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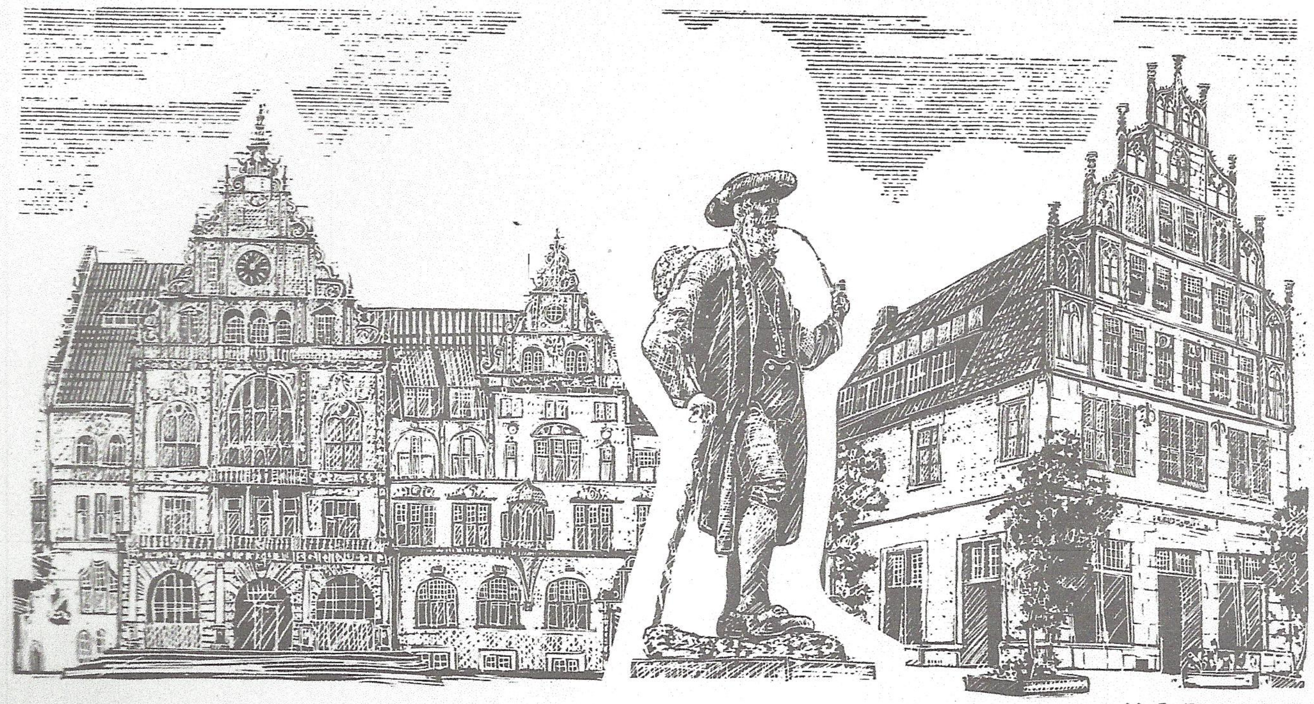

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CONVOLUTIONS, INERTIA SUPERGAMES, AND OLIGOPOLISTIC EQUILIBRIA

T. Marschak and R. Selten

I. INTRODUCTION

Consider an oligopolistic economy, where firms choose prices and productions and each has reason to worry about how its choices influence other firms' choices. In a general equilibrium of such an economy all markets are cleared, and given what it knows and observes, every firm and household is content with its choice. "Contentment" of firms in such an equilibrium has long been an elusive prey for theorists. In this paper we pursue the hunt further; we shall propose and apply a new equilibrium concept for a class of game situations into which oligopolistic economies naturally fit. We shall view the oligopolistic economy as a game and its firms as the players. Each player chooses his actions independently. Whatever agreements occur must be tacit ones since, as the law requires in most real market economies, no institution is available to enforce agreements.[†]

Each firm will be free to change its action, but when it does so it will evoke responses from other firms. In our approach, it will be the situation <u>after these responses are completed</u> which will matter to a firm should it decide to change its action. Any transitory profits which the firm might earn before the responses are complete will not influence its decision to change its action.

[†]A cooperative theory is developed in [13].

The neglect of transitory profits seems to fit the bulk of classical oligopoly literature, although much of it is simply unclear on the matter. In "conjectural variation" and "reaction function" discussions[†] the firm appears either to imagine an instantaneous response to any change it makes, so that transitory profits cannot occur, or else it simply ignores the intermediate stage at which its action is a newly changed one but the other firms' actions are still unchanged.

Outside the classic negligible-transitory-profit tradition are more recent studies in which the firms make their choices at each of an infinite sequence of time periods. They play a supergame--a regularly repeated game--and a firm which changes its action in one repetition may expect others to respond in the next.[††] Each firm collects a payoff stream; the payoff a firm earns in the period when its action is a changed one but others' are still unchanged <u>does</u> enter its payoff stream and hence may influence its decision to make a change.

We shall deal here with an <u>inertia</u> supergame which differs in one important respect from the supergame of the recent literature: a player incurs a cost when he changes his action. The periods are short and the cost of change is high enough to outweigh any transitory gains which he might obtain in one period by changing his action. On the other hand, if a player can obtain a higher profit in all future periods by switching forever to a new action, then he considers the change cost negligibly

[†]The most extensive criticisms and interpretations of both concepts appear still to be those of Fellner [3].

[††]A supergame framework in which oligopoly behavior could be studied is given by J. Friedman [5], [6]. The second of these papers includes the possibility that action in one period affects payoffs attainable in later ones. Such period-wise interdependence also occurs in [15]. The framework we develop here could be extended to deal with some forms of period-wise interdependence (see the final footnote of the paper). Other multiperiod approaches include Shubik's treatment [16] of ruin games and Cyert and DeGroot's duopoly model [2]

small in relation to the improvement in his long-run profit.

We shall develop, for the inertia supergame, an equilibrium behavior with certain persuasive properties. This behavior is best introduced, however, without reference to the inertia supergame. It will be presented first in the setting of an ongoing-game situation, where the timing of choices and payoffs is not formally specified.

The equilibrium presented here is not an equilibrium of the traditional Nash-Cournot type, wherein each firm is content with its action because it imagines that if it changed its action others would make no change at all in theirs. In our concept such implausible behavior is not generally attributed to others. Moreover, all attempts made so far to argue the existence of a general equilibrium of such a traditional type, with all markets cleared, have required quite strong ad hoc assumptions on the economy's consumers and producers.† To argue existence of an economic state which is a game equilibrium in our sense, and is a market-cleared state as well, is a very different mathematical task and requires less extreme assumptions.

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in which players alternate choices. Extending the latter notion to n oligopolists, however, seems somewhat artificial. Telser [18, Chapter VI] studies a model which is not a supergame since each firm has to commit itself in advance to a sequence of choices and cannot revise its sequence as the others' sequences unfold.

†Arrow and Hahn [1], Negishi [11], Fitzroy [4], Marschak and Selten [10, Chapter 2], Lamont and Laroque [9], Gabscewicz and Vial [7]. The assumptions are needed to ensure that the set of choices which are optimal for a firm, given the choices of all other units in the economy, is always a convex nonempty set. This permits use of the appropriate fixed-point theorem. The difficulty arises whether the economy has oligopolists, monopolists, or both. See Roberts and Sonnenschein [12] for a demonstration that the required condition may be violated in simple monopolistic economies with "normal" tastes and technologies, and for a conclusion that nevertheless any attempt to replace the condition with another one less easily violated appears futile for general economies.

Part II of the present paper introduces the new equilibrium concept in the on-going game setting just referred to. Part III studies it in the setting of the inertia supergame, wherein the opportunity to change his action confronts each player at precisely dated points of time.

In Part IV we apply the equilibrium concept to an oligopolistic economy wherein each firm has a nondecreasing-returns technology and may enter or leave a number of industries. In doing so, we also deal with another challenge that has to be faced in finding a reasonable concept of oligopolistic equilibrium: the information on which each firm in a large economy bases its actions ought to be less than total knowledge of the entire economy and its current state. We shall develop a limited-information equilibrium for the oligopolistic economy studied, but the concept can also be stated abstractly, for noncooperative games in general.

II. CONVOLUTIONS: RESPONSE FUNCTIONS WHICH PRESERVE RATIONALITY

A duopoly game

Suppose duopolists 1 and 2 face a common demand curve and each is to choose a selling price. If one price is lower than the other, the entire quantity demanded is sold by the setter of the lower price. If both set the same price, each sells half the demand. Cost conditions are the same for both. The same four prices are possible choices for each, namely Low (L), Medium (M), High (H), and Very High (V). The two profits for each pair of choices are given in the table, where the upper number in each box is 1's profit and the lower number is 2's. The table defines a game and we shall freely refer to the duopolists as "players."

Duopolist 2

Duopolist 1	L	M	H	V
L	-1 -1	-2 0*	-2 0*	-2 0
M	0* -2	2* 2*	6* 0	6* 0
H	0* -2	0 6*	4* 4*	7* 0
V	0* -2	0 6*	0 7*	1 1

What price pair could we reasonably expect to persist? It is possible at any instant of time for each duopolist to deviate from the prevailing pair. As a matter of economic realism, such a deviation will be <u>observed</u> and will evoke some sort of response from the other. Imagine, as does the traditional oligopoly literature just referred to, that the response evoked occurs <u>at once</u>, or after a negligible time. It is the new situation, following the response, which matters; transitional profits, experienced after the deviation but before the response, are unimportant. What price pair, then, once established, could be expected to endure because each duopolist, contemplating a deviation, reasonably attributes to the other a response behavior which would make the deviation and the response no better for the deviator than the original price pair? The classic answer is a pair displaying the Nash property of noncooperative games --a price pair such that each price is best against the other. The pair (M,M) is the (unique) Nash solution; we would call it the Cournot solution if the table described quantity duopolists rather than price duopolists.

The classic discussion does not provide a very convincing reason to view the pair (M,M) as a stable situation, from which neither player wants to deviate. For each must repeatedly ask, "What will be his response if I deviate from (M,M)?" If the answer is "He will make no change," then each will be content to stay at (M,M). But the deviation would be observable by the other player and there are responses which seem more believable than "No change." For example, 1 might say: "If I were to go to V, then he will observe that and will want to go to H rather than to make no change. If he did so, then we would be in a completely new situation, and a further deviation might make me better off than I am now, at (M,M)."

Consider, on the other hand, the following <u>response function</u>, for either player i (i = 1,2), an alternative to the classic "no change" response function. For every possible value of the status quo (every price pair), and every possible deviation by $j \neq i$, the function prescribes a response for i. We shall call the proposed response function the "kinked" response function. To describe it in words: <u>the responder never raises his price; he follows the deviator down but never below M</u>. The proposed function is of the kinked-demand-curve type (price rises are not matched but price drops are) except that it avoids the matched position (L,L) (costs cannot be covered there). Formally, let a_i (i = 1,2) be i's four-valued action, let a denote (a_1,a_2), and let $\phi_i(a,a_j')$ denote i's response when the status quo is a and j deviates to

$a_j' \neq a_j$. Then the "kinked" response function ϕ_i for player i is the following:

If, at a, a_i =	and a_j' =	then $\phi_i(a,a_j')$ =
L	L,M,H,V	L
M	L,M,H,V	M
H	L,M	M
H	H,V	H
V	L,M	M
V	H,V	H

We can then ask: if each player attributes the kinked response function† to the other, then at what values of the status quo—at what price pair—is each stable? If 1, say, is stable at a given pair $\bar{a} = (\bar{a}_1,\bar{a}_2)$, then 1 cannot benefit by deviating from $\bar{a}_1$ to a_1' because at the resulting situation $\bar{\bar{a}} = [a_1',\phi_2(a,a_1')]$ 1's profit is no higher than it was at $\bar{a}$. Moreover, a further deviation by 1, from a_1' to a_1'', leads to $[a_1'',\phi_2(\bar{\bar{a}},a_1'')]$, at which 1's profit is again no higher than it was at $\bar{a}$. And similarly for a third deviation, a fourth one, and so on. It is easily checked that for the kinked response function, the price pairs with an asterisk in the upper left (lower right) corner of the box are stable†† for 1 (for 2). The pairs (H,H) and (M,M) are stable for both. Many response functions (ϕ_1,ϕ_2), besides the kinked one, can be written down. But notice that the kinked response function has a distinctive property: when a player i is stable, then if he applies ϕ_i he regains his stability following any deviation by the other player. This is a property which many response functions lack;

†The function ϕ_i happens to depend on the status-quo vector a only through the a_i-component of a. It also happens that ϕ_1 and ϕ_2 are symmetric in the two players. Response functions without these properties can certainly be constructed.

††At (H,H), for example, 2 has profit 4 and is stable since a deviation (by 2) to L, M, or V leads, respectively, to (H,L), (M,M), (H,V), at each of which 2's profit is less than 4. It is also clear that no chain of deviations can yield, when it ends, a new price pair at which 2's profit is higher than 4.

in particular the classic no-change function ($\phi_i(a,a_j') = a_i$) clearly lacks it in the duopoly game. We shall argue next that the "restabilizing" property ought to be displayed by an established response function, which players expect to be applied. If this is so, then the kinked function is a good candidate for an established response function, and both (H,H) and (M,M) are reasonable candidates for a price pair which can be expected to persist, for at both pairs each player cannot improve himself if he attributes the kinked response to the other player.

Response functions for general games

We shall argue for the restabilizing property in the context of a general n-player game, returning to the duopoly game later. Let there be n players, comprising the set N. Player i chooses an action a_i from the set A_i. For the action n-tuple $a = (a_1,\ldots,a_n)$, which we shall also call a state, i's payoff is $H_i(a)$. The pair[†] $(\{A_i\}_{i\in N}, \{H_i\}_{i\in N})$ defines a game. The symbol A will denote the set of possible action n-tuples, i.e., the set product $A_1 \times A_2 \times \cdots \times A_n$. We shall assume throughout the paper that in the games discussed either the sets $\{A_i\}_{i\in N}$ are finite or they are compact and the functions $\{H_i\}_{i\in N}$ are continuous on A. (In particular, then, every player always possesses a payoff-maximizing best reply to given actions of the others.) A response function for player i, denoted ϕ_i, assigns an action $a_i = \phi_i(a,a_j')$ in A_i to every pair (a,a_j'), where j is in N, a_j' is in A_j, and a is in A. The state a in such a pair will often be called the "status quo" or "prevailing state," while a_j' will often be called a "deviation." The n-tuple of individual response functions $\{\phi_i\}_{i\in N}$ is denoted ϕ and will be called simply a response function for the game.[††]

[†]Some notational conventions: Suppose S is a set, s is an element of S, and with every element s we associate one variable, denoted x_s. Then the vector of such variables is denoted x_S; an alternative symbol for x_S is $\{x_s\}_{s\in S}$. If $\bar{x} = \bar{x}_S$ is a value of the vector, then the symbol $\bar{x}/x_s'$ denotes the value obtained from $\bar{x}$ when x_s' becomes the value of the variable x_s (where s belongs to S). If r is an element of the set R, then $R \setminus r$ denotes the set which remains when r is removed from R.

[††]The function ϕ_i is from $(A \times \bigcup_k A_k)$ to A_i and the function ϕ from $(A \times \bigcup_k A_k)$ to A.

A response function ϕ has two properties:

(II.1) if $a = a/a_i = a/a_j'$, then $\phi_i(a,a_j') = a_i$.

(II.2) $\phi_j(a,a_j') = a_j'$, all j in N.

(II.1) says that i's response to j's deviation is to make no change if j's deviation is not a proper one (if $a_j' = a_j$), and (II.2) makes the symbol ϕ more versatile by defining a deviator's "response" to his own deviation to be the deviation itself. The symbol $\phi(a,a_j')$ means the n-tuple $\{\phi_i(a,a_j')\}_{i \in N}$.

Associated with ϕ is the extended response function $\hat{\phi}$, which specifies the outcome of a sequence of deviations by some player j; $\hat{\phi}$ is defined recursively by

$$\hat{\phi}(a,\{a_j^1,\ldots,a_j^t\}) = \phi[\hat{\phi}(a,\{a_j^1,\ldots,a_j^{t-1}\}),a_j^t], \qquad t > 1$$

$$\hat{\phi}(a,a_j^1) = \phi(a,a_j^1),$$

where $\{a_j^1,\ldots,a_j^t\}$ is a sequence with terms in A_j. Then player i is stable at the state a with respect to ϕ if no sequence of deviations can benefit him given that ϕ dictates the responses, i.e., for any sequence $\{a_i^1,\ldots,a_i^t\}$ with terms in A_i, $H_i[\hat{\phi}(a,\{a_i^1,\ldots,a_i^t\})] \leqq H_i(a)$.

The time required by a deviation sequence and by the responses it evokes is viewed as negligible when a player contemplates a deviation. Only the final outcome and its payoff matters, not the fleeting, transitory payoffs due to the states between initial deviation and final response. Moreover, simultaneous deviation from a prevailing state by two or more players is treated as an abnormal and negligible occurrence; a response function does not allow for it.

Having defined stability, we are now in a position to impose conditions on the response function ϕ so as to capture partially the intuitive idea

The "reaction function" in most of the literature has only $\bigcup_k A_k$ as its domain. An exception is the work of Krelle [8], who appears to have been the first to consider the dependence of reactions on the status quo.

of a response function which has become established. If ϕ is established, then every player who contemplates a deviation expects ϕ to be applied in response and expects that a deviation by himself or others would not destroy the general belief that ϕ will be applied in response to still further deviations; and a player i who is rational--who never chooses an action which could be improved upon--finds that it is in his own interest to follow ϕ_i should another player deviate.

We propose as a minimum requirement for an established response function that it possess the restabilizing (or rationality-preserving) property:

> ϕ *is restabilizing if* "a *is stable for* i *with respect to* ϕ" *implies* "$\phi(a,a_j')$ *is stable for* i *with respect to* ϕ" *for all* a *in* A, a_j' *in* A_j, j *in* N.

If ϕ did not possess this property, then for some pair (a,a_j') some rational player i would find it in his own interest to violate ϕ_i. Such a player is stable at a and so demonstrates--or gives us no reason to doubt--his rationality. But given the deviation a_j' he would not be stable at $\phi(a,a_j')$. We can therefore not expect him to follow ϕ_i and then rest content. Under our assumption that response and deviation times are negligible it makes no difference whether we view him as first choosing the action $\phi_i(a,a_j')$ and then deviating to $\bar{a}_i \neq \phi_i(a,a_j')$ or as immediately applying a response function $\bar{\phi}_i \neq \phi_i$ for which $\bar{\phi}_i(a,a_j') = \bar{a}_i$. In either case, he would be treated by the other players as a deviator from the state $\phi(a,a_j')$. Since ϕ is established, i expects any other player k, including the original deviator j, to use ϕ_k. Player i therefore expects the choice of $\bar{a}_i$ to yield him the payoff $H_i(\phi[\phi(a,a_j'),\bar{a}_i])$ and either this payoff or the payoff reached after still further deviations is larger than the payoff $H_i[\phi(a,a_j')]$, which he would receive if he followed ϕ_i.

We could not view ϕ as established, then, if it lacked the restabilizing property. To put it another way, if and only if ϕ has the property, does a player i, who is stable at a, find, following a deviation to a'_j by $j \neq i$, that if he expects every other player k to use ϕ_k (now and subsequently), then i's use of ϕ_i is a best reply to the others' use of $\{\phi_k\}_{k \in N \setminus i}$ and is also, therefore, credible to other players. A player i who is not stable at a, may not be stable at $\phi(a,a'_j)$ and for him the use of ϕ_i may not be best given that the others use $\{\phi_k\}_{k \in N \setminus i}$. But such a player has forfeited his claim to be judged rational at a, and others cannot expect that he display rationality in his response.

If a response function is to be of any interest in explaining the persistence of some state, it must meet a second requirement: there must be at least one state at which all players are jointly stable. At such a state there is no reason to doubt any player's rationality and if all expect ϕ to be followed none wants to deviate. We shall call a response function which displays both properties a convolution:

> *ϕ is a convolution if (i) it is restabilizing and (ii) there exists a jointly stable state a^* in A, at which every player is stable with respect to ϕ.*

The kinked response function for the duopoly game is a convolution, with jointly stable states (M,M) and (H,H). For games in general, the "no-change" response function is very seldom a convolution. In a game in which it is one, whenever an action a_i is best (for i) against $a_{N \setminus i}$ it is also best against $a_{N \setminus i}/a'_j$ for all j in $N \setminus i$, all a'_j in A_j. That would be a very peculiar property for a game to display.

The convolution requirement may be a strong one in many games.[†] A weakened requirement, but still interesting in economic contexts, is the following:

> *The response function* ϕ *is a* <u>*weak convolution*</u> *if there exists for each player* i *a set* S_i*, a subset of* A*, such that the sets* S_i *have a nonempty intersection and such that if* a *is in* S_i*, then (i)* a *is stable for* i *with respect to* ϕ*, (ii)* $\phi(a,a_j')$ *is in* S_i *for all* i,j *and all* $a_j' \in A_j$*. The set* S_i *is called* i's <u>*normality set*</u> *and a state which lies in all the normality sets is* <u>*jointly normal*</u> *with respect to* ϕ.

Player i's deliberate rationality is displayed, in the judgment of other players, only at all states in S_i. If i is stable with respect to ϕ elsewhere, then that is an abnormal "accident" and ϕ is only required to restabilize i when there is a deviation from a "normal" state, i.e., a state in S_i. When that happens, ϕ takes i back into S_i, so that his rationality, as judged by others, is preserved. The "best reply" property discussed above still holds for a player i given that some player j has deviated from a state in S_i. A <u>strong</u> convolution ϕ--a response function which meets the previous definition--is also a weak one. We need only take as the normality set for i the set

$$\{S_i = a \mid a \text{ is stable for } i \text{ with respect to } \phi\}.$$

[†]A two-player game for which no convolution exists is

1 \ 2	a_2	b_2
a_1	1, 4	2, 5
b_1	7, 3	0, 0

If a response function ϕ were a convolution, there would be a jointly stable action pair. But (b_1,b_2) cannot be such a pair, since each player benefits by deviating from it no matter what the other's response. (b_1,a_2) cannot be jointly stable with respect to a restabilizing response function, since, if 2 is to be deterred from deviating, 1 must respond by **staying at b_1**, which leads again to (b_1,b_2). (a_1,a_2) cannot be jointly stable since 1 can be deterred from deviating only if 2's response is b_2, which leads again to (b_1,b_2). Finally, (a_1,b_2) cannot be, since deviation by 1 must evoke no change from 2, so that (b_1,b_2) is again reached. Note that the game does possess two Nash points: (b_1,a_2) and (a_1,b_2).

Then any state which is jointly stable with respect to ϕ is jointly normal.

To illustrate a weak but not strong convolution, consider the following symmetric two-player game:

1 \ 2	α	β	γ	δ
α	5, 5	1, 6	0, 0	0, 0
β	6, 1	2, 2	2, 0	0, 0
γ	0, 0	0, 2	1, 1	0, 0
δ	0, 0	0, 0	0, 0	0, 0

The simple response function given by

$$\phi_i(a,a_j') = \begin{cases} \beta & \text{if } a_i = \alpha \text{ or } a_i = \beta; \\ \delta & \text{otherwise} \end{cases}$$

is a weak but not strong convolution with normality sets

$$S_1 = \{(\alpha,\alpha),(\beta,\alpha),(\beta,\beta),(\beta,\gamma),(\beta,\delta)\}$$

and

$$S_2 = \{(\alpha,\alpha),(\alpha,\beta),(\beta,\beta),(\gamma,\beta),(\delta,\beta)\}$$

(player 1's action is the first member of each pair). Player 1 is stable at any pair in S_1 and is brought back to S_1 should 2 deviate; and analogously for player 2. The pairs (α,α) and (β,β) are jointly normal. The pair (γ,γ) is also stable for each player but at that pair each player is judged to be "accidentally" stable, and is not expected to restabilize himself following a deviation. Thus if the prevailing state happens to be (γ,γ) and 1 deviates to α, then the resulting new pair is (α,δ), at which 2 is no longer stable (a deviation from there to β, say, would benefit him). If the players start at one of the jointly normal pairs, however, and if only one of the players ever deviates while the other only changes his action when he is a responder who uses ϕ, then the accidentally stable pair (γ,γ) can never be reached; nor, in fact, can any "abnormal" pair--any pair which lies outside $S_1 \cup S_2$. In general, it is easier to construct a weak but not strong convolution for a game than a strong convolution, since not all stable states need to be investigated to see whether stability is regained following a deviation.

III. INERTIA SUPERGAMES AND CONVOLUTIONS

One could leave the convolution concept at that: restabilizing is a minimal property for established response functions and if the response function is to be of any interest in studying persistent states it must have a jointly stable state. But our defense of the minimal property, the genesis of an established response function, and the "timeless" setting of the ongoing game, with its "instantaneous" deviations and responses--all these have been rather sketchy and intuitive. Fortunately, we need not simply let the concept rest there, standing on its own feet, to be accepted or rejected depending on whether one feels that it captures an essential part of what one means by an established response function and a persistent state. We can, instead, explicitly relate the concept to the repeated playing of a game at precisely dated points of time and to the choice of strategies by players of the repeated game. This will provide a more precise defense of the convolution properties.

As before, the sets $\{A_i\}_{i\in N}$, and the functions $\{H_i\}_{i\in N}$, are given, but we now specify that the game defined by this pair is to be played at each period of an infinite sequence of periods, starting at period 1. In each period, each player chooses an action. The n-tuple of choices define that period's state. If a is a period's state, then player i collects $H_i(a)$ for that period. But if player i's action in period t is a_i and in t + 1, it is $a_i' \neq a_i$, then he incurs a <u>change cost</u> $M(a_i',a_i)$; and for period t + 1, i collects $H_i(a') - M(a_i',a_i)$, where a' is the state in t + 1. The function M is positive but bounded from above. For most of what follows we shall also assume that

$$\text{(III.1)} \quad \max_{a\in A} [H_i(a/a_i') - H_i(a)] < M(a_i,a_i'), \qquad \text{all } i \in N, \text{ all } a \in A, \text{ all } a_i' \in A_i.$$

In this way we give a precise rationale for our earlier informal notion of "instantaneous" deviation and response, wherein we ignored any payoffs due to the states of the game after an initial deviation in a deviation sequence but before completion of the final response. If (III.1) holds, a player's net gain from changing his action can never be a transitory (one-period) gain; a change can only be beneficial to him as a result of the payoff collected after he has maintained the new action for at least two periods. If the periods are short enough, a change cost satisfying (III.1) is realistic for economic games, since all changes of action cost something.

Let $a^t = \{a_i^t\}_{i \in N}$ denote the action n-tuple chosen in the t-th period, $s^t = \{a^1,\ldots,a^t\}$, a t-period sequence of action n-tuples, and $s = \{a^1,a^2,\ldots\}$ an infinite sequence of action n-tuples. We shall suppose that player i is interested in the long-run average payoff which he obtains from the sequence $\{a^1,a^2,\ldots\}$ which occurs, specifically in[†]

$$H_i^\infty(s) = \liminf_{v\to\infty} \frac{1}{v} \sum_{t=1}^{v} H_i^t(a^t,a^{t-1}),$$

where

$$H_i^t(a^t,a^{t-1}) = \begin{cases} H_i(a^t) & \text{if } a_i^t = a_i^{t-1} \text{ or } t = 1, \\ H_i(a^t) - M(a_i^t,a_i^{t-1}) & \text{if } a_i^t \neq a_i^{t-1} \text{ and } t > 1. \end{cases}$$

Then the triple $(\{A_i\}_{i \in N}, \{H_i\}_{i \in N}, M)$ defines a supergame with change cost, wherein each player i chooses an infinite sequence $s_i = \{a_i^1,a_i^2,\ldots\}$ and payoffs are given by the functions $\{H_i^\infty\}_{i \in N}$. If (III.1) is satisfied by M, the supergame with change cost is called an inertia supergame.

Each player follows a strategy in his playing of the supergame. Let S denote the set composed of all possible t-period sequences s^t for all values

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[†]Other measures of i's gain from an infinite sequence present difficulties. The simple limit of average payoff might not exist. Using the discounted sum of the H_i^t's could not, as we shall see, lead to the propositions we obtain about strategies for the repeated game.

of $t \geq 1$, together with an empty sequence s^0. Then a strategy for player i, denoted π_i, is a function from S to A_i, where $\pi_i(s^0)$ is an initial action to be used in period 1. If s^t has been the history of play up to and including period t, then player i chooses the action $\pi_i(s^t)$ for period t + 1. A strategy n-tuple $\pi = \{\pi_i\}_{i \in N}$ will be called a combination. If π is specified and followed, then the history of play from period 1 on is determined: it will be an infinite sequence $\sigma(\pi)$ of action n-tuples with $\pi(s^0)$ as its first term. Extending our use of the symbol H_i, we shall let $H_i(\pi)$ stand for $H_i^\infty[\sigma(\pi)]$.

Note that a combination π must specify, among other things, what action a player i chooses if some other player j stops following the strategy π_j, which is the j-component of π, or if i himself has just violated π_i. A combination $\bar{\pi}$ is called an equilibrium combination if $\bar{\pi}_i$ is a best reply to $\bar{\pi}_{N\setminus i}$, i.e., $\bar{\pi}_i$ is a maximizer of $H_i(\bar{\pi}/\pi_i)$.

It would be a poor theory of the supergame which went only this far. The equilibrium property is important but there may be many equilibria.† Some of

†There is always at least one equilibrium combination in an inertia supergame as long as the sets $\{A_i\}_{i \in N}$ are finite or are compact with $\{H_i\}_{i \in N}$ continuous. For any player i, let $A_{N\setminus i}$ be the set of action (n - 1)-tuples $a_{N\setminus i} = \{a_k\}_{k \in N\setminus i}$, with $a_k \in A_k$. Let $\mu_{N\setminus i} = \{\mu_k\}_{k \in N\setminus i}$ be a function which assigns to any a_i in A_i a value of $a_{N\setminus i}$ in $A_{N\setminus i}$ which minimizes $H_i[(a_i, a_{N\setminus i})]$ on $A_{N\setminus i}$, where $(a_i, a_{N\setminus i})$ denotes the n-tuple with the indicated components. Let the players in N be numbered (if i is in N, then i is a positive integer). Call player j the last deviator from π in the sequence $s^t = \{a^1,\ldots,a^t\}$ if for some $d < t$, (i) $a_j^{d+1} \neq \pi_j(\{a^1,\ldots,a^d\})$, (ii) $a_i^{\tau+1} = \pi_i(\{a^1,\ldots,a^\tau\})$ for $i < j$ and $d \leq \tau < t$, (iii) $a_i^{\tau+1} = \pi_i(\{a^1,\ldots,a^\tau\})$ for $i > j$ and $d < \tau < t$. Let a_i^* be a maximizer of $H_i([a_i, \mu_{N\setminus i}(a_i)])$ on A_i. Then the combination defined by

$$\pi_i(s^t) = \begin{cases} \mu_i(a_j^t) & \text{if j is the last deviator in } s^t \\ a_i^* & \text{otherwise} \end{cases}$$

is an equilibrium, leading to perpetual repetition of $a^* = \{a_i^*\}_{i \in N}$. If i violates π_i, then the first period in which he does so (by choosing $a_i' \neq a_i^*$), he achieves no gain because of the inertia condition (III.1). Subsequently, the others' actions them may require of a player enormously detailed memory of the preceding history. Others may be unappealing once the history of play has taken a certain turn. We shall consider a special class of equilibria which has compelling properties and is closely related to convolutions in the game $(\{A_i\}_{i\in N}, \{H_i\}_{i\in N})$.

First, a combination π is a <u>low-memory</u> combination if, for $t \geq 1$, $\pi_i(s^t)$ depends only on a^t and on $\pi(s^{t-1})$, where $s^t = \{a^1,\ldots,a^{t-1},a^t\}$ and $s^{t-1} = \{a^1,\ldots,a^{t-1}\}$. To determine his next action, a player i who follows π_i needs to remember only what action every player k chose last period and what action k <u>should</u> have chosen if he had followed π_k. Low memory does not exclude wide observation--every player may have to observe every other's current action--but that is reasonable in the economic conflicts we wish to model.

Second, a combination π is <u>conservative</u> if for all $t \geq 1$ and all $i \in N$

$$\pi_i(s^t) = a^t \text{ when } a_j^t = \pi_j(s^{t-1}), \text{ for all } j \in N \setminus i.$$

In a conservative combination a player makes no change in his action if all others have followed the combination. Roughly speaking, a player must have a "good reason" to change.

A third property for combinations has to do with the infinite-period <u>subgame</u> which lies ahead following whatever particular history of play up to period t occurs, out of the many possible histories. The initial period for the subgame is t + 1. A combination π induces a combination π' for the subgame, i.e., π' is commanded by π once the history preceding t + 1 is specified. A combination has to take care of <u>all</u> histories, "unexpected" or "abnormal" ones included. We could now require that the combination be a <u>perfect</u> equilibrium, that it

minimize his payoff for a_i' and he collects no more than $H_i(a^*)$. A further action change cannot benefit i for the same two reasons.

The combination described is unappealing, being an equilibrium of extreme caution. Each player acts as if he expected others to punish him maximally, even at great loss to themselves, for any deviation.

induces an equilibrium combination in every subgame which could lie ahead in every period.[†] No matter what the preceding history, no matter how strange the behavior of some players in that history, the full rationality of equilibrium is always required of every player in the subgame which lies ahead. The slate is wiped clean, in a sense, and regardless of some players' "deviant" behavior, the persuasiveness of the equilibrium property for what lies ahead is viewed as undiminished.

We consider instead a less stringent requirement. We shall call a sequence $s^t = \{a^1,\ldots,a^t\}$ normal for player i with respect to the combination π if for $\tau = 1,\ldots,t-1$, (i) $a_i^\tau = \pi_i(s^{\tau-1})$ and (ii) there is at most one player $j \neq i$ with $a_j^\tau \neq \pi_j(s^{\tau-1})$. Player i, in other words, has not deviated from the strategy π_i in the sequence s^t and in no period has more than one other player deviated.[††] Following such a sequence, there lies ahead a subgame normal for i with respect to π. Then the combination π is paraperfect if in every subgame normal for a player i, π induces a best reply for i to the strategies which π induces for the other players. If π is paraperfect then it is, clearly, an equilibrium.

If π is both conservative and paraperfect, then it provides each player with a model of both "deviators" and "nondeviators." Suppose a conservative and paraperfect combination $\bar{\pi}$ is always followed by all players and no player changes his action from the initial action n-tuple, say $\bar{a}^1$, which $\bar{\pi}$ dictates. Then, when all follow $\bar{\pi}$, i perpetually repeats $\bar{a}_i^1$, and $\bar{\pi}_i$ is a best reply to the others' strategies $\bar{\pi}_{N\setminus i}$. Now suppose instead that in some period player $j \neq i$ violates $\bar{\pi}_j$ and deviates to another action $\tilde{a}_j \neq \bar{a}_j^1$, and that j is viewed

[†]This version of perfectness is developed in [13] and is reconsidered, together with other versions, in [14].

[††]Part (ii) drops out for two-person games.

by i as a "serious" deviator, not an erratic or frivolous one, who will stick with his new action as long as other players continue to follow $\bar{\pi}$, in the sincere if mistaken belief that doing so will benefit him. Then the strategy which the conservative $\bar{\pi}_j$ induces for j in the subgame which lies ahead tells j to stick with his deviation, as long as no one else subsequently violates $\bar{\pi}$, and so it fulfills i's view of j. If i is normal at the period when j deviates, then the strategy which $\bar{\pi}_i$ induces for i is best against the believable strategy which $\bar{\pi}_j$ induces for j and against the strategy which $\bar{\pi}_k$ induces for every other player $k \neq i$. A normal player, then, will not want to abandon $\bar{\pi}$, and neither will a deviating player who fulfills the general view as to how deviators behave. No claims are made for $\bar{\pi}_i$ in the abnormal and unimportant case in which two or more players deviate simultaneously. Nor is it claimed that the strategy induced by $\bar{\pi}_j$ is a "good" one for a player j who has deviated. We only claim that the "conservative" picture of a deviator is a simple and natural one for other players to form, and that given this picture a normal player i will find, if $\bar{\pi}$ is paraperfect, that $\bar{\pi}_i$ provides a good strategy. Since $\bar{\pi}$ is an equilibrium combination, moreover, a player j cannot benefit by deliberately becoming a deviator (a violator of $\bar{\pi}_j$), knowing that the other players will follow $\bar{\pi}$ in all subgames. If the combination did not display paraperfectness, or some other version of the perfectness property, a player j might, by a well-chosen deviation, give other players a good reason to abandon the combination in the resulting subgame, and might benefit from doing so.†

We turn now to response functions and convolutions. Given any low-memory combination π we shall say that the function†† $\phi = \{\phi_i\}_{i \in N}$ <u>represents</u> π if for any $t \geq 1$, any sequence $s^{t-1} = \{a^1, \ldots, a^{t-1}\}$, any j in N, and any a_j' in A_j

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†See the discussion of the duopoly game in the next section.

††As before, ϕ (ϕ_i) is from $(A \times \bigcup_k A_k)$ to A (to A_i).

(α) $\pi(s^t) = \phi[\pi(s^{t-1}), a'_j]$

whenever†

(β) $a^t = \pi(s^{t-1})/a'_j$,

where $s^t = \{a^1, \ldots, a^{t-1}, a^t\}$. The equality (β) says that in the n-tuple chosen at t, all players other than j followed π. Player j may or may not have done so; in any case he chose a'_j.

Any low-memory combination π which is also conservative can be represented by a function φ which is also a response function in our earlier sense and therefore satisfies (II.1) and (II.2). The low-memory property means that for a sequence s^t satisfying (β), $\pi(s^t)$ depends only on the n-tuple actually chosen at t--namely, $\pi(s^{t-1})/a'_j$--and on the n-tuple which should have been chosen if every player had followed π--namely, $\pi(s^{t-1})$. To put it equivalently, $\pi(s^t)$ depends only on $\pi(s^{t-1})$ and a'_j--precisely the two arguments of φ in (α). If π is also conservative then, first, if (β) holds, then that part of (α) which refers to j himself also holds and takes the form

$$\pi_j(s^t) = \phi_j[\pi(s^{t-1}), a'_j] = a'_j,$$

for at t + 1 the conservative π_j commands j to choose a'_j again, since the others have followed π at t. Second, if, in fact, $a'_j = a^t_j = \pi_j(s^{t-1})$, so that j's "deviation" to a'_j is not a proper deviation at all, then--since <u>all</u> players followed π at t--π commands that they repeat their action at t + 1, i.e., (α) becomes

$$\pi(s^t) = \phi[a^t/a_j, a_j] = a^t/a_j.$$

So if π is conservative; if φ represents π; and if, to use the langauge of Part II, we interpret the first argument of φ as a "prevailing state" (which would continue

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†We again use the "slash" notation described in the "notational conventions" footnote of Part II.

at t if no one deviated from it), then ϕ obeys the conditions (II.1) and (II.2) required of a response function.

To construct a response function ϕ which represents a given conservative low-memory combination π, we first consider *all n-tuples* a *in* A *which equal* $\pi(s^{t-1})$ *for some sequence* $s^{t-1} = \{a^1,\ldots,a^{t-1}\}$ *whose first term is* $\pi(s^0)$. Then for any j in N and any a_j' in A_j we take $\phi(a,a_j')$ to equal $\pi(s^t)$, where $s^t = \{a^1,\ldots,a^{t-1},a^t\}$ and a^t satisfies (β). For all other n-tuples a in A--those which are not "reachable" for π in the sense just given--we let ϕ be any arbitrary function satisfying (II.1) and (II.2)

To summarize: a low-memory conservative combination π, and a response function ϕ which represents it, equivalently describe the other players' response to a given player j's proper deviation from a prevailing state. At such a state a, in period t - 1, all responses to the preceding history--the responses commanded by π--have been completed, so that a would also prevail in period t if everyone followed π. But in t, player j makes an "unprovoked" deviation--a change of action not commanded by π_j--to a_j'. At t + 1 every player i $\neq$ j chooses as his response the action given by both ϕ_i and π_i, while j continues to choose a_j', as π_j commands. Our previous "timeless" concepts of prevailing state, deviation, and response have been given, then, a precise timing.

Next, if we are <u>given</u> a response function ϕ, we can then construct a low-memory conservative combination representable by ϕ, once we have specified an initial action n-tuple, say, $\bar{a}$. We shall say that π *is the* $\bar{a}$*-combination for* ϕ if for all i in N

$$\pi(s^t) = \begin{cases} \bar{a} & \text{for } t = 0 \\ \phi[\pi(s^{t-1}),a_j^t] & \text{if } t > 1, \text{ and for some j in N, } a_j^t \neq \pi_j(s^{t-1}) \\ & \qquad \text{while } a_k^t = \pi_j(s^{t-1}), \quad \text{all k in } N \setminus j \\ a^t & \text{in all other cases} \end{cases}$$

Note that this π tells a deviator (a violator of π) to stick with his action unless another player has just deviated, and tells all players to do so in the abnormal case of simultaneous deviations by two or more players.

If a conservative, low-memory combination π is also paraperfect, then more can be said.

<u>Theorem A</u>. *If π is a low-memory, conservative, paraperfect combination for a supergame with change cost, then a response function which represents π must be a weak convolution.*

<u>*Proof*</u>. Suppose ϕ represents the combination π. Consider, for each player i, the set

$S_i = \{a \in A|$ for some sequence $s^t = \{a^1,\ldots,a^t\}$, $a^t = a$ and s^t is normal for i with respect to $\pi\}$.

(S_i is the set of all states which can be reached by sequences normal for i. The state a never leaves the set S_i as long as i is only a responder who applies the function ϕ_i and as long as there is no simultaneous deviation by two or more players.) Then the function ϕ is a weak convolution with normality sets S_i (the sets clearly have a nonempty intersection). At any state a in S_i, i is stable with respect to ϕ. For consider such a state and a deviation sequence $\{a_i^1,\ldots,a_i^m\}$ for i. Suppose

(III.2) $$H_i(\hat{\phi}[a,\{a_i^1,\ldots,a_i^m\}]) > H_i(a),$$

where $\hat{\phi}$ is the extended response function for ϕ. Consider, in the supergame, the subgame which begins at t + 1, is normal for player i, and is preceded by $a^t = a$. Let π' be the combination which π induces in this subgame. In the subgame i could follow a strategy $\tilde{\pi}_i$ defined by:

$\tilde{\pi}_i$: In periods t+1,...,t+m choose $a_i^1,\ldots,a_i^m$, respectively, regardless of what other players have chosen. Apply the strategy π_i'', induced by π_i', in the subgame which follows t + m.

If (III.2) were true, then, for the subgame in question, $H_i(\pi'/\tilde{\pi}_i) > H_i(\pi')$. The strategy $\tilde{\pi}_i$ would be a better reply to π' than π_i', contradicting paraperfectness. So S_i is indeed a normality set for i and ϕ, which represents π, always leads to a new state in S_i after a deviation from a state in S_i; ϕ is indeed a weak convolution.||

There is also a proposition which is close to a converse of Theorem A.

Theorem B. Let ϕ be any weak convolution (in particular, ϕ may be a strong convolution) for the game defined by $(\{A_i\}_{i\in N},\{H_i\}_{i\in N})$, *and suppose* $a^* = \{a_i^*\}_{i\in N}$ *is a jointly normal state for ϕ. Then, in the inertia supergame defined by* $(\{A_i\}_{i\in N},\{H_i\}_{i\in N},M)$ *(with M satisfying (III.1)), the* a^**-combination for ϕ is paraperfect.*

Proof. Let π denote the a*-combination for ϕ and let $\{S_i\}_{i\in N}$ be the normality sets for ϕ. Suppose the t_0-period history of the game is a sequence $s^{t_0} = \{a^1,\ldots,a^{t_0}\}$ normal for player i. Such a sequence cannot leave the normality set S_i, since i applies ϕ_i throughout the sequence. Suppose that in the subgame which follows t^0 every player $k \neq i$ uses the strategy π_k' induced by π_k but player i uses a strategy $\tilde{\pi}_i$ distinct from the induced strategy π_i'. The distinction appears in particular at $t_0 + 2$, i.e., $a_i^{t_0+2} \neq \pi_i'(s^{t_0+1})$ (s^{t_0+1} is the $(t_0 + 1)$-period history). Then every period $t > t_0 + 1$ must be in one of two classes:

(i) Period $t > t_0 + 1$, with history s^{t-1}, is a <u>violation period for</u> i if $\tilde{\pi}_i(s^{t-1}) \neq \pi_i'(s^{t-1})$. In that case, i must change his action from t - 1 to t and, because of the change cost, collects a net payoff in t which does not exceed $\min_{a\in A} H_i(a)$. (ii) Period $t > t_0 + 1$ is a <u>nonviolation period for</u> i if $\tilde{\pi}_i(s^{t-1}) = \pi_i'(s^{t-1})$. Then there is no change cost, for π_i' cannot command a response (an application of ϕ_i) which is an action change for i, since no player $k \neq i$ deviates at t - 1. But one or more earlier periods were violation periods for i, i.e., there is a sequence $\{b_i^1,\ldots,b_i^m\}$, with terms in A_i and with:† $a^t = \hat{\phi}[\bar{a}^{t_0+1},\{b_i^1,\ldots,b_i^m\}]$, where $\bar{a}^{t_0+1} = \pi(s^{t_0})$. But because i followed π_i up to period t_0, and because ϕ is a weak convolution, $\bar{a}^{t_0+1}$ is in S_i and i is stable at $\bar{a}^{t_0+1}$, so that $H_i(a^t) \leqq H_i(\bar{a}^{t_0+1})$. So whether period $t > t_0 + 1$ is of the first or second kinds, player i does not collect a higher net payoff at t than if he had never violated the strategy π_i at any previous period following $t_0 + 1$.

Hence for any sequence involving violations by i after $t_0 + 1$ but not before, whether the violations are finite in number or not, i's long-run average payoff in the infinite subgame which follows $t_0 + 1$ cannot exceed his long-run average payoff if he uses the strategy π_i' induced by π_i from $t_0 + 2$ on. As for period $t_0 + 1$, it is possible for i to engage in a once-only violation of π_i' there, by sticking with his previous action, not incurring a change cost, but failing to follow ϕ_i in response to some player's deviation at t_0. But any one-period gain from doing so plays no role in i's long-run average payoff.†† If i violates π_i at $t_0 + 1$ and again later, then the preceding two-case argument applies again.

Paraperfectness of π is then established. ||

It is easy to see why Theorem A could not be strengthened to say that a convolution representing π must be a strong one, and to see why it is that Theorem B holds for all weak convolutions, not just for strong ones. Consider the weak but not strong convolution ϕ illustrated in Part II. The (α,α)-combination for ϕ is a paraperfect combination--call it $\bar{\pi}$--represented by ϕ. The response function ϕ is a weak convolution and $\bar{\pi}$ does not provide player 2, say, with a best reply if the pair (γ,γ) has been reached and 1 deviates to α, even though 2 is stable at (γ,γ) with respect to ϕ. But the pair (γ,γ) can never be reached by sequences which are

†We use conditions (II.1) and (II.2) which a response function satisfies.

††If it were the discounted sum of payoffs which measured i's satisfaction in the supergame, rather than long-run average payoff, then such a one-period gain might be significant and π might not be paraperfect.

normal for 2 with respect to $\bar{\pi}$, since in such a sequence 2 always applies ϕ_2 and the sequence starts with the jointly normal pair (α,α). So the fact that ϕ represents the paraperfect combination $\bar{\pi}$ does not imply that ϕ is a strong convolution--that ϕ restabilizes each player for every prevailing pair at which he is stable. And even though ϕ is weak and not strong, we can find a paraperfect combination, namely $\bar{\pi}$, which ϕ represents.

The duopoly game again

Return now to the duopoly game and consider the inertia supergame associated with it. One equilibrium combination for the supergame would be $\hat{\pi}$, defined by

$$\hat{\pi}_i(s^t) = \begin{cases} V & \text{if } t = 0 \\ L & \text{if } a_j^t \neq \hat{\pi}_j(s^{t-1}) \\ a_i^t & \text{in all other cases} \end{cases}$$

This is a low-memory, conservative combination and is the (V,V)-combination for the response function given by

$$\phi_i(a,a_j') = L, \qquad a_j' \neq a_j,\ i \neq j.$$

If $\hat{\pi}$ is followed, the result is perpetual choice of (V,V). But $\hat{\pi}$ is not paraperfect. If, for example, 1 deviates to H, then in the subgame which follows, 2's $\hat{\pi}_2$-induced strategy tells him to choose L forever if 1 makes no further change. But that is not a best reply to 1's subgame strategy (induced by $\hat{\pi}_1$) of making no further change provided 2 himself does not deviate. Against the latter subgame strategy, a best reply for 2 is rather "choose H forever." If 2 thinks of 1 as a "serious," conservative, deliberate deviator, then 2 expects 1 to stick with his deviation provided 2 does what he is expected to do. Given this view of 1, 2 will wish to abandon $\hat{\pi}_2$ in the subgame.

On the other hand, the (H,H)-combination for the kinked convolution is paraperfect and so is robust against a deviation by 1; given the same view of the deviator 1, 2 will <u>not</u> wish to abandon the combination and will apply the kinked response function. Moreover, the perpetual choice of (V,V) can <u>never</u> result from use of any convolution, or rather from use of the (V,V)-combination for a convolution whose jointly stable state is (V,V). For (V,V) can never be jointly stable for a convolution: to discourage 1 from deviating from (V,V) to M, 2 must reply with L; but that is not a restabilizing response for 2, since no matter what 1's response, 2 is better off deviating from (M,L).

The kinked convolution has jointly stable pairs at (H,H) and (M,M); (M,M) happens also to be the Nash solution. In the inertia supergame, the (H,H)- or the (M,M)-combination for the kinked convolution results in perpetual choice of that price pair. Whether one thinks of the inertia-supergame setting or the "timeless" ongoing-game setting of Part II, we now have a stronger argument for persistence of either of these states than the classic appeal to the Nash solution provides.

<u>Summary</u>

The goal of parts II and III has been to describe a state which persists because players expect that deviations from that state will be observed and will evoke certain responses, according to a response behavior which is credible to other players, is good for the responders, and could reasonably become established over time. A prospective deviator is concerned only with his payoff in the state which prevails when he has stopped deviating and the responses to his deviations have been completed; and players ignore the possibility of two or more simultaneous deviators. The reasonableness of the response behavior rests on some particular plausible "psychology" of a deviator as he is viewed by the other players. In the convolution the simple view is that a deviator, like all players, expects the response function to be followed and deliberately deviates because of a serious intent to improve himself; since he is serious, he will stick with his deviation but is prepared himself to follow the response function should other players subsequently deviate. If the other players accept this psychology, then the convolution prescribes good responses for them; each responder finds the prescribed behavior of the other responders to be credible and his own prescribed behavior to be good. Alternative deviation psychologies can be explored, though the psychology of a frivolous, erratic deviator would be hard to model.†

The inertia supergame setting provides us with a precise statement of the chosen deviation psychology, and permits us to link response functions with strategies. The use of a response function is associated with the low-memory and conservative properties of a combination, and the use of a convolution with the paraperfect property. In constructing a convolution, one has then also constructed in a convenient and compact form a defensible combination for the inertia supergame. This compactness is a principal appeal of the convolution, since, in general, a supergame strategy is an extremely complex object to study.

Having provided the inertia supergame setting, we need not repeat it when studying convolutions for an economic game and shall not do so in what follows. The inertia supergame remains in the background as an interpretation. An economic game may possess a number of convolutions and some, of course, are more plausible than others. A formal theory for choosing among them remains to be constructed.

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†One alternative nonfrivolous deviation psychology is as follows: A deviator from an action n-tuple which is jointly stable with respect to a response function he expects others to obey would realize, after he deviates, that he has made an error. Knowing that others will apply the response function to each of his further action changes, he then engages in a <u>corrective sequence</u> of action changes--a sequence leading to the best possible state for him--but reverts to the role of responder (and applies the response function himself) should other players start deviating or violating the response function. For some response functions it may be that such a "corrective" strategy, in the subgame of the inertia supergame, is best for the deviator against a strategy for the others which tells them to apply the response function as he makes his changes, <u>and</u> their strategy is best against his. If such a response function is also a convolution, then it provides good responses to deviations whether the deviator is viewed as a serious but mistaken deviator <u>or</u> as a deviator who recognizes his mistake and rationally corrects it, using a strategy which is in equilibrium with the others' strategies. Response functions having this ambitious dual property would be worth studying.

IV. OLIGOPOLISTIC EQUILIBRIA

A convolution for a "small" oligopolistic game with passive variables

We consider now a "small" oligopolistic economy or economic game. Firms are the players of the game (we refer to a firm as "he"), and they do not sell to each other. All firms are capable of producing the same set of products, using the same nondecreasing-returns technology with set-up requirements. Inputs are obtained from households and products are sold to households, but households are not players in the game; only firms are players. Each firm chooses which products to produce and sets a price for each of them. A firm produces exactly what he sells. The quantity of a product sold is positive only if the firm is one of those who set the lowest price for the product; if he is, then his share in the total sales of the product is proportional to his market potential for the product. A firm is endowed at the start with a market potential for each product. The model does not explain this magnitude further; it reflects previous advertising expenditures, location, and so forth.

The profit which a firm earns from the sale of a product is then a function not only of all firms' action variables (product choices and prices), whose values are chosen by firms, but also of the total sales of the products he chooses to produce and the prices of the inputs he needs to produce them. Total sales and input prices are passive variables. The values are not chosen by the players of the game (the firms). Passive variables bear a complex relation to the behavior of households as well as firms. Firms, in our model, are ignorant of this relationship and of households' behavior. They therefore make the simplest possible assumption about passive variables--they assume them to remain unchanged. How satisfactory this assumption is in economic models depends on how inaccurate it makes each player's picture of the economic setting in which the game is played. For the case of total sales, it seems quite reasonable to suppose that the main effect of a firm's price change is its effect on the prices chosen by other firms, that this is far more important than any effect on total sales, and that the firms are willing to neglect change in total sales when making their choices.

We seek a state of affairs for such an economy--a value for product choices, chosen prices, sales, and input prices--at which the players (firms) are content in the sense developed and defended in the preceding section: the players' action n-tuple is jointly stable with respect to a convolution. It is therefore also the result of applying a paraperfect strategy combination in the inertia supergame which stretches before the players when they assume sales and input prices to remain unchanged. The introduction of passive variables will require us to restate in a straightforward way the notions of response function and convolution.

Notation, and the elements of the game, now follow. There are n players (firms), comprising the set N. Each is capable of producing the set Π of products, using the set B of inputs to do so. B and Π have an empty intersection. Every firm requires $z_{jv} \geqq 0$ units of input v to set up for production of input j; in addition, $f_{jv}(q)$ units of v are required when q units of j are produced. The function f_{jv} is increasing and satisfies

$$\text{(IV.1)} \quad f_{jv}(0) = 0; \quad \text{if } q' > q > 0, \text{ then } \frac{f_{jv}(q')}{q'} \leqq \frac{f_{jv}(q)}{q}$$

(nonincreasing average input requirements). The market potential of firm t with respect to product j is denoted α_{tj}, with $\alpha_{tj} > 0$. Total sales (by all firms) of product j are denoted y_j. If R is the set of all firms selling j, and if firm t belongs to R, then firm t sells $y_j e_{tj}(R)$, where $e_{tj}(R) \equiv \alpha_{tj} / \sum_{s \in R} \alpha_{sj}$.

The symbol Δ_i denotes the set of products for which firm i sets up, and p_{ij} is the price firm i sets for j. He sets a price for j if and only if he sets up to produce j. But it will save notation if we say that when he declines to set up for the product j, then he sets an infinite price for j. Then $p_j \equiv \min_i p_{ij}$ is the <u>prevailing price</u> of j. The symbol p_v denotes the price paid by all firms for an input $v \in B$. The action of firm i (whose two components are called i's action variables) is the set-up-and-price pair $a_i = (\Delta_i, p_{i\pi})$, where $p_{i\pi} = \{p_{ij}\}_{j \in \pi}$ denotes an assignment of value to every product price; a_i lies in the set $A_i = \{(\Delta_i, \{p_{ij}\}_{j \in \pi}) \mid \Delta_i \subseteq \pi,\ p_{ij} \geqq 0 \text{ if } j \in \Delta_i,\ p_{ij} = \infty \text{ if } j \notin \Delta_i\}$.

The action n-tuple $a = (a_1,\ldots,a_n)$ and the passive variables $s = (y_\pi, p_B)$ comprise the <u>state</u> $x = (a,s)$. For a fixed value of x, firm i's payoff (profit) is

$$H_i(x) = \begin{cases} 0, & \text{if } \Delta_i \text{ is empty,} \\ \sum_{\{j \mid j\in\Delta_i; p_{ij}=p_j\}} \{p_j y_j e_{ij}(R_j) - \sum_{v\in B} p_v f_{jv}[y_j e_{ij}(R_j)]\} - \sum_{\substack{v\in B \\ j\in\Delta_i}} p_v z_{jv}, & \text{otherwise,} \end{cases}$$

where $R_j = \{i \in N \mid j \in \Delta_i; p_{ij} = p_j\}$ is the set of <u>sellers</u> (prevailing-price setters) of product j. Payoff, then, is the profit i anticipates earning given that product sales and input prices are fixed and that he can obtain all the inputs he needs to meet his share of product sales.

A <u>response function</u> $\phi = (\phi_1,\ldots,\phi_n)$ prescribes a new value of each action given an existing state and a deviation from it by some firm. Firm i's response function ϕ_i assigns an action $\bar{a}_i = \phi_i(x,a_j')$ to every pair composed of an existing state $x = (a,s)$ and a deviation a_j' by some $j \in N$. For a fixed value of s, the passive variables, ϕ is then a response function, in our original sense, for the game defined by the payoff functions $\{H_i\}_{i\in N}$ and the sets $\{A_i\}_{i\in N}$. Hence the earlier definitions of "extended response function," "i is stable at x with respect to ϕ," and "convolution" are adapted in an obvious way to allow for the addition of passive variables. In particular, "i is stable at $x = (a,s)$" now means "i is stable (in the previous sense) for s fixed"; and "convolution" is redefined using "stable" in the new sense: ϕ is a convolution for a fixed value, $\bar{s}$, of the passive variables, if there exists a value $\bar{a}$ of the action variables such that $\bar{x} = (\bar{a},\bar{s})$ is jointly stable with respect to ϕ and for any i, j "i is stable at $x = (a,\bar{s})$" implies "for any a_j' in A_j, i is stable at $\phi(x,a_j')$."

Suppose, for the economic game just defined, we have found a response function $\bar{\phi}$, and a value $\bar{s}$ of the passive variables, such that $\bar{\phi}$ is a convolution and the state $(\bar{a},\bar{s})$ is jointly stable with respect to $\bar{\phi}$. Then for the reasons developed in parts II and III we have found a state which we may reasonably expect to persist, as far as the contentment of players is concerned.†

We now construct a response function $\bar{\phi}$ for the "small" economic game. We shall show that for any fixed value of the passive variables the function $\bar{\phi}$ is a convolution. We have assumed that there are separate set-up requirements for each product a firm produces. The function $\bar{\phi}$ will be similarly separable with respect to products. When a firm k deviates in his set-up-and-price action with respect to a product j, a responding firm r may change his action with respect to j but not with respect to any other product. We can then speak, in connection with $\bar{\phi}$, of a firm being stable at a given state with respect to the product j: changing his set-up-and-price action with regard to j leads to responses also involving j which do not improve his payoff. Then a firm is stable at a given state if and only if he is stable with respect to product j for every j in Π. Effectively, each firm i divides himself into as many independent "players" as there are potential products, and each such "player" follows that part of $\bar{\phi}_i$ which deals with the product in question. To check that $\bar{\phi}$ has the convolution property, it then suffices to check that (1) if a firm is stable with respect to a product j he regains his stability with respect to j following a deviation (by another firm) which involves j; and (2) for every product j there is an n-tuple of set-up-and-price actions with respect to j, which is jointly stable with respect to $\bar{\phi}$.

The function $\bar{\phi}$ specifies first that if at the state $x = (a,s) = [a,(y_{\pi},p_B)]$ the sales y_j of a product j are zero, then no deviation involving j evokes any change in any responder's action. If the sales of j are positive then the table

†But $\bar{x}$ may not be economically feasible, since markets may not be cleared. To study feasibility, passive variables must be explained—e.g., by the behavior of households who supply inputs and choose demands (sales). We turn to feasibility (general equilibrium) at the end of the next section.

which follows describes $\bar{\phi}$. Each box describes a responder r's set-up-and-price action with respect to product j following a deviation from the existing state with respect to j by a deviating firm $k \neq r$. In some boxes the word "rank" appears. Imagine players to be numbered, in a totally arbitrary way, the numbering known to all. A player's number is called his rank. Knowing a pair $(a, a_j^!)$ every player knows, for example, who was the lowest-ranked player among those who chose certain values of their action variables at the state a. Such information can then enter the forming of the response $\bar{\phi}(a, a_j^!)$. In some boxes of the table this happens, since a particular kind of response has to be assigned arbitrarily to one or more of the responders but not to all, and it is convenient to assign the responses by means of rank. It will prove easy to check that neither the convolution property of the response function $\bar{\phi}$, nor its jointly and individually stable states, are affected by a renumbering of the players.

The term "insider" means a firm who is set up to produce j and "outsider" a firm who is not. A "nonselling insider" sets a price above the prevailing (i.e., minimum) price for j; a "seller" sets the prevailing price for j. A "nonseller" is, then, either a nonselling insider or an outsider. A sole seller is an "accompanied" or "unaccompanied" monopolist depending on whether he is or is not in the presence of nonselling insiders. The response function $\bar{\phi}$ is, as in the example of part II, partly suggested by the "kinked demand curve" model, in which price cuts are matched but price rises are not. If the deviator k is a seller of j who raises his price or leaves the industry (ceases to be set up), then the other sellers (if there are any) make no change. But suppose k is a seller who lowers his price below the prevailing price, or an outsider or nonselling insider who now sets the prevailing price (first setting up if he has not yet done so). Then the other previous sellers match k if they can do so without loss—if they form what the table calls a "qualified set". But

THE RESPONSE FUNCTION $\bar{\phi}$ AS IT APPLIES TO A GIVEN PRODUCT j WHOSE SALES ARE POSITIVE

	Responder's (r's) status in existing state x		
Deviator's (k's) status at x and his new action	r sells j	r is a nonselling insider	r is an outsider
k is a nonseller who now sets prevailing price or lower, or a seller or a.m.† who undercuts prevailing price (but keeps price above m.b.e.† price)	(1) n.a. if k is an a.m.; otherwise, all insiders other than k (r included) match deviator if at his price they form a qualified set;† if not, r takes over as "policeman" if he has lowest rank among responders and makes no change otherwise.	(2) all insiders other than k (r included) match deviator if at his price they form a qualified set;† if not, r takes over as "policeman" if he has lowest rank among responders and makes no change otherwise	(3) no change unless policeman is required [see box (1)] and r has lowest rank, in which case he becomes the policeman
k is a seller who deviates upward or leaves industry	(4) r sells j at previous prevailing price if sellers other than k plus some "replacers" (previous nonsellers) can form a qualified set at that price; if not, r takes over as policeman if he has lowest rank; otherwise, no change	(5) r sells j at prevailing price if r is one of the "replacers" who together with previous sellers can form a qualified set at that price; if no such set exists, r takes over as policeman if he has lowest rank; otherwise, no change	(6) same as (5)
k is any firm who deviates to the m.b.e. price or lower	(7) r leaves industry (ceases to be set up for j)	(8) no change	(9) no change
k is a u.m.;† he deviates downward but keeps price above m.b.e. price	(10) n.a.	(11) n.a.	(12) no change

†A set of selling firms is a "qualified set" at a given prevailing price if they are the only sellers and each makes nonnegative profit on j. "M.b.e." price means monopoly break-even price; "u.m." stands for "unaccompanied monopolist" and "a.m." for "accompanied monopolist"; "n.a." means not applicable. "No change" means that r continues to take his previous action with respect to the product j.

if they cannot, then, in effect, a price war ensues. We model a price war by means of a "policeman." He is the lowest ranking of the firms other than the deviator and he is the only responder who makes a change. In his response he sets up to produce j if he was not set up in the existing state and he now sets the monopoly break-even price for j, computed using the passive variables of the existing state $x = [a;(y_\pi,p_B)]$ (at which sales y_j are positive). This is the price $p_j^{\#} = \{\sum_{v\in B} p_v[f_{jv}(y_j) + z_{jv}]\}/y_j$. The policeman is to be interpreted as the survivor of a price war. His arbitrary selection by rank expresses the fact that the survivor of a price war is unpredictable. The deviator himself cannot, in our response function, be the survivor of the price war. But that seems to matter little, if we want to capture the essence of the price-war threat, especially in our inertia super-game setting, wherein a deviator gains nothing until he ceases deviating: the deviator is indifferent between being the sole survivor of the price war and having someone else be the survivor, since in either case his profits at the end of the war are zero.

If the deviator k is a seller of j who raises his price above the prevailing price or leaves the industry (ceases to be set up) then, as already stated, the other sellers (if any) make no change. The "departure" of k may, however, make room for certain previous nonsellers. The previous sellers (if any) are joined, at the preceding price, by "replacers"—nonsellers who are added to the group of previous sellers in the order of their market potentials (with ties broken by rank) until the addition of another nonseller to the selling group would make the group no longer a qualified set at the preceding price (i.e., the additional firm would make negative profit†). If a qualified set

†Condition (IV.1) implies that an "earlier" replacer (one with higher market potential) makes no less profit than a "later" one (with lower market potential), since smaller sales can never yield more profit than larger sales.

cannot be formed in this way following k's "departure", then, again, a price war ensues (a policeman takes over).[†]

The proof that $\bar{\phi}$ has the restabilizing property is given in the Appendix. For convenience we shall sometimes call $\bar{\phi}$ the matching response.

Which states $x = [a;(y_{\Pi},p_B)]$ are jointly stable with respect to $\bar{\phi}$? There are a number of them. We shall consider the following two extreme states: (1) a "totally monopolized" state in which one and only one firm sets up to produce every product in Π and sells a positive amount of it at zero profit (i.e., sets the monopoly break-even price corresponding to (y_{Π},p_B)); (2) a "totally oligopolized" state--any state in which every firm produces and sells a positive amount of every product and does so at nonnegative profit. A state of the first kind is jointly stable with respect to $\bar{\phi}$, since if the monopolist lowers price, his profit becomes negative and if he raises price he loses all sales; and since nonsellers can only join him as sellers by setting a price at or below the monopoly break-even price. No one can benefit, therefore, by deviating from the state. A state of the second kind is jointly stable with respect to $\bar{\phi}$ since undercutting leads only to matching or a price war, while raising price or leaving the industry yields no gain. It is clear that given any value of the passive variables (y_{Π},p_B), there is a value of the action n-tuple which achieves a state of the first kind and also one which achieves a state of the second kind.[††]

[†] In particular, k may be a monopolist who raises his price. This "abuse" of his power is, so to speak, punished--either by replacers who set his previous price, thus depriving him of all sales, or, if this cannot be managed, by a policeman. Or, interpreting it another way, the monopolist's greed and its potential fulfillment attracts other firms who previously stayed outside. His replacement, or a price war, ensues. The cases in the last two rows of the table need no comment.

[††] For the second kind of state one simply picks for each product j some prevailing price p_j^* for which profit on j for the firms with lowest market potential--i.e., $p_j^* y_j \min_{t \in N} e_{tj}(N) - \sum_{v \in B} p_v(f_{jv}[y_j \min_{t \in N} e_{tj}(N)] + z_{jv})$--is positive. Condition (IV.1) implies that profit on j for the remaining firms is then also positive.

To summarize, we have

<u>*Theorem C*</u>. *Let any value* $(\bar{y}_{\Pi},\bar{p}_{B})$ *of the passive variables be given. Then there exists a value* a^* *of the action* n*-tuple such that every firm sells every product in* Π *at a nonnegative profit, using* $(\bar{y}_{\Pi},\bar{p}_{B})$ *to calculate profit. There also exists a value* a^{**} *of the action* n*-tuple such that every product is sold by one and only one firm, who does so at zero profit. Moreover, (1) at* $x^* = [a^*;(\bar{y}_{\Pi},\bar{p}_{B})]$ *and also at* $x^{**} = [a^{**};(\bar{y}_{\Pi},\bar{p}_{B})]$ *every firm in N is stable with respect to* $\bar{\phi}$*, the "matching" response function defined by the table; (2)* $\bar{\phi}$ *is a convolution.*

We turn next to a "large" economy or economic game, wherein each firm visualizes a small game, possibly a game of the sort just studied.†

<u>A large oligopolistic economy as a game of limited information</u>

The "large" economic game has again price-making firms as players, but now they may buy from each other, their technologies and potential products differ, and their set-up requirements may be "nonseparable," depending in a nonadditive way on the collection of products produced. To specify the game, moreover, we must now state what variables--active and passive--each player observes, for it is a game in which players' information is limited, <u>in the specific sense that a player either correctly observes a given variable or</u>

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†One could study general oligopolistic equilibria for the small economy. Assume households to be the source of the inputs in B and the demanders of the products in Π. One would seek conditions on households' preference and endowments such that there exist prices (p_{Π},p_{B}) which evoke demands (sales) y_{Π} and input offers sufficient to produce these demands and which at the same time satisfy the conditions of one of the two types of stable states. In such an equilibrium markets are cleared and firms are content in the sense we have developed.

he does not observe it at all, so that it plays no role in his choice.

Notation, and the elements of the game follow.

N is the set of firms (players). U, V, W are sets of final, intermediate, and primary commodities, respectively. The set $\Pi_i \subseteq U \cup V$ is firm i's set of potential products and $B_i \subseteq V \cup W$ his set of potential inputs. Firm i's action is, as before, the set-up-and-price pair $a_i = (\Delta_i, p_{i\Pi_i})$; again, infinity is taken to be i's price for a product for which he does not set up. So a_i is chosen from the set $A_i = \{(\Delta_i, p_{i\Pi_i}) | \Delta_i \subseteq \Pi_i;\ p_{ij} \geq 0 \text{ if } j \in \Delta_i,\ p_{ij} = \infty \text{ if } j \notin \Delta_i\}$. To produce q units of product $j \in \Pi_i$, firm i requires f_{ijk} units of input $k \in B_i$. If firm i sets up to produce the collection $\Delta \subseteq \Pi_i$ he requires $f_{i0k}(\Delta)$ units of input $k \in B_i$. As in the small economic game, the function f_{ijk} is increasing and displays nonincreasing average input requirements.

The passive variables of the game are sales of produced (final and intermediate) goods and primary-good prices, comprising the nonnegative vector $t = (\{y_j\}_{j \in U \cup V \cup W}, \{p_j\}_{j \in W})$. The symbol Π_k^i denotes the set of k's products, a subset of Π_k, which are observable by i: if $j \in \Pi_k^i$, then i observes k's setting up for j as well as k's chosen price p_{kj}. (We may formally express i's observing of the set-up decision by saying that i observes whether the price p_{kj} is finite or infinite.) The set of products whose sales i observes is denoted $Y^i \subseteq U \cup V$ and the set of primary inputs whose prices he observes is denoted $W^i \subseteq W$. Firm i observes total sales of his own products, his own (primary and intermediate) input prices, and the prices other firms set for products which i can also produce, so that

$$\text{(IV.2)} \quad \Pi_i^i = \Pi_i,\quad B_i \cap V \subseteq \{\Pi_k^i\}_{k \in N},\quad \Pi_i \subseteq Y^i,\quad B_i \cap W \subseteq W^i;\ \text{and}\ \Pi_i \cap \Pi_k \subseteq \Pi_k^i,\ \text{all } k \text{ in } n.$$

A state of the game is $x = (a,t)$. The aspect of the state x observable by i comprises the triple $x_i \equiv (\{p_{i\Pi_k^i}\}_{k \in N}, y_{Y^i}, p_{W^i})$. As before, $\{\alpha_{ij}\}_{i \in N, j \in \Pi_i}$

are market potentials. The payoff to firm i at the state x depends only on the i-observed aspect x_i and is given by†

$$\text{(IV.3)}\quad H_i(x_i) \equiv \sum_{j\in\Delta_i} \{p_j y_j e_{ij}(R_j) - \sum_{v\in B_i} p_v f_{ijv}[y_j e_{ij}(R_j)]\} - \sum_{v\in B_i} p_v f_{i0v}(\Delta_i).$$

Define $H = \{H_i\}_{i\in N}$, $\alpha = \{\alpha_{ij}\}_{i\in N, j\in\Pi_i}$, $J = (\{\Pi_k^i\}_{i\in N,k\in N}, \{Y^i\}_{i\in N}, \{W^i\}_{i\in N})$, $F = (\{f_{ijk}\}_{i\in N, j\in\Pi_i, k\in B_i}, \{f_{i0k}\}_{i\in N, k\in B_i})$. Then a game G in the class $\mathscr{G}$ of large economic games is defined by a quintuple $G = (N,J,\alpha,H,F)$, for which (IV.2), (IV.3), and the condition of nonincreasing average input requirements are satisfied. The aspect of G known to firm i comprises the septuple††
$G_i \equiv (\{\Pi_k^i\}_{k\in N}, W^i, Y^i, \{k \in N \mid \Pi_k^i \neq \Phi\}, \{\alpha_{kj}\}_{j\in\Pi_i\cap\Pi_k, k\in N}, \{f_{ijk}\}_{j\in\Pi_i, k\in B_i},$
$\{f_{i0k}\}_{k\in B_i})$. Firm i knows his own technology and market potentials. He knows the identity of firms whose decisions he observes. He knows the identity of the commodities whose

prices or sales he observes. He knows the market potential of every firm with respect to any product which i can also produce. But he does not know other market potentials or the technologies of other firms.

Firm i, then, does not know the complete game G, and he observes only x_i, not x, the complete current state of the game. Moreover, taking fully into account all of G_i and all of x_i may be burdensome. For both reasons we shall permit firm i to construct a model. Player (firm) i's model comprises (1) a game $\bar{G} = (\bar{N},\bar{J},\bar{\alpha},\bar{H},\bar{F})$ which also belongs to the class $\mathscr{G}$, but may be simpler than G,

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†The symbols e_{ij}, R_j have the same meaning as in the previous section and again p_j means $\min_{i\in N} p_{ij}$ for j in U ∪ V. Note that if i's production (to meet sales) requires a nonprimary input which no firm is selling, then H_i takes the value $-\infty$.

††Φ denotes the empty set.

since it may have fewer players and variables and some of G's action variables may become passive variables in $\bar{G}$; (2) a current state, $\bar{x}$, of the game $\bar{G}$; and (3) the response functions, forming the collection $\{\bar{\phi}^k\}_{k\in\bar{N}\setminus i}$, which each player $k \neq i$ in $\bar{G}$ will use to respond to a deviation. To obtain the model, player i uses a modeling rule $\mathcal{M}$ which assigns† a triple $(G,x,\{\phi\}^k_{k\in\bar{N}\setminus i}) = \mathcal{M}(G_i,x_i,i)$, with $\bar{G}$ in $\mathcal{G}$, to every triple (G_i,x_i,i), where, for some game G in $\mathcal{G}$, G_i is the aspect of G known to i and x_i is the i-observed aspect of a state in G.

We shall impose restrictions on the modeling rule. A modeling rule $\mathcal{M}$ is an <u>admissible modeling rule</u>--an AMR--if it has five properties, which we shall state in such a way as to suggest informally that they can be viewed as general properties of a modeling rule in abstract games, not only in the present economic game.††

<u>First</u>, the set $\bar{N}$ includes the modeler i himself and includes only players whom i observes (with respect to one or more of their action variables) in the original game G. <u>Second</u>, except for the possibility that some action variables in G may become passive variables in $\bar{G}$, the triple $(\bar{J},\bar{\alpha},\bar{F})$ neither contradicts nor goes beyond the game aspect G_i. To be specific: $\bar{\Pi}_i \subseteq \Pi_i$;

$\bar{\Pi}_k \subseteq \Pi_k$ for all $k \in \bar{N}$;

$\bar{B}_i \subseteq B_i$, $\bar{Y}^i \subseteq Y^i$, $\bar{W}^i \subseteq W^i \cup \{\Pi^i_k\}_{k\in N}$, $\{\bar{\alpha}_{ij}\}_{j\in\bar{\Pi}_i} = \{\alpha_{ij}\}_{j\in\bar{\Pi}_i}$, $\{\bar{f}_{ijk}\}_{j\in\bar{\Pi}_i,k\in\bar{B}_i} = \{f_{ijk}\}_{j\in\bar{\Pi}_i,k\in\bar{B}_i}$, $\{\bar{f}_{i0k}\}_{k\in\bar{B}_i} = \{f_{i0k}\}_{k\in\bar{B}_i}$; and for every $k \in \bar{N}$ with $k \neq i$, $\bar{B}_k \subseteq W^i \cup \{\Pi^i_k\}_{k\in\bar{N}}$, $\bar{\alpha}_{kj} = \alpha_{kj}$ for all $j \in \bar{\Pi}_k \cap \Pi_i$, $\bar{Y}^k \subseteq \{\Pi^i_k\}_{k\in\bar{N}}$, $\bar{W}^k \subseteq W^i$.

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†Formally, $\mathcal{M}$ is <u>from</u> the set $\{(G_i,x_i,i) \mid i \in N;\ G_i$ is the aspect known to i of a game G in $\mathcal{G}$; x_i is an i-observed aspect of a state of G$\}$ <u>to</u> the set $\{(\bar{G},\bar{x},\{\bar{\phi}^k\}_{k\in\bar{N}\setminus i}) \mid \bar{G} \in \mathcal{G}$; $\bar{N}$ is the set of players in $\bar{G}$; $\bar{x}$ is a state of $\bar{G}$; $\bar{\phi}^k$ is a response function for $k \in \bar{N} \setminus i\}$.

††A formal abstract version is given in [10, Chapter IV].

Then[†] every variable (passive or action) in $\bar{G}$ is also an i-observed variable in G and no player k in $\bar{G}$ has an action variable he does not have in G. But some intermediate products, whose prices were action variables in G, may in $\bar{G}$ become primary inputs, whose prices are passive variables.

Third, if x_i^{**} is the i-observed aspect of some state in $\bar{G}$, if x_i^* is the i-observed aspect of a state in G, and if x_i^{**} and x_i^* coincide with respect to those variables they have in common (namely, the variables in x_i^{**}), then $\bar{H}_i(x_i^{**}) = H_i(x_i^*)$. The modeler does not discard, in other words, any variable which determines his true payoff. Specifically, this implies that $\bar{\Pi}_i = \Pi_i$ and $\bar{B}_i = B_i$. Fourth, the value $\bar{x}_i$, which i gives to the i-observed aspects of $\bar{G}$'s current state, must coincide (as regards common variables) with his observations, $x_i^\cdot$, on the current state of the true game G.

Fifth, the response functions $\{\bar{\phi}^k\}_{k\in\bar{N}\setminus\{i\}}$ which i attributes to other firms in his model must be credible to i, in the sense developed in Parts II and III. A player k, in i's model, is himself a modeler, using, like all players, the admissible modeling rule $\mathcal{M}$. Player k sees only those aspects of a deviation which the game $\bar{G}$ assigns to him as k-observed, and these are the arguments in the response function $\bar{\phi}^k$ which the modeler i attributes to him. The function $\bar{\phi}^k$ yields new values of the action variables which the game $\bar{G}$ assigns to k. If the functions $\{\bar{\phi}^k\}$ are to be credible to the modeler i, then, following any deviation by some player $s \in \bar{N}$, two conditions must hold: (a) The modeled responding players k ($k \neq s$, $k \neq i$) must, as modelers in their own right, not surprise each other when they use the functions $\bar{\phi}^k$; otherwise the responses

[†]We may wish to restrict the rule still further with respect to i's modeling of the technology of other firms. We may, for example, want to require that for any firm $k \neq i$ in $\bar{N}$ which has a potential product j in common with i, and for any input ℓ in $\bar{B}_i \cap \bar{B}_k$, the functions $\bar{f}_{kj\ell}$ and $\bar{f}_{k0\ell}$ are not too different, in some precise sense, from i's own functions $f_{ij\ell}$, $f_{i0\ell}$.

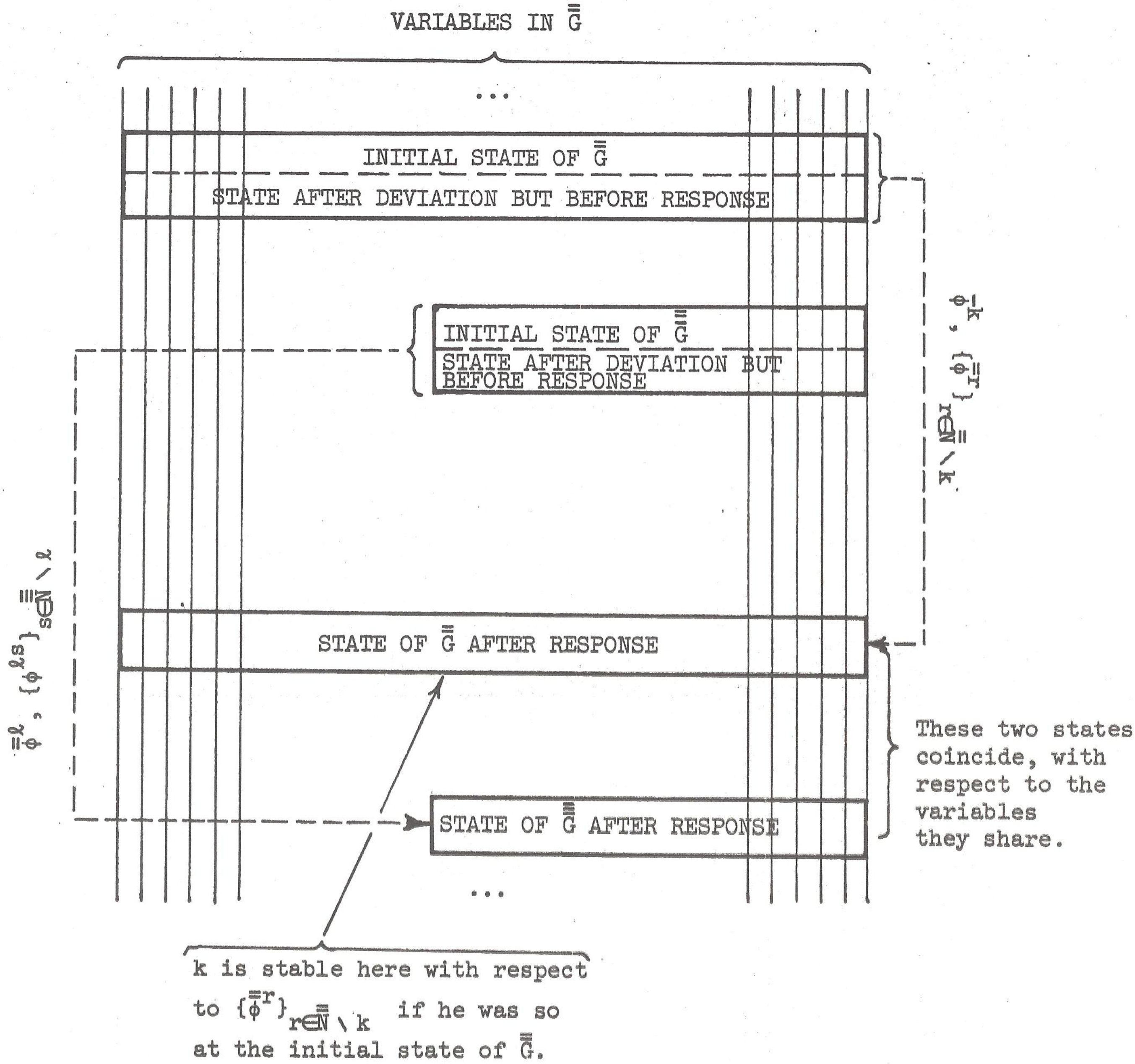
generated by these functions would themselves be viewed as deviations and would evoke further responses. (b) With respect to the response functions which k in $\bar{N}$ attributes to the other players in $\bar{G}$, the function $\bar{\phi}^k$ must restabilize k if he was stable before the deviation.

Requirements (a) and (b) must be met by the response functions assigned by an AMR. Schematically, the requirements may be portrayed as on the following page.

The figure deals with a particular player $k \neq i$ in i's modeled game $\bar{G}$ and portrays the credibility of the response function $\bar{\phi}^k$, given that all players use the given AMR. To put it another way, the figure shows i that if he were to "propose" $\bar{\phi}^k$ to k, then k would find it an acceptable response function. Player k's own modeled game, based on $\bar{G}_k$, the aspects of $\bar{G}$ known to k, is $\bar{\bar{G}}$; and it is in $\bar{\bar{G}}$ that k contemplates using the proposed function $\bar{\phi}^k$.

Player $\ell \neq k$ is any other player in $\bar{\bar{G}}$; ℓ, using $\bar{\bar{G}}_\ell$ and the given AMR, forms the modeled game $\bar{\bar{\bar{G}}}$. Columns of the figure correspond to (passive and action) variables in $\bar{\bar{G}}$. In the lower and upper parts of the top two boxes of the figure, and in the two bottom boxes, imagine every column to contain a value of that column's variable. There is an initial state of $\bar{\bar{G}}$ and some deviation from it by a player who is neither k nor ℓ. But ℓ perceives the initial state as an initial state of $\bar{\bar{\bar{G}}}$; ℓ's version of the initial state, of the deviation, may (as in the figure) involve fewer variables than k's version, but the two versions agree with respect to the variables they have in common, player k expects every fellow player r in $\bar{\bar{G}}$ to use the function $\bar{\bar{\phi}}^r$ given by the AMR. On the other hand, k knows that ℓ expects that in <u>his</u> modeled game $\bar{\bar{\bar{G}}}$ all other players s, <u>including</u> s = k, will use $\bar{\bar{\bar{\phi}}}^{\ell s}$. And because the modeling rule is admissible, k knows that the response of k using the proposed $\bar{\phi}^k$, and the

Modeler i's view as to why $\bar{\phi}^k$ is a credible response function for k in i's modeled game $\bar{G}$, when all players use a given AMR



$\bar{\bar{G}}$ is k's model of the game, based on $\bar{G}_k$; k attributes $\{\bar{\bar{\phi}}^r\}_{r\in\bar{\bar{N}}\setminus k}$ to the other players in $\bar{\bar{G}}$; ℓ is a player in $\bar{\bar{G}}$; $\bar{\bar{\bar{G}}}$ is ℓ's model of the game, based on $\bar{\bar{G}}_\ell$; ℓ attributes $\{\phi^{\ell s}\}_{s\in\bar{\bar{\bar{N}}}\setminus\ell}$ to the other players in $\bar{\bar{\bar{G}}}$.

responses of k's fellow players r, using the functions $\bar{\bar{\phi}}^{r}$, cannot surprise ℓ; the functions $\bar{\phi}^{k}$, $\{\bar{\bar{\phi}}^{r}\}_{r\in\bar{\bar{N}}\setminus k}$ agree with the functions $\bar{\bar{\phi}}^{\ell}$, $\{\phi^{\ell s}\}_{s\in\bar{\bar{N}}\setminus \ell}$ as regards the values they assign to common variables. Hence k knows that these responses will evoke no further ones. In addition, k verifies that if he was stable at the initial state of $\bar{\bar{G}}$ with respect to the functions $\{\bar{\bar{\phi}}^{r}\}_{r\in\bar{\bar{N}}\setminus k}$ which he expects others to use, then if he uses the proposed $\bar{\phi}^{k}$ he is stable once again following the responses of all players in $\bar{\bar{G}}$. The proposed $\bar{\phi}^{k}$ is therefore acceptable to k and the modeler i can credibly attribute $\bar{\phi}^{k}$ to k. That concludes the statement of the five admissibility conditions.

Suppose now that x is a state of a game G in $\mathcal{G}$, and suppose all players use a given AMR. Suppose that at x every player i finds that in his modeled game $\bar{G}$ he is stable at the modeled current state $\bar{x}$, with respect to the attributed response functions $\{\bar{\phi}^{r}\}_{r\in\bar{N}\setminus i}$. Then we shall say that x is a sustainable state of G with respect to the given AMR.

Consider the special case of a full-information game G in $\mathcal{G}$, where every player observes every variable. For the modeling rule used, let each player's model of the game and its current state be the true game itself and its true current state. Then if the modeling rule is to be admissible, all players must attribute the same response function to a given player; otherwise, his responses could, for some states and deviations, surprise some player. Moreover, when each player uses the response function unanimously attributed to him he restabilizes himself with respect to the response functions unanimously attributed to all the others. Hence the collection of unanimously attributed response functions form a convolution for the game; and a state of the game which is sustainable with respect to the AMR is also stable with respect to that convolution. So the concept of a state sustainable with respect to an AMR is a true generalization--from the special case of full information and full modeling to the general case of limited information and partial modeling --of our original concept of a state which is stable with respect to a convolution. It is a state from which no player wants to deviate given what he knows and observes and what he credibly attributes to other players. We now consider two examples of AMR's and their sustainable states.

First example: *The separable-fixed-cost economy; each firm's model contains only his competitors, who are technologically like him and use the "matching" response.* In an economic game of the class $\mathcal{G}$, suppose (a) that set-up requirements are separable, i.e., firm i, producing the product collection $\Delta \subseteq \Pi_i$, requires $f_{i0k}(\Delta) = \sum_{j \in \Delta} z_{ijk}$ of input k to set up, where $k \in B_i$ and $z_{ijk} \geq 0$. Assume also (b) that there are at least two potential producers of every product.(i.e., if $j \in \Pi_i$, then $j \in \Pi_s$ for some $s \in N$ with $s \neq i$); and (c) if two firms have one potential product in common then their sets of potential products and inputs are the same (if i and j are in N, with $\Pi_i \cap \Pi_j$ nonempty, then $\Pi_i = \Pi_j$ and $B_i = B_j$). Let $\mathcal{G}'$, a subclass of $\mathcal{G}$, be the class of economic games for which (a), (b), (c) are true.

Now consider the following modeling rule. A player i, using his current-state observations x_i and his knowledge of the game aspects G_i, constructs the modeled game $\bar{G} = (\bar{N},\bar{J},\bar{\alpha},\bar{H},\bar{F})$ which belongs to $\mathcal{G}'$, a current state $\bar{x}$ of $\bar{G}$, and response functions $\{\bar{\phi}^k\}_{k \in \bar{N} \setminus i}$ to be used by the other players in $\bar{N}$. The players $k \neq i$ in $\bar{N}$ are i's competitors and have i's own technology; i.e., $\bar{\Pi}_k = \Pi_i$, $\bar{B}_k = B_i$, and for all $j \in \bar{\Pi}_k$, $v \in \bar{B}_k$, $\bar{f}_{kjv} = f_{ijv}$, $\bar{f}_{k0v} = f_{i0v}$. The players in $\bar{N}$, however, retain their true market potentials ($\bar{\alpha}_{kj} = \alpha_{kj}$, all $k \in \bar{N}$, all $j \in \bar{\Pi}_k$). The prices of nonprimary inputs in B_i were action variables in G but become passive variables in $\bar{G}$. A state of $\bar{G}$ specifies prices of all inputs in B_i, and set-up decisions, prices, and sales for all products in Π_i. Every player in $\bar{G}$ observes the entire state (so that, for all k, s in $\bar{N}$, $\bar{\Pi}_k^s = \bar{Y}^k = \Pi_i$, $\bar{W}^k = B_i$). The payoff function for a player in $\bar{N}$

is the same function as in (IV.3), with bars placed over B_i, f_{ijv}, and f_{i0v}. The game $\bar{G}$, then, is precisely the "small" economic game studied earlier.

In the modeled current state $\bar{x}$ of $\bar{G}$, set-up decisions, prices, and sales are exactly those of x for the products in Π_i, and so are the prices of the inputs in B_i. Clearly, the first four conditions of an AMR are fulfilled.

Finally, the modeler i--as well as every player k in $\bar{G}$ in <u>his</u> role of modeler--attributes to every other player k in $\bar{N}$ the "matching" response function defined in the previous section. But then the fifth condition of an AMR is also fulfilled. The modeler i easily verifies that <u>every</u> player k in $\bar{G}$ must, when k applies the rule, take $\bar{G}$ itself to be k's model of the game and $\bar{x}$ as k's model of the game's current state. Moreover, i knows that each attributes the matching response to every other player in $\bar{G}$. Since each observes what all the others observe it follows immediately that no player's response to a deviation can surprise any other player. As Theorem C showed, moreover, each player who uses the matching response in $\bar{G}$ restabilizes himself.

What states of the original game G are sustainable with respect to the AMR just described? There are at least two kinds of sustainable state of G, corresponding to the two kinds of jointly stable states of $\bar{G}$ described in Theorem C. First, consider a "totally oligopolized" state x* of G wherein every product is sold in positive quantity by every one of its potential producers at nonnegative profit. Then every player i models a state $\bar{x}^*$ of the modeled game $\bar{G}$ at which i and every other player sell a positive quantity of each of i's potential products and does so at nonnegative profit, using the relevant sales and input prices of x^* to compute profit. Theorem C then tells us that i is stable with respect to the matching response function which he attributes to the players in $\bar{G}$. So the state x^* of G is sustainable with respect to our AMR.

Second, consider a "totally monopolized" state x^{**} of G in which every product j is produced by only one of its possible producers, who does so at zero profit; it is also true at x^{**} that if any of the other possible producers were to join the producer of j as seller of j at its current price (producing and selling the share appropriate to his market potential), then he would do so at zero or negative profit (if he were to replace the producer of j at the replacer's monopoly break-even price or less, he would also do so, of course, at zero or negative profit). The each player i models a state $\bar{x}^{**}$ of $\bar{G}$ in which each product is produced by a monopolist. If the monopolist is i himself, then in the modeled state he makes zero profit and gains nothing by changing his price. If the monopolist is some player other than i, then i finds that he would make zero or negative profit were he to join the monopolist (a replica of himself) at the current price and *a fortiori* (since sales are constant) if he were to undercut him and be matched. (The nonproducing firm i expects that he could replace the monopolist by setting a price less than or equal to i's monopoly break-even price, but that cannot interest i either.) So x^{**} is also sustainable with respect to the AMR.

Second example: *The nonseparable-fixed-cost economy; each firm's model contains only his competitors, whose technology is a "superior" separable version of his and who use the matching response.* Consider a game G in the original class of games in $\mathcal{O}$ which satisfies conditions (b) and (c) of the preceding example but not necessarily (a): input requirements need not be separable. Assume, on the other hand, that (d) every firm displays <u>economies of diversification</u>. This means that for every firm i, and for two product collections Δ, $\bar{\Delta}$, with $\bar{\Delta} \subset \Delta \subset \Pi_i$, where $\bar{\Delta}$ may, in particular, be the empty set Φ,

$$0 \leqq f_{i0k}(\Delta \cup \{j\}) - f_{i0k}(\Delta) \leqq f_{i0k}(\bar{\Delta} \cup \{j\}) - f_{i0k}(\bar{\Delta})$$

for every j in the set $\Pi_i \setminus \Delta$ and for every input $k \in B_i$. We define $f_{i0k}(\Phi)$ to be zero.

Let $\mathcal{G}''$ denote the subclass of games satisfying (b), (c), and (d). The AMR we now consider differs from the preceding one <u>in one respect only</u>. Any player $k \neq i$ in i's modeled game $\bar{G}$ has, though i himself may not, separable set-up requirements. Specifically, $\bar{f}_{kjv} = f_{kjv}$ for every j in $\bar{\Pi}_k = \Pi_i$; but for any collection $\Delta \subseteq \bar{\bar{\Pi}}_k$, $\bar{f}_{k0v}(\Delta) = \sum_{j \in \Delta} \bar{z}_{kjv}$, where $\bar{z}_{kjv} \equiv f_{i0v}(\Pi_i) - f_{i0v}(\Pi_i \setminus j)$. The modeler i, then, in view of condition (d), conservatively pictures every competitor k (every fellow player in $\bar{G}$) as possessing a separable technology superior to his own: k, in augmenting his collection of goods produced, never adds more to his set-up requirements than i himself would have to add. For a firm which, like the modeled firm k, is already separable, the function $\bar{f}_{k0v}$ is identical to the original requirement function. Therefore, the game $\bar{G}$ is in $\mathcal{G}''$ and each modeled firm $k \neq i$ in $\bar{G}$ models a game $\bar{\bar{G}}$ (also in $\mathcal{G}''$) which is precisely like the game $\bar{G}$ of the preceding example, wherein all players are separable firms. Once again, then, i knows that k, whose model has only separable firms, finds (using the given modeling rule himself) that the matching response function, when used by k and k's fellow players, surprises no one and restabilizes k.

Our second modeling rule, then, is also admissible. What states of G does it render sustainable? First, a totally oligopolized state $x^{*'}$ in which every firm sells a positive amount of each of his potential products and, at the prices and sales of $x^{*'}$ cannot increase his profit by dropping any subset of them, in particular, any one-element subset. We are then automatically assured that a firm i has no interest in changing his <u>prices</u>: raising a price simply means, in i's model, abandoning that market to other producers; lowering the price of a product j either means selling the same amount at the lower price or it means take-over of the industry by a policeman (a price war). The policeman sets the monopoly break-even price $p_j^{\#}$ appropriate to the superior separable version of i, the technology which, in i's model, the policeman possesses. But if it is true that a (separable) firm with that technology makes zero profit on product j when he supplies the entire demand at $p_j^{\#}$, then it is also true that the (possibly nonseparable) firm i himself would find, if he were selling any portion of the demand at $p_j^{\#}$, that his profit could be increased by dropping j (since the set-up inputs he saves by doing so are at least as great as the modeled separable policeman's set-up requirements for j). If, therefore, firm i does not want to drop j at the prices of $x^{*'}$, then i has no interest in taking over the entire j-market by setting a price less than his break-even price. We conclude that if i does not want to drop any subset of products at the prices of $x^{*'}$--and therefore no single product--then he also is content with his price choices at $x^{*'}$.

A second type of state sustained by the AMR is $x^{**'}$, at which every product is produced and sold by only one of its potential producers. In this state the seller i of a product j cannot increase his profit, given the prices and sales of $x^{**'}$, by dropping either j or any of the other products he sells. Further, i cannot increase his profit by adding some other product j' to his collection of products sold. Given $x^{**'}$, the AMR indicates to i two possible ways of adding j' to his collection: first, in i's model, he could join the seller of j' by matching the price set by that seller, provided that seller (a superior separable version of i) can survive the matching; if the seller of j' were to make negative profit when i matches him, then a policeman would take over the industry. Second, i could replace the seller of j', taking over the j'-market himself, by setting a price lower than the model's monopoly break-even price (i.e., the break-even price for the superior separable version of i). Clearly, the second method can have no interest for i. Since, at the state x**', no firm wants either to drop products or to add products by joining their current sellers, it follows from (d) that a firm has a fortiori no wish to first drop a product and then to add some other products (by joining these products' current sellers); the additional set-up inputs required to add the new products to the smaller collection are at least as great as the additional set-up inputs required to add the new products to the original collection. Just as in the preceding discussion of x*', moreover, firm i has no wish to make any price change for the products in his collection: he is deterred by the expected responses of the policeman in his model; the policeman's break-even price for a product is not higher than any price that can interest firm i.

We conclude that if x**' is such that each product has one seller and no firm wants to drop products or to add new ones by joining their sellers, then x**' is also such that no firm wants to make any other of his possible changes.

General equilibria

If a state of an economic game G is sustainable with respect to an AMR and is also a state in which all markets are cleared, then it is an oligopolistic general equilibrium of G for that AMR. For a class of economic games, definition of the set of market-cleared states of a game requires that household demand functions and endowments, which have so far played no role, be specified for every game.†

The existence of oligopolistic general equilibria for specified AMR's can be studied. In particular, several further conditions on the technologies of firms in games of classes $\mathcal{G}'$ and $\mathcal{G}''$, together with conditions on household demand functions and endowments, imply the existence of market-cleared states which are also sustainable states of the totally monopolized types x** and x**'. These conditions, then, guarantee the existence of oligopolistic general equilibria for the two AMR's which we have illustrated.††

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†In a game G of the class $\mathcal{G}'$ or $\mathcal{G}''$, a state $x = (\{\Delta_i, p_{\Pi_i}\}_{i \in N}, \{y_j\}_{j \in U \cup V \cup W}, \{p_j\}_{j \in W})$ is a market-cleared state if

$$y_k = h_k\left(\frac{1}{L \cdot p_W + \sum_{i \in N} H_i(x)} \cdot p_{U \cup W}\right) + \sum_{\substack{j \in \Delta_i \\ i \in N}} f_{ijk}[y_j e_{ij}(R_j)] + \sum_{i \in N} f_{i0k}(\Delta_i) - L_k,$$

$$\text{all } k \in U \cup V \cup W.$$

Here h_k is a households' demand function, assigning a total household demand for a primary or final commodity k to every value of $\frac{1}{\tau} p_{U \cup W}$, with $\tau > 0$, where τ denotes total household income (endowment income plus firms' profits); $h_k = 0$ when k is an intermediate commodity ($k \in V$). $L = \{L_k\}_{k \in U \cup V \cup W}$ denotes households' total commodity endowments, with $L_k = 0$ for $k \in U \cup V$ and $L_k > 0$ for $k \in W$.

††For an existence proof see [10, Chapter V].

In both the separable and nonseparable economies there are other convolutions, which could provide the attributed response functions for other Admissible Modeling Rules. The matching response functions and the AMR's studied have been illustrations, though the matching response function does prescribe a behavior discussed, but never rationalized, in the oligopoly literature,† and perhaps crudely captures a few aspects of real oligopolistic behavior. One attraction of the approach developed here is precisely that it makes the institutionalized behavior of firms the object of study. The approach seems closer in spirit to the field of "industrial organization" than other fully formalized theories of oligopoly.

We have made each firm, in our illustrations, view only his competitors as fellow players, not his customers and suppliers, and have made him model his competitors in a very simple way, so that they differ from each other in market potentials only. As a result, we have achieved the "nonsurprising" property of the attributed response functions (the fifth property of an AMR) in a very easy manner. It remains a challenge to construct restabilizing response functions and associated AMR's for oligopolistic games so as to permit a richer assortment of players, both in the game itself and in the models which players construct.

Our concept of limited information has been a special one, which a thoroughgoing Bayesian, for example, might view as degenerate and unacceptable. But it would immensely complicate matters, and would impose far more burdensome computations on the players, if we were to introduce probabilities, so as to generalize our notions that a player either accurately observes a variable or else ignores it altogether, and that he forms a unique model of the game and its current state.

The concept of a state stable with respect to a convolution, and its generalization to a state sustainable with respect to an AMR, appear to be more plausible

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†Moreover, the states stabilized by the matching response function include, as we saw, states in which a form of "stay-out pricing" is practiced by the producing firms. See Sylos-Labini [17] for a discussion of this concept.

and to hold more promise for a noncooperative oligopoly theory than the classic Cournot-Nash equilibrium. Convolutions can also be constructed for monopolistically competitive economies,† and simple associated AMR's can probably be developed there as well.

At the same time, a convolution is a compact representation of a generally very cumbersome object, namely a supergame strategy combination, and to check whether a certain empirically observed or plausible behavior has the convolution property may be a fairly simple matter. The convolution, and the associated supergame strategy combination, paint a picture of a deviator which is reasonable--he is serious and sticks with his deviation as long as others behave as he expects them to. Reasonable as it is, it is still arbitrary and so is any other picture which might be painted by another equally defensible strategy combination. Yet <u>every</u> defensible combination must face the question how a deviator is expected to behave. It is empirical study, whether of subjects in a laboratory or firms in an economy, which eventually will have to settle the usefulness of the concept developed here.††

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†See [10, Chap. 3].

††Two suggestive extensions of the preceding framework ought to be mentioned. <u>First</u>, it is possible to handle some truly dynamic supergames (economies), with one period's actions affecting the payoffs subsequently attainable by players. To do so, one adds to the action variables and the passive variables what is usually called a "state variable" in dynamic discussions. Let σ denote such a variable. Then the inertia supergame is defined by a quintuple: (i) the players and their action variables; (ii) the passive variables; (iii) a transition function, which assigns a current value of σ to a pair composed of the previous value of σ and the current value of the passive and action variables; (iv) current-payoff functions, which assign a current payoff for each player to given current values of σ and of the passive and action variables; (v) a change-cost function defined on changes in the action variables. If the possible values of σ and the other variables form a compact set and the payoff functions are continuous, then the extension of all our concepts, and of Theorems A and B, is straightforward, and a reasonable persistent value for all variables (including σ) can be described. If, on the other hand, one wanted to treat action variables like investment, and a growing (unbounded) state variable like capital stock, then a more radical reinterpretation would be required. Long-run average payoff could no longer measure a player's

satisfaction in the supergame, since that quantity might not be finite. As before, summed discounted payoff is ruled out, since that might place sufficient weight on the current period's payoff so as to invalidate Theorem

B. A suitable measure might be one which coincides with equilibrium growth rate whenever the actions chosen in all subsequent periods of the supergame are those of equilibrium growth. The persistent state of the preceding theory would be replaced by an equilibrium growth path, with each player content to repeat forever his equilibrium growth action. The details remain open.

Second, the static equilibrium which is the persistent state of the theory we have developed could remain the object of study, but one could associate with it a comparative-statics analysis. To do so, one enlarges the domain of a player's response function to include the passive variables. Such a function, then, assigns a new value of the player's action to existing values of all variables and to a deviation from them by an action variable <u>or</u> by one or more passive variables. The definitions of a "state stable for a player with respect to a given response function" and of "convolution" are adapted in an obvious way. We again want the response function to have the restabilizing property. Suppose the players (the economy) are in an equilibrium state (a jointly stable state) of the sort we have described and a passive variable now changes (consumers' demand shifts, for example). Then the action variables (e.g., prices) respond and there is a new state, at which players are again content. If one is interested in general equilibrium in economic games, then the question is whether, starting from a market-cleared jointly stable state, such a readjustment also brings with it the reclearing of markets. Response functions which are "reclearing" as well as restabilizing would need to be found.

APPENDIX: PROOF THAT $\bar{\Phi}$ IS A CONVOLUTION

First, consider the ways in which a firm r can find himself stable with respect to $\bar{\phi}$ as far as his decisions with respect to a product j are concerned --no sequence of deviations with regard to this decision can leave r with a higher profit on j.

In Type 1 stability of firm r with respect to product j, firm r is one of a group of two or more who set the prevailing price. Firm r makes non-negative profit. Raising price or abandoning the industry does not increase his profit. (Firm r cannot, clearly, be a nonselling insider.)

In Type 2 stability of firm r with respect to j, firm r is a monopolist-- the sole seller at the prevailing price. Firm r might have become such as a result of having previously been assigned to the "policeman" role following some firm's deviation. Firm r is making nonnegative profit on j, and if his price is above the break-even price, he cannot benefit if he lowers price while keeping it above the break-even price, even though no response would be evoked in the others if he did so. Firm r cannot benefit by raising price, for this would mean entry of a policeman at the break-even price (a price war).

In Type 3 stability of r with respect to j, firm r is an outsider who cannot benefit from entering the empty j-industry at any price (even though this would evoke no changes) or from entering the nonempty j-industry at the prevailing price or lower (the nonempty j-industry might, in particular, have zero sales). If the industry is nonempty, the result of entering would be a matching that yields nonpositive profit to r; or entry of a policeman at the break-even price; or--if j has been solely sold by a monopolist charging the break-even price or less, or if sales have been zero--negative profit for r.

It is verified easily that no other types of stability are possible and that if r exhibits any of the above types of stability with respect to j, then no chain of deviations can leave r with higher profit than he now receives.

Now consider the deviations with respect to product j which could be observed by firm r, who exhibits Type 1 stability with respect to j. Firm r could observe some firm k (previously as an insider or an outsider) setting a price lower than the prevailing one. The responses to this deviation establish a new situation. There are three possibilities: (1) firm r is again one of a group of firms making nonnegative profit with none desiring to raise or lower price--so that r again exhibits Type 1 stability; (2) a policeman has taken over the industry as sole seller (or the sole seller is the deviator, who charges the break-even price or less), so that r now exhibits Type 3 stability; (3) firm r has himself been assigned the policeman role, so that he now exhibits Type 2 stability.

The deviator k could, instead, be a seller who raises his price or who leaves the industry (ceases to be set up). The responses will either leave r --together, perhaps, with some new sellers who replace k--as one of a group of firms with nonnegative profit and no wish to raise or lower price; or a policeman (possibly r himself) will take over. In any case r's stability is restored. The deviator could be a previous nonselling insider who leaves the industry or raises his price. Then no responder changes his actions and r remains Type-1 stable.

Next,suppose r exhibits Type 2 stability with respect to j. Suppose a deviator (a previous outsider or nonselling insider) matches or undercuts r's price. Then the result is either two firms with nonnegative profit and no wish to lower price (so that r is Type-1 stable) or a single firm set up for production and selling, namely either a policeman (possibly r himself) or else the deviator, selling at the break-even price or less (so that r is Type-2 or Type-3 stable). If the deviator sets a price above r's, then there is no change by r and r remains Type-2 stable.

Finally, let r exhibit Type 3 stability with respect to j. A deviator may be an insider, or a previous outsider (like r) who now enters the industry, setting in either case a price at or below the prevailing one. The result, following the responses, is either that all the previously selling firms continue to sell, but at the deviator's price or less, while r continues to have no wish to enter the industry (for if he could achieve positive profit by entering now he could also have done so before the deviation); or there is a sole selling insider who is either a policeman (possibly r himself) or else is the deviator selling at the break-even price or lower; or there is a sole selling insider who was also previously sole seller (with r having no desire to match or undercut him) and continues to be such but at a lower price (so that r's attitude persists). Hence r exhibits Type 3 stability or (if he is, in fact, the policeman) Type 2 stability.

The deviator may be a previous seller who raises price (above the break-even price) or leaves the industry. As a result r may be one of the replacers of k, making nonnegative profit with no desire to raise or lower price (Type 1 stability); or he may be excluded from the set of replacers while the industry retains at least one seller, in which case he knows† that entering and then matching or undercutting could not benefit him (Type 3 stability), as was true before the deviation; or he may observe, once again, a sole seller who may be a policeman, possibly himself (Type 2 or Type 3 stability).

The deviator may be a previous nonselling insider again ineffectually setting a new price still above the prevailing one, or he may be an insider who raises price but to a level less than or equal to the break-even price. No change is evoked and r retains his Type 3 stability. The deviator may be

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†The condition of nonincreasing average input requirements is used here.

an entrant into an empty industry. Since r had no wish to enter the empty industry himself (an action which would have evoked no change from any responder) he cannot be interested in entering the industry now that it contains one firm, and so he retains his Type 3 stability.

We have shown, then, that if a firm r is stable at a state x with respect to $\bar{\phi}$ (which means he exhibits one of the three types of stability with respect to each product), then he continues to be stable following a deviation by some other firm. Existence of a jointly stable state was shown in the text. It follows that $\bar{\phi}$ is a convolution.

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E r r a t a

for

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p.12, line 13, replace "i" by "the state" and replace "his" by "i's"

p.17, line 12, after "if" insert "for every i in N"

p.17, footnote, line 5 from bottom, replace "if j" by "if some j $\neq$ i"

p.30, line 9, "input", at end of line, should be "product"