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On the Regions of Linearity for the Nucleolus and their Computation

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1. Introduction

The concept of the nucleolus has been introduced by D. Schmeidler. It is the purpose of this paper to give a method for computing the linearity regions of the nucleolus.

Therefore it was necessary to generalize the notion of balanced sets (introduced by L.S. Shapley) to that of B_o-balanced sets. In section 3 we give some properties about these sets and about unions of B_o-balanced sets. After having introduced the notion of B_o-extensions for given B_o-balanced sets we achieve results about the linear independence of certain incidence vectors corresponding to coalitions out of B_o-balanced sets. Main results are theorem 3.2 and theorem 3.5.

In section 4 these results are used to achieve similar statements about maximal coalition arrays (called B-finest coalition arrays by J.H. Grotte). The main conclusion is theorem 4.10.

In section 5 we then introduce an equivalence relation on the set of all maximal coalition arrays. The notion of normalized coalition arrays enables us to find suitable representatives for each of the equivalence classes. Thus we are able to determine regions in the game space, characterized by a system of inequalities, on each of which the nucleolus is a linear function (note that the nucleolus is a piecewise linear function on the game space). The linearity regions determined by theorem 5.11, theorem 5.14 and lemma 5.17 are greater than those of E. Kohlberg. We conjecture that our regions are the greatest possible ones. We have also found a practical method to compute the nucleolus for games with a small number of players.

In section 6 we describe a procedure for constructing normalized coalition arrays and compute the nucleolus for the general 3-person-

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game. (J.H. Grotte has already computed the nucleolus for the superadditive 3-person-games.) Furthermore the author has evaluated the nucleolus for all superadditive 4-person-games. This will be published as a separate working paper.

2. Notations and basic definitions

A characteristic function game is a pair $(N;v)$ consisting of a set $N = \{1,\dots,n\}$ of n players and a characteristic function v, which maps each subset S of N, called a coalition, to a real number $v(S)$. In addition it is assumed that $v(N) \geq 0$ and that $v(S) = 0$ for all one-person coalitions, as well as for the empty set. The power set of N is denoted by $P(N)$.

We only consider games $(N;v)$ with $v(N) > 0$; the set of all these games is denoted by V_N. For the sake of simplicity let v be an element of V_N.

A pay-off vector (imputation) is a n-tupel $x \in R^n$ such that $x_i \geq 0$ for all $i=1,\dots,n$ and $\sum_{i=1}^{n} x_i = v(N)$. $X_N(v)$ is the set of all imputations.

For such games D. Schmeidler [D. Schmeidler 1969] has defined the nucleolus which is described as follows. If $x \in X_N(v)$, define $\Theta(x)$ to be the vector in R^{2^n-2} with the components $e(S,x) := v(S) - \sum_{i \in S} x_i$ for all $S \in P(N) \setminus \{N,\emptyset\}$ arranged in descending numerical order. The term $e(S,x)$ is called the excess of S with respect to x.

The nucleolus of the game $v \in V_N$, denoted by $N(v)$, is that unique imputation x for which $\Theta(x) \leq \Theta(y)$ lexicographically for all imputations $y \in X_N(v)$.

If B is a set, then the symbol $|B|$ specifies the number of elements of B. In the following let $N = \{1,\dots,n\}$ be an arbitrary but fixed set of n players. To avoid trivial cases we further assume the number n to be not smaller than three.

The concept of balanced sets has been introduced by L.S. Shapley [L.S. Shapley 1967]. Here we give a slight generalization of this concept.

In the following let B_o be an arbitrary but fixed subset of $\{\{1\},\dots,\{n\}\}$ such that $|B_o| \leq n-1$. If $B = \{b_o, b_1, \dots, b_k\}$ is a set of non-empty subsets of N, the incidence matrix that corresponds to B is the matrix

$$Y := (y_{ij}) \qquad i = 0,1,\dots,k \; ; \quad j = 1,\dots,n$$

where $$y_{ij} := \begin{cases} 1 & \text{if } j \in b_i \\ 0 & \text{if } j \notin b_i \end{cases} .$$

The vector $y_i := (y_{i1},\dots,y_{in}) \in R^n$ is called the incidence vector belonging to $b_i \in P(N) \setminus \{\emptyset\}$. A balancing vector of B is a vector $c^B := (c_o, c_1, \dots, c_k)$ that satisfies

$$\sum_{i=o}^{k} c_i y_{ij} = 1 \quad \text{for all} \quad j = 1,\dots,n$$

and $$c_i > 0 \qquad \text{for all} \quad i = 0,1,\dots,k.$$

Obviously, we get the equivalent formulation

$$\sum_{i=o}^{k} c_i y_i = c^B Y = e_N$$

where $e_N \in R^n$ is the vector $(1,\dots,1)$ and $c^B > 0$.

We denote by $C(B)$ the set of all balancing vectors of B. Then the set B is called B_o-balanced, if there exists a set $B_o^B \subset B_o$ such that $C(B \cup B_o^B) \neq \emptyset$.

Remarks 2.1

1. In the case of $B_o = \emptyset$ this exactly coincides with L.S. Shapley's definition of a balanced set ($\emptyset$-balanced sets). It is clear that for all possible sets $B_o \subset \{\{1\},\dots,\{n\}\}$ a $\emptyset$-balanced set is also B_o-balanced.

2. Since $|B_o| \leq n-1$ the empty set can never be B_o-balanced.

3. The set B_o^B , which is needed for the set B to be B_o-balanced, is not necessarily unique. Note that B_o^B may be empty. In the following it is assumed without loss of generality that the set $B_o^B \subset B_o$ has been chosen so that there does not exist a set $\hat{B}_o^B \subset B_o$ which satisfies $|\hat{B}_o^B| < |B_o^B|$ and $C(B \cup \hat{B}_o^B) \neq \emptyset$. Thus we obviously have $B \cap B_o^B = \emptyset$.

4. If $B \subset P(N)$ is B_o-balanced, then the set $B \cup B_o^B$ is $\emptyset$-balanced and conversely.

5. Let B_1, B_2 be subsets of $P(N)$; if B_1 and B_2 are B_o-balanced, then the set $B_1 \cup B_2$ is B_o-balanced, too.

It is obviously clear that the sets $\{N\}$ and $P'(N) := P(N) \setminus \{\emptyset, N\}$ are B_o-balanced. Moreover, $\{N\}$ is the only $\emptyset$ -balanced set with one element. In the following we consider only B_o-balanced sets with elements out of $P'(N)$.

The concept of minimal balanced sets has been originated by L.S. Shapley, too. We call a set $B \subset P'(N)$ <u>B_o-minimal</u> , if it is B_o-balanced and if it does not contain a proper subset B' which is also B_o-balanced. The set of all B_o-minimal subsets of $P'(N)$ is denoted by $M_N^{B_o}$. It follows directly from the above definition and from remark 2.1.3 that if $B \subset P'(N)$ is B_o-minimal, then the set $B \cup B_o^B$ is $\emptyset$-minimal.

The following example shows that in general the converse is not true:
Let N be the set $\{1,2,3,4\}$ and $B_o := \{\{3\},\{4\}\}$; the set $\{\{1,2\},\{1,3\},\{2,3\},\{4\}\}$ is $\emptyset$-minimal, but the set $\{\{1,2\},\{1,3\},\{2,3\}\}$ is not B_o-minimal because it contains the B_o-balanced set $\{\{1,2\}\}$.

In the following we use the symbol $I(B)$ for the set of incidence vectors of a subset B of $P'(N)$; the subspace of R^n, which is generated by the vectors of $I(B)$, is denoted by $L(B)$ or $L(I(B))$.

The next results are due to L.S. Shapley [L.S. Shapley 1967] :
Let B be a $\emptyset$-balanced set. Then B is $\emptyset$-minimal if and only if the set of the rows of the corresponding incidence matrix is linearly independent (i.e. $I(B)$ is linearly independent). Furthermore, B is the union of the $\emptyset$-minimal sets that it contains.

Remarks 2.2

1. A B_o-balanced set, which contains only one element, is B_o-minimal.

2. We have $|B| \leq n$ for all B_o-minimal sets $B \subset P'(N)$.

3. If B is $\emptyset$-balanced, then B is $\emptyset$-minimal if and only if $|C(B)| = 1$.

4. If B is B_o-minimal, then the set $I(B \cup B_o^B)$ is linearly independent.

The next definition serves the purpose to extend a given B_o-balanced set to a greater one with certain properties, which will become clear in the following sections.

Suppose $B_1, B_2 \subset P'(N)$; let B_1 be B_o-balanced and $B_2 \not\subset B_1$.
If there exists a set $B_2' \subset P'(N)$ which satisfies
$B_1 \cup B_2' \subsetneq B_1 \cup B_2$, $B_2' \not\subset B_1$ and $C(B_1 \cup B_2') \neq \emptyset$, we call the the set B_2' a <u>reduction</u> of $B_1 \cup B_2$ with respect to B_1 . Furthermore, let B_1, B_2 be given as above. Then we call the set $B_1 \cup B_2$ a <u>B_o-extension</u> of B_1 , if we have

i) $B_1 \cup B_2$ is B_o-balanced

ii) $B_1 \cup B_2$ does not possess a reduction with respect to B_1 .

Remarks 2.3

1. The subsets of B_o , which are necessary for the sets B_1 resp. $B_1 \cup B_2$ to be B_o-balanced, are denoted by B_o^1 resp. B_o^2 .

If it is clear which reduction resp. B_o-extension is meant, then the phrase "with respect to B_1" is omitted. In addition to the requirement for B_1 in remark 2.1.3 we shall always require that the set $B_o^2 \subset B_o$ has been chosen in such a manner that there does not exist a set $\hat{B}_o^2 \subset B_o$ which satisfies $|\hat{B}_o^2| < |B_o^2|$ and $C(B_1 \cup B_o^1 \cup B_2 \cup \hat{B}_o^2) \neq \emptyset$. Obviously, this can be done without loss of generality. In what follows this requirement will be always assumed to be satisfied. Now it follows that
$B_1 \cap B_o^2 = \emptyset$, $B_o^1 \cap B_o^2 = \emptyset$ and $B_2 \cap B_o^2 = \emptyset$.

2. Let B_1, B_2 be subsets of $P'(N)$ with $B_2 \not\subset B_1$. If both B_1 and $B_1 \cup B_2$ are B_o-balanced and if $|B_1 \cup B_2| = |B_1| + 1$, then $B_1 \cup B_2$ is a B_o-extension with respect to B_1.

The notion of coalition arrays has been introduced by E. Kohlberg [E. Kohlberg 1971]. A <u>coalition array</u> (in the following: "array") is a sequence $B_o, B_1, \dots, B_q$ of non-empty subsets of $P'(N)$ such that B_o contains only one-member coalitions, and may by empty, and $B_1, \dots, B_q$ is a partition of the set $P'(N)$. An array B we denote by $B = [B_o, B_1, \dots, B_q]$; the sets B_i are called <u>array parts</u>.

In accordance with the above definitions an array $B = [B_o, B_1, \dots, B_q]$ is called <u>array balanced</u> ("property II" by E. Kohlberg), if for all $k = 1, \dots, q$ the sets $\bigcup_{l=1}^{k} B_l$ are B_o-balanced. In the following we call such an array "balanced".

Remark 2.4

Similarly to the former remarks 2.1.3 and 2.3.1 in the following we shall always require that the set $B_o^t \subset B_o$, which is necessary for the set $\bigcup_{l=1}^{t} B_l$ $(t \in \{1, \dots, q\})$ to be B_o-balanced, has been chosen in such a manner that there does not exist a set $\hat{B}_o^t \subset B_o$ which satisfies $|\hat{B}_o^t| < |B_o^t|$ and $C(\bigcup_{l=1}^{t-1} B_l \cup \bigcup_{l=1}^{t-1} B_o^l \cup B_t \cup \hat{B}_o^t) \neq \emptyset$.

A set $B_o^t \subset B_o$ selected in this way is called the <u>"B_o-subset corresponding to the array part B_t"</u>. In what follows this requirement will be always assumed to be satisfied.

We shall use the following notations: let $B = [B_o, B_1, \dots B_q]$ be a balanced array. For $i \in \{0,1,\dots,q\}$ the array part B_i is the set $\{b_o^i, b_1^i, \dots, b_{k_i}^i\}$ and similarly the set $I(B_i) := \{y_j^i : 0 \leq j \leq k_i\}$ is the set of incidence vectors corresponding to $b_j^i \in B_i$. The union of those array parts B_{i_1}, which possess more than one element, is denoted by A_B, i.e.

$$A_B := \bigcup_{\substack{l=1 \\ |B_{i_l}| \geq 2}}^{m} B_{i_l} \; , \; i_l \in \{1,\dots,q\} \; .$$

If $A_B \neq \emptyset$, then i_1^B is the smallest index in the set $\{1,\dots,q\}$ which satisfies $|B_{i_1}| \geq 2$, and the index i_m^B is the greatest index in $\{1,\dots,q\}$ which satisfies $|B_{i_m}| \geq 2$ (if there is no danger of confusion we omit the subindex B). The set $\{i_1,\dots,i_m\} \subset \{1,\dots,q\}$ is the set of those indices whose corresponding array parts have more than one element. The set $K_B := A_B \cup B_o$ is called the <u>"set of critical coalitions"</u> belonging to the array B; furthermore, the union of the set $L_B := \{B_{i_1},\dots,B_{i_m}\}$ and the set $\{B_o\}$ is called the <u>"set of critical array parts"</u> belonging to the array B.

For $k = 1,\dots,q$ the vector $\bar{c}^k \in C(\bigcup_{l=1}^{k} B_l \cup \bigcup_{l=1}^{k} B_o^l)$ is the balancing vector of the set $\bigcup_{l=1}^{k} B_l \cup \bigcup_{l=1}^{k} B_o^l$ with the components c_j^i $(1 \leq i \leq k)$ corresponding to the incidence vectors $y_j^i \in I(B_i)$ and c_j^o corresponding to the incidence vectors belonging to $\bigcup_{l=1}^{k} B_o^l$. The exesses of coalitions $b_j^i \in B_i$ are denoted by $e_i(x)$ (with respect to a vector $x \in R^n$). Finally, we call the sequence $(B_o, B_1, \dots, B_l)$ <u>the l-truncation</u> of the array B $(l \in \{0,1,\dots,q\})$.

An array $C = [C_o, C_1, \ldots, C_r]$ is said to be derived from the array $B = [B_o, B_1, \ldots, B_q]$, if there exists indices $0 = k_o < k_1 < \ldots < k_r = q$ such that $C_o \supset B_o$ and $C_v = B_{k_{v-1}+1} \cup \ldots \cup B_{k_v}$ for all $v = 1,\ldots,r$. E. Kohlberg has shown that if an array is balanced, then any array derived from it is balanced, too.

The following definition has been introduced by J.H. Grotte [J.H. Grotte 1972]; it is used here in a modified way.

An array $B = [B_o, B_1, \ldots, B_q]$ is called maximal (B-finest by J.H. Grotte), if it is balanced and $|B_o| \leq n-1$ and if the only array from which it is derived and which is balanced is itself. (The second condition is useful in view of applications in the next sections.)

Finally two arrays $B = [B_o, B_1, \ldots, B_q]$ and $C = [C_o, C_1, \ldots, C_r]$ are called equal if they satisfy $q = r$ and $B_i = C_i$ for all indices $i = 0,1,\ldots,q$.

In the following let $N = \{1,\ldots,n\}$ be an arbitrary but fixed set of n players; to avoid trivial cases we assume $n \geq 3$. Some proofs in the next sections are somewhat lengthy. Therefore they will be omitted if they are not difficult.

3. B_o-balanced sets

For the whole section let B_o be an arbitrary but fixed subset of $\{\{1\},\ldots,\{n\}\}$ which satisfies $|B_o| \leq n-1$.

We now want to examine the set of incidence vectors corresponding to unions of B_o-balanced sets. For this purpose the incidence vectors of a set $B_t := \{b_o^t, b_1^t, \ldots, b_{k_t}^t\} \subset P'(N)$ are denoted by y_i^t, $i \in \{0,1,\ldots,k_t\}$, i.e. $I(B_t) = \{y_i^t : 0 \leq i \leq k_t\}$.

The following lemma is a direct consequence of the definition of B_o-balanced sets.

Lemma 3.1

Let B_1, B_2 be subsets of $P'(N)$; further let B_1 be B_o-balanced and $B_2 \neq \emptyset$. If $B_1 \cup B_2$ is B_o-balanced (i.e. there is a set $B_o^2 \subset B_o$ such that $C(B_1 \cup B_o^1 \cup B_2 \cup B_o^2) \neq \emptyset$) , then we have for all $y_i^2 \in I(B_2)$: $y_i^2 \in L(B_1 \cup B_o^1 \cup B_o^2 \cup B_2 \setminus \{b_i^2\})$.

The next theorem will be important for the whole paper, expecially for some of the proofs still to follow. It gives an algebraic characterization of those subsets of $P'(N)$ whose union with a given B_o-balanced set is again B_o-balanced.

Theorem 3.2

Let B_1, B_2 be subsets of $P'(N)$; further let B_1 be B_o-balanced and $B_2 \neq \emptyset$. Then the set $B_1 \cup B_2$ is B_o-balanced if and only if there is a set $B_o^2 \subset B_o$ and a real-valued vector $v := (v_o^2, v_1^2, \ldots, v_{k_2}^2 \,;\, v_1^{o2}, \ldots, v_{|B_o^2|}^{o2})$ with $v > 0$ such that

$$\sum_{j=o}^{k_2} v_j^2 y_j^2 + \sum_{y_j^o \in I(B_o^2)} v_j^{o2}\, y_j^o \in L(B_1 \cup B_o^1) .$$

Proof:

Obviously we can assume $B_1 \cap B_2 = \emptyset$. The theorem will be proved for the special case $B_o = \emptyset$; the general case can be reduced to this case.

Suppose first the set $B_1 \cup B_2$ to be B_o-balanced. Then the assertion is immediately clear (note that $e_N \in L(B_1 \cup B_o^1)$).

Conversely, we first have a vector $c^1 \in C(B_1)$ which satisfies

$c^1 > 0$ and $\sum_{j=o}^{k_1} c_j^1 y_j^1 = e_N$. Further there is a subset I' of $I = \{0, 1, \ldots, k_1\}$ such that $\{y_i^1 : i \in I'\}$ is a basis of $L(B_1)$. Now we define d to be the term $e_N - \sum_{j \notin I'} c_j^1 y_j^1$. This implies

$d = \sum_{i \in I'} c_i^1 y_i^1$ and $d \in L(B_1)$. If Y' is a matrix with rows $y_i^1 (i \in I')$ and if $\hat{c}^1 := (c_i^1)_{i \in I'}$ is a row vector, then we have the following equation: $\hat{c}^1\, Y' = d$.

The matrix Y' characterizes an injective, continuous linear function from $R^{|I'|}$ into R^n, which we call Y', too. Using the continuity of Y'^{-1} and the assumption of the theorem we can find a real number ϵ, $\epsilon > 0$, and a vector $x \in R^{|I'|}$ with $x > 0$ such that $xY' = d - \epsilon \sum_{j=o}^{k_2} v_j^2 y_j^2$ (note that $d - \epsilon \sum_{j=o}^{k_2} v_j^2 y_j^2 \in L(B_1)$).

We now eliminate the term d and get

$$\sum_{j \in I'} x_j y_j^1 + \sum_{j \notin I'} c_j^1 y_j^1 + \sum_{j=o}^{k_2} \epsilon v_j^2 y_j^2 = e_N$$

where all coefficients x_j, c_j^1, ϵv_j^2 are strictly positive. Thus we have found a balancing vector for the set $B_1 \cup B_2$.

For the general case $B_o = \emptyset$ we have only to note that the assumption that $B_1 \cup B_2$ is B_o-balanced is equivalent to saying that there is a set $B^2 \subset B_o$ such that $(B_1 \cup B_o^1) \cup (B_2 \cup B_o^2)$ is $\emptyset$-balanced.

Remarks 3.3

1. The above theorem is also true for sets

$$B_o' \subset \{\{1\},\ldots,\{n\}\} \quad \text{with} \quad |B_o'| = n.$$

2. Let B_1 be B_o-balanced, i.e. $(B_1 \cup B_o^1)$ is $\emptyset$-balanced. Then $\dim L(B_1 \cup B_o^1) = n$ implies that the set $(B_1 \cup B_o^1) \cup \{b\}$ is $\emptyset$-balanced for all sets $b \in P'(N)$.

Our next aim is to look at the linear independence of sets of incidence vectors of B_o-balanced sets. For this purpose it is appropriate to start with B_o-minimal sets which are the "smallest"

B_o-balanced sets and whose sets of incidence vectors are linearly independent. Step by step we combine these B_o-minimal sets with other subsets of $P'(N)$ such that the union is again B_o-balanced. Then we examine the set of incidence vectors of this greater set.

Lemma 3.4

Let B_1, B_2 be subsets of $P'(N)$ such that $B_1 \cap B_2 = \emptyset$ and $|B_2| \geq 2$; further let B_1 be B_o-balanced. If $B_1 \cup B_2$ is a B_o-extension of B_1, then we have $y \notin L(B_1 \cup B_o^1)$ for all vectors $y \in I(B_2 \cup B_o^2)$.

Using theorem 3.2 we immediately obtain an indirect proof.

In view of what has been said about the extension of B_o-balanced sets it is obviously clear that the assumption $B_1 \cap B_2 = \emptyset$ is only a technical one. Furthermore, in the case of $|B_2| \geq 2$ we can assume without loss of generality that, if $B_1 \cup B_2$ is a B_o-extension of B_1, the set $B_o^1 \cap B_2$ is empty.

The following theorem shows exactly which set of incidence vectors of a B_o-extension is linearly independent.

Theorem 3.5

Let B_1, B_2 be subsets of $P'(N)$ such that $B_1 \cap B_2 = \emptyset$ and $B_2 \neq \emptyset$; further let B_1 be B_o-balanced. Suppose the set $S_1 \subset I(B_1 \cup B_o^1)$ is a basis of $L(B_1 \cup B_o^1)$ and denote the set $I(B_o^2 \cup B_2 \setminus \{b_o^2\})$ by S_2. Now, if $B_1 \cup B_2$ is a B_o-extension of B_1, it follows:

i) $S_1 \cup S_2$ is linearly independent

ii) $y_o^2 \in L(B_1 \cup B_o^1 \cup B_o^2)$ if $|B_2| = 1$

Proof:

The second assertion follows directly from theorem 3.2. To prove the first assertion suppose that $S_1 \cup S_2$ is linearly dependent.

Then there are real-valued coefficients $\alpha_j, \beta_j, \gamma_j$ which do not all vanish such that

$$\sum_{j=1}^{k_2} \beta_j y_j^2 + \sum_{y_j^o \in I(B_o^2)} \gamma_j y_j^o + \sum_{y_j^1 \in S_1} \alpha_j y_j^1 = 0 .$$

Because of the linear independence of S_1 we can further assume without loss of generality that the coefficients β_j, γ_j do not all vanish. Thus

$$\text{x)} \quad \sum_{j=1}^{k_2} \beta_j y_j^2 + \sum_{y_j^o \in I(B_o^2)} \gamma_j y_j^o = - \sum_{y_j^1 \in S_1} \alpha_j y_j^1 \in L(B_1 \cup B_o^1)$$

Proposition:

We can assume $\beta_j \geq 0$ for all coefficients β_j in equation x)

Proof:

Suppose there is an index $l \in \{1,\dots,k_2\}$ such that $\beta_l < 0$.

In view of the B_o-balancedness of the set $B_1 \cup B_o^1 \cup B_2 \cup B_o^2$ there is a vector $c \in C(B_1 \cup B_o^1 \cup B_2 \cup B_o^2)$ such that $c > 0$ and

$$\text{xx)} \quad \sum_{j=o}^{k_2} c_j^2 y_j^2 + \sum_{y_j^o \in I(B_o^2)} c_j^{o2} y_j^o + \sum_{j=o}^{k_1} c_j^1 y_j^1 + \sum_{y_j^o \in I(B_o^1)} c_j^{o1} y_j^o = e_N .$$

We get:

$$\beta_l y_l^2 = \frac{\beta_l}{c_l^2} e_N - \frac{\beta_l}{c_l^2} \sum_{y_j^o \in I(B_o^2)} c_j^{o2} y_j^o - \frac{\beta_l}{c_l^2} \sum_{j=o}^{k_1} c_j^1 y_j^1 -$$

$$- \frac{\beta_l}{c_l^2} \sum_{y_j^o \in I(B_o^1)} c_j^{o1} y_j^o - \frac{\beta_l}{c_l^2} \sum_{\substack{j=o \\ j \neq l}}^{k_2} c_j^2 y_j^2$$

This last equation and equation x) imply

$$\text{xxx)} \quad \sum_{\substack{j=1 \\ j \neq l}}^{k_2} (\beta_j - \frac{\beta_l}{c_l^2} c_j^2)\, y_j^2 - \frac{\beta_l}{c_l^2} c_o^2 y_o^2 + \sum_{y_j^o \in I(B_o^2)} (\gamma_j - \frac{\beta_l}{c_l^2} c_j^{o2}) y_j^o \in L(B_1 \cup B_o^1)$$

(note that $e_N \in L(B_1 \cup B_o^1)$).

The above replacement procedure (P) guarantees that the coeffients of the vectors y_j^2 and y_j^o on the left side of x) will all increase by a strict positive amount

(notice that $(-\frac{\beta_1}{c_1^2} c_j^2) > 0$ and $(-\frac{\beta_1}{c_1^2} c_j^{o2}) > 0$ for all indices j).

Furthermore, coefficients of vectors y_j^2 which are positive in x), are positive in equation xxx), too. If necessary we have to repeat the procedure (P), now beginning with equation xxx), etc. After at most k_2 steps we get a linear combination similar to that of equation xxx), which is an element of $L(B_1 \cup B_o^1)$ and which possesses only positive coefficients of vectors $y_j^2 \in I(B_2)$.

Without loss of generality we can assume that this is the case in xxx). Now, if all coefficients of vectors $y_j^o \in I(B_o^2)$ are positive in equation xxx), we get a contradiction owing to the fact that $B_1 \cup B_2$ is a B_o-extension, because theorem 3.2 states the existence of a B_o-reduction of $B_1 \cup B_2$. (Note that at least the term $-\frac{\beta_1}{c_1^2} c_o^2$ is strictly positive).

If, on the other hand, there are coefficients of vectors $y_j^o \in I(B_o^2)$ which are strictly negative, then we apply the same procedure (P) on these coefficients and equation xxx). After at most $|B_o^2|$ steps we get a term of the following form:

$$\sum_{j=o}^{k_2} v_j y_j^2 + \sum_{y_j^o \in I(B_o^2) \setminus \{y_t^o\}} w_j y_j^o \in L(B_1 \cup B_o^1) = L(S_1)$$

where all coefficients v_j, w_j are strictly positive and $y_t^o \in I(B_o^2)$. Now, let $\{t\}$ be the subset of N corresponding to the incidence vector y_t^o. Theorem 3.2 states that the set $B_1 \cup B_o^1 \cup B_2 \cup B_o^2 \setminus \{t\}$ is B_o-balanced, which contradicts the fact that the set B_o^2 has been "minimally" chosen in accordance with remark 2.3.1. Thus the proposition has been proven.

An analogous proof shows that we can assume all coefficients y_j (in equation x)) to be equal or greater than zero.

For the proof of the theorem we now have to look at two cases.

Case 1: the coefficients β_j in equation x) do not all vanish. Then theorem 3.2 yields an immediate contradiction owing to the fact that $B_1 \cup B_2$ is a B_o-extension of B_1.

Case 2: all coefficients β_j vanish (this includes the case $|B_2| = 1$). Without loss of generality we assume all coefficients γ_j to be strictly positive. Now we choose any vector $y_t^o \in I(B_o^2)$ and evaluate the term y_t^o out of equation xx). Instead of y_t^o we insert the new term in equation x) and get a term of the following form:

$$\sum_{y_j^2 \in I(B_2)} \delta_j y_j^2 + \sum_{y_j^o \in I(B_o^2) \setminus \{y_t^o\}} \delta_j' y_j^o \in L(B_1 \cup B_o^1)$$

with the appropriate coefficients $\delta_j, \delta_j' \in \mathbb{R}$. (In the case of $|B_2| = 1$ we can assume $\delta_j > 0$.) Again applying procedure (P), if necessary, and theorem 3.2 we get a contradiction owing to the fact that $B_1 \cup B_2$ is a B_o-extension resp. that the set B_o^2 has been "minimally" chosen in accordance with remark 2.3.1.

Now the proof of the theorem is complete.

Thus, if $B_1 \subset P'(N)$ is B_o-minimal, we have the equality $I(B_1 \cup B_o^1) = S_1$ (remark 2.2.4). Together with the remarks 2.1.3, 2.3.1 and lemma 3.1 we immediately get the following corollary:

Corollary 3.6:

Let B_1, B_2 be subsets of $P'(N)$ such that $B_1 \cap B_2 = \emptyset$ and $B_2 \neq \emptyset$; further let B_1 be B_o-minimal. If $B_1 \cup B_2$ is a B_o-extension of B_1, it follows:

i) $I(B_1 \cup B_o^1 \cup B_o^2 \cup B_2 \setminus \{b_o^2\})$ is a basis of $L(B_1 \cup B_o^1 \cup B_o^2 \cup B_2)$

ii) $\dim L(B_1 \cup B_o^1 \cup B_o^2 \cup B_2) = |B_1| + |B_o^1| + |B_2| + |B_o^2| - 1$

Now we have reached our first aim : the property of being a B_o-extension is a sufficient condition for a certain subset of the incidence vectors of an extension of a B_o-minimal set to be linearly independent. Thus, we have a method for constructing larger sets of linearly independent incidence vectors.

The last lemma of this section gives necessary conditions for the B_o-extension of a B_o-balanced set. Both assertions are important for the construction of maximal coalition arrays considered in the next section.

Lemma 3.7

Let B_1, B_2 be subsets of $P'(N)$ such that $B_1 \cap B_2 = \emptyset$ and $B_2 \neq \emptyset$; further let B_1 be B_o-balanced. If $B_1 \cup B_2$ is a B_o-extension of B_1 , it follows:

i) $(B_1 \cup B_o^1) \cup (B_2 \cup B_o^2)$ is a $\emptyset$-extension of $(B_1 \cup B_o^1)$

ii) there is a $\emptyset$-minimal set $V \in M_N^{\emptyset}$ such that $B_2 \cup B_o^2$ is a subset of V.

Both assertions can be proven indirectly. The first proof uses theorem 3.2 and the second uses the fact that every $\emptyset$-balanced set is the union of the $\emptyset$-minimal sets that it contains [L.S. Shapley 1967] .

4. Coalition arrays

The first aim of this section is to show that in the case of $A_B \neq \emptyset$ the space $L(K_B)$ has full dimension n ; furthermore, we shall specify a basis of $L(K_B)$.

Lemma 4.1

Let $B = [B_o, B_1, \ldots, B_q]$ be a balanced array.
Then:

i) $\dim L(K_B) = \dim L(B_o) \geq n-1$ if $A_B = \emptyset$

ii) $\dim L(K_B) = n$ if $A_B \neq \emptyset$

Proof :

To prove the first assertion we have only to show that the set B_o has n-1 elements; using theorem 3.2 this can be done indirectly. For the proof of the second assertion it is sufficient to show that

$L(\bigcup_{i=1}^{q} B_i) \subset L(K_B)$. This is by induction on i (i=1,...,q) , again mainly using theorem 3.2. The full proof will not be given in this paper.

Obviously this lemma is also true in the case of maximal arrays which will be considered in the following. The study of maximal arrays is convenient because for every balanced array there is a maximal array from which the former can be derived. (in general this maximal array is not unique.

The following remarks are intuitively clear:

Remarks 4.2

1. If $B = [B_o, B_1, \ldots, B_q]$ is a balanced array, then $B_v \cap B_o^v = \emptyset$ for all $v=1,\ldots,q$ and $B_o^j \cap B_o^k = \emptyset$ for all indices $j \neq k$; $j,k \in \{1,\ldots,q\}$ (note remark 2.4).

2. If $B = [B_o, B_1, \ldots, B_q]$ is a maximal array we have $\bigcup_{v=1}^{q} B_o^v = B_o$. This reflects the fact that the set B_o is not allowed to be larger then necessary to balance the array.

3. Let $B = [B_o, B_1, \ldots, B_q]$ be a maximal array. For an index $k \in \{2,\ldots,q\}$ let the vector y_j^k be an element of the set $I(B_k)$. If $y_j^k \in L(B_o)$ then $|B_k| = 1$.

4. Let $B = [B_o, B_1, \ldots, B_q]$ be a maximal array. If for an index $k \in \{2,\ldots,q\}$ the set $B_k \cap B_o$ is not empty then we have $B_k \subset \bigcup_{v=1}^{k-1} B_o^v$ and $|B_k| = 1$.

The next lemma gives a connection between the notion of a B_o-extension and the maximal array now to be discussed.

Lemma 4.3

Let $B = [B_o, B_1, \dots, B_q]$ be a balanced array. Then a necessary and sufficient condition for B to be maximal is that the following statements are satisfied:

i) $B_o = \bigcup_{v=1}^{q} B_o^v$

ii) B_1 is B_o-minimal

iii) $\bigcup_{v=1}^{k-1} B_v \cup B_k$ is a B_o-extension of $\bigcup_{v=1}^{k-1} B_v$ for all indices $k=2,\dots,q$.

The proof of this lemma is straightforward.

Now let $B = [B_o, B_1, \dots, B_q]$ be a maximal array. We shall use the following notations: suppose S_1 is the set $I(B_1 \cup B_o^1)$ and S_v is the set $I(B_o^v \cup B_v \setminus \{b_o^v\})$ for all indices $v=2,\dots,q$. With these definitions we get the following

Lemma 4.4

Let $B = [B_o, B_1, \dots, B_q]$ be a maximal array. Then $\bigcup_{v=1}^{q} S_v$ is a basis of $L(P^c(N))$ and we have $\left|\bigcup_{v=1}^{q} S_v\right| = n$.

Proof:

Using lemma 3.1, lemma 4.3 and theorem 3.5 the proof is by induction on $k=1,\dots,q$, for it is sufficient to show that $\bigcup_{v=1}^{k} S_v$ is a basis of $L(\bigcup_{v=1}^{k} B_o^v \cup \bigcup_{v=1}^{k} B_v)$.

Thus, if $B = [B_o, B_1, \dots, B_q]$ is a maximal array with $|B_1| \geq 2$, the set $\bigcup_{v=1}^{q} S_v$ is a basis of $L(K_B) = L(P'(N))$ which only consists of incidence vectors corresponding to coalitions out of the set of critical coalitions K_B .

Realizing the fact that $y_o^1 \in L(B_o \cup B_{i_1})$ we have only to use a well-known result about changing of a basis for vectors to get the following lemma for the case $|B_1| = 1$:

Lemma 4.5

Let $B = [B_o, B_1, \ldots, B_q]$ be a maximal array such that $A_B \neq \emptyset$ and $|B_1| = 1$. Then the set $(\bigcup_{v=1}^{q} S_v \cup \{y_o^{i_1}\}) \setminus \{y_o^1\}$ is a basis of $L(P'(N))$.

Thus for all maximal arrays B with $A_B \neq \emptyset$ we have found a basis of $L(P'(N))$ which consists only of elements of $I(K_B)$. In the following we are always using the sets determined in lemma 4.4 resp. lemma 4.5 as a basis of the space $L(P'(N))$, according as $|B_1| \geq 2$ (lemma 4.4) or $|B_1| = 1$ (lemma 4.5). Now we have achieved the first aim of this section; the result can be seen in connection with L.S. Shapley's results concerning $\emptyset$-minimal sets and their incidence vectors (mentioned in section 2).

The following lemma specifies the number of elements of the set A_B and the number of array parts of a given maximal array B.

Lemma 4.6

Let $B = [B_o, B_1, \ldots, B_q]$ be a maximal array. Suppose m is the number of array parts B_j $(1 \leq j \leq q)$ with $|B_j| \geq 2$ and T is the number of all array parts of B (including the array part B_o). Then we have :

i) $|A_B| = n + (m-1) - |B_o|$

ii) $T = 2^n - n + |B_o|$

Proof :

To prove the first assertion we have to note that $S_i \cap S_j = \emptyset$ for all indices $i \neq j$; $i,j \in \{1,\dots,q\}$. This follows directly from the definition of arrays and from the remarks 2.4 and 4.2 . Again using remarks 4.2 and lemma 4.4 resp. lemma 4.5 we can conclude:

$$n = \left|\bigcup_{v=1}^{q} S_v\right| = \sum_{v=1}^{q} |S_v| \quad \text{and} \quad \sum_{v=1}^{q} |S_v| = \sum_{v=1}^{q} |S_v \setminus I(B_o^v)| + \sum_{v=1}^{q} |I(B_o^v)| \ .$$

In the case of $A_B \neq \emptyset$ we now have the equations :

$$|A_B| = \sum_{j:|B_j| \geq 2} |B_j| = \sum_{j=1}^{q} |S_j \setminus I(B_o^j)| + (m-1) =$$

$$= \sum_{j=1}^{q} |S_j| - \sum_{j=1}^{q} |B_o^j| + (m-1) =$$

$$= \sum_{j=1}^{q} |S_j| - |B_o| + (m-1) = n + (m-1) - |B_o|$$

(notice that $\bigcup_{j=1}^{q} B_o^j = B_o$) . In the case of $A_B = \emptyset$ we have $m = 0$ and $|B_o| = n-1$ (see lemma 4.1) . It follows:

$$0 = |A_B| = n + (0-1) - |B_o| = n - 1 - n + 1 = 0.$$

This shows that the assertion is also true in the last case.

To prove the second assertion we note that

$$r = |\{B_j : |B_j| = 1\}| + |\{B_j : B_j \subset A_B\}| + 1 \quad (1 \leq j \leq q) \ .$$

Together with the first assertion and the fact that there can be at most 2^n-2 array parts B_j with $|B_j| = 1$ we now obtain :

$$r = (2^n-2) - (n + (m-1) - |B_o|) + m+1$$

$$= 2^n - n + |B_o| \ .$$

This lemma shows that the number of array parts of a given maximal array B only depends on the number of players n and the array part B_o.

Until now we have obtained some general results on maximal arrays which we now want to specialize. By these specializations we are able to compute the nucleolus of a given game and the linearity regions of the nucleolus.

For a given array $B = [B_o, B_1, \ldots, B_q]$ we shall use the following notations:

$$Z_i := \begin{cases} \emptyset & \text{if } |B_i| = 1 \\ \{y_o^i - y_1^i, \ldots, y_o^i - y_{k_i}^i\} & \text{if } |B_i| > 1 \end{cases}$$

for all indices $i=1,\ldots,q$. Furthermore, we define Z_B to be the set $\bigcup_{i=1}^{q} Z_i \cup I(B_o)$.

Lemma 4.7

If $B = [B_o, B_1, \ldots, B_q]$ is a maximal array, it follows:

i) $Z_i \cap I(B_o) = \emptyset$ for all indices $i=1,\ldots,q$

ii) $Z_i \cap Z_j = \emptyset$ for all indices $i \neq j$; $i,j, \in \{1,\ldots,q\}$

Proof:

To prove the first assertion we can assume without loss of generality $A_B \neq \emptyset$. Suppose there is an index $i \in \{1,\ldots,q\}$ with $Z_i \neq \emptyset$, a vector $y \in I(B_o)$ and an index $t \in \{1,\ldots,k_i\}$ such that

x) $$y_o^i = y_t^i + y .$$

In the case of $i=i_1$ this equation is directly inconsistent with the fact that the set $S_{i_1} \cup I(B_o)$ is linearly independent. Therefore we have only to consider the case $i > i_1$ with $|B_i| \geq 2$ (note that $Z_i \neq \emptyset$) . Then there is a vector

$$\bar{c}^i \in C\left(\bigcup_{v=1}^{i} B_o^v \cup \bigcup_{v=1}^{i} B_v\right) \quad \text{with} \quad \bar{c}^i > 0$$

such that

$$y_o^i = \frac{1}{c_o^i} e_N - \frac{1}{c_o^i} \sum_{j=1}^{k_i} c_j^i y_j^i - \sum_{l=1}^{i-1} \frac{1}{c_o^i} \sum_{j=o}^{k_l} c_j^l y_j^l -$$

$$- \frac{1}{c_o^i} \sum_{y_j^o \in I\left(\bigcup_{v=1}^{i} B_o^v\right)} c_j^o y_j^o =$$

$$= - \frac{1}{c_o^i} \sum_{j=1}^{k_i} c_j^i y_j^i + \alpha$$

where $\alpha \in L\left(\bigcup_{v=1}^{i-1} S_v \cup I(B_o^i)\right) \subset L\left(\bigcup_{v=1}^{i} S_v\right)$. Note the fact that $e_N \in L\left(\bigcup_{v=1}^{i-1} S_v\right)$ and that the set $\bigcup_{v=1}^{i-1} S_v$ is a basis of $L\left(\bigcup_{v=1}^{i-1} B_o^v \cup \bigcup_{v=1}^{i-1} B_v\right)$.

Because of linear independence of the set $\bigcup_{v=1}^{i} S_v$ the last equation is unique. Now, if we compare the coefficient of the vector y_t^i with that in equation x) we get

$$1 = - \frac{1}{c_o^i} \cdot c_t^i < 0$$

and this gives the desired contradiction.

To prove the second assertion we assume $i < j$ and $Z_i, Z_j \neq \emptyset$. Suppose there are indices t and m with $1 \leq t \leq k_i$ and $1 \leq m \leq k_j$ such that

xx) $$y_o^j = y_o^i - y_t^i + y_m^j \ .$$

Again we have a unique representation of the vector y_o^j :

$$y_o^j = \frac{1}{c_o^j} e_N - \frac{1}{c_o^j} \sum_{s=1}^{k_j} c_s^j y_s^j - \sum_{l=1}^{j-1} \frac{1}{c_o^j} \sum_{s=1}^{k_l} c_s^l y_s^l -$$

$$- \frac{1}{c_o^j} \sum_{y_s^o \in I(\bigcup_{v=1}^{j} B_o^v)} c_s^o y_s^o =$$

$$= - \frac{1}{c_o^j} \sum_{s=1}^{k_j} c_s^j y_s^j + \beta$$

where $\beta \in L(\bigcup_{v=1}^{j-1} S_v \cup I(B_o^j)) \subset L(\bigcup_{v=1}^{j} S_v)$.

If we compare the coefficients of the vector y_m^j we get a contradiction and the proof is complete.

Theorem 4.8

If $B = [B_o, B_1, \ldots, B_q]$ is a maximal array then

i) $\bigcup_{i=1}^{q} Z_i \cup I(B_o)$ is linearly independent

ii) $|\bigcup_{i=1}^{q} Z_i \cup I(B_o)| = n-1$

Proof:

To prove the first assertion it is sufficient to show that the set $\bigcup_{i=1}^{k} Z_i \cup \bigcup_{i=1}^{k} I(B_o^i)$ is linearly independent for all indices $k=1,\ldots,q$.

The proof is by induction on k, $1 \leq k \leq q$.

Because of lemma 4.3.2 the assertion is easily verified in the case of $k = 1$. Suppose the assertion has been proven for all indices l with $1 \leq l \leq k-1 < q$. We now have to show:

$\bigcup_{i=1}^{k-1} Z_i \cup \bigcup_{i=1}^{k-1} I(B_o^i) \cup Z_k \cup I(B_o^k)$ is linearly independent.

Case 1) $Z_k = \emptyset$

First we have $|B_k| = 1$; without loss of generality suppose the set $I(B_o^k)$ is not empty. Now, if $k < i_1$ there is nothing to show because of the fact that $\bigcup_{i=1}^{k} Z_i \cup \bigcup_{i=1}^{k} I(B_o^i) = \bigcup_{i=1}^{k} I(B_o^i)$.

Therefore assume $k > i_1$. Then there is a highest index $i_l \in \{i_1,\dots,i_m\}$ such that $|B_{i_l}| \geq 2$ and $i_l < k$. In order to get a contradiction suppose that there are real-valued coefficients $\alpha_j, \beta_j, \gamma_j^{i_1}, \dots, \gamma_j^{i_l}$ which do not all vanish such that

$$x) \quad \sum_{y_j^o \in I(B_o^k)} \alpha_j y_j^o + \sum_{y_j^o \in I(\bigcup_{i=1}^{k-1} B_o^i)} \beta_j y_j^o + \sum_{j=1}^{k_{i_1}} \gamma_j^{i_1} (y_o^{i_1} - y_j^{i_j}) + \dots$$

$$\dots + \sum_{j=1}^{k_{i_l}} \gamma_j^{i_l} (y_o^{i_l} - y_j^{i_l}) = 0$$

In view of the facts that the set $\bigcup_{i=1}^{k-1} S_i$ is a basis of $L(\bigcup_{v=1}^{k-1} B_o^v \cup \bigcup_{v=1}^{k-1} B_v)$ and that $\bigcup_{i=1}^{k-1} S_i \cap I(B_o^k) = \emptyset$ there is a vector $\delta \in L(\bigcup_{i=1}^{k-1} S_i)$ such that we can write equation x) in the following form:

$$\sum_{y_j^o \in I(B_o^k)} \alpha_j y_j^o + \delta = 0$$

The linear indipendence of $I(B_o^k) \cup \bigcup_{i=1}^{k-1} S_i = \bigcup_{i=1}^{k} S_i$ implies that all coefficients α_j must vanish. By the induction assumption all other coefficients in equation x) must vanish, too. This contradicts our assumption.

Case 2) $Z_k \neq \emptyset$

First we have $k \geq i_1$, for the set B_k has more than one element. In the case of $k = i_1$ there is nothing to show because of linear independendence of the set $S_{i_1} \cup \bigcup_{v=1}^{i_1-1} I(B_o^v)$. Therefore we can assume $k > i_1$. Again there is a highest index $i_l \in \{i_1,\ldots,i_m\}$ such that $|B_{i_l}| \geq 2$ and $i_l < k$. In order to give an indirect proof assume the existence of real-valued coefficients $\alpha_j, \beta_j, \delta_j$ and $\gamma_j^{i_1},\ldots,\gamma_j^{i_l}$, which do not all vanish such that

$$\text{xx)} \quad \sum_{y_j^o \in I(B_o^k)} \alpha_j y_j^o + \sum_{j=1}^{k_k} \beta_j (y_o^k - y_j^k) + \sum_{y_j^o \in I(\bigcup_{i=1}^{k-1} B_o^i)} \delta_j y_j^o +$$

$$+ \sum_{j=1}^{k_{i_1}} \gamma_j^{i_1} (y_o^{i_1} - y_j^{i_1}) + \ldots + \sum_{j=1}^{k_{i_l}} \gamma_j^{i_l} (y_o^{i_l} - y_j^{i_l}) = 0 .$$

Now we define a to be the term $\sum_{j=1}^{k_k} \beta_j$. Let us begin with the case $a = 0$.

Then in the same way as in case 1) there is a vector $\lambda \in L(\bigcup_{i=1}^{k-1} S_i)$ such that we can rewrite equation xx) in the following form :

$$\sum_{y_j^o \in I(B_o^k)} \alpha_j y_j^o + \sum_{j=1}^{k_k} (-\beta_j) y_j^k + \lambda = 0$$

Because of linear independence of the set $\bigcup_{i=1}^{k} S_i$ and because of the fact that $(\bigcup_{i=1}^{k-1} S_i) \cap S_k = \emptyset$ all coefficients α_j and $(-\beta_j)$ vanish and so do all other coefficients in equation xx) (note the induction assumption) .

Now we consider the case $a := \sum_{j=1}^{k_k} \beta_j \neq 0$. In view of B_o-balancedness of the set $\bigcup_{v=1}^{k} B_v$ there is a vector $\bar{c}^k \in C(\bigcup_{v=1}^{k} B_v \cup \bigcup_{v=1}^{k} B_o^v)$ with $\bar{c}^k > 0$ such that

$$a y_o^k = \frac{a}{c_o^k} e_N - \sum_{j=1}^{k_k} \frac{a}{c_o^k} \cdot c_j^k y_j^k - \sum_{y_j^o \in I(B_o^k)} \frac{a}{c_o^k} \cdot c_j^o y_j^o + \mu$$

where $\mu \in L(\bigcup_{v=1}^{k-1} B_v \cup \bigcup_{v=1}^{k-1} B_o^v) = L(\bigcup_{i=1}^{k-1} S_i)$.

Instead of $a y_o^k$ we now insert the last term in equation xx) .

Noting the facts that $i_1 < k$ and thus $\frac{a}{c_o^k} e_N \in L(\bigcup_{i=1}^{k-1} S_i)$ we get the following expression:

$$\sum_{y_j^o \in I(B_o^k)} (\alpha_j - \frac{a}{c_o^k} \cdot c_j^o) y_j^o + \sum_{j=1}^{k_k} (-\beta_j - \frac{a}{c_o^k} \cdot c_j^k)\, y_j^k + \eta = 0$$

with the appropriate vector $\eta \in L(\bigcup_{i=1}^{k-1} S_i)$. The linear independence of the set $\bigcup_{i=1}^{k} S_i$ implies :

$$\frac{c_j^k}{c_o^k} \sum_{j=1}^{k_k} \beta_j = -\beta_j \quad \text{for all indices } j=1,\ldots,k_k .$$

Now, the case $\sum_{j=1}^{k_k} \beta_j < 0$ implies $\beta_j > 0$ for all indices $j=1,\ldots,k_k$; $\sum_{j=1}^{k_k} \beta_j > 0$ implies $\beta_j < 0$ for all indices $j=1,\ldots,k_k$. In both cases we get the desired contradiction and thus the first assertion has been proved.

To prove the second assertion suppose m is the number of array parts B_i with $|B_i| \geq 2$. Remembering lemma 4.6.1 and lemma 4.7 and considering the fact that $Z_i = \emptyset$ if and only if $|B_j| = 1$ we obtain the following equations:

$$|\bigcup_{i=1}^{q} Z_i \cup I(B_o)| = |\bigcup_{i=1}^{q} Z_i| + |B_o| = \sum_{i\,:\,|B_i| \geq 2} |Z_i| + |B_o| =$$

$$= \sum_{B_i \subset A_B} (|B_i| - 1) + |B_o| = \sum_{B_i \subset A_B} |B_i| - m + |B_o| =$$

$$= (n + (m-1) - |B_o|) - m + |B_o| = n-1 .$$

This proves the second assertion.

Lemma 4.9

If $B = [B_o, B_1, \ldots, B_q]$ is a maximal array, then the set $Z_B \cup \{e_N\}$ is linearly independent.

Proof:

In view of the fact that $|B_o| \leq n-1$, the assertion is immediately clear in the case of $\bigcup_{i=1}^{q} Z_i = \emptyset$. Therefore let us assume $\bigcup_{i=1}^{q} Z_i \neq \emptyset$.

It is sufficient to show that $\bigcup_{i=1}^{k} Z_i \cup I(\bigcup_{i=1}^{k} B_o^i) \cup \{e_N\}$ is linearly independent for all indices $k=1,\ldots,q$. This will be done by induction on k, $1 \leq k \leq q$.

Consider the case $k = 1$. Contrary to the assertion assume $e_N \in L(Z_1 \cup I(B_o^1))$. In the case of $Z_1 = \emptyset$ this yields an immediate contradiction owing to the fact that $|B_o| \leq n-1$. Thus we can assume $Z_1 \neq \emptyset$. Then there are coefficients $\alpha_j, \beta_j \in R$ such that

$$e_N = \sum_{j=1}^{k_1} \alpha_j (y_o^1 - y_j^1) + \sum_{y_j^o \in I(B_o^1)} \beta_j y_j^o =$$

$$= (\sum_{j=1}^{k_1} \alpha_j) y_o^1 + \sum_{j=1}^{k_1} (-\alpha_j) y_j^1 + \sum_{y_j^o \in I(B_o^1)} \beta_j y_j^o \, .$$

On the other hand there is a vector $\bar{c}^1 \in C(B_1 \cup B_o^1)$ with $\bar{c}^1 > 0$ and a unique representation of the vector e_N (note that $B_1 \cup B_o^1$ is $\emptyset$-minimal) such that

$$e_N = c_o^1 y_o^1 + \sum_{j=1}^{k_1} c_j^1 y_j^1 + \sum_{y_j^o \in I(B_o^1)} c_j^o y_j^o \, .$$

Comparing the coefficients of the vector y_o^1 we have $\sum_{j=1}^{k_1} \alpha_j = c_o^1 > 0$. On the other hand we have $\alpha_j = -c_j^1 < 0$ for all indices $j=1,\dots,k_1$. This yields a contradiction and therefore the assertion is true in the case $k = 1$.

Suppose that the assertion has been proven for all indices l with $1 \leq l \leq k-1 < q$. Suppose further that $e_N \in L(\bigcup_{i=1}^{k} Z_i \cup I(\bigcup_{i=1}^{k} B_o^i))$.

Case 1) $Z_k = \emptyset$

Then there are real-valued coefficients α_j, β_j and γ_j such that

$$e_N = \sum_{z_j \in \bigcup_{i=1}^{k-1} Z_i} \alpha_j z_j + \sum_{y_j^o \in I(\bigcup_{i=1}^{k-1} B_o^i)} \beta_j y_j^o + \sum_{y_j^o \in I(B_o^k)} \gamma_j y_j^o$$

In view of $|B_o| \leq n-1$ we immediately have a contradiction in the case $k < i_1$; furthermore, $Z_k = \emptyset$ implies $k \neq i_1$. Therefore we can assume $k > i_1$. Because of $e_N \in L(\bigcup_{i=1}^{k-1} S_i)$ there is a vector

$$\lambda \in L(\bigcup_{i=1}^{k-1} S_i) \quad \text{such that} \quad \sum_{y_j^o \in I(B_o^k)} \gamma_j y_j^o + \lambda = 0$$

The linear independence of the set $\bigcup_{i=1}^{k} S_i$ implies that all coefficients γ_j vanish and this implies $e_N \in L(\bigcup_{i=1}^{k-1} Z_i \cup I(\bigcup_{i=1}^{k-1} B_o^i))$, which contradicts our induction assumption.

Case 2) $Z_k \neq \emptyset$

Consider first the subcase $k = i_1$. In accordance with the above assumption there are real-valued coefficients α_j, β_j such that

$$e_N = (\sum_{j=1}^{k_k} \alpha_j) y_o^k + \sum_{j=1}^{k_k} (-\alpha_j) y_j^k + \sum_{y_j^o \in I(\bigcup_{i=1}^{k} B_o^i)} \beta_j y_j^o$$

(Note that the coefficients are uniquely determined because of linear independence of the incidence vectors). On the other hand we know from lemma 4.4 that $y_o^k \in L(B_1 \cup B_k \setminus \{b_o^k\} \cup \bigcup_{i=1}^{k} B_o^i)$ and that $e_N \in L(B_1 \cup B_o^1)$ because of B_o-balancedness of B_1. Therefore we can rewrite the above equation in the following form :

$$\sum_{j=1}^{k_k} \gamma_j y_j^k + \sum_{y_j^o \in I(B_o^k)} \delta_j y_j^o \in L(B_1 \cup \bigcup_{i=1}^{k-1} B_o^i)$$

with appropriate coefficients $\gamma_j, \delta_j \in R$.

This expression is similar to equation x) in the proof of theorem 3.5. Therefore we can assume without loss of generality that $\gamma_j, \delta_j \geq 0$ (otherwise we can apply procedure (P)). Similar to the proof of theorem 3.5 we now get a contradiction either to the fact that $(\bigcup_{i=1}^{k-1} B_i) \cup B_k$ is a B_o-extension of $\bigcup_{i=1}^{k-1} B_i$ or to the fact that the set B_o^k has been "minimally" chosen in accordance with remark 2.4 .

Next we consider the subcase $k > i_1$; again there is a highest index $i_1 \in \{i_1,\dots,i_m\}$ such that $|B_{i_1}| \geq 2$ and $i_1 < k$. In accordance with the above assumption there are real-valued coefficients $\alpha_j, \beta_j, \gamma_j^{i_1}, \dots, \gamma_j^{i_1}$ and δ_j such that

$$x) \qquad e_N = \sum_{j=1}^{k_k} \alpha_j (y_o^k - y_j^k) + \sum_{y_j^o \in I(B_o^k)} \beta_j y_j^o + \sum_{y_j^o \in I(\bigcup_{i=1}^{k-1} B_o^i)} \delta_j y_j^o +$$

$$+ \sum_{j=1}^{k_{i_1}} \gamma_j^{i_1} (y_o^{i_1} - y_j^{i_1}) + \dots + \sum_{j=1}^{k_{i_1}} \gamma_j^{i_1} (y_o^{i_1} - y_j^{i_1}) .$$

Suppose first that the term $a := \sum_{j=1}^{k_k} \alpha_j$ is zero. In view of the fact that $e_N \in L(\bigcup_{i=1}^{k-1} S_i)$ there is a vector $\mu \in L(\bigcup_{i=1}^{k-1} S_i)$ such that we can rewrite equation x) in the following form :

$$\sum_{j=1}^{k_k} (-\alpha_j) y_j^k + \sum_{y_j^o \in I(B_o^k)} \beta_j y_j^o + \mu = 0$$

Because of the linear independence of the set $\bigcup_{i=1}^{k} S_i$ all coefficients $(-\alpha_j), \beta_j$ must vanish and this implies

$e_N \in L(\bigcup_{i=1}^{k-1} Z_i \cup I(\bigcup_{i=1}^{k-1} B_o^i))$, which contradicts our induction assumption.

Now suppose $a := \sum_{j=1}^{k_k} \alpha_j \neq 0$.

Because of B_o-balancedness of the set $\bigcup_{v=1}^{k} B_v$ there is a vector

$$\bar{c}^k \in C(\bigcup_{v=1}^{k} B_v \cup \bigcup_{v=1}^{k} B_o^v) \quad \text{with} \quad \bar{c}^k > 0$$

such that

$$a y_o^k = \frac{a}{c_o^k} e_N - \sum_{j=1}^{k_k} \frac{a}{c_o^k} c_j^k y_j^k - \sum_{y_j^o \in I(B_o^k)} \frac{a}{c_o^k} c_j^o y_j^o + \eta$$

where η is the appropriate vector out of $L(\bigcup_{i=1}^{k-1} S_i)$. Inserting this term in equation x) we get the following expression (note that $e_N \in L(\bigcup_{i=1}^{k-1} S_i))$:

$$\sum_{j=1}^{k_k} (-\alpha_j - \frac{a}{c_o^k} c_j^k)\, y_j^k + \sum_{y_j^o \in I(B_o^k)} (\beta_j - \frac{a}{c_o^k} c_j^o) y_j^o + \rho = 0$$

where $\rho \in L(\bigcup_{i=1}^{k-1} S_i)$. The linear independence of the set $\bigcup_{i=1}^{k} S_i$ implies $-\alpha_j - \frac{a}{c_o^k} c_j^k = 0$ for all indices $j=1,\dots,k_k$. Both in the case $a < 0$ and in the case $a > 0$ we have now the desired contradiction similar to that at the end of the proof of the first assertion of theorem 4.8 . This completes the proof by induction.

Summarizing we get the following

Theorem 4.10

If $B = [B_o, B_1, \dots, B_q]$ is a maximal array, then the set $Z_B \cup \{e_N\}$ is linearly independent and the space $L(Z_B \cup \{e_N\})$ has dimension n .

This result enables us to compute the nucleolus and its linearity regions.

5. On the computation of the nucleolus and of its linearity regions

Let N be the set $\{1,\dots,n\}$ of n players. Then any O-normalized game such that $0 \leq v(S) \leq 1$ for all $S \subset N$ can be associated with a unique point $v \in R^{2^n-n-1}$. Without loss of generality we can assume $v(N) > 0$ as otherwise the nucleolus is always the zero-vector.

If $v \in V_N$ is a game and $x \in R^n$ an imputation, let $B_1(v,x)$ be the set of those coalitions in N for which $\max e(S,x)$ for $S \in P'(N)$ is attained; $B_2(v,x)$ the set of those $S \in P'(N)$ for which $\max e(S,x)$, $S \notin B_1(v,x)$ is attained, and so forth. Let $B_o(v,x)$ be the set of all $\{i\}$ $(i \in N)$ such that $x_i = 0$. The collection $B(v,x) := [B_o(v,x), B_1(v,x),\dots,B_q(v,x)]$ is called the <u>coalition array</u> or the array which belongs to the game v and the imputation $x \in R^n$.

Theorem 5.1 [E. Kohlberg 1971]
Suppose $v \in V_N$ and $x \in R^n$; then we have $x = N(v)$ if and only if the array which belongs to (v,x) is balanced.

Obviously we have $|B_o(v,x)| \leq n-1$ for any array $B(v,x)$ belonging to a game $v \in V_N$ and an imputation $x \in R^n$. This enables us to apply the results of the previous chapters.

For any given balanced array we now compare the excesses of those coalitions which are in the same array part. Together with the condition $\sum_{i=1}^{n} x_i = v(N)$ (for any given game $v \in V_N$) we get a system of n linear equations which are linearly independent and therefore possess a unique solution $x \in R^n$. Then we have to investigate whether this solution is the nucleolus of the given game $v \in V_N$. Thus, if T and S are coalitions of the same array part and if y_T and y_S are the corresponding incidence vectors, the equation $e(T,x) = e(S,x)$ $(x \in R^n)$ is equivalent to: $(y_S - y_T) \cdot x = v(S)-v(T)$.

The maximal arrays, introduced by J.H. Grotte [J.H. Grotte 1972], enables us to subdivide the set of all balanced arrays into subsets of arrays which can be derived from the same maximal array (note that this subdivision is not a partition). Clearly, all coalitions, which are in the same array part of a maximal array B, are also in the same array part of an array C which can be derived from B. In view of our aim this justifies restricting our attention to maximal arrays.

Now let $B = [B_o, B_1, \dots, B_q]$ be a maximal array and $v \in V_N$. In accordance with the notions of the previous sections we denote by $v(y_j^k)$ the value $v(b_j^k)$, where y_j^k is the incidence vector corresponding to the coalition $b_j^k \in B_k (\overline{I}(B_o) = \{y_1^o, \dots, y_{k_o}^o\})$. Consider the following system of equations (P_B) :

$$\begin{pmatrix} y_o^{i_1} - y_1^{i_1} \\ \vdots \\ y_o^{i_1} - y_{k_{i_1}}^{i_1} \\ \vdots \\ y_o^{i_m} - y_1^{i_m} \\ \vdots \\ y_o^{i_m} - y_{k_{i_m}}^{i_m} \\ y_1^o \\ \vdots \\ y_{k_o}^o \\ e_N \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} v(y_o^{i_1}) - v(y_1^{i_1}) \\ \vdots \\ v(y_o^{i_1}) - v(y_{k_{i_1}}^{i_1}) \\ \vdots \\ v(y_o^{i_m}) - v(y_1^{i_m}) \\ \vdots \\ v(y_o^{i_m}) - v(y_{k_{i_m}}^{i_m}) \\ 0 \\ \vdots \\ 0 \\ v(N) \end{pmatrix}$$

Obviously the rows of the matrix are just the elements of the set $Z_B \cup \{e_N\}$. See the definitions before lemma 4.7. Therefore theorem 4.10 implies that the matrix is quadratic and that (P_B) has an unique solution. This justifies the following definition.

Definition 5.2

If $B = [B_o, B_1, \ldots, B_q]$ is a maximal array and if $v \in V_N$, we call the solution $x =: N_B(v) \in R^n$ of the system (P_B) the <u>pseudo-nucleolus</u> of the game v relative to B.

Remarks 5.3

1. The pseudo-nucleolus of a game $v \in V_N$ relative to a maximal array B depends only on values of coalitions $S \in K_B$, i.e. on values of the "critical coalitions".

2. $N_B(v)$ may not be an imputation for there can be negative components.

Now we have to answer the question under what conditions the pseudo-nucleolus is equal to the nucleolus of a given game. For this purpose let F_N be the set of all possible maximal arrays B such that there exists a game $v \in V_N$ for which the array belonging to $(v, N(v))$ can be derived from B. Again, the set F_N gives rise to a classification of the set of all balanced arrays which have to be considered as belonging to any game $v \in V_N$ and its nucleolus $N(v)$.

Definition 5.4

Let $B = [B_o, B_1, \ldots, B_q]$ be a maximal array. Then we call the set

$$D_B := \{v : v \in V_N;\ N_B(v) \geq 0;\ e_1(N_B(v)) \geq \ldots \geq e_q(N_B(v))\}$$

the <u>game region</u> belonging to the array B.

Now we can answer our question.

Theorem 5.5

If $B = [B_o, B_1, \ldots, B_q]$ is a maximal array out of F_N and if $v \in V_N$, then

i) $D_B \neq \emptyset$

ii) $v \in D_B$ implies $N_B(v) = N(v)$

Proof:

To prove the first assertion we notice that $B \in F_N$ implies that there is a game $v_o \in V_N$ such that the array $C = [C_o, C_1, \ldots, C_t]$ belonging to $(v_o, N(v_o))$ can be derived from B. The excesses of coalitions (with respect to $N(v_o)$), which are in the same array part C_i, are equal. Thus, because of the definition of derivations of arrays and in view of the uniqueness of the solution of (P_B) we have $N_B(v_o) = N(v_o)$. Clearly all conditions of the definition of D_B are satisfied; this proves $v_o \in D_B$.

To prove the second assertion we notice that if $v \in D_B$, the vector $N_B(v)$ is an imputation. Obviously the array belonging to $(v, N_B(v))$ can be derived from B and therefore is balanced. Theorem 5.1 states $N_B(v) = N(v)$.

Remark 5.6

The converse of the second assertion of theorem 5.5 is not true. Consider the following example in the case $N = \{1,2,3\}$ (in the following we omit the brackets when describing the coalitions) :

Suppose $v(N) = 1$ and $v(i) = 0$ for all $i \in N$; suppose further $v(23) = 1/2$ and $v(12) = v(13) = 1/6$. Then we get $N(v) = (1/4;3/8;3/8)$ and the array $C = [C_o, C_1, C_2, C_3]$ belonging to $(v, N(v))$ is obviously balanced ($C_o = \emptyset$, $C_1 = \{1,23\}$, $C_2 = \{2,3\}$, $C_3 = \{12,13\}$) . Now it is easy to see that both of the following maximal arrays $A = [A_o, A_1, A_2, A_3, A_4]$ and $B = [B_o, B_1, B_2, B_3, B_4]$ satisfy $N(v) = N_A(v) = N_B(v)$:

$A_o = \emptyset$, $A_1 = \{1,23\}$, $A_2 = \{2,3\}$, $A_3 = \{12\}$, $A_4 = \{13\}$;

$B_o = \emptyset$, $B_1 = \{1,23\}$, $B_2 = \{12,13\}$, $B_3 = \{2\}$, $B_4 = \{3\}$.

But the array C can only be derived from A and we have $v \in D_A$, but $v \notin D_B$.

For any given maximal array $B \in F_N$ the game region D_B is easy to compute: after solving the system (P_B) we get a vector $N_B(v) \in R^n$ whose components are terms $v(T)$, $T \in P'(N)$. The conditions in the definition of D_B give rise to a system of inequalities in terms of $v(T)$, $T \in P'(N)$, which determine the game region D_B .

If a given game $v_o \in V_N$ satisfies all these inequalities, we have $N_B(v_o) = N(v_o)$ in accordance with theorem 5.5 . Thus we now can compute the nucleolus of all games $v \in D_B$.

Remarks 5.7

1. E. Kohlberg has defined the set $\varphi_B := \{v : v \in V_N$; the coalition array that belongs to $(v,N(v))$ is derived from $B\}$, where B is any maximal array of the set F_N [E. Kohlberg 1971] . Obviously, we have $\varphi_B = D_B$ which is now easily to compute. In particular, the linearity of the nucleolus on φ_B and the convexity of the region φ_B follow directly from the above definitions and from theorem 5.5 . Note that these results are also from E. Kohlberg.

2. For any game $v \in V_N$ let C be the array belonging to $(v,N(v))$. If C is not maximal and if $F' \subset F_N$ is the set of all maximal arrays from which C can be derived, then v is an element of the common border of all game regions D_B with $B \in F'$.

3. In the case of $n \geq 4$ the set of all maximal arrays $B = [B_o,B_1,\ldots,B_q]$ such that $B_o = \emptyset$ is a proper subset of F_N . Note that all corresponding game regions include J.H. Grotte's Central Game.

J.H. Grotte has mentioned that the set of all maximal arrays $B \in F_N$ give rise to a subdivision of the game space V_N into a finite number of regions D_B , on each of which the nucleolus is a linear function (note that $|F_N| < \infty$) . Therefore, we now can theoretically compute the nucleolus for any game $v \in V_N$: first we evaluate all game regions D_B belonging to maximal arrays $B \in F_N$. For a given game $v_o \in V_N$ we then have to look for a "fitting" game region $D_{B'}$. This region exists and $N_{B'}(v_o)$ will then be the nucleolus of the game v_o.

There is a practical difficulty in the fact that even for a small number of players the number $|F_N|$ is very large. Therefore, we now extend the game regions D_B $(B \in F_N)$ such that the nucleolus remains to be a linear function on these larger regions. In view of the nucleolus being a piecewise linear function we hope to achieve the largest connected regions of linearity. This will also reduce the number of game regions to be evaluated for computing the nucleolus.

First we give an example in the case $N = \{1,2,3\}$. Consider the following maximal arrays A and B :

$$A_o = \emptyset \qquad B_o = \emptyset$$
$$A_1 = \{1,23\} \qquad B_1 = \{1,23\}$$
$$A_2 = \{2,13\} \qquad B_2 = \{2,13\}$$
$$A_3 = \{12\} \qquad B_3 = \{3\}$$
$$A_4 = \{3\} \qquad B_4 = \{12\}$$

Obviously, for any game $v \in V_N$ we obtain $N_A(v) = N_B(v)$. Furthermore, both game regions possess a common border because they have common derivations. Therefore it is convenient to consider the union of both game regions. (Note that both arrays only differ in the arrangement of their one-coalition array parts.) In accordance with the notations of section 2 we now give the following definition.

Definition 5.8

Two maximal arrays B and C are called <u>strongly-similar</u> if and only if the following conditions are satisfied:

i) $B_o = C_o$

ii) $A_B = A_C$

iii) if $A_B = A_C \neq \emptyset$, then $B_{i_1^B} = C_{i_1^C}$,

$B_{i_2^B} = C_{i_2^C}, \dots, B_{i_m^B} = C_{i_m^C}$.

We write: $B \sim C$

By the above definition we obtain a collection of those maximal arrays which possess the same set of critical coalitions and for which the arrangement of their more-coalition array parts is the same. Obviously, relation " $\sim$ " is an equivalence relation on the set of all maximal arrays. For any maximal array B we denote the corresponding equivalence class by $[B]$.

Lemma 5.9

Let $B = [B_o, B_1, \dots, B_q]$ and $C = [C_o, C_1, \dots, C_r]$ be maximal arrays of the set F_N . If $B \sim C$ then

i) $q = r$

ii) $N_B(v) = N_C(v)$ for any game $v \in V_N$.

Proof:

The first assertion follows directly from lemma 4.6.2 and from the fact that $|B_o| = |C_o|$. The second assertion is an immediate consequence of remark 5.3.1 and of the fact that $Z_B = Z_C$ (see the notations before lemma 4.7).

The following theorem is important for the practical computation of the nucleolus. Its proof is immediately. For any array $B \in F_N$ let $D_{[B]}$ be the set $\bigcup_{C \in [B]} D_C$.

Theorem 5.10

If $B \in F_N$ and $v \in D_{[B]}$, then $N_B(v) = N(v)$.

For any array $B \in F_N$ E. Kohlberg's game region $\varphi_B = D_B$ (see remark 5.7.2) is obviously a subset of the now achieved larger game region $D_{[B]}$. The nucleolus is a linear function on $D_{[B]}$; furthermore, $D_{[B]}$ is a closed set. To compute the nucleolus we have mainly used these game regions.

In addition to definition 5.8 we now abstract from the arrangement of the more-coalition array parts (see the notations in section 2) .

Definition 5.11

Two maximal arrays B and C are called <u>weakly-similar</u> if and only if the following conditions are satisfied:

i) $B_o = C_o$

ii) $L_B = L_C$

i.e. the sets of the critical array parts are the same. We write: $B \overset{*}{\sim} C$.

Again, by relation " $\overset{*}{\sim}$ " we obtain an equivalence relation on the set of all maximal arrays. The corresponding classes are denoted by $[B]^*$. Obviously, if $B,C \in F_N$, then $B \sim C$ implies $B \overset{*}{\sim} C$. Moreover, $B \overset{*}{\sim} C$ implies $K_B = K_C$, i.e. the sets of critical coalitions are equal. Similar to lemma 5.9 we now obtain

Lemma 5.12

Let $B = [B_o,B_1,\dots,B_q]$ and $C = [C_o,C_1,\dots,C_r]$ be maximal arrays of the set F_N . If $B \overset{*}{\sim} C$ then

i) $q = r$

ii) $N_B(v) = N_C(v)$ for any game $v \in V_N$.

The proof is immediate.

For $B \in F_N$ let $D_{[B]^*}$ be the set $\bigcup_{C \in [B]^*} D_C$. Analogously to theorem 5.10 we get the following theorem:

Theorem 5.13

Let B be a maximal array of the set F_N . Then it follows:

i) $v \in D_{[B]^*}$ implies $N_B(v) = N(v)$

ii) $D_{[B]^*}$ is a closed set

iii) $N(v)$ is a linear function on $D_{[B]^*}$

iv) If for $C \in F_N$ and $C \overset{*}{\nsim} B$ a game $v \in V_N$ is an element of $D_{[B]^*} \cap D_{[C]^*}$, then v lies on the common border of some regions $D_{B'}$ and $D_{C'}$, where $B' \in [B]^*$ and $C' \in [C]^*$.

Proof:

The first assertions are immediately clear. To prove the fourth assertion we notice that there are maximal arrays B' and C' such that $v \in D_{B'} \cap D_{C'}$, where $B' \in [B]^*$ and $C' \in [C]^*$. Further we have $N_{B'}(v) = N_{C'}(v)$ because of the uniqueness of the nucleolus. Then in view of $B \overset{*}{\not\sim} C$ the array belonging to $(v,N(v))$ is not maximal. Therefore this array can be derived both from B' and C'. This proves the assertion.

We conjecture that these game regions $D_{[B]^*}$ $(B \in F_N)$ are the largest regions on which the nucleolus behaves like a linear function, but we do not have a proof.

The following notation of "normalized arrays" enables us to achieve a better description of the game regions $D_{[B]}$. But first we explain a procedure which allows us, for any given array $B \in F_N$, to construct a corresponding maximal array B' such that $B' \sim B$ and such that the array B' is normalized in view of the following definition 5.14 .

Let $B = [B_o, B_1, \dots, B_q]$ be an array of the set F_N such that $B_o = \emptyset$. If there are indices $t \in \{2,\dots,q\}$ such that $1 = i_1 < t < i_m$ and $|B_t| = 1$, then the following array B' which we obtain by shifting the array parts B_t with $|B_t| = 1$ on places with indices $l > i_m$, is balanced, too (see theorem 3.2). Because there is no change of the set of critical array parts and of their arrangement, the array B' is maximal and we have $B' \sim B$.

Now let $B = [B_o, B_1, \dots, B_q] \in F_N$ be an array such that $B_o \neq \emptyset$. In the case of $A_B = \emptyset$ we define the index i_m to be zero. In addition let $k_o^B \in \{1,\dots,q\}$ be the smallest index such that for all array parts B_i with $i > k_o^B$ the B_o-subset B_o^i, corresponding to the array part B_i, is empty. If there is no danger of confusion we omit the subindex B. Again, if there are indices $t \in \{2,\dots,q\}$ such that $1 < t < \max(k_o^B, i_m)$, $|B_t| = 1$ and the B_o-subset B_o^t is empty, then we obtain a balanced array B' by shifting those array parts B_t on places with indices $l > \max(k_o^B, i_m)$. Obviously, B' is maximal and we have $B' \sim B$.

Definition 5.14

1. A maximal array $B = [B_o, B_1, \ldots, B_q]$ with $B_o = \emptyset$ is called <u>normalized</u>, if there are no one-coalition array parts B_i such that $i_m > i$ $(i, i_m \in \{1, \ldots, q\})$.

2. A maximal array $B = [B_o, B_1, \ldots, B_q]$ with $B \neq \emptyset$ is called <u>normalized</u>, if there are no one-coalition array parts B_i with an empty B_o-subset B_o^i such that $\max(k_o^B, i_m) > i$ $(i, i_m, k_o^B \in \{1, \ldots, q\}$ with k_o^B as defined above).

Example:

The following maximal array B $(N = \{1,2,3,4\})$ is not normalized, because the B_o-subsets B_o^3 and B_o^4 are empty. We only write down the first array parts:

$B_o = \{1,2\}$, $B_1 = \{234\}$, $B_2 = \{134\}$, $B_3 = \{12\}$, $B_4 = \{34\}$, $B_5 = \{13,24\}$, ...

Obviously we have $k_o = 2$ and $i_1 = i_m = 5$. By the above described method we obtain the following corresponding normalized array $B' \sim B$:

$B'_o = \{1,2\}$, $B'_1 = \{234\}$, $B'_2 = \{134\}$, $B'_3 = \{13,24\}$, $B_4 = \{12\}$, $B'_5 = \{34\}$, ...

As remarked above the set F_N contains all maximal arrays B with $B_o = \emptyset$, and particularly all normalized arrays C with $C_o = \emptyset$. The following example shows that this is not true in the case of a normalized array C with $C_o \neq \emptyset$ $(N = \{1,2,3\})$:

$C_o = \{1\}$, $C_1 = \{2,3\}$, $C_2 = \{1\}$, $C_3 = \{12\}$, $C_4 = \{13\}$, $C_5 = \{23\}$.

Obviously C is a normalized array, but there is no game $v \in V_N$ such that the array belonging to $(v, N(v))$ can be derived from C.

(Note that $\sum_{i=1}^{3} x_i = v(N)$ must be greater than zero.)

Remarks 5.15

1. If B is a maximal array of F_N, then by the above method we can always find an array B' which is normalized and which can serve as a representative of the class $[B]$.

2. For normalized arrays $B = [B_o, B_1, \ldots, B_q]$ it follows directly from theorem 3.2 that by changing the arrangement of one-coalition array parts on places with indices out of the set $\{i_m+1, \ldots, q\}$ resp. out of $\{\max(k_o, i_m) + 1, \ldots, q\}$ we again obtain another normalized array $B' \sim B$. (Note that $L(B_o \cup \bigcup_{i=1}^{i_m} B_i)$ resp. $L(B_o \cup \bigcup_{i=1}^{\max(k_o,i_m)} B_i)$ has full dimension n.)

3. If $B = [B_o, B_1, \ldots, B_q]$ is a normalized array of F_N with $B_o = \emptyset$, then all normalized arrays $B' \in [B]$ have the same i_m-truncations (note that $i_m = m$). The analogous statement is not true for normalized representatives of an equivalence class $[C]$, where $C \in F_N$ and $C_o \neq \emptyset$. See the following example $(N = \{1,2,3,4\})$:

$C_o = \{1,2\}$, $C_1 = \{234\}$, $C_2 = \{13,24\}$, $C_3 = \{14\}$, ...

$C'_o = \{1,2\}$, $C'_1 = \{234\}$, $C'_2 = \{13,24\}$, $C'_3 = \{134\}$, ...

Both arrays are normalized and belong to the same equivalence class with respect to relation " $\sim$ ". Note that in both cases we have $\max(k_o, i_m) = 3$.

4. If two normalized arrays B and C with $B_o = C_o$ possess the same i_m-truncation resp. the same $(\max(k_o, i_m))$-truncation, then $B \sim C$.

For arrays $B \in F_N$ with $B_o = \emptyset$ we now are able to describe the game region $D_{[B]}$.

Lemma 5.16

If $B = [B_o, B_1, \ldots, B_q]$ is a normalized array of F_N such that $B_o = \emptyset$, then it follows:

i) $D_{[B]} = \{v : $ 1. $v \in V_N$ and $N_B(v) \geq 0$

2. $e_{i_1}(N_B(v)) \geq \ldots \geq e_{i_m}(N_B(v))$

3. $e_k(N_B(v)) \leq e_{i_{1_k}}(N_B(v))$ for all indices

$k \in \{1,\ldots,q\}$, $k > i_m$; $i_{1_k} \in \{i_1,\ldots,i_m\}$

is the smallest index such that

$(\bigcup_{v=1}^{i_{1_k}} B_v) \cup B_k$ is $\emptyset$-balanced$\}$.

ii) $D_{[B]}$ is convex .

Proof:

When the first assertion has been proven the second is immediately clear. To prove the first assertion we now assume $v \in D_{[B]}$. Then there is a maximal array $B' = [B'_o, B'_1, \ldots, B'_q] \in [B]$ such that the array B'' belonging to $(v, N_B(v))$ can be derived from B' and therefore is balanced. Further we have $N_B(v) \geq 0$ and $e_{i_1}(N_B(v)) \geq \ldots \geq e_{i_m}(N_B(v))$ because of $B' \sim B$. Now, if for all coalitions of the one-coalition array parts of B' the corresponding excesses are smaller or equal than $e_{i_m}(N_B(v))$, then the above third condition is satisfied. Suppose, on the other hand, that there is a coalition T out of the one-coalition array parts of B' such that without loss of generality its excess α satisfies the following inequalities:
$e_{i_1}(N_B(v)) \geq \alpha > e_{i_1+1}(N_B(v)) \geq e_{i_m}(N_B(v))$. Then, because of the balancedness of B'' and in view of the fact that $B' \sim B$, the set $(B_{i_1} \cup \ldots \cup B_{i_1}) \cup \{T\}$ is $\emptyset$-balanced. If $k \in \{i_m+1, \ldots, q\}$ is the index with $B_k = \{T\}$, the $\emptyset$-balancedness of $(B_{i_1} \cup \ldots \cup B_{i_1}) \cup T$ implies $i_{1_k} \leq i_1$ and we have
$\alpha = e_k(N_B(v) \leq e_{i_1}(N_B(v)) \leq e_{i_{1_k}}(N_B(v))$.

Conversely, let $v \in V_N$ be an element of the above described set.

We now have to specify an array $B' \in [B]$ such that $v \in D_{B'}$. First we notice that $N_B(v)$ is an imputation. If for all indices $k \in \{i_m+1,\dots,q\}$ the inequality $e_k(N_B(v)) \leq e_{i_m}(N_B(v))$ is satisfied, then obviously there is a maximal array B' with $B' \sim B$ such that the arrays B' and B only differ in the arrangement of their one-coalition array parts and such that the array C belonging to $(v,N_B(v))$ can be derived from B' (see remark 5.15.2) . Thus C is balaced and we have $N_B(v) = N(v)$. In view of remark 5.7.1 we now have $v \in D_{B'} \subset D_{[B]}$. If, on the other hand, there is an index $t_o \in \{1,\dots,q\}$ with $t_o > i_m$ such that the inequalities $e_{i_1}(N_B(v)) \geq e_{t_o}(N_B(v)) > e_{i_1+1}(N_B(v)) \geq e_{i_m}(N_B(v))$ are satisfied, then the third condition of the above definition states the existence of a maximal B'' with $B'' \sim B$ from which the array belonging to $(v,N_B(v))$ can be derived. This array B'' can be obtained from B by an appropriate change of the arrangement of the one-coaltion array parts and by inserting the array part B_{t_o} between the parts B_{i_1} and B_{i_1+1} . Again we have $N_B(v) = N(v)$ and this implies $v \in D_{B''} \subset D_{[B]}$. This completes the proof.

For small $n \in N$ (e.g. $n = 3$, $n = 4$) the above description of the game region $D_{[B]}$ is useful to compute the nucleolus for such games and to determine the regions $D_{[B]}$. Notice that, if B is a maximal array of the set F_N with $B_o = \emptyset$, we only have to know the i_m-truncation of any normalized representative of the class $[B]$. This is an essential reduction of the set of all maximal arrays to be considered: e.g. if we have a normalized array B with $B_o = \emptyset$ and $i_m = 1$, then $|B_1| = 4$ (lemma 4.6.1) and $D_{[B]}$ contains about 10 ! game regions $D_{B'}$ with $B' \in [B]$.

For any array $B \in F_N$ with $B_o = \emptyset$ there is in general no description of the region $D_{[B]}{}^*$ like that in lemma 5.16 . But we have the following

Lemma 5.18

For any array $B \in F_N$ with $B_o = \emptyset$ the game region $D_{[B]}{}^*$ is connected.

To prove this lemma it is sufficient to note that J.H. Grotte's Central Game is an element of all game regions $D_{B'}$ with $B' \in \lfloor B \rfloor^*$

To compute the game regions $D_{\lceil B \rfloor}$ $(B \in F_N$ and $B_o \neq \emptyset)$ we first have to evaluate partial regions as follows:
We devide the set of all normalized arrays of the class $[B]$ into disjoint sets of arrays with identical $(\max(k_o, i_m))$-truncations (see remark 5.15.3) . For each of these subsets we choose a representative $B' \in [B]$ and evaluate the region

$$d_{B'} := \{v : \ 1.\ v \in V_N \text{ and } N_{B'}(v) \geq 0$$

$$2.\ e_1(N_{B'}(v)) \geq \ldots \geq e_{\max(k_o, i_m)}(N_{B'}(v))$$

$$3.\ e_k(N_{B'}(v)) \leq e_{l_k}(N_{B'}(v)) \text{ for all indices}$$

$$k \in \{1, \ldots, q\} \text{ and } k > \max(k_o, i_m) \ ;$$

$$l_k \in \{1, \ldots, \max(k_o, i_m)\} \text{ is the smallest}$$

$$\text{index such that the set}$$

$$(\bigcup_{v=1}^{l_k} B'_v) \cup B'_k \text{ is } B_o\text{-balanced}\} .$$

Then $d_{B'}$ is convex and $D_{\lfloor B \rceil}$ is the union of all these regions $d_{B'}$, $B' \in [B]$. We conjecture that for $B \in F_N$ and $B_o \neq \emptyset$ the regions $D_{[B]}$ are also convex and that the regions $D_{[B]^*}$ are connected, too.

Again we see that for computing the regions $D_{[B]}$, $B \in F_N$ and $B_o \neq \emptyset$ we only have to consider the normalized arrays $B' \in [B]$ with different $(\max(k_o, i_m))$-truncations, i.e. we have to know all $(\max(k_o, i_m))$-truncations of arrays of the set $[B]$ (see remark 5.15.4) .

Therefore, if we want to compute the nucleolus or its regions of linearity, we can reduce our interest on normalized coalition arrays.

6. Construction of normalized arrays and computation of the nucleolus for 3-person-games

6.1 Construction of normalized arrays

In view of lemma 4.3 and lemma 3.7.2 we first have to evaluate the set $M_N^{\emptyset}$ of all $\emptyset$-minimal sets ($N = \{1,2,\dots,n\}$). In the case of $N = \{1,2,3,4\}$ these sets have been determined by L.S. Shapley [L.S. Shapley 1967]. Furthermore, B. Peleg has given a procedure to evaluate the set $M_{N+1}^{\emptyset}$, if the set $M_N^{\emptyset}$ is known [B. Peleg 1965].

To construct normalized arrays B with $B_o = \emptyset$ we proceed as follows:
First we choose a $\emptyset$-minimal set of $M_N^{\emptyset}$ and denote this set by B_1. If $|B_1| = n$ we have $\dim L(B_1) = n$ and all following array parts have to be one-coalition parts, arranged in any order, because we want the array to be maximal (see remark 3.3.2). Thus, according to lemma 4.3 the arrays are maximal, normalized and belong to the same equivalence class with respect to the relation " $\sim$ ".

Now assume $|B_1| < n$. Then the array to be constructed must possess a further more-coalition array part. Furthermore, the set $B_1 \cup B_2$ has to be a $\emptyset$-extension of B_1 (lemma 4.3). Therefore we have to form the union of the set B_1 with another set $B' \in M_N^{\emptyset}$ with $B' \not\subset B_1$ such that there is no set $B_2'' \in M_N^{\emptyset}$ with $B_2'' \not\subset B_1$ and $B_1 \cup B_2'' \underset{\neq}{\subset} B_1 \cup B_2'$ (see lemma 3.7.2). Then the set $B_1 \cup B_2'$ is $\emptyset$-balanced and we denote the set $B_2' \setminus B_1$ by B_2. In addition we have to choose the set B_2' in such a way that $|B_1 \cup B_2'| \geq |B_1| + 2$, because we want the array to be normalized. This implies $|B_2| \geq 2$. If we have $|S_1 \cup S_2| = n$ we are ready and all further array parts have to be one-coalition parts (see the notations in lemma 4.4 and remark 3.3.2). Otherwise we have to form further $\emptyset$-extensions in the above described manner, until we have $\left|\bigcup_{i=1}^{l} S_i\right| = n$ for an appropriate integer $l \leq 2^n - n$ (see lemma 4.6.2). Note that this procedure must end because of corollary 3.6.2.

The whole array we obtain by addition of one-coalition array parts in any ordering. Obviously the arrays achieved are normalized and belong to the same equivalence class with respect to the relation " $\sim$ " .

The construction of normalized arrays B' with $B'_o \neq \emptyset$ can be reduced to the above case: for each of these arrays B' with $B'_o \neq \emptyset$ we in the following construct a corresponding unique normalized array $C^{B'}$ with $C_o^{B'} = \emptyset$. By a converse procedure we then can get the original array B' .

Let $B = [B'_o, B'_1, \ldots, B'_q]$ be a normalized array with $B'_o \neq \emptyset$. Then we withdraw all array parts B'_i with $B'_i \cap B'_o \neq \emptyset$; note that for these array parts B'_i we have $i \neq 1$ and $|B'_i| = 1$ according to remark 4.2.4 . Now we define $C_o^{B'}$ to be the empty set, and the array parts $C_j^{B'}$ to be the union of $B'_{i_{(j)}}$ and $B_o'^{\,i_{(j)}}$ $(i_{(j)} \in \{1,\ldots,q\})$, this in accordance with the ordering of the remaining parts of the given array B' .

In view of remark 4.2.2 , lemma 3.7.1 and lemma 4.3 the array $C^{B'}$ is maximal. Furthermore, the array $C^{B'}$ is normalized because the $(\max(k_o,i_m))$-truncation of the normalized array B' does not possess one-coalition array parts B'_i with empty corresponding B'_o-subsets $B_o'^{\,i}$.

Example for the case $N = \{1,2,3,4\}$:
$B'_o = \{1,2\}$, $B'_1 = \{234\}$, $B'_2 = \{34\}$, $B'_3 = \{13,24\}$, ...
Obviously B' is normalized and we have $\max(k_o,i_m) = 3$. The following normalized array $C^{B'}$ we have achieved by the above described procedure:
$C_o^{B'} = \emptyset$, $C_1^{B'} = \{234,1\}$, $C_2^{B'} = \{34,2\}$, $C_3^{B'} = \{13,24\}$, ...

Thus, if we know all normalized arrays B with $B_o = \emptyset$, we now can obtain all normalized arrays B' with $B'_o \neq \emptyset$ by the reverse procedure described above. Note that for the practical computation of the game regions $D_{[B]}$ $(B \in F_N)$ we only have to know all possible different kinds of i_m-truncations resp. $(\max(k_o,i_m))$-truncations of normalized arrays.

6.2 Computation of the nucleolus for the general 3-person game

To give an example for the previous sections we now compute the nucleolus of the general 3-person-game and its regions of linearity. Note that J.H. Grotte has already evaluated the nucleolus for all superadditive 3-person-games [J.H. Grotte 1970].

By a suitable permutation of the three players we can assume $v(23) \geq v(13) \geq (12)$ for all games $v \in V_3$. Then for such a game the components of $N(v) := (x_1,x_2,x_3)$ satisfy $x_3 \geq x_2 \geq x_1$ and this implies $e(\{3\},N(v)) \leq e(\{2\},N(v)) \leq e(\{1\},N(v))$ (see [M. Maschler; B. Peleg 1966]). Note that this fact has to be considered when constructing the necessary normalized arrays. For the purpose of shortening we only write down the i_m-truncations resp. the $(\max(k_o,i_m))$-truncations of normalized arrays $B \in F_3$. The inequalities determining the linearity regions $D_{[B]}$ are already reduced as much as possible (see lemma 5.16).

6.2.1 $\emptyset$-minimal sets for $N = \{1,2,3\}$ up to permutations of the players

$\{1,2,3\}$; $\{1,23\}$; $\{12,13,23\}$; $\{123\}$

6.2.2 Normalized arrays

$A_o = \emptyset$ $B_o = \emptyset$
$A_1 = \{1,2,3\}$ $B_1 = \{12,13,23\}$

$C_o = \emptyset$	$D_o = \emptyset$	$E_o = \emptyset$	$F_o = \{1\}$
$C_1 = \{1,23\}$	$D_1 = \{1,23\}$	$E_1 = \{1,23\}$	$F_1 = \{23\}$
$C_2 = \{2,13\}$	$D_2 = \{12,13\}$	$E_2 = \{2,3\}$	$F_2 = \{2,13\}$
$G_o = \{1\}$	$I_o = \{1\}$	$J_o = \{1,2\}$	$K_o = \{1,2\}$
$G_1 = \{23\}$	$I_1 = \{23\}$	$J_1 = \{23\}$	$K_1 = \{13\}$
$G_2 = \{12,13\}$	$I_2 = \{2,3\}$	$J_2 = \{13\}$	$K_2 = \{23\}$

Note that for the corresponding arrays we have $J \sim K$; all other corresponding arrays belong to different equivalence classes with respect to relation " $\sim$ " . In view of the previous mentioned permutation of the players all possible normalized arrays possess one of the above truncations.

6.2.3 Computation of the game regions $D_{[B]}$ (see lemma 5.16)

For any 3-person-game $v \in V_3$ we define

$a := v(12)$

$b := v(13)$

$c := v(23)$

a) $N_A(v) = (\frac{v(N)}{3}, \frac{v(N)}{3}, \frac{v(n)}{3})$

$$D_{[A]} = \{v : 0 \leq c \leq \frac{v(N)}{3}\}$$

b) $N_B(v) = (\frac{v(N)+a+b-2c}{3}, \frac{v(N)+a+c-2b}{3}, \frac{v(N)+b+c-2a}{3})$

$$D_{[B]} = \{v : a+b \geq \frac{v(N)+c}{2} ;\ a+b \geq 2c - v(N)\}$$

In the case $c \geq v(N)$ only the last inequality is relevant, otherwise only the first one.

c) $N_C(v) = (\frac{v(N)-c}{2}, \frac{v(N)-b}{2}, \frac{b+c}{2})$

$$D_{[C]} = \{v : a \leq \frac{v(N)-c}{2} ;\ b \geq \frac{v(N)-c}{2} ;\ c \leq v(N)\}$$

d) $N_D(v) = (\frac{v(N)-c}{2}, \frac{v(N)+2a+c-2b}{4}, \frac{v(N)+2b+c-2a}{4})$

$$D_{[D]} = \{v : a \geq \frac{v(N)-c}{2} ;\ a+b \leq \frac{v(N)+c}{2} ;\ c \leq v(N)\}$$

e) $N_E(v) = (\frac{v(N)-c}{2}, \frac{v(N)+c}{4}, \frac{v(N)+c}{4})$

$$D_{[E]} = \{v : b \leq \frac{v(N)-c}{2} ;\ \frac{v(N)}{3} \leq c \leq v(N)\}$$

Up till now we have achieved the game regions necessary for super-additive 3-person-games.

f) $N_F(v) = (0, \frac{v(N)- b}{2} , \frac{v(N)+ b}{2})$

$D_{[F]} = \{v : a \leq 0 ; b \leq v(N) ; c \leq v(N)\}$

g) $N_G(v) = (0, \frac{v(N)+ a - b}{2} , \frac{v(N)+ b - a}{2})$

$D_{[G]} = \{v : a + b \leq 2c - v(N) ; b \leq v(N)+ a ; c \leq v(N)\}$

h) $N_I(v) = (0, \frac{v(N)}{2} , \frac{v(N)}{2})$

$D_{[I]} = \{v : a \leq b \leq 0 ; c \geq V(N)\}$

In view of the general assumption $v(S) \geq 0$ $(S \subset N)$ the game region $D_{[G]}$ contains the regions $D_{[F]}$ and $D_{[I]}$.

i) $N_J(v) = (0,0,v(N))$

$D_{[J]} = \{v : b \geq v(N)+ a ; c \geq v(N)\}$

Note that $D_{[J]}$ is the union of the partial regions d_J and d_K (see the end of the previous section).

In view of the above mentioned permutation of the players we think that these game regions are at the same time the largest regions on which the nucleolus is a linear function.

" Recipe "

For any given 3-person game we first have ensure by a suitable permutation that $a \leq b \leq c$.

If $c \leq \frac{v(N)}{3}$, then case a) and $N_A(v)$ will give the nucleolus ; if $\frac{v(N)}{3} \leq c \leq v(N)$, then we have to look at the cases b) to e) . Furthermore, if $c \geq v(N)$ then one of the regions of the cases b), g) or i) will be the "fitting" game region.

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