics and biases. *science 185*, 4157 (1974), 1124–1131.

UMBHAUER, G. Forward induction, consistency and rationality of-perturbations. Tech. rep., mimeo, Universit e Louis Pasteur, Strasbourg, 1991.

VAN DAMME, E. Strategic equilibrium. *Handbook of Game Theory with Economic Applications 3* (2002), 1521–1596.

Von Neumann, J., and Morgenstern, O. *Theory of games and economic behavior*. Princeton university press, 2007.

Wagner, E. O. The dynamics of costly signaling. *Games 4*, 2 (2013), 163–181.