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SELECTION OF VALUES FOR NON-SIDEPAYMENT GAMES

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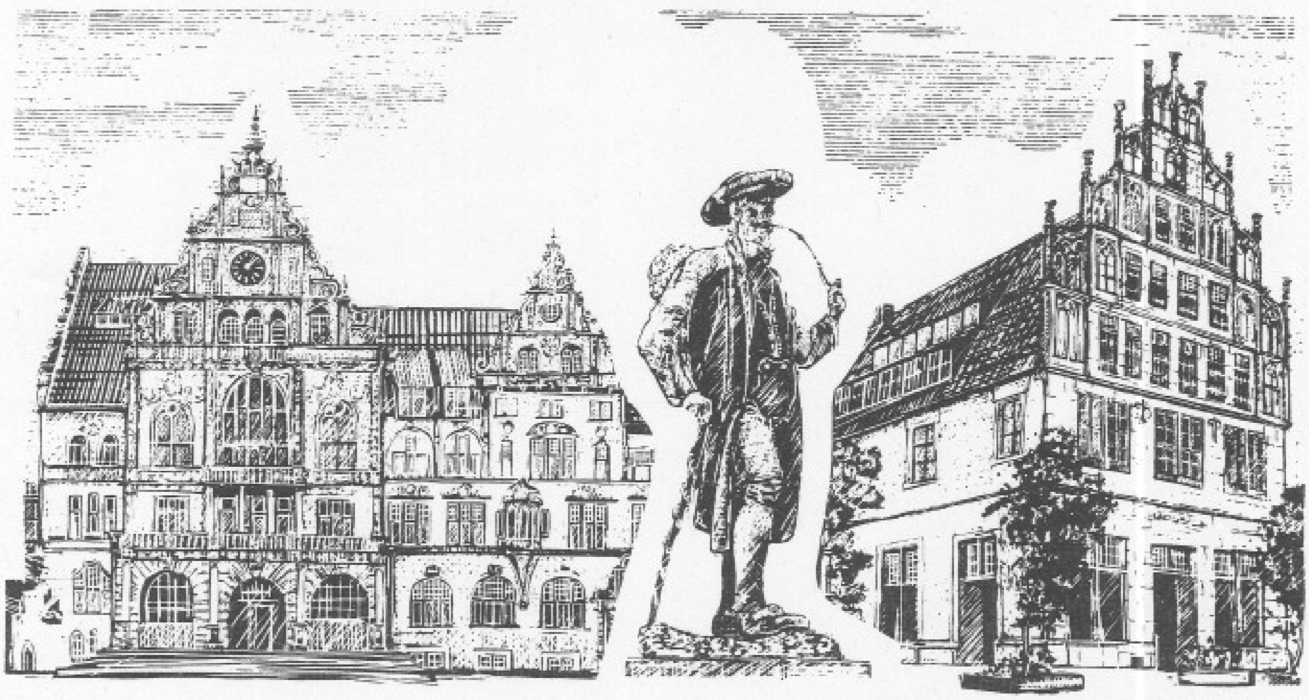

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P r e f a c e

A game without side payments essentially is given by a set valued mapping which assignes a system of utility vectors to each coaliton of players. Many authors have defined the concept of a value for non side payment games; let us mentione Harsanyi [1,2], Miyasawa [3], Nash [4], Owen [5], Selten [8], Shapley [9, 10]. While Nash's concept of a value is restricted to a class of "Unanimous Games",most other authors define their value concept for games where proper subcoalitions may achieve something.

Frequently it is necessary to apply a fixed point theorem in order to ensure the existence of a value (cf. [2], [10]) . Such a theorem in general yields serveral solutions implying that the value is a correspondence and not a function. By Nash's original paper we know that a value should have invariance properties under the application of a permutation of the players as well as of an affine transformation of utility. Now, if the value is a correspondence, it will usually be invariant as such, however, does this mean, that an invariant selection of this correspondence does exist?

Even if the answer is affirmative (as it happens for instance with Shapley's value, cf. [10], [7]), it may turn out that some of the players trivially rescale their utility to be identically 0 without any obvious reason. Moreover, the application of a fixed point theorem is sometimes rather hard to explain if a bargaining procedure is thought of to be underlying the definition of the value.

In this paper, we shall attempt to define a value which is somewhat more constructive (no fixed point theorem being used), which is a function, and which in addtion enjoys all necessary invariance properties. Intuitivly, some thoughts will rest upon ideas of Harsanyi in the sense that coalitions sucessively are looking to the power of theirproper subcoalitions, adding up all these quantities, and,using the result as a starting point, compute their own power. However, there will be no global rate of transfer of utility - instead each coalition obtains its own transfer rate by computing an extended version of the Nash value.

Miyasawa's value should also be mentioned as to be similar to the concept presented here. But,as is readily seen, the value obtained in [3] is a correspondence the values of which may sometimes cover large parts of the Pareto surface.

Finally, the present value depends only on the "characteristic function " version of a game, i.e., a set valued mapping defined on coalitions. Thus no normal form (strategy sets and payoff-functions) will enter the discussion.

Notations and definitions are being taken from 7 . However, section 0 contains a short introduction.

SECTION 0 Notations

Let $\Omega = \{1,\dots,n\} (n \geq 2)$ denote the „set of players", $\underline{\underline{P}} = \mathfrak{P}(\Omega)$ (power set of Ω) the system of „coalitions". The mapping $\mathrm{Proj}_S : \mathbb{R}^n \to \mathbb{R}^n$ is defined via

$$\mathrm{Proj}_S(x)_i = \begin{cases} x_i & i \in S \\ 0 & i \notin S \end{cases}$$

we shall write $x_S = \mathrm{Proj}_S(x)$ $(x \in \mathbb{R}^n)$ and $A_S = \mathrm{Proj}_S(A)$ $(A \subseteq \mathbb{R}^n)$. Consider a mapping $V : \underline{\underline{P}} \to \mathfrak{P}(\mathbb{R}^n)$ satisfying the conditions

(1) $V(S) \subseteq \mathbb{R}^n_S$ $(S \in \underline{\underline{P}})$

(2) $V(S) \neq \emptyset$, closed, convex $(S \in \underline{\underline{P}})$

(3) $V(S) \dot{-} \mathbb{R}^n_S \subseteq V(S)$ (algebraic difference) $(S \in \underline{\underline{P}})$

(4) $V_i := V(\{i\})$ bounded from above

We may define $\underline{x}(V) \in \mathbb{R}^n$ by

$\underline{x}_i(V) = \max\{t \mid t e^i \in V_i\}$

(the „threat point" of V)

Then the mapping V may in addition satisfy the conditions

(5) $\underline{x}_S(V) \in V(S)$ $(S \in \underline{\underline{P}})$

(6) $V_{\underline{x}}(S) = \{x \in V(S) \mid x \geq \underline{x}_S(V)\}$ bounded from above

Definition 0.1. Let $\mathbb{V} := \{V : \underline{\underline{P}} \to \mathcal{P}(\mathbb{R}^n) \mid V \text{ satisfies } (1) - (6)\}$. If $V \in \mathbb{V}$, then $(\Omega, \underline{\underline{P}}, V)$ is said to be a game (without side payments). $\underline{x}(V)$ is said to be the threat point (of V) and

$I_S(V) := \{x \in V(S) \mid x \geq \underline{x}_S(V),\ x \text{ Pareto optimal in } V(S)\}$

is the set of S-imputations (of V) (subscript S will be omitted if $S = \Omega$).

Consider also the system

$$\mathbb{V} := \{v : \underline{\underline{P}} \to \mathbb{R} \mid v(\emptyset) = 0, \sum_{i \in S} v(\{i\}) \leq v(S) (S \in \underline{\underline{P}})\}$$

(the „weakly superadditive set functions"). Given $v \in \mathbb{V}$

$$V^v(S) := \{x \in \mathbb{R}^n_S \mid \sum_{i \in S} x_i \leq v(S)\}$$

defines $V^v \in \mathbb{V}$. Given $V \in \mathbb{V}$,

$$v^V(S) := \max \{\sum_{i \in S} x_i \mid x \in V_{\underline{x}}(S)\}$$

defines $v^V \in \mathbb{V}$. If $v \in \mathbb{V}$, then $(\Omega, \underline{\underline{P}}, v)$ is a side payment game while $(\Omega, \underline{\underline{P}}, V^v)$ is a game with transferable utility („without side payments").

Finally, let $\mathbb{A} := \{m \in \mathbb{V} \mid m \text{ additive}\}$. Writing $v_i := v(\{i\})$ $(v \in \mathbb{V}, i \in \Omega)$ we may identify $m \in \mathbb{A}$ with the vector $(m_1, \ldots, m_n) \in \mathbb{R}^n$; on the other hand we shall always write $x(S) := \sum_{i \in S} x_i$ $(S \in \underline{\underline{P}})$ for $x \in \mathbb{R}^n$;

this kind of notation will be used freely, so that vectors of $\mathbb{R}^n$ and additive set functions $m \in \mathbb{A}$ will always be used synonymously.

SECTION 1 Transformations and values

A mapping $L = L^{\alpha}_{\beta} : \mathbb{R}^n \to \mathbb{R}^n$, $L(x)_i = \alpha_i x_i + \beta_i$ $(i=1,\dots,n)$ $(\alpha = \alpha_1,\dots,\alpha_n) > 0$, $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{R}^n)$ is an affine transformation of utility (a.t.u.) (linear if $\beta = 0$, $L = L^{\alpha}$). Of course,

$$LV(S) := L_S\,(V(S)),\; L_S := \mathrm{Proj}_S \circ L$$

defines a mapping $LV \in \mathbb{V}$ if $V \in \mathbb{V}$, i.e., we have $L\mathbb{V} = \mathbb{V}$.

Similarly, if $\pi : \Omega \to \Omega$ is a permutation of Ω, than $\pi : \mathbb{R}^n \to \mathbb{R}^n$, $\pi(x)_i = x_{\pi^{-1}(i)}$, induces also $\pi V \in \mathbb{V}$ for any $V \in \mathbb{V}$ via

$$\pi V(S) = \pi\,(V\,(\pi^{-1}(S)))$$

i.e. we have

$$\pi\mathbb{V} = \mathbb{V}.$$

Note that $\underline{x}\,(LV) = L(\underline{x}(V))$, $\underline{x}(\pi V) = \pi(\underline{x}(V))$, as is easily checked. Let $\mathbb{L}$ ($\mathbb{L}_0$) denote the system of a.t.u.'s (l.t.u.'s) and Π the permutations.

Definition 1.1. 1. $\mathbb{V}^0 \subseteq \mathbb{V}$ is invariant if $\pi\mathbb{V}^0 = \mathbb{V}^0$, $L\mathbb{V}^0 = \mathbb{V}^0$ $(\pi\in\Pi,\ L\in\mathbb{L})$.

2. Let $\mathbb{V}^0$ be invariant. A mapping $\varphi : \mathbb{V}^0 \to \mathbb{R}^n$ is

(1) feasible if $\varphi(V) \in V(\Omega)$ $\qquad (V \in \mathbb{V}^0)$

(2) Pareto optimal if $\varphi(V)$ is P.O. in $V(\Omega)$ $(V \in \mathbb{V}^0)$

(3) individually rational (i.r.) if $\varphi(V) \geq \underline{x}(V) (V \in \mathbb{V}^0)$,

(4) $\mathbb{L}$-invariant if $\varphi(LV) = L(\varphi(V))$ $(V \in \mathbb{V}^0,\ L \in \mathbb{L})$,

i.e., $\varphi \circ L = L \circ \varphi$, φ commutes with all a.t.u.'s ,

(5) Π-invariant if $\varphi(\pi V) = \pi(\varphi(\pi V))$ $(V \in \mathbb{V}^0,\ \pi \in \Pi)$,

i.e., $\varphi \circ \pi = \pi \circ \varphi$, φ commutes with all permutations ,

a value (for $\mathbb{V}^0$) if (1) - (5) are satisfied.

Note that we are not prepared to accept correspondences as „values". It is one of our aims to show that a single point value causes quite specific problems in its definition compared to correspondences.

Example 1.2

Let $\mathbb{V}^c := \{V \in \mathbb{V} \mid \exists\, a \in \mathbb{R}^{n+},\ a > o,\ v \in \mathbb{W}:$

$$V(S) = \{x \in \mathbb{R}^n_S \mid ax \leq v(S)\}\}$$

Note that any a.t.u. $L = L^{\alpha}_{\beta}$ and any $v \in \mathbb{W}$ yield

$$L^{-1} V^v(S) = \{x \in \mathbb{R}^n_S \mid \sum_{i \in S} \alpha_i x_i \leq v(S) - \beta(S)\}$$

(using $\beta(S) = \sum_{i \in S} \beta_i$, see SEC. 0). Hence

$$\mathbb{V}^c = \{LV^v \mid v \in \mathbb{V},\ L \in \mathbb{L}\}$$

(because $v - \beta \in \mathbb{V}$, $\beta \in \mathbb{A}$). Obviously,

if $V \in \mathbb{V}^c$, then $(\Omega, \underline{\underline{P}}, V)$ is a game with constant rate of utility transfer (this rate being represented by the normal a (or α) of the hyperplanes bounding $V(S)$).

Suppose that for some $v, w \in \mathbb{V}$, $L \in \mathbb{L}$ we have

(6) $$LV^v = V^w$$

Then it is easy to show that $L = L^{\alpha}_{\beta}$ satisfies $\alpha_1 = \alpha_2 = \dots = \alpha_n$ while

(7) $$w = \alpha_1 v + \beta$$

holds true. („Strategic equivalence"). Let $\phi : \mathbb{V} \to \mathbb{R}^n$ denote the „Shapley value" (Shapley [9]). We claim,

(8) $$\text{if } LV^v = L'V^w \text{ then } L(\phi(v)) = L'(\phi(w))$$
$$(v, w \in \mathbb{V};\ L, L' \in \mathbb{L})$$

Indeed in this case we have

$$L'^{-1} \circ L\, V^v = V^w$$

thus, by (6) and (7)

(9) $$w = \tilde{\alpha}_1 v + \tilde{\beta},\ \ \tilde{L} := L^{\tilde{\alpha}}_{\tilde{\beta}} := L'^{-1} \circ L,$$
$$\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_n).$$

Because ϕ is additive

$$\phi(w) = \tilde{\alpha}_1 \phi(v) + \tilde{\beta} = \tilde{L}(\phi(v))$$
$$= L'^{-1} \circ L(\phi(v))$$

i.e.

$$L'(\phi(w)) = L(\phi(v)).$$

proving (8).

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Now, let us define for $V \in \mathbb{V}^c$, $V = LV^v$

(1o) $$\Psi(V) := \Psi(LV^v) : = L(\phi(v))$$

Clearly, (1o) is noncontradictory in view of (8) that is, when computing $\Psi(V) = L(\phi(v))$, we may take an arbitrary representation $V = LV^v$ of $V \in \mathbb{V}^c$.

As is easily seen, Ψ satisfies (1) and (2) (i.e., Ψ is feasible and P.O.). Property (3) (i.r.) is in general only ensured, if v is monotone; thus $\Psi(V)$ is an imputation on a proper subclass of $\mathbb{V}^c$. On thus subclass, however, it is a value, more generally, Ψ is $\mathbb{L}$- and Π- invariant on a $\mathbb{V}^c$. We proceed as follows:

Ψ is $\mathbb{L}$-invariant: Indeed, for $V = LV^v$ and $\hat{L} \in \mathbb{L}$:

$$\Psi(\hat{L}V) = \Psi(\hat{L}LV^v) = \Psi(\hat{L}oLV^v)$$
$$= \hat{L}oL(\phi(v)) = \hat{L}(L\phi(v)) = \hat{L}(\Psi(LV^v))$$
$$= \hat{L}(\Psi(V))$$

(where the third equation follows from our above remark)

Ψ is Π-invariant: let $\pi \in \Pi$, $L = L^{\alpha}_{\beta} \in \mathbb{L}$.
Define $L^{\pi} \in \mathbb{L}$ by $L^{\pi(\alpha)}_{\pi(\beta)}$ and verify that

(11) $$\pi o L = L^{\pi} o \pi$$

-/-

Now, for $V = LV^v \in \mathbb{V}^c$:

$$\begin{aligned}\Psi(\pi V) &= \Psi(\pi L V^v) = \Psi(L^\pi \pi V^v) \\ &= \Psi(L^\pi V^{\pi v}) = L^\pi(\phi(\pi v)) \\ &= L^\pi(\pi(\phi(v)) = L^\pi o \pi(\phi(v)) \\ &= \pi o L(\phi(v)) = \pi(\Psi(LV^v)) \\ &= \pi(\Psi(V))\end{aligned}$$

Here, of course, $\pi v \in \mathbb{V}$ is defined via $\pi v(S) = v(\pi^{-1}(S))$ and the 5^{th} equation essentially uses the invariance properties of ϕ. Moreover, the 3^{th} equation ($\pi V^v = V^{\pi v}$) must be verified.

This example has been straightforward; we have performed our pedantical computations in view of a further example which is not quite straight forward (SEC 2).

The next example however is also well known (but we shall again make some fuss in order to have a rigorous definition).

<u>Example 1.3</u>

Let $T \in \underline{\underline{P}}$, $V \in \mathbb{V}$ is <u>T-unanimous</u> if

(12) $$V(S) = \begin{cases} \sum_{i\in S}^{\oplus} V_i & S \not\supseteq T \\ V(T) \oplus \sum_{i \in T-S}^{\oplus} V_i & S \supseteq T \end{cases}$$

($\oplus$: direct sum); call $S \in P$ <u>flat</u> if $V(S) = \sum_{i\in S}^{\oplus} V_i$ and call V <u>flat</u> if all S are flat.

The following is not hard to see: if $V \in \mathbb{V}$ is T- and S- unanimous ($S \neq T$), then V is flat. It follows that T is either uniquely determined or V is T- unanimous for all $T \in \underline{\underline{P}}$ and flat.

Given a closed convex set $A \subseteq \mathbb{R}^n_T$, that is T-comprehensive ($A - \mathbb{R}^{n+}_T \subseteq A$) and a vector $\hat{x} \in \mathbb{R}^n$, such that

$$A_{\hat{x}_T} := \{x \in A \mid x \geq \hat{x}_T\}$$

is bounded from above, define $V = E_{T,A,\hat{x}}$ by

$$\text{(13)} \qquad V(S) = \begin{cases} \hat{x}_S - \mathbb{R}^{n+}_S & S \not\supseteq T \\ A \oplus (\hat{x}_{T-S} - \mathbb{R}^{n+}_{T-S}) & S \supseteq T \end{cases}$$

Provided $\hat{x}_T \in A$, we have indeed $V \in \mathbb{V}$ and it is not hard to see that

$$V = E_{T,V(T),\, \underline{x}(V)}$$

for any T- unanimous $V \in \mathbb{V}$. Let $\mathbb{V}^1 \subseteq \mathbb{V}$ denote the T- unanimous (for some T) functions. $\mathbb{V}^1$ is invariant and it turns out that (see [7])

$$\text{(14)} \qquad \pi E_{T,A,\hat{x}} = E_{\pi(T),\, \pi(A),\, \pi(\hat{x})} \qquad (\pi \in \Pi)$$

$$\text{(15)} \qquad L E_{T,A,\hat{x}} = E_{T,\, L_T(A),\, L(\hat{x})} \qquad (L \in \mathbb{L})$$

Given $V = E_{T,A,\hat{x}}$ let

$T_o = T_o(V) = \{i \mid \exists\, x \in A,\ x_i > \hat{x}_i\}$

and define $g^V : \mathbb{R}^n \to \mathbb{R}$ by

$$g^V(x) = \prod_{i \in T_o} (x_i - \underline{x}_i(V))$$

g^V has a unique maximizer $\nu(V)$ relative to

$V_{\underline{x}}(\Omega)$; ν: $\mathbb{V}^1 \to \mathbb{R}^n$ is <u>a value for $\mathbb{V}^1$</u>

and called the <u>Nash - value</u> (NASH [4]).

Of course, $T_o \subseteq T$ and $\nu_i(V) = \underline{x}_i(V)$ $(i \notin T_o)$.

<u>Remark 1.4</u>

It will be necessary to have ν also defined if $V(S)$ does not necessarily contain the "threat point" $\underline{x}_S$. Now, if we omit our condition (5) of SEC 0,we may still consider V as defined by (13) without asking for "$\hat{x}_T \in A$". All remarks of Example 1.3 apply, so $V = E_{T,A,\hat{x}}$ is well defined and T is well defined unless V is flat. However, the Nash value is <u>not</u> well defined because there is in general no unique maximizer of g_V ond $V(\Omega)$. If $\underline{x}(V) \notin V(\Omega)$ one might of course take <u>any</u> maximizer of g_V, say, on the Pareto set of $V(\Omega)$. But , this defines just a correspondence. This correspondence is e.g. symmetric or permutation invariant - but that does not imply, that we may find a permutation invariant selection (a point valued function of V). This problem is delt with in the following section.

SECTION 2 Extending the Nash-value

Lemma 2.1 Let A be a closed set and let L be an a.t.u. having a fixed point $\bar{t} \in \mathbb{R}^n$.

Assume

1. $\bar{t} \notin A$
2. $\{s \mid \bar{t} + se^i \in A\}$ is non-empty and bounded from above.

If $L(A) = A$ then $L = id.$ (i.e. $\alpha = (1,\dots,1)$; $\beta = 0$)

Proof The fixed point $\bar{t}$ has to satisfy

(1) $\alpha_i \bar{t}_i + \beta_i = \bar{t}_i$.

Supose for some $i_0 \in \Omega$ we have $\alpha_{i_0} \neq 1$.

Then, by (1)

(2) $$\bar{t}_{i_0} = \frac{\beta_{i_0}}{1-\alpha_{i_0}} .$$

Now, let

$$\bar{s} := \sup \{s \mid \bar{t} + s\, e^{i_0} \in A\} < \infty$$

such that $\bar{t} + \bar{s}\, e^{i_0} \in A$.

-/-

Because $L(A) = A = L^{-1}(A)$, we have also

$$\bar{s} = \sup\{s \mid \bar{t} + se^{i_0} \in L^{-1}(A)\}$$

$$= \sup\{s \mid L(\bar{t} + se^{i_0}) \in A\}$$

$$= \sup\{s \mid \bar{t} + s\alpha_{i_0} e^{i_0} \in A\}$$

$$= \frac{\bar{s}}{\alpha_{i_0}}$$

Thus $\bar{s} = o$ and $\bar{t} \in A$, a contradiction. Hence $\alpha_i = 1$ $(i \in \Omega)$ and $\beta_i = 0$ $(i \in \Omega)$ follows from (1); this means that $L = id$.

In what follows, we shall consider mappings of the type $V = E_{\Omega,A,\hat{x}}$ as defined by (13) SEC 1. Here $A \subsetneq \mathbb{R}^n$ is assumed to be closed, comprehensive, convex. Moreover, we want the following conditions to be satisfied

(3) $\hat{x} \notin A$

(4) $\{x \notin A \mid x \leq \hat{x}\}$ is bounded.

Note, that it follows from (3) and (4) that

(5) For $i \in \Omega$ there is $s \in \mathbb{R}$ such that $\hat{x} - se^i \in A$

(6) There is $x \notin A$ such that $x < \hat{x}$

holds true since A^c is open.

The set of mappings $V = E_{\Omega,A,\hat{x}}$, as defined by (3), (4) and (13), SEC 1, is tentatively called $\mathbb{V}^2_\Omega$. Clearly $\mathbb{V}^2_\Omega$ is invariant.

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Corollary 2.2. Let $V \in \mathbb{V}^2_\Omega$. If $LV = V$ then $L = id$.

Indeed, as $\underline{x}(V) = \hat{x}$ and $L(\hat{x}) = \hat{x}$, we have a fixed point of L. The conditions of Lemma 2.1 are obviously satisfied for $A = V(\Omega)$.

Now, it is our aim to define a solution concept like the one of NASH for games of the type $E_{T,A,\hat{x}}$, where $\hat{x} \notin A$. Intuitively,we may imagine that coalition T finds itself in a position where everybody has specified certain ideas about the magnitude of his utility coordinate. This specification may be justified by bargaining in proper subcoalitions. However, the utility vector defined by everybody's claims is not available to coalition T. Hence, the players are forced to find a fall-back position within the utility vectors coalition T commands. A "fair" fall-back position, within good tradition of the NASH value would be a vector which gives equal utility to all players in T provided utility is transfered at a rate, which is intrinsincally defined by the solution vector and the initial starting point or "bliss" point. In view of NASH's result, this is equivalent to maximizing the function g_V on the Pareto surface of A. Another version is that we are to find a normal at the Pareto surface of A in some point x the coordinates of which are inverse to $\hat{x}_i - x_i$.

The main problem arriving is now that maximizing the function g_V in general is not a unique procedure, there may be more than one maximizer and again we have a selection problem at hand.

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Theorem 2.3 Let $V \in \mathbb{V}^2_\Omega$. Then there is $\bar{x} \in A = V(\Omega)$ and $\bar{h} \in R^{n+}$ such that

1. $\bar{x}$ P.o., $\bar{x} < \hat{x} = \underline{x}\ (V)$.

2. $\bar{h}x \leq \bar{h}\bar{x}$ $(x \in A)$ (i.e., $\bar{h}$ is normal to a tangency hyperplane at A in $\bar{x}$).

3. $\bar{h}_i = \dfrac{-1}{\bar{x}_i - \hat{x}_i}$ $\qquad (i \in \Omega)$.

4. Whenever for some $L \in L$, $\pi \in \Pi$

$$L\pi V = V ,$$

then it follows that

$$L o \pi\ (\bar{x}) = \bar{x}$$

Proof 1.st Step

Let $G: = \{l = (L,\pi) \mid L \in \mathbb{L},\ \pi \in \Pi,\ L\pi V = V\}$

For $l = (L,\pi)$, $l' = (L',\pi')$ define

$$lol' = (L,\pi) \circ (L',\pi'): = (LoL'^{\pi},\ \pi o \pi')$$

Here $L^{\pi} = L^{\pi(\alpha)}_{\pi(\beta)}$ for $L = L^{\alpha}_{\beta}$ as in SEC 1, Example 1.2

If we write $l(x): = (L, \pi)\ (x): = L o \pi\ (x)$,

then (11), SEC 1, indicates that o can always be interpreted as composition of mappings. Clearly, (G, o) is a group.

-/-

2nd Step (G,o) is finite:

Indeed, Π is finite and if (G,o) is not,

then

$$L\pi V = L'\pi V$$

for some L, L' $\in \mathbb{L}$, $\pi \in \Pi$. But then $L'^{-1}L\pi V = \pi V$ and $L' = L$ follows from Corollary 2.2. since $\pi V \in V^2_\Omega$.

3rd Step The set

$$B: = \{x | x \leq \hat{x},\ x \notin A\}$$

is bounded. The set

$$H^0: = \{x \in \mathbb{R}^n \mid l(x) = x\ (l \in G)\}$$

is an affine subspace of $\mathbb{R}^n$. Clearly, $\hat{x} \in B \cap H^0$ but there are also vetors x^0 such that

(8) $$x^0 < \hat{x},\ x^0 \in B \cap H^0$$

Indeed, we find $x^* \notin A$, $x^* < \hat{x}$ by the condition $V \in \mathbb{V}^2_\Omega$ (cf (6)). Clearly

(9) $$x^0 = \frac{1}{|G|} \sum_{l \in G} l(x^*) \in H^0 ,$$

every l being an affine mapping. Since $l(x^*) < l(\hat{x}) = \hat{x}$, we have also $x^0 < \hat{x}$, and if x^* is sufficiently close to $\hat{x}$, then so is x^0, implying $x^0 \notin A$ (Since A is closed).

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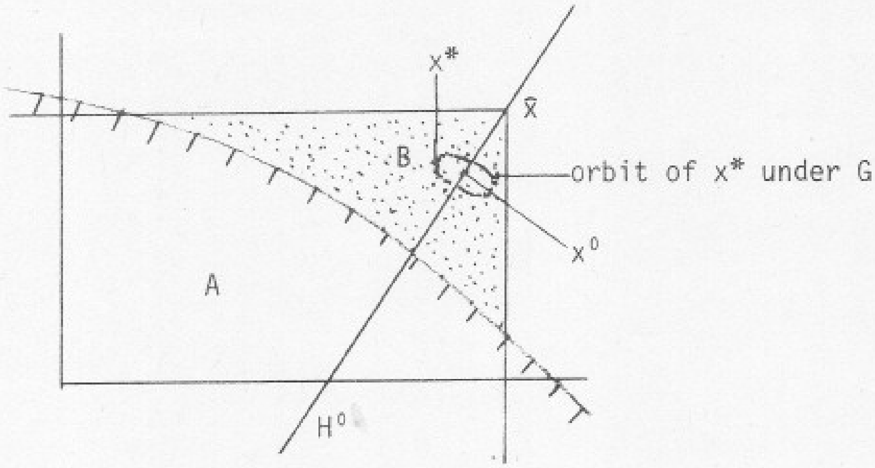

4th Step   Define $g^V : \mathbb{R}^n \to \mathbb{R}$ by

$$g^V(x) = \prod_{i \in \Omega} (\hat{x}_i - x_i)$$

g^V takes a maximizing value $\bar{x}$ on the compact set $\bar{B} \cap H^0$ and since there are vectors x^0 satisfying (8), we know that $\bar{x} < \hat{x}$. In fact, since $B \cap H^0$ is (relatively) open, it is seen, that $\bar{x} \in A$, more specifically, $\bar{x}$ is a boundary point of $A \cap H^0$.

Now, as $A \cap H^0$ is convex and closed and $\bar{x}$ is a boundary point, there is $\hat{h} \in \mathbb{R}^n$ such that

(1o) $$x\hat{h} \leq \bar{x}\hat{h} \qquad x \in A \cap H^0$$

i.e., $\hat{h}$ is normal to a suporting hyperplane at $A \cap H^0$ in $\bar{x}$. The affine subspace H^0 is generated by some linear subspace

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M^0 via $H^0 = n + M^0$ (n a normal to H^0) and we may assume $\hat{h} \in M^0$.

If $G^0 := \{x \in H^0 \mid g^V(x) \geq g^V(\bar{x})\}$, then, for some spherical neighborhood U of $\bar{x}$,

$$G^0 \cap U \subsetneq A \cap H^0 \cap U$$

and hence

$$x\hat{h} \leq \bar{x}\hat{h} \quad (x \in G^0 \cap U)$$

from (1o). This means that $\hat{h}$ is normal at G^0 in $\bar{x}$ (al relative to H^0). However, $G^0 \subsetneq H^0$ has a unique ("in M^0") normal in $\bar{x}$, namely

$$h^0 := \mathrm{Proj}_{M^0} \frac{\partial g^V}{\partial x} (\bar{x})$$

up to a constant, hence $\hat{h} = \text{const.} \ \mathrm{Proj}_{M^0} \frac{\partial g^V}{\partial x} (\bar{x})$

and

$$\text{const.} \ xh^0 \leq \text{const.} \ \bar{x}h^0 \qquad (x \in H^0 \cap A \cap U)$$

Define

(11) $$\bar{h} := \text{const.} \frac{\partial g^V}{\partial x} (\bar{x})$$

then, if $x \in H^0 \cap A \cap U$, then $x - \bar{x} \in M^0$

and

$$\text{const} \ (x-\bar{x})h^0 = (x-\bar{x}) \ \mathrm{Proj}_{M^0}\bar{h} = (x-\bar{x}) \ \bar{h}$$

that is

(12) $$x\bar{h} \leq \bar{x}\bar{h} \qquad (x \in H^0 \cap A \cap U)$$

<u>5^{th} Step</u> Let W be a spherical neighborhood of $\bar{x}$ such that $l(x) \in U$ ($x \in W$, $l \in G$); this is possible since G is finite and $l \in G$ continuous.

For any $x \in W$ let

$$\tilde{x} := \frac{1}{|G|} \sum_{l \in G} l(x)$$

Then $\tilde{x} \in U$. If $x \in A$, then $\tilde{x} \in A$ (A being convex), moreover $\tilde{x} \in H^0$ is easily checked. Hence

$$\tilde{x} \in H^0 \cap A \cap U \qquad (x \in A \cap W)$$

By (12)

$$\tilde{x}\bar{h} \leq \bar{x}\bar{h} \qquad (x \in A \cap W) \tag{13}$$

<u>6^{th} Step</u> By (11)

$$\bar{h}_i = \frac{\text{const.}}{\bar{x}_i - \hat{x}_i}$$

and since we find $x \in H^0 \cap A \cap U$, $x < \hat{x}$, the constant must be negative and can be normalized to -1.

Now, by $\bar{x}$, $\hat{x} \in H^0$ we have

$$l(\bar{x}) = \bar{x},\ l(\hat{x}) = \hat{x} \qquad (l \in G)$$

and

$$\bar{x}_i = l\ (\bar{x})_i = \alpha_i \bar{x}_{\pi^{-1}(i)} + \beta_i$$

$$\hat{x}_i = l\ (\hat{x})_i = \alpha_i \hat{x}_{\pi^{-1}(i)} + \beta_i$$

whenever

$$l = (L,\pi),\ L = L_\beta^\alpha\ .\ \text{Thus}$$

$$\bar{h}_i = \frac{-1}{\bar{x}_i - \hat{x}_i} = \frac{-1}{\alpha_i(\bar{x}_{\pi^{-1}(i)} - \hat{x}_{\pi^{-1}(i)})}$$

It follows that for $x \in A \cap W$

$$\bar{h}(l(x) - \bar{x}) = \bar{h}(l(x) - l(\bar{x}))$$

$$= \sum_{i\in\Omega} \bar{h}_i\ \alpha_i(x_{\pi^{-1}(i)} - \bar{x}_{\pi^{-1}(i)})$$

$$= \sum_{i\in\Omega} \frac{-1}{\bar{x}_i - \hat{x}_i}\ \alpha_i(x_{\pi^{-1}(i)} - \bar{x}_{\pi^{-1}(i)})$$

$$= \sum_{i\in\Omega} \frac{-1}{\alpha_i(\bar{x}_{\pi^{-1}(i)} - \hat{x}_{\pi^{-1}(i)})}\ \alpha_i\ (x_{\pi^{-1}(i)} - \bar{x}_{\pi^{-1}(i)})$$

$$= \sum_{i\in\Omega} \frac{-1}{\bar{x}_i - \hat{x}_i}\ (x_i - \bar{x}_i) = \bar{h}\ (x - \bar{x})$$

for all $l \in G$. Consequently

$$\bar{h}(\tilde{x} - \bar{x}) = \bar{h}\ \frac{1}{|G|} \sum_{l\in G}(l(x) - l(\bar{x}))$$

$$(14) \qquad = \frac{1}{|G|} \sum_{l\in G} \bar{h}(l(x) - l(\bar{x}))$$

$$= \frac{1}{|G|} \sum_{l\in G} h(x-\bar{x}) = h(x-\bar{x})$$

-/-

Hence, whenever $x \in A \cap W$, then

$$\bar{h}\,(x - \bar{x}) = \bar{h}\,(\tilde{x} - \bar{x}) \leq 0$$

using (13) and (14). However, this means now that $\bar{h}$ defines a suporting hyperplane at A in $\bar{x}$; as A is convex, we may forget about the neighborhood W of $\bar{x}$ thus having

$$\bar{h}x < \bar{h}\bar{x} \qquad (x \in A)$$

Clearly,2.,3.,4.of our theorem are satisfied, it remains only to establish "$\bar{x}$P.O.".
However,as $\bar{h} > 0$ any $\bar{\bar{x}} \in A$, $\bar{\bar{x}} \gneq \bar{x}$ would yield $\bar{h}\bar{\bar{x}} > \bar{h}\bar{x}$, q.e.d.

Of course, the definition of a "value" (cf 1.1) has to be changed for $\mathbb{V}^2_\Omega$. Suitably, the term "i.r." should now mean "$x \leq \hat{x} = \underline{x}(V)$". Bearing this in mind, we have

<u>Theorem 2.4</u>

There exists a value ν on $\mathbb{V}^2_\Omega$ and a mapping $\bar{h}(\cdot) : \mathbb{V}^2_\Omega \to \mathbb{R}^n$ such that for every $V \in \mathbb{V}^2_\Omega$ the following holds true

1. $\nu(V) < \underline{x}(V)$
2. $\bar{h}_i(V) = \dfrac{-1}{\nu_i(V) - \underline{x}_i(V)}$
3. $\bar{h}(V)x \leq \bar{h}(V)\nu(V) \qquad (x \in V(\Omega))$
   (i.e., $\bar{h}(V)$ is normal at $A = V(\Omega)$ in $\nu(V)$)

4. $\bar{h}(\pi V) = \pi(\bar{h}(V))$ $(\pi \in \Pi)$

5. $\bar{h}(L^{\alpha}_{\beta}V) = (L^{\alpha})^{-1}\bar{h}(V)$ $(L = L^{\alpha}_{\beta} \in \mathbb{L})$

Proof 1st Step

Pick $\hat{V} \in \mathbb{V}^2_{\Omega}$ and define

$$\hat{\mathbb{V}} := \{L\pi\hat{V} \mid L \in \mathbb{L}, \pi \in \Pi\} \subseteq \mathbb{V}^2_{\Omega}$$

Clearly, $\hat{\mathbb{V}}$ is invariant. Define

$$\nu(\hat{V}) := \bar{x}, \; \bar{h}(\hat{V}) := \bar{h}$$

by means of Theorem 2.3. Our first aim is to extend the definition of ν to all of $\hat{\mathbb{V}}$. To this end, consider $V^0 \in \hat{\mathbb{V}}$ and assume that it satisfies

(15) $$L'\pi'\hat{V} = V^0 = L\pi\hat{V}$$

(i.e. the representation is not unique). Then

$$\hat{V} = \pi^{-1}L^{-1}L'\pi'\hat{V} = L^{-1\pi^{-1}}\pi^{-1}L'\pi'\hat{V}$$

i.e. $$\hat{V} = L^{-1\pi^{-1}}L'^{\pi^{-1}}\pi^{-1}\pi'\hat{V}$$

which, by Theorem 2.3. implies

$$L^{-1\pi^{-1}} \circ L'^{\pi^{-1}} \circ \pi^{-1} \circ \pi'(\bar{x}) = \bar{x}$$

or

(16) $$L' \circ \pi'(\bar{x}) = L \circ \pi(\bar{x})$$

Therefore, the definition

$$\nu(V^0) := \nu(L\pi V) := L \circ \pi(\bar{x}) = L \circ \pi(\nu(\hat{V}))$$

is noncontradictory in view of "(15) implies (16)". That is, the definition of ν on V^0 is independent on the particular L and π used to represent V^0.

-/-

From this, the invariance properties of ν follow at once. For, if $V^0 = L\pi\hat{V} \in \hat{\mathbf{V}}$ and $\tilde{L},\tilde{\pi}$ are arbitrary, then

$$\nu(\tilde{L}\tilde{\pi}V^0) = \nu(\tilde{L}\tilde{\pi}L\pi\hat{V})$$

$$\nu(\tilde{L}L^{\tilde{\pi}}\tilde{\pi}\pi\hat{V}) \overset{(17)}{=} \tilde{L}oL^{\tilde{\pi}}o\tilde{\pi}o\pi(\bar{x})$$

$$\tilde{L}o\tilde{\pi}(Lo\pi(\bar{x}) \overset{(17)}{=} \tilde{L}o\tilde{\pi}(\nu(V^0))$$

We have thus established that ν is a value on $\hat{\mathbf{V}}$.

2nd Step Given $V^0 = L\pi\hat{V} \in \hat{\mathbf{V}}$, define $\bar{h}(V^0)$ by

$$\bar{h}_i(V^0) = \frac{-1}{\nu_i(V^0) - \underline{x}_i(V^0)}$$

Then properties 4. and 5. of our theorem follow immediately from the invariance of ν on $\hat{\mathbf{V}}$. To check property 3., pick $y \in V^0(\Omega)$ and let $x \in \hat{V}(\Omega)$ be such that $y = Lo\pi(x)$.
A simple computation reveals that $(L = L^{\alpha}_{\beta})$

$$\bar{h}(V^0)\cdot(y - \nu(V^0)) = \bar{h}(V^0)\ (Lo\pi(x) - Lo\pi(\bar{x}))$$

$$= (L^{\alpha})^{-1}o\pi(\bar{h})\cdot(L^{\alpha}o\pi(x - \bar{x}))$$

$$= \pi(\bar{h})\cdot(\pi(x - \bar{x}) = \bar{h}(x - \bar{x}) < 0$$

(since $\bar{h}$ and $\bar{x}$ where chosen by Theorem 2.3.)

Hence $\bar{h}(V^0)$ has the desired normal property. We have thus established all claims of Theorem 2.4 as far as the subclass $\hat{\mathbf{V}}$ is concerned.

-/-

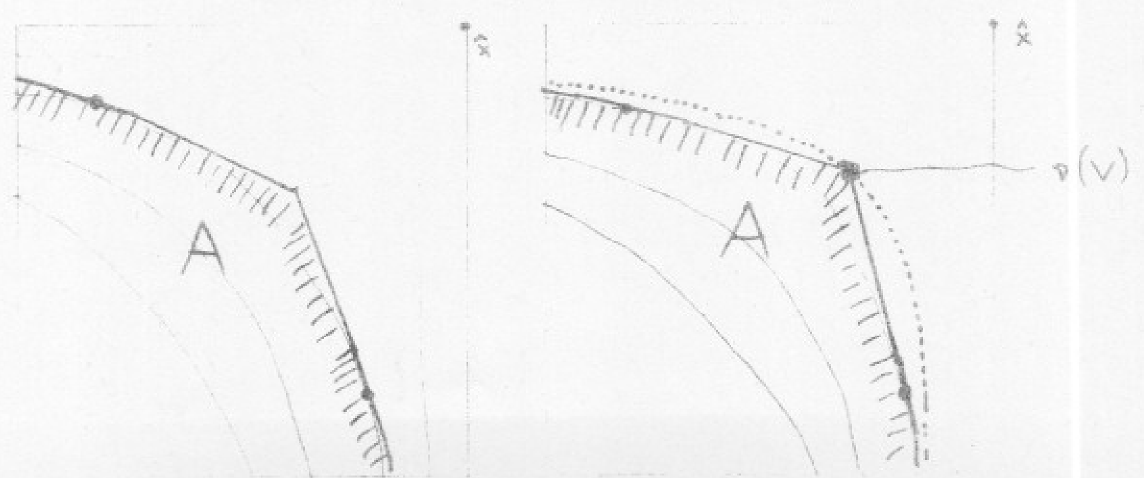
3rd Step — Now consider $\hat{W} \in \mathbb{V}^2_\Omega$ and let $\hat{\mathbb{W}} \subseteq \mathbb{V}^2_\Omega$ be defined accordingly. As is easily seen, either $\hat{\mathbb{W}} = \hat{\mathbb{V}}$ or $\hat{\mathbb{W}} \cap \hat{\mathbb{V}} = \emptyset$. Thus, ν and $h(\cdot)$ can be defined on all of $\mathbb{V}^2_\Omega$ by a straightforward procedure, q.e.d.

Remark 2.5 — In view of Remark 1.4 we may define funktions $V = E_{T,A,\hat{x}}$ whenever $A \subseteq \mathbb{R}^n_T$ is (relatively) closed, T-comprehensive, convex, and does not contain $\hat{x}_T$. In addition we shall require that

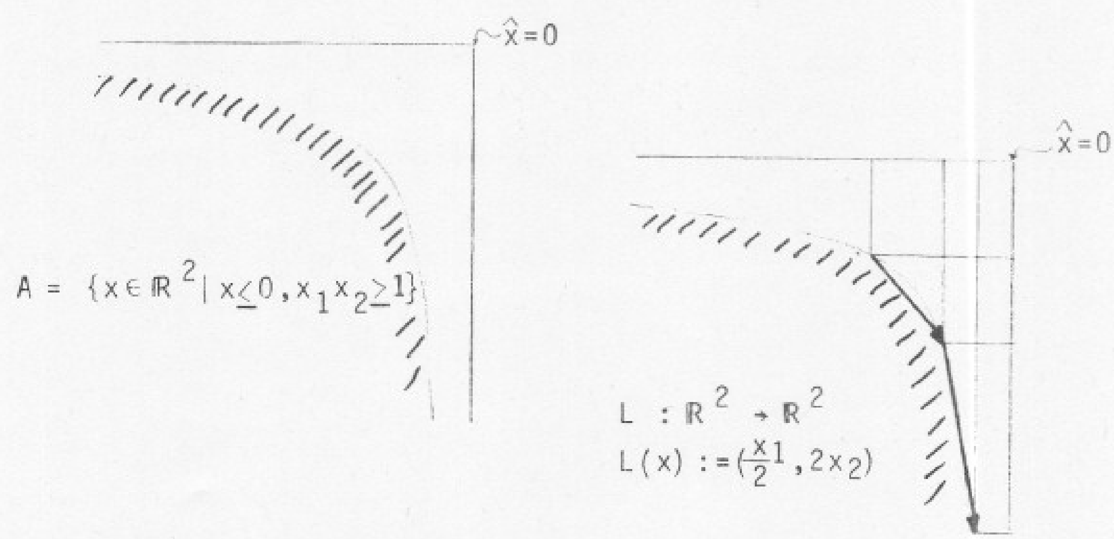
$$\{x \in \mathbb{R}^n_T \mid x \notin A, \; x \leq \hat{x}\}$$

is bounded. The set of all these functions $V = E_{T,A,\hat{x}}$ is denoted by $\mathbb{V}^2$. It is then a purely technical matter to define ν, a value on $\mathbb{V}^2$, with the properties of Theorem 2.4 modified appropriately.

Remark 2.6 — The following sketches illustrate that Theorem 2.4 is not quite trivial. The first drawing (n = 2) shows $\hat{x}$ and $A = V(\Omega)$ as well as constancy curves of g^V. Clearly, arbitrary selection of a maximizer of g^V in $\overline{\{x \mid x \notin A, \; x < \hat{x}\}}$ might yield an "asymmetric" value - despite the game being completely symmetric. However, there is a symmetric "local" maximizer, which is a (the only) candidate for $\nu(V)$.



The next drawing (n = 2) shows that (4) is a necessary condition to be imposed on the class $\mathbb{V}^2(\mathbb{V}^2_\Omega)$. Here, $\hat{x}$ and $A = V(\Omega)$ are both invariant under the linear mapping L (i.e. in particular, L(A) = A). But there is no invariant vector $x \in A$) (i.e. $L(x) \neq x$, $(x \in A)$), hence the choice of an invariant maximizer of g^V (to serve as $\nu(V)$) is impossible.



As Theorem 2.4 tells us, it is sufficient to avoid this latter pathology in order to have a symmetric selection.

SECTION 3 Tight games

The idea of a marginal value of a player within the frame-work of a cooperative game is already explicitely stated in Shapley's paper [9]. Given an increasing system of coalitions, every player receives the marginal value he is adding to a coalition the moment he enters the process (and this procedure is then randomized). We shall now specify a version of marginal value for non side-payment games: the moment a player enters a coalition the Nash value is being computed, using the value of the previous coalition as to be the threat point. However, because of the limited possibilities of utility transfer, "randomizing" this procedure makes no sense. Hence we can define a value by this procedure only for a very limited class of games.

Definition 3.1.1 $\underline{\underline{S}} = \{S_0, S_1, \ldots, S_n\} \subseteq \underline{\underline{P}}$ is said to be a tight system of coalitions if

(1) $\emptyset = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n = \Omega$

(2) $|S_k - S_k| = 1 \qquad (k = 1, \ldots, n)$

3.1.2 $V \in \mathbb{V}$ is said to be attached to a tight system $\underline{S}$, or just tight, if, for $S \in \underline{\underline{P}}$

$$V(S) = V(S_{\varkappa}) \oplus \overline{\sum_{i \in S - S_{\varkappa}}} {}^{\oplus} V_i$$

where

$$\varkappa = \max \{k \mid S_k \subseteq S\}$$

and if

$$V(S_k) \subseteq V(S_{k+1}) \quad (k=1, \ldots, n-1)$$

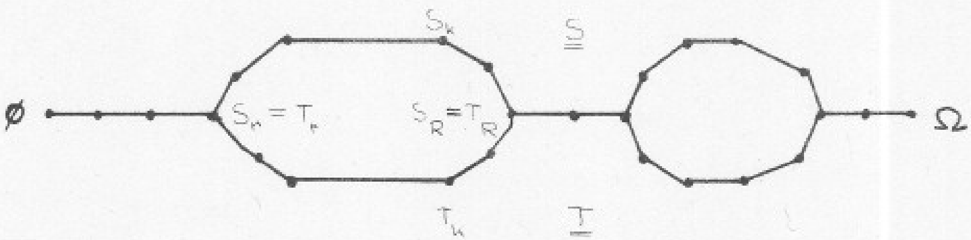
Let us use the term "S reduces to T" (w.r.t.V) if

$$V(S) = V(T) \oplus \sum_{i \in S-T}^{\oplus} V_i \qquad (T \subseteq S)$$

Now, suppose, that $V \in \mathbb{V}$ is attached to $\underline{\underline{S}}$ and $\underline{\underline{T}}$. If $S_k \neq T_k$ for some k, then pick

$$r = \max \{l < k \mid S_l = T_l\}$$

$$R = \min \{l > k \mid S_l = T_l\}$$



Because $S_l \neq T_l$ $(r < l < R)$ we have $S_l \notin \underline{\underline{T}}$, $T_l \notin \underline{\underline{S}}$ and obviously S_l, T_l reduce to $S_r = T_r$. In particular

$$V(T_l) = V(T_r) \oplus \sum_{i \in T_l - T_r}^{\oplus} V_i$$

$$= V(S_r) \oplus \sum_{i \in T_l - S_r}^{\oplus} V_i$$

From this it follows easily, that V is also attached to any $\underline{\underline{R}}$, say, such that

$$S_r = R_r, \; S_R = R_R$$

$$S_l = R_l \qquad (o \leq l \leq r, \; R \leq l \leq n)$$

Consequently, if

$$S_{r_1}, \ldots , S_{r_n} \quad ,$$

say, are the "irreducible" sets of $\underline{\underline{S}}$, then V is attached to all $\underline{\underline{T}}$ such that $T_{r_l} = S_{r_l}$ $(l = 1,\ldots,n)$.

V is uniquely determined by its value on the S_{r_l}. Therefore we have

Lemma 3.2 $V \in \mathbb{V}$ is tight, if and only if there is a uniquely defined increasing system

$\underline{\underline{T}} \subseteq \underline{\underline{P}}$ such that

1. $T \in \underline{\underline{T}}$ is not flat
2. $V(S) = V(T_0) \oplus \sum^{\oplus}_{i \in T-T_0} V_i$

   where T_0 is the largest set of $\underline{\underline{T}}$ included in S.
3. $V(S) \subseteq V(T)$ $(S,T \in \underline{\underline{T}},\ S \subseteq T)$

If $\underline{\underline{T}}$ is empty, then V is flat.

Let $\mathbb{V}^t$ denote the system of tight functions, $(\Omega,\underline{\underline{P}}, V)$ is tight if $V \in \mathbb{V}^t$. If $V \in \mathbb{V}^t$, then $\underline{\underline{T}} = \underline{\underline{T}}(V)$ as defined by Lemma 3.2 is the system of irreducible sets of V.

Note that $\mathbb{V}^t$ is invariant. In fact, if V is attached to $\underline{\underline{S}}$, then LV is attached to $\underline{\underline{S}}$ and πV is attached to $\pi S = \{\pi(S) \mid S \in \underline{\underline{S}}\}$

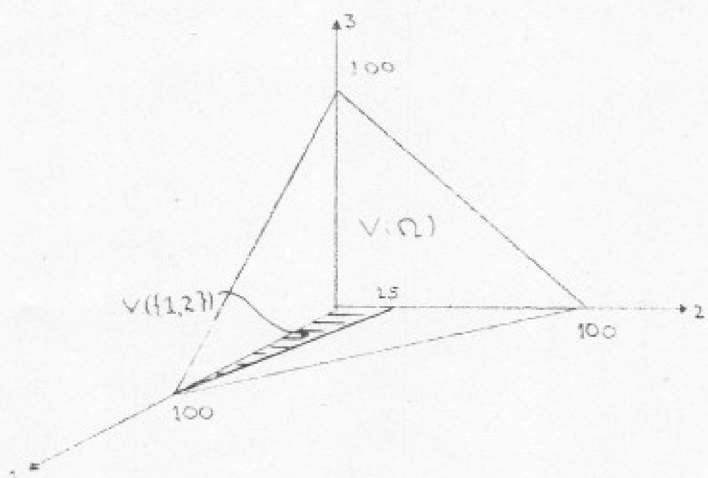
Example 3.3 In [5] OWEN considers the game $(\Omega, \underline{B}, V)$ defined via $n = 3$ and

$V_i = \{0\} \dot{-} \mathbb{R}_i^{n+}$

$V(\{i,j\}) = \{0\} \dot{-} \mathbb{R}^{n+}_{\{i,j\}} \qquad \{i,j\} \neq \{1,2\}$

$V(\{1,2\}) = \{x \in \mathbb{R}^{n+}_{\{1,2\}} \mid x_1 + 4x_2 \leq 100\}$

$V(\Omega) = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 100\}$



This is interpreted as follows:

The game is taking place somewhere in South America. Coalition {1,2} may obtain an amount of loo units by cooperation, but the money is being paid to player 1 and must be transfered to player 2. Because of the very unreliable mail system, a unit of money may vanish with a certain probablility and hence it is not worth a unit of utility for player 2. However, they may take in a banker, who, for a small fee, will transfer the money - in which case the utility transfer rates are constant and equal for everybody involved.

OWEN computes the following values: SHAPLEY's [1o] value is ($^1/_2$, $^1/_2$, 0), HARSANYI's [2] value is (4o, 4o, 2o). OWEN's [5] value is, as he claimes, (51,85; 47,57; 0,58). To which he rightly remarks that player 1 should get more than player 2 and player 3 should get something. It could be added that OWEN uses an approximating procedure in order to compute his own value (since V does not satisfy the necessary differentiability assumptions), the uniqueness of which is not obvious.

Since V is tight and attached to $\underline{\underline{S}} = \{\emptyset, \{1\}, \{12\}, \Omega\}$ ($\underline{\underline{T}}(V) = (\{12\}, \Omega\})$ we propose the following procedure: coalition {12} computes its Nash value. Then Ω computes its Nash value, using the result of {12} as threat point. We have

$$\nu(E_{\{1,2\}}, V(\{1,2\}), 0) = (5o, \frac{5o}{4}, 0)$$

and

$$\nu(E_{\Omega}, V(\Omega), (5o, \frac{5o}{4}, 0)) = (5o + \frac{5o}{4}, \frac{5o}{2}, \frac{5o}{4})$$

This is also a value satisfying the above requirements.

Definition 3.4 For $V \in \mathbb{V}^t$ define

$$\zeta^1(V) := \underline{x}(V)$$

$$\zeta^k(V) := \nu(E_{S_k}, V(S_k), \zeta^{k-1}(v))$$

$$\zeta(V) := \zeta^n(V)$$

Note, that the definition of ζ does not depend on the choice of $\underline{\underline{S}}$

Theorem 3.5

1. ζ is a value on $\mathbb{V}^t$
2. If $V \in \mathbb{V} \hat{\cap} \mathbb{V}^t$, then $\zeta(V) = \nu(V)$
3. If $V \in \mathbb{V}^c \cap \mathbb{V}^t$, then $\zeta(V) = \Psi(V)$

Proof: We shall only indicate a proof for 1. but leave 2. and 3. to the reader (actually, 2 and 3. follow from Theorem 4.13.)

Now, clearly, ζ is P.O. and feasible (since ν is). Moreover, ζ is i.r.: for ζ^1 is trivially i.r. and ζ^k is (inductively) satisfying $\zeta^k(V) \geq \zeta^{k-1}(V)$ since ν is i.r.

In order to check invariance properties observe that ζ^1 is $\mathbb{L}$ invariant (since $\underline{x}(\cdot)$ is. Now, if, by induction, ζ^k has been established to be $\mathbb{L}$-invariant then

$$\zeta^{k+1}(LV) = \nu(E_{S_{k+1}}, LV(S_{k+1}), \zeta^k(LV))$$

$$= \nu(E_{S_{k+1}}, L_{S_{k+1}}(V(S_{k+1})), L(\zeta^k(V)))$$

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$$= \nu(LE_{S_{k+1}}, V(S_{k+1}), \zeta^k(V)) \quad \text{(by (15) SEC. 2)}$$

$$= L(\nu(E_{\ldots, \ldots, \ldots})) \quad \text{(since } \nu \text{ is } L\text{-invariant)}$$

$$= L\zeta^{k+1}(V)$$

Π- invariance runs similarly, q.e.d.

SECTION 4 Extending the SHAPLEY value

Next, let us define a further value. Some of the ideas are similar to the ones of HARSANYI and some of the following computations have been used in other contexts. However, our value rests heavily upon the results of section 2 and moreover, it is a function, invariant under permutations of the players and affine transformations of utility. This will be proved in a rigorous and may be sometimes pedantical way.

Lemma 4.1 Let $(z^S)_{S\in\underline{\underline{P}}}$, $(w^S)_{S\in\underline{\underline{P}}}$ be families of vectors of $\mathbb{R}^n$.

Then

(1) $$z^T = \sum_{S\subseteq T} w^S \qquad (T \in \underline{\underline{P}})$$

if, and only if

(2) $$w^T = \sum_{S\subseteq T} (-1)^{t-s} z^S \qquad (T \in \underline{\underline{P}})$$

This is well known.

We shall denote by $e^T \in \mathbb{V}$ the function defined by

$$e^T(S) = \begin{cases} 1 & S \supseteq T \\ 0 & S \not\supseteq T \end{cases}$$

We have then

Lemma 4.2 The SHAPLEY value ϕ: $\mathbb{V} \to \mathbb{R}$ satisfies

(3) $$\phi(v_T) = c_T\phi(e^T) + \sum_{S\subsetneq T} (-1)^{t-s+1}\phi(v_S)$$

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Here, c_T is the unique coefficient of the representation

(4) $$v = \sum_{S \in \underline{\underline{P}}} c_S \; e^S$$

i.e.

(5) $$c_T = \sum_{S \subseteq T} (-1)^{t-s} \; v(S)$$

while v_T is the restriction of v on T. Also, we write $t = |T|$, $s = |S|$ etc.

Proof: We have

$$v_T = (\sum_{S \in \underline{\underline{P}}} c_S \; e^S)_T = \sum_{S \subseteq T} c_S \; e^S$$

and if $\mu^T \in \mathbb{A}$ is uniform distribution on T, then

$$\phi(v_T) = \sum_{S \subseteq T} c_S \phi(e^S) = \sum_{S \subseteq T} c_S \; \mu^S$$

using Lemma 4.1 we find

$$c_T \phi(e^T) = c_T \mu^T \overset{4.1}{=} \sum_{S \subseteq T} (-1)^{t-s} \; \phi(v_S)$$

$$= \phi(v_T) - \sum_{S \subsetneq T} (-1)^{t-s+1} \phi(v_S)$$

which proves (3).

Remark 4.3 Let $m \in \mathbb{A} \subseteq \mathbb{V}$. Because $m = \sum_{i \in \Omega} m_i e^{\{i\}} = \Sigma m_i e^i$ is the unique representation of m by the basis $(e^T)_{T \in \underline{\underline{P}}} \subseteq \mathbb{V}$, we have

$$c_T = 0 \qquad (|T| \geq 2) \quad .$$

Lemma 4.2. now yields

$$m_T = \phi(m_T) = 0 + \sum_{S \subsetneqq T} (-1)^{t-s+1} \phi(m_S)$$

(6) $$m_T = \sum_{S \subsetneqq T} (-1)^{t-s+1} m_S \qquad (|T| \geq 2) .$$

Consider a family $(z^S)_{S \in \underline{\underline{P}}}$ such that $z^S \in \mathbb{R}^n_S$ and an a.t.u. $L = L^{\alpha}_{\beta}$. Let us write

$$L(x) = L^{\alpha}_{\beta}(x) = L^{\alpha}(x) + \beta$$

Then, for $|T| \geq 2$, since L^{α} is linear

$$L_T\Big(\sum_{S \subsetneqq T} (-1)^{t-s+1} z^S\Big) = L^{\alpha}_T(\ldots) + \beta_T$$

$$= \sum_{S \subsetneqq T} (-1)^{t-s+1} L^{\alpha}_T(z^S) + \beta_T$$

$$= \sum_{S \subsetneqq T} (-1)^{t-s+1} L^{\alpha}_S(z^S) + \beta_T \qquad (\text{since } z^S \in \mathbb{R}^n_S)$$

$$= \sum_{S \subsetneqq T} (-1)^{t-s+1} (L^{\alpha}_S z^S + \beta_S) \qquad (\text{by } (6))$$

$$= \sum_{S \subsetneqq T} (-1)^{t-s+1} L_S(z^S) \quad .$$

As the first line and last line are equal also for

-/-

$|T| = 1,0$ (trivially), we have

(7) $$L_T\,\Big(\sum_{S\subsetneq T}(-1)^{t-s+1}z^S\Big) = \sum_{S\subsetneq T}(-1)^{t-s+1}L_S(z^S)$$

which, in view of Lemma 4.1, will be useful later on.

Remark 4.4 Define, for $v \in \mathbb{V}$

$$t^T: = t^T(v): = \sum_{S\subsetneq T}(-1)^{t-s+1}\phi(v_S) \qquad (T \in \underline{\underline{P}})$$

according to Lemma 4.2

(8) $$\phi(v_T) = c_T\phi(e^T) + t^T = \phi(c_T e^T + t^T)$$

(where $t^T \in \mathbb{R}^n$ is regarded as an element of $\mathbb{A}$).

Now a computation analogue to the one of 4.2 is as follows

$$c_T \overset{(5)}{=} \sum_{S\subseteq T}(-1)^{t-s}v(S)$$

$$= v(T) - \sum_{S\subsetneq T}(-1)^{t-s+1}v(S)$$

$$= v(T) - \sum_{S\subsetneq T}(-1)^{t-s+1}\,\phi(v_S)\,(T)$$

because $\phi(v_S)(T) = \phi(v_S)(S) = v_S(S)$

$$= v(T) - \Big(\sum_{S\subsetneq T}(-1)^{t-s+1}\phi(v_S)\Big)\,(T)$$

$$= v(T) - t^T(T)$$

i.e.

(9) $$c_T = v(T) - t^T(T)$$

-/-

Plugging this into (8) yields

$$\phi(v_T) = \phi((v(T) - t^T(T))e^T + t^T) \qquad (10)$$

Definition 4.5 Given $\alpha \in \mathbb{R}$, $x^0 \in \mathbb{A} = \mathbb{R}^n$

let

$$e^{T,\alpha,x^0} := (\alpha - x^0(T))e^T + x^0$$

i.e.

$$e^{T,\alpha,x^0}(S) = \begin{cases} \alpha + x^0(S-T) & (S \supseteq T) \\ x^0(S) & (S \not\supseteq T) \end{cases}$$

Clearly e^{T,α,x^0} represents a side payment game, where coalition T will distribute α while everybody else can only command his x^0 coordinate. This seems to be the appropriate version of E_{T,A,x^0} in the frame work of $\mathbb{W}$ (note that $\alpha - x^0(T)$ is not prohibited from being negative). Note however, that E_{T,A,x^0} is in general not of transferable utility type.

Corollary 4.6 For any $v \in \mathbb{W}$

$$\phi(v_T) = \phi(e^{T,v(T),t^T}) \qquad (11)$$

$$= \phi(e^{T,v(T),\sum_{S \subsetneq T}(-1)^{t-s+1}\phi(v_S)})$$

$$\phi(e^{T,\alpha,x^0}) = (\alpha - x^0(T))\mu^T + x^0 \qquad (12)$$

and of course, (11) and (12) define ϕ uniquely.

Clearly, we could as well state that $\phi(e^T) = \mu^T$ and linearity define ϕ uniquely together with (11). However, it is nice to oberserve, that Corollary 4.6 does not need the linearity condition (which is rather bad to interprete economically). Instead we have a boundary condition (12) and an "extension rule" (11), which determines ϕ recursively. It is our aim to show that the NASH value has a similar property and that this procedure clearly works for a larger class of mappings V.

Remark 4.7 1. If $V = E_{T,A,x^0}$, then $v^V = e^{T,\alpha,x^0}$

where

$$\alpha = v^V(T) = \max \{x(T) \mid x \in A\}$$

which may be written

(13) $$v^{E_{T,A,x^0}} = e^{T,v^V(T),x^0}$$

2. However, if $v = e^{T,\alpha,x^0}$ then in general

$$V^v \neq E_{T,A,x^0}$$

where

$$A = \{x \in \mathbb{R}^n_T \mid x(T) \leq \alpha\}$$

3. If we write $A^\alpha_T = \{x \in \mathbb{R}^n_T \mid x(T) \leq \alpha\}$

then

$$\nu(E_{T,A^\alpha_T,x^0}) = x^0 + (\alpha - x^0(T))\mu^T$$

(14) $$= x^0 + (\alpha - x^0(T))\phi(e^T)$$

$$= \phi(e^{T}, \alpha, x^{0})$$

If $M^T := \{x \in \mathbb{R}^m_T \mid x \leq \mu^T\}$ and

$E^T := E_{T,M^T,0,}$ then (14) may be completed to

(15) $$\ldots \quad = \phi(e^{T}, \alpha, x^{0})$$

$$= \nu(E_{T,A^{\alpha}_{T},x^{0}}) = x^{0} + (\alpha - x^{0}(T)\nu(E^{T})$$

Clearly, this holds true for $V = E_{T,A^{\alpha}_{T},x^{0}} \in \mathbb{V}^1 + \mathbb{V}^2$ (disjoint union)

4. Combining (13) and (15) we find

$$\phi(v^{E_{T}, A^{\alpha}_{T}}, x^{0}) = \phi(e^{T}, v^{V}(T), x^{0})$$

$$= \phi(e^{T}, \alpha, x^{0}) = \nu(E_{T,A^{\alpha}_{T},x^{0}})$$

Meaning that - as far as unanimous games are concerned - NASH value and SHAPLEY value coincide , if side payments utility transfer is permitted in V(T).

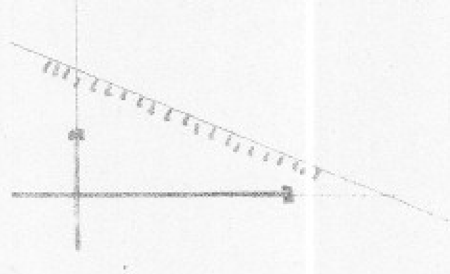
5. As $\mathbb{V}^1 \cap \mathbb{V}^2 = \phi$ there is no problem to consider ν as to be defined on $\mathbb{V}^1 + \mathbb{V}^2$. However, we want to compare ν and Ψ (cf. Example 1.2).

-/-

Consider $V \in (\mathbb{V}^1 + \mathbb{V}^2) \cap \mathbb{V}^C$, say $V = E_{T,A,x^0}$.

Whenever $S \not\supseteq T$, then $V(S) \doteq x^0_S \doteq \mathbb{R}^{n+}_S$ should be bounced by a hyperplane in $\mathbb{R}^n_S$ which can only happen for $|S| = 1$. Thus, $S \not\supseteq T$ implies $|S| = 1$. It follows that $|T| = |\Omega| = 2$ and V is necesserily of the type

$$|\Omega| = 2,\ V = LE_{\Omega, A^\alpha_\Omega, x^0}$$



In view of the definition of ψ (cf. Example 1.2.) we observe that in this particular case

$$E_{\Omega, A^\alpha_\Omega, x^0} = V^{e^{\Omega,\alpha,x^0}} .$$

Hence

$$\Psi(V) \underset{}{\overset{(1o)SEC1}{=\!=}} L\phi(e^{\Omega,\alpha,x^0}) \overset{14}{=} L\psi(E_{\Omega, A^\alpha_\Omega, x^0})$$

$$= \nu(LE_{\Omega, A^\alpha_\Omega, x^0}) = \nu(V)$$

Hence Ψ and ν coincide on $(\mathbb{V}^1 + \mathbb{V}^2) \cap \mathbb{V}^C$.

we may therefore state the following definition:

Definition 4.9 Let $\mathbb{V}^3 := (\mathbb{V}^1 + \mathbb{V}^2) \cup \mathbb{V}^C$

then

$$\psi : \mathbb{V}^3 \to \mathbb{R}^n$$

is defined via

$$\psi \,|_{\mathbb{V}^1 + \mathbb{V}^2} = \nu, \quad \psi \,|_{\mathbb{V}^C} = \Psi$$

Remark 4.1o ψ is a value on $\mathbb{V}^3$. Clearly

$$\psi(E_{T,A_T^\alpha,x^0}) = \nu(E_{T,A_T^\alpha,x^0}) = \phi(v^{E_{T,A_T^\alpha,x^0}})$$

$$= \phi(e^{T,\alpha,x^0}) = \psi(V^{v^{E_{T,A_T^\alpha,x^0}}})$$

by our previous remark.

Theorem 4.11 For $V \in \mathbb{V}^3$, $T \in \underline{\underline{P}}$

$$(16) \qquad \psi(V_T) = \psi(E_{T,V(T),\sum_{S\subsetneq T}(-1)^{t-s+1}\psi(V_S)})$$

Moreover

$$(17) \qquad \psi(E_{T,A,x^0}) = \nu(E_{T,A,x^0})$$

(where V_S is the appropriately defined restriction of V on $S\in\underline{\underline{P}}$).

Comparing this with (14) and (15) of Remark 4.7 we argue that ψ is uniquely defined by a recursive extension procedure and a boundary condition.

Proof: First let $V \in \mathbb{V}^c$, say $V = LV^v$. As is easily seen

$$(18) \qquad V_T = L_T V^{v_T} \quad (T \in \underline{\underline{P}})$$

Now we have

$$\Psi(E_T, V(T), \sum_{S \subsetneq T} (-1)^{t+s-1} \psi(V_S))$$

$$= \Psi(E_{\ldots}, \ldots, \sum_{\ldots} \ldots \psi(L_S V^{v_S})) \qquad \text{(by (18))}$$

$$= \Psi(E_{\ldots}, \ldots, \sum_{\ldots} \ldots (L_S \phi(v_S))) \qquad \text{(by definition of } \Psi)$$

$$= \Psi(E_{\ldots}, \ldots, L_T(\sum_{S \subsetneq T} (-1)^{t-s+1} \phi(v_S)) \qquad \text{(Remark 4.3, (7))}$$

$$= \Psi(E_{\ldots}, L_T V^v(T), L_T(\ldots)) \qquad \text{(by (18))}$$

$$= \Psi(L_T E_{\ldots}, V^v(T), \sum_{S \subsetneq T} \ldots) \qquad \text{(by (15) of SEC 1)}$$

$$= L_T \, \Psi(E \ldots, \ldots, \ldots) \qquad \text{(since } \Psi \text{ is invariant)}$$

$$= L_T \phi(e^T, v(T), \sum_{S \subsetneq T} (-1)^{t-s+1} \phi(v_S)) \qquad \text{(by Remark 4.1o)}$$

$$= L_T \phi(v_T) \qquad \text{(by Corollary 4.6)}$$

$$= \Psi(L_T V^v T) \qquad \text{(by definition of } \Psi)$$

$$= \Psi(V_T) \qquad \text{(by (18))}$$

This proves half of our theorem since $\psi = \Psi$ on $\mathbb{V}^c$. It remains to check the case that $V \in \mathbb{V}^1 + \mathbb{V}^2$.

-/-

Suppose that $V = E_{R,A,x^0} \in \mathbb{V}^1$, say.

First note that for $x \in \mathbb{R}^n$

$$\sum_{\substack{S \supseteq R \\ S \subsetneqq T}} (-1)^{t-s+1} x_S = \sum_{\substack{S \supseteq R \\ S \subsetneqq T}} (-1)^{t-s+1} x_R + \sum_{\cdots} \cdots x_{S-R}$$

$$= x_R \underbrace{\sum_{\substack{S-R \subsetneqq T-R \\ S \supseteq R}} (-1)^{t-r-(s-r)+1}}_{1} + \sum_{\substack{S-R \subsetneqq T-R \\ S \supseteq R}} (-1)^{t-r-(s-r)+1} x_{S-R}$$

(19)

$$= \begin{cases} x_R + x_{T-R} & (|T-R| \geq 2) \\ x_R + 0 & (|T-R| = 1) \end{cases} \qquad \text{(by Remark 4.3)}$$

$$= \begin{cases} x_T & (|T-R| \geq 2) \\ x_R & (|T-R| = 1) \end{cases}$$

Moreover, we have for any R-unanimous $V = E_{R,A,x^0}$ in general

$$V_T = E_{R,A,x^0_T} \qquad (T \supseteq R)$$

(2o) $$V_T = E_{T,x^0_T - \mathbb{R}^{n+}_T, x^0_T} \qquad (T \not\supseteq R)$$

From (20) it is easily concluded that

(21) $$\nu(V_T) = \begin{cases} \nu(V)_T & T \supseteq R \\ x^0_T & T \not\supseteq R \end{cases}$$

Clearly, whenever $T \not\supseteq R$, then

$$\sum_{S \subsetneq T} (-1)^{t-s+1} v(V_S) = \sum_{S \subsetneq T} (-1)^{t-s+1} x^0_S$$

$$= x^0_T = v(V_T)$$

using (21) and Remark 4.3, thus

$$v(E_{T,V(T)}, \sum_{S \subsetneq T} (-1)^{t-s+1} v(V_S))$$

(22) $$= v(E_{T,V(T)}, v(V_T)) = v(V_T) \qquad (T \not\supseteq R)$$

For $T \supseteq R$, the argument runs as follows:

we have for $|T-R| \geq 2$

$$\sum_{S \subsetneq T} (-1)^{t-s+1} v(V_S) = \sum_{\substack{S \subsetneq T \\ S \supseteq R}} \ldots + \sum_{\substack{S \subsetneq T \\ S \not\supseteq R}} \ldots$$

$$= \sum_{\substack{S \subsetneq T \\ S \supseteq R}} \ldots v(V)_S + \sum_{\substack{S \subsetneq T \\ S \not\supseteq R}} \ldots x^0_S \qquad \text{(by (21))}$$

$$= v(V)_T + \underbrace{\sum_{S \subsetneq T} \ldots x^0_S}_{x^0_T} - \underbrace{\sum_{\substack{S \subsetneq T \\ S \not\supseteq R}} \ldots x^0_S}_{x^0_T} \qquad \text{(by Remark 4.3) (and (19))}$$

$= v(V_T)$. Hence (22) holds true for $T \supseteq R, (T-R) \geq 2$

Next, for $|T-R| = 1$, $T \supsetneq R$

$$\sum_{S \subsetneq T} (-1)^{t-s+1} \nu(V_S) = \nu(V_R) + \sum_{\substack{S \subsetneq T \\ S \not\supseteq R}} \ldots x_S$$

$$= \nu(V_R) + \sum_{S \subsetneq T} \ldots x_S^0 - \sum_{\substack{S \subsetneq T \\ S \supseteq R}} \ldots x_S^0$$

$$= \nu(V_R) + x_T^0 - x_R^0 = \nu(V_R) + x_{T-R}^0 = \nu(V_T)$$

so we may again refer to (22).

Finally, for $T = R$

$$\sum_{S \subsetneq T} (-1)^{t-s+1} \nu(V)_S = \sum_{S \subsetneq T} \ldots x_S^0 = x_T^0$$

and

$$\nu(E_{T,V(T),\sum_{S \subsetneq T} (-1)^{t-s+1} \nu(V_S)})$$

$$= \nu(E_{T,V(T),x_T^0}) = \nu(V_T)$$

completes our proof since $\nu = \psi$ on $\mathbb{V}^1$ ($\mathbb{V}^2$ runs similarly, if not identically)

Lemma 4.12. Let $\mathbb{V}^0 \subseteq \mathbb{V}$ be invariant, $\mathbb{V}^0 \supseteq \mathbb{V}^1 + \mathbb{V}^2$, and suppose $\varphi: \to \mathbb{R}^n$ is a mapping satisfying

$$\varphi(V_T) = \varphi(E_{T,V(T),\sum_{S \subsetneq T} (-1)^{t-s+1} \varphi(V_S)}).$$

If φ is invariant on $\mathbb{V}^1 + \mathbb{V}^2$, then it is invariant on $\mathbb{V}^0$.

Proof: By induction. Let $L \in \mathbb{L}$ and observe that

$$= \varphi(E_{T}, LV(T), \sum_{\dots} \dots \varphi(LV_S))$$

$$= \varphi(E_{T}, L_T V(T), \sum_{\dots} \dots L_S \varphi(V_S)) \qquad \text{(By induction)}$$

$$= \varphi(E_{T}, L_T V(T), L_T \sum_{\dots} \dots \varphi(V_S)) \qquad \text{(By (7), Remark 4.3)}$$

$$= \varphi(L_T E_{T}, V(T), \sum_{\dots} \dots \varphi(V_S)) \qquad \text{(By (15), SEC 1)}$$

$$= L_T \varphi(E_{T}, V(T), \sum_{\dots} \dots \varphi(V_S)) = L_T \varphi(V_T)$$

serves as induction step for $|T| \geq 2$. For $T = \Omega$ this proves $\mathbb{L}$-invariance. $\mathbb{T}$-invariance runs analogously (using linearity of π instead of (7), Remark 4.3).

Theorem 4.13 There is a unique maximal $\mathbb{V}^4 \subseteq \mathbb{V}$, $\mathbb{V}^4 \supseteq \mathbb{V}^3$, which admits a unique mapping

$$\chi: \mathbb{V}^4 \to \mathbb{R}^n$$

such that

$$\chi(V_T) = \chi(E_{T}, V(T), \sum_{S \subsetneq T} (-1)^{t-s+1} \chi(V_S)) \qquad (V \in \mathbb{V}^4,\ T \in \underline{P}) \tag{23}$$

(24) $\chi(V) = \nu(V) \qquad (V \in \mathbb{V}^1 + \mathbb{V}^2)$.

χ has the following properties

(25) $\chi|_{\mathbb{V}^1 + \mathbb{V}^2} = \nu$

(26) $\chi|_{\mathbb{V}^c} = \Psi$

(27) $\chi|_{\mathbb{V}^3} = \psi$

(28) $\chi|_{\mathbb{V}^t} = \zeta$

(29) χ is $\mathbb{L}$ and Π-invariant ;

(3o) χ is feasible and Pareto optimal .

Proof: Define $\chi = \nu$ on $\mathbb{V}^1 + \mathbb{V}^2$. $\mathbb{V}^4$ is the set of all those $V \in \mathbb{V}$ such that χ my defined inductively via (23), i.e., whenever $\chi(V_S)$ is defined for $|S| \leq t - 1$, then $E_{T,\mathbb{V}(T), \sum_{S \subsetneqq T} (-1)^{t-s+1} \chi(V_S)}$ has to be an element of $\mathbb{V}^1 + \mathbb{V}^2$ and thus $\chi(V_T)$ is given by (23); completing the induction step of our definition. (25), (26), (27) is clear, (29) follows form Lemma 4.12 and (3o) form the properties of ν . It remains to show (28). This will be postponed, for we need some auxiliary theorems.

Given $V \in \mathbb{V}^{+}$, define for $T \in \underline{P}$

(31) $$\bar{t}^T := \bar{t}^T(V) := \sum_{S \subsetneqq T} (-1)^{t-s+1} \chi(V_S)$$

(32) $$\bar{\upsilon}^T := \bar{\upsilon}^T(V) := \sum_{S \subseteqq T} (-1)^{t-s} \chi(V_S) = \chi(V_T) - \bar{t}^T$$

such that $\bar{t}^T(V^v) = t^T(v)$ (cf. Remark 4.4)

Theorem 4.14: Let $V \in \mathbb{V}^{+}$. Define rekursively

1. $\hat{\upsilon}^{\{i\}} = \underline{x}_i(V)\, e^i$

2. If $\hat{\upsilon}^S$ is defined for $|S| < t$, then for $|T| = t$

$$\hat{t}^T := \sum_{S \subsetneqq T} \hat{\upsilon}^S$$

$$\hat{x}^T := \nu(E_{T,V(T)}, \hat{t}^T)$$

$$\hat{\upsilon}^T := \hat{x}^T - \hat{t}^T.$$

Then $\hat{t} = \bar{t} = \bar{t}(V)$, $\hat{\upsilon} = \bar{\upsilon} = \bar{\upsilon}(V)$ and

$\hat{x}^T = \Psi(V_T) \qquad (T \in \underline{P})$

Proof: We have

$$\hat{x}^T = \hat{t}^T + \hat{\upsilon}^T = \sum_{S \subsetneqq T} \hat{\upsilon}^S + \hat{\upsilon}^T = \sum_{S \subseteqq T} \hat{\upsilon}^S$$

and hence by Lemma 4.1,

$$\hat{\upsilon}^T = \sum_{S \subseteq T} (-1)^{t-s} \hat{x}^S$$

-/-

i.e.

$$\hat{t}^T = \hat{x}^T - \hat{\upsilon}^T = \sum_{S \subsetneqq T} (-1)^{t-s+1} \hat{x}^S$$

Therefore $\hat{x}^{\cdot}$ satisfies

$$\hat{x}^T = \nu(E_{T,V(T)}, \sum_{S \subsetneqq T} (-1)^{t-s+1} \hat{x}^S)$$

which defines χ uniquely.

We may, therefore, take $\bar{\upsilon}, \bar{t}$, and $\bar{x}^T = \chi(V_T)$ also as to be the quantities defined via 4.14

Remark 4.15 1. Note that for $V = V^v \in \mathbb{V}^c$:

$$\bar{\upsilon}^T = \bar{x}^T - \bar{t}^T = c_T \mu^T = c_T \phi(e^T)$$

where all quantities except μ^T and $\phi(e^T)$ are functions of V or v respectively

2. Clearly

$$\chi(V_T) = \bar{x}^T = \sum_{S \subseteq T} \bar{\upsilon}^S$$

Corollary 4.16 Let $V \in \mathbb{V}^4$. Define recursively the quantities $(\bar{\upsilon}^S)_{S \in \underline{\underline{P}}}$ by $\bar{\upsilon}^{\emptyset} = 0$ and

1. $\bar{\upsilon}^{\{i\}} = \underline{x}_i(V) e^i$

2. If $\bar{\upsilon}^S$ is defined for $|S| < t$,

then

$$\bar{\upsilon}^T = \sum_{S \subsetneqq T} \bar{\upsilon}^S + \nu\left(E_{T,V(T)}, \sum_{S \subsetneqq T} \bar{\upsilon}^S\right)$$

Then $\chi(V) = \sum_{T \subseteq \Omega} \overline{\upsilon}^T$

The proof is trivial, it runs inductively via

$$\overline{\upsilon}^T = \overline{x}^T - \overline{t}^T$$

$$= \nu(E_{T,V(T)}, \overline{t}^T) - \overline{t}^T$$

$$= \nu(E_{T,V(T)}, \sum_{S \subsetneqq T} \overline{\upsilon}^S) - \sum_{S \subsetneqq T} \overline{\upsilon}^S$$

Now, 4.16 suggests an interpretation similar to the one given in [1], [2]: every coalitions somehow fixes a guaranteed payoff $\overline{\upsilon}^S$ (not necessarily $\in V(S)$). If coalition T cooperates, it takes all guaranteed values $\sum_{S \subsetneqq T} \overline{\upsilon}^S$ of all proper subcoalitions and computes its own guaranteed payoff $\overline{\upsilon}^T$; the final value is obtained by adding up all these payoffs, i.e., $\sum_{T \subseteq \Omega} \overline{\upsilon}^T$.

Remark 4.17

It is not hard to verify by means of (31), (32) and Lemma 4.3 that

$$\overline{t}^T(\pi V) = \pi(\overline{t}^T(v), \quad \overline{t}^T(LV) = L_T(\overline{t}^T(V))$$

$$\overline{\upsilon}^T(\pi V) = L_T(\overline{\upsilon}^T(v), \quad \overline{\upsilon}^T(LV) = L_T(\overline{\upsilon}^T(v))$$

holds true for $\pi \in \Pi$, $L \in \mathbb{L}$, and $V \in \mathbb{V}^4$.

Proof of (28): Pick $V \in \mathbb{V}^t$ such that V is attached to $\underline{\underline{S}}$. Recall that $\underline{\underline{T}} = \underline{\underline{T}}(V)$ is the system of "irreducible" sets (Lemma 3.2.) Since ζ and ψ are $\mathbb{L}$ invariant, we may assume that $\underline{x}(V) = 0$ w.l.g.

1st Step We are going to show that

(34) $$\bar{\upsilon}^S = 0 \quad \text{if S is reducible}$$

This is done inductively. Suppose, (34) is proved for $|S| < t$. Pick T, $|T| = t$, T reducible; we are going to show that $\bar{\upsilon}^T = 0$.

Observe that

(35) $$\bar{t}^R = \sum_{S \subsetneqq R} \bar{\upsilon}^S = \sum_{\substack{S \subsetneqq R \\ S \text{ irred.}}} \bar{\upsilon}^S$$

for $|R| \leq t$ follows from induction hypothesis. Therefore, if $T_0 \subsetneqq T$ is the largest irreducible set in T, then

$$\bar{t}^T = \sum_{\substack{S \subsetneqq T \\ S \text{ irred.}}} \bar{\upsilon}^S$$

$$= \bar{\upsilon}^{T_0} + \sum_{\substack{S \subsetneqq T_0 \\ S \text{ irred.}}} \bar{\upsilon}^S$$ (since the irred. sets form an increasing sequence, Lemma 3.2)

$$= \bar{\upsilon}^{T_0} + \bar{t}^{T_0}$$ by (35)

$$= \bar{x}^{T_0}$$

Therefore

$$x^T = \Psi(V_T) = \nu(E_{T,V(T)}, \bar{t}^T)$$

$$= \nu(E_{T,V(T_o)} + \sum_{T-T_o} V_i,\ \bar{x}^{T_o})$$

$$= x^{T_o} + x_{T-T_o}V \qquad \text{(since } \bar{x}^{T_o} \text{ is P.O. in } V(T_o))$$

$$= \bar{x}^{T_o} \qquad \text{(since } \underline{x}(V) = 0)$$

and consequantly

$$\bar{\upsilon}^T = \bar{x}^T - \bar{t}^T$$

$$= \bar{x}^{T_o} - \bar{x}^{T_o} = 0$$

which proves, by induction (34)

<u>2^{nd} Step</u> Given any $T \in \underline{\underline{P}}$ and $T_o \subseteq S$, the maximal irreducible set, it follows from the first step, that

$$\bar{t}^T = \sum_{\substack{S \subsetneq T \\ S \text{ irred.}}} \bar{\upsilon}^S$$

$$= \begin{cases} \bar{t}^{T_o} & T_o = T_o \\ \bar{\upsilon}^{T_o} + t^{T_o} & T_o \subsetneq T \end{cases}$$

3rd Step: Now, since V is attached to $\underline{\underline{S}}$ and $\underline{\underline{S}} \subseteq \underline{\underline{T}}$, this means

$$\bar{t}^{S_{k+1}} = \sum_{l=0}^{K} \bar{\upsilon}^{S_l}$$

$$= \upsilon^{S_k} + \sum_{l=o}^{k-1} \upsilon^{S_l}$$

$$= \bar{\upsilon}^{S_k} + \bar{t}^{S_k}$$

$$= \bar{x}^{S_k}$$

Hence

(36) $$\chi(V_{S_k}) = \nu(E_{S_k}, V(S_k), \bar{t}^{S_k})$$

$$= \nu(E_{S_k}, V(S_K), \bar{x}^{S_{k-1}}$$

$$= \nu(E_{S_K}, V(S_k), \chi(V_{S_{k-1}}))$$

However, $\chi(V_{S_1}) = \underline{x}_{S_1}(V) = \zeta^1(V)$ by definition and thus,

(37) $$\chi(V_{S_k}) = \zeta^k(V)$$

follows via induction. Hence, $\chi(V) = \chi(V\Omega)$

$$= \zeta^n(V) = \zeta(V), \text{ q.e.d.}$$

-/-

As has been announced, we now have a value defined as a function; no fixed point theorem is involved. The computation in general requires a maximization procedure that might be solved be means of Lagrange - or Kuhn - Tucker principles. In addition the value has all the desired invariance properties.

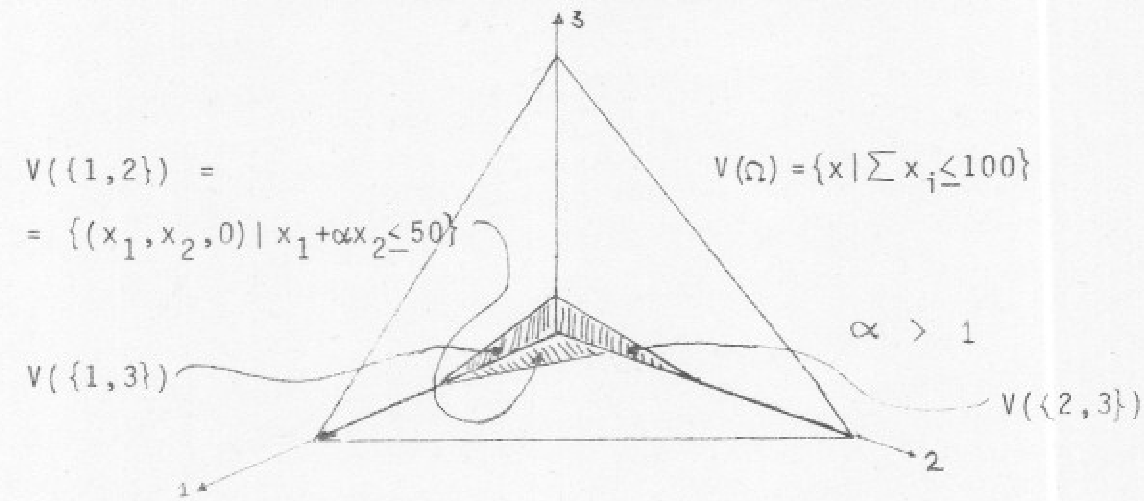
Note that despite of being a function, it is not necessarily unique; this is so, because Theorem 2.4 allows for se veral definitions of υ. It would be easy, to define a subclass of V^4, such that χ is unique: by appropriate conditions on the possibilities of proper subcoalitions one can ensure, that the procedure of summing up the values of all subcoalitions of T will never lead out of V (T). In this case only the classical NASH-value would be involved, hence the theory of section 2 is not required and χ is unique.

Simmilarly it would be easy to define a proper subclass of $\mathbb{V}^4$ by appropriate conditions on the mappings V, such that this subclass is nontrivial and ensures the existence of χ.

Note, however, that there is also a disadvantage: unless further conditions are imposed, χ is not necessarly individually rational - a property that it shares with other values.

It is a good custom to present an example, which exhibits the merits of a newly defined value. The following example is a slight extension of Example 3.3. Here, in every two-person coalition the rate of transfer of utility is given by some constant $\alpha > 1$; while, within the grand coalition, utility transfers are being performed at a rate 1:1:1. However, the game is such that player 1 is better off than everybody else and player 2 is better off than player 3 - at least, if we adapt the ideas of Example 3.3.Hence a value should rate all 3 players differently putting player 1 in the best position and player 2 exactly in the middle position.

<u>Example 4.18</u> : Let us consider a slight modification of Example 3.3 as indicated by the following figure (n = 3)



Here, SHAPLEY's Value yields $(\frac{100}{3}, \frac{100}{3}, \frac{100}{3})$

HARSANYI's Value yields $(\frac{100}{3}, \frac{100}{3}, \frac{100}{3})$

While, on the other hand we have

$$\chi(V) = (\frac{100}{3} + 25(1-\frac{1}{\alpha}), \frac{100}{3}, \frac{100}{3} - 25\ (1-\frac{1}{\alpha}))$$

-/-

L I T E R A T U R E

[1] HARSANYI, J.C.: A bargaining model for the cooperative n-person game. Ann. Math. Stud. 4o(1959), pp. 325 - 356

[2] HARSANYI, J.C.: A simplified bargaining model for the n-person cooperative game. Int. Ec.-Review 4, No.2, (1963) pp. 194 - 22o

[3] MIYASAWA, K.: The n-person bargaining game Ann. Math. Studies 52 (1964), pp. 547 - 575

[4] NASH, J.F.: The bargaining problem Econometrica 18 (195o), pp. 152 - 162

[5] OWEN, G.: Values of games without side payments. Int. J. Game Th. 1 (1971), pp. 95 - 1o9

[6] ROSENMÜLLER, J.: Einstimmigkeitsspiele 2. Art, Standortspiele und Fortsetzungsmöglichkeiten des Nash-Wertes, 1977. To appear

[7] ROSENMÜLLER, J.: Lectures in the theory of games of markets. Manuscript. Inst. of Statistics and Math. Ec., Karlsruhe, University, December 1977.

[8] SELTEN, R.: Valu ation of n-person games. Ann. Math. Studies 52 (1964), pp. 577 - 625.

[9] SHAPLEY, L.S.: A value for n-persons games. Ann. Math. Stud. 28 (1953), pp. 3o7 - 318

[10] SHAPLEY, L.S.: Utility comparison and the theory of games. La decision (NRS Paris (1969))