Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

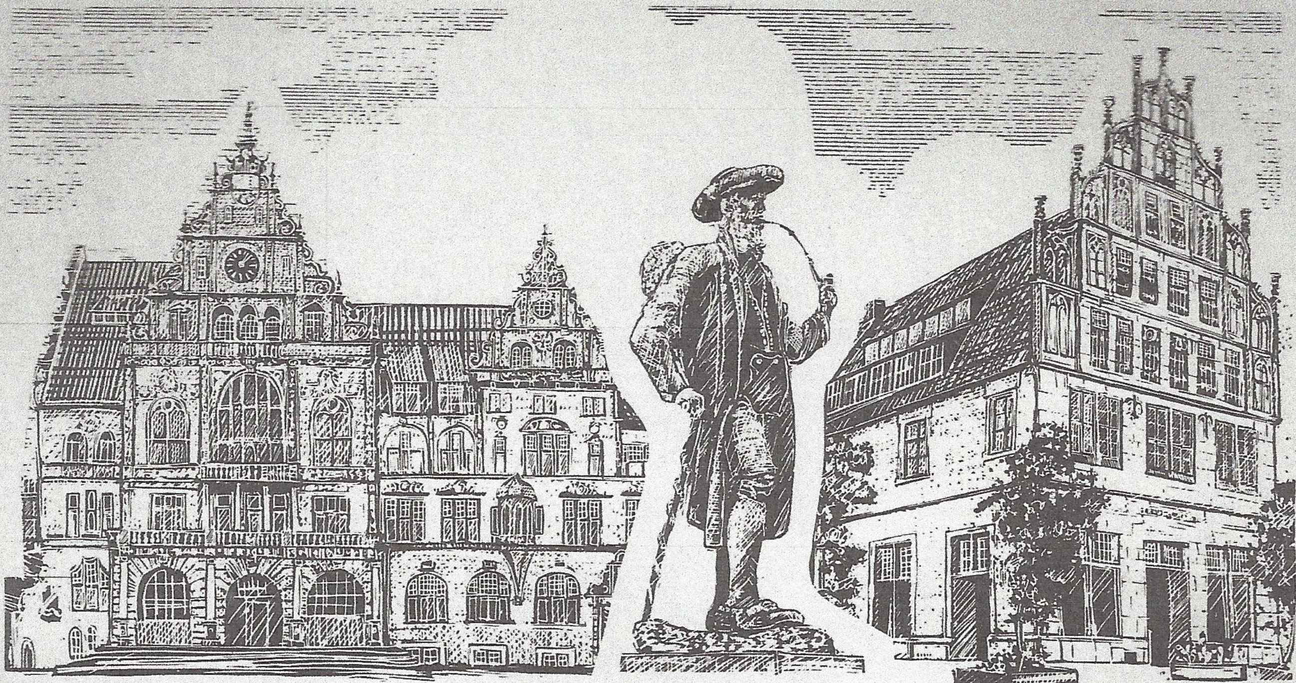
Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 41

Dierk Bitter

The Kernel for the Grand Coalition of the Four-Person Game

January 1976



H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

Dierk Bitter

The kernel for the grand coalition of the four-person game

1. Introduction

The purpose of this paper is to give a method for computing the kernel for the grand coalition, whenever the kernel and the nucleolus do not coincide.

It is well-known that the kernel of a cooperative game with side-payments includes the nucleolus $\mathcal{N}$.

Further the kernel of a coalition structure $\mathcal{B}$ in a four-person game consists of a line segment which may shrink into a single point.

We know by Peleg (see (7)) that the kernel for the grand coalition and the nucleolus are not identical if and only if for a suitable naming of the players the following sentence is true:

$$(1.1) \quad \exists x_1 \exists x_2 \exists x_3 \exists x_4$$

$$(x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_4 > 0,$$

$$x_1 + x_2 + x_3 + x_4 = v(N),$$

$$e(x,12) = e(x,34) > F(x),$$

$$e(x,14) = e(x,23) > F(x),$$

$$\text{where } F(x) = \max\{e(x,1),\ e(x,2),\ e(x,3),\ e(x,4),\ e(x,13),\ e(x,24),\ e(x,123),\ e(x,124),\ e(x,134),\ e(x,234)\}$$

Using the method of (3) Peleg has found that (1.1) is equivalent to

---

This work formed part of the author's M.S. thesis, which has been supervised by Professor Dr. R. Selten.

$$
\begin{aligned}
(1.2)\quad \min \Big\{ & v(N) - f_{12},\ v(N) - f_{23},\ v(12) - f_{12} + f_{14}, \\
& v(34),\ \tfrac{1}{2}\Big[f_{23} + v(14) + v(34) - v(23) - v(13)\Big], \\
& v(12) - v(123) + f_{23} - f_{12} + f_{14},\ v(14), \\
& f_{23} + v(14) + v(34) - v(134) - v(23), \\
& v(23) + f_{14} - f_{12},\ v(14) + f_{23} - v(123), \\
& \tfrac{1}{2}\Big[v(14) - v(13) + f_{23} - f_{12} + f_{14}\Big], \\
& v(14) - v(134) + f_{23} - f_{12} + f_{14} \Big\} > \\
\max \Big\{ & 0,\ f_{14} - f_{12},\ v(N) - v(12) - f_{23}, \\
& f_{12} - v(12),\ v(124) - v(12),\ v(234) - v(23), \\
& v(234) + \tfrac{1}{2}\Big[v(14) - v(23) - v(12) - v(34)\Big], \\
& \tfrac{1}{2}\Big[v(24) - f_{23} + v(N) - v(12)\Big], \\
& f_{14} - f_{12} - f_{23} + v(N) - v(14), \\
& v(124) - v(14) - f_{12} + f_{14}, \\
& \tfrac{1}{2}\Big[v(24) - f_{23} + f_{14} - f_{12} + v(N) - v(14)\Big], \\
& v(N) - f_{23} - v(14) \Big\},
\end{aligned}
$$

where $f_{ij} = \frac{1}{2}\Big[v(ij) + v(N) - v(N - \{i,j\})\Big]$.

Thus (1.2) characterizes all four-person games for which the kernel for the grand coalition (the kernel of N) consists of a non-degenerate interval.

Therefore we can restrict our attention to those games for which the property (1.2) is true.

If (1.2) does not hold one has to compute the nucleolus in order to find the kernel.

In section 3 we shall see that for many four-person games (including all proper four-person games) the kernel of N is included in the core, whenever the kernel consists of a non-degenerate interval.

Furthermore the kernel of N occupies there a kind of a central position.

In section 4 we describe algebraic properties of the kernel, whenever the kernel and the nucleolus are not identical.

These properties of the kernel enable us to give a method for computing the kernel, if it differs from the nucleolus. This will be done in section 5.
In section 6 we offer a second method for classifying games according to their kernels for all four-person games and in the last section we give some examples.

2. Definitions

Let $N = \{1,2,3,4\}$ be a set with four elements.
A characteristic function is a non-negative real function v defined on the subsets of N satisfying $v(\emptyset) = 0$ and $v(i) = 0$ for all i in N. The pair (N,v) is a four-person game.
The members of N are called players, subsets of N are called coalitions.
The game will be called proper, if its characteristic function v satisfies

$$v(S) + v(T) \leq v(S \cup T) \text{ for all } S, T \subset N \text{ with } S \cap T = \emptyset.$$

We call a four-person game (N,v) pseudoproper, if its characteristic function v satisfies

$$v(ij) + v(N - \{i,j\}) \leq v(N) \text{ for all } i,j \text{ in } N,\ i \neq j.$$

We remark that a proper four-person game is pseudoproper.
A simple game is one whose characteristic function satisfies $v(S) = 0$ or $v(S) = 1$ for all $S \subset N$,
then S is called winning if $v(S) = 1$, and losing if $v(S) = 0$.
Let (N,v) be a four-person game.
A coalition structure is a partition of N.
An individually rational payoff configuration is a pair $(x, \mathcal{B})$, where $\mathcal{B}$ is a coalition structure and the payoff vector $x = (x_1, x_2, x_3, x_4)$ is a quadruple of real numbers, satisfying

$$x_i \geq 0,\ i = 1,2,3,4 \text{ and } x(B) = \sum_{i \in B} x_i = v(B) \quad \text{for all } B \in \mathcal{B}.$$

We call a payoff vector an imputation, if it satisfies

$x_i \geq 0$, $i = 1,2,3,4$ and $x(N) = v(N)$.
$X(N)$ denotes the set of imputations.

Let i and j be two distinct players.
By T_{ij} we denote the set of all coalitions containing player i but not player j, thus

$$T_{ij} = \{ B \,/\, B \subset N,\ i \in B,\ j \notin B \} .$$

Let $(x, \mathcal{B})$ be an individually rational payoff configuration and let D be an arbitrary coalition.
The excess of D with respect to $(x, \mathcal{B})$ is

$$e(x,D) = v(D) - x(D) .$$

The maximum surplus of i over j with respect to $(x, \mathcal{B})$ is

$$s_{ij}(x) = \max \{ e(x,D) \,/\, D \in T_{ij} \} .$$

If i and j belong to the same coalition $B \in \mathcal{B}$, then i is said to outweigh j with respect to x, if

$$s_{ij}(x) > s_{ji}(x) \text{ and } x_j > 0 .$$

x is balanced if there exists no pair of players h and k, such that h outweighs k.

The kernel of the coalition structure $\mathcal{B}$ is the set of all balanced payoff vectors.
By $\mathcal{K}$ we denote the kernel of N.
The nucleolus $\mathcal{N}$ of a game (N,v) (for the grand coalition) is that unique imputation x such that $\theta(y)$ is not smaller than $\theta(x)$ lexicographically for all imputations y, where $\theta(x)$ is that vector in R^{2^n} (n is the cardinality of N) with components $e(x,S)$ for all $S \subset N$, arranged in descending numerical order.
The core $C(N,v)$ of a game (N,v) is the set of all payoff vectors that give rise only to non-positive excesses:

$$C(N,v) = \{ x \,/\, x \in X(N) \text{ and } e(x,S) \leq 0 \text{ for all } S,\ S \subset N \} .$$

3. The geometric structure of $\mathcal{K}$

In this section we shall see that for $\mathcal{K} \neq \mathcal{N}$ the kernel (for the grand coalition) of a pseudoproper four-person game can be described completely by the core.

Lemma 3.1

Let (N,v) be a four-person game satisfying $\mathcal{K} \neq \mathcal{N}$. Then the core of (N,v) is not empty if and only if (N,v) is pseudoproper.

Proof:

It is obvious that the game must be pseudoproper if it has a non-empty core.

We have to show that the core is not empty if (N,v) is pseudoproper.

$\mathcal{K} \neq \mathcal{N}$ implies by (1.1) that for a suitable naming of the players the following is true:

there exists $x_0 \in X(N)$ such that

$$e(x_0, 12) = e(x_0, 34) > F(x_0)$$

$$e(x_0, 13) = e(x_0, 24) > F(x_0),$$

where $F(x_0)$ is defined as in (1.1) by the maximum of the remaining ten excesses (without the excesses of N and $\emptyset$).

For all $x \in X(N)$ we have

$e(x, 12) + e(x, 34) \leq 0$ and

$e(x, 13) + e(x, 24) \leq 0$, because (N,v) is pseudoproper.

Thus $e(x_0, 12) = e(x_0, 34) \leq 0$ and

$e(x_0, 13) = e(x_0, 24) \leq 0$, so

$e(x_0, S) \leq 0$ for all $S \subset N$.

Thus x_0 is an element of the core of (N,v).

Theorem 3.2

Let (N,v) be a pseudoproper four-person game satisfying $\mathcal{K} \neq \mathcal{N}$.

Let y be a vector in the core C of (N,v).

Denote by $d_{ij}(y)$ the maximum amount which can be transferred from i to j while remaining in C, thus

$$d_{ij}(y) = \max \{ d \;/\; (y_1, \ldots, y_i - d, \ldots, y_j + d, \ldots, y_4) \in C \}$$

Then

$x \in \mathcal{K}$ if and only if

$x \in C$ and $d_{ij}(x) = d_{ji}(x)$ for all i,j in N, $i \neq j$.

Proof:

(i) By lemma 3.1 the core of (N,v) is not empty.

(ii) $C \neq \emptyset$ implies $\mathcal{K} \subset C$ (see (7), cor. 3.3).

(iii) $\mathcal{N} \cap C = \mathcal{K}$ implies the theorem (see (5), lemma 3.1 and lemma 3.5 for $\varepsilon = 0$).

Remarks:

(i) It is obvious that theorem 3.2 holds too, if (N,v) is a four-person game, satisfying $\mathcal{K} = \mathcal{N}$ and having a non-empty core.

(ii) The theorem shows that $\mathcal{K}$ depends only on the geometric shape of the core. Therefore we get: two different four-person games, having the same non-empty core, have the same kernel for the grand coalition.

(iii) Lemma 3.1 implies that a pseudoproper four-person game (and therefore also a proper four-person game) with an empty core must satisfy $\mathcal{K} = \mathcal{N}$.

4. Algebraic properties of $\mathcal{K}$

The following theorem describes the kernel $\mathcal{K}$ of a four-person game completely, whenever $\mathcal{K}$ consists of a non-degenerate interval. We remark that the game need not be pseudoproper.

In section 5 theorem 4 will be the basis for a method to compute the kernel.

Theorem 4

Let (N,v) be a four-person game satisfying $\mathcal{K} \neq \mathcal{N}$ and let $x \in X(N)$.

Then

$x \in \mathcal{K}$ if and only if for a suitable naming of the players (I) holds:

$$\text{(I)}: \quad e(x,13) = e(x,24) \geq F(x) \quad \text{and}$$
$$e(x,14) = e(x,23) \geq F(x), \quad \text{where}$$
$F(x)$ is defined as before.

Proof:

Let $x, y \in \mathcal{K}$, $x \neq y$

Define $t = y - x$

Then for a suitable naming of the players

$$t_1 \leq t_2 \leq t_3 \leq t_4 \quad \text{with} \sum_{i=1}^{4} t_i = 0 \quad \text{is true.}$$

Inview of $x \neq y$ we get

$t_1 < 0$ and $t_4 > 0$, thus $x_1 > 0$ and $y_4 > 0$,

so $s_{14}(x) \geq s_{41}(x)$ and $s_{41}(y) \geq s_{14}(y)$. (a)

$t_1 < 0$ and $t_1 + t_2 < 0$ and

$t_1 + t_3 \leq 0$ and $t_1 + t_2 + t_3 < 0$ implies $t(D) \leq 0$ for all $D \in T_{14}$, so

$e(x,D) \leq e(y,D)$ for all $D \in T_{14}$, thus

$$s_{14}(x) \leq s_{14}(y) \qquad \text{(b)}$$

analogeously, $e(x,S) \geq e(y,S)$ for all $S \in T_{41}$,

$$\text{so } s_{41}(x) \geq s_{41}(y) \qquad \text{(c)}$$

By (a) and (b) and (c) we get

$$s_{41}(y) = s_{41}(x) = s_{14}(x) = s_{14}(y)$$

From the fact that we have $e(x,S) \geq e(y,S)$ for all $S \in T_{41}$

it is clear that

$s_{41}(x) = s_{41}(y)$ implies that there exists an $S' \in T_{41}$,

such that

$$e(x,S') = s_{41}(x) = s_{41}(y) = e(y,S'),$$

so $x(S') = y(S')$, thus $t(S') = 0$,

on account of $t(S') = 0$ and by $S' \in T_{41}$ we get $S' = \{2,4\}$.

Analogeously,

$s_{14}(x) = s_{14}(y)$ implies that there exists a $D' \varepsilon T_{14}$,

such that

$$e(x,D') = s_{14}(x) = s_{14}(y) = e(y,D'),$$

so $x(D') = y(D')$, thus $t(D') = 0$.

So we get $D' = \{1,3\}$.

Together

$$e(x,13) = e(x,24) = e(y,24) = e(y,13)$$

Furthermore

$$t_1 = -t_3 \text{ and } t_2 = -t_4 \text{ , thus } t_1 - t_4 = t_2 - t_3$$

and because of $t_1 \leq t_2 \leq t_3 \leq t_4$ we get $t_1 = t_2$ and $t_3 = t_4$.

So

$$e(x,13) = e(x,24) = s_{14}(x) = s_{41}(x) = s_{23}(x) = s_{32}(x),$$

analogeously (on account of $t_1 = t_2$ and $t_3 = t_4$)

$$e(y,13) = e(y,24) = s_{14}(y) = s_{41}(y) = s_{23}(y) = s_{32}(y),$$

$$e(x,23) = e(x,14) = s_{24}(x) = s_{42}(x) = s_{13}(x) = s_{31}(x) \text{ and}$$

$$e(y,23) = e(y,14) = s_{24}(y) = s_{42}(y) = s_{13}(y) = s_{31}(y)$$

and therefore

$$e(x,13) = e(x,24) \geq F(x),$$
$$e(x,14) = e(x,23) \geq F(x) \text{ and}$$
$$e(y,13) = e(y,24) \geq F(y),$$
$$e(y,14) = e(y,23) \geq F(y).$$

Now let

a) $e(x,13) = e(x,24) \geq F(x)$ and

b) $e(x,14) = e(x,23) \geq F(x)$ be true

a) implies $s_{14}(x) = e(x,13) = e(x,24) = s_{41}(x)$

and $s_{23}(x) = e(x,24) = e(x,13) = s_{32}(x)$

b) implies $s_{13}(x) = e(x,14) = e(x,23) = s_{31}(x)$

and $s_{42}(x) = e(x,14) = e(x,23) = s_{24}(x)$

Furthermore we have

$$s_{12}(x) = \max\{e(x,13), e(x,14)\}$$
$$s_{21}(x) = \max\{e(x,24), e(x,23)\},$$

and $s_{34}(x) = \max \{ e(x,23),\ e(x,13) \}$

$s_{43}(x) = \max \{ e(x,14),\ e(x,24) \}$

Thus

$$s_{ij}(x) = s_{ji}(x)\ , \quad \text{so} \quad x \in \mathcal{K}\ .$$

Remarks:

(i) If a four-person game satisfies $\mathcal{K} \neq \mathcal{N}$, then by reason of (1.1) it is necessary that there be a suitable naming of the players which varifies the inequalities

$$v(13) + v(24) > v(12) + v(34)$$

and

$$v(14) + v(23) > v(12) + v(34)$$

(ii) It is obvious that theorem 4 is equivalent to:
Let (N,v) be a four-person game satisfying $\mathcal{K} \neq \mathcal{N}$ and let $x \in X(N)$. Then

$x \in \mathcal{K}$ if and only if (I) or (II) or (III), where

$$\text{(I)}:\ e(x,13) = e(x,24) \geq F(x)\ ,\quad e(x,14) = e(x,23) \geq F(x)$$

$$\text{(II)}:\ e(x,14) = e(x,23) \geq F(x)\ ,\quad e(x,12) = e(x,34) \geq F(x)$$

$$\text{(III)}:\ e(x,13) = e(x,24) \geq F(x)\ ,\quad e(x,12) = e(x,34) \geq F(x)$$

($F(x)$ is defined as the maximum of the remaining ten excesses without the excesses of N and $\emptyset$).

(iii) The kernel is not empty, therefore one of the conditions in (ii) must hold.
On account of (i) exactly one of them must hold.

(iv) It is obvious, too that the condition (II) originates from (I) by exchangeing player one and player four, the condition (III) originates from (I) by exchangeing player two and player four.

5. A method for computing $\mathcal{K}$

The conditions (I), resp. (II) and (III) are equivalent to a system of two equations and 20 weak inequalities.

For the present we transform (I), the conditions (II) and (III) merely stand for another naming of the players.
The system which is equivalent to (I) :

$$v(13) - v(12) \geq x_3 - x_2$$
$$v(13) - v(34) \geq x_1 - x_4$$
$$v(23) - v(12) \geq x_3 - x_1$$
$$v(23) - v(34) \geq x_2 - x_4$$
$$v(13) \geq x_1$$
$$v(13) \geq x_3$$
$$v(23) \geq x_3$$
$$v(23) \geq x_2$$
$$v(13) \geq x_1 + x_3 - x_2$$
$$v(13) \geq x_1 + x_3 - x_4$$
$$v(23) \geq x_2 + x_3 - x_1$$
$$v(23) \geq x_2 + x_3 - x_4$$
$$v(13) - v(123) \geq - x_2$$
$$v(13) - v(124) \geq x_3 - x_2 - x_4$$
$$v(13) - v(134) \geq - x_4$$
$$v(13) - v(234) \geq x_1 - x_2 - x_4$$
$$v(23) - v(123) \geq - x_1$$
$$v(23) - v(234) \geq - x_4$$
$$v(23) - v(124) \geq x_3 - x_1 - x_4$$
$$v(23) - v(134) \geq x_2 - x_1 - x_4$$

$$v(N) + v(13) - v(24) = 2x_1 + 2x_3$$
$$v(N) + v(14) - v(23) = 2x_1 + 2x_4$$

Now we are able to substitute x_1 , x_2 and x_4 by x_3 .
(The third equation is $x(N) = v(N)$, for x is an imputation).

So we get

$$2x_1 = v(N) + v(13) - v(24) - 2x_3$$

$$2x_2 = v(N) + v(23) - v(14) - 2x_3$$

$$2x_3 \leq v(13) - v(12) - \frac{v(14)}{2} + \frac{v(23)}{2} + \frac{v(N)}{2} = a_1$$

$$2x_3 \leq v(23) - v(12) - \frac{v(24)}{2} + \frac{v(13)}{2} + \frac{v(N)}{2} = a_2$$

$$2x_3 \leq 2v(13) = a_3$$

$$2x_3 \leq 2v(23) = a_4$$

$$2x_3 \leq v(13) + v(23) + v(24) - v(14) = a_5$$

$$2x_3 \leq v(23) + v(13) + v(14) - v(24) = a_6$$

$$2x_3 \leq v(N) - 2v(123) + 2v(13) + v(23) - v(14) = a_7$$

$$2x_3 \leq v(N) - 2v(123) + 2v(23) + v(13) - v(24) = a_8$$

$$2x_3 \leq v(N) - 2v(124) + v(13) + v(24) = a_9$$

$$2x_3 \leq v(N) - 2v(124) + v(23) + v(14) = a_{10}$$

$$-2x_3 \leq -\frac{v(N)}{2} - v(34) + v(24) - \frac{v(23)}{2} + \frac{v(14)}{2} = -b_1$$

$$-2x_3 \leq -\frac{v(N)}{2} - v(34) + v(14) - \frac{v(13)}{2} + \frac{v(24)}{2} = -b_2$$

$$-2x_3 \leq -v(N) + v(13) + v(24) = -b_3$$

$$-2x_3 \leq -v(N) + v(23) + v(14) = -b_4$$

$$-2x_3 \leq -v(N) + 2v(24) + v(14) - v(23) = -b_5$$

$$-2x_3 \leq -v(N) + 2v(14) + v(24) - v(13) = -b_6$$

$$-2x_3 \leq 2v(24) - 2v(234) = -b_7$$

$$-2x_3 \leq v(14) + v(23) + v(24) - v(13) - 2v(234) = -b_8$$

$$-2x_3 \leq v(24) + v(13) + v(14) - v(23) - 2v(134) = -b_9$$

$$-2x_3 \leq 2v(14) - 2v(134) = -b_{10}$$

It is obvious that this system of 20 inequalities and 2 equations is equivalent to:

$$x_1 = \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} - x_3 ,$$

$$x_2 = \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - x_3 ,$$

$$\frac{1}{2}\max\{b_1, b_2, \ldots, b_{10}\} \leq x_3 \leq \frac{1}{2}\min\{a_1, a_2, \ldots, a_{10}\},$$

$$x_4 = v(N) - x_1 - x_2 - x_3$$

and we have

$b_i \leq b_{i+1}$ $\quad i = 1, 3, 5, 7, 9$ if and only if $v(14) + v(23) \leq v(13) + v(24)$

and

$a_j \leq a_{j-1}$ $\quad j = 2, 4, 6, 8, 10$ if and only if $v(14) + v(23) \leq v(13) + v(24)$

Summing up we get

Theorem 5.1

Let (N,v) be a four-person game satisfying $\mathcal{K} \neq \mathcal{N}$.
Let $x \in X(N)$.

(a) Let $v(12) + v(34) < v(14) + v(23) \leq v(13) + v(24)$, then $x \in \mathcal{K}$ if and only if

$$x_1 = \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} - x_3,$$

$$x_2 = \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - x_3,$$

$$x_3 \in \left[\frac{1}{2}\max\{b_2, b_4, b_6, b_8, b_{10}\}, \frac{1}{2}\min\{a_2, a_4, a_6, a_8, a_{10}\}\right],$$

$$x_4 = v(N) - x_1 - x_2 - x_3$$

(b) Let $v(12) + v(34) < v(13) + v(24) \leq v(14) + v(23)$, then $x \in \mathcal{K}$ if and only if x_1, x_2 and x_4 are defined as in (a) and

$$x_3 \in \left[\frac{1}{2}\max\{b_1, b_3, b_5, b_7, b_9\}, \frac{1}{2}\min\{a_1, a_3, a_5, a_7, a_9\}\right].$$

Remark:

(b) originates from (a) by exchangeing player one and player two.

Using remark (iv) in section 4 we get theorem 5.2

<table>
<tr><th></th><th>(a)</th><th>(b)</th></tr>
<tr><td>property(*)</td><td>$v(12)+v(34) < v(23)+v(14) \leq v(24)+v(13)$</td><td>$v(12)+v(34) < v(24)+v(13) \leq v(23)+v(14)$</td></tr>
<tr><td>A =</td><td>$\{\frac{v(N)}{2}+\frac{v(13)}{2}-\frac{v(24)}{2}-v(12),$
$v(N)+v(14)-2v(124),$
$v(N)+v(13)+v(23)-v(24)-2v(123),$
$v(14)+v(13)-v(24),$
$v(23)\}$</td><td>$\{\frac{v(N)}{2}+\frac{v(23)}{2}-\frac{v(14)}{2}-v(12),$
$v(N)+v(24)-2v(124),$
$v(N)+v(13)+v(23)-v(14)-2v(123),$
$v(24)+v(23)-v(14),$
$v(13)\}$</td></tr>
<tr><td>B =</td><td>$\{\frac{v(N)}{2}+\frac{v(13)}{2}-\frac{v(24)}{2}+v(34),$
$v(N)-v(23),$
$2v(134)-v(14),$
$v(N)+v(13)-v(14)-v(24),$
$2v(234)+v(13)-v(24)-v(23)\}$</td><td>$\{\frac{v(N)}{2}+\frac{v(23)}{2}-\frac{v(14)}{2}+v(34),$
$v(N)-v(13),$
$2v(234)-v(24),$
$v(N)+v(23)-v(14)-v(24),$
$2v(134)+v(23)-v(14)-v(13)\}$</td></tr>
<tr><td>I =</td><td>$\left[-\frac{v(14)}{2}+\frac{\max B}{2}, \frac{v(23)}{2}+\frac{\min A}{2}\right]$</td><td>$\left[-\frac{v(24)}{2}+\frac{\max B}{2}, \frac{v(13)}{2}+\frac{\min A}{2}\right]$</td></tr>
<tr><td>$x \in \mathcal{K}$ if and only if</td><td colspan="2">$x \in X(N)$ satisfying $x_3 \in I$ and
$x_1 = \frac{v(N)}{2}+\frac{v(13)}{2}-\frac{v(24)}{2}-x_3,$
$x_2 = \frac{v(N)}{2}+\frac{v(23)}{2}-\frac{v(14)}{2}-x_3,$
$x_4 = \frac{v(14)}{2}+\frac{v(24)}{2}-\frac{v(13)}{2}-\frac{v(23)}{2}+x_3$</td></tr>
</table>

Figure 1

(see theorem 5.2)

	(c)	(d)
property(*)	$v(13)+v(24) < v(14)+v(23) \le v(12)+v(34)$	$v(13)+v(24) < v(12)+v(34) \le v(14)+v(23)$
A =	$\{ \frac{v(N)}{2} + \frac{v(34)}{2} - \frac{v(12)}{2} - v(24),$ $v(N)+v(14)-2v(124),$ $v(N)+v(34)+v(23)-v(12)-2v(234),$ $v(14)+v(34)-v(12),$ $v(23) \}$	$\{ \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - v(24),$ $v(N)+v(12)-2v(124),$ $v(N)+v(23)+v(34)-v(14)-2v(234),$ $v(23)+v(12)-v(14),$ $v(34) \}$
B =	$\{ \frac{v(N)}{2} + \frac{v(34)}{2} - \frac{v(12)}{2} + v(13),$ $v(N)-v(23),$ $2v(134)-v(14),$ $v(N)+v(34)-v(14)-v(12),$ $2v(123)+v(34)-v(12)-v(23) \}$	$\{ \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} + v(13),$ $v(N)-v(34),$ $2v(123)-v(12),$ $v(N)+v(23)-v(12)-v(14),$ $2v(134)+v(23)-v(34)-v(14) \}$
I =	$\left[-\frac{v(14)}{2} + \frac{\max B}{2}, \frac{v(23)}{2} + \frac{\min A}{2} \right]$	$\left[-\frac{v(12)}{2} + \frac{\max B}{2}, -\frac{v(34)}{2} + \frac{\min A}{2} \right]$
$x \in \mathcal{K}$ if and only if	$x \in X(N)$ satisfying $x_3 \in I$ and $x_1 = \frac{v(14)}{2} + \frac{v(12)}{2} - \frac{v(23)}{2} - \frac{v(34)}{2} + x_3,$ $x_2 = \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - x_3,$ $x_4 = \frac{v(N)}{2} + \frac{v(34)}{2} - \frac{v(12)}{2} - x_3$	

Figure 2

(see theorem 5.2)

	(e)	(f)
property(*)	$v(14)+v(23) \leq$ $v(12)+v(34) \leq$ $v(13)+v(24)$	$v(14)+v(23) \leq$ $v(13)+v(24) \leq$ $v(12)+v(34)$
$A =$	$\Big\{\frac{v(N)}{2}+\frac{v(13)}{2}-\frac{v(24)}{2}-v(14),$ $v(N)+v(12)-2v(124),$ $v(N)+v(13)+v(34)-v(24)-2v(134),$ $v(12)+v(13)-v(24),$ $v(34)\Big\}$	$\Big\{\frac{v(N)}{2}+\frac{v(34)}{2}-\frac{v(12)}{2}-v(14),$ $v(N)+v(24)-2v(124),$ $v(N)+v(34)+v(13)-v(12)-2v(134),$ $v(34)+v(24)-v(12),$ $v(13)\Big\}$
$B =$	$\Big\{\frac{v(N)}{2}+\frac{v(13)}{2}-\frac{v(24)}{2}+v(23),$ $v(N)-v(34),$ $2v(123)-v(12),$ $v(N)+v(13)-v(12)-v(24),$ $2v(234)+v(13)-v(34)-v(24)\Big\}$	$\Big\{\frac{v(N)}{2}+\frac{v(34)}{2}-\frac{v(12)}{2}+v(23),$ $v(N)-v(13),$ $2v(234)-v(24),$ $v(N)+v(34)-v(24)-v(12),$ $2v(123)+v(34)-v(13)-v(12)\Big\}$
$I =$	$\left[-\frac{v(12)}{2}+\frac{\max B}{2}, \frac{v(34)}{2}+\frac{\min A}{2}\right]$	$\left[-\frac{v(24)}{2}+\frac{\max B}{2}, \frac{v(13)}{2}+\frac{\min A}{2}\right]$
$x \in \mathcal{K}$ if and only if	$x \in X(N)$ satisfying $x_3 \in I$ and $x_1 = \frac{v(N)}{2}+\frac{v(13)}{2}-\frac{v(24)}{2}-x_3,$ $x_2 = \frac{v(12)}{2}+\frac{v(24)}{2}-\frac{v(13)}{2}-\frac{v(34)}{2}+x_3,$ $x_4 = \frac{v(N)}{2}+\frac{v(34)}{2}-\frac{v(12)}{2}-x_3$	

Figure 3

(see theorem 5.2)

<u>Theorem 5.2</u> (see figure 1, 2, 3)

Let (N,v) be a four-person game satisfying $\mathcal{K} \neq \mathcal{N}$.
Let $x \in X(N)$.
Then one of the properties (*) in the table must hold and the kernel of N is given by the corresponding column in the table.

The method described in the table enables us to compute the kernel for the grand coalition of the general-sum four-person game, whenever the kernel and the nucleolus do not coincide. But before computing the kernel we have to prove that the kernel really differs from the nucleolus.

6. A characterization of all four-person games for which $\mathcal{K}$ consists of a non-degenerate interval

Inequality (1.2) characterizes all four-person games for which the kernel of N consists of a non-degenerate interval. But it is usually a lengthy process to find a suitable naming of the players for which the inequality could be true.

For that reason the following theorems may be useful to characterize the four-person games for which the kernel for the grand coalition consists of a non-degenerate interval.

<u>Theorem 6.1</u>

Let (N,v) be a four-person game, satisfying

$$v(12) + v(34) < v(14) + v(23) \leq v(13) + v(24).$$ Then

$\mathcal{K} \neq \mathcal{N}$ if and only if $\min A' > \max B'$,

where

$$A' = \Big\{ \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} - v(12) + v(23),$$
$$v(N) + v(14) - 2v(124) + v(23),$$
$$v(N) + v(13) + 2v(23) - v(24) - 2v(123),$$
$$v(14) + v(13) + v(23) - v(24),$$
$$2v(23),$$

$$v(N) + v(23) - v(14),$$
$$v(N) + v(13) - v(24) \}$$ and

$$B' = \{ \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} + v(34) - v(14),$$
$$v(N) - v(23) - v(14),$$
$$v(N) + v(13) - 2v(14) - v(24),$$
$$2v(134) - 2v(14),$$
$$2v(234) + v(13) - v(14) - v(24) - v(23),$$
$$0,$$
$$v(13) + v(23) - v(14) - v(24) \} .$$

Corollary:

It is obvious that the last two elements of A' and the last two elements of B' may be omitted, if (N,v) is pseudoproper.

Proof:

$v(12) + v(34) < v(14) + v(23) \leq v(13) + v(24)$ is the case (a) (see figure 1, 2, 3).

$\mathcal{K} \neq \mathcal{N}$ implies that $\min A + v(23) > \max B - v(14)$,

thus
$$\min \{ \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} - v(12) + v(23),$$
$$v(N) + v(14) - 2v(124) + v(23),$$
$$v(N) + v(13) + 2v(23) - v(24) - 2v(123),$$
$$v(14) + v(13) + v(23) - v(24),$$
$$2v(23) \} >$$
$$\max \{ \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} + v(34) - v(14),$$
$$v(N) - v(23) - v(14),$$
$$v(N) + v(13) - 2v(14) - v(24),$$
$$2v(134) - 2v(14),$$
$$2v(234) + v(13) - v(14) - v(24) - v(23) \}$$
and
$$x_1 = \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} - x_3$$

$$x_2 = \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - x_3$$

$$x_3 \in I$$

$$x_4 = v(N) - x_1 - x_2 - x_3$$

$\mathcal{K}$ is a line segment, so we find $x^0 \in \mathcal{K}$, such that $x_3^o > \frac{1}{2}\max B - \frac{v(14)}{2}$

$x_3^o < \frac{1}{2}\min A + \frac{v(23)}{2}$

$x_3^o > 0$.

So
$x_1^o + x_3^o > 0$ implies $\frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} > 0$

and $x_2^o + x_3^o > 0$ implies $\frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} > 0$,

thus $v(N) + v(13) - v(24) > 0$

and $v(N) + v(23) - v(14) > 0$ 1)

Furthermore we have

$$\frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} = x_1^o + x_3^o \geqslant x_3^o > \frac{1}{2}\max B - \frac{v(14)}{2}$$

and

$$\frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} = x_2^o + x_3^o \geqslant x_3^o > \frac{1}{2}\max B - \frac{v(14)}{2}$$

thus $v(N) + v(13) - v(24) > \max B - v(14)$

and $v(N) + v(23) - v(14) > \max B - v(14)$ 2)

Now we have to prove:

$$v(13) + v(23) - v(14) - v(24) < v(N) + v(13) - v(24) .$$

This is equivalent to

$$v(23) - v(14) < v(N) .$$

By (1.1) we get for

$$v(12) + v(34) < v(23) + v(14) \leqslant v(24) + v(13)$$

and $\mathcal{K} \neq \mathcal{N}$ that there be an imputation x, satisfying

$x_i > 0$ for all i in N and

$e(x,14) = e(x,23)$ (see (1.1))

Thus $v(23) - v(14) = x_2 + x_3 - x_1 - x_4 < v(N)$

Analogeously,

$$e(x,13) = e(x,24)$$

Thus $v(13) - v(24) = x_1 + x_3 - x_2 - x_4$,

thus $v(13) - v(24) < v(N)$,

which is equivalent to

$$v(N) + v(23) - v(14) > v(13) + v(23) - v(14) - v(24)$$

So we get

$$v(N) + v(13) - v(24) > v(13) + v(23) - v(14) - v(24) \quad \text{and}$$
$$v(N) + v(23) - v(14) > v(13) + v(23) - v(14) - v(24) \; . \qquad 3)$$

We have

$$x_1^o + x_2^o = v(N) + \frac{v(13)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - \frac{v(24)}{2} - 2x_3^o \; , \quad \text{so}$$

$$x_3^o = x_4^o + \frac{v(13)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - \frac{v(24)}{2}$$

$$\geqslant \frac{v(13)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - \frac{v(24)}{2} \; , \quad \text{thus}$$

$$\frac{1}{2}\,\text{minA} + \frac{v(23)}{2} > x_3^o \geqslant \frac{v(13)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - \frac{v(24)}{2} \; , \quad \text{so}$$

$$\text{minA} + v(23) > v(13) + v(23) - v(14) - v(24) \qquad 4)$$

$$\frac{1}{2}\,\text{minA} + \frac{v(23)}{2} > x_3^o > 0 \quad \text{implies}$$

$$\text{minA} + v(23) > 0 \qquad 5)$$

Alltogether we get by r), r = 1, 2, 3, 4, 5 :

$\mathcal{K} \neq \mathcal{N}$ implies $\text{minA}' > \text{maxB}'$.

Now let $\text{minA}' > \text{maxB}'$

Set: $\text{minA}' - \text{maxB}' = d$. Then $d > 0$

Set:

$$x_1 = \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} - \frac{\text{minA}'}{2} + \frac{d}{4}$$

$$x_2 = \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - \frac{\text{minA}'}{2} + \frac{d}{4}$$

$$x_3 = \frac{\text{minA}'}{2} - \frac{d}{4}$$

$$x_4 = v(N) - x_1 - x_2 - x_3$$

$$= \frac{v(24)}{2} + \frac{v(14)}{2} - \frac{v(13)}{2} - \frac{v(23)}{2} + \frac{\text{minA}'}{2} - \frac{d}{4}$$

Then

$$x_1 \geqslant \frac{d}{4} > 0$$

$$x_2 \geqslant \frac{d}{4} > 0$$

$$x_3 = \frac{\text{maxB}'}{2} + \frac{d}{4} \geqslant \frac{d}{4} > 0$$

$$x_4 = \frac{v(24)}{2} + \frac{v(14)}{2} - \frac{v(13)}{2} - \frac{v(23)}{2} + \frac{\text{maxB}'}{2} + \frac{d}{4} \geqslant \frac{d}{4} > 0 \; ,$$

and

$$e(x,13) = v(13) - \frac{v(N)}{2} - \frac{v(13)}{2} + \frac{v(24)}{2} = -\frac{v(N)}{2} + \frac{v(13)}{2} + \frac{v(24)}{2}$$

$$e(x,24) = v(24) - \frac{v(N)}{2} - \frac{v(23)}{2} + \frac{v(14)}{2} - \frac{v(24)}{2} - \frac{v(14)}{2} + \frac{v(13)}{2} + \frac{v(23)}{2}$$

$$= -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}$$

$$e(x,14) = v(14) - \frac{v(N)}{2} - \frac{v(14)}{2} + \frac{v(23)}{2} = -\frac{v(N)}{2} + \frac{v(14)}{2} + \frac{v(23)}{2}$$

$$e(x,23) = v(23) - \frac{v(N)}{2} - \frac{v(23)}{2} + \frac{v(14)}{2} = -\frac{v(N)}{2} + \frac{v(14)}{2} + \frac{v(23)}{2}$$

$$e(x,1) = -\frac{v(N)}{2} - \frac{v(13)}{2} + \frac{v(24)}{2} + \frac{\min A'}{2} - \frac{d}{4}$$

$$\leq -\frac{v(N)}{2} - \frac{v(13)}{2} + \frac{v(24)}{2} + \frac{v(14)}{2} + \frac{v(13)}{2} + \frac{v(23)}{2} - \frac{v(24)}{2} - \frac{d}{4}$$

$$< -\frac{v(N)}{2} + \frac{v(14)}{2} + \frac{v(23)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}$$

$$e(x,2) = -\frac{v(N)}{2} - \frac{v(23)}{2} + \frac{v(14)}{2} + \frac{\min A'}{2} - \frac{d}{4}$$

$$\leq -\frac{v(N)}{2} - \frac{v(23)}{2} + \frac{v(14)}{2} + v(23) - \frac{d}{4}$$

$$< -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}$$

$$e(x,3) = -\frac{\max B'}{2} - \frac{d}{4} \leq -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} - \frac{d}{4}$$

$$< -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}$$

$$e(x,4) = \frac{v(13)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - \frac{v(24)}{2} - \frac{\max B'}{2} - \frac{d}{4}$$

$$\leq \frac{v(13)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - \frac{v(24)}{2} - \frac{v(N)}{2} - \frac{v(13)}{2} + v(14) + \frac{v(24)}{2} - \frac{d}{4}$$

$$< -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}$$

$$
\begin{aligned}
e(x,12) &= v(12) - v(N) - \frac{v(13)}{2} - \frac{v(23)}{2} + \frac{v(14)}{2} + \frac{v(24)}{2} + \\
&\qquad + \min A' - \frac{d}{2} \\
&\leq v(12) - v(N) - \frac{v(13)}{2} - \frac{v(23)}{2} + \frac{v(14)}{2} + \frac{v(24)}{2} + \frac{v(N)}{2} + \\
&\qquad + \frac{v(13)}{2} - \frac{v(24)}{2} - v(12) + v(23) - \frac{d}{2} \\
&< -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}
\end{aligned}
$$

$$
\begin{aligned}
e(x,34) &= v(34) - \frac{v(24)}{2} - \frac{v(14)}{2} + \frac{v(13)}{2} + \frac{v(23)}{2} - \max B' - \frac{d}{2} \\
&\leq v(34) - \frac{v(24)}{2} - \frac{v(14)}{2} + \frac{v(13)}{2} + \frac{v(23)}{2} - \frac{v(N)}{2} - \frac{v(13)}{2} + \\
&\qquad + \frac{v(24)}{2} - v(34) + v(14) - \frac{d}{2} \\
&< -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}
\end{aligned}
$$

$$
\begin{aligned}
e(x,123) &= v(123) - \frac{v(N)}{2} - \frac{v(13)}{2} + \frac{v(24)}{2} + \frac{\min A'}{2} - \frac{d}{4} - \frac{v(N)}{2} - \\
&\qquad - \frac{v(23)}{2} + \frac{v(14)}{2} \\
&\leq v(123) - v(N) - \frac{v(13)}{2} + \frac{v(24)}{2} - \frac{v(23)}{2} + \frac{v(14)}{2} + \frac{v(N)}{2} + \\
&\qquad + \frac{v(13)}{2} + v(23) - \frac{v(24)}{2} - v(123) - \frac{d}{4} \\
&< -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}
\end{aligned}
$$

$$
\begin{aligned}
e(x,124) &= v(124) - v(N) + \frac{\min A'}{2} - \frac{d}{4} \\
&\leq v(124) - v(N) + \frac{v(N)}{2} + \frac{v(14)}{2} - v(124) + \frac{v(23)}{2} - \frac{d}{4} \\
&< -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}
\end{aligned}
$$

$$
\begin{aligned}
e(x,134) &= v(134) - v(N) + \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - \frac{\max B'}{2} - \frac{d}{4} \\
&\leq v(134) - \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - v(134) + v(14) - \frac{d}{4} \\
&< -\frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq -\frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}
\end{aligned}
$$

and

$$e(x,234) = v(234) - v(N) + \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} - \frac{\max B'}{2} - \frac{d}{4}$$

$$\leq v(234) - \frac{v(N)}{2} + \frac{v(13)}{2} - \frac{v(24)}{2} - v(234) - \frac{v(13)}{2} +$$

$$+ \frac{v(24)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} - \frac{d}{4}$$

$$< - \frac{v(N)}{2} + \frac{v(23)}{2} + \frac{v(14)}{2} \leq - \frac{v(N)}{2} + \frac{v(24)}{2} + \frac{v(13)}{2}$$

Thus $x_i > 0$ and $x(N) = v(N)$ and

$e(x,13) = e(x,24) > F(x)$ and

$e(x,14) = e(x,23) > F(x)$

where $F(x)$ is defined as the maximum of the remaining ten excesses (without $e(x,N)$ and $e(x,\emptyset)$), so we get by (1.1) $\mathcal{K} \neq \mathcal{N}$.

In an analogeous way we get the following theorem (by exchangeing player one and player two).

Theorem 6.2

Let (N,v) be a four-person game, satisfying

$v(12) + v(34) < v(13) + v(24) \leq v(14) + v(23)$. Then

$\mathcal{K} \neq \mathcal{N}$ if and only if $\min A' > \max B'$

where

$$A' = \Big\{ \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} - v(12) + v(13) ,$$
$$v(N) + v(24) - 2v(124) + v(13) ,$$
$$v(N) + v(23) + 2v(13) - v(14) - 2v(123) ,$$
$$v(24) + v(23) + v(13) - v(14) ,$$
$$2v(13) ,$$
$$v(N) + v(13) - v(24) ,$$
$$v(N) + v(23) - v(14) \Big\}$$ and

$$B' = \Big\{ \frac{v(N)}{2} + \frac{v(23)}{2} - \frac{v(14)}{2} + v(34) - v(24) ,$$
$$v(N) - v(13) - v(24) ,$$
$$v(N) + v(23) - 2v(24) - v(14) ,$$
$$2v(234) - 2v(24) ,$$
$$2v(134) + v(23) - v(14) - v(13) - v(24) ,$$
$$0 ,$$
$$v(13) + v(23) - v(14) - v(24) \Big\} .$$

Remark:

Let (N,v) be a four-person game satisfying the property $(*)$ for the cases (c), (d), (e), or (f) (see figure 1, 2, 3). Then we get the corresponding theorems by:

(1) exchangeing player one and player four in theorem 6.1, if (c) holds

(2) exchangeing player one and player four in theorem 6.2, if (d) holds

(3) exchangeing player two and player four in theorem 6.1, if (e) holds

(4) exchangeing player two and player four in theorem 6.2, if (f) holds.

7. The simple four-person game

Using theorem 6.1 we find a characterization for the class of the simple four-person games.

Let (N,v) be a simple four-person game.
W. l. o. g. we name the players in a way, such that

$$v(12) + v(34) \leq v(14) + v(23) \leq v(13) + v(24).$$ Then

1) Let $v(N) = 0$. Then $\mathcal{K} = \mathcal{N}$, because $x = (0,0,0,0)$ is the only imputation.

2) Let $v(12) + v(34) = v(14) + v(23)$. Then $\mathcal{K} = \mathcal{N}$. (see remark (i), section 4)

3) Let $v(12) + v(34) < v(14) + v(23) \leq v(13) + v(24)$, $v(N) = 1$. Then $\mathcal{K} \neq \mathcal{N}$ if and only if $v(14) = v(23) = v(13) = v(24) = 1$.

Proof:

Let $\mathcal{K} \neq \mathcal{N}$

By theorem 6.1 we get $v(23) > 0$ and
$v(N) + v(13) - v(24) > v(13) + v(23) - v(14) - v(24)$,

so

$v(N) > v(23) - v(14)$, thus $v(14) = 1$

$v(14) + v(23) \leq v(13) + v(24) \leq 2$ implies

$$v(14) = v(23) = v(13) = v(24) = 1 .$$

Now let $v(14) = v(23) = v(13) = v(24) = 1$. Then

$\min A' \geq \frac{1}{2}$ and $\max B' \leq 0$, if $v(12) = 1$, $v(34) = 0$

$\min A' \geq 1$ and $\max B' \leq \frac{1}{2}$, if $v(12) = 0$, $v(34) = 1$

$\min A' \geq 1$ and $\max B' \leq 0$, if $v(12) = v(34) = 0$,

thus $\min A' > \max B'$, so $\mathcal{K} \neq \mathcal{N}$.

It is obvious that any proper simple four-person game must satisfy $\mathcal{K} = \mathcal{N}$.

In the following we compute the kernel for the grand coalition of the simple for-person game, whenever $\mathcal{K}$ and $\mathcal{N}$ do not coincide.

(i) Let $v(N) = v(13) = v(14) = v(23) = v(24) = v(34) = 1$, $v(12) = v(i) = 0$, $i = 1, 2, 3, 4$, $v(S) \in \{0, 1\}$ otherwise (for $v(ijk) = 1$ $\mathcal{K}$ has been computed in (8)).

The game satisfies

$v(12) + v(34) < v(13) + v(24) = v(14) + v(23) = 2$,

thus $\mathcal{K} \neq \mathcal{N}$

$$\min A = \min\left\{\frac{1}{2}, 2 - 2v(124), 2 - 2v(123), 1\right\}$$

$$\max B = \max\left\{\frac{3}{2}, 0, 2v(234) - 1, 2v(134) - 1\right\} = \frac{3}{2}$$

1) $1 \in \{v(124), v(123)\}$,

then $\min A = 0$, $\max B = \frac{3}{2}$, thus $I = \left[\frac{1}{4}, \frac{1}{2}\right]$

2) $1 \notin \{v(124), v(123)\}$,

then $\min A = \frac{1}{2}$, $\max B = \frac{3}{2}$, thus $I = \left[\frac{1}{4}, \frac{3}{4}\right]$

$x_1 = \frac{1}{2} - x_3$, $x_2 = \frac{1}{2} - x_3$, $x_4 = x_3$ for $x \in \mathcal{K}$,

so we get $x_3 \in \left[\frac{1}{4}, \frac{1}{2}\right]$.

Thus $\mathcal{K} = \left\{ x/x = (\frac{1}{2} - z, \frac{1}{2} - z, z, z), z \in \left[\frac{1}{4}, \frac{1}{2}\right] \right\}$ does not depend on the coalitions S containing three players.

(ii) Let $v(N) = v(13) = v(24) = v(23) = v(14) = 1$
$v(12) = v(34) = v(i) = 0$, $i = 1, 2, 3, 4$,
$v(S) \in \{0, 1\}$ otherwise.

This game also satisfies
$v(12) + v(34) < v(13) + v(24) = v(14) + v(23) = 2$,
thus $\mathcal{K} \neq \mathcal{N}$

$$\min A = \min\left\{\frac{1}{2}, 2 - 2v(124), 2 - 2v(123), 1\right\}$$

$$\max B = \max\left\{\frac{1}{2}, 0, 2v(234) - 1, 2v(134) - 1\right\}$$

1) $1 \in \{v(124), v(123)\}$ and $1 \in \{v(234), v(134)\}$,
then $\min A = 0$, $\max B = 1$, thus $I = \left[0, \frac{1}{2}\right]$

2) $1 \in \{v(124), v(123)\}$ and $1 \notin \{v(234), v(134)\}$,
then $\min A = 0$, $\max B = \frac{1}{2}$, thus $I = \left[-\frac{1}{4}, \frac{1}{2}\right]$

3) $1 \notin \{v(124), v(123)\}$ and $1 \in \{v(234), v(134)\}$,
then $\min A = \frac{1}{2}$, $\max B = 1$, thus $I = \left[0, \frac{3}{4}\right]$

4) $1 \notin \{v(124), v(123)\}$ and $1 \notin \{v(234), v(134)\}$,
then $\min A = \frac{1}{2}$, $\max B = \frac{1}{2}$, thus $I = \left[-\frac{1}{4}, \frac{3}{4}\right]$

$x_1 = \frac{1}{2} - x_3$, $x_2 = \frac{1}{2} - x_3$, $x_4 = x_3$ for $x \in \mathcal{K}$,
so we get $x_3 \in \left[0, \frac{1}{2}\right]$

Thus $\mathcal{K} = \left\{ x/x = (\frac{1}{2} - z, \frac{1}{2} - z, z, z), z \in \left[0, \frac{1}{2}\right] \right\}$
does not depend on the coalitions S containing three players.

(iii) Let $v(N) = v(13) = v(24) = v(14) = v(23) = v(12) = 1$
$v(34) = v(i) = 0$, $i = 1, 2, 3, 4$, $v(S) \in \{0, 1\}$ otherwise.
$v(12) + v(34) < v(13) + v(24) = v(14) + v(23) = 2$ holds,
thus $\mathcal{K} \neq \mathcal{N}$.

$$\min A = \min\left\{-\frac{1}{2}, 2 - 2v(124), 2 - 2v(123), 1\right\} = -\frac{1}{2}$$

$$\max B = \max\left\{\frac{1}{2}, 0, 2v(234) - 1, 2v(134) - 1\right\}$$

1) $1 \in \{v(134), v(234)\}$,

then $\min A = -\frac{1}{2}$, $\max B = 1$, thus $I = \left[0, \frac{1}{4}\right]$

2) $1 \notin \{v(134), v(234)\}$,

then $\min A = -\frac{1}{2}$, $\max B = \frac{1}{2}$, thus $I = \left[-\frac{1}{4}, \frac{1}{4}\right]$

$x_1 = \frac{1}{2} - x_3$, $x_2 = \frac{1}{2} - x_3$, $x_4 = x_3$ for $x \in \mathcal{K}$,

so we get $x_3 \in \left[0, \frac{1}{4}\right]$

Thus $\mathcal{K} = \left\{x/x = (\frac{1}{2} - z, \frac{1}{2} - z, z, z), z \in \left[0, \frac{1}{4}\right]\right\}$

does not depend on the coalitions S containing three players.

The examples (i), (ii) and (iii) describe the kernel for the grand coalition, whenever the kernel and the nucleolus do not coincide, for the values v(12) and v(34) characterize $\mathcal{K}$ completely.

We find

for $v(12) = v(34) = 0$

$$\mathcal{K} = \left\{x/x = (\tfrac{1}{2} - z, \tfrac{1}{2} - z, z, z), \quad z \in \left[0, \tfrac{1}{2}\right]\right\},$$

for $v(12) = 1$, $v(34) = 0$

$$\mathcal{K} = \left\{x/x = (\tfrac{1}{2} - z, \tfrac{1}{2} - z, z, z), \quad z \in \left[0, \tfrac{1}{4}\right]\right\},$$

and for $v(12) = 0$, $v(34) = 1$

$$\mathcal{K} = \left\{x/x = (\tfrac{1}{2} - z, \tfrac{1}{2} - z, z, z), \quad z \in \left[\tfrac{1}{4}, \tfrac{1}{2}\right]\right\}.$$

Remark:

For the cases

$v(13) + v(24) < v(12) + v(34) = v(14) + v(23) = 2$ and

$v(14) + v(23) < v(12) + v(34) = v(13) + v(24) = 2$ we get

the characterization of $\mathcal{K}$ in an analougeous way.

In the following we give some examples of general-sum four-person games.

8. Examples

(i) An example of a kernel consisting of a non-degenerate interval is the well-known game (N,v) (see (7)), satisfying

$v(12) = v(23) = v(34) = v(14) = 1, \ v(N) = 2,$

and $v(B) = 0$ otherwise.

(N,v) is not proper, but pseudoproper.

The kernel of N is $\mathcal{K} = \{x/x = (z, 1-z, z, 1-z), z \in [0,1]\}$.

(ii) Now we give an example for computing a class of games containing the game computed above.

Let $v(13) = v(24) = 0, \ v(N) = 2,$

$v(12) = v(34) = v(14) = v(23) = 1 ,$

$v(123) = v(124) = v(134) = v(234) = s ,$

so $v(13) + v(24) < v(12) + v(34) = v(14) + v(23) = v(N),$

which enables us to apply the corollary to theorem 6.1 .

So

$\min A + v(34) > \max B - v(12)$ which is equivalent to

$\min\{1, 3-2s, 3-2s, 1, 1\} + 1 >$

$\max\{1, 1, 1, 2s-1, 2s-1\} - 1$.

This is equivalent to

$\min\{2, 4-2s, 4-2s, 2, 2\} >$

$\max\{0, 0, 0, 2s-2, 2s-2\}$ $\quad$ *)

We have to distinguish three cases:

1. let $s \leq 1$, then $2 \leq 4 - 2s$ and $0 \geq 2s - 2$,

   thus $2 > 0$, so $\mathcal{K} \neq \mathcal{N}$ and $I = [0, 1]$

2. let $s \in (1, \frac{3}{2})$, then $2 > 4 - 2s$ and $0 < 2s - 2$

   $4 - 2s > 2s - 2$ if and only if $6 > 4s$,

   thus $\mathcal{K} \neq \mathcal{N}$ and $I = [s-1, 2-s]$

3. let $s \geq \frac{3}{2}$, then *) does not hold, thus $\mathcal{K} = \mathcal{N}$.

Now we are able to compute the kernel of N for the first and the second case.

1. $s \leq 1$, then

   $x \in \mathcal{K}$ if and only if

   $x_1 = x_3 , \ x_2 = 1 - x_3 , \ x_3 \in [0, 1] , \ x_4 = 1 - x_3$

2. $s \in (1, \frac{3}{2})$, then
   $x \in \mathcal{K}$ if and only if
   $x_1 = x_3$, $x_2 = 1 - x_3$, $x_3 \in \left[s - 1, 2 - s \right]$, $x_4 = 1 - x_3$

For this example we see that the kernel does not depend on the value of a coalition S containing three players, unless $v(S) > 1$.
On the other hand we see that the kernel becomes smaller, if the value for the coalitions S (containing three players) increases and for $s \geq \frac{3}{2}$ we get $\mathcal{K} = \mathcal{N}$.
This is quite clear from another point of view, too.
For $s = \frac{3}{2}$ the game has a core containing only one point: $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and the kernel is included in the core, whenever the latter is not empty.
For $s > \frac{3}{2}$ the core is empty, thus the game must satisfy $\mathcal{K} = \mathcal{N}$, because it is pseudoproper (see remark (iii), sec. 3).

(iii) Another example is the game (N,v), satisfying
$v(N) = 20$, $v(123) = 14$, $v(124) = 13$, $v(134) = 13$,
$v(234) = 13$, $v(12) = 13$, $v(13) = 14$, $v(14) = 12$,
$v(24) = 12$, $v(23) = 15$, $v(34) = 16$ and
$v(i) = 0$ for $i = 1, 2, 3, 4$

It is obvious that
$v(13) + v(24) < v(14) + v(23) \leq v(12) + v(34)$ and
$\min A' > \max B'$, thus $\mathcal{K} \neq \mathcal{N}$.
Using the method of theorem 5.2
we get $I = \left[\frac{27}{4}, \frac{29}{4} \right]$, thus
$x \in \mathcal{K}$ if and only if $x_3 \in \left[\frac{27}{4}, \frac{29}{4} \right]$, and
$x_1 = x_3 - 3$, $x_2 = x_4 = 11.5 - x_3$

(iv) In the last example we use a method of Gillies (see (2)).
By theorem 3.2 we have seen that the kernel of a pseudoproper game satisfying $\mathcal{K} \neq \mathcal{N}$ can be described by the core completely.

Let $v(N) = 20$, $v(123) = 8$, $v(124) = 10$, $v(134) = 6$,
$v(234) = 3$, $v(12) = 9$, $v(13) = 6$, $v(14) = 8.5$,
$v(23) = 4.5$, $v(24) = 1$, $v(34) = 4$, then

minA' > maxB' holds for this game, so $\mathcal{K} \neq \mathcal{N}$.

The game is pseudoproper , so we get by lemma 3.1 that the core is not empty.

Let u(N) = 20, u(123) = 9, u(124) = 10, u(134) = 8.5 ,
u(234) = 4.5 , u(12) = 9, u(13) = 6, u(14) = 8.5 ,
u(23) = 4.5 , u(24) = 1, u(34) = 4

The game (N,u) originates from (N,v) by setting

$$u(S) = \max_{\mathcal{T}} \sum_i v(S_i) \quad \text{for all} \quad S \subset N ,$$

where $\mathcal{T}$ is the class of partitions of S into non-empty subsets S_i .

It is obvious that (N,v) is not proper, whereas (N,u) is a proper game.
From the fact that we have v(N) = u(N) and the definition of (N,u) we get

C(N,v) = C(N,u) , i. e. the core of (N,v) and the core of (N,u) are identical.

Therefore their kernels for the grand coalition are identical, too and we get

$x \in \mathcal{K}$ if and only if

$x_1 = 4.5 + x_3$, $x_2 = 8 - x_3$, $x_3 \in [3.5, 4]$, $x_4 = 7.5 - x_3$

Remarks:

(i) Let (N,v) be a pseudoproper four-person game satisfying $\mathcal{K} \neq \mathcal{N}$ and $v(N) = \max\{v(S) / S \subset N\}$.
Then we find a proper four-person game (N,u) and the kernels for the grand coalition of (N,v) and (N,u) do not differ.

(ii) If we compute the nucleolus for the games described above, we find that the nucleolus is the middle of the kernel, i.e., $x = (x_1 , x_2 , x_3 , x_4)$ is the nucleolus of (N,v) if and only if x_i is the middle of that interval which presents the payoff for the player i .

(iii) Let (N,u) be the game defined above, but instead of $u(14) = 8.5$ set $u(14) = 8$, then $\min A' = \max B'$, thus $\mathcal{K} = \mathcal{N}$.

(iv) Two players in a game are called symmetric, if the game remains invariant when these players exchange roles.

This concept was generalized (see (4)) by defining a player k to be more desirable than a player l ($k \succsim l$), if player k always contributes not less than player l by joining coalitions which contain none of these players.

The game (N,u) is an example for the fact (see (4)) that the payoffs in the kernel always preserve the order determined by the desirability relation and that the desirability relation is transitive, for we have

$$1 \succsim 2 \succsim 4 \quad \text{and} \quad x_1 > x_2 > x_4 .$$

But the payoffs in the kernel do not strictly preserve desirability, for we have seen in the example (iii) that in spite of

$2 \succsim 1$ but not $1 \succsim 2$ the imputation

$$x = \left(\frac{17}{4} , \frac{17}{4} , \frac{29}{4} , \frac{17}{4} \right)$$

is an element of the kernel, i.e., player two is strictly more desirable than player one, but $x_1 = x_2$ holds for an imputation in the kernel.

References:

(1) Davis, Maschler
The kernel of a cooperative game,
Nav. Res. Logist. Quart. 12 (1965)

(2) Gillies
Solutions to general non-zero sum games,
Ann. of Math. Studies No. 40
Princeton University Press,
Princeton 1959

(3) Hodes
Dense linear order and linear inequalities,
Thomas J. Watson Research Center
Yorktown Heights, New York
Research Paper RC - 908 (1963)

(4) Maschler, Peleg
A characterization, existence proof and dimension bounds for the kernel of a game,
Pacific Journ. of Math., Vol 18, No. 2 1966

(5) Maschler, Peleg, Shapley
The kernel and the nucleolus of a cooperative game as locuses in the strong ε-core,
Dep. of Mathematics
The Hebrew University of Jerusalem,
Jerusalem, Israel (1970)

(6) Maschler, Peleg, Shapley
The kernel and bargaining set for convex games,
International Journal of Game Theory 1
1972

(7) Peleg
The kernel of the general-sum four-person game,
Canadian J. Math. 18 (1966)

(8) Aumann, Peleg, Rabinowitz
A method for computing the kernel of n-person games,
Math. of Computation 19 (1965)

(9) Schmeidler
The nucleolus of a characteristic function game,
Siam Journal of Applied Math.
Vol 17, No. 6 1969