Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

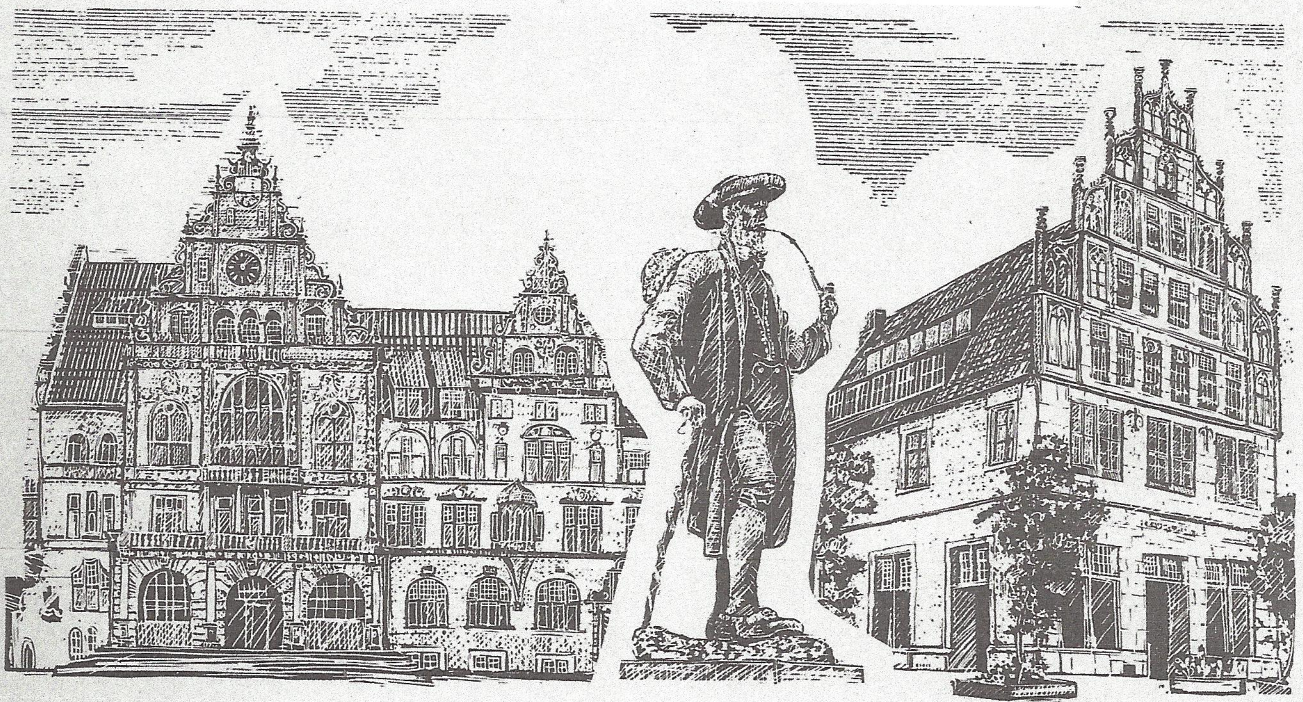
Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 51

Alfred Zauberman

Planning under Indeterminacy

December 1976



H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

PLANNING UNDER INDETERMINACY

by

Alfred Zauberman

London School of Economics,

Institut für Mathematische Wirtschaftsforschung,
Universität Bielefeld

Contents

2. Kinship of Theories of Fuzzy Sets and of Indeterminacy in Planning
3. Pattern Recognition as Instrument in Planning under Indeterminacy

This "working paper" is the result - initial and preliminary - of my research for a monograph on economic planning under conditions of indeterminacy. Its focus is on theory rather than practice. It is concerned essentially with formalized (mathematical) rather than the more "traditionalist" methodology of plan construction.

The proposed monograph is expected to be an analytical survey of theoretical developments in the field - in both East and West (while its title indicates concern with planning so described this is understood sensu largo, that is encompassing economic policy formulation when sufficiently systematized). The present working paper deals essentially with Soviet theoretical developments; however, as the reader will easily notice, there are almost in every context indications of links with Western thought; these make, as it were, a bridge with the proposed next working paper dealing with Western theoretical developments.

While the body of the working paper is concerned with planning approaches which rely on more "conventional" formalization its area is extended in the Appendix. There we very briefly discuss - from the angle of our topic - some "modern" approaches and methods (theories of automata, of pattern recognition, of fuzzy sets) to which the planning theorist is nowadays looking for help in coping with the unplannable, unformalizable, untractable elements of reality.

Initial research for this paper was carried out at the London School of Economics. It was written while I was visiting professor with the Institut für Mathematische Wirtschaftsforschung, Universität Bielefeld. A grant from the Social Science Research Council, United Kingdom, is thankfully acknowledged.

1. Introductory Remarks

§1. Economic planning has developed over decades as largely a deterministic operation. This is true in particular of the country where it originated, the Soviet Union. The philosophy behind the attitude will not be discussed here, interesting as the subject may be. But we are inclined to point at least to the underdevelopment of the probabilistic thought in planning and of the formal apparatus as one of the elements crucial for the determistic orientation over decades of theory as well as practice.

It is the advances in both fields that might account, at least in part, for the change in attitudes. (The traditional brilliance of the country's scholarship in probabilistic theory is obviously of relevance for the change in the intellectual climate.) A crucial background to these developments is the general progress in the mathematization of planning theory and computational technology. In the context of the latter what deserves particular mention is the drive towards the "automated system of control" (in Soviet usual reference "avtomaticheskaya sistema upravlenya" ("ASU") - a goal aimed at for some years (to be sure a goal which has proved more difficult and far slower in materializing than expected and indeed elusive, although pursued with increasing determination). Its idea postulates a network - a system integrated in the three dimensions of "substance" of territory and time - automated throughout - to carry out both plan construction and implementation, one supported by an expanded range of algorithms. By the 1970s the viewpoint has crystallized that planning is by its very nature, and therefore inescapably, an operation performable only under conditions of <u>indeterminacy</u>; that therefore its formalization must allow for this characteristic. And, that consequently allowance for it must be emodied in the mathematized support for the ASU. Thus the drive towards the ASU and the shift away from determinism in planning have become intertwined. The concept of "indeterminacy" is taken <u>sensu largissimo</u>. It denotes imperfectness of information on events past and present, let alone the obviously precarious predictability of the future ones. Here come as related the implications - in particular the informational implications - of the organizational structure of the planning system, especially its centralism.[1] It is this aspect which, as it appears, has drawn the attention of the center for computation (Vychislitelnyi Institut) of the USSR Academy of Sciences. The conclusion arrived at in its inquiries would be that in the limiting case centralization in both collecting and processing information leads at best to a degree of indeterminacy which seriously affects the quality of plan making (This facet of indeterminacy is directing the student

---

1) See Moiseyev's paper in <u>Kibernetika</u> No. 6,1973

to a search of effective adjustments in the system's structure (on this see in III).

A vitally important domain of indeterminacy, with some problems of its own, is that of technological progress. Coping with risk - involved in promoting, gestating and implanting it - presents itself in both of its facet: of its "uncertainty-engulfed" nature, and of the specific organizational forms. (The matter of this risk has been incisively discussed by Dudkin [2)] at a recent symposium on control of scientific-technical progress).

Finally the concept of indeterminacy encompasses the nature of the planning and controlling processes "as such". Under this heading fall the limitations in "capturing" the relevant facets of economic reality, in particular where these are intertwined with extra-economic factors very often at best only imperfectly quantifiable. Also - the effects of technical imperfectness of the planning operations and of error entailed;again - error in the widest sense:additionally to its more conventional notion, covering the lack of exactness to which the planner has to reconcile himself. Such reconciliation is but one of the factors with enforce an approximational sub-optimal (sub-efficient) approach in plan construction and implementation. To anticipate further discussion here is a situation where the approximationist possibilities of stochastic methods and techniques come to the fore.

---

2) cf. L.M. Dudkin in symposium of the conference edited by L.D. Davidovich and E. Lomenov, publication of Moscow University 1975" cf. also the same in Eko, Siberian Branch, USSR Academy of Sciences, No. 5, 1976

Very generally speaking what is of relevance is the change of intellectual climate in mathematical planning, in particular in longer term planning: a change largely influenced by the sobering experience with mathematical plan modelling. At some stage there was the prevalent assumption that the state of the system at any time is quantitatively determinable. The revised attitude has been interestingly put as an analogy with the world of physics from which so much has been borrowed in mathematized economic dynamics (by Albert Vainshtein [3]). The stress is on consequences to be drawn from the fact that economics has not reached, as yet, the level of theory and methodology of physics where it can be pivoted on the "quantification triplet". (Substance, time and space;as a matter of fact - we would argue - even in physics it rests on some hypothesizing where no stage of definitiveness has been reached; as an example we may mention the peripatea in the theory of the Brownian motion that had been assimilated by mathematical theory of policy-making[4]). It is an acceptable view that economics does share with physics the phenomenon of difficulties increasing with the increase of the time perspective (and aspirations to exactness). The more or less explicit idea is that the analogous triplet - time, space,"cost" (resource input) - in longer-term economic policy-making evades simultaneous handling; only if time and resource

---

3) in T.S. Khachaturov Ed., Methods of Long-term Planning and Forecasting,Macmillan Press London 1976, pp. 86ff

4) cf. E. Nelson, Dynamical Theories of Brownian Motion, Princeton 1967

input were fixed, the result would appear as something within a range of stochastic evaluation or within a zone of indeterminacy expressible by a probability distribution rather, that is, than exact quantified parameters.

§2. It is such lines of thinking and experience (observation of the mathematical and the "real life" (economic, organizational, institutional,operational) facts that have inspired in Soviet theory, in recent years, the concept of the planning system of an economy as one that is immanently (to use the term en vogue) "probabilistically indeterminate". Thence the tenet - ranked as central in the approach to planning - on the "objective" existence of what is termed the "indeterminacy zone" in an economy: specifically in its optimal development. This implies the relevance of a set of solution variants - each of which "best" for some possible materialization of certain conditions: to a large extent - endogenous conditions. It would follow from this proposition that "detecting" and delimiting the economy's "zone of indeterminacy" is facing its planners and controllers as their prime task. It is indeed the contention of some planning theoreticians that the usual deterministic methods of plan optimization can be but an auxiliary instrument in revealing the content of the variants which form the zone of indeterminacy. Furthermore, it is being argued that the probabilistic probings into the zone of indeterminacy tend to yield a relatively large range of equi-economical solutions based on mean values (mathematical expectations). This and other limitations induce (as we shall see) the planning theorist to seek additional mathematical support in some new areas of formalization. Yet he has quite often to fall back on the more "traditionalist" instrumentarium; "conservatism" apart, the reason is simply (as, very à propos of our subject, has been noted by Athans) because alternate approaches have not as yet reached the theoretical sophistication of the probabilistic one.

Note 1 Two appraches have been favoured in established practice by the planners in dealing with indeterminacy with respect to exogenous variables with values not determined by the decision processes or structural relationships.[5)] One would be to build up the plan in such a way that it would still be not too far from the expected efficiency - where less likely values would materialize. The other, complementary to the former, has been termed one of "conditional planning"; its idea is to bring in some policy instruments and secondary decision criteria with the view of putting them into effect only if and when some of the prognosticated values materialized. Technically that would call for specifying a set of alternatives and/or by making the values in question functionally dependent on the successive realization of some other variables; the matter would then boil down to the expliciting of certain priorities. The first of the two appraches, while obviously appealing to the practitioner,has evaded adequate theoretical statement. The other - is clearly dependent on the state of techniques.

---

5) cf. P. Sevaldson, W. Trzeciakowski, "Construction and application of macro-economic models", E.C.E., U.N., Macro-Economic Models for Planning and Policy Making, Geneva 1967

§3. While being concerned with the theory as it has originated and developed on Soviet ground, we by no means suggest that this has no tie-up, with the Western thought. Indeed it has naturally a good deal in common with Western-developed theories of macro-decision making under uncertainty; especially - where these theories work at the junction of such decision-making with forecasting (see Note 7 below). That includes technically inquiries into non-mandatory planning; as we proceed the reader will easily observe affinity in some of the basic conceptual elements. And still closer are the analogues in the technical approaches and instrumentarium. (To anticipate,say,the treatment of uncertainty of the future as one of the probabilistic type - replacement of probability distributions by mean values; the adoption of certainty equivalents and so on and so forth).

But it does seem tenable that the dimensions of Soviet-type macro-decision-making - the proportion of contrallable to uncontrallable variables: both the range and the directness and operational intensity of controlling-policy instruments in the macro-policy-maker's hands - all they - do affect,to some extent,the qualitative as well as quantitative characteristics of the decision processes: In some sense they, as it were, "itensify" the problems involved and,thereby, also from the point of view of the Western inquiry, make a suitable "seminar case".

§ 4. Finally something may be said in these introductory observations about the stand of the attempts to relate the attitude of planning theory vis-à-vis contemporary currents in the philosophy of probabilistic thinking. Our reference will be on this matter to a paper on "undeterminacy and probability" by Yefimov and his associate[6] - strong protagonists of the tenet that mathematical probability (nonnegative, additive, normed measure) is a necessary means for quantitative handling indeterminacy. Under the conditions of indeterminacy probabilistic models are believed by them to be adequate for optimal planning models. The question would be then what is to be understood as the correct approach in such modelling.

To begin with, probabilistic logic is considered insufficiently developed, at least as yet, to permit the appraisal of indeterminacy: for generally speaking it fails to provide adequate formal language for rigorous description of an economic system. The issue poses itself then as the familar one of an objective versus subjective probability. (But whatever else can be said on the attitude to Kolmogorov's position, his axiomatics is in any case accepted as valid under the subjectivistic just as well as under objectivistic philosophy).

When the issue is posed in this fashion, the stand of the school under discussion has a strong subjectivistic inclination. This however is not without qualifications. (We will note that qualifications are by now formulated by quite a few scholars on either side of the grand dividing line. Thus, to refer oneself e.g. to Carnap[7] there is no incompatibility between the objecti-

---

6) V.M. Yefimov, V.A. Spivak, " O neopredelennosti i veroyatnosti", Ekonomika i Matematicheskiye Metody, No. 5, 1972

7) Rudolf Carnap, "The Aims of Inductive Logic" in E.Nagel, P. Suppes, S. Tarski, eds., Logic, Methodology, Philosophy of Science, 1962

vistic and the subjectivistic doctrine when carefully defined. For while the concept of objective or statistical probability is singularly definable as related to frequencies in mass phenomena, the subjective - or personal one - encompasses two versions, one of which represents the "actual" degree of belief while the other - the "rational" degree of belief. Or, take von Wright's [8] epistemological inquiry into subjective probability: his contention is that whereas such probability is not definable in Ramseyan terms, a conception of probability does lend itself to defining in a way which combines some features of a subjectivist with some of an objectivist view of this notion).

Now, even all this granted, the school we are concerned with would stress that inasmuch as the objectivistic concept presupposes an objective characteristic - measurable approximately by means of frequency of mass phenomena - it lends itself to employment to but some restricted class of planning; in fact a narrow class of planning problems only: in operational rather than current and long-range planning. As a rule in non-mass cases the probabilistic characteristics would be determinable either by expert opinion or some inductive rule such as for instance the principle of maximum entropy, or the principle of invariance which is itself a "core" of the old principle of insufficient reason. (Note that the principle of invariance is rejected a limine by leading representattives of the subjectivist interpretation, to name Ramsey, De Finetti, Savage).

---

8) G.H. von Wright, "Remarks on the Epistemology of Subjective Probability" in Nagel, Suppes, Tarski, Eds., op. cit.

(Parenthetically we may note that the concept of statistical probability would rest - in the view presented here - on a misunderstanding. A misunderstanding - inasmuch as it is concerned with determinacy rather than indeterminacy; to be specific - it concerns itself with the statistical species of determinacy).

To repeat then, for what is the prevalent focus of planning, it is the subjectivistic approach that is believed to be most congenial to the planner and important to both the theory and the practice of planning. This is so in particular where it develops within the framework of decision analysis. This relates specifically to the matter of preferences; in terms of this doctrine,when preference relation in decision-making, (under conditions of indeterminacy) satisfies certain axioms which characterize the coherence of this relation, then there exists a utility function, unique (with exactness up to a linear transformation) which induces this preference relation; and this function is one of mathematical expectation of utilitites, for various "states of nature" of subjectively probabilistic distribution on the set of the possible states of nature; and this subjective-probabilistic measure too agrees with Kolmogorov's axiomatics [9].

Incidentally, the essentially sceptical stand on the relevance of the "objectivistic" theories for planning does not affect the interest in the more recent direction of Kol-

---

9) A.N. Kolmogorov, in Voprosy Peredachi Informatsyi, no. 1, 1965 and ibidem, no. 3, 1969

mogorov's work in which random sequence is determined with the use of the apparatus borrowed from the theory of algorithms (Algorithmic approach is seen by Kolmogorov as the third alternative approach - the other approaches being the combinatorial and the probabilistic - in determining the notion of information quantity). The concept of random sequence is in Kolmogorov closely connected with the problem of quantifying information (as a matter of fact this concept - in the case of infinite sequences - is analogous to that of von Mises). Incidentally, the lines of approach of Kolmogorov - and of Martin-Löf attempting to build up a novel theory of random mass phenomena - are credited with a promise.

To conclude the few remarks one should perhaps note the strong emphasis on the crucial role of the Bayesian approach and methodology for planning as the frame within which the subjective-probabilistic conception progresses. It is the Bayesian theory that is seen to correspond to the logic of man-machine system on which contemporary plannning is bound to rely. For here the operational researcher can cooperate and make use of the objective - "explicit" and subjective information. It is the methods of Bayesian theory of statistical decisions that permit to apply the optimality principle through sequential decision-making.

2. Contribution from Theory of Choice-making under Conditions of Incomplete Informativeness

§1. A crucial ramification of - conceptual support for - theory of planning in conditions of indeterminacy is that of optimal choice-making with incomplete information.

The memoryless Markovian system is "naturally" congenial to the planner inasmuch as it takes as a point of departure, for his formulation, the situation as it exists at the opening of the plan period; that is, abstracts from the "prehistory". The inclination for sliding planning adds to the attraction of the Markov approach; indeed virtually all plan-oriented work in the field discussed accepts this approach. The range of models coming under this heading is too large to be fully presented here. Most of their problematics is reducible to the familiar question of optimally "stopping" the Markov chain. We may say that the range opens with the class where the solution is interpretable as that of a dynamic programming (here again there is the common doctrine of abstracting from the past beyond the neighbouring stage; and the stoppage problem is concerned with the chain formed by the moments of appearance and of the "leads" i.e. "objects" superior to all antecedents, with the number of steps limited and a final number of states. In that case the quasi-game's value - the mean value of the criterion when optimal strategy is followed and the strategy itself are computed from the terminal instant - backwards.

§2. The broad class of problems considered in Arkin/ Pressman/Sonin [10] as the umbrella construct, the class "A", is relatively uncomplicated. We have there a finite, known set of n "objects", P, and the consideration process is cost-

---

10) V.I. Arkin, E.L. Pressman, I.M. Sonin, Ekonomika i Matematicheskiye Metody, no. 3, 1975

less; the procedure is constrained by the proviso of no-return to objects previously considered; there exists some k^*_n such that optimal is strategy of k*-threshold type meaning that the first stage takes time k^*_n-1, the second - a random time from k^*_n to the moment of appearance of an object better than the standard one. As indicated the average time taken by second state equals that of the first stage (approximately, that is n/e). The solutional framework is broadly speaking that of dynamic programming.

Of interest is within the "A" the case with choice-making carried out simultaneously by several agents. Here we have in particular the problem "A^m_n" and its variant $A^m(\nu,\varphi)$, on which - later on.

Speaking generally in A^m_n we have m agents and n choice "objects" - the agents receive no information about others' behaviour, and the objective of each of them is to maximize the probability of "stopping" first at the best object. What results in substance is an m-person symmetric game. Its solution is assumed, on familiar lines, to be the finding of the Nash equilibrium points in the usual sense; i.e. collection of strategies such that shifting to another strategy is unfavourable to a player whereever the others refuse to change their own strategies.; we have then a collection of strategies termed the point of ε-equilibrium where such a shift in strategy would yield a gain in payoff not greater than ε.

Where n is very large we have a variant: some "limiting" quasi-game on the "**interval**" [0,1] (which can be treated as a particular case of the game problem $A^m(\nu,\varphi)$. Over that interval **there** are m identical independent Poisson flows with variable intensity $\nu(x)$, $0 \leq \nu(x) \leq \tilde{\nu}$. Say, player j, with j=1 ... m, observes the j-th flow of events. Suppose at the moment x the sequential event appears, and the game is not concluded by that time; the alternatives appear then as open to the player - to continue observations or to stop; in the latter situation he is the "victor" with probability $\varphi(x)$ and with complementary probability $(1-\varphi(x))$ he drops out of the game, which is however not reported to his co-players. Where $\nu=1/x$ we have the case of the "limiting" problem for A_n^m. As to $A^m(\nu,\varphi)$ it is thought of as the mathematical statement of the situation with the following three features. Firstly, each player can reach some defined solution, associated with some payoff, only at some random time instants; secondly, in the absence of other participants the player's payoff would rise over time; thirdly, reaching a decision by a participant at time instant x reduces the payoff of those who have not done so by that time instant, and it does not influence the payoffs of remaining ones. It is established that for the problem $A^m(\nu,\varphi)$ there exists a unique equilibrium point formed of "threshold" strategies. The value of the root appears as a root of a certain equation.

To restate the main characteristics of the A_n^m and of the $A^m(\nu,\varphi)$, for the m-th player the possible moments of decision making - of the appearance that is of the "leads" in the former and of the Poisson events' flow in the latter - are independent of either the observations by, or behavior of the remaining players. The link between participants materializes exclusively through their payoffs, in other words, gains from the stopping. Essentially, then, the finding of optimal strategy for the m-th player reduces to the solution of the optimal stopping of the Markow chain - the same as in the absence of remaining participants with a changed payoff function. Denote $\varphi(x)$ the payoff in the absence of remaining players; then with the fixed strategies $S=(S^1 \, .. \, S^{m-1})$ the payoff function of the m-th player will be $g(x|S)=\varphi(x) \prod_i \alpha_i(x|S^i)$ where $\alpha(x|S)$ is the probability that the player who applies the strategy S will not be declared "victor" up to the moment x in $A^m(\nu,\varphi)$, that is in the absence of other players.

The reasoning is this. Assume there exists an equilibrium point formed of identical z*-threshold strategies. Then the treshold **according to the basic equation for the game's value**

$$S(x) = \max\{f(x), TS(x)\}$$

must be a point where gain from stopping equals that from continuing. However, the latter gain from continuation is equal for all the players, implying that it equals the probability of at least one of them being declared the "victor", divided by m. The probability that nobody will be declared victor if each uses z*-threshold strategies equals the m-th degree of probability that the player using the z*-threshold strategy will not be declared the victor in $A^m(\nu,\varphi)$. It follows that the latter probability is $1 - \int_{z^*}^{1}\varphi(y)p(z^*,y)dy$, where $p(z,y)$ is the transition density of the Markow chain of the appearance of the events in the Poisson flow, whence z^* satisfies

$$\varphi(z) - \frac{1-(1-\int_{z}^{1}\varphi(y)p(z,y)dy)^m}{m} = 0.$$

§2. It has been established (by Pressman/Sonin in their paper on equilibrium points[11]) that this equation has a unique root and that collection of the z^*-threshold strategies is an equilibrium, and indeed - unique equilibrium,point. Furthermore, it has been shown (by Sonin[12]) for the problem A_n^m that for any $\varepsilon>0$, where the n is sufficiently large, the collection of $[nz^{*(m)}]$-threshold strategies is the ε-equilibrium point (the $z^{*(m)}$ is equilibrium point for the problem $A^m(1/x,x)$); for A_n^m there exists for some values of m and n the unique equilibrium point, for others several equilibria all of which, however, are concentrated over a small integer-valued interval.

Finally, for cases with large n it is asserted that firstly there exist numbers $k^*(n,m)$ and $l^*(n,m)$ such that if m-1 players use (k^*+1)-threshold strategies then the unique optimal strategy for the remaining player's strategy will be a k^*-threshold and if the m-1 players use k^*-threshold strategies then the strategy for the remaining one will be the (k^*+l^*)-threshold one. Secondly - $k^*(n,m)$ grows with growing n.

---

11) E.L. Pressman, I.M. Sonin in Teoriya Veroyatnostey i Yeye Primenenya, No. 4, 1975

12) I.M. Sonin, Kibernetika, No. 4,1973

Thirdly any equilibrium point is such that there is no need for a player to stop up to k* and he needs to do so after (k*+l*). Where l*(n,m) ≠ 0 there exists always an equilibrium point such that there is no need to stop up to k* and there is a need to do so after k*; the stopping at k* is necessary with some probability.

§3. A more complicated class among problems investigated in Arkin/Pressman/Sonin [13] is one with the *P* **unknown** ("B" problem). Here too the problem reduces to following up the moment of appearance of the leads - the stopping takes place where the moment of the lead's appearance belongs to the "stopping set" (where integers k .. m belong to that set while the k-1 , m+1 do not; the **interval** (k,m) **is termed** the "island" of stopping).

Under "B" if χ is a random number of objects in *P* and $\lim_{\lambda\to\infty} P\{\chi_\lambda/\lambda<x\}=F(x)$, **then** limiting for B is the problem of optimal stopping the Markovian chain with transitional density

$$p(x,y)=xy^{-2}(1-F(y))/(1-F(x))$$

and payoff function

$$g(x)=x(1-F(x))^{-1}\int_x^\infty s^{-1}dF(s).$$

If optimal state in the limiting problem is of k*-threshold type, then the λk*-threshold strategy is the approximation of the optimal strategy in the "up-to-threshold" problem (the exactness of the approximation is examined by Pressman and

---

13) V.I. Arkin, E.L.Pressman, I.M. Sonin, Ekonomika i Matematicheskiye Metody, No. 3, 1975, op. cit.

Sonin [14] in a 1972 paper), generalizing the problem of best choice of objects appearing at times 1,2, ... in random order with all n being equiprobable.

To anticipate further discussion, where in problem B the number of the objects in *P* is random but has a known distribution {p}, the transition probabilitites and the payoff are, respectively,

$$p(k,l) = \frac{k \quad \pi_1}{l(l-1) \quad \pi_k},$$

$$g(k) = \sum_{s=k}^{\infty} \frac{kp_s}{s\pi_k}, \quad (\pi_k = \sum_{s=k}^{\infty} p_s).$$

The troublesome aspect is here that, unlike A, in B there is no point in the space of states of the Markovian chain such as to make the starting point for computation of the game's value.

The characteristics of optimal strategy and results are re-tabulated from the papers for the three specific distributions, depending on a given parameter; the relation of $k^*/_\lambda$ is examined (the integers k,k+1 .... m) are members of the stopping set, but neither k-1 or m+1 is.

---

14) E.L. Pressman, I.M. Sonin, in Teoriya Veroyatnosti i Yeye Priminenya, No. 4, 1972

Distribution	Para-meter	Valuation of k^*_λ	lim s_λ
$A, p_n = 1$	$\lambda=n$	$\lambda/e<k^*_\lambda<\lambda/e+2$	$1/e$
Uniform on (m,n) $p_m=p_{m+1}=\ldots=$ $=p_n=(n-m+1)^{-1}$	$\lambda=n$	$\lim_{\lambda\to\infty} k^*_\lambda/\lambda=e^{-1}\max[e^{-1},\sqrt{a}]$, $a=\lim m/n$	$2/e^2(1-a)$ with $a\leq 1/e^2$, $\frac{\sqrt{a}}{e(1-a)}\log\frac{1}{a}$ with $a>1/e^2$
Poisson $p_i=\frac{\lambda^{i-1}e^{-\lambda}}{(i-1)}$, $i=1,2,\ldots$	λ	$\lim_{\lambda\to\infty} k^*_\lambda/\lambda=1/e$	$1/e$
Geometric $p_i=p(1-p)^{i-1}$, $i=1,2,\ldots$	$\lambda=1/p$	$\lim_{\lambda\to\infty} k^*_\lambda/\lambda=\gamma_o=0,17\ldots$, ($\gamma_o$, root of eq.) $\int_1^\infty e^{-\gamma s}(1-\log s)\frac{ds}{s}=0$	$\gamma_o\int_1^\infty e^{-\gamma s}s^{-1}ds$

Note 2 The planning-control system with incomplete information on the current state with the task "evading" may be exemplified by Zhelinin's construct [15]. The approach adopted for a nonlinear deterministic system reduces the control problem with limited information on the current phase-states to a game

15) Yu. N. Zhelinin "Ob optimalnom upravlenii pri nepolnoy informatsii" Doklady, USSR Academy of Sciences, 1971, Vol.199, No. 1, see for discussion Zauberman, Differential Games and other Game-theoretic Topics in Soviet Literature, 1975

problem with constraints on phase coordinates in a certain extended space.

The system's behaviour is described by

$$\dot{x}=f(x,u,v,\xi),\ x(t_o) = x^o \in X^o,$$

$$y_u(t)-\varphi_u(x,u,v,\xi,w_u) = 0,\quad y_v(t) - \varphi_v(x,u,v,\xi,w_v) = 0$$

$$I = \int_{t_o}^{T} f^o(x,u,v,\xi)dt + F[x(T)].$$

where $u(u_1,\ldots,u_r), \in U$, $v(v_1,\ldots,v_q) \in V$ are control vectors of the two players, pursuing and evading, respectively, P and E - the latter possibly conceived as "nature"; $\xi(\xi_1,..,\xi_p)\in\Xi$ - vector of disturbances; measurement vectors $y_u(y_{ui},..,y_{ui}),y_v(y_{vi},..,y_{vj})$ characterize, the players' information on the game's current state; $w_u(w_{ui},..,w_{uk})\in W_u$, $w_v(w_{vi},..,w_{wl})\in W_v$ are vectors of "inexactness" in information. The functional I appraises the game's quality; it is respectively minimized and maximized by P and E.

Vectors of phase-state appraisals are

$\hat{x}_u(t):\hat{x}_u^o,u,\hat{v}_u\ \hat{\xi}_u,\hat{w}_u$; $\hat{x}_v(\hat{t}):x_v^o,\hat{u}_v,v,\hat{\xi}_v,\hat{w}_v$

The appraisals of the functionals $\hat{I}_u$, $\hat{I}_v$ correspond to them respectively; these are determined by each of the players with exactness characterized by functions ρ_u and ρ_v. The lambda coefficients are determined from $\rho - \bar{\rho} = 0$ where the $\bar{\rho}$ is some given value; then the point-of-departue of game problem is stated $\dot{x}=f, \hat{x}_u=\hat{f}_u, \hat{x}_v=\hat{f}_v, \varphi_u-\hat{\varphi}_u = 0, \varphi_v-\varphi_v=0$,

$$I^{\bullet}_u = \min_{a_p} \max_{\beta_E} \hat{I}_u + \lambda^o_u \rho_u, y_u(t_o) = Y^o_u,$$

$$I^{\bullet}_v = \max_{a_E} \min_{\beta_p} \hat{I}_v + \lambda^o_v \rho_v, \; y_v(t_o) = y^o_v.$$

The symbols $\alpha_p \beta_p \alpha_E \beta_E$ denote values at the disposal respectively of P, E.

This game with constraints on phase coordinates and controls has possibly certain variants: one of them is the case where $\xi(t)$ is treated as a third player antagonistic to both P and E.

A possible approach is also this. We differentiate the relation $y(t) - \varphi(x,u,\xi,w) = 0$ and with the employment of the equation $\dot{x} = f(x,u,\xi)$, we have a differential equation which is being satisfied by $y(t)$; thereafter the game problem of control is reformulated in the space of coordinates of the vector $y(t)$.

3. Hierarchy for Reducing Uncertainty in Large System

In our introductory remarks I have singled out among sources of indeterminacy in planning an economic system's structural rigidities. As hinted at here before, remedies are being sought in structurally adjusting the system; essentially - in a build-up of a hierarchic decentralization. (In Western literature the issue of indeterminacy is generalized in the context by that of complexity [16)]; as put by Gottinger, hierarchy is a heuristic device to approach complexity of a large system: as a matter of fact the same author comes close to the conception of "indeterminacy" when contending that it is "uncertainty" associated with large scale that leads to unreliability in performance. The source is seen to lie in the distributional pattern of informational structure - which may call for some reshuffling mechanism - or a limited divisibility of information).

The prevalent opinion then is that large scale and its impact on the system's informativeness - rooted in certain characteristica of centralist circulation and/or collection and/or processing of information point to an alternative mechanism: one which would have an in-built "paralleling" of information processes.

Such being the motivation in designing the hierarchic decentralized models, the Soviet planning theory starts from the concept of "subordination". Thus in a study of a decade ago, by Krasovskiy and Moyseyev a system is defined to be hierarchic with such a property where the latter is expressible by

$$x = A_x y^1,..,y^N , \quad y = A_y z^1,.,z^n ,$$

The A_x, A_y are aggregation operators. Changes of the variables over time are governed by some equations which contain free functions-controls. The simplest case of a hierarchic system

---

16) H.W. Gottinger, "Complexity and Catastrophe", Working Paper 246 (1976) Western Nanagement Science Institute, University of California, Los Angeles

would be

$$x_i = \sum_{l=1}^{N_i} \alpha_{il} y_l^i ,$$

$$\dot{y}_j^{s_j} = \sum_{k=1}^{n_j} \beta_{jk} y_s^k + f_j^{s_j}(u_j^{s_j}),$$

$$i = 1,2,\ldots,n; \quad s_j = 1,2,\ldots,n_j; \quad j = 1,2,\ldots,m.$$

In such case the aggregation operator is simply the summation operator. The hierarchic system's objective functions would be then stated in terms of the hierarchically higher level; say, for explicitness, $I = \int_0^T F(x_1,..,x_n)\,dt$ which is to be extremized. Such presentation implies the possibility of eliminating the variables x_i by means of the aggregation operator x_i. In the course of this operation a $\Sigma_{j=1}^m y_j$-dimensional Lagrangean problem would emerge. As Krasovskiy-Moiseyev[17)] observed at the time, this formulation leaves open the key-issue which is defining the lower level's criterial functions, or in other words - their tasks. (The authors remarked at the time that the example by itself demonstrates the inevitability of combining rigorous and heuristic methods; that indeed such combining would appear to be one of the fundamental characteristics of large systems).

Here we note in particular Soviet mathematicians'work on systems where hierarchic interaction is organized in a differential game.

---

17) N.N. Krasovskiy, N.N. Moiseyev, Tekhnicheskaya Kibernetika, No. 3, 1967

We may first refer to Fatkin as representing the studies which rely on the more standardized antagonistic game [18]. As in Krasovskiy et al the basic concept is that of subordination. The system - in conventional notation - has the form

$$\dot{x}_i = f_i(x_j, u_r), \quad L = L(x_i(T)) \quad (i,j=\overline{1,n};\ r=\overline{1,m}).$$

$$F_{w1}(x,u,t) \leq 0, \quad F_{w2}(x,t_o) \leq 0, \quad F_{w3}(x,T) \leq 0,$$

$$(w_1=\overline{1,\Omega_1};\quad w_2=\overline{1,\Omega_2};\quad w_3=\overline{1,\Omega_3},\quad x=\{x_i\},\quad u=\{u_r\}).$$

The functional L is the extremand. It is assumed that that where x_j enters n into the r.h.s. of the equation for x_i the coordinate x_i is subordinate to x_j. Whence the mapping rule

$$x_i \in \Gamma x_j, \quad x_j \in \Gamma^{-1} x_i \Longleftrightarrow \frac{\partial f_i}{\partial x_j} \neq 0, \quad \dot{x}_i = f_i(x_j),$$

Γ^1 and its inverse denote sets of coordinates which are respectively hierarchic subordinates and superiors. Like several other students in both the West (cf on them an instructive study by Warfield [19])) and the East,Fatkin resorts for description of coordinates to a finite directed graph (Berge's familiar conceptualization and terminolgy). The hierarchic structure is a graph with vertices placed at levels such that each of them ensures subordination to levels above and - in particular - to an overall majorant.

---

18) M. Fatkin, Avtomatika i Telemekhanika, No. 10, 1973

19) op. cit. See also for exercise in a graph approach and a good discussion of some structural problems in complex systems under hierarchic control Richeton's "Analyse Structurale des Systèmes Complexes en vue d'une Commande Hiérarchisée," Université de Toulouse, doctoral thèsis No. 674/1975

More specifically, the structure is described by the following relationships

(a) $\exists i \in [1,n] (x_i \in \Gamma^s x_i)$ $(s=1,2,\ldots,\leq n)$.

(b) $\exists i_\circ \neq i (i, i_\circ \in [1,n]) : (\Gamma^{-1} x_i = \emptyset, \Gamma^{-1} x_i = \emptyset)$

$$\exists i \in [1,n] (\Gamma x_i = \emptyset),$$

$$\forall i \in [1,n'] (\Gamma x_i = \emptyset) \Rightarrow n' \leq (n-1)$$

(a) indicates that for the degree of the mapping s=1 the i-th vertex has a loop; for s equal at least 2 - the graph possesses a contour with s vertices; (b) indicates absence of a majorant, (c) reflects the finitness of the multigraph and (d) - connectivity of the multigraph.

The system's hierarchic structure as graphed is then "translated" into the differential game for which it is being proved by Fatkin that

$$\max_{w_I^g, w_I^S} \min_{w_{II}^g w_{II}^S} \quad L = \max_{w_I^S} \min_{w_{II}^S} \left[\int_{t_o}^{T} \varphi_o(x^S, w_I^S, w_{II}^S) dt + \right.$$

$(g=\overline{1,G_s}; s=\overline{1,(S-2)})$

$$\left. + \left(\sum_{g=1}^{G_{S-1}} \max_{w_I^g} \min_{w_{II}^g} J^g + \left(\sum_{g=1}^{G_{S-2}} \max_{w_I^g} \min_{w_{II}^g} J^g + \left(\ldots + \left(\sum_{g=1}^{G_1} \max_{w_I^g} \min_{w_{II}^g} J^g \right) \ldots \right) \right) \right) \right]$$

Here s stands for the level, $s=\overline{1,S}$; g - vertex at a given level, thus $g=\overline{1,G_s}$. W is a vector function with components being elements of three matrices U^g, A^g , B^g; of the latter matrix A^g is one of control functions, matrix B^g - of constraints, U is a diagonal matrix of constants. When the rows of these matrices are partitioned into two components, one minimizing and the other maximizing the functional we have the two parts subscripted respectively by II and I. $J^g = \int_{t_o}^{T} \varphi\, dt$, and $\dot{x}^s = \varphi(x^s,u^s)$. The sequence in determining the optimal strategies (and phase coordinates) follows that of the parentheses in the maxmin.

While Fatkin and some others have been working on hierarchy with the antagonistic games a school of thought has turned to the nonantagonistic type. (Games with nonconflicting interests). It is associated with Germeyer and his associates.

To take the simplest case we consider in Germeyer the type of game [20] of two players α,β with respective strategies x and y. The α player is entitled to the first move: selects x and communicates it to β who accordingly determines his $y = \hat{y}$:

$$\Phi(x,\hat{y}) = \max \Phi(x,y)$$

the $\hat{y}$ being function of $\hat{y}(x)$.

Next α adopts his strategy of best guaranteed result

$$\max_{x \in X} F(x,\hat{y}(x)) = F(\hat{x},\hat{y}(\hat{x})).$$

We may take that $\hat{y}(x)$ is single-valued. But α can notify to β his choice as a function x(y) with a possible meaning of a positive (negative) reward (penalty function).

Then β chooses the element y^* from condition

$$\max_{y \in Y} \Phi(x(y),y) = \Phi(x(y^*),y^*)$$

here the element y^* will be an operator

$$y^* = y[x(y)].$$

Now, for α the decision-making entails the choice of a

---

20) as restated in Moiseyev, reference below

function $x^*(y)$ which maximizes the functional

$$F(x(y^*),y^*)$$

Next, a comparison with "antagonistic" games, say

$$\Phi(x,y) = - F\ (x,y),\ y \in Y.$$

Suppose α can influence the structure of β's constraints: say he can replace Y by $Y_1 < Y$ to his gain since

$$\max_x \min_{y \in Y_1} F(x,y) \geq \max_x \min_{y \in Y} F(x,y).$$

But in a Germeyer game α may find it to be to his benefit to <u>enlarge</u> rather than reduce the set of strategies admissible to β.

<u>Remark</u> A way of handling indeterminacy by means of game corresponds to Germeyer's systematization of game conceptions[21]. It encompasses situations with the system's uncontrollable elements of three categories, i.e. those with (1) random elements determinable by laws of distribution; (2) elements belonging to "nature" such that only their domain of change is known, (3) elements in control of "reasonable", active "adversaries" whose objectives in action may be known either inexactly or not at all.

The system's states are taken to be adequately described by means of phase coordinates - the subsystems' efficiency criteria W_i, $i=0,1,\ldots$; the objective is taken to be maximization of W_i. Denote x_i controllable elements of the i-th subsystem; then the system's states are described by $W_i(x_o,x_1,\ldots,x_n,y_i,\lambda_i)$, $i=0,1,\ldots n$. The conditions $x=(x_o,x_1,\ldots,x_n) \in X$ reflect the possibilities of varying the controllable elements x_i which in a

---

21) Yu.B. Germeyer, <u>Igrovyie Kontseptsii i Issledovanye Sistem</u>

general case are interconnected; the arguments y_i indicate factors exogenous to the system (uncontrolled by it), and the λ_i parameters of an efficiency-criterion type which may remain in general indeterminate or unknown to the system's "observer", or perhaps reflect the "indeterminacy of desires" of the i-th player (the "observer" in the general case, rather than being an umpire, is "taking sides" in the game). The presence of the "unpurposed" y_i and λ_i, as well as the "observer's" belonging to one side makes the game "untraditional"

To summarize the line of advance. It is tenable that the theory of nonantagonistic game has generalized the formalism of the investigation in hierarchically controlled systems and also widened it. The claims seem to be warranted at least on two points. Firstly because the theory has sought closer links with information theory. Secondly because it has at least initiated the formal inquiry into the hitherto rather neglected problem of the build-up of a hierarchy (see our reference to Warfield on p. 27).

We single out the school which has focused on group properties of controlled dynamic systems; in particular one bears in mind the "natural" concept of divisibility possessed by the group. (In fact in our submission the analysis could be fruitfully carried out by employing the construct of semigroups (cf. Clifford-Preston [22]) for a rigorous discussion in Western writing of potentialities of the group and semigroup formalism in handling the problem of system decomposition see Gottinger [23]).

---

22) A.H. Clifford, G.R. Preston, <u>On Archimedean Semigroups in The Algebraic Theory of Semigroups</u>, Providence 1961, pp.131 ff

23) H. Gottinger, "Complexity and Catastrophe", <u>op.cit</u>.

That line of approach has led at the same time to a new handling of the time-honoured (and never satsifactorily resolved) matter of a system's aggregation (cf. my Aspects of Planometrics 1967, also here para 1, ch.V). The problem of aggregation has an obvious impact on the degree of determinacy.

[On the trends in the approach to aggregation as hitherto developed we may quote a passage form an illuminating study by Vakhutinskiy, Dudkin, Khomyakov [24] " ... having passed the stage of using means containing inevitable biases and attempts to reduce them with the aid of methods of classical aggregation, mathematical economics arrived at the construction of methods of iterative aggregation that made it possible to obtain values of aggregated and detailed indices of a plan with any specified accuracy ... . The methods of classical aggregation, in turn, from a tool for reducing the biases in the mean planning indices, become methods of speeding up processes of calculating the aggregated and detailed indices in planning schemes that are based on processes of iterative aggregation. "

One may note that in addition to the "substantive" aggregation the planner faces the problem of "information aggregation" in the,quite usual, situation where he obtaines disparate informations and has to reach some single probability; this is a new area of study; thus see Winkler and Murphy [25] where the two methodologies are considered purely judgmental and based on Bayes's theorems.]

---

24) I.Ya. Vakhutinskiy, L.M. Dudkin, V.A. Khomyakov, in Metron, 31.XII., 1974

25) R.L. Winkler, A.H. Murphy, in IEEE Transactions on Systems, Man and Cybernetics, Vol. SMC-3, No.2, 1973

§3. In Pavlovskiy distinctly original approach [26] we are considering the dynamic systems

$$\frac{dy^i}{dt} = f^c\ (t, y^1, .., y^n, u^1, ..., u^r),\ i = 1, .., w \qquad (1)$$

y^i-phase variables; u^α, α =1,..,r controls, piecewise continuous functions of time.

Set of collections $u^1(t), .., u^r(t)$ denoted U, and a closed domain in r-dimensional space

$$U^\sigma (u^1, ..., u^r) \geq 0,\ \sigma = 1, .., \chi \qquad (2)$$

is termed <domain of controls .>

System (1) is defined to admit regular aggregation of order n-m, w.r. to phase variables and of order r-s, w.r. to controls, if

$$\exists z^k = I^k (t, y^1, ..., y^k),\quad K = 1, .., m \qquad (3)$$

($m<n$, and rank of mx $\| \partial I^k / \partial y^i \|$ equals m),

$$\exists v^\beta = v^\beta (t, u^1, .., u^r)\ .\ \beta = 1, ..., s \qquad (4)$$

($s \leq r$, and rank of mx $\| \partial v^\beta / \partial w^k \|$, equals s),

$$\exists \varphi^k (t, z^1, ..., z^m, v^1, .., v^s) \text{ such that} \qquad (5)$$

relations $dz^k/dt = \varphi^k (t, z^1, .., z^m, v^1, .., v^s)$ are (6)

identically satisfied w.r. to (3), (4) and (1).

For the purposes of the reasoning - with the view to operating controls encompassed in (1) - a special group ($\hat{G}$) is being defined - such as to permit treating these controls "as if" they were constants. An infinitesimal operator is then devised:

---

26) Yu. N. Pavlovskiy, Zhurnal Vychislitelnoy Matematiki i Matematicheskoy Fiziki, No. 5, 1971

$$X = \xi(t,y,u)\frac{\partial}{\partial t} + \eta^i(t,y,u)\frac{\partial}{\partial y^i} + w^a(t,y,u)\frac{\partial}{\partial u^a} \qquad (\S)$$

Theorems fundamental for the results of the inquiry lend themselves to this summarizing statement:

The <u>necessary and sufficient condition</u> for the system (1) to admit of "<u>regular aggregation</u>" of order n-m, w.r.:

I) to phase variables and of order r-s, w.r. to <u>controls</u> is that there exists in group G, admitted by (1), a <u>sub</u>group $\bar{H}$ with operators (§) such that complete collection of their functionally <u>in</u>dependent invariants is definable as

$$I^i(t,y),\dots,I^m(t,y),V^1(t,u),\dots,V^s(t,u)$$

II) to <u>phase-variables</u> is that there exists a complete system of linearly <u>in</u>dependent operators of form:

$$Z_a = b_a^i(t,y)\frac{\partial}{\partial y^i} \qquad \alpha = 1,\dots,n-m \quad (*)$$

such that

$$(X_o, Z_a) = h_a^b(t,y,u)Z_b \quad (**)$$

Now consider the full set S' of complete systems with linearly independent operators

$$Z_a^{(*)} = b_a^i(t,y)\frac{\partial}{\partial y}, \quad a = 1,2,\dots,n-m = R, \quad R \leq n$$

Two such systems Z_a and Y_b of S' consisting of the same number R of operators are considered to be <u>equivalent</u> ($Z_a \sim Y_b$) if

$$Z_a = g_a{}^b Y_b$$

Systems Z_a and Y_b of S' for which the last eq.holds will yield the same method of regular <u>aggregation</u>. The equivalence relation introduced is symmetric and transitive. Thus set S' splits into a set S of equivalence classes.

Then we introduce into set S' a precedence relation: system $Z_a, a=1,2,\ldots,R$, precedes a system Y_b, $b=1,2,\ldots,R_1$, $Z_a \leq Y_b$ if $R_1 \geq R$ and $Z_a = g_a^b Y_b$, $a = 1,2,\ldots,R$, $b = 1,2,\ldots,R$, consequently $Z_a \leq Y_b$, $Z_a \sim Z_a'$, $Y_b \sim Y_b'$ entails $Z_a' \leq Y_b'$.
This precedence relation introduced into S' induces a similar relation into S: this relation transforms S into a partially ordered set (the relation crucial for hierarchy).
With the property established - precedence relation, that is, as above - the partially ordered set S appears to be a lattice. This lattice is the "phase-organizational structure of the dynamic system" or "structure" S of system (1). With finite substructures of the structure S of (1) we may now link organizations controlling processes defined by (1). It is these finite substructures which reflect what is intensively termed "structure of an organization"; and they are termed "natural hierarchical organizations."

Note The study of informational processes in hierarchically controlled systems has been carried out - by some students - also within the framework of learning systems (see Appendix §1). Among alternative mechanisms considered one of relatively greater realism is that of "unreliable teacher": the complicating element is the question of convergence at each level: there arises also the more general issue of the learning performance of multilogarithm systems (cf. inter al. Fu [27]) and Pugachev [28])).

---

27) K.S. Fu, in IEE Transactions in Automatic Controls, April 1970

28) V.S. Pugachev, Engineering Cybernetics, November-December 1967

§ 2. Formalism favoured is that of multiperson game or to be more specific a metagame where the metaplayer - the planning center - defines the rules of the game between the participants and thereby determines the control regime. The regime would be such as to ensure the reaching of an equilibrium point which maximizes the overall (the center's) payoff. This precept comes up against the difficulties [29] in selecting the type of solution as between, inter al., that of the Nash, of the Pareto optimum of the maximum-guaranteed-result equilibria. It would appear that, theoretical considerations apart, results obtained in experiments in " gaming " account for distinct favouring of the Nash-type equilibrium point. The theoretical considera-tion supporting this favour are broadly this: the tactics of indi-vidual elements are influenced by technical difficulty in coping with the complexity of the game; this is in particular due to the lack of knowledge of other participants' objective functions; and it is presumed that each of the elements tends to increase its payoff with respect to its variable. In the presence of rather general conditions the tactics is taken to bring about the convergence of "social behaviour" toward the Nash point [30].

[As Blaquière et al. note [31] the established dychotomic categorization of nonzero game into the cooperative and noncoope-rative classes with the largely associated concepts of, respective-ly, the Pareto and the Nash equilibria, does not hold in the area of our interest. The trouble is that in reality, specifically in economic reality one is confronted with a whole gamut of inbetween

---

29) N.V. Burkov, V.I. Opoytsev, Avtomatika i Telemekhanika, No.1,1974

30) A.V. Malishevskiy, Avtomatika i Telemekhanika, Nos. 11,12,1972

31) A.Blaquière, L.Juricek, K.E. Wiese, in A.Blaquière Ed., Topics in Differential Games, 1973

situations. Hence the necessity of, and difficulty in, expliciting assumptions as to the interplayer communications.

Now, it is well-known that so long as it is sure that a player's partners or rivals will play Nash equilibrium, it is to his advantage to stick to the same. (Otherwise he may be well advised to minimize his cost against the worst possible strategies others may adopt). And - we would suggest - it belongs to the structure of hierarchy that security is provided for on this count.]

§3. The conception of mitigating indeterminacy by means of a hierarchic structure (implying a degree of "decentralization" - see page) is pursued in Moiseyev's [32] game theoretic inquiry. We have a dynamic system

$$\dot{x}_i = f_i(x_i, u_i, v_i, t), \quad i = 1,\dots,N \qquad (1)$$

s.t. constraints on phase z control variables

$$\forall^t x_i(t) \geq 0, \ \forall^t \{u_1,\dots,u_N\} \varepsilon G_u \qquad (2)$$

$$\forall^t \phi_i(x_i, v_i, w_i) \geq 0 \qquad (3)$$

associated functionals $\dot{I}_i$:

$$\dot{I}_i = \dot{I}_i(x_i, v_i, w_i(x_i, t)) \qquad (4)$$

and "general function", $\dot{I}$

$$\dot{I} = \dot{I}(x_1,..,x_N, u_1,..,u_N, w_1,..,w_N, G_u) \qquad (5)$$

Mathematical and economic interpretation: x_i - vector-valued function (fixed and circulating capital stock of plant i)

u_i, v_i - vector-valued functions (resp. "external" and internal"

---

32) N.N. Moiseyev, Kibernetika, No. 6, 1973

input of resources): resp. control excercised by firm ("centre") and by "plant." G_u - a set; $x_i(t) \epsilon E_{ni}$, $u_i(t) \epsilon E_{mi}$; $w_i(t)$ reward (positive, negative); $w_i(x_i,t)$ thought of as <u>synthesist control</u>, taken from E_{ki} . $\dot{I}_i$ - performance criterion (possibly: profit)

<u>Description of game</u>

Denote firm and plant resp. player α, β_i. Focus on sequence of moves - in enforcing hierarchic precedence.

First move by α: notifying to $\forall\beta_i$ functions $w_i(x_i(t))$ and $u_i(t)$ and set G_u. [At t = O vector $x_i(O)$ fixed, also notified to all β_i].

<u>Behavioural hypothesis</u>: β_i's choose functions $v_i(t)$ s.t. $\dot{I}_i \to max$; α chooses functions u_i, G_u s.t. $\dot{I} \to max$.

T-period of game (fixed or free). Same applies to phase variable $x_i(T)$.

Take reformulation of the original problem - one that approximates solution - yet its parameters and functions are <u>no longer constrained</u>.

Here α (again <u>in the first move</u>) notifies $\forall\beta_i$ of the $u_i(t)$ and $w_i(x_i,t)$ s.t.

$$I^* = \int_O^T F^*(x_1,..,x_N, u_1,..,u_N, w_1,..,w_N)dt \to max \quad (6)$$

β_i adopts strategy $v_i(t)$ s.t.

$$I^* = \int_O^T F_i^*(x_i,v_i,w_i)dt \to max \quad (7)$$

* <u>accounts for penalty functions permitting to discard constraints</u>.

Approach to solution of reformulated problem rests on necessary conditions for extremum as familiar from optimal control theory:

Start with problem facing β_i: Here the Hamiltonian is written

$$H_i = [\lambda_i, f_i(x_i, u_i, v_i, t) + F^*(x_i, v_i, w_i)] \tag{8}$$

Now, considering that β_i was notified at the start of $u_i(t)$, $w_i(x_i t)$; and - that * has no constraints on $u_i(t)$, the necessary conditions for max of the Hamiltonian can be stated in the form

$$\frac{\partial H_i}{\partial v_i} \equiv (\lambda_i \frac{\partial f_i}{\partial v_i}) + \frac{\partial F_i^*}{\partial v_i} = 0 \tag{9}$$

or restated, to read

$$\Psi_i(\lambda_i, x_i, v_i, t) = 0 \tag{10}$$

which is a vector-valued condition (with dimension of vector-valued function $v_i(t)$).

Lagrange multipliers λ satisfy the vector equation

$$\dot{\lambda} = -\frac{\partial H_i}{\partial x_i} \equiv -\lambda_i \frac{\partial f_i}{\partial x_i} - \frac{\partial F_i^*}{\partial x_i} - \frac{\partial F_i^*}{\partial w_i} \frac{\partial w_i}{\partial x_i} \tag{11}$$

Since r-h-end of tragetory **is unconstrained we assert:** < to be optimal for β_i it is necessary for function $v_i(t)$ to satisfy condition (10), where, in turn, λ_i, x_i meet boundary conditions $x_i(0) = x_{io}$, $\lambda_i(T) = 0$ (the x_{io} being known) as well as the evolution equation (1) and Lagrange equation (11) >.

Because, in addition to the standard-form control function $u_i(t)$, α operates also the functions $w_i(x_i, t)$, α will be expected to try to solve a synthesis - kind of problem for which we have no simple necessary conditions.

In the theory of hierarchic control systems the reward function is often given the shape

$$w_i(x_i,t) = a_o^i + a_1^i x_i + a_2^i x_i^2 \qquad (12)$$

or expressed as

$$w_i(x_i,t) = b_1^i x_i + b_2^i (x_i - x_i^*)^2 \qquad (13)$$

(terms-functions $b_1^i(t)$, $b_2^i(t)$, $x_i^*(t)$ easily definable in terms of $a_o^i(t)$; economic translation: $x_i^*(t)$ - the plant's output target, $b_1^i x_i$ (where $b_1^i \geq 0$) - is the reward and $b_2^i(x_i - x_i^*)^2$ penalty for below-the-target performance. The player's α optimal control problem is here:

find $x_i(t)$ and $\lambda_i(t)$ satisfying equations (1) and (11) and functions $v_i(t)$, $u_i(t)$, $a_o^i(t)$, $a_1^i(t)$, $a_2^i(t)$ whether maximize functional (6) and satisfy constraints (10) and boundary conditions $x_i(0) = x_{io}, \lambda_i(T) = 0$.

Generalization

Consider situation with dynamic process described by equations

$$\dot{x}_i = f_i(x_i, u_i, v_i, \xi_i(t), t), \quad i = 1,.,N$$

Notation as before with added function à priori known as

$$\xi_i(I) \varepsilon G_{\xi i} \quad , \quad G_{\xi i} \text{ a set.}$$

This is taken to let the only information available to center (it becomes known to the plant in the course of operation). Postulated, however, possibilities of à priori estimation.

We retain the hypotheses as to the maximand and degree of information . The center takes the first move, i.e. communicating to plant the value of "external" resources $u_i(t)$ and reward function $w_i(x_i)$. The plan (the plant can make) at this stage is of a <u>synthesis</u> type. It schedules its controls and its trajectory respectively as:

$$v_i = v_i(\xi_i, u_i, w_i, t) \quad , \quad x_i = x_i(t, \xi_i, u_i, w_i).$$

The overall functional (of the firm) accordingly (in an analogous fashion) **is written:**

$$I = I[x_1(t, \xi_1, i_1, w_1), .., x_N(t, \xi_N, u_N, w_N), u_1, ..., u_N, w_1, .., w_N, G_u]$$

Likewise, we obtain for the firm's control <u>a synthesis</u>

$$u_i = u_i(t, \xi_1, .., \xi_N), \quad w_i = w_i(t, x, \xi_1, . \xi_N), \quad G_u = G_u(t, \xi_1, .., \xi_N).$$

A priori <u>estimation</u> of ξ_i (<u>à priori</u> unknown) taken to be "natural" that this be a <u>guaranteed result</u>. And guaranteed result is here a number calculated by following this rule:

$$I^* = \min_{\xi_i \varepsilon G_{\xi i}} \max_{u_i, w_i, G_u} I(x_1(t, \xi_1, u_1, w_1), ..., x_N(t, \xi_N, u_N, w_N),$$
$$, u_1, ..., u_N, w_1, ..., w_N, G_u)$$

Now, turn to a system with a single-centre ("centralist") controls, i.e. <u>not only</u> u_i, w_i, G_u but v_i as well are determined by the firm.

The guaranteed result for a system completely centralized is then

$$\hat{I} \max_{u, v, w,} \min_{\xi_i \varepsilon G_{\xi i}} I(x_1, ..., x_N, u_1, ..., u_N, w_1, .., w_N)$$

The relative performance under "decentralization" (as formalized) and centralization is appraised by comparison

$$\hat{I} - I^{*} \gtreqless 0$$

where $\Delta = \hat{I} - I^{*} > 0$ hierarchy deteriorates / improves performance

$\Delta < 0$

$$\left.\begin{matrix} \Delta = \hat{I} - I^{*} > 0 \\ \Delta < 0 \end{matrix}\right) \text{hierarchy} \left\{\begin{matrix} \text{deteriorates} \\ \text{improves} \\ \text{performance} \end{matrix}\right.$$

The Δ is taken to be the measure of the benefit from bringing in hierarchy into the system, and to this extent - from decentralization.

There is in Moiseyev a hint on the possible stochastic reformulation of the problem of hierarchic decentralization; it has not been attacked in his illuminating inquiry.

It seems well tenable that the problematics of hierarchic structure is still a theoretically underdeveloped area. It has been defensibly remarked by Varayia that the mathematical theory of hierarchic form has not crystallized (though it may be not justified to maintain, as he does that there has been as yet no serious attempt to explain why hierarchic forms are worthwhile). Nor would we share the surprise of another author in the field that it has not been possible to find a work that deals explicitly with methods for forming hierarchies. But we do share his point that it seems "impossible for one person to grasp the breadth and intensity with which hierarchies pervade science in society" (J.N.Warfield) 33)

33) P. Varayia, "Trends in the Theory of Decision-making in Large Systems", Annals of Economic and Social Measurements, No. 4, 1972; J,N. Warfield, IEEE; Vol. SMC-3,1973

Remark 1. In the context of the discussion of Moiseyev's paper we may draw the student's attention to Gottinger's (as it appears pioneering) study of an information-saving stochastic decomposition for a hierarchic system. [34] Unlike other authors' (treating separately the aspects of a system's hierarchic build-up and its decomposition) Gottinger's paper handles them jointly which deepens the analysis of their link-up. A restatement of the problem in terms of probability densities and measurement functions helps the insight into informational requirements and also the role of a coordinator in meeting them. What is postulated in Gottinger is a "complete decentralization" of decision making. Unlike Moiseyev then, Gottinger is not concerned with the question of its degree.

Remark 2. It may be noted that having suggested a method of measuring relative benefit from decentralization versus contralism Moiseyev contends [35] that the advances in productive potential, growing sophistication of technology, increasing role of the plan-element in modern economy, all tend to intensify the role of its centralist steering. And that while the optimum of "decentralization" is conditional upon several factors, progress in the technology of generating and processing information is bound to raise the level of centralism.

It is not only in the Soviet literature that the issue has been brought up by analysts. In the West it has been

---

34) H.W. Gottinger, in Angewandte Informatik, No. 1, 1976

35) N.N. Moiseyev, Kibernetika, No. 6, 1973

raised by students of the team construct. [A constract of relevance for the subject of systems working under indeterminacy; parenthetically its very concept with a definitionally postulated uniformity of preference scales is a kin to a system of mandatory planning; the Soviet student's lack of interest in it is rather surprising].

It is contended that the computational effort entailed in calculating an optimal rule for a team is considerably greater than in a centralized decision-making, with a single decision-maker that is. The finding (by Chong and Athans) has been generalized - by Varaiya's hunch - for a stochastic, a Gausssian linear quadratic situation: the supposition that decentralization would be still more effective where "enforced" by physical or instutional constraints. This supposition has given a stimulus to investigation of some specified situations (including Varaya's own) for maximizing a flow of commodity through a capacitated network where an agent is located at each node of the network; he observes the capacities and the flows in branches incident to that particular node and partially controls the flow of these branches; each agent sends messages to neighbours and updates decisions . While interesting, the results of such particularized designs they do not allow "distilling" some general - more abstract - theory.

It seems to us that studies à la Moiseyev and the work in the West alluded here permit to draw an intuitive conclusion that one is faced by a tradeoff between the two aspects: the informational and the computational: and (also intuitively) that more often than not it is the former, as a generator of indeterminacy, that outweighs the latter.

4. Plan-Modelling under Conditions of Uncertainty

§ 1. Very broadly models dealing with the problematics of planning under conditions of indeterminacy - or to be more specific, under conditions uncertainty in the more conventional sense - can be classified according to their logic in four groups.

One is devoted to analysis of some basic theoretical issues. Among these dominating is the relationship between the established results of deterministic optimal planning and the characteristics of the planning affected by uncertainty. The transplant from the sphere of determinism to that of stochastic seems to be a shortcut to the latter's theory. Dynkin's work has been understandably considered as leading in the area.

Next comes the theory of algorithms designed for the solution of a probabilistic plan problem. We take as representative here the work of Yudin (which in fact has an immediate nexus with the study of choices under indeterminacy, we discussed in ch. III.)

A third class is formed of fully-fledged models of a stochastic instrument such as could be employed as prototype in the planner's workshop. We choose Yefimov's design as a good prototype.

Models bringing into the stochastic plan-models some additional important features may be grouped in the last class.

Such feature would be in the first place the coordination of multilevel plans. With an eye to this feature we have selected Gadzhiyev's construct.

The representative constructs thus selected - and in this order - will be found in this chapter.

§ 2. The Dynkin [36] stochastic control-theoretic model of a planned economy - as it appears in his recent writings - has as its point of departure Gale's well familiar deterministic design. The broad framework is then this.

We have as given an arbitrary set Z - the space of states with the initial $z_o \varepsilon Z$; objective functions $u_t(z)$; the mappings a_t $(t=0,\ldots,N-1)$ which bring into correspondence to $\forall z \varepsilon Z$ a nonempty subject $a_t(z)$ of Z. To meet the existence conditions for an optimal plan it would be required that Z be a metric space, the u_t - upper semicontinuous and the mappings a_t - quasicontinuous. In addition it is postulated that Z be a closed convex set in Euclidean space, u_t and a_t - concave. Translated into elements of an economic system a_t is the state of technology. Moreover, we have in Z, as given, subsets $\mathcal{T}_1,\ldots,\mathcal{T}_N$, functions $g_1,\ldots,g_N$ and $h_o,\ldots,h_{N-1}$ the values of which are nonnegative ℓ-dimensional vectors. The set $a_t(z)$ is given as an ensemble of all $z \varepsilon \mathcal{T}$ for which $g_t(z') \leq h_{t-1}(z)$ where $g_t(z)$, $h_t(z)$ are respectively input and output vectors at time t in the state z. For $a_t(z)$ to be nonempty for any z it is sufficient that among g_t on $\mathcal{T}_t$ there should exist a null vector; and for a

36) E.B. Dynkin, "Nekotoryie vyeroyatnostnyie modeli razvivayushcheysia ekonomiki Doklady USSR Academy of Sciences 1971, Vol. 200, No. 3

to be quasicontinuous it is sufficient that the functions g_t, h_t be upper semicontinuous and T_t compact. For convexity of a_t it is sufficient that T_t be convex and g_t, h_t concave.

The stochastic element is brought in by a random parameter $s \varepsilon S$ to describe the state of technology or of the trade cycle with probability distribution of $s_1, \ldots, s_N$ known, and a_t, u_t-dependent on s^t, and measurable on s (measurable, that is, with respect to $F \times \ldots \times F$ where F is a fixed σ-algebra in the space S; measurability of a(s) means that for any $z \varepsilon Z$ the function $\rho(z\ a(s))$ is measurable (ρ being distance in Z)); further u_t is constrained from above.

For this stochastic situation the sequence of measurable functions $\zeta_t(s^t)$ (t=1,2,..,N) is a plan if for any values of s^t

$$\zeta_t(s^t) \in a_t(s_t, \zeta_{t-1}(s^{t-1})$$

assuming by definition $\zeta_o = z_o$. It is an <u>optimal plan</u> where $U(\zeta) = E\Sigma u_t(s^t, \zeta_t)$ max. It is asserted that assuming the existence of the optimal plan, $\zeta_t(s^t) = F_t(s^t, \zeta_{t-1}(s^{t-1})$, the function F_t which determines it can be found from the equation $v_t(s^t, z) = \Psi_t(s^t, F_t)$, the v_t, Ψ_t given by the recursive formula

$$v_{N+1} = \Psi_{N+1} = 0, \quad v_t(s^t, z) = \sup_{z' \in a_t(s^t, z)} \Psi_t(s^t, z)$$

$$\Psi_t(s^t, z) = u_t(s^t, z) + \int_S p_t(ds_{t+1} | s^t) v_{t+1}(s^{t+1}, z).$$

The $p_t(ds_{t+1}|s^t)$ is conditional distribution of s_{t+1} where s^t is known (for its existence the sufficient condition is that (S,F) be a standard Borel space). The maximum value of $U(\xi)$ is $Ev_t(s_t, z_o)$. This assertion has a proof by the standard method of dynamic programming which the formulae clearly express. It is also evident that where the $(s_1, \ldots, s_N)$ process is Markovian, the functions F_t would be dependent only on s_t (and z) but not on $s_1, \ldots, s_{t-1}$.

For any optimal plan $\bar{\xi}_t$ and any $\xi_t \in Z_t$ we have

$$E\Sigma[u_t(\xi_t) + (\pi_t, h_{t-1}(\xi_{t-1}) - g_t(\xi_t))] \leq \bar{E}\Sigma u_t(\xi_t)$$

where $\pi_t(s^t)$ are the dual prices. For these prices, assuming $\xi_t = \bar{\xi}_t$ and considering that $g_t(\xi_t) \leq h_{t-1}(\xi_{t-1})$, w.p. 1 we have $(\pi_t, g_t(\bar{\xi}_t)) = (\pi_t, h_{t-1}(\bar{\xi}_{t-1}))$.

It is then proved that for $\xi_t \in Z_t$

$$E[u_t(\xi_t) + (\bar{\pi}_{t+1}, h_t(\xi_t)) - (\pi_t, g_t(\xi_t)) \leq$$

$$\leq E[u_t(\xi_t) + (\bar{\pi}_{t+1}, h_t(\xi_t)) - (\pi_t, g_t(\xi_t))].$$

When the function u_t is strictly concave on z these conditions are equivalent to

$$u_t(\bar{\xi}_t) + (\bar{\pi}_{t+1}, h_t(\bar{\xi}_t)) - (\pi_t, g_t(\bar{\xi}_t)) =$$

$$= \sup_{t \in T_t} [u_t(z) + (\bar{\pi}_{t+1}, h_t(z)) - (\pi_t, g_t(z))]$$

with probability 1 and $\bar{\pi}_{t+1} = E(\pi_{t+1} | s^t)$.

The main conclusion appears to be that assuming "good" behaviour of the relevant functions (convexity-concavity), the main characteristics of the stochastic and, in particular, the Markovian optimal plan parallel those of a deterministic one.

[The model is germane to the fundamental elaboration of Dynkin's [37] model for controlled random sequences. Dynkin focus is on incomplete information: the problem is defined with respect to regular conditional probabilities - stochastic kernels such that, jointly with a control law, they determine a distribution on the sample law.

The theory of probablistic optimal control economic plan outlined above has been developed in subsequent writings of Dynkin and members of his school in particular I.B.Yevstigneyev. Cf. their contributions to the Symposium on mathematical economics at the Institute of Mathematics of the Polish Academy of Sciences.]

§3. By the latter 1970s Yevstigneyev evolved - as a contribution to the strategic planning under uncertainty - a stochastic analogue of turnpike theorem.[38]

In a briefest sketch the fundamental theorem and its framework can be presented as follows. We have a set T with elements which make pairs of n-dimensional vectors (x,y) and function u(x,y) defined on it. The former are production processes (x,y resp. consumption and output), T is technoloy; u - utility function. T is convex, compact, u(x,y) - continuous, concave.

---

37) cf. E.B. Dynkin, "Optimal programming and stimulating prices, in the stochastic models of economic growth" in J.Los,M.Los, Mathematical Models in Economics, Amsterdam-London 1974; also Teoria Veroyatnostey i yeye Primenenya, 1965. For an interesting equivalent model see Charlotte Striebel, Optimal Control of Discrete Time Stochastic Systems, 1975.

38) I.V. Yevstigneyev, in Matematicheskiye Zamyetki, No. 2, 1976

We have further a random process over: $t, \ldots, s_{-1}, s_0, s_1 \ldots$ in measurable space; they describe the economy's states. For T we have $T_t = {}_t(s^t), u_t = u_t(s^1; x, y)$.

A set of assumptions is adopted

1) function $U_t(s^t, z)$, equal to $u_t(s^t, z)$ for $z \in T_t(s^t)$ and infinity for $z \notin T_t(s^t)$, is measurable w.r. to (s^t, z)

2) $\exists q_t(s^t)$ such that $\exists q_t(s^t)$ is finite and
$|u_t(s^t, z)| \leq q_t(s^t)$ for $\forall t, s^t$ and $z \in T_t(s^t)$

3) $T_t(s^t) \subseteq \{z : \|z\| \leq C\}$ with C independent of t and s^t

4) the system is stationary this implying that process s_t is stationary and technology and utility are time-independent. It is proved however that the assumption is weakened for a stationarily expanding economy where technology and utility are reducible (with lambda positive) to:

$$T_t = \lambda(s^1) \ldots \lambda(s^t) T(s^t)$$

$$u_t = u(s^t, z \cdot [\lambda(s^1) \ \ldots \ \lambda(s^t)]^{-1}), \text{ where } \lambda(,) > 0$$

A programme $\zeta = (z_1, z_2, \ldots)$ is defined as stationary where z_t does not depend explicitly on t. If $\zeta = z\{t\}$ is stationary, then $F(\zeta) = Eu_t(z_t)$ is independent of t. It is proved by Yevstingeyev that among all programmes stationary on his definition $\exists \zeta = \{\bar{z}_t\}$ maximising F.

5) $u(s^t, z)$ is strictly concave w.r. to z. Follows that programme $\bar{\zeta}$ is unique; it is termed <u>turnpike</u>; and for $\forall N$ and $\forall y_0$ there exist only one optimal programme with length N and initial vector y_0

We then let ε be positive and let $\xi^N = \{z_1^N, \ldots, z_N^N\}$ $N = 1,2,\ldots.$ be optimal programme with initial vector ${}_o(s^o)$ nonnegative. And it is proved that if three further conditions are met viz.

6) all coordinates of $y_o(s^o)$ are uniformly distanced from zero,

7) $\exists$ vector $(\overset{o}{x},\overset{o}{y})$ independent of s^t and endowed with property $\overset{o}{x} < \overset{o}{y}$, $(\overset{o}{x},\overset{o}{y}) \varepsilon T(s')$ for $\forall s^t$

8) for some positive μ the set $\{z: \|z - \bar{z}\ (s')\| < \mu\} \subset T(s')$ for $\forall s^t$,

then firstly $\exists H$ such that $E\|z_t^N - \bar{z}_t\| \leq \varepsilon$ for $H \leq t < N-H$

and secondly where u is uniformly and strictly concave, $\exists L$ constant s.t.

$$p\ \{\|z_t^N - z_t\| \leq \varepsilon \text{ for } L \leq t \leq N-L\} \geq 1-\delta$$
$$t$$

The meaning of the first assertion is then that functions forming optimal programming can differ relevantly from $\bar{z}(s^t)$ in the metric of L_1 only at the start and/ or the end of the period of the plan-programme. The meaning of the second - a stronger - assertion is that optimal programme can depart from the ε-neighbourhood of turnpike with probability near one only at the start and the end of that period. That patently corresponds to the concept of a turnpike.

Of immediate significance for price-directed planning is another theorem to the effect that $\exists p(s^t) \geq 0$, an integrable function, with values in R^n, s.t. with probability one, firstly $z(s^t)$ maximizes $v(s^t,z) = u(s^t,z) + E(p(s^{t+1}, |s^t)\ y - p(s^t)x$ w.r. to $\forall z = (x,y)\varepsilon T(s^t)$; and secondly $p(s^t)\cdot(\bar{y}(s^{t-1}) - x(s^t) = 0$.

Patently the $p(s^t)$ is interpretable as price vector and the expectation of the difference $E(.)$ - as profit obtained from production (x,y) when valued at $p(s^t)$. Consequently the $v(s^t,z)$

is the sum of utility and profit in these terms. The two conditions state respectively that firstly it is along the turnpike that the reduced utility is maximized and secondly that the price of a resource which is not used is zero. The two conditions when satisfied insure the realization of $\bar{\xi}$.

§ 4. To reemphasize,the close analogy in Dynkin's findings with the familiar tenets from the deterministic theory is pinned on the good behaviour of the relevant elements of the system. Note in the context the Arkin/Levin recent work[39)] on necessary and sufficient optimality conditions for the reformulated Maximum Principle (generalizing A.A. Lyapunov's results [40)] on convexity of a vector valued integral where integration is carried out on all possible combinations of the relevant variables). And also - the Arkin/Levin application of this new apparatus in a design of plans - in "substantive", time and space dimensions - for a centralized economy with a global criterion (its constraints being formalized in the shape of operator inequalities). The significance of the findings on the existence of an optimal plan solution - its sufficient and necessary conditions - lies in the absence of any convexity assumptions. Thus the utility function is not assumed to be concave with respect to the control function; the technology functions are permitted to be nonconvex with respect to controls and even noncontinuous; and local constraints on controls - nonconvex and discrete. The assumed continua of sectors and regions forming

---

39) V.I. Arkin, V.Levin, "Varyatsyonnyie zadachi s funktsyami mnogikh peremennykh i model raspredelenya resursov", B.S. Mityagin Ed., Matematicheskaya e Ekonomika i Funktsyonalnyi Analiz, 1974

40) A.A.Lyapunov, "O vpolne additivnykh vektor funktsyakh" Izvestya USSR Academy of Sciences Mathemat. ser. No. 4, 1940

spaces with nonatomic measures (specifically Lebesgue measures) correspond to the disaggregate structure of the system with non-relevance of individual contributions. And it is the continuum of participants that allows to discard in both types of models analysed - the equilibrium and the optimization models - the "goodness" postulates for the curves. In the context of the crucial issue of an economy's controllability with prices derived from optimizations plans the results should deserve due attention. For, indeed, there is in Arkin/Levin the conjecture of legitimacy for an interpretation to the effect that, in the case of a "very large number" of an economy's agents, a value-term mechanism may be safely employable for an optimally steered resource allocation where the participants' "properties" such as their utility functions, local constraints and technology are arbitrarily designed. Certainly, the conjecture waits for a rigorous probing: its line would be to test the asymptotic behabiour of a sequence of problems with finite numbers of sectors and regions obtained from overall plan when analysed by some "regular" aggregation process (the positing of the problem is close to that of Hildenbrand and Schmeidler [41]).

The probing into the significance of findings and conjectures just indicated may be possibly extendable into the area of probabilistic plan-probing - in the light of Dynkin's assertion of the overall analogy. This is in fact in agreement with the view increasingly acceptable to Soviet mathematicians (Volkonskiy et al.) that some of the familiar limitations of plan-optimization constructs can be alleviated by a shift from

---

41) See discussion in A. Zauberman, Mathematical Theory in Soviet Planning, 1976

deterministic to probabilistic statement of problems. This is in particular true of the stability aspect of optimal solution in some formulations. (The related point has been asserted also in Western studies; cf Tintner/Sengrupta[42] on the mathematical feature of this; i.e. that differential equations models though unstable inasmuch as they have characteristic roots with positive real part - in deterministic formulation, may be, possibly, quite stable in a stochastic framework.)

Note 3 The last section of para 3 indicates the relevance of stochastic stability from the angle of planning theory. The issue and its theoretical background have not escaped the Soviet theorist's attention - the more understandably as in the beginning of the 1960s the substance of the nexus between dynamic programming and stability formalism was explored by Krasovsky and his followers. To be more specific - the nexus between the functions and functionals of Bellman and Lyapunov: its substance is that for certain problems with special sign-determined criteria of optimality, the Bellman function satisfying the relevant equations - subject to some conditions-is also the Lyapunov function. (Explorations in this direction have been related, from the early stage,to that on the application of methods of the Lyapunov functions and the corresponding optimal impulse). Naturally then the methods of Lyapunov are of interest to the optimizing planner; in the sense indicated the solution of optimal plan-control problem reduces to the optimal Lyapunov function cf. (Zauberman)[43].

---

42) G. Tintner, K.Sengupta, Stochastic Economics, 1972, p. 15

43) A. Zauberman, Mathematical Theory in Soviet Planning, 1976

As is well-known the theory of a stochastic system's stability suffers from "amibquity" in some fundamental concepts, starting from the definition of its very substance. Enough to refer oneself to the definitional alternatives such as Lyapunov stability with probability one, that of the m-th mean and so on. Mutatis mutandis that is also true of stochastic stability of controlled systems. (To single out some of the basic works in the field we may refer to those of Khazminskiy and Kushner).[44)]

Of immediate interest from our particular angle is the work on analytical construction - optimal or suboptimal of a controller (seminally initiated by N.N. Krasovskiy, A.M. Lyetov, R. Kalman - with a host of followers).

It has been demonstrated by several students - for a class of linear stationary systems - that with the help of the Lyapnov function the variational synthesis of the optimal "regulator" is a solvable problem. Now, in Soviet literature Kuntsevich and Lychak have produced a discrete analogue of this solution for a stochastic (as well as deterministic) system which may be of closer pertinence to planning.

Kuntsevich-Lychak[45)] work with a class of a fully controlled stochastic system governed by a vector-matrix stochastic difference equation $X_{n+1}=A_{\lambda}(n)X_n+BU_n$, $X_o=X^o(n=0,1,\dots)$.[(*)]

where X_n is m-dimens. vector of phase cordinates, U_n - q-dimens.

---

44) R.Z. Khazminskiy, Ustoichivost Sistem Differentsyalnykh Uravneniy pri Sluchainykh Vosmushchenyakh, 1969

H.H. Kushner, Stochastic Stability and Control, 1967

45) V.M. Kuntsevich, M.M. Lychak, Avtomatika i Telemekhanika, No. 1, 1967

vector of controls, A,B. Constant matrices more specifically; $A_\lambda(n)$ describes stationary random processes s.t. their values in two neighbouring discrete time intervals are statistically independent (as is often postulated for random discrete processes), so that (denoting the ensemble in averaging)

$$E\{A_\lambda^{\cdot}(n)A_\lambda(k)\} = E\{A_\lambda^{\cdot}(n)\}E\{A_\lambda(k)\}; \quad k \neq n,$$

then denoting $\bar{A}$ the constant matrix $\bar{A} = E\{A_\lambda(n)\}$ - ,

$$E\{X_{n+1}\} = \bar{A}\, E\{X_n\} + BE\{U_n\},$$

What is sought then is the control $U_n = U(X_n)$ which for all initial conditions X_o secures the system's stochastic stability and minimizes

$$\bar{J} = \sum_{n=0}^{\infty} E\{X_n^{\cdot}QX_n + 2X_n^{\cdot}HU_n + U_n^{\cdot}RU_n\},$$

where we have

$$\begin{pmatrix} Q & H \\ H & R \end{pmatrix} > 0$$

Now, suppose the optimal sought, $U_n=U(X_n)$, does belong to the set of controls which secures the stochastic stability of the system. Then (with reference to Lorenz)[46] the statistical analogues of Lyapunov's theorem for discrete system would entail the existence of a positive definite function $v(X_n,U_n)$ such that

$$E\{\Delta v_n\} = - E\{X_n^{\cdot}QX_n + 2X_n^{\cdot}HU_n + U_n^{\cdot}RU_n\},$$

(here the Δv_n is the first difference of the Lyapunov function taken on the basis of[(*)].

---

46) J.Lorenz in <u>Archiwum Automatyki i Telemechaniki</u>, Vol. XIV, No. 3, 1969

Thence

$$\bar{J} = \sum_{n=0}^{\infty} E\{-\Delta v_n\} = E\{v(X_o,U_o)\}$$

and as shown by Kuntsevich-Lychak

$$\frac{\partial \bar{J}}{\partial U_n} = \frac{\partial E\{v(X_n,U_n)\}}{\partial U_n} = 0,$$

whence $v(X_n,U_n) = v(X_n)$. And the existence of these requirements is established by Kuntsevich-Lychak in a proof we ommit here.

Note 4[47] Dynkin's theory of planning under conditions of indeterminacy has been supplemented by his theory of preference relations under such conditions. In substance it expands Arrow's necessary conditions by offering sufficiency conditions as well. The preference relations are described by means of some utility function U and some probability distribution P. Pairs (P,U) are considered such that P is the probabilistic measure on S, and U is a constrained measurable function on E; the utility functions f(s), $s \varepsilon S$, are appraised by means of mathematical expectation U(f), that is the integral U(f(s)) on measure P. The totality of all measurable f(s) with values in E is designated X.

Following Arrow, it is additionally postulated that P is nonatomic in the sense that for any event A with positive probability an event $B \subset A$ can be found such that $P(A) > P(B) > 0$. By setting in correspondence to $\forall f(s)$ on X, an integral of U(f(s)) on P-a numerical function on X-termed a "Bernoulli indicator" - is obtained. And the theorem is formulated to the effect that the preference relation $\succcurlyeq$ on the set X is definable

47) E.B. Dynkin, A.I. Ovseyevich, "Ob otnoshenyakh predpochtenya v uslovyakh neopredelyonnosti", Ekonomika i Matematicheskiye Metody, No. 2, 1975

iff a set of conditions is met.

The substance of these conditions is in short this:

1. Assume that $A = \{s:f(s) = g(s)\}$, $f = f'$ and $g = g'$ on complement to A, $f' \neq g'$ on A, then $f \succcurlyeq g$ iff $f' \succcurlyeq g'$ (this property permits to introduce the notion of "local" preference).
Let B be some measurable subset of S and f,g - functions. Substitute arbitrarily f' and g' for f and g over B, and equal over $S \setminus B$. We posite by definition $f \succcurlyeq g|B$ iff $f' \succcurlyeq g'$.

2. Let $f > g$ and $\{f_n\}$ be a sequence of functions - one converging to the function f in the sense that $\{s:f_n(s) \neq f(s)\} \downarrow \emptyset$. Then $f_n > g$ for a sufficiently large n.

3. If S is partitioned into a numerable family of nonintersecting sets A_n and $f \geq g|A_n$ (f not inferior to g on $\forall A_n$), then $f \geq g$. If in addition $f > g|A_1$ then $f > g$.

4. The preference relation between functions permits to introduce a "natural" preference relation in the space E; we write $x \geq y$ if the f identically equal x, is not inferior to g identically equal y'. If for $\forall s$, the $f(s) \geq g$ then f is preferable to g.

5. The preference relation of functions indicates the preference relation between the subsets of the space S: determine two elements $x > y$ in E, and set in correspondence to $\forall A$ the function f_A equal x on A; write $A \geq B$ if $f_A \geq f_B$. The relation $A \geq B$ is independent of the choice of elements $x > y$.

6. The preference relation over events is non-atomic in the sense that $A > \emptyset$, then $\exists B$ such that $A > B > \emptyset$.

<u>Remark</u>. Note in the context that the problem of indeterminacy caused by changes in preferences in time dimension is still a

subject neglected by the theory of planning. The limitations of approximated certainty equivalence in the prevalent approach are discussed in an interesting paper by Witsenhausen [48] (where the suggested utility function contains parameters whose evolution is described by a system of stochastic difference equations).

§ 4. Yudin's [49] contribution to the general theory of multistage planning under uncertainty is made again with the view to the requirements of the ASU. It is Yudin that has been one of the strongest protagonists of the view that providing the mathematical support for ASU calls for elaboration of an expanded apparatus of probabilistic algorithms such as would permit a stagewise solution of the plan problems: Stagewise, that is, allowing for a continuous corrective of the informational basis of the plan decisions; the Bayesian type approach is thought to be logically inherent in plan construction. In Yudin's design we have a preparatory stage which concentrates on defining the solutional rules and distributions: formalizing the task of automated control with any admissible information input: at this stage the knowledge of the problems' structure and of some statistical characteristics of the random parameters of conditions would suffice.

Speaking generally, the solution of the plan problem is a collection of functions, x_k, of the materialized and observed random parameters of this problem. The problem is being solved in a posteriori rules where solution is reached after the random parameters w^k had materialized and been observed; the solutional

---

48) H.S. Witsenhausen, in Annals of Economic and Social Management, No. 1, 1974

49) D.B. Yudin "Monogoetapnoye planorovanye v uslovyakh nyepolnoty informatsyi" Tekhnicheskaya Kibernetika, No. 6, 1972

a posteriori rules have then the form $x_k=x_k(w^{k-1})$; and the problem is solved in a priori rules where the decision is taken after the materialization and observation of w^{k-1}, but before the observation of w_k (here the a priori rules are $x=x_k(w^{k-1})$; additional notation is given further on)

Where the overall plan-programme in terms of expectations is

$$E\Psi_o(w^n,x^n)\to\inf,$$
$$E\Psi_k(w^k,x^h)\geq b_k,$$
$$x^k\in G_k,$$
$$k = 1,\dots,n$$

The multistage constrained plan problem in pure strategies is stated as

$$\int_{\Omega^n\times X^n}\Psi_o(w^n,x^n)\,dF_{w^n}\to\inf$$

$$\int_{\Omega_k\times X_k}\psi_k(w^k,x^k)\,dF_{w^k|w^{k-1}}\geq b_k(w^{k-1}),\quad (*)$$
$$x^k\in G_k(w^k),$$
$$k = 1,\dots,n$$

The notational framework is this. We have Ω_i, $i = 0,1,\dots,n$ the set of prime ("elementary") events w_i at stage i. Ω^k is the cartesian product of Ω_i, $i = 1,\dots,k$; $w^k = (w_i,\dots,w_k)\in\Omega^k$; $\Omega^n\equiv\Omega$.

Let P be probability measure on Ω. The probability measure P^k is determined as follows: if $A\subset\Omega^k$, then $P^k = P(\Lambda\times\Omega_{k+1}\times\dots\times\Omega_n)$. Then P_k, conditional probability measure on Ω_k: for $\forall A\subset\Omega_k B\subset\Omega^{k-1}$

$$P_k(A|w^{k-1}\in B) = \frac{P^k(A\times B)}{P^k(\Omega_k\times B)} .$$

Further - a sequence $X_i, i=0,1,\ldots,n;\ x_i\in X_i$, with a cartesian product of $X_i.i = 1,\ldots,k;\ x^k = (\ \ ,\ldots,x_k);\ \ X^n \equiv X$. For each random $w^k\in\Omega^k$ and $x^k\in X^k$, $k=1,\ldots,n$, there is a given vector valued random function $\Psi_k(w^k,x^k)$. $G_k = G_k(w^k)$, $k = 1,\ldots,n$, - are some, generally - random sets and $b_k = b_k(w^{k-1})$ random m-dimenional bounded measurable vector-valued functions of w^{k-1}. Correspondingly $b^k(w^{k-1}) = (b_1,b_2(w^k),\ldots,b_k(w^{k-1})$. Lastly $F_w, F_{w,x}, F_x, F_{x|w}$ stand, respectively, for distribution function of w, for joint distribution function for w and k, and for the unconditional and conditional (assuming, that is, that the materialization of w is known) distribution function for x.

Take stage i of plan construction. First a set K_i corresponding to the region of the problem of the i-th stage is being introduced ($K_1\neq\emptyset$ is the sufficient as well as necessary condition for the solvability of the problem) :

$$K_i = \{x_1\in G_i^O|\exists[y_{i+1}\in G_{i+1}^O,\ldots,y_n\in G_n^O],\ E_{w_i}[\psi_i(w^i,x^i)|w^{i-1}]\geq b_i(w^{i-1}).$$

$$E_{w_{i+s}}[\psi_i(w^{i+s},x^i,y_{i+1},\ldots,y_{i+s})|\geq b_{i+s}(w^{i+s-1}),$$

$$\forall w_{i+s-1},\ldots,w_{n-1},\quad S = 1,\ldots,n-i\},^{(*)}$$

The term G_i^O is the mapping of G_i on the coordinate hyperplane determined by the components of the vector x_i. The requirement of existence of vectors y_{i+s}, $S = 1 .. n-i$, that would satisfy the conditions of (*) is an analogue of the induced constraints in the classical Dantzig-Madansky two-stage problem.

Further on, an objective function $Q_i(x_i)$ for the i-th stage is being defined. It is the conditional expectation of $\psi_o(w^n,x^n)$ under the assumption that at stages preceding stage i the ensemble w^{i-1} of random parameters of the problem's conditions had materialized and decisions x^{i-1} had been reached, and also that at stages following the i-th, optimal decisions $x^*_{i+1},\ldots,x^*_n$ would be taken

$$Q_i(x_i) = E_{wn|w^{i-1}}(w^n,x^{i-1},x_i x^{\cdot}_{i+1},\ldots,x^{\cdot}_n)$$

Thus determining the solution rule at the i-th stage of the determining multistage stochastic problem reduces to the solution of the mathematical programming problem $\inf\limits_{x_i \in K_i} Q_i(x_i)$

where the posterior and prior rules are respectively

$$x_i = x_i(w^i),\ y_{i+s} = y_{i+s}(w^{i+s}),$$

$$x_i = x_i(w^{i-1}),\ y_{i+s} = y_{i+s}(w^{i+s-1}),\ s = 1,\ldots,n-i.$$

For a separable objective function with $\psi_o(w^n,x^n) = \sum\limits_{j=1}^{n} \psi_{oj}(w^j,x^1)$.

$$Q_i(x_i) = E_{w_i|w^{i-1}}\{\psi_{oi}(w^i,x^i)+Q^{\cdot}_{i+1}(w^i,x^i\},$$ should be $E_{wi|w^{i-1}}\{\psi_{oi}\ldots$

where

$$Q^{\cdot}_i\ (w^{i-1},x^{i-1})=\inf_{i\in k_i} E_{w_i|w^{i-1}}\{\psi_{oi}(w^i,x^i)+Q^{\cdot}_{i+1}(w^i,x^i)\},\ i=1,\ldots,n-1$$ should be $E_{wi|w^{i-1}}\{\psi_{oi}\ldots$

with i=n

$$Q^{\cdot}_n(w^{n-1},x^{n-1}) = \inf_{x_n\in k_n} E_{w_n|w^{n-1}}\psi_{on}(w^n,x^n).$$ should be $E_{wn|w^{n-1}}\{\psi_{on}\ldots$

<u>Note 5</u> While the Yudin-type of reasoning helps to expand the theoretical basis for indetermincy in planning, its suggested Bayesian approach has some more immediate implications for one of the most central and least clarified - theoretically as well as empirically - issues: of setting a plan's horizon. It is because it remains insufficiently explored that Kantorovich/Makarov[50] suggest as the relatively least risky approach adopting a very distant horizon for a plan-programme - with the presupposed abandoning of the plan at some a priori uncertain point of implementation; the underlying idea being that the long distance to the goal by itself tends to "neutralize" the impact of the chosen objective function: a proposition that Kantorovich/Makarov have made theoretically plausible. But it is the question of determining the time-point of discarding the plan that has become most controversial. (By itself the conception of a sliding horizon for the plan has found wide acceptance in Soviet theory; of Western writing on the matter, Goldman's results of the late 1960s [51] - within the framework of a Pontriagin-type optimization - have given noteworthy support to and formalism for the conception; in Soviet literature Smirnow[52] - also within the Pontriaginian framework - has rigourized an idea of a plan horizon as a function of changing valuation of time, i.e. the discount rate).

More direct contribution to the matter in the context of indeterminacy has been by writers of the probabilistic school.

---

50) V.L. Kantorovich, L.V. Makarov, in <u>Primemenye Matematicki v Ekonomicheskich Issledovanyakh</u>, Nemchinov Ed., Vo. 3,1965

51) S.M. Goldman, <u>Journal of Political Economy</u>, 1969

52) A.D. Smirnov, "Optimal Interbranch Model of Socialist Reproduction", in A. Carter, A.Brody, <u>Input-Output Techniques</u> 1970

Thus Pressman [53] asserts that under some conditions of smoothness for random parameters of his plan model (dynamic perspective plan with elements of indeterminacy), where current correction of the plan is sufficiently **frequent**, the error resulting from adopting in plan construction mean values of such parameters tends toward zero independently of the plan period. Furthermore, for an operative Bellman-type stochastic equation of his model he shows that in principle it is <u>not</u> reducible to one-stage optimization due to "non-shiftability" of the operations of the mean-taking and maximization; from which the conclusion is drawn that for the plan of this class optimality is obtainable if and only if it is designed as a multistage sliding system.

<u>Note 6</u> An asymptotic treatment of the infinite-horizon control problem is suggested by Moiseyev [54] for an economic growth plan i.e. finding the the control u(t) which minimizes the functional

$$I = \int_{t_0}^{\infty} (c,x)\,p(t)\,dt.$$

s.t.

$$\dot{x} = f(x,u,t), \quad (x,u) \in G_u,$$

$$0 \leq u(t) \leq \varphi(x)$$

with discount function $p(t) \to 0$ for $t \to \infty$ (Conventional notation).

The impact of the horizon's infinity is well demonstrable to be inhibitive in the application of the Maximum Principle.

---

53) L.S. Pressman, doctoral thesis "Dinamicheskiye modeli perspektivnogo planirovanya s uchetom neopredelennosti" (as cited by Lavrov-Makarov, see reference below p. 63.

54) N.N. Moiseyev, <u>Zhurnal Vychislitelnoy Matematiki i Matematicheskoy Fiziki</u> 14, 4, 1874

The computational difficulties (the solution of the boundary value problem in the infinity case) **apart**, this is so also owing to difficulties on the two scores: 1) the lack of a priori information as to the behaviour of the costate multiplier in the infinity's neighbourhood and 2) the building-up, **at infinity**, of the cones of admissible variations.

The approach rests on shifting the boundary condition to a finite point and defining a certain asymptotic form of the unknown functions.

In principle a function which minimizes the functional

$$I_1 = \int_{t_o}^{T} F(x,u,t)\,dt + R(x_T)$$

would be an approximate asymptotic solution of the infinity problem. For the free-end problem the Pontryagin principle is definable with the Hamiltonian of the form:

$$H = (\Psi, f) - F^{\cdot}(x,u,t)$$

Here

$$F^{\cdot}(x,u,t) = F(x,u,t) + \frac{dR}{dx} f,$$

$$R(x_r) = \int_r^{\infty} F(\hat{x},\hat{u},t)\,dt$$

By motivation familiar to mathematical economics it is assumed that the differential relations are linear so that

$$\dot{x} = A(t)x + B(t)u$$

with constraints $0 \le u_i \le k_i x_i$, $k \in [0,1]$.

It is stipulated furthermore that in the neighbourhood of infinity, matrices A, B lend themselves to statement in the form

$$A = A_o + \frac{A_1}{t} + \cdots + \frac{A_N}{t^N} + o\left(\frac{1}{t^N}\right).$$

Next control of the form $u=\Theta x$ is substituted for $u(t)$, the Θ being a diagonal matrix where the elements satisfy $0 \leq \Theta_i \leq k_i$. Then

$$\dot{x}=(A+\Theta)x. \qquad (\&)$$

We postulate Θ_o to be known and accept as plausible the asymptotic form

$$\Theta=\Theta_o + \frac{\Theta_1}{t} + \ldots + \frac{\Theta_N}{t^N} + o\left(\frac{1}{t^N}\right).$$

With reference to the Hukuhara theorem [55], under the assumption that the $(A_o + B_o\Theta_o)$ matrix has only simple eigen-values, the general solution of (&) is written

$$x = \sum_{s=i}^{n} C_s y(s,t),$$

and

$$I=\int_{t_o}^{r} \left(C, \sum C_s y(s,t) \right) p(t)dt = I(C_i,\ldots,C_n,\Theta_{ik}), (\&\&) \text{ s}$$

where $y=(s,t)$ is of the form (with λ denoting eigenvalue)

$$y(s,t)=\exp\{\lambda_s t+\lambda_s{}^o \ln t\}\{x_o(s)+x_i(s)t^{-i} + \ldots + x_N(s)t^{-N}+o(t^{-N})\}.$$

One disregards $o(1/t^N)$ and finds admissible control in the solution of a nonlinear programme wherein the C and Θ_{ik} are sought such as to minimize (§§) s.t. $0 \leq \Theta_i \leq K_i$ and

$$x_T = \sum C_s y(s,t).$$

---

55) M. Hukuhara, "Sur les point singuliers des équations différentielles linéaires, Journal of the Faculty of Sciences, Hokkaido University, Ser. I. II, 1934

Remark For presentation of some new ideas on optimal control on an infinite horizon see inter al a recent paper by Haurie [56]. It explores inter al, the existence of optimal trajectories as well as a system's controllability and asymptotic behaviour. In some sense it advances recent findings of Halkin [57] where the general conditions for the solution of the infinite horizon optimal control problem do not contain transversality conditions thereby impeding the full characterization of the system's extremal trajectories. But in Haurie the turnpike property and nice asymptotic behaviour of optimal trajectories provide the conditions; they lead to a fairly general existence conditions for a solution.

Also of late the problem of infinite horizon for an optimally controlled system has been taken up by Brock and Scheinkman [58]. The study - which is in particular concerned with application of the findings to a model of global asymptotic stability for a growing economic system - formulates a set of sufficient conditions on the Hamiltonian of the system to converge to a steady state as time tends to infinity.

§ 5. In a very broad outline the Yefimov model [59] of stochastic "perspective" planning for an economy can be presented in these terms.

---

56) A. Haurie, Journal of Mathematical Economics, No. 1, 1976

57) H. Halkin, Econometrica, No. 2, 1974

58) W.A. Brock, J.A. Scheinkman, Journal of Economic Theory, No. 1, 1976

59) V.M. Yefimov,"Stokhasticheskaya model perspektivnogo planirovanya", in Teoriya Optimalnykh Resheniy, Issue 3. Institute of Cybernetics, Academy of Sciences of the Ukrainian Academy of Sciences, Kiev 1969

Let production be formed of m kinds of items and b^k - an m-dimensional random vector describing the demand for the output over the k-th period, A^k - an $m \times n^k$ dimensional matrix with (generally, random) elements characterizing input-output coefficients when technologies are used with a unit intensity in the k-th period . Further c^k is a random n^k-dimensional vector whose components describe the cost of employment of technologies with a unit intensity in the k-th period; q_+^k and q_-^k are m-dimensional deterministic vectors whose components describe resp. the prices of output produced in the (k-1)-th period and the cost of storing products over this period. Further x^k is an n-dimensional deterministic vector of intensities in the employment of technologies over the k-th period where X^k is the set of admissible technologies, $X^k \subset R^{n^k}$; X^k is convex, closed.

The overall planned production time is formed of N periods (k=0,...,N-1). The q_+^k , q_-^k , c^k, as defined, are reduced to the 0-th period,allowance made for some accepted scale of time preferences. The some admissible plan $(x^0,...,x^{N-1})$ is chosen and the mathematical expectation of profit, thus time-reduced, is computed for the plan. For the purpose, complementing variables y_+^k, y_-^k are introduced; these make m-dimensional non-negative vectors describing respectively the excess demand over supply (deficit) and excess supply over demand (surplus) over the (k-1)-th period. It is assumed that over the N period the range of products is constant; hence the supply excess may be stored and sold in a subsequent period. Let D^k be a diagonal deterministic $m \times n$ matrix, with elements greater than zero and not greater than one, which show the proportion of output stored at the beginning of the k-th period. Now the mathematical expectations of profits time-reduced, are computed for all the N periods starting with the 0-th , y^o being initial stocks (given). Their sum appears then as

$$E[\sum_{k=o}^{N-1} (b^k, q_+^{k+1}) - (y_+^{k+1}, q_+^{k+1}) - (y_-^{k+1} q_-^{k+1}) - (c^k, x^k)], \tag{1}$$

$$y_t^{k+1} - y_-^{k+1} + D^k y_-^k = b^k - A^k x^k, \tag{2}$$

$$(y_t^{k+1}, y_-^{k+1}) = 0, \tag{3}$$

$$y_+^{k+1} \geq 0; \; y_-^{k+1} \geq 0, \tag{4}$$

$$k = 0, \ldots, N-1$$

The problem is to find an admissible plan maximizing the time-reduced profit as above. The solutional approach is that of a two-stage linear stochastic programming.

First a linear programme of the following form is considered, with b^k, A^k, x^k constant

$$\sum_{k=1}^{N} (y_+^k, q_+^k) + \sum_{k=1}^{N} (y_-^k, q_-^k) \rightarrow \min, \tag{5}$$

$$y_+^{k+1} - y_-^{k+1} + D^k y_-^k = b^k - A^k x^k, \tag{6}$$

$$y_+^{k+1} \geq 0; \; y_-^{k+1} \geq 0, \; k=0, \ldots, N-1 \tag{7}$$

The dynamics of prices is determined by a superior controlling agency and we postulate that q_+^k satisfies

$$q_+^{k+1} D^k - q_+ < q_-^k, k=1,\dots,N-1, \tag{8}$$

$$q_+^N + q_-^N > 0, \tag{9}$$

meaning that the storing cost per unit of product over (k+1)th period is greater than the possible additional profit (obtainable through increase in pricing) from a unit of product produced in (k+1)th period. If (8) is met, it is reasonable to store production only if there is no demand for it (from (8) it fofows that unrealized production on which there is a demand, would entail loss). It is then proved that if (5), (9) are met, then optimal solution (5) - (7) satisfies (3).

Next the stochastic model of perspective planning is given the form

$$E\left(\sum_{k=0}^{N-1} (c^k, x^k) + \{\min \sum_{k=1}^{N} (q_+^k, y_+^k) + (q_-^k, y_-^k)\}\right) \tag{10}$$

$$y_+^{k+1} - y_-^{k+1} + D^k y_-^k = b^k - A^k x^k \quad k = 0,\dots,N-1, \tag{11}$$

$$y_+^k \geq 0; \quad y_-^k \geq 0 \quad k = 1,\dots,N, \tag{12}$$

$$x^k \in X^k \quad k = 0,\dots,N-1 \tag{13}$$

Here the $\{.\}$ is the optimal value of (5) - (7) for some fixed b^k, A^k, x^k ; $k = 0,\dots,N-1$.

Finally, the dual to (5) - (7) is formed

$$\sum_{k=0}^{N-1} (\pi^k, b^k - A^k x^k) \to \max, \tag{14}$$

$$\pi^k \leq q_+^{k+1} \quad k = 0,\dots,N-2, \tag{15}$$

$$\pi^{k+1} D^{k+1} - \pi^k \leq q_-^{k+1} \quad k = 0,\dots,N-2, \tag{16}$$

$$\pi^{N-1} \leq q_+^N, \tag{17}$$

$$\pi^{N-1} \leq q_-^N \tag{18}$$

Denote the vector $(b^o,b^1,\dots,b^{N-1})$ as b, matrix $[A^o,A^1,\dots,A^{N-1}]$ as A and vector $(x^o,x^1,\dots,x^{N-1})$ as x. Let the

$$\pi(x,A,B) = [\pi^o(x,A,b),\ \pi^1(x,A,b),\dots,\pi^{N-1}(x,A,b)]$$

be the optimal solution (14) - (18).

Then with reference to the duality theorem for linear programming the (10) can be re-written as

$$E(\sum_{k=0}^{N-1}(c^x,x^k) + \sum_{k=0}^{N-1}\pi^k(x,A,b)(b^k-A^kx^k))\rightarrow \max, \qquad (18)$$

this expression is a function of x only and it is demonstrable that it is a convex function.

Note the equivalence definition: a deterministic programming problem is defined as equivalent to a stochastic problem of the type (10) - (12) where the optimal values of the criterial functionals coincide for both problems and where so do also the sets of the optimal solutions for x. Hence (18), (13) is the problem of convex programming equivalent to (10) - (13).

Note 7. When the planner resorts to stochastic formulation of his plan-control task,adaptive methods and techniques offer themselves as an abvious aid. There are several noteworthy Soviet contributions thus oriented. Perlmutter's[60] control design for a nonlinear dynamic system commends itself as a good specimen.

---

60) V.M. Perlmutter, Avtomatika i Telemekhanika, No. 3, 1975

The extremand is

$$J = E\{\lambda_o(x(T) + \int_o^r [\lambda_x(x)+\lambda_u(u)]dt\} \to \min$$

for the system describable by $\frac{dx}{dt} = f(x,\beta) + \varphi(x,\beta)u, \quad y=Hx+v.$

The x and y are, respectively, m and n dimensional state and output vectors, H is an $m \times n$ mx, $\nu(t)$ is Gaussian white noise; $u \in U$ - scalar control.

In the criterion the three analytic functions, the lambdas, $\lambda_o, \lambda_x, \lambda_u$, are expanded in series with respective convergence points $x = 0$, $u = 0$:

$$\lambda_o = x^T(T)F_2x(T) + \sum_{i=3} F_i(x(T)),$$

$$\lambda_x = x^T(t)Q_2x(t) + \sum_{i=3} Q_i(x(t)),$$

$$\lambda_u = \sum_{i=2} g_i u^i$$

The f, φ - analytic functions in y-are expanded into series converging in the neighbourhood of $x = 0$:

$$f=A(\beta)x + \sum_{i=2} A_i(x,\beta), \quad \varphi=b(\beta) + \sum_{i=1} B_i(x,\beta)$$

The g_2 is positive; F_2, Q_2 are positive semidefinite matrices; F_i, Q_i - homogenous forms of degree i in x_j. The A is an $n \times n$ mx; b - n-dim. vector; A_i, B_i homogenous forms of degree i in x. The x_j components of x depend on the vector of parameters β (unknown). A priori distribution of β is normal with average β_o and correlation mx p_{x_o}. Initial value $x(0)$ is analogously distributed (normal law, average x_o, correlation matrix P_{x_o}).

The joint a posteriori distribution of x and β are considered to be sufficient statistics of the control process. And in the first approximation that control is posited to be a function exclusively of the conditional average, $\hat{x},\hat{\beta}$ of these two statistics, for given observation of y(t) where $0 \le t \le t$.

The procedure is then one of separation of estimation and control. The stochastic control is calculated as "deterministic" with estimates substituted for random magnitudes, i.e.

$$P_x = E[\Delta x \Delta x^T], \quad P_{x\beta} = E[\Delta x \Delta^T], \quad P_\beta = E[\Delta\beta\Delta^T],$$

here $\Delta x = x - \hat{x}$, $\Delta\beta = \beta - \hat{\beta}$;

Conditional averages and correlation matrices are being found by some procedure from the differential equations of the first approximation. In Perlmutter they are borrowed from Bryson and Ho

$$\frac{d\hat{x}}{dt} = f(\hat{x},\hat{\beta}) + \varphi(\hat{x},\hat{\beta})u + P_x H^T R^{-1}\varepsilon, \quad \hat{x}(0) = x_o,$$

$$\frac{d\hat{\beta}}{dt} = P_{x\beta}^T H^T R^{-1}\varepsilon, \quad \varepsilon = y - H\hat{x}, \hat{\beta}(0) = \beta_o,$$

$$\frac{dP_x}{dt} = (f'_x+\varphi'_x u)P_x + P_x(f'_x+\varphi'_x u)^T + (f'_\beta+\varphi'_\beta u)P_{x\beta} + P_{x\beta}(f'_\beta+\varphi'_\beta u)^T - P_x H^T R^{-1} H P_x =$$

$$= \alpha_x, P_x(0) = P_{xo},$$

$$\frac{dP_{x\beta}}{dt} = (f'_x+\varphi'_x u)P_{x\beta} + (f'_\beta+\varphi'_\beta u)P_\beta - P_x H^T R^{-1} H P_{x\beta} = \alpha_{x\beta} P_{x\beta}(0) = P_{x\beta o},$$

$$\frac{dP_\beta}{dt} = - P_{x\beta}^T H^T R^{-1} H P_{x\beta} = \alpha_\beta, \quad P_\beta(0) = P_{\beta o}.$$

§6. The Gadzhiyev construct [61] deserves attention on many counts. It has an interesting mathematical formulation of the control in terms of a differential game; it realistically captures a problem of hierarchic coordination of decision-making in a multi-criterial system - in generalized situation the criteria are permitted to be conflicting. And it explicitly treats indeterminacy in giving the problem a stochastic form. (It also offers a solution for a case of regressive control where payoffs are Gaussian and so is the characteristic of the object controlled - this we omit in our presentation, for brevity's sake). The model is formed of 4 systems: (A) controlled, (B) controls, (C) observational, (D) command. Controlled is described by probability distribution $P_a(f|\lambda)$ of the output vector $f = (f_1,...,f_\tau)$ depending on a set of parameters $\alpha \doteq (a_1,...,a_m)$ which are controlled and on a set of non-controlled parameters $\lambda=(\lambda_1,...,\lambda_s)$; the values of the latter form, in general, materialization of a random vector whose probability distribution depends on vector a. Controls with controlling agencies. Each agency has a payoff f. $u_i(a_i,\{f\}_i,d_i)$, formed of 3 groups of arguments; a_i- extremize the f. u_i; the $\{f_i\}$ which is supplied by the controlled system to the ith agency's input (specially each agency maximizes its payoff by means of its controls). d_i-a variable whose value is selected by the command system. Observational: it processes and supplies to agencies, information on non-controlled parameters. Information is supplied either as a direct realization of λ or - in a general case - as a realization of random magnitudes statistically connected with realization of the uncontrolled parameters.

---

61) M.Yu. Gadzhiyev in Avtomatika i Telemekhanika, No. 5, 1972

In the latter case it is assumed that each agency observes the realizations from two sources: (1) directly from the observation system and (2) through channels of circulation of information - from agencies of the same level. It is assumed that structural parameters of the observational system, and, in particular, the choice of agencies exchanging information and ways of exchanges - are determined by command system. Thus $\forall_i, (i=\overline{1,m})$, agency observes the realizations y_i, z_i-random components of Y_i, Z_i- whose probability distribution $Q\ (y_i, z_i|\lambda)$ depends on realization of λ; here y_i - realizations are coming from the observational system: z_i - are coming from remaining agencies. When using realizations $y_i, z_i, (i=\overline{1,m})$ obtained in the leaving process, agencies are averaging their payoff fs. (allowance made for the form of Q).

Command: coordinates the operations of agencies which are pursuing their own interests; in this it follows "common" (overall) objectives. The D also evolves the desirable modus of the model's functioning, the appraisal of its functioning is being carried out in the corresponding space of characteristics. It is assumed that any of the vectors of this space makes a realization, $g = (g_1, \ldots, g_n)$. of a random vector with the probability distribution $G(y|f)$.

As a rule - in the general case - the D does not receive directly any information on **current realizations of the non-controlled vector.**

The coordinating operations of D are described in this way: The coordination vector $\Theta=(\theta, \delta)$ is formed of two subvectors. Subvector $\theta=(\theta_1, .., \theta_h, .., \theta_l), (0\leq l\leq m)$ - of "corrections; $\forall\theta_n$ has a nontrivial set of possible values $T^{(h)}$. It is assumed that corrections $\theta_n\ (h=\overline{1,l})$ are being substituted into the payoff fs. of those agencies whose numbers form a set J_n ; thus if the no. is i system, and in particular, the choice of agencies exchanging information and ways of exchanges - are determined by command system. Thus $\forall_i$, $(i = \overline{1,m})$ agency observes the realizations y_i, z_i - random components of Y_i, Z_i - whose probability distribution $Q(Y_i, Z_i | \lambda)$ depends on realizations of λ; here Y_i - realizations are coming from the observational system; z_i - are coming from the remaining agencies.*)

When using realizations Y_i, Z_i $(i = \overline{1,m})$ obtained in the learning process, agencies are averaging their payoff fs. (allowance made for the forms of Q)

D) - command: co-ordinates the operations of agencies which are pursuing their own interests; in this it follows "common" (overall) objectives. The D also evolves the desirable modus of the model's functioning; the appraisal of its functioning is being carried out in the corresponding space of characteristics. It is assumed that any of the vectors of this space makes a realization, $g = (g_1, \ldots g_n)$, of a random vector with the probability distribution $G(g/f)$. As a rule - in the general case - the D does not receive directly any information on current realizations of the non-controlled vector.

The co-ordinating operations of D are described in this way: the co-ordination vector $\Theta = (\theta, \theta)$ is formed of 4 systems - controlled, controls, observational, command;

controlled is described by probability distribution $P_a(f|\lambda)$ of the output vector $f = (f_1, \ldots f_\tau)$ depending on a set of parameters $a = (a_1, \ldots, a_m)$ which are controlled and on a set of non-controlled parameters $\lambda = (\lambda_1, \ldots, \lambda_s)$; the values of the latter form, in general, materialization of a random vector whose probability distribution depends on vector a.

controls with m controlling agencies. Each agency has a payoff f. formed of 3 groups of arguments $U_i(a_i, \{f\}_i, \delta_i)$; a_i - extremize the f. U_i; the $\{f_i\}$ which is supplied by the controlled system to the i-th agency's input (specifically each agency maximizes its payoff by means of its controls). δ_i - a variable whose value is selected by the command system.

observational: it processes and supplies to agencies, information on non-controlled parameters. Information is supplied either as a direct realization of λ, or - in a general case - as a realization of random magnitudes statistically connected with realization of the non-controlled parameters. In the latter case it is assumed that each agency observes the realizations from two sources: (1) directly from the observation system and (2) through channels of circulation of information - from agencies of the same level. It is assumed that structural parameters of the observational consist of two subvectors. Subvect $\theta = (\theta_1, \theta_k, \ldots, \theta_\ell)'$; $(0 \leq \ell < m)$ - $(\theta_1, \theta_k, \ldots \theta)$of "corrections"; $\forall\theta_h$ has a nontrivial set of possible values $T^{(h)}$. It is assumed that corrections $\delta_h, (h = 1, \ell)$ are being substituted into the payoffs fs of those agencies whose numbers for a set J_h.; thus if the no. is i and $i \varepsilon J_h$, then $\delta_i^h = \theta_h$. Suppose $s \leq m$ is the no. of agencies in set $J = UJ_i$. The second subvector $\theta = (\nu_1, \ldots, \nu_k, \ldots, \nu_e)$, is "subvector of constants" $\ell = 0$ if $s = m$ and $1 \leq t \leq m - s$ if $s < m$. The const. ν_k $(K = 1,t)$ is substituted by command in place of δ_i into payoff fs of agencies with numbers forming a set I_k; thus if $i \varepsilon I_k$ then $\delta_i = \nu_k$.

There are two procedures in which D carries out co-ordination. (1) a procedure based on the choice of components of the subvector of corrections from the set $T^{(h)}$ and (2) that by which D determines the model's structural parameters.

The structural adjustments ("corrections") at D's disposal form the set Ω. The adjustment ("correction") factors may be in particular: (a_1) the selection of the structure for the distribution of the controlled system - as between agencies; (a_2) the choice of the sets $T^{(h)}$ of possible values of corrections issued by the Command system; (a_3) the choice of the $s \leq m$ agencies to which the corrections $\theta_1, \ldots, \theta_\ell$ are being directed and distribution of these agencies in groups $I_1, \ldots, J_i, \ldots, J_\ell$ such that on each group a constant $\nu_k(k = \overline{1,t})$ is produced.

---

*) M. Yu.Gadzhiyev in Avtomatika i Telemekhanika, No.5, 1972.

Positing the problem

D formulates the extremization problem corresponding to the mode of "desired" behaviour of the system as a whole. The set of agencies is guided by equilibrium strategies maximizing payoffs for ($i = \overline{1,m}$). The task is to co-ordinate the solution of the command's extremization problem with the agencies' equilibrium strategies with the use of corrections θ' and, if necessary, of structural correction factors of Ω

General values of solution

We average the payoff f_i of each agency

$$\int \ldots \int u_i (a_i, \{f\}_i\ \delta_i)\ P_a(f|\lambda)\ df =$$

$$\bar{u}_i (a_i, a^i, \delta_i, \lambda, \Omega) \quad , \qquad (i = \overline{1, m})$$

$$a^i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, am) \qquad \ldots (*)$$

Assume that all agencies can observe the realizations of components λ. From (1) $\Longrightarrow$: to find controls a_i extremizing u_i, ($i = 1, \ldots, m$) it is necessary to solve the m-person game (it is so since the solution of the i's probably depends not only on a_i but also a^i of controls by the remaining agencies. To specify: agencies carry out a non-coalitional game of m-persons with complete information.

Command formalizes the objective f_c for some extremization problem for components of the vector Ψ, (5) possibly as some overall problems A

$$\max \Psi (a_1, \ldots, a_m, \lambda)$$

$$a \in \Pi (a)$$

where $\Psi(.) = F(\Psi_{i1}, \ldots, \Psi_{ik})$ is the given function of selected components ($\Psi_{i1}, \ldots, \Psi_{ik}$) (being extremized) of the vector Ψ, the $\Pi(a)$ - is admissible set of extremization problem, determined by constraints $\Psi_{jk} \geqslant 0$ ($j_k \neq i_s$; $S = \overline{1, K}$) on possible values of remaining (n-K) components of vector Ψ.

In fact, the D can solve the problem $\mathcal{A}$ without the agencies. Suppose the solution is deterministic; then D has to find vector $a^*(\lambda) = (a_1^*(\lambda), \ldots, a_m^*(\lambda))$ that maximizes $\Psi(a,\lambda)$ s.t. $\Pi(a)$. Suppose $\Psi^*(\lambda) = \Psi(a_i^*, \lambda)$, the corresponding extremal value for problem $\mathcal{A}$. [When solution is randomized one has to find distribution $\eta(a|\lambda)$ which maximizes the fnal $\int\ldots\int\Psi(a,\lambda)\,\eta(a|\lambda)\,da$].

Applicationally more tractable is co-ordination with the use of the 'agencies' equilibrium strategies. The extremization problem has the form max $\hat{\Psi}(\theta;\lambda,\Omega)$ s.t.

$$\theta \in \hat{\Pi}(\theta)$$

$$\hat{\Psi}(\theta, \lambda, \Omega) = \int\ldots\int\Psi(a,\lambda)\;\xi(a|\theta: \lambda, \Omega)\,da,$$

$$\hat{\Pi}(\theta) = \int\ldots\int\Pi(a)\cdot\xi(a|\theta; \lambda, \Omega)\,da$$

where Ξ is the agencies' equilibrium vector.

In the process of co-ordination the m-dimensional space of controls a (on which the problem $\mathcal{A}$ of the command system is posed), will be transformed into an ℓ-dimensional space of subvectors, θ, of corrections.

Now let $\tilde{\theta} = (\tilde{\theta}_i, \ldots, \tilde{\theta}_\ell)$ be the deterministic solution of B. For its extremal value take

$$\tilde{\Psi}(\lambda,\Omega) = \hat{\Psi}(\ddot{\theta}; \lambda, \Omega) = \max\ \hat{\Psi}(\theta; \lambda, \Omega)\ \theta \in \hat{\Pi}(\theta)$$

The $\tilde{\theta}$ is termed the "co-ordinated vector of corrections" and its components - "co-ordinated corrections". The $\tilde{\phi} = \tilde{\phi}(\lambda, \Omega)$ - term the "co-ordinated extremum value of $\mathcal{A}$". Components of the co-ordinated equilibrium vector $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_i, \ldots, \tilde{\xi}_m)$ are determined as $\tilde{\xi}(a_i|\lambda, \Omega) = \xi_i(a_i|\theta;\lambda,\Omega)$, $(i=\overline{1,m})$ making the "co-ordinated equilibrium strategies".

Efficiency of "co-ordinated solution" is determined as the difference between Ψ^* and $\tilde{\Psi}$: generally we presume $\Psi^*(\lambda) \gtrsim \tilde{\Psi}(\lambda, \Omega)$.

In any case we have

$$U_i(\lambda, \Omega) = \tilde{U}_i(\tilde{\xi}),\quad (i = \overline{1,m})$$

as the co-ordinated value of the game of m organs. This can be varied by adjusting the structural correcting factors. As the measure of the total payoff of m agencies the Zukhovitskiy-Polak-Primak[62)] "normalized payoff" $U = \sum_i u_i(\lambda, \Omega)$ is accepted (of Zukhovitskiy/Polak/Primak*) theory of a multi-person concave games and their method of searching the "normalised equilibrium point" based on, and generalizing Nash's concept. A solutional method is offered by the same authors. Cf my discussion in my book Differential Games ...)

Finally, the case is considered where, realizations of λ are not observed by agencies. Assume that for the agencies we have a learning process formed by n "beats" over the time of which each agency obtains the realization, $\{y_i\}^n = \{yi_1, \ldots, y_{in}\}$, $\{Z_i\}^n = \{Zi_1, \ldots Zi_n\}$ of random vectors Y_i, Z_i with conditional probability $Q_i\ (Y_i, Z_i|\lambda)$. By making use of this information the $\forall i$ = th, $(i=\overline{1,m})$ constructs conditional probability $[R_i(\lambda_i\ \{Y_i\}^n, \{Z_i\}^n, \Omega)]$ of the vector λ.

The payoff functions are now averaged overall the values of :

$$M_i\ (a_i, a^i; (\delta_i^i, \Omega^i) =$$

$$= \int \ldots \int R_i\ (\lambda, \ldots)\ \tilde{U}_i\ (a_i\ (Y_i, Z_i)\ a^i\ (Y^i, Z^i); \lambda, \ldots)\ .$$

$$.\ [\Pi Q_i\ (Y_i, Z_i|\lambda)\ dy_i\ dz_i]\ d\lambda$$

there $a^i = (y^i, z^i) = (a_1\ (y_1, z_1) \ldots a_{i-1}\ (Y_{i-1}, Z_{i-1})\ a_{i+1}\ (Y_{i+1}, Z_{i+1}),$ $\ldots, a_m\ (y_m, z_m)$, the Y_i, Z_i being "examinational" realizations (as in Pugachev's statistical theory of learning-automata systems).

The equilibrium strategies of the corresponding game of m agencies have the form $\xi_i\ (a_i\ Y_i, Z_i, \theta, \Omega)$, $(i=\overline{1,m})$. It is these strategies that are made use of by D in formulating the extremization problem in the space of corrections θ.

Remarks

One has in Gadzhivev an algorithmic example of regressive control for a Gaussian system, i.e. with both the payoff fs and the controlled object's characterstics being Gaussian which naturally makes the handling of the problem somewhat easier.

Gadzhivev, op.cit., p.111.

We assume that the controlled object is describable by the r-dimensional normal probability distribution

$$P_a(f|\lambda) = \frac{1}{(\sqrt{2.\Pi}^{\lambda}\sqrt{|D|}} \exp - \frac{1}{2} \{\Sigma_{qij}^{r} d^{qj} (f_q - \Sigma_t^m b_{qt} a_t$$

$$- \Sigma_\ell^s h_{q\ell} \lambda_\ell)(f_j - \Sigma_n^m b_{jn} a_n - \Sigma_k^s h_{jk} \lambda_k)\}$$

$|D|$ - determinant of covariational mx D of output vector f; d^{ij} - element of mx D^{-1}, $b_{q\ell}$ - element of mx B of weighting coefficients; $h_{q\ell}$ - element of mx H which describes contribution of components of the non-controlled vector. Components: $\{f\}_i = \{f_{i1}, \dots fi_m\}$ of vector f are given on input of i-th agency. Payoff f. of i-th agency is taken to have form

$$u_i(\{f\}_i, \delta_i) = \frac{1}{(\sqrt{2.\Pi})^m\, i\sqrt{|G(i)|}} \exp - \frac{1}{2} [\Sigma_{k,h}^{mi} g^{kh}(i)(f_{ik} - \delta_i)(f_{ih} - \delta_i)]$$

$$i = \overline{1, m}$$

G(i) mx, m.m, is describing "connection" of components of $(f_{i_1}, \dots, f_{i_m})$ for i-th agency. The average payoff f_φ is taken as

$$\tilde{u}_i(a_i, a^i; \delta_i) = \int \dots \int u_i(\{f\}_i, \delta_i) P_a(f|\lambda)\, df =$$

$$= \frac{1}{(\sqrt{2.\Pi})^{mi}\sqrt{|Q^{(t)}|}} \exp - \frac{1}{2} | \Sigma_{h,t}^{mi} q^{nt}(i)(\Sigma_v^m b_{i_{nv}} a_v +$$

$$+ \Sigma_k^s h_{i_n k} \lambda_k^f - \delta_i)(\Sigma_r^m b_{i_q r} a_r + \Sigma_\ell^s h_{i\ell}\lambda_i - \delta_i)]$$

$|Q^{(i)}| = |G^{(i)} + D^{(i)}|$ is determinant of sum of mx $G^{(i)}$ and a submx of the mx D which corresponds to chosen components$(f_{i_1}, \dots, f_{i_{mi}})$ of vector f; $q^{nt}(i)$ is an element of inverse mx $Q^{-1}(i)$.

Nash equilibrium strategies are defined as solution of the equations system, w.r. to components $(a_1, .., a_m)$.

62) S.I. Zukhovitskiy, R.A. Polak, M.E. Primak in USSR Academy of Sciences Doklady, vol. 163, No.2, 1965 and ibid. vol. 191, No.6, 1970.

$$\sum_r^m c_{ir} a_r = \rho_i \delta_i - \sum_i^s \eta_{it} \lambda_t, \quad (i = \overline{1, m}) \quad (*)$$

$$a_{ir} = \sum_{j,f}^{mi} q^{jf} (i) \, b_{ifi} \, b_{ij} \, r; \; \rho = \sum_{j,f}^{mi} q^{jf} (i) \, b_{ij} (i)$$

$$\eta_{it} = \sum_{j,f}^{mi} q^{jf} (i) \, b_{ifi} \, h_{ifr}, \quad (1, r = \overline{1, m}; \; t = \overline{1, s})$$

We substitute for $\delta = (\delta_1, .., \delta_m)$ the vector of co-ordination $\theta = (\theta, \nu)$ and assume that mx C, formed of elements c_{ir}, has inverse, and write solution of

$$a_i = \sum_r^\ell \theta_r \, w_{ir} + w_{i\theta} \qquad (i = \overline{1, m})$$

$$w_{ir} = \sum_{jr \varepsilon J} c^{ijr} \rho_{jr}$$

$$w_{i\theta} = \sum_k^t \nu_k \sum_{h_k \varepsilon I_k} c^{ih_k} \rho h_k - \sum_r^m \sum_k^s c^{ir} \eta_{rk} \lambda_k$$

The equilibrium strategies are pure strategies.

From this reformulation proceeds the Gadzhiyevian approximative solution which we omit.

Note 8 The next note introduces three constructs in stochastic planning which supplement - from some specific angle - those discussed in the preceding text.

Thus a design by Yudin exemplifies his theoretical considerations (§4, IV above) by modelling the operations of an intermediate-level planning and control agency - from the angle of inter-level communications.

Against this a model by Mitryashin claims to formalize stocha-

---

Gadzhiyev, op.cit, p.111

stically the actual practice of such an agency.

Finally,a model by Yefimov branches out from the dual solution of his stochastic plan-programme (§5,IV above) and offers an approach to stochastic approximative pricing under conditions of indeterminacy.

We cannot expand - within the scope of this essay on the all-important problem of optimal (or near-optimal) price dependable for steering the mandatory planned economy. One may be referred to the relevant chapters of my books Aspects of Planometrics (1967) and Mathematical Theory in Soviet Planning (1976) where the issue of such price is discussed at some length. It is my tenet that the problem of this price remains still unresolved inspite of crucial and seminal clarification of some of its elements. It is also our point that it is this state of the problem that handicaps - inspite of what is required by the growing complexity of the economy - a shift from a predominantly physical-term to a price-term planning and control with a price-type instrument playing (via a market-type mechanism)any significant role in the economy's allocational and distributive processes. It may be conjectured that attempts towards approximative stochastic pricing might help in handling this cucial problem.

The reader will be advised to acquaint himself in the present context with Aoki's [63] major inquiry into stochastic adjustment schemes. It is general enough to encompass pricing by a "market authority" as an agency of centralized planning.

---

63) M. Aoki in Annals of Economic and Social Management, No. 1, 1974; and the same, contribution to Conference on Optimization Techniques, Rome 1973 as quoted therein

It deals with the situation where excess demand in response to price p is modelled by $x(p) = f(p;\theta) + \zeta$, with f(.) known function of p, θ unknown parameter, ζ a random element; the alternative methods in handling this matter would consist in the pricing agent's "subjective" estimate of θ and his adopting a Bayesian approach; or in treating θ as an unknown constant vector and resorting to stochastic approximation or a programming algorithm e.g. the familiar stochastic gradient method (Aoki adopts the Bayesian method in price adjustment).

Note 9 (I) What follows is a rough outline of application of a stochastic plan model for an intermediate-level planning and control agency in actual Soviet practice. (The particular case described is that of sugar industry) [64].

The operation to be optimized with the use of the model is the supply of sugar beet root from n "collecting units" to m users (sugar plants. The plants requirements are A_j (j=1,...,m) over the plan period; the volume of sugar beet at each collecting point is B_i (i=1,...,n); further C_{ij} and X_{ij} are, respectively, the per-carriage cost of transporting a volume unit of sugar beet from collection point i to plant j, and the planned supplies from i to j.

In the past the plans were taking A_j, C_{ij}, B_i as exactly known. In recent planning for the branch uncertainty is being allowed for. The B_i at i depends patently on the acrage under the beet root and the average crop.

---

64) N.P. Mitryashin, "Opyt reshenya stokahsticheskoy modeli optimalnogo planirovanya", Ekonomika i Matematicheskiye Metody, No. 5, 1968

It has been found empirically that average-crop b_i data have a continuous - specifically a normal - distribution with some means and dispersions. Each random value of b_i with continuous distribution is approximated by means of finite b_{ik}, $k=1,\dots,k_i$, with known probabilities p_{ik}. The index k and the B_{ik} are observed to appear in the order of their increasing values: the probability of the volume of sugar beet at i exceeding B_{ik} is denoted $q_{ik} = 1 - \sum_{\mu=1}^{t} p_{i\mu}$.

The problem is stated as that of minimizing the expectation of the total cost of transporting the beet-root from the i's to the j's plus penalty for the amount left untransported in collection point (which has not reached that is the plant during the"campaign").

The stochastic optimal-problem for the "trust" embracing the n plants is modelled in this way

$$E\left(\sum_{i,j} C_{ij}X_{ij} + \sum_i h_i y_i^+\right) \rightarrow \min$$

$$\sum_i X_{ij} = A_j,$$

$$\sum_j X_{ij} + y_i^+ - y_i^- = B_i,$$

$$X_{ij} \geq 0; y_i^+, y_i^- \geq 0; \; i=1,\dots,n; \; j=1,\dots,m,$$

$$\left.\begin{array}{ll} y_i^+ = B_i - \sum_j X_{ij}, y_i^- = 0, & \\ & B_i \\ y_i^+ = 0,\; y_i^- = \sum_j X_{ij} - B_i, & \end{array}\right\} \begin{array}{l} \geq \sum_j X_{ij}, \\ \\ < \sum_j X_{ij}. \end{array}$$

(The variable y_i with superscript + or - denotes, respectively, the quantity of sugar beet left uncollected from the collection points, and the amount of sugar beet by which the volume of sugar beet at i falls below the plan target $\Sigma_j x_{ij}$).

(2) The Yudin theoretical approach has been made use of in his recent elaboration of an operational "instruction" for the "authority" ("association") which controls in the existing institutional set-up a group of production enterprises. Realistically for Soviet conditions it is assumed that such an "association" steers the operations of several units with the similar outputs - units which may employ some jointly held resources, such as specialized manpower. It is the task of the "association" to prescribe for each of the component units some operational variants most rational from the overall, that is in practice, the association's viewpoint.

We may give a broad idea of the set of instructions with reference to one of Yudin's several constructs. It is designed as a two-stage model for a quasi-Bayesian process where in the course of plan-implementation some indeterminate parameters acquire greater exactness possibly entailing some additional expenditure on restructuring the unit's production to accord with the new (fuller/more exact) information. The formula is then

---

65) B.B. Yudin, "Stokhasticheskiye metody razrabotki plana funktsyonirovanya proizvodstvennogo obyedinenya" <u>Ekonomika i Matematicheskiye Metody</u>, No. 6, 1974

$$E\{v \sum_{k=1}^{S} c^h(w)x^h + \min \sum_{k=1}^{S} [(1-v)c^k(w) + \sum_{k=1}^{n_k} e^h Q^k(w)] y^k(w)\} \to \min$$

$$\sum_{h-1}^{S} A^k(w)[vx^k + (1-v)y^k(w)] \geq b(w)$$

$$x^h \varepsilon X^k, \; y^k(w) \varepsilon X^k, \; k=1,\dots,s, \; \forall w$$

The solution of the problem consists of two vectors $x^* = \{x^{*k}_j\}$ and $y^{(w)} = \{y^{*k}_j(w)\}$. The deterministic vector x determines the plan at the first stage which is being implemented over the period T (the T-plan horizon). After the random parameter had materialized the conditions of the "Association's" problem are being adjusted and the remaining time periond (1-v)T it works according to the plan $y^*(w)$.

Notation

The agency ("association") (with m types of outputs ℓ types of resources) is formed of some enterprises, for each n_k plan variant being elaborated (k index of enterprise); c^k_j inputs (cost); a^k_{ij} output of i-th product of k working with j plan-variant; b_i demand for product; $\tilde{a}^k_{rj}$ input; $\tilde{b}_r$-volume of the r-th kind of resources at the disposal of the Association; x^k_j - parameter of control (1 or 0 respectively when the k does or does not work on the j variant); $y^k(w)$ is the vector of control parameters of the second stage, its component $y^k_j(w) = 1$ or $= 0$, respectively, when under a materialization of the random parameters w the k-th enterprise, at the 2nd stage, would work according to the j-th variant, or - otherwise; $Q^k(w)$ are matrices whose elements show additional cost of transfer of the k-th enterprise to the new plan at the same state of w; v is fraction of period worked under the original plan; e - unit row vector of dimention j_k with 1 at the t-th place.

The point of departure for Yefimov's [66] model of pricing under conditions of indeterminacy is an existential (equilibrium) theorem to the effect that where criterion of optimization described for a branch by model

$$E \sum_{j-1}^{N} (c_j, x_j) + E\{\min[(q^1, y^1) + (q^2, y^2)]\} \to \min, \tag{1}$$

$$\text{s.t.} \quad y^1 - y^2 = b - \sum_{j=1}^{N} A_j x_j, \tag{2}$$

$$y^1 \geq 0; \; y^2 \geq 0, \tag{3}$$

$$x_j \in X_j, \; j = 1, \ldots, N. \tag{4}$$

is expectation of the maximum profit, there exists prices $\Pi(x^*, A, b)$ for the branch's output such that a plan optimal for the enterprises is also optimal for the branch as a whole. A more general stochastic formulation of the plan is

$$E(c,x) + E\{\min(q,q)\} \to \min \tag{5}$$

$$W_y = b - Ax, \tag{6}$$

$$y \geq 0, \tag{7}$$

$$x \in X. \tag{8}$$

Notation is the same as in Yefimov's basic model (our §5,IV); c,b,A are random, and q and W are deterministic; W is a full matrix of dimensions $m \times \ell$. Discrepancies are removed by a matrix of "break-down-removal technologies" collected in matrix W and y is a vector of intensity in the use of these

66) V.M.Yefimov, Ekonomika i Matematicheskiye Metody, No. 3,1970

technologies.

Now the dual problem for the stochastic plan as above will take the form

$$(\pi, b - Ax) \rightarrow \max, \tag{9}$$

$$\pi W \leq q. \tag{10}$$

Suppose the random oscillations of demand are changing such that balance equation of the stochastic plan, as above (6) becomes

$$W_y = b + \Delta b - A\tau, \tag{11}$$

where $\Delta b = (\Delta b_1, \ldots, \Delta b_m)$ is a random vector. Thus e.g. if Δb_i is independent of random parameters of the stochastic plan programme,then the demand for the i-th product will have the expectation $Eb_i + E\Delta b_i$ and the dispersion $Db_i + D\Delta b_i$ (where D is symbol of dispersion). Denote the optimal value of the functional in the stochastic plan problem ((5),(11), (7),(8)) as $p(\Delta b)$: then the optimal value of the functional of the problem (5)-(8) will be $\rho(0)$.

It is then asserted that

$$\rho(\Delta b) = \rho(0) + E(\pi(x^*, A, b), \Delta b) + \varepsilon(\Delta b), \tag{12}$$

$$\frac{|\varepsilon(\Delta b)|}{\sqrt{E\left(\sum_{i=1}^{m} \Delta b_i^2\right)}} \rightarrow 0 \text{ with } \sqrt{E\left(\sum_{i=1}^{m} \Delta b_i^2\right)} \rightarrow 0, \tag{13}$$

where x^* is the optimal solution of (5)-(8) and $\Pi(x^*, A, b)$ is optimal sulution of (9),(10) when $x = x^*$. (In substance the

theorem asserts that $\Pi(x^*,A,b)$ is the Fréchet derivative **) $\rho(\Delta b)$ at the point 0).
The equation in the theorem referred to by us just now means that $\Pi(x^*,A,b)$ is the measure of the increment in the optimal value of the functional (5) for small changes in the right-hand sides of (b).

It is of crucial relevance for the possibilities of the model that the stochastic prices $\Pi(x^*,A,b)$ arrived at here for $x = x^*$ have properties analogous to those of dual valuations in problems of convex and linear programming(cf Dynkin, our 2, ch. IV).

It is claimed that the $E(\Pi(x^*,A,b)\Delta)$ can be computed by a Monte Carlo method.

**) The formalism of the Fréchet derivative may be recalled in these terms:

$$F \ni U(\zeta) \quad : \quad \exists U_{\zeta}(\zeta,t) \, \varepsilon \tilde{F}^{n-1}_{n-1}$$

<u>δU variation:</u>

a) linear in $\delta\zeta$

b) $U(\zeta+\delta\zeta) - U(\zeta) = \delta U + o(\delta\zeta)$

$$\delta U = \int U'_{\zeta}(\zeta,t)\,\delta\zeta(t)\,dt$$

5. The Doctrine of the Indeterminacy Zone

§ 1. We have been so far concerned with the work on the build - up of the foundations for the theory of planning which while oriented towards "indeterminacy" has been largely confined to the "established" stochastic theory. This we have treated here, as it were, as a prolegomenon to the inquiry on the "zone of indeterminacy" proper. Theories devoted explicitly to latter are the subject of the present chapter. In some sense - and taken the matter as a whole, it can be defensibly said that their constructions are of lesser rigour (and mathematical elegance) than some of those hitherto discussed. But the theories - we shall present here two of them - one by Lavrov and Markov, the other by Makarov, Makarova and Zeylger - stand out as an expression of the intensity of the planner's present day need of, and search for some practicable ways of replacing his "certainty-pretending" apparatus by a more realistic one; they are empirically rooted (in particular this is true of the second of the two mentioned) and are not unoriginal in their basic ideas.

§ 2. Something which may validly aspire to the status of an outline of a general theory of planning under conditions of indeterminacy will be found in a recent (1975) paper by Lavrov and Markov [67].

Its broad point-of-departure proposition is that indeterminacy in plan construction has its "objective" existence in the shape of possible states of nature, S, and actions A.

---

67) N.G. Lavrov, Yu.G. Markov "Planirovanye v uslovyakh neopredelennosti", Ekonomika i Matematicheskiye Metody, No. 2, 1975

Such being the background, the fundamental approach to planning must be based on expectations. This, though, is not an unqualified proposition. For it is granted that the practice of "perspective" (that is longer-run) planning points to situations where planning which relies on mean values of mathematical expectations would not guarantee the rationality of actions.

Another fundamental assumption is the existence of a criterial function (this by-passes such issues as the nature of the system of choices):the criterial function postulated is to measure whatever is accepted as the index of performance.

Under these assumptions the problem of planning appears as one of selection extremizing criterion function $F(x,w)$, possibly nonlinear and discontinuous, for $x \varepsilon X(w)$; and hypotheses are adopted with respect to the unknown distribution of the random w, the vector of initial data or parameters of the problem, in a region Ω. The plan (state) is described by a vector x from some n-dimensional space. The Ω is assumed to be known; as to w all that may be known is that $w \varepsilon \Omega$; or, at best - what is its law of distribution.

Formally, we would have the criterial $F_{is}=F(x_i,w_s)$, $i \varepsilon A$, $s \varepsilon S$, from which the payoff matrix $\|F_{is}\|$ is built up with columns and rows respectively w and x.

The forming of the finite set of admissible actions is handled as one analogous to a uniform distribution of the possible combination in a given domain. By discarding non-admissible actions a set B would be eventually built up. As a matter of fact in the present-day plannning practice the construction of the function F_{is} is considered too tricky. Hence the path followed is rather this: for a considered state of nature, an optimal strategy is discovered by means of some (deterministic) treatment of the problem of the economy's functioning: each of these strategies is then entered into B as admissible and relevant. Then each of them is probabilistically appraised. Specifically for the case of minimizing the F(.). In the process of selection, first inadmissible actions x_j are being eliminated such that $F_{js} \geq F_{is}$ for $\forall s \epsilon S$ with $i,j \epsilon A$. The remaining ones being adopted for the set $B \subset A$ each of them being better than others for at least one state of external conditions w. In "sieving out" the ultimate choice familiar criteria of

Wold: $$\min_{i \epsilon B} F_i^{max} = \min_{i \epsilon B} \max_{s \epsilon S} F_{is},$$

Laplace: $$\min_{i \epsilon B} \bar{F}_i = \min_{i \epsilon B} \frac{1}{k} \sum_{s=1}^{k} F_{is},$$

Savage: $$\min_{j \epsilon B} R_i^{max} = \min_{i \epsilon B} \max_{s \epsilon S} R_{is},$$

$$R_{is} = F_{is} - \min_{i \epsilon B} F_{js}$$

are taken to be, each of them, suitable in some situations - but only with qualifications in general. However, their <u>combined</u> application is taken to lead to "economically equivalent" strategies - in practice indifferent from the angle of the criterion. For the set thus formed $C \subset B \subset A$ the ultimate recommendation for "rational" action would take into account any additional points - in particular those evading formalization or related to extra-economic considerations.

We noted heretofore that the space of states exogenous to the system is postulated to be known. The way of acquiring this knowledge is of considerable interest both theoretical and practical. Patently, the number of possible states is, in principle, infinite and the plan-builder's task is to discover a limited number of points sufficiently "representative" of the whole - The matter is being investigated in several directions.

One of the directions pursued is by Saneyev, based on the Monte Carlo procedure. Where the vector of possible states is given by an interval of values, a large number of random combinations, within it, is being formed.

The task is to form a non-self-contradictory set of random combinations of conditions for the economic system within some given confidence limits. Then its elements are to be "sorted out" into K groups ($K \subset M$, where M is the system) by means of pattern recognition; the "central representatives" of the groups are treated as if they made up all the sets of probable conditions of the economy's advance (i.e. Ω; $k=1..K$).

Another direction rests on a method elaborated by Belov/Belayev/Maskin: one of optimally inscribing a given number of balls into an n-dimensional unit cube. It is applied by Makarov/Makarova/Zeyliger (see below our §3,V). Its substance is this: The set Ω is being normalized and reduced to the unit cube; depending on the problem's conditions, on the task of investigation and on computing capacities, a finite number of "states of nature" is adopted to represent the whole of the Ω: the relevant points are being distributed in the cube uniformly: then the centers of balls uniformly distributed over the cube are taken to represent the situation for the whole.

A third direction - by Lavrov and Markov - is in substance some adjustment of that of Saneyev, with an eye to the possibilities of the planning practice.

The probabilistic analysis of the plan solutions largely rests with Lavrov/Markov on parametric programming. Say, demand for all output components is a function of some random magnitude ζ with a known distribution: under full indeterminacy all of its values are taken to begin with, as equiprobable.

The analytical procedure starts from the solution of the parametric problem with some $\lambda \varepsilon \Omega$ and constructing the function $F(\lambda)$ which characterizes the criterion depending on λ: it is taken to be piecewise linear, monotonic, but possibly - depending on the problem's conditions, convex (concave) - properties believed to agree with reality in most cases of actual planning and making it easier to determine the probability of its not exceeding (or, as the case may be, equalling at least) some postulated C. Take domain Ω as the interval $[a,b]$ and $F(\lambda)$-monotonic, convex; the probability $P(F(\zeta) \leq C)$ would equal that of ζ belonging the segment $[a,\lambda(C)]$; the value of parameter $\lambda(C)$ to be found from the condition $F(\lambda)C$. If ζ is uniformly distributed over [a,b] then

$$P(F(\zeta) \leq C) = \frac{\lambda(C)-a}{b-a} \quad .$$

Within the critrical points of intervals the $\lambda(C)$ and $P(F(\zeta) \leq C)$ are varying linearly; beyond it with the rising of the threshold value C the probability of P(.) rises at an accelerating pace.

The method has various applicational possibilities under not too exacting circumstances. One of them is plan analysis, in conditions of uncertainty, of intermediate inputs and possibly requirements. A coordinate-wise procedure may be rewarding (provided the number of random elements is not too large.)

Another applicational possibility is the case of some minimum benefit (say profit) to be secured. Suppose demand for output is ζ (taken to be uniformly distributed over [a,b] as above). The C, minimum benefit, guaranteed with given probability p_o appears to be

$$\lambda(C) = (1-p_o)b+p_o a.$$

$$\zeta = \lambda(C).$$

Technically, the probabilistic element would be handled by means of a two-stage stochastic programming (in Soviet application success is claimed for it, in particular, in optimization of plans for the all country's system of energy production and distribution). First the sets of possible states of nature $\{w_s | s \epsilon S\}$ and of admissible actions $\{k_i | i \epsilon B\}$ are formed; in the process K (k=1,...,K) optimal values of Φ^i_{opt} are calculated corresponding to K states of nature. Where deviation of reality, s, from the optimal plan-version is detected, additional costs $\Delta\Phi^i_s$ incurred to implement action, i (allowance made for actual situation (s)) are calculated, again with the use of the two-stage model. Thus we take $\Delta\Phi^i_s = \Phi^i_s - \Phi^i_{opt}$ where Φ^i_s is the index of cost involved in implementing the i-th action under the various conditions of "nature", s. Indices of the Φ^i_s make the elements of the payoff matrix i.e. $F_{is} = \Delta\Phi^i_s$; and for each action x_i the $|S|$ values of economic risk F_{is} are determined; the payoff matrix thus obtained is meant to show the possible economic consequences of ignorance of actual conditions of the economy's development.

Finally, we find in Lavrov/Markov an attempt to cope with the planner's "curse of dimensions" - the problem of choosing the suitable degree of aggregation (with references to Chirba,Hoeffding, Cortillot).

Indeterminacy for the multidimensional case is handled starting from the simplest plan-programming case

$$F = cx \rightarrow max,$$
$$Ax-b \leq 0,$$
$$x \geq 0$$

with A,b,c random magnitudes of dimensions respectively m.n, m and n. Rather than trying to obtain the distribution of maximum value of the F(.) functional from the (known) distribution functions of A,b,c, the less exacting path is being followed: to secure some values of the distribution of the objective function. Suppose that for A,b,c the expectations and the dispersions are respectively A_o, b_o, c_o, and α, β, γ. (The random elements are numbered by elements of the sets I with $|I| = m.n+m+n$ and each of the elements is supposed to be located within the interval $[a_i, b_i], i\varepsilon I$); and denote

$$F_{\pm s} = F(A_o \pm s\alpha,\ b_o \pm s\beta, c_o \pm s\gamma).$$

Then for independent random magnitudes the lowest value for probability p(s) is shown to be

$$p(s) = \prod_{i\varepsilon T} [1-2e^{-2s^2/(b_i-c_i)^2}]$$

and in the normal-distribution case $p(s) \geq (2\Phi(s)-1)^{|I|}$,

where $\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-u^2/2} du.$

In Chirba [68] for the case of independent random variables from (A,b,c), normally distributed the lowest value for probability p(s) appears as $p(s) \geq [2\Phi(s)-1]^{|I|-N} (\Phi(s))^{\bar{N}-N}$

Where only the vector c (with normally distributed components) is random he gives

$$[\Phi(s)]^{|I|} \leq P(F(c) \leq F^*) \leq \frac{1}{\sqrt{2\pi}\ \delta} \int_{-\infty}^{F^*} e^{-(u-F_o)^2/2\delta^2} du,$$

$$F^* = F(c_o+s\gamma), F_o = (c_o, \hat{x}),\ \delta^2 = \sum_{j=i}^{n} \hat{x}_j^2 \gamma_j^2,\ \hat{x}\text{-optimal plan}$$

with $c=c_o$. With rising dimension, number I and p(s) repidly decline.

68) S.S. Chirba, "Nekotoryie voprosy passivnogo stokhasticheskogo Lineynogo programmiroyanva" in Voprosy Ekonomicheskogo Matematicheskogo Modelirovanya, 1972 (as cited in Lavrov and Makarov, op.cit.)

The exercise we present here carries but the extremely simple cases. And the workability of the apparatus is anything but certain. But what is being done points to some potentially fruitful direction in the planner's struggle with dimensionality. For it is a credible supposition that aggregation of indices does secure a degree of exactness through averaging and mutual compensation of the random factors.

Remark 1 The Monte-Carlo method still enjoys the favour of students and practicians of planning under indeterminacy (inspite of some reservations, such as in Chow, see reference in Note 7.) It does so at least on two counts. Firstly,because of the wide range of the random-values generators that can be employed in working with it (among others that can be a suitably instructed computer). Secondly, because of the acceptability of probability distribution with an inexacting level of explicitness. However, there have appeared in Soviet literature some more sophisticated elaborations of the method, such as e.g. a combination of difference (iterative) algorithms with Monte-Carlo method [69]: the effectivness of that method is being defined as the mathematical expectation of the total number of arithmetic operations necessary for securing - with some postulated probability - a given degree exactness.

The 1970s have seen also effective efforts for application of Monte-Carlo techniques in handling optimal information processing. An interesting generalization is due to Zaritskiy,

---

69) cf B.S.Yelepov, V.P.Ilyin, Zhurnal Vychislitelnoy Matematiki i Fiziki, No. 2, 1975

Svyetnik and Shimelevich [70]. Their central ideas is to bring into the operation an auxiliary Markovian process: a process wherein the a posteriori characteristics are formulated in terms of expectations for some functionals of possible values. The Zaritskiy et al construct is designed for both discrete and continuous time cases; our very rough sketch is limited to the latter only. Here Markovian diffusion process is

$X_t=\{U_t,V_t\}$; $U_t=\{U_{\alpha t},\alpha=\overline{1,m}\}$; $V_t=\{V_{\rho t};\ \rho=m+1,n\}$.

In the subvector $V\rho$ components $V_{\rho t}$ are subject to measure (over the processing interval) $V_\tau=Y_\tau$ for $\forall\tau\varepsilon[0,t]$. Process X_t is stated in terms of initial density $f_o(x_o)$ and local parameters $\{A(x_t,t),\ F(x_t,t)$ $A=\{A_\alpha;\alpha=\overline{1,m}\}$, $F=\{F_\rho;\rho=\overline{m+1,n}\}$.

Thus $X_t = U_t{'}V_t$ satisfies the system of the Jtô-type stochastic differential equations

$$\mathring{U}_t=A(U_t,V_t,t) + W_t,\quad \mathring{V}_t=F(U_t,V_t,t)+\Omega_t,$$

Here (W_t,Ω_t) is the vector of Gaussian white noise with zero mean. For the subvector of the nonmeasured components U_t

$$\hat{U}_t(Y_o^t) = \int u_t w_t(u_t)du_t,$$

the $w_t(u_t) = f(u_t|Y_o^t)$ being the a posteriori density of subvector U_t.

The solution rests on a sublimit model, with respect to time, of the X_t process and the moving to the limit. For discreitization we consider the system of finite difference equations

---

70) V.S. Zaritskiy, V.B. Svyetnik, L.I.Shimelevich, <u>Avtomatika i Telemekhanika</u>, No. 12, 1975

$$U_{t+\Delta} = U_t+\Delta A+\theta_t, \quad V_{t+\Delta} = V_t+\Delta F+\Lambda_t,$$

If U is the set of admissible control functions, then

$$w_1(x(0)) = \max_{u\varepsilon U} \min_{s\varepsilon S}(c,x(N)) =$$

$$=\max_{u\varepsilon U} \min_{s\varepsilon S}[c,y(N)+(c,z(N))]=\max_{u\varepsilon U}(c,y(N))+\min_{s\varepsilon S}(c,z(N)).$$

Consequently, the initial problem is replaced by a dynamic linear programming problem the substance of which is finding that u* which extremizes the performance index max $I_1=(c,y(N))$. Now to secure the guaranteed value of the performance index the z-related problem $\min_s(c,z(N))$ subject to the respective constraints (S) must be solved.

The solution is obtained by formulating two Hamiltonians [71] one for each problem in order to obtain the necessary and sufficient conditions of optimal control. (There is the well-known Rozenoer paper on the relationship between Maximum principle and dynamic programming, of 1957, which points to such a partition. In such partitioning of the optimization problems both linear dynamic problems are endowed with the same structure and do not entail special constraints on the state variables).

---

[71] the Hamiltonian function has the form

$$\max_{s(i)\in S_i} H(p(i+1),s(i)=H(p(i+1),s^*(i)),$$

We note also a reformulation of the problem

$$x(k+1)=A(k)x(k)+B(k)u(k)-s(k),$$
$$G(k)x(k)+D(k)u(k)\geq d(k),$$
$$s(k)\in S_k \quad (k=0,\ldots,N-1),$$
$$I=\Phi(x(N))\to\max.$$

Φ is a quasiconvex upward C^1-class function, s_k are convex compact. And the constraint s(k) is acting here as a "player" in a game choosing the u.

[Another paper by the same authors, in Avtomatika i Telemekhanika No. 3, 1974, reworks the dynamic programming problem into one of feedback, with the instantaneous values of the state known at each step k].

Remark 2 From its very beginnings traditionalist planning relied on the technological "normative" - per unit input of resources (factors or intermediate products, in physical and/or money terms) as its cornerstone. To be sure long before its mathematization the "normative" established itself as the principal element in plan construction and instrument for checking the plan's discipline implementation and appraisal of performance(plan-underfulfilment, fulfilment, overfulfilment) on both the micro and macro plane.

Starting from very primitive non-formalized quesses, the dynamization of "normatives" has gradually absorbed some mathematical methods. One,claiming adequate result in prognostication (designed by Olivson and Levin [72]), is this.

---

72) L.M. Olivson, V.F.Levin, Ekonomika i Matematicheskiye Metody, No. 6, 1973

Statistical series are employed in building up a production function y giving the dependence of each factor x_i on time intervals (s=1,...,m) over period t_s, T_s . All functions are built as approximative, say by a least-squares method

$$f,(g_1(\tau_s),...,g_n(\tau)) = \max_{t\varepsilon[t_s T_s+\tau]} f_s(g_1(t),...,g_n(t)).$$

$$x_i=g_i(t), \quad i=1,...,n.$$

Then approximation to a maximum of the function is sought over $[t_s, T_s+r]$ where r relates to a period permitting extrapolation.

Ultimately the formulae for y_s and x_i are arrived at:

$$y_s=\alpha(s)\prod_{i=1}^{n} x_i^{\beta_i(s)} e^{\gamma_i(s)x_i} \qquad (*)$$

$$x_i=a_i t^{b_i} e^{c_i t}+d_i. \qquad (**)$$

Under given tendencies of technical progress, organizational conditions the pseudo-optimal factor relations are arrived at: these are described by parameters of factors in function (**), - a_i, b_i, c_i, d_i - and parameters of functions (*) a(s); $\beta_i(s), \gamma_i(s)$, at points $\hat{y}_s(\tau_s)=\alpha(s)\prod_{i=1}^{n}(x_i(\tau_s))^{\beta_i(s)} e^{\gamma_i(s)x_i(\tau_s)}$. (Again least sqare method is employed).

§ 3. It is with the work of Makarov/Makarova/Zeyliger (and in particular their 1970 paper on the "zone of indeterminacy in development of complex economic systems" [73])that the conception (and term) of such a zone can be identified. The "zone" is being

---

73) A.A.Makarov, A.S. Makarova, A.I. Zeyliger, "Issledovanye zony neopredelennosti optimalnogo razvitya slozhnykh ekeonomicheskikh sistem", <u>Ekonomika i Matematicheskiye Metody</u>, No. 6, 1970

delimited first by setting out "by hand" the economy's basic development alternatives and next, forming by the Monte-Carlo method, a sufficiently large collection of initial data for each of the alternative. In this the multiplicity of random factors, their synchronic changes and individual characteristics of dispersion variance coefficients (errors) and possible distribtuion laws of all the influencing factors are taken into consideration. (The form of these laws would be possibly derived from the principle of maximal entropy, based on intuitive considerations as to the probabilistic characteristics of a random magnitude; where only the interval of its variation is known, uniform distribution would be accepted; where the most likely value of the magnitude can be estimated - normal or an asymptotic-beta distribution could be adopted depending on the location of the most probable value with respect to the center of the interval). The hypotheses as to the distribution - none too trustworthy - are taken as a preliminary technical device to be descarded at later stages of the analysis. For the collection of alternatives thus formulated, and suitably grouped in the process, some "essential" parameters are "sieved out".

As the crucial characteristics of the situational varaints we have in Makarov/Makarova/Zeyliger risk: risk of possible increase of cost entailed in ignorance of the future.

Note that in appraising the significance of quantitative differences of solutionsqualitatively belonging to the same class,methods of pattern recognition are employed. Optimal solutions are interpreted as points in an n-dimensional space with coordinates determined by values of each of the m parameters.

Next regions of density indicating relatively lesser discrepancy serve to identify groupings.

For cases (1) of randomness of the coefficients of the model's functional(the value of economic risk λ_r^n in the variant r with a p^n combination of the initial data) (2) of randomness of the constraint vectors (3) of change in both the economic and "natural" indicators, the risk is defined respectively as

$$(1) \quad \lambda_r^n = p^n X_r - p^n X_n = \Phi_r^n - \Phi_{opt}^n$$

$$(2) \quad \lambda_r^n = \varphi_r B^n - \varphi_n B^n = \Phi_r^n - \Phi_{opt}^n$$

$$(3) \quad \lambda_r^n = p^n U_\tau B^n - p^n U_n B^n = \Phi_r^n - \Phi_{opt}^n$$

(Φ denotes total cost; subscript "opt" - the optimality case)

In (1) the X_r, X_n are the development variants under the optimal and the actual p^n; in (2) the φ_r, φ_n are optimal prices with the B^n being the values of the economy's requirements (and constraints in resources) as respectively planned and materialized; in (3) the U_r, U_n are inverted matrices of the optimal base for the proposed and materialized values of initial data.

Inevitably the calculation of economic risk, as described here, makes sense, prevalently, at initial stages of the plan gestation - before, that is, committed resources had been "frozen" in some assets beyond the possibility of effective adjustment. For each variant r N values of economic risk are determined.

From the results an orthogonal matrix of values for the economic risk is being compiled with columns and rows respectively denoting the considered random combinations of initial data and the optimal development variants. Here it is this matrix that is intended to provide the generalized characteristics of economic consequences resulting from ignorance of conditions for the economy's development - one of particular help in comparing development variants in an environment affected by indeterminacy.

The matrix as a whole is thus thought of as the image of the zone of indeterminacy

	P_1	P_2	...	P_N
X_1	λ_1^1	λ_1^2	...	λ_1^N
X_2	λ_2^1	λ_2^2	...	λ_2^N
⋮	⋮	⋮	⋮	⋮
X_R	λ_R^1	λ_R^2	...	λ_R^N

ColumnP - random combinations of the system's development paths

rows X - optimal variants of these paths.

In a final choice-making the matrix of risk indices would be interpreted as a two-person zero-sum game and solved as a linear programming problem

$$\min\left\{\sum_{n=1}^{N} y_n \,\middle|\, \sum_{n=1}^{N} \lambda_r^n y_n \geq 1; y_n \geq 0\right\};$$

Then a new variant is formed

$$\left(\tilde{\lambda} = \sum_{r=1}^{R} \gamma_r \lambda_r\right) > \quad (\lambda = \min_r \lambda_r); \; v < (\bar{\lambda} = \min_r \lambda_r).$$

Inequalities above in the second line indicate relationship with alternative appraisals of variants. The intention is to have a somewhat higher value of average risk, $\bar{\lambda}$, than yielded by the criterion of minimum mean value, but - on the other hand - one more optimistic (with lower maximum ν) than one secured by that of minimax.

As it appears, the approaches and methods have been tested empirically (in the planning of the all-national electricity system), and merit attention. The more so do the conclusions - as to desirable line of further research in the field - deserve attention. We would single out two observations.One commends intensifying the inquiry into the relative potential and significance measure of criteria for the planner's operation in the indeterminacy zone:relative in the sense of account being taken of errors in initial data. (It is maintained that, generally speaking, the optimization approach has been empirically proved to be on balance profitable when an undue technical complexity of the apparatus employed is avoided). The second concerns the inquiry into the probabilistic properties of initial economic information and the elaboration of methodology for quantification. What is meant here first and foremost is the need for tests in other systems: tests to discover whether the Makarov/Makarova/Zeyliger finding as to the weak influence (on the basic results for a system's optimization) of the shape of the distribution law of the initial data can be credited with any general validity.

The truthfulness and the generality of the proposition is indeed a subject with calls for scrutiny.

§ 4 A particular, narrowed case of indeterminacy in planning is discussed in Soviet literature by Propoy and Yadykin [74]. It is one in which indeterminacy is "encapsulated" in some vectors for which only the domain of variation is known to the planning agency.

Very loosely in the problem formulation the performance index is $I = (c,x(N) \to \max$ subject to

$$x(k+1)=A(k)x(k) + B(k)u(k)+s(k) \quad (k=0,\dots,N-1)$$

$$x(0)=a$$

$$G(k)x(k) + D(k)u(k) \geq d(k) \quad (k=0,\dots,N-1)$$

Vectors $x(k)$, $u(k)$ are respectively the system's state and control function; k indexes the step. Matrices A,B,G,D are given. Of vector $s(k)$ all that is known is its domain of variation $s(k) = S$

Then we introduce the system

$$y(k+1) = A(k)y(k)+B(k)u(k),$$

$$y(0) = x(0) = a$$

(and consider the "error" dynamics of both systems, i.e. the initial and the last indecated); and - also a system of vectors $R(k)$ such that we have

---

74) A.I. Propoy, A.B. Yadykin, Avtomatika i Telemekhanika, parts I and II, nos. 2, 1974 and 3,1974.

$$z(k+1)=A(k)z(k)+s(k),$$
$$z(0)=0, \quad s(k) \in S_k, \qquad (\S)$$
$$z(k)=x(k)-y(k).$$

$$R_j(k) = \min_{s(0),\dots,s(k-1)} (g_j(k), z(k))$$

$$D(k)u(k)+G(k)y(k) \geq d(k)-R(k);$$

$g_j(k)$ is the j-th row of matrix $G(k)$; j and k are fixed in the minimand.

Here $\{\Theta_t, \Lambda_t\}$ is discrete Gaussian white noise with zero mean and covariance matrix ΔE (Δ denotes a step of discretization). The formula for the sublimit optimal estimate is of the form

$$\hat{U}_k(Y_o^k) = \frac{E_{\tilde{U}_o^k}\{\tilde{U}_k \varphi_k(\tilde{U}_o^{k-1}, Y_o^k)\}}{E_{\tilde{U}_o^{k-1}}\{\varphi_k(\tilde{U}_o^{k-1}, Y_o^k)\}}.$$

with the limit for $\Delta \to 0$ being the optimal estimate $\hat{U}_t(Y_o^t)$. The assumption - that has a proof in Zaritskiy et al.- is that in going to the r.h. firstly the aposteriori Markovian sequence which corresponds to the sublimit model is actually transformed into a Markovian diffusion process, and secondly that the sequence does actually possess a continuous analogue.

The employment of the Monte-Carlo technique in calculating U_t confines to modelling N possible values $Z_o t[j]$ of the a posteriori process $\tilde{U}_t$ from the differential stochastic equations

$$\widetilde{U}_t = \widetilde{A}(\widetilde{U}_t, t) + \widetilde{W}_t \ ,$$

solving for these values the relevant equation; and then - approximate evaluating the expectation by following the equation

$$\hat{U}_t(Y_o{}^t) \approx \sum_{j=1}^{N} z_t[j]\psi_t[j] \Big/ \sum_{j=1}^{N} \psi_t[j].$$

The promise carried by the method is claimed for three virtues: estimation of accuracy on line, its independence of dimensions of the modelled process and universality.

Note 10. While there is a fast rising sophistication of the theoretical inquiry there is at the same a growingly strong call for simple instruments in the handling - in the planning practice - of elements of indeterminacy. Among such approaches and techniques, those broadly labelled "certainty equivalence", take prominent place in the directly planning-oriented Soviet work.

It is only right to note that, while born in Western literature, the concept was being, from its initiation, related to planning. To quote from the co-pioneer's Herbert Simon's paper of two decades' ago "... when the criterion function is quadratic, the planning problem (our emphasis) for the case of uncertainty can be reduced to the problem for the case of certainty simply by replacing, in the computation of the optimal first-period action, the "certain" future values of variables by their unconditional expectations. In this sense the unconditional expected values of these variables may be regarded as a set of sufficient statistics for the entire joint probability distribution, or alternatively, as a set of "certainty equivalents." [75] (p.74). (Theil's paper of one year later relates in its very title the certainty equivalent to "dynamic planning" [76]).

In Simon's study the idea had a control-theoretic link-up: the construct was Bellman's dynamic programming (with the quadratic criterion function). There is then the seeming paradox that whereas the analysis is performed in terms of certainty "as if" the future was perfectly known it is optimal in the sense of extremizing the expected value of the decision criterion. The explanation naturally lies in that <u>linear</u> decision rule have been demonstrated (by both the co-pioneers Theil and Simon) to have the property of certainty equivalence.

It would exceed the scope of this essay to discuss in detail the well-known methodology. We may just observe that it has made - and very desiredly so - a striking career in planning under indeterminacy in both East and West; and that its theory has reached (as it is validly claimed) the stage of definitiveness.

As a matter of fact there are by now several decision-analytical models which fall under the heading. We have then the Theil approach resorting for the purpose of decision opimizing to matrix inversion. Subsequently, Theil and others directed their work also to problems of stability. The use

---

75) H.A. Simon, "Dynamic Programming under Uncertainty," <u>Econometrica</u>, 1956

76) H. Theil, <u>Econometrica</u>, 1957

of the Bellmanian dynamic programming is characteristic of the approach of Kalman, Lapidus and Shapiro. Holt, Modigliani, Muth and Simon [77] have done celebrated work in the field of planning of production, inventories and manpower. Finally of late (1975) Hay and Holt [78] produced a new improved version of the original Holt-Modigliani construct. In this the z-transform is being made use of. We have here exogenous variables explicited in a fashion such that (unlike what prevails in control-theoretic designs) the forecasting problem lends itself to isolating. The z-transform when applied obviates (the often onerous) inversion of matrices. On the other hand that entails a numerical search for a polynomial's roots; and the Kalman matrix iteration is being carried out. The decision problem in Hay-Holt has a formulation close to that of the real-life planner. Its content is roughly this: find values of controls extremizing the decision-maker's welfare subject to constraints which stem from the relationship between variables - those controlled, those partly controlled and those uncontrolled, the last being taken to make his environment; and his task is to try to find the best "response" by suitably adjusting the controlled variables (in the design the partly and indirectly controlled variables which appear in the criterion function are taken to be determined by a linear relationship with the controlled and uncontrolled variables).

---

77) C.C. Holt, C.C.F. Modigliani, J.F. Muth, H.A. Simon, Planning Production Inventories and Work Force, 1960

78) G.A. Hay, C.C. Holt, "A General Solution for Linear Decision Rules": An Optimal Dynamic Strategy Applicable under Uncertainty", Econometrica, No. 2, 1975

In Theil's [79] writing we may draw attention to his rigorous discussion on replacing stochastic consequences by their mean value with respect to the welfare function as the planner's optimand. There the argument leads to conclusions which do justify the use of the mean value as the predictor of random variables: the unbiased point predictions being shown as the certainty equivalents; but the reasoning demonstrates also the assumptions behind the arguments; where these assumptions fail to apply it is the certainty bias rather than its equivalence that must be expected.

Note 11. A rather new issue in Soviet thinking on planning in conditions of indeterminacy is that of the place in it of what more or less conventionally can be labelled as econometric modelling. In a sense one can rather loosely describe the trend as one of integration of planning (policy-making) and forcasting in particular in control-theoretic terms - an integration progressing actually from opposite directions in both West and East, with the latter rather lagging behind the former.

In the West a conscious work has been undertaken to reach a "rapprochement" between the two areas of theoretical work. Thus admirable work has been carried out by Mehra [80] to close, or at least to narrow, the gap between the control and the econometric studies on the system identification: a gap which exists

---

79) H. Theil, Economic Forecasts and Policy, 1970, pp. 414 ff; note also the reference J. Durbin, "The effects of forecasting errors in dynamic programming with a quadratic cost function", (mimeographed, London School of Economics, 1959)

80) R.K. Mehra, Annals of Economic and Social Measurement, No. 1, 1974.

although both rely largely on the theories of probability and statistical inference, in particular least squares and Bayesian inference.

The last ten years or so have also recorded interesting "integrational" efforts in modelling. It is noteworthy though that at the same time a discussion has developed in which some students have aired their doubt as to practicability limits of a stochastic-control, type of optimization under the present state of art. Thus Pindyck [81] while granting that in any case econometric modelling is in some sense stochastic: yet - so he argues-solving a stochastic optimal control problem in policy-making, one that takes into account both the implicit additive error terms and the statistical properties of the estimated coefficients, may be extremely difficult, especially in the case of a large model. (His reservations would not seem to apply to deterministic treatment - linear-quadratic deterministic control of a non-linear stochastic system - inasmuch as the self-correcting nature of the linear control regime would ensure not only a better computability but also a tolerable suboptimality.) In what may appear as in a sense supplementing the inquiry, Chow reports [82] about the results of his exercise - measuring the welfare gains by following an optimal stochastic control policy as compared with suboptimal policy which only permits a constant rate of growth for each policy variable: when breaking up the impact into two parts, it is shown that the gain from optimal deterministic control policy over the suboptimal policy

81) R.S. Pindyck, Annals of Economic and Social Measurement, No.4, 1972

82) G.C. Chow, Annals of Economic and Social Measurement, No. 4, 1972

is of far lesser consequence than the gain from optimal stochastic control over optimal deterministic control.

A few more lines about the results obtained in Western experimentation in modelling. Thus Fair has demonstrated the solvability of a large problem of optimal control for an econometric model treated as an unconstrained nonlinear maximization, with the stochastic problem being solved by means of stochastic simulation.[83] Of late Chow[84] generalized

---

83) R.C. Fair , Annals of Economic and Social Measurement, No.1, 1972

To expand on the point made by Fair, it is that while the stochastic closed-loop feedback control does obtain the optimal first-period control values in the case of a linear model with additive error terms and quadratic criterion function, through solving the deterministic control problem (setting the error terms to their expected values), the first-period certainty-equivalence does not hold for non-linear systems. For a nonlinear system the mean values of endogenous variables could be obtained by stochastic simulation. For a relatively small problem it would be feasible to secure optimal open-loop controls for non-linear stochastic model in a procedure similar to that for deterministic model. (As to the applicability - in economics, as against engineering - of the Athans designed procedure (solving deterministic control problem and then linearizing around deterministic-control paths to obtain linear feedback equations around the paths) there appears to be an intuitively-based controversy).

Note the Keleijan/Howrey assertion that in the case of nonlinear systems, endogenous variables' values-yielded from solution of a model where all stochastic variables have been set equal to their expected value - would not be, in general, equal to expected values of such endogenous variables (cf. E.P. Howrey, H.H. Keleijan, "Simulation versus Analytical Solutions, the Case of Econometric Models" in T.H. Naylor, Ed., Computer Simulation Experiment with Models of Economic Systems, 1971).

84) G.S. Chow, Econometrica, July 1976

his well-known previous results - in a search for an approximate solution, relying on dynamic programming, for an optimal control of a system of nonlinear structural equations in econometrics with unknown parameters. It makes an improvement over the method of certainty equivalence replacing the unknown parameters by their expectations. (The solution is given in the form of feedback control equations helping the treatment of nonlinear system by the method designed - a method which also permits calculation of the expected loss by analytical techniques: it is taken to be a virtue that costly Monte-Carlo calculations are avoided).

[Well representative of attempts, during the last decades or so, in the West to build into econometric models effective optimization, essentially - a stochastic control-theoretic construct - is the attractive 1975 exercise by Cooper and Fischer [85]. In that case - this is a method for controlling the stochastic - "St. Louis"-model: a method which consists in estimating the means and variances of the dynamic responses of the nonlinear model to changes in instrument variables - in order to produce the model's linear representation with random coefficients, and then using this linear, stochastic version of optimization. (Cf. also Adams and Burmeister on exercises with the Wharton US economy model [86].) A related type of exercise

---

85) Ph.C. Cooper, S. Fischer, Econometrica, 1975; cf. also their paper in Journal of Money, Credit and Banking, Febr. 1974

86) F.G. Adams, E. Burmeister, in IEEE Transactions on Systems, Man and Cybernetics, Vol. SMC-3, No. 1, 1973

are simulations by Bray [87] of actual attempts to control a stochastic model of an economy - in the case of the actual controlling policies of the United Kingdom economy - by an "informal predictive" control system. We may also cite here an illuminating exercise by Tintner [88] in fitting results from his econometric inquiry for India's agriculture into a stochastic, control-theoretic, Maximum Principle based model.]

Now, in mandatorily planning countries, philosophical attitudes apart, what appears to have hampered until not long ago the acceptance of econometric modelling as such was the fastness and irregularity of the pace of the structural changes in the economy: its slow-down might be thought of as one of the factors which contribute to the growing interest in such modelling. Be this as it may there is now a strong feeling through the area of mandatory planning that methods of econometric prediction can provide a very important auxiliary tool for planning authorities (cf. Pawlowski [89]); although as a recent contributor to Ekonomika i Matematicheskiye Metody, Kotas, notes [90] "the intimate link of planning methods and forecasting with econometric modelling still fails to be understood", forecasting being applied, as a rule, to uncontrollable processes such as mateorological facts, foreign trade, rates of technological progress - an observation which corres-

---

87) J. Bray, in Annals of Economic and Social Management, No.1, 1974

88) cf. G. Tintner, American Journal of Agricultural Economics, Vol. 51, No. 2 May 1969

89) Z. Pawlowski, "On some applications of econometric predictions in economic planning" in IFAC/IFORS International Conference on Dynamic Modelling, University of Warwick, 1973

90) M. Kotas, Ekonomika i Matematicheskiye Metody, No. 5, 1973

ponds to the prevailing position (patently unqualified lumping together the three examples of uncontrollability may be questioned). However, even those who accept the integration of econometric methods into optimizing planning still maintain that theoretical arguments as well experience suggest that econometric prediction is especially suitable for short term and medium term (up to five years) inference; the argument rests again on its obsolescence where structural change is fast. In Pawlowski's view the use of econometric methods could be most helpful in prediction of non-plannable elements, prediction of future values of technological and economic coefficients, of effects of alternative policy measures, comparison of effects of plan variants, prediction of degree of fullfilment and deviation from plans.

Technically (cf. Kotas) the interaction between planning and econometric prognostication can take various forms. One would be to employ econometric estimates as a means for widening the sphere of factual data: this object would be served by an absorption of a more sophisticated and precise system of interrelation indices (more dependable than the traditional "direct" derivation from statistical data or with regression estimated on time series). An alternative possibility would consist in limiting the area of the planned-target indices to predetermined variables, and - in forecasting endogenous variables my means of a model. However - to come back to the issue mentioned - it is the indeterminacy of structural changes that creates the main, most difficult dilemma in forecasting with the help of an econometric model. As a matter of fact, in addition to difficulties in synchronization this - so we are told - proved to be the most troublesome point in the (Hungarian) planners' experiments with this method: as it appears in that case the cohesion of forecast and plan had to be sought by **means** of " backward simulation".

<u>Remark</u> This remark touches upon the wider issue as to whether - as some students maintain - advances in mathematical-economic methods and techniques, in particular in so far as they can be based on stochastic optimal control (or to be still more specific on Kalman type-filter),have not obviated the employment of "traditional" econometric techniques in plan-programming. It seems to us that the question can be answered in the negative; and in this we find convincing Athans [91] argument broadly to this effect. To apply the Kalman-filter based algorithm one must know: 1) the statistics of the measurement white noise, 2) the system's structural elements in particular some informational lags, 3) some initial guess-timates for parameters, in particular for the prior mean-expectation for state and for the initial value of the parameter variances and covariances in the respective matrix. And the information can be expected to be supplied by the "traditional" econometric model.

[The work of the Soviet Academy's Computation Center has been outlined of late broadly in these terms [92]. At this stage, they consider three stages in plan designing 1) forecast based on specially processed information on possible alternatives

---

91) M. Athans, "The importance of Kalman filtering methods for economic systems", <u>Annals of Economic and Social Measurement</u>, No. 1, 1974

92) cf. N.N. Moiseyev's contribution to discussion in <u>Methods of Long-term Planning and Forecasting</u> edited by T.S. Khachaturov, Macmillan Press 1976

aimed at formulation of goals, 2) development programmes which set objectives and rely on "clear understanding" of the possibility of reaching the goals, 3) plan proper for resource allocation. The three elements have different degrees of authority and detail. The whole exercise is rooted in the assumption of "manageability": the prognosis starts by discovering the extent of manipulable factors whereas the plan is a synthesis of the three elements. The forecasts result in a cone of possible trajectories of the system (rather than a single design).

As to forecasting proper two procedures are taken into consideration. One rests on expert valuation and thus continues the planning tradition. The other is based on direct modelling and mathematical analysis. This procedure is not accepted to be universal if only because it entails astronomical quantities of computation time. At present, attempts are made to combine the two procedures in one synthetic approach which has been labelled the "emittance system". It is believed that the work carries some analogies with the Forst exercise at M.I.T.; the differences relate to substance and the structure of models.]

6. A Few Concluding Remarks

A few remarks may seem to be not out of place at this concluding stage or our essay.

The proposition that the planner operates in what is describable as a "zone of indeterminacy" has been if anything rather overdue - both in theory and in practice. (We have made at the outset some allusions as to the factors which have contributed to the delay in its acceptance). On the other hand, the articulation of the planner's dilemma has anything but strengthened the "traditionalist" disinclination to formalized planning - and for that matter economic decision-making pro futuro - as such. On the contrary, it is a formal statement of its problem that unravels the elements of uncertainty, especially in the longer-term planning (decision making).

The next question which arises is this. Should one foresee a widening or a narrowing of the indetermincy zone faced by economic planning? This is itself a matter affected by indeterminacy. On the one hand, it seems tenable that a developed economy's progress tends to reduce its vulnerability with respect to "nature" (say, climatic conditions, in particular). But at the same time, there is the growing and crucial impact of two factors: one relating to the increasing organizational complexity of society, the other to the pattern and pace of technological progress . (It seems to be not accidental that formalized treatment of uncertainty in planning has been most strongly promoted with an eye to the need of practice - by one of the sectors whose longer term planning crucially depends on the trends in techniques, i.e. that of power generation - see chapter 5).

§ 1. It is safe to foresee that the new tendencies will have an increasing impact on both the theory and practice of planning. By general consensus the "traditional" deterministic methods of plan optimization have to undergo a radical change in their status. At best they can serve as an auxiliary instrument for revealing the alternatives embedded in the "zone of indeterminacy." And, it is the employment of stochastic methods and techniques that is recognized to be imperative.
But - at least for one of the schools of thought - emphasizes use of not too complicated practice-oriented methods of decision making at least over foreseable future. Under this heading would come a) simplest devices for the use of mathematical **expectations** b) reducing the decision processes to a matrix game with some pre-selected finite set of alternative states of nature c) some multi-stage procedures - with the use of stochastic programming and control such as would permit to build up a "careful" policy in planning d) methods for analysis of plan stability devised with the view to ultimate solution being reached by experts.

The last point links up to a more fundamental proposition gaining ground of late. It is that the idea of planning within the "zone of indeterminacy" implies impossibility of a complete formalization of plan-construction. The tenet has close nexus with the conception of planning as a man-machine dialogue.

Once such conceptions prevail the issue of delimiting the frontier of formalization - feasible and effective - has to be faced. It is foreseable that advance in the state of art -

in formal apparatus (and computational technology) - is likely to make that frontier shiftable.

Note 12 Intensive work is being carried out by a team of mathematicians, economists and technologists to design an overall model of all economy automated planning and control and to elaborate its mathematical support. The system has an overall metaplan: it ramifies into a set of subsystems with their plans of different degree of mathematical sophistication. (Incidentally, just as in the case of the French system "Fifi" [93], the idea behind it is that as various complexes blocks of the system mature for more exacting formalism their plans will be moving to higher levels of mathematization). The overall system is being equipped with - variously detailed - schemas. The schema has very briefly and broadly this shape [94].

The object controlled is presented by some operators. To start with, we have F, an operator for the regime determining current values for the vector of phase-coordinates of the object, $x(t)$: it determines them, that is on the control actions $u(t)$, disturbances ("noises"), $\zeta(t)$, the object's initial state, and time t:

$$F : \{u(t), \zeta(t), x_o t\} \rightarrow x(t)$$

Further we have, as given, the rule, D, by which random disturbances, the $\zeta(t)$, is generated. In the general case, the probability characteristics of the disturbance are taken to depend

93) See the outline of "Fifi" presented by the French delegation to the December 1974 Seminar of the United Nations ECE, in the paper EC.AD/Sem.,2/R3

94) See inter al. A.N. Dyukalov, Yu.N. Ivanov, V.V.Tokarev in Avtomatika i Telemekhanika, No. 12, 1973

on time, phase-coordinates and control

$$D : \{t, x(t), u(t)\} \rightarrow \zeta(t)$$

The objective of control, and quality indices of its implementation which are formalized as constraints (for the vector of phase coordinates to belong to some given set over time x(t) and "constrain" a collection of functionals J, respect.

$$x(t) \varepsilon x(t) \text{ with } t \varepsilon [t_o, t_1]$$
$$J = \{J_k(x, u, \zeta), \quad k = 1, \ldots, k\}$$

the first-line-expression includes also boundary constraints on the phase-coordinates.

Now, the control mechanism is represented by two operators simulating two basic blocks of the system: the operator of measurement, μ, and the operator for control generation, π : for an economic system that would be the operator of planning. Operator of measurement, μ, puts the actual values of phase coordinates of the object x(t) and disturbances $\zeta(t)$ into correspondence with some observed magnitudes $\mu(t)$

$$M: \{x(t), \zeta(t)\} \rightarrow \mu(t)$$

Operator of control generation, π, puts the observable magnitudes into correspondence with control actions u(t)

$$\pi_{F,D,X,J} : \mu(t) \rightarrow u(t)$$

The operators' subscripts characterize the object model which can be employed in the elaboration of controls.

The problem faced by the architects of the meta-model is this. Too detailed presentation of the real object in SU complicates the elaboration of controlling processes, a too tight simplification affects quality of the construct. Hence one of the aims of modelling is to determine the tradeoff.

The methodology is transplanted into economic planning from technology. But naturally, the fact complicating the former is the participation of man in control. (One of the tendencies in Soviet planning is to apply the method of "business game", a Soviet-pioneered type of simulative gaming (a version of "evolutive" simulative, stochastic plan modelling which might be here of help is presented in a Note attached).

At present there is inclination towards working with closed or nearly-closed modelling complexes of "object + system control". Intervention of man is envisaged only in the parts which are maximally difficult for algorithmization such as changing structurally the measurements - the "language", variations of the planning schema, forming additional or elimiating some established feedback connections and so on. Those control functions of man which do lend themselves to algorithmization are carried out with the help of computer. Here belongs the problem of operative control and of designing the plan itself.

The effort is conceptually and technically of interest. Patently its success depends largely on the ability of the models' designers to schematize the man's behavioural characteristics.

<u>Note 13</u>. As pointed out (in Note 8) a noteworthy feature in the present-day development of planning theory is the emphasis on simulation modelling; and one of the most attractive contri-

bution in this field is the design by Likhtenshtein [95] for "evolutive-simulative" planning (the "ES" model. (On some points we note Pressman's influence (cf. our ch. II)). As against the traditional approach it does allow for risk and indeterminacy.

In Likhtenshtein terminology a set

$$\beta_{\tilde{p}^o}\alpha = \begin{cases} \alpha' \varepsilon D \,|\, P_\varphi(\varphi(\alpha) \geq \varphi(\alpha')) \geq P^o\}, \varphi(\alpha) \to \min, \\ \alpha' \varepsilon D \,|\, P_\varphi(\varphi(\alpha) \leq \varphi(\alpha')) \geq P^o\}, \varphi(\alpha) \to \max \end{cases}$$

is that of "sufficiently realistic" plans. Thus $\bar{\beta}_{p^o\alpha} = D \backslash \beta_{p^o\alpha}$ is the set of those "insufficiently realistic". Here D is the domain of admissible plans; the planner's problem is to find the plan a on which the objective function, that maps D into some linearly ordered set R, reaches its extremum. The objective function is a random magnitude. The domain of its values is $\varphi = \{\varphi(\alpha) | \alpha \varepsilon D\}$; the existence of the law of its distribution P_φ guaranteed by the postulated uniqueness of the mapping. The P^o is the "a priori [postulated] level of realism".

In the ES evolutive algorithms when simulation is carried out as a statistical test with two possible results (i.e. $\alpha^e \varepsilon \beta_{p^o\alpha}$, $\alpha^e \varepsilon \bar{\beta}_{p^o\alpha}$,) we find that in N simulations at least one plan belongs to $\beta_{p^o\alpha}$ with probability $P^* = 1-(1-P^o)^N$ whence

$$N \cong \left] \frac{\ln(1-P^*)}{\ln(1-P^o)} + 0{,}5 \right[$$

and it is easily shown that P* should be taken equal 0.9. The al-

95) V.E. Likhtenshtein "Evolyutsyonno-simulyativnyi metod planirovanya", in Ekonomika i Matematicheskiye Metody, No. 6 1971

gorithmic procedure is then this 1) from given P^o (from a table) find N; 2) carry out N simulations to find (α^e,φ^e), e=1,..,N; 3) adopt a plan such that $\hat{\varphi}(\hat{\alpha}) = \max(\min)\{\varphi|e=1,..,N\}$. This a is termed "the <u>pseudo-best</u> plan".

Next consider convergent algorithms (with the assumption that the overall number of simulations N is known). Then we have (1) to carry out N simulations and to find (α^e,φ^e), e=1,..,N; (2) to order the results (α^e,φ^e) with reference to n=1,..,N which would correspond to the rising φ^e such that for all n $\varphi^e_{(n+1)} \geq \varphi_{(n)}{}^e$ and thus $\varphi(\alpha^e_{(n+1)}) \geq \varphi(\alpha_{(n)}{}^e)$ is met; (3) here $\varphi(\alpha)\to\max\ \alpha^e_{(n)}$ is taken to be the pseudo-best plan with $n' =]N(1-P^o)+0,5[$; if $(\varphi(\alpha)\to\min$ then it is taken to be plan $\alpha^e_{(n')}$ with $n' =]NP^o+0,5[$.

Further the generalized ES model is built up on the conception of penalty for a deviating from the "accepted" plan α^π. The expectation of loss due to such deviation is

$$\Phi_M(\alpha^\pi) = \left| \int_{\alpha\varepsilon D} \Phi(\alpha^\pi,\alpha)\,dP_D \right| =$$

$$= \left| \left| \int_{\alpha\varepsilon\beta_{P^o}} F(\varphi(\alpha)-\varphi(\alpha^\pi))\,dP_D \right| - \left| \int_{\alpha\varepsilon\beta_{P^o_\alpha}} F(\varphi(\alpha)-\varphi(\alpha^\pi))\,dP_D \right| \right| .$$

Here P_D is the probability measure on D, determined by the laws of P, $i\varepsilon I$, and the simulator. Thus the generalized ES model reduces to the optimality condition $\Phi_M(\alpha^\pi)\to\min$.

The generalization of the ES algorithms rests on this procedure; 1) carry out N simulations and find (α^e,φ^e),e=1,..,N; 2) with reference to numbers 1..,m order the set φ^e,e=1,..,N, assigning the same number to equal values of φ^e ; 3) find the frequencies $P^{(n)}$, $n=1,..,m$ of the appearance of plans which secure to the objective function the value $\varphi^e_{(n)}$. Then $P^{(n)}=\frac{\text{card}\{\alpha^e/\varphi^e=\varphi^e_{(n)}\}}{N}$. 4) choose some n', say, $n'=m/2$ and $=(m+1)/2$ where m is, respectively, even and odd; 5) find $\varphi^e_{(n')}-\varphi^e_{(n)}$, $n=1,...,m$; 6) compute $F(\varphi^e_{(n')}-\varphi^e_{(n)})$, $n=1,...,m$; 7) compute $\underline{P}=\sum_{n=1}^{n'-1} F(\varphi^e_{(n')}-\varphi^e_{(n)})P^{(n)}$; $\bar{P}=\sum_{n=n'+1}^{m} F(\varphi^e_{(n')}-\varphi^e_{(n)})\times P^{(n)}$;

if $\underline{P}\simeq\bar{P}$ **for a plan, or any of the plans, securing to the objective** function the value $\varphi^e_{(n')}$ adopt it as the pseudo-best; if $\underline{P}<\bar{P}$ then posit $n''=n'+1$ and switch to algorithm 5; if $\underline{P}>\bar{P}$ then posit $n''=n'-1$ and switch to algorthm 5.

The algorithms generalized in this fashion are converging: $\varphi(\hat{\alpha})\to\varphi(\alpha^{\pi})$ for $N\to\infty$; and they permit to find the value of P^o. When $\varphi(\alpha))\to$ max, or $\to$ min then, respectively, $P^o\cong\sum_{n=n'}^{m} P^{(n)}$ or $P^o=\sum_{n=1}^{n'} P^{(n)}$ (n' being the number of the pseudo-best plan).

In substance the converging and generalized Likhtenshtein algorithms are an application of the Monte-Carlo method in the search for the best plan while the evolutive **algorithms are an** application of the random search in the solution of a stochastic problem.

APPENDIX

As we have seen the planning theorist has become intensely conscious of the presence of the "unplannable", "unformalizable" and untractable elements he is confronted with in the steering of the economic system. This accounts for antithetic tendencies in the mathematical methods and techniques to be employed. On the other hand, there is a continuous call for operating the simplest possible of the "conventional" mathematical instruments. But,at the same time, there has been of late an observable looking out for help to"modern","unconventional" - as often as not, abstract and sophisticated - areas of formalization. This is true especially where the "unformalizable" phenomena come to the fore.

With this in mind I have appended this brief sketch as no more than a hint at the three areas of "modern" formalism in which help is being,or can be expected to be, sought by the planning thought.

Of the three newly disciplined bodies of scholarly inquiry, the first, concerned with the theory of automata and their interaction is the relatively most intensely abstract; and it is motivated toward crossing the threshold of the formalizable; it is hardly surprising then that here the Soviet planning-oriented scholarship has been making the initiating and major contribution. The last of the three is, as will be seen, devoted to a methodology - pattern recognition - which is acknowledged as one of a possibly more immediate aid to the planner and his theory. The middle one discusses from our angle a discipline - that of fuzzy sets - to which Soviet scholars have, surprisingly, done hardly any contribution at all, yet - one whose the formative concept - "fuzziness" and inexorable inexactness - are acknowledged by them to be very close relatives of the indeterminacy zone.

What I am saying now may justify, I hope, the inclusion of matters (even if only scratching the surface) of matters which while appearing rather remote or peripheral are pertinent for the trends in thinking on our subject.

1. Theory of Automata as Focused on Planning under Indeterminacy

Coping with indeterminancy with an eye to the ASU system has turned the attention - indeed pioneering attention - of the Soviet student to the theory of automata specifically to that of automata in random environment. On the technical plan the approach appears as an extension of, and yet - to some extent - as an alternative to the control theoretic one (be it classical or post-classical). The point stressed by protagonists is that essentially the application of the latter presuppose a sufficient "listing" of situations in which a system would be controlled in pursuit of an optimum. But the situations to be coped with are such that adequate mathematical description as often as not proves exceedingly difficult; say, one where the "players" lack knowledge of other participants' payoff functions, their strategies and choices; that frustrates the "feeding" of actions into the control element.

In this approach each automaton is being presumed to have perception only of the payoff value in response to its own action; and to be possessed of ability to react by adjustment. Hence the model's architect - when without a priori information - must secure the control element with adequate adoptability to environment, possibly with a certain lag, the automaton would be then expected to assimilate within the system.

To expand on the point, generally a postulate of sufficient a priori knowledge of result of actions appears to be unrealistic: hence the control has to be constructed on the basis of random outcomes of a process which analyses the control signal at the input point and secures the corresponding output observed during the system's operation. +)

In particular, inasmuch as controling and planning an economy is treated as large-scale processes whose complete and exact description is intractable (even if the problematics is theoretically advanced) recourse must be taken to methods which try to bypass the practicability barrier. Decomposition of the all-system problem is taken to mitigate it, installing at suitably chosen stations automata - as a way in a multi-level dynamic game - gains favour as such an approach+).

§1. In Tsetlin pioneering design ++) the finite automaton's (A) canonical equations appear as

---

+) Yn. A. Seton in Journal of Cybernetics, No. 1, 1973

++) M.L. Tsetlin, Avtomatika i Telemekhanika, No. 10, 1961

$$\varphi(t+1) = \Phi(\varphi(t), s(t+1)),$$
$$f(t) = F(\varphi(t)) \quad (t=1,2...).$$

The automaton's input variable has values s=0 and s=1; its output is action with the function of time $f(t) = f(1,..,k)$; the number of its states is m; the states are then $\varphi_1^{(t)}..\varphi_m^{(t)}$; the number m measures the memory capacity. The Tsetlin automaton performs in a random environment where the automaton records as its reaction either penalty or non-penaltly, that is s=0 or s=1 with respective probabilities p, or q=(1-p). Among Tsetlin's assertion is one to the effect that for a class of automata with linear tacties, the expectation of penalty decreases as the memory capacity increases to reach the limit with the possible minimum expectation. Thus for a sufficiently large memory capacity the automaton of this class would carry out only an action with minimum penalty probability (automata endowed with such properties are those known as asymptotically optimal).

Tsetlin's fundamental study has been extended by Varshavskiy and Vorontseva [+). In this extension the concept of finite automaton - now a stochastic automaton - is taken to have a variable structure. In their design an automaton is taken to operate in some medium when at each time point the value of output depends on that in the anteceding time instant $x(t) = C[y(t-1)]$. Measure $E(A,C)$ - measuring that is the automaton's "behaviour expediency" - is then the average of penalties

---

+) V.I. Varshavskiy, I.P. Vorontseva, Avtomatika i Telemekanika, No. 3, 1963

$$E(A,C) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} x(t).$$

The automaton is considered to operate in a stationary random medium $C = C(p_1, \dots, p_r)$ where its input and action have this relation $y_\nu (\nu=1,\dots,r)$, carried out at t, yields the value x=1 termed penalty at t+1 with probability p_ν, and the value x=0, termed non-penalty, with probability $q_\nu = 1-p_\nu$.

Suppose at t the automaton is in state $a(t) = a_i$ where its action $y(t) = y_\nu = F(a_i)$ is being carried out. Now the probability p_{ij} of the automaton being transferred from a_i to a_j is $p_{ij} = p_\nu \pi^1_{ij} + q_\nu \pi^o_{ij}$, s.t.

$$\sum_{j=1}^{n} p_{ij} = \sum_{j=1}^{n} p_\nu \pi^1_{ij} + \sum_{j=1}^{n} q_\nu \pi^o_{ij} = p_\nu + q_\nu = 1.$$

Thus the system (medium + automaton) presents itself as a Markovian homogenous chain with finite number of states. Where an addition assumption is ergodicity we have the system final probabilities (denoted Q_j). Then for the penalty expectation for automaton medium A in C

$$E(A,C) = \sum_{j=1}^{n} Q_j p_{\nu j} \quad (\nu_j = 1,2,\dots,k).$$

$$E_{max} = \max(p_1, p_2, \dots, p_k), E_{min} = \min(p_1, p_2, \dots, p_k) =$$

$$= \frac{p_1 + p_2 + \dots + p_k}{k}$$

Further, the automaton A is termed to be of expedient behaviour where E(,) is smaller than E_o: disparity of E(,) and E_{min} measures the expediency stochastic automata for which the transfer probabilities π^{ℓ}_{ij} vary over time are considered to be of variable structure. Thus for this kind of automata the probability π^{ℓ}_{ik} rises where after transfer from a_i to a_k brought about by input x_1 a non-penalty results; the converse applies in the case of penalty.

Very generally Varshavskiy/Vorentseva have established that the fundamental results obtained originally by Tsetlin for his finite automata do hold also for the stochastic ones with variable structure: that is such that the transition probabilities change in the process over time; but they have demonstrated that even in that situation the concept of final probability remains pertinent. To be more specific it is demonstrated that stochastic automata with such a structure are of "good" ("expedient") behaviour in random environment, and are of optimal behaviour in a stationary random one; in a "switching" random environment as defined a stochastic automaton approximates one with linear tactics; and for a sufficiently large number of original states the average number of unit-penalties equals - at the limit for the stochastic automaton with variable structure - the penalty expectation for a finite automaton with linear tactics and optimal number of states.

Note 14 For a method of designing - in Soviet literature - automata performing a required set of experiments with decreasing indeterminacy we may refer e.g. to studies by Spivak and by Trofimchuk.[+)]

+) R.N. Trofimchuk, Kibernetika, No. 1, 1975
M.S. Spivak, Kibernetika, No.6, 1966

We have as the finite initial automaton the sixtuplet $A(H,J,A,\delta,\lambda,a_o)$. Here H,J,A designate resp. finite input, output and alphabets. The $H=\{x_i\}_{n=1}^{n}$, $J=\{y_1\}_{1=1}^{m}$, $A=\{a_k\}_{k=o}^{r}$; δ and λ - denote extended transition and output function; the A's initial state is a_o.

The nature of the automaton - deterministic and non-deterministic resp. - depends on functions δ and λ (on them being uniquely defined or not).

Recursive $\varphi_A(e) = e$ (e stands for "empty set"), $\varphi_A(ix) = \varphi_A(t)\lambda(\delta(a_o,t)x)$, $x\varepsilon H$, $t\varepsilon F(H)$. The $\varphi:F(H)\to F(J)$ when met, permits the build-up of automaton M. Its input and output alphabets are H and J, and set of states is now taken to be the semigroup F(H). The transition and the output functions are described by $\delta(p,x) = px$, $\lambda(p,x)=\Psi(px)$, $x\varepsilon H$, $p\varepsilon F(H)$; here $\varphi(x_{i_1},x_{i_2},\ldots,x_{i_s}) = y_{i_s}$ with $\varphi(x_{i_1},\ldots,x_{i_s}) = y_{j_1},\ldots,y_{i_s}$; $s = 1,2,\ldots$ For a non-deterministic automaton $\varphi(p)$ is a set of output alphabet available from A when "p" is added to the input.

Now define d(p) - length, and by $\pi_k(p)$ initial segment of p with length k; $(d(e) = 0, \pi_o(p) = e)$.

On Trofimchuk's-Spivak definition, for our purposes, the "experiment" is the pair (p,g) with $p\varepsilon F(H)$ and $g\varepsilon F(J)$ when $d(p)=d(g)$; then a set of experiments $\Phi_A=\{(p,g)/\varphi_A(p)=q\}$ is taken to correspond to automaton A (the automaton set of A).

The task problem is then stated to be the designing of a finite automaton such that the required finite automaton set $E = \{(p_i, q_i)\}_{i=1}^{N}$ is realized.

It is the method of designing the sequence of nondeterministic automata A_k (k=1,2,...), every one of which solving the E task problem, that is offered in Trofimchuk. Two of his theorems proved assert that firstly a rising value of k is accompanied by a declining degree of indeterminacy in the automaton A_k (with increasing number of states though), and secondly that a minimal deterministic automaton realizing the postulated finite automaton set of experiments E is secured when k is sufficiently large.

§ 2. The model for games between automata - in a deterministic framework - originated in the Krylov/Tsetlin work+). It translates the Tsetlin construct (see above) into the von Neumann/Morgenstern framework. In a game made of reiterated plays, for each of the plays the automaton-actions and outcomes are respectively the strategies and the unit loss-win; the probability of the latter is determined by the strategy. The information on wins-losses is the input variable and it determines the choice of strategy, and thereby of an action in the next play: data on individual plays' outcomes (as functions of strategies) form the automaton's sole information in the game. It is demonstrated that the game of automata as defined is describable by a finite Markov chain. The focus is here on the two-automaton zero-sum games and results which are shown to parallel closely the classical von Neumann-Morgenstern construct. Among the Krylov/Tsetlin +) variants of particular applicational im-

---

+) V.Yu. Krylov, M.L. Tsetlin, Avtomatika i Telemekhanika, No. 7,1963 portance seem to be those of a game between an automaton and an opponent applying mixed strategy (in this class are also the automaton games against nature).

Since the Tsetlin/Krylov contribution several noteworthy developments have been recorded in this area.

Firstly, Chandrasekaran and Shen have designed models for stochastic automata of variable structure (with results not dissimilar from those of Tsetlin/Krylov) and further on Viswanathan/Narenda [+)] investigated competitive and cooperative games of such automata similarly structured.

Yet another generalization of construction is that by Vaisbord [++)]: a game of N-automata with differing memory depths. One of its interesting results indicates that the optimal set of actions may not necessarily be adopted as is the case for a single automaton with the greatest frequency, for the minimum penalty probability. Rather, the choice frequencies are, in certain way, influenced (in addition, that is, to penalty probabilities of the given sets) by the averaged penalty probabilities for sets for which the actions of automata with "deepest" memory coincide with those of that given sets.

Various aspects of games between automata have been further studied of late by Flerov [+++)] et al. Thus in par-

---

+) R. Viswanathan, K.S. Narenda, "Competitive and Cooperative Games of "Variable-structure Automata", Journal of Cybernetics, No. 3, 1973

++) Vaisbord, Avtomatika i Telemekhanika, No. 12, 1968

+++) Y.A. Flerov, "Multilevel Dynamic Games", Journal of Cybernetics", No. 1, 1973

ticular we have a simple model of a two-level game with an analysis of the automaton's payoff in terms of the disparity with that payoff in an equilibrium solution. In the process of simplification of the game each level is taken to consist of one position ; the three vertices-positions and edges connecting positions directly follow each other making ultimately a graph with each of the levels formed of one position with two players of identical class.
For the automaton with linear strategy - on basis of the payoff value a, $|a| \leq 1$, the probabilities p and q of penalty (-1) and reward (+1) inputs, are respectively $p=(1-a)/2$ and $q=(1+a)/2$; the output response actions are in an one-to-one correspondence to the pure strategies of the automaton.

Thus for this kind of game the >>criterion on closeness >> and the index of "purposefulness of behaviour" for the automaton are respectively

$$\delta_A = \frac{|\varphi-\varphi^*|}{\varphi^*} , \qquad \bar{\delta} = \frac{\sum_i \delta_{A_i}}{N} ,$$

(the φ and φ^* are respectively average payoff obtained by the automaton and the payoff at equilibrium where optimally stationary strategies of behaviour are resorted to; the δ denotes the average of relative deviations of N automata "A" in the game.

§ 3. Yet another step toward realism has been made in the design of what has become known as the G-type automaton. They were defined originally by Sragovich +) as a subclass

---

+) V.G. Sragovich, "Automata with Multivalued Input and their Behaviour in Random Environment", Journal of Cybernetics, No. 2, 1972 of those with multivalued inputs (in this lies the contrast with the Tsetlin classical design - see above) and provide a gradation for the environment's reaction to the automaton's action. Their behaviour has been analysed in a computer simulation in a random environment,+) both stationary and "switching". In the latter - more general and realistic - the automata emit signals to the plant and modify their strategy, in response, so as to increase the probabilities of performing actions yielding a higher average payoff. By definition the G automaton has no a priori knowledge of the environment i.e. "competitive situation" in which it operates (the supposition is, however, that some experimental and historical information may be available and built into the automaton's design).

The automaton's efficiency is measured by the payoff accumulated: it is determined by the dynamics of returns corresponding (in most cases) to the transition in Markov chain; and the problem of adaptability of a G-automaton relates to such chains. (In the theory of Markov chains,

---

+) On the adopted defintion a random environment is a stochastic automaton C, where $C=\{y,\mu,X\}$, the $Y=\{y_1,\ldots,y_k\}$ and $X=\{x_1,\ldots,x_m\}$ are respectively finite input and output alphabets; $\mu=\mu(x,y)$ is conditional probability distribution on the output alphabet: give input signal y_{t-1} the environment responds at time t, the choice of element being controlled by the distribution $\mu(x_t|y_{t-1})$. A random environment is stationary if the probability distribution, μ, does not depend on t; and it is "switching" if the probability distribution of its responses does depend explicitly on t.

the average payoff in plays is expressible by partial sums of the respective functionals; therefore the asymptotic behaviour of these sums would be taken to provide indication as to which type of the automata functions better than some other).

The behaviour of G automata in games has been studied by several students: in a multiperson game - by Sragovich/Shapiro; in a matrix game - by Ivanov +). In Ivanov the G-type's behaviour is observed in game with an automaton-opponent with linear strategy, $L_{m,\lambda}$,(capable of m moves and λ states for each moves) - with the view to discover the most effective performer such as to be a relevance in an automation design for a controlled system.

The Ivanov results seem to suggest the allround superiority of the G automaton.(Note that Flerov ++) carried out an exercise of comparing a game of an G automaton with others belonging to class of multivalued inputs; there,too,the G type appeared as the winner).

+) V.I. Ivanov, "Behaviour of G-Type Automata in a Matrix Game against Automata with Linear Strategy", Journal of Cybernetics, No. 2, 1972

++) Yu. Flerov, "Some Classes of Multi-Input Automata", Journal of Cybernetis, No. 3, 1972

§ 4. In the 1970s a school of thought has concentrated more directly on possibilities of automata in the operation of stochastic optimization of systems; in particular economic systems. This applies specifically to control of random search.

Thus Poznyak +) (extending earlier results of Rastrigin and Ripa) has shown that such control can be secured by the use of stochastic automaton which determines any nonzero distribution of possible directions of motion. This does apply to situations of a random search in conditions of incomplete information - where the objective function makes some mean with distribution unknown; also to situations with time-variable probability distribution of direction of the motion. The assumption is that the variation regime for the distribution corresponds to, and indeed is generated by the criteria; where this is assumed a stochastic automaton makes a learning automaton structurally variable.
This kind of functional - as demonstrated in Posnyak - is the average penalty which corresponds to the number of steps proving a "failure" (the learning functional may be some nonlinear variety of average penalty). All the functionals correspond to a certain probabilistic distribution; and they provide representation of averaged responses over the whole range of all feasible trajectories of motion.

Of late the Poznyak ++) inquiry has progressed in a direction of still greater immediacy for plan-programming application: relating the behaviour of learning automata - (as cha-

---

+) A.S. Poznyak, <u>Avtomatika i Telemekhanika</u>, No. 12, 1972
++) A.S. Poznyak, <u>Avtomanika i Telemekhanika</u>, No. 10, 1973

racterized above) in random environment subject to constraints - to stochastic programming.

To sketch it out very briefly and broadly the formal framework within which the learning automata are taken by Poznyak to operate is a stochastic programme

$$I_o = \int_X E\{f_o(x,b)\,|\,x\}p(x)dx \to \inf_{p(x)},$$

$$I_j \equiv \int_X E\{f_j(x,b)\,|\,x\}p(x)dx \leq 0, \quad (j=1,\ldots,m).$$

Here the functions $Ef_j(x,b)$ are measurable on the sets $X\times B$ $(x\varepsilon X;\ b\varepsilon B)$. The vector $x = (x_1,\ldots,x_n)$ denotes the control actions vector $b = (b_1,\ldots,b_m)$ is an index of the problem's indeterminacy expressing, that is, exogenous information: the distribution density of $p_b(b)$ is taken to be unknown *a priori*. The magnitudes $\{p_i\}$ meets the constraints

$$\sum_{i=1}^{s} p_i=1, \quad p_i\geq 0 \ (i=1,\ldots,s).$$

In what appears to be the principal problem of variants considered the distribution density $p(x)$ of the control vector x is given the form

$$p(x) = \sum_{i=1} p_i\delta(x-x^i),$$

the variables p_i satisfy the constraints - as above and $\delta(.)$ is a generalized delta function.

We additionally postulate that

$$E\{f_j(x,b)\,|\,x\} \equiv f_j(x)\,(j=0,\ldots,m)$$

be continuous on X which is taken to be a closed bounded set.

It is then asserted that solution of the programme lies in a subclass of functions

$$\sum_{k=1}^{m+2} E\{f_o(x,b)\,|\,x^k\}p_k \to \min_{(p_k,x^k)},$$

$$\sum_{k=1}^{m+2} E\{f_j(x,b)\,|\,x^k\}p_k \leq 0 \qquad (j=1,\ldots,m)$$

One then carries out an ε-partition of the set X into subsets $\{X_\ell\}$, $X_\ell \leq \varepsilon$, $X_\ell \cap X_k = \emptyset$, $1 \neq k$, $\cup_\ell X_\ell = X$; and it is assumed that there are altogether s (ε) subsets. The stochastic problem with ε-exactness is then restated as

$$\sum_{\ell=1}^{s(\varepsilon)} E\{f_o(x,b)\,|\,x \in X_\ell\}p_\ell \to \min_{(p_\ell)}$$

$$\sum_{\ell=1}^{s(\delta)} E\{f_j(x,b)\,|\,x \in X_\ell\}p_\ell \leq 0 \qquad (j = 1,\ldots,m).$$

It is then demonstrated that as the decreases the handling of the automaton becomes intractable owing to the steeply rising number of s(ε) the required number the internal automaton states. To evade this a team of stochastic automata is resorted to : Considering that the minimum number of internal states is two. (For a given ε-partition the reasonable maximum of team members is $\left[\frac{s(\varepsilon)}{2}\right]$.

§ 5. Learning automata belong to the class of discrete systems randomly positioned in each internal state with certain distribution; and its state distribution is being controlled by adjusting the transition probability matrix (see above - on Poznyak's original design). It is in this way that the learning automata are employed for direct implementation of mixed strategies in stochastic programming. One is being offered in Poznyak algorithms which,on the one hand, secure the learning automaton's behaviours asymptotically optimal over time - in environment with mean constraints; and, on the other hand, ensure minimization of the penalty function: one corresponding to the stochastic programming problem. (Note that improving accuracy of stochastic programming is likely to raise the number of international states of the automaton to the point of intractability; to cope with this Poznyak ressorts to a controlled team of automata (Optimization by a team of independent automata is a subject which has accumulated of late considerable literature;) one is referred,in particular, to the writings of Grigorenko, Neymark, Rapoport and Ronin+).)

As indicated here the theory of automata is focused on processes in large-scale complex systems with low level of informativeness and low susceptibility to complete and exact description. All this justifies the interest of planning theorists; the more or so as its progress does record a growing absorption of real-life situations. As matters stand now it offers but little scope for an analytical investigation; yet, it provides increasingly valuable help in computer simulation. This help is especially important where the planner-controller has to examine qualitatively the re-

---

+) see V.P. Grigorenko, Y.I. Neymark, A.N. Rapoport, Y.I. Ronin, "Optimization of Cybernetics", Journal of Cybernetics, No. 3 1973 and references actions of the agents' behaviour and the sensivity of agents to exogenous stimuli and counter-stimuli.

Note 15 1. Stochastic automata provide but one of learning-system schemes. As presented in Fu's +) excellent overview of the filed at least four more can be classified under this headline; these trainable controllers using pattern recognition, reinforcement, learning control systems, Bayesian estimation, stochastic approximation. In Soviet literature, as far as self-teaching systems go, the present-day thought could link up with Feldman's seminal work of the early 1950s on "mixed" systems. In cotemporary literature we refer in particular to Krasovskiy et al.

Thus we may exemplify the problematics with reference to the simpliest case presented by Krasovskiy/Gavrilov/Lyetov/Pugachev ++) - the case where the control is performed by changing one (scalar) parameter w. We have here z=x-y (the x, y denoting respectively the required and actual output signals). As often as not a good-quality control will be secured by postulating the control parameter in the form of a linear function of the descrepancy z and its first time derivative

$$w = k_o z + k_i \dot{z}.$$

where the two r.h. terms are meant respectively to reduce the discrepancy and to extrapolate it with the view to damping

+) K.S. Fu, "Learning Control System-Review and Outlook" in IEEE, Transactions on Automatic Control, April 1970

++) N.N.Krasovskiy, M.A.Gavrilov , A.M.Lyetov, V.S.O.Pugachev, "Obshchiye problemy upravlenya", Vestnik Akademii Nauk, No. 8, 1970 down the oscillations. In the classical theory of regulation in order to obtain the parameters k the exact knowledge of the equations of motion and of all its coefficients is assumed. When the control law is added, the closed system of equations is investigated for stability; from this, first the admissible, and next the acceptable values for the parameters k are secured. As against this in the theory of learning systems one postulates the system's being equipped with measuring device, this device is expected to determine the parameters k by processing information forthcoming in the course of the teaching. That, say, could be the following - as in Krasovskiy/Gavrilov/Lyetov/Pugachev: The regime of change for the control parameter $w = w_y(t)$, $0 \leq t \leq T$, and the $z = z(t)$ are brought into the measuring device which determines the k_o, k_1 from the condition of least-square error of the approximation equation

$$w_y(t) \approx k_o z(t) + k_1 z'(t) \qquad (0<t<T),$$

that is from the condition

$$\int_o^T [w_y(t) - k_o z(t) - k_1 z'(t)]^2 dt = \min.$$

Alternatively, the value of the parameters may be found in the process of the control system's self-teaching. For this purpose one would insert into the measuring device a valuation programme for the quality of the control and purposeful change of the parameters. Say, we can organize within the control system a continuous computation of the integral quadratic estimate $\varepsilon(t) = \int_{t-T}^{t} z^2(t)dt$ over the sliding interval $(t, t-T)$, and the search of values for the k_o, k_1 at which ε is minimized.

2. Kinship of Theories of Fuzzy Sets and of Indeterminacy in Planning.

In this appendix we turn to another area of inquiry one which shows a remarkable conceptual kinship with the conceptual background of the theory of planning in a "zone of indeterminacy. We have in mind the theory of fuzzy system; fuzzy -

Our second excursus points to yet another field with certain potentialities, in which Soviet scholarship rather surprisingly has not been very active as yet (although here and there a hint may be found in contributions on the zone of indeterminacy). We have in mind the theory of fuzzy systems in the sense of Zadeh (and more recent formulations in Zadeh/Bellman [+)]: the fuzzy set being understood, that is, as a class of objects with a continuum of grades of membership: the set whose characteristic function assigns to each object a grade of membership ranging between zero and one). Definitionally we find a noteworthy East-West convergence in thinking: the concern about inexactness (or "vagueness") of systems, specifically as it faces the decision-maker for an economic system, be it "planned" or "unplanned", be it "micro" or "macro" - but in planning it is dominant; the leading idea is to have "fuzzy mathematics to represent exactly the inexact state of knowledge"[++)].

+) R.E. Bellman, L.A. Zadeh, "Decision-making in a fuzzy environment", Management Science, Vol. 17, pp. B-141 ff, 1970

++) S.L. Chang, L.A. Zadeh, "On Fuzzy Mappings and Control", IEEE, Transactions on Systems, Man and Cybernetics, Vol. SMC-2, No. 1, 1972

This goal appears to be pursued with noteworthy tenacity. Since its inception not much more than a decade ago the doctrine of fuzziness has moved distinctly toward crystallization. To begin with it has been evolving its formal languages (thus see e.g. Kling's +) "fuzzy planner: reasoning with inexact concepts in a procedural problem-solving language"); it has extended to the areas of learning theory, automata, pattern classification; it has been extended of late by Chang and Zadeh++) to fuzzy mapping - with a declared pertinence to problems of social and economic systems; this extension carries that of fuzzy function and its inverse, fuzzy parametric functions, fuzzy observation and control; importantly it is demonstrated that under some conditions a precise control goal can be secured with fuzzy observation and control provided the observation becomes sufficiently exact as the coal is approached. There are extension also to the realm of algorithmization.

[A field where workable techniques have been by now developed such as fuzzy-mathenatical programming (Tanaka/Okuda/Asai+++)). Or to go back to the context of our discussion in sect. we may draw attention to the relevance of certain findings (by Aubin ++++)) on the extension of the concept of fuzziness to game-theoretical constructs specifically in the treatment of

---

+) R. Kling, "Fuzzy Planner ...", Journal of Cybernetics, No. 4, 1974

++) Chang and Zadeh, op.cit.

+++) H. Tanaka, T. Okuda, K. Asai, "On Fuzzy Mathematical Programming", Journal of Cybernetics, No. 4, 1974

++++) J. Aubin, "Fuzzy Games", Technical Summary Report No.1480, University of Wisconsin/Madison Mathematical Research Center, 1974 (?)

equilibria . Also we may note that Blin/Fu/Whinston/Moberg +) adopt the fuzzy system approach as a basis for applying pattern recognition which is, as we have noted, also one of the techniques considered by Soviet theory of the "indeterminacy zone"].

Finally crucial for our theme is the relationship between the fuzziness and the probability theories. It seems valid to say that originally when the former was trying to establish its identity, it is the differences rather than links that tended to be underscored. (Here lies a point of distinction from the doctrine of planning under indeterminacy).

Thus, to refer oneself to the argument in Blin-Fu-Whinston-Moberg while akin to cumulative probability distribution, the fuzzy-set membership function has some very distinct characteristics. Thus, as against the case of probability-distributions, for determining the membership function of the fuzzy set there is no need to invoke infinite or very large numbers of observations. Also while there is some resemblance to the idea of subjective probability, the point is that for a fuzzy system one may confine oneself to estimating or giving a partial ordering of unknown events, rather than "speculating" on a particular *a priori* distribution. The contention is then that the fuzzy-system approach is more "natural": natural in the sense that one can proceed straight on without being involved in the estimating-problems entailed either in the objective or subjective probability theory; and very generally

---

+) J.M. Blin, K.S. Fu, A.B. Whinston, K.B. Moberg, "Pattern-Recognition in Micro-Economics", *Journal of Cybernetics*, No.4, 1974 the fuzzy set is claimed to rest on a more "reasonable" representation of the idea of uncertainty in the "everyday" sense. To exemplify, the social choice problem is used in Blin to provide a completely nonprobabilistic interpretation of the theory of fuzzy sets: a clear cut distinction is made for that field between the interpretation of probabilistic concepts and fuzzy set concepts.

Whether such stand is the only fruitful one - that may be in our submission open to question. Theorists in this field - who feel that somehow the theory has failed to provide, as yet, workable tools for handling real-world phenomena - have turned to the more established stochastic armoury. Thus, with such explicit motivation, Jacobson now resorts to a stochastic control-theoretic handling of fuzzy-system problems +). He demonstrates that a quadratic performance criterion of optimal control can be treated as a particular "confluence" of fuzzy goals and constraints - in Bellman-Zadeh sense - with membership functions of exponential type (the definition of fuzzy "confluence" of goals and constraints by the product rule - rather than intersection rule-appears to be preferable in the stochastic control-system context). The maximizing decision also introduced by following Bellman-Zadeh is found to be adequate in the deterministic case:a minimizing alternative is adopted as one suiting better a stochastic setting ++).

+) D.H. Jacobson, in Journal of Mathematical Analysis and Applications, August 1976; see also his paper on optimal stochastic linear systems with exponential performance criteria (related to deterministic games) in IEEE, Transactions, AC, 1973

++) for an attempt at empirical application see the paper by H. Tanaka, T.Okuda, K. Asai (Kybernetes, Vol. 5, 1976) which formulates a fuzzy decision problem and its application to investment.

Quite possibly, the new trend in fuzzy-theoretic approaches will make it more congenial to the empirically oriented theorist of planning under indeterminacy.

3. Pattern Recognition as Instrument in Planning under Indeterminacy

Pattern recognition is yet another discipline on which hope are staked by almost every Soviet theorists in the field. Now, to begin with, it is not easy to see what exactly is the help expected from it since so far it makes a conglomeration - with little coherence - of ideas borrowed from wide range of disciplines (although claim has been put on its behalf that to represent one of the most ambitious scientific ventures of our century - inasmuch as it makes and explicit endeavour at mechanization of the most fundamental human function of perception and concept formation (Watanabe +)). However, when its area is narrowed down to concern with mathematical techniques of optimal decision-making under uncertainty - as in Ho and Agravala ++) - its potentialities in aiding the planner under indeterminacy are more discernible; and they are still more or so when such narrowing down reduces its area largely to classification processes (in particular in support of aggregational methods, cf Blin, Fu and Whinston +++)).

+) S.Watanabe, Ed., Methodologies of Pattern Recognition, 1969

++) Y.C. Ho, Ak.Agravala, in Proceedings IEEE, 56 (12),1968

+++) J.M. Blin, K.S. Fu, A.W. Whinston, Journal of Cybernetics, No. 4, 1974

The formalism of the new discipline is still immature (Simon has offered for it, of late,[+] a formal language, similar to that of logic for the description of pattern-recognition algorithms; reinterpretation would have to be made in other languages: "natural", machine and programming) .

When - as in our submission - the area of application for our purposes is reduced to classification, one finds a design - a very helpful one - in Fu[++] (here supporting learning controls). To restate very loosely it is this. We have to deal with "features" (measurements) $x_1, \ldots, x_k$ of phenomena representable by a k-dimens. vector $x \varepsilon \Omega_x$. Assume that the possible m classes are $w_1, \ldots, w_2$. In pursuing his task of assigning class memberships the classifier - with the help of discriminants - partitions the Ω_k into m non-overlapping regions, or carries out the mapping from space Ω_k to decision space. Then a discriminant function $d_i(X)$, $i=1,\ldots,m$ is adopted s.t. if $X \varepsilon$ class w_i then $d_i(X) > d_j(X), j \neq i$ and the decision surface $w_i - w_j$ is $d_i(X) - d_j(X) = 0$.

We have in Fu three discriminant functions, viz.

1) linear, of feature measurements $x_1, \ldots, x_k$

$$d_i(X) = \sum_{r=1}^{k} w_{i'} x_{r'} + w_{i,k+1}$$

2) polynomial, chosen as an n-th order polynomial; thus for $n = 2$

$$d_i(X) = \sum_{r=1}^{k} w_{rr} x_r^2 + \sum_{r=1}^{k-1} \sum_{q=r+1}^{k} w_{rq} x_r x_q + \sum_{r=1}^{k} w_r x_r + w_{N+1}$$

$$N = k+k(k-1)/2+k = k(k+3)/2.$$

+) J.C. Simon, in Pattern Recognition, Vol. 7, 1975

++) K.S. Fu, IEEE, Transactions on Automatic Control, April 1970

3) statistical - (where noise affects measurements)

$$d_i(X) = P_i p(X|w_i), \ i = 1,\ldots,m$$

Here P_i is a priori probability of class w_i and $p(X|w_i)$ - multivariate conditional probability density function of X(usually vector-valued, random) given w_i. The discriminant function in that case would correspond to the Bayesian decision rule with 1-0 loss function.