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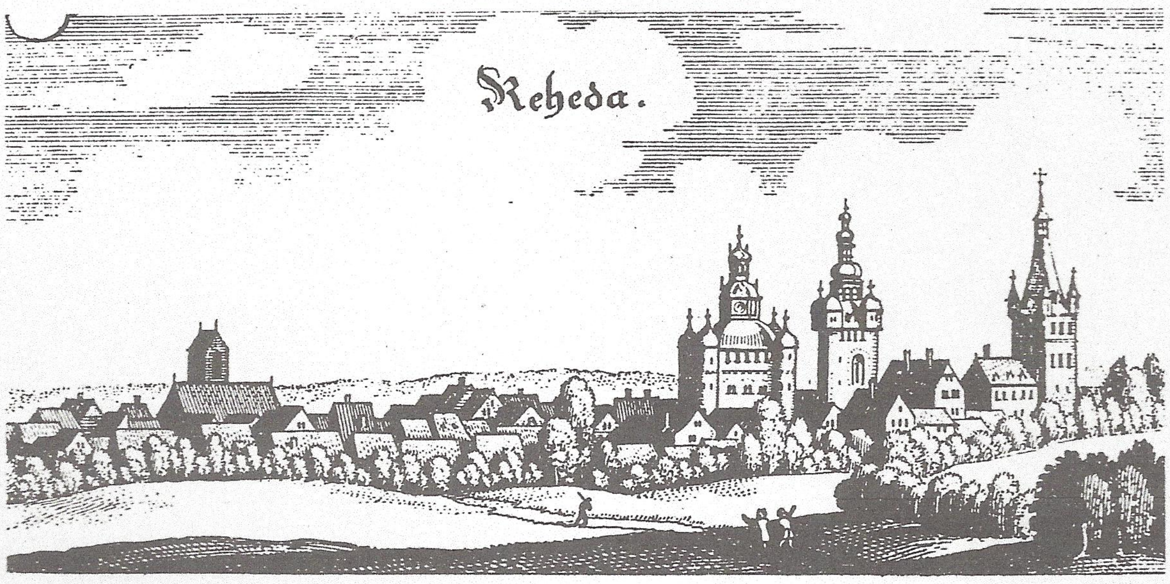
INTERNATIONAL WORKSHOP ON BASIC PROBLEMS OF GAME THEORY

Bad Salzuflen, September 2 to 17, 1974

Collection of Abstracts



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Preface

In the last years game theory has grown into several different directions but there are still many unsolved problems of non-cooperative and cooperative theory which already arise in the basic context of finite games with a small number of players. It was the aim of the International Workshop on Basic Problems of Game Theory to stimulate research in this area.

Most of the work summarized in this collection of abstracts was presented at formal meetings. For the sake of completeness we also include abstracts of research which was reported only in more informal discussion groups, even if not strongly related to the subject area of the workshop.

Towards the end of the workshop there was a discussion session about the prevailing trends in the field. The main result was that no agreement was reached on the most fruitful directions of future research. To my mind this shows that game theory is still a developing field with a wide open future.

This collection of abstracts gives a fair picture of present research interests in basic game theory. Refined concepts emerge in non-cooperative theory and deepen our understanding of equilibrium points. Robert Aumann's work on "partially correlated equilibria" points and John Harsanyi's papers on the "tracing precedure" are excellent examples of this kind of work. Some researchers hope to develop a promising new area which may be called "non-cooperative models of cooperative games". R.J. Weber's work belongs to this category.

The research interests in cooperative theory have narrowed down to a smaller number of solution concepts. The von-Neumann-Morgenstern solution, the bargaining set with its offshoots, the kernel and the nucleolus and the Shapley-value with its modifications get most of the attention.

An exciting area of research is the study of dynamic transfer processes approaching various cooperative solutions of characteristic function games. The abstracts of the talks by G. Kalai, G. Owen and M. Maschler and by M. Maschler and B. Peleg indicate the impressive amount of progress which has been achieved since the publication of the pioneering work of R. Stearns.

Ordinarily, game theory is considered to be a normative discipline. Nevertheless, there seems to be a rising interest in the results of laboratory experimentation.

Game theory has been successfully applied to various theoretical social sciences. Undoubtedly, there are still many open opportunities in this area. Up to now more concrete applications to practical problems have been relatively rare but new attempts are being made in this direction. Surprisingly, the Shapley-value and the nucleolus can be used in order to solve problems of cost accounting. The Shapley-value and the Banzhaf index have been applied to problems of constitutional law.

Obviously, these brief remarks cannot do justice to the great variety of stimulating new results reported at the workshop. Let the abstracts speak for themselves.

Reinhard Selten

List of Abstracts

A b s t r a c t

by

R.J. Aumann

Partially Correlated Equilibria

We report on and extend the theory presented in our paper "Subjectivity and Correlation in Randomized Strategies", Journal of Mathematical Economics 1, 1974. In particular, we present the following new theorem, a key element of whose proof was provided by L.S. Shapley: Let a 2-person game be defined by payoffs (h^2_{ij}, h^2_{ij}). Then a necessary and sufficient condition that (g^1, g^2) be a payoff to a correlated equilibrium point is that there exist probabilities $p_{ij} \geqq 0$, $\sum_i \sum_j p_{ij} = 1$,

such that

$$\sum_j p_{ij}(h^1_{ij} - h^1_{i'j}) \geqq 0 \quad \text{all } i,i'$$

j

and

$$\sum_i p_{ij}\ (h^2_{ij} - h^2_{ij'}) \geqq \quad \text{all } j,j'$$

A similar theorem can be proved for n-person games. This yields a linear characterization of correlated equilibrium payoffs, and leads one to conjecture that there might exist an existant proof for correlated equilibrium payoffs that is elementary in the sense that it requires no fixed point theorems.

Robert J. Aumann
The Hebrew University of Jerusalem

Abstract

by

R.J. Aumann, S. Hart and E. Kohlberg

Obstacle Tag

A pursuer who runs faster than an evader in the plane should pursue him along the straight line joining them, and the evader should run away along the same straight line (the payoff is assumed to be capture time). But what if there is a cicular hole between them? This problem was posed by Isaacs some 2 decades ago, and appears in his book "Differential Games" as a Research Problem. In this paper a complete solution for the problem is obtained.

Sergiu Hart, Tel-Aviv University

Elon Kohlberg, Harvard University,
Cambridge, Mass. 02138

Abstract

" Markov Games"

by

Truman Bewley and Elon Kohlberg

We consider multiple stage, two person, zero-sum games in which at each stage the payoff is determined by one of a finite set of matrices. This payoff consists of a current payment from one player to the other and a lottery on the set of payoff matrices. The outcome of the lottery is the payoff matrix at the next stage of play. If the game is played for finitely many stages, the value of the game is well defined as a function of the payoff matrix played at the first stage. This value divided by the number of stages is termed the average value function. We ask whether the average value converges as the number of stages played goes to infinity. The difficulties arise when some matrices or a subset of matrices may be absorbing in the sense that one of the players may hold subsequent play to that matrix or subset. We have succeeded in proving that the average value converges when there are no more than two non-absorbing matrices.

Truman Bewley & Elon Kohlberg
Harvard University
Cambridge, Ma.02138

Abstract

Structure of Equilibrium in N-Person Non-cooperative Games

by
H.H. Chin, T.Parthasarathy and T.E.S. Raghavan

Here we study the structure of Nash equilibrium points for N-person games. For two person games we observe that exchangeability and convexity of the set of equilibrium points are synonymous. This is shown to be false even for three person games. For completely mixed games we get the necessary inequality constraints on the number of pure strategies for the players. Whereas the equilibrium point is unique for completely mixed two person games, we show that it is not true for three person completely mixed game without some side conditions such as convexity on the equilibrium set. It is a curious fact that for the special three person completely mixed game with two pure strategies for each player, the equilibrium point is unique; the proof of this involves some combinatorial arguments.

T,E,S, Raghavan, University of Illinois

T. Parthasarathy, University of Illinois, Chicago, Ill. 60680

Title

Elementary Proof of the Existence of a Nash-Equilibrium Point for the 2x2 Bimatrix Games

by Hubert Chin, City University New York

Abstract

We give an elementary proof of the existence of a Nash-Equilibrium point for the special case of 2x2 bimatrix games that could be used for students who have no calculus background.

Abstract

General Bargaining Sets for Cooperative Games Without Side Payments

by

Claude d'Aspremont

CORE, Heverlee/Belgium

L.J. Billera has extended the existence property of the classical bargaining set, as defined for side-payment games and games of pairs, to a class of general bargaining sets for cooperative games without side payments. Several definitions of general bargaining sets belonging to this class shall be introduced and some of their properties analysed, mainly the property of reducing to the classical bargaining set for side-payment games or games of pairs. Also, their behavioral interpretation shall be discussed.

A b s t r a c t

by

Pradeep Dubey, Cornell University, Ithaca,New York

Values of Non-atomic Production Economies

A non-atomic model of an economy with transferable utilities is considered, similar to the one in Chapter 6 of "Values of Non-atomic Games" by R.J. Aumann and L.S. Shapley, but with production added to the model, and a result analogous to proposition 31.5 is proved, i.e. in a finite-type economy the core of the derived game coincides with its Shapley value.

SOME PROPERTIES OF THE BANZHAF POWER INDEX FOR SIMPLE GAMES

P. Dubey, Cornell University, Ithaca, New York

L. S. Shapley, Rand Corporation, Santa Monica, California

A power index for simple games may be defined by counting the number of "swings" for each player, that is, the number η_i of coalitions S such that S wins and S - $\{i\}$ loses. This index $\eta = (\eta_1, \ldots, \eta_n)$ and its normalized version $\eta / \Sigma \eta_i$ (the latter was proposed by J. F. Banzhaf III about 1965 and applied by him to a number of practical problems) can be based on a plausible probability model or on a set of axioms quite similar to those recently discovered by Dubey for the more familiar Shapley-Shubik power index (i.e. the "Shapley value" restricted to the domain of simple games). A theory of the asymptotic behavior of the Banzhaf index for large weighted-majority games has been developed, offering some rather unexpected contrasts to the corresponding theory for the Shapley-Shubik index.

ABSTRACT

The Tracing Procedure: A Bayesian Approach to Defining a Solution for n-Person Noncooperative Games.

By John C. Harsanyi,
University of California, Berkeley,
and
University of Bielefeld.

The paper proposes a Bayesian approach to selecting a particular equilibrium point s^* of any given finite n-person noncooperative game Γ as solution for Γ. It is assumed that each player i starts his analysis of the game situation by assigning a subjective prior probability distribution p_j to the set of all pure strategies available to each other player j. (The prior distributions p_j used by all other players i in assessing the likely strategy choice of any given player j will be identical, because all these players i will compute this prior distribution p_j from the basic parameters of game Γ in the same way.) Then, the players are assumed to modify their subjective probability distributions p_j over each other's pure strategies systematically in a continuous manner until all of these probability distributions will converge, in an appropriate sense, to a specific equilibrium point s^* of Γ, which, then, will be accepted as solution.

A mathematical procedure, to be called the <u>tracing procedure,</u> is proposed to provide a mathematical representation for this intellectual process of convergent expectations. Two variants of this procedure are described. One, to be called the <u>linear</u> tracing procedure, is shown to define a unique solution in "almost all" cases but not quite in all cases. The other variant, to be called the <u>logarithmic</u> tracing procedure, always defines a unique solution in all possible cases. Moreover, in all cases where the linear procedure yields a unique solution at all, both procedures always yield the same solution. For any given game Γ, the solution obtained in this way heavily depends on the prior probability distributions $p_1, \ldots, p_n$ used as a starting point for the tracing procedure. In the last section, the results of the tracing procedure are given for a simple class of two-person variable - sum games, in numerical detail.

Asymptotic Value of Non-Atomic Games

Abstract

by
Sergiu Hart, Tel-Aviv University, Tel-Aviv

We consider the class of all non-atomic games in pNA' (i.e., the closure of all polinomials in non-atomic measures with respect to the supremum norm - see Aumann & Shapley : "Values of Non-Atomic Games", 1974) which are superadditive and homogeneous of degree one. First result is that the asymptotic value of such a game, whenever it exists, belongs to the core. Second result is that the asymptotic value exists only if the core has a center of symmetry (which is the value). As a corollary we get that in a non-atomic exchange market, any (asymptotic) value allocation is competitive - and this is true in the side-payments case as well as in the non-side-payments one, with no differentiability assumptions made on the utilities.

ON A GAME THEORETIC NOTION OF COMPLEXITY

by

E. Kalai and M. Smorodinsky,
Statistics Department, Unversity of Tel-Aviv

ABSTRACT

The notion of complexity of a compact convex subset of R^n was introduced by Billera and Bixby in their paper "A Characterization of Pareto Surfaces" (Proc. Amer. Math. Soc. 41 (1973), 261-267). In their paper they delt with the question: what are the conditions that **a subset** C of R^n must satisfy so that each point of C would represent a combination of utilities of n people when these n people exchange among themselves a finite number of commodities and their utility functions defined on bundles of these commodities are assumed to be concave? They showed that a necessary and sufficient condition that C must satisfy is that C is convex and compact and proceeded to define the complexity of such a set C as the minimal number of commodities under which this representation is possible. They showed that the complexity of any such C is not greater than n(n-1) and demonstrated an example of such a set in R^3 of complexity 2. They naturally raised a second question which was: What is the maximal complexity attained by some compact convex subset of R^n.

In this paper we show an example of a subset of R^n $(n \geqslant 3)$ of complexity n. We also prove that any compact convex subset of R^n can be represented by $(n-1)^2-(n-2)$ commodities. Thus, we reprove Billera-Bixby's result of the representability and we improve the upper bound on the complexity to $(n-1)^2-(n-2)$ instead of n(n-1).

Thus, with regard to their second question the following partial answer is now known. The maximal complexity attained in R^n is o, 1, 3 for n=1, n=2 and n=3

respectively. For $n \geq 4$ the maximal complexity attained in R^n is not less than n and not greater than $(n-1)^2-(n-2)$. An unproven conjecture is that this maximal complexity is $n(n-1)/2$, the number of unordered pairs of players.

Linear-Transformation Invariance of Solutions to Cooperative Games

by Ehud Kalai

ABSTRACT

Part I : For the two-person bargaining problem it is shown that there is one and only one symetric, linear-transformation invarient solution which satisfies a certain axiom of monotonicity. This solution is shown to be necessarily the eventual outcome of a certain process of bargaining.

Part II : Linear-transformation invariant excess functions for non-sidepayment cooperative n-person games are demonstrated. The resulting nucleolus has the same linear-transformation invariance property. For two such excess functions the nucleolus is a generalization of Nash's solution and of the monotonic solution (of the two-person case) and in both cases it coincides with the nucleolus of the sidepayment games.

ABSTRACT

ASYMPTOTIC STABILITY AND OTHER PROPERTIES OF TRAJECTORIES AND TRANSFER SEQUENCES LEADING TO THE BARGAINING SETS

Gill Kalai
Michael Maschler
Guillermo Owen

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The foundation of a dynamic theory for the bargaining sets started with R. Stearns, when he constructed transfer sequences which always converge to appropriate bargaining sets. A continuous analogue was developed by L. Billera, where sequences were replaced by solutions of systems of differential equations.

In this paper we show that the nucleolus is locally asymptotically stable both with respect to Stearns' sequences and Billera's solutions if and only if it is an isolated point of the appropriate bargaining set. No other point of the bargaining set can be locally asymptotically stable.

As by-products of the study we derive the results of Billera and Stearns in a different fashion. We also show that along the non-trivial trajectories and sequences, the vector of the excesses of the payoffs, arranged in a non-increasing order, always decreases lexicographically, thus each bargaining set can be viewed as resulting from a certain monotone process operating on the payoff vectors.

Abstract

Partition Function Games without Side Payments

by

William L. Lucas, Ithaca
Cornell University

A model for n-person cooperative games without side payments is described by means of a discrete partition function which assigns an individual imputation to each partition of the set of n players. Some classical solution concepts in the finite set of imputations are investigated for n = 3 and for some special cases of arbitrary n. Some ideas for dynamical theories in this context are considered, including concepts on coalition formation.

Abstract

Measuring Power in Weighted Voting Systems

by

William F. Lucas, Dept. of OR, Cornell University,
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This paper gives a discussion of the Shapley-Shubik and Banzhaf power indices for measuring power in voting schemes in which some individuals or blocs cast more votes than others. A review of several applications of these indices to real-world voting situation is given, and several additional potential projects are suggested.

THE INERTIA SUPERGAME AND ITS CONNECTION TO THE CONVOLUTION CONCEPT

by Tom Marschak, University of California, Berkeley,
and Reinhard Selten, University of Bielefeld.

Like an ordinary supergame an inertia supergame results from the repetition of a game in normal form but with the additional feature that a change of strategy causes costs which are greater than any possible short run gain. This has the interpretation that the periods of the supergame are very short in comparison with the costs of change of strategy. It is shown that every infinite inertia supergame has at least one equilibrium point in pure strategies. The convolution concept which has been introduced by T. Marschak and R. Selten has a connection with the inertia supergame. Every convolution can be interpreted as a condensed description of an equilibrium point of an infinite inertia supergame.

Pareto Minimal Sets as Stable Sets of Bargaining Sets

by

Michael Maschler and Bezalel Peleg

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Jerusalem, Israel

ABSTRACT

Let X be a subset of E^n, the n-dimensional Euclidean space. An algorithm on X is a set-valued function φ from X to X (i.e., for each $x \in X$ $\varphi(x) \subset X$) which satisfies $\varphi(x) \neq \phi$ for $x \in X$. An algorithm φ on X generates the following φ-sequences

(1) $x^{(t+1)} \in \varphi\,(x^{(t)})$, $t = 0,1,2,\ldots$, $x^{(0)} \in X$.

We study the stability properties of the dynamic process (1). Necessary and sufficient criteria for stability of sets and points relative to (1) are given. Our main result is, essentially, that a subset of X is stable iff it is a Pareto minimal set of a vector-valued function which decreases along φ-sequences.

As a corollary we obtain a characterization of all stable sets and points of Stearns' transfer schemes as generalized nucleoli. In particular, the "lexicographic kernel", (a new solution concept due to Gill Kalai) is always a stable set of the kernel which may not include the nucleolus.

COMPOSITIONS OF COOPERATIVE GAMES

Nimrod Megiddo

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Given m games over disjoint sets of players, and another m-player game, one defines a compound game over the union of the m disjoint sets. The m-player game is called the quotient and the m other games are the components.

A decomposition theory for the above compounding is introduced. The composition structure of a game is shown to be advantageous for the calculation of the nucleolus and the kernel of the game. Specifically, projections of vectors in the kernel of a game are proportional to vectors in the kernels of its components. A similar result is shown with respect to the nucleolus.

ON A REPEATED GAME WITHOUT A RECURSIVE STRUCTURE

by

J.F. MERTENS and S. ZAMIR

ABSTRACT.

The existing theory of repeated 2-person 0-sum games of incomplete information is confined to games in which either the information functions are independent of the actual state of nature or they are the same for both players and they include the two players' moves as part of the information revealed by both players after each stage. A class of games that satisfy neither of the above requirements is analyzed and resolved by finding the lower and the upper value of the infinite game and the ε-optimal strategies that ε-guarantee them. As a byproduct we have a classification of these games into two subclasses according to whether the infinite game has or has not a value. When the value of the game exists it equals also the limit of the values of the truncated (finite) games. It seems that repeated games with information functions satisfying neither of the above stated conditions loose their recursive structure that enables to give a simple relation between an n-stage game and an n+1-stage game of the same type. The loss of this property which has been the main building stone in the theory, so far, seems to provide serious difficulties and complications in the study of these games.

J.P. Mertens, CORE, Heverlee

Shmuel Zamir, Hebrew University of Jerusalem

Extension of 2 person zero sum game value functions

by

H. Moulin, University Paris 9.
Mathématiques de la Decision

Let X and Y be the pure strategy sets of 2 pleayers Xavier and Yves. Then, every payoff function $g: X \times Y \rightarrow R$ defines a 2 person zero sum game where Xavier maximizes and Yves minimizes. An extension of the games with pure strategy sets X and Y is a "way of playing" the game which associates to every payoff function g a "value" in the duality interval:

$$\left[\sup_{x\in X} \inf_{y\in Y} g(2,y),\ \inf_{y\in Y} \sup_{x\in X} g(x,y)\right]$$

The classical example of extension is the mixed one. We describe a large class of extension - the iterated extensions - which contains the mixed one as a particular case (part 2).
Finally we characterize all the value functions by 4 "natural" properties. (part 3).

A note on the simple game with ordinal preferences

Kenjiro Nakamura

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Abstract

Michael Dummett and Robin Farquharson [Stability in Voting, Econometrica Vol.29.1 (1961)] provided a sufficient condition for an n-person simple majority game with ordinal preferences to have a nonempty core. In the present paper we generalize this result to an arbitrary proper simple game. It is proved that their condition is also sufficient for this game to have a nonempty core. Our proof of this theorem is much simpler than the proof given by Dummett and Farquharson. Finally some applications of the theorem are presented.

TWO-PERSON BARGAINING: AN EXPERIMENTAL TEST OF THE NASH AXIOMS

R. V. Nydegger
G. Owen
Rice University, Houston, Texas

ABSTRACT

Tests were carried out on thirty pairs of subjects, using three different sets of conditions, for the purpose of validating Nash's axioms (and also, incidentally, certain other hypotheses of two-person bargaining). Under validation, it was found that subjects' responses conformed to both the symmetry and independence of irrelevant alternatives axioms. On the other hand, the axiom of invariance under linear transformations of utility was constantly violated. This may be due to the fact that subjects, whenever possible, try to effect an interpersonal comparison of utility.

ABSTRACT

COMPUTATION OF THE NUCLEOLUS FOR A CLASS OF n-PERSON GAMES

Guillermo Owen
Rice University
Houston, Texas

An inductive procedure is given for the computation of the nucleolus in n-person games which have the property that all coalitions with less than $n-1$ players are totally defeated. An example is worked out in detail.

ABSTRACT

EVALUATION OF A PRESIDENTIAL ELECTION

Guillermo Owen
Rice University
Houston, Texas

The presidential election is analyzed as an n-person simple game. The Shapley value is approximated by the method of multilinear extensions, and compared with the Banzhof ratio. Some discussion of the implications is included.

EXISTENCE OF EQUILIBRIUM PAIRS IN CONTINUOUS GAMES

Guillermo Owen
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ABSTRACT

It is shown that all continuous two-person general-sum games over the square have equilibrium pairs of mixed strategy. A slight extension of Helly's Second Theorem, which may be of some slight interest in its own right, is used. The result is then generalized to all continuous n-person games on the cube.

Research supported by Army Research Office-Durham under grant DA-ARO-D-31-124-72-G30 to Rice University

The Extended Bargaining Set

by

Bezalel Peleg

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ABSTRACT

A modification of the Davis-Maschler bargaining set $M_1^{(i)}$, the extended bargaining set E, is presented in this paper. Two properties of E deserve emphasis:

(1) It is non-empty for every cooperative game without side payments.

(2) It seems always to yield "intuitively acceptable" results for games with a small number of players.

$M_1^{(i)}$ is always a subset of E; furthermore, $M_1^{(i)}$ may be a proper subset of E even for a game with side payments.

Solving Zero Sum Sequential Games with Incomplete Information

Abstract

Zero sum sequential games with incomplete information are formulated as linear programs in which the players' behavioral strategies appear as primal and dual variables. Using the theory of linear programming one derives the "concave hull" property for these games and the "Bayesian nature" of the behavioral strategies at the optimum (or more appropriately at the equilibrium of the associated Selten model of these"I-games").

Decomposition of the linear program makes it possible to study non optimal play, a legitimate concern for games in extensive form. It is shown that the equilibrium concept results in non-uniqueness "outside" the equilibrium path. This non-uniqueness if interpreted in a Bayesian context, may produce paradoxical behavior. An extension of the equilibrium concept outside the equilibrium path is proposed: each personal information set in the game tree should be reached with a small probability ε,letting ε go to zero one obtains a limit solution in which the non-uniqueness associated with non optimal play is ordinarily resolved. Properties of this extension are discussed in the general context of games in extensive form (including non zero sum).

Jean-Pierre Ponssard

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A b s t r a c t

by
T.E.S. Raghavan

On the Equilibria of continuous two-person games

Some sufficient conditions are given to show the existence of equilibrium points with finite spectrum for non-zero-sum two person continuous games on the unit square. We also examine the uniqueness of the equilibrium point for such games. In particular for continuous payoffs $k_1(x,y,)$, $k_2(x,y)$ on the unit square the following extensions of the theorems of Bohnenblust, Karlin and Shapley are true.

Theore : Let $k_1(x,y)$, $k_2(X,y)$ be continuous payoffs on the unit square for two-players in a non-zero-sum game. Let k_2 be concave in the variable y for player II when x is fixed. Then there is a Nash equilibrium point (F^o, G^o) in mixed strategies such that G^o is degenerate for II und F^o has a spectrum with at most two points.

Theorem: Let k_1, k_2 be continuous on the unit square and further let $\frac{\partial^n}{\partial y^n} k_2(x,y) \leq o$. Then there is a Nash equilibrium point in mixed strategies using at most finitely many pure strategies.

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A b s t r a c t

by

T.E.S. Raghavan

Existence of p-Equilibrium strategies in Stochastic Games

In the language of a two-person stochastic games we have five objects namely the state S, action spaces A,B for two players, q the law of motion, $r = (r_1, r_2)$ the immediate reward after the choice of actions by the two players. By introducing a discount factor $0 < \beta < 1$ one has a total expected discounted income. A stationary strategy π for player I is a map f : $S \to P_A$ where P_A is the space of probability measures on A. Stationary Γ for II is similarly defined. Let p be a fixed probability measure on S. We call (π^*, Γ^*) a p-equilibrium pair if for the expected discounted rewards I_1 and I_2 we have

$$p\{s: I_1(\pi^*, \Gamma^*)(s) \geq I_1(\pi, \Gamma^*)(s)) \text{ for all } \pi$$
$$I_2(\pi^*, \Gamma^*)(s) \geq I_2(\pi^* \Gamma)(s)) \text{ for all } \Gamma\} = 1$$

We establish the following

Theorem: Let S be the unit interval , A & B finite sets and q, r_1, r_2 satisfy necessary measurability conditions. Then there exists a p-equilibrium pair.

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Prof. Dr. J. Rosenmüller

Extreme Points and Cooperative Game Theory

- Abstract -

Methods of convex analyses play an important rôle in Operations Research and related topics, but apparently not so much in equilibrium theory: extreme economies, at least in a deterministic model, have not been defined and are not important as the set of equilibrium points is basically non-convex.
If one is willing to settle cooperative game theory between Operations Research and equilibrium theory, it might be interesting to consider the question as to whether the study of extreme games makes sense and can be used in order to establish the existence and structure of solution concepts.

To this end, I am going to use the results of "representation theory": every convex game is "max" taken over "affine games" (or "generalized production"); every superadditive game is "max" taken over "voting games" (or "generalized voting"). Using representation theory, it is possible to deal with extreme point theory of convex and superadditive games. I would like to interprete the results in the following sense: extremality can be interpreted in a social way such that extreme games admit of core payoffs suggesting a partition of the players in classes (convex games). On the other hand, extreme superadditive games admit of stable sets (von Neumann/Morgenstern-solutions) that are of similar shape. The characterizing property for an extreme game in both cases is a type of nondegeneration of the measures representing the game that are obtained by the representation theory. On the other hand, nondegeneration is related to problems of elementary number theory and integer programming, and a few solutions for these problems, as have been discussed in previous papers, will be mentioned.

Cores and Lindahl Equilibria in Economies with Public Goods

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In economies with public goods, the Lindahl equilibrium has many properties which are analogous to those of the competitive equilibrium in economies with only private goods. One might expect that some equivalence theorem could be obtained between the core and the set of Lindahl allocations in economies with public goods. For the core concept in which each blocking coalition must produce all of the public goods it desires with its own resources, this is known to be false. The question remains open, however, as to whether for some other natural core concept (perhaps representing aspects of political power) the core and the set (or some subset) of Lindahl allocations coincide when the measure space of agents is atomless. Some work on this problem will be described.

THE POSSIBILITY OF A CHEAT PROOF SOCIAL CHOICE FUNCTION:

A THEOREM OF A. GIBBARD AND M. SATTERTHWAITE

by

David Schmeidler, Tel Aviv University
and the Foerder Institute
Hugo Sonnenschein, Northwestern University

Let A and N be finite nonempty sets and let Σ denote the family of quaziorderings of A (i.e. every R in Σ is a transitive, reflexive and total binary relation on A). A social choice function, F, is a function for Σ^N to A. In the social choice theory context we interpret A as the set of alternatives the society face, N is the set of individuals who compose the society, R_i in Σ with i in N denotes a possible preference relation of individual i among the alternatives in A and F abstracts the decision making institutions of the society which picks the alternative $F(\{R_i\}_{i\in N})$ when the preferences of the individuals in the society are expressed by (the so-called profile) $\{R_i\}_{i\in N}$.

The social choice function F is said to be manipulable at $\{R_i\}_{i\in N}$ if there is an individual, say j, in N and a profile $\{R_i'\}_{i\in N}$ with $R_i' = R_i$ for $i \neq j$ so that, denoting $x = F(\{R_i\}_{i\in N})$ and $y = F(\{R_i'\}_{i\in N})$, we have $y R_j x$ and not $x R_j y$. F is called cheatproof if there is no profile at which it is manipulable.

From game theoretic point of view the function F describes a collection of N-person games in normal form. The set Σ denotes the (pure) strategies of every player in each of the games. The range of F is the set of outcomes in each game. A game from the collection is specified by a profile $\{R_i\}_{i\in N}$ in Σ^N (the quaziordering R_i of A induces a quaziordering on the range of F). The statement "F is cheatproof" means that in each game from the collection, if every player chooses as his strategy in the game his "true" preferences, a Nash equilibrium of this game obtains.

The social choice function F is called dictatorial if there is an individual, say j, so that for every profile $\{R_i\}_{i\in N}$ and every alternative x in the range of F, we have $F(\{R_i\}_{i\in N})R_j x$.

THEOREM *(Gibbard-Satterthwaite):*

If a social choice function F is cheatproof and the range of F contains at least three alternatives, then F is dictatorial.

The paper consists of a simple and short proof of this theorem.

Abstract

Ehud Kalai, Elisha A. Pazner and David Schmeidler,
Tel-Aviv University

Collective Choice Correspondences as Admissible Outcomes of Social Bargaining Processes.

This paper deals with the social choice problem in the context of Arrow's social welfare function model. Its purpose is to define a choice correspondence which is an extension of the Condorcet criterion.

Let A denote a finite set of alternatives and let N denote a finite set of individuals. The institution of decision making in the society is described by a list of ordered pairs of disjoint sets of individuals which is denoted by $\mathcal{RM}$ (relative majority). A pair (S,T) is in $\mathcal{RM}$ means that S has a relative majority over T in the sense that if the members of S unanimously wish to enforce a decision, the members of T are unanimously opposed to it, and the members of $N \setminus (S \cup T)$ are indifferent to it, the decision will be enforced. Given the preferences of each individual over the alternatives, we characterize a set of alternatives (the admissible set), one of which will be finally chosen by the society. For two distinct alternatives x and y we say that a bargaining process moves from x to y with a positive probability if and only if the inddividuals who prefer y to x have a relative majority over those who prefer x to y. The admissible set consists of the union of the minimally closed sets of the Markov chain thus defined.

The paper contains discussion of this solution, and of its properties including cheatproofness.

Collusive Oligopolies: An Economic Interpretation of Convex Games

by

Gerhard Schwödiauer
(Institute for Advanced Studies, Vienna)

Abstract

A convex game is a pair (I,v) where I is a finite set of players and v is a real-valued characteristic function fulfilling the condition that $v(K) + v(L) \leqq v(K \cup L) + v(K \cap L)$ for all $K, L \subset I$. It is shown that an oligopoly in which the profit-maximizing producers have continuous, increasing and convex cost functions and the other side of the market is represented by a demand schedule with the price being a continuous, decreasing and convex function of total output (the quantity produced being the only strategic variable of a firm), and where unrestricted cooperation among the oligopolists is allowed, can be modelled as a convex game. This result can be generalized to the case where prices are the relevant strategic variables of the oligopolists and the demand side is, because of product differentiation, described by a system of demand functions which display a necessary convexity property. As the solutions to convex games have rather unusual properties (the core of a convex game, for instance, is always non-empty and coincides with its unique von Neumann-Morgenstern solution) these results show the exceptional inherent stability and viability of collusive arrangements in such oligopolistic markets.

Solutions of Two-Sided Market Games

(work in progress)

by

Oskar Morgenstern
(New York University)

and

Gerhard Schwödiauer
(Institute for Advanced Studies, Vienna)

A b s t r a c t

The two-sided market games studied in the present paper (which is not yet completed) are generated by markets made up of two different groups of traders - sellers and buyers. Every seller is endowed with the same quantity of the only non-monetary commodity that is traded in the market (or, has the same production capacity) and possesses the same continuous, increasing and convex cost (disutility) function; every buyer owns the same amount of ideal "money" (which can be used for transferring utility) and is characterized by the same continuous, increasing and concave utility function. The purpose of the paper is to determine (stable set) solutions of such very elementary and highly symmetrical market games and to give those solutions an economic interpretation.

Unverschränkte Graphen und strategische Spiele in kombinatorischer Form

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ABSTRACT

Non-interlocking graphs and n-person games in combinatorial form.

Games are described by simple combinatorial structures. The reduced combinatorial form is the same for a game in extensive form and for its complete inflation. The condition of bilateral complete information in the complete inflation, essential for the existence of pure strategies in equilibrium points of extensive games, becomes a simple relationship between graphs.

In the first chapter of the paper a theory of decomposition of structures of graphs with the same point set is developed. The induced decomposition of the maximal independent sets of the graphs and of their randomizations is treated in some detail.

In the second chapter the concept of combinatorial games is introduced. A general theorem shows the existence of equilibrium points under assumptions which are weak enough to include the well-known theorems referring to extensive games. By a generalized equivalence for structures of extensive games it is made plausible that the reduced combinatorial form of an extensive game contains not more information about the extensive structure than what is necessary for our generalization. For games in extensive form we study still another characterization of the concept of bilateral complete information in the complete inflation.

THE CHAIN STORE PARADOX

by Reinhard Selten, University of Bielefeld.

The paper presents the example of a game where game theory prescribes a unique solution which is hardly acceptable as a practical recommendation for a player of the game, even if one assumes that all players are as rational as human beings can be supposed to be and that they have full knowledge of game theory. The problem is in some way similar to that of the finitely often repeated prisoner's dilemma. Reasoning by induction is important in both cases. Otherwise the situation is quite different in the chain store paradox. A three level theory of human decision making is proposed as a possible explanation of the phenomenon. Other psychological explanations which might apply are discussed, too.

THE INCOMPATIBILITY OF A SET OF AXIOMS FOR A NON-COOPERATIVE SOLUTION FUNCTION

by Reinhard Selten, University of Bielefeld.

The paper presents some plausible axioms for a noncooperative solution function which selects one equilibrium point for every game. An axiom system is introduced which determines a solution function for the class of all 2x2-games. This solution function has a relationship with Nash's solution of the bargaining problem with fixed threats. If this axiom system is complemented by some further plausible axioms on games in extensive form, it can be shown that for a class of extensive 2-person games no solution function can be found which satisfies the axioms.

REEXAMINATION OF THE PERFECTNESS CONCEPT FOR EQUILIBRIUM POINTS IN EXTENSIVE GAMES

by Reinhard Selten, University of Bielefeld

In earlier pbulications the author introduced the following concept of a perfect equilibrium point: An equilibrium point in behavior strategies is perfect if it induces an equilibrium point on every subgame. It is proposed to refer to this kind of perfectness as "subgame perfectness". Subgame perfectness does not exclude all cases of intuitively unreasonable equilibrium points in extensive games. Therefore a new concept of a perfect equilibrium point is introduced. The definition is based on a "model of slight mistakes". The idea to use a model of slight mistakes is due to John C. Harsanyi. Existence is proved for every finite extensive game with perfect recall. It is shown that perfect equilibrium points can be characterized by local optimality conditions at the information sets. An agent normal form is introduced where the players are the agents of the original extensive game with perfect recall. It is shown that the agent normal form contains all the information needed for the computation of the perfect equilibrium points of the original game. Counterexamples show that this is not true for the ordinary normal form.

Abstract

by

Reinhard Selten and John C. Harsanyi

A Solution for n-Person Noncooperative Games: Inductive Definition of the Prior Distributions Used in the Tracing Procedure

The tracing procedure always selects a specific equilibrium point as solution for my given noncooperative game if the prior probability distributions over the various players' pure strategies are given. This paper will define these prior distributions by means of an inductive procedure.

Any given game will be considered in *agent normal form* which is a normal form in which the agents (one agent for each information set) are considered to be the players. Our inductive procedure is based on the cell structure of the game. A *cell* is obtained by restricting the game to a subset of all players (agents), called the *cell players* for the cell, and by restricting each cell player to a subset of his original pure strategies, called his *cell strategies*. It is required that each cell player should always have a best reply to the other players' strategies among his cell strategies, and that it should depend only on the other cell players' strategies whether a given strategy of his is a best reply or not. We define a solution (subsolution) for each cell by means of the tracing procedure. In each cell, including the whole game, the prior distributions over the cell players' pure strategies are based on the subsolutions of the maximal subcells of the cell in question. Thus, the solution of the game is defined inductively, proceeding from the *primitive* cells (i.e.. cells containing no subcells) to larger and larger cells and eventually to the game as a whole.

Abstrct

by

L.S. Shapley, Rand Corporation, Santa Monica, Cal.

Disconneted Bargaining Sets

A class of games based on replicated two-sided exchange economies with transferable utility is considered, from the viewpoint of the bargaining set $\mathcal{B}$ or $\mathcal{M}_1^{(i)}$ (see below). A specific 10-person example is used to illustrate the techniques of analysis and to provide an illustration of a bargaining set whose symmetric part is not connected. It is conjected that if the nonsymmetric part is included, the set remains disconnected. Generalizing the method used in constructing this example yields, for each n>1, a 2n-person game of which the symmetric part of the bargaining set $\mathcal{B}$ or $\mathcal{M}_1^{(i)}$ consists of

$$2n - 2d(n) - 2d(n-1) + 3$$

disjoint components, where d(x) denotes the total number of divisors of x (i.e., including x and 1).

The bargaining set $\mathcal{B}$, which is conceptually more closely related to the core, differs from $\mathcal{M}_1^{(i)}$ in two respects: (1) no condition of individual rationality is imposed; and (2) no individual player is named in the objection to be the leader of the counterobjection - instead, the counterobjection may be made by any (nonempty) coalition not including the leader of the objection. In general, $\mathcal{B} \supset \mathcal{M}_1^{(i)}$. For replicated-market games it can be shown that the symmetric parts of $\mathcal{B}$ and $\mathcal{M}_1^{(i)}$ coincide, and that under suitable conditions of differentiability $\mathcal{B}$ (and hence $\mathcal{M}_1^{(i)}$) converges to the core as the number of replications increases without limit.

NONCOOPERATIVE GENERAL EXCHANGE

by
L. S. Shapley, Rand Corporation, Santa Monica, Cal.
M. Shubik, Yale University, New Haven, Conn.

Given the familiar data of a simple exchange economy (finite numbers of goods and traders, nonnegative initial endowments, continuous concave nondecreasing utility functions), one can define a noncooperative game in strategic form by arbitrarily designating one good as "trading money" and letting a (pure) strategy for each player be an allocation of his initial supply of trading money, less whatever he wants to hold back, among bids for the other goods. Transactions are then carried out according to prices determined by dividing the total amount of trading money bid for each good by the total initial supply of that good. The payoffs of the game are the utilities of the final bundles so obtained. (Many structural variations of this model are possible, including one suggested by Scarf that is "money free".)

Theorem. For each good other than the trading money, assume that there are at least two players who (1) have strictly increasing utility for at least a small amount of that good, and (2) have a positive initial supply of trading money. Then a noncooperative equilibium point in pure strategies exists.

CORE STABILITY IN SYMMETRIC GAMES

L. S. Shapley, Rand Corporation, Santa Monica, California

Definitions. An n-person game v is a function from the subsets of $I_n = \{1, 2, \ldots, n\}$ to the reals with $v(\emptyset) = 0$. The dual v^* of v is given by

$$v^*(S) = v(I_n) - v(I_n - S).$$

The cover $\bar{v}$ of v is given by $\bar{v}(\emptyset) = 0$ and, for $S \neq \emptyset$,

$$\bar{v}(S) = \max_{\gamma} \left\{ \sum_{T \subseteq S} \gamma(T) v(T) \ : \ \gamma \geqq 0 \text{ and for all } i \in S, \sum_{T : i \in T \subseteq S} \gamma(T) = 1 \right\}.$$

We say v is balanced if $v(I_n) = \bar{v}(I_n)$ and totally balanced if $v = \bar{v}$. The core of v is the set of real vectors $(x_1, \ldots, x_n)$ such that $x(I_n) = v(I_n)$ and for all $S \subseteq I_n$, $x(S) \geqq v(S)$, where $x(S)$ denotes the sum of x_i for i in S; the core is nonempty if and only if the game is balanced. We say the core is stable if for every $(y_1, \ldots, y_n)$ not in the core but satisfying $y(I_n) = v(I_n)$ there is an $(x_1, \ldots, x_n)$ in the core and a set $S \neq \emptyset$ such that $x(S) = v(S)$ and for all $i \in S$, $x_i > y_i$; the core is stable if and only if it is a solution in the sense of von Neumann and Morgenstern. Finally, a game is symmetric if $|S| = |T|$ implies $v(S) = v(T)$ for all S and T in I_n.

Theorem. The core of a balanced, symmetric game v is stable if and only if $-\bar{v}^*$ is totally balanced.

Remarks. 1. There is a fairly simple pictorial representation of the condition on $-\bar{v}^*$, in terms of the "graph" of the symmetric game v.

2. The theorem holds true if "symmetric" is replaced by "three-person", but for $n > 3$ we do not know if our condition is either necessary or sufficient. Kulakovskaya (Sov. Math. Doklady 12 (1971), 1231-1234) has given another, more complex necessary and sufficient condition for the core to be stable in the general case.

ORIENTATION INDICES FOR BIMATRIX EQUILIBRIUM POINTS

L. S. Shapley, Rand Corporation, Santa Monica, California

An index, +1, -1, or 0, can be defined intrinsically for each pair of mixed strategies in a bimatrix game, in such a way that only equilibrium points (EP) can have nonzero index; if the game is "nondegenerate" then all EP in fact have nonzero index. The pair O = ((0, ..., 0), (0, ..., 0)) of "ideal" strategies (which is of course not an EP) has the pro forma index +1. Every "primary" Lemke-Howson path (i.e., starting from O) terminates in an EP with index -1, while the remaining, "secondary" L-H paths each join two EP having opposite index. Moreover, we can calculate directly (i.e. without having to trace the path to its end) the orientation of the individual links of a L-H path (or loop), and these orientations are consistent when projected on the separate mixed-strategy spaces.

There is a considerable analogy between these results and the "strong" form of Sperner's Lemma and its extension (Ky Fan, J. Comb. Theory 2 (1967), 588-602) to pseudomanifolds.

ON THE POSSIBILITY OF AN ORDINAL VALUE FOR n-PERSON GAMES

L. S. Shapley, Rand Corporation, Santa Monica, Cal.

A construction is described that determines a unique, centrally-located point on any strictly monotonic three-person bargaining surface, using operations that are invariant under all order-preserving transformations of the individual utility scales. (This is impossible for strictly monotonic two-person bargaining curves.) Possibilities for generalization will be discussed.

A CONJECTURE ON THE CORE AND COMPETITIVE EQUILIBRIUM

by Martin Shubik, Yale University

Although we know a considerable amount about the relationship between the core and competitive equilibrium for replicated market games, or for games with measure spaces of players, there is an open question concerning the relationship between the core and the competitive equilibrium for market games with large cores.

It is conjectured that the set of competitive equilibria associated with the set of all market games which have the same core will completely cover the core.

GAMES, MARKET GAMES, THE CORE AND THE VALUE

by Martin Shubik, Yale University

What fraction of all games of a given size possess a core? What fraction of games with a core are market games? What fraction of market games have the value of the game lying within the core?

In general, it appears that for large games the chances for a core are small and the chances for a market game are considerably smaller.

A way of characterizing these chances is by considering characteristic function in 0-1 normalization and then considering the other values of the characteristic function to be picked from rectangular distributions whose range is bounded by the requirements of super-additivity. In particular, in the simplest non-trivial case, the 3-person game, 5/6 of all of the games have a core. For games of size 5 or more, brute-force computation does not appear to be particularly easy or interesting. However, for n equals 3 or 4, it appears possible to ask not only what are the percentages of games with cores and market games but also what percentages of these games contain the value within the core.

EXPERIMENTAL GAMES IN CHARACTERISTIC FUNCTION FORM

by Martin Shubik, Yale University

There has been a certain amount of interest by both social psychologist and game theorist on the playing of games in characteristic function form. The literature is somewhat confusing and there is a considerable gap between the mathematical approach and in game theory approaches and the social psychological experimentation. In 1973, I ran a series of experiments on 3-person games. There seems to be some indication that the normalization of the game makes a difference. The core value nucleous and competitive equilibrium were considered as predicters.

EXPERIMENTAL GAMES, BIDDING, LOW COMMUNICATION AND NON-COOPERATIVE BEHAVIOR

by Martin Shubik, Yale University

A two-sided market model was considered and examined as a non-cooperative game (Cowles Foundation Discussion Paper 368). It is related to the work of both E. H. Chamberlin (1948) and the recent work on experimental games of V. Smith.

The key feature of this model is that it is not a tatonnement process. Once a bid is made, it is either filled or not filled. The market mechanism stresses low levels of communications. Some pilot experiments indicate it generates prices close to the competitive prices but the spread in the quantitiy sold is large and tends to be lower than the competitive quantity.

THE THEORY OF MONEY AND FINANCIAL INSTITUTIONS

by Martin Shubik, Yale University

It is suggested that a natural way to investigate the role of money and financial institutions is as a game in extensive form using, in some instances as a solution concept, the perfect equilibrium points.

It is my belief that attempts to incorporate money and financial institutions into the competitive general equilibrium model can not be satisfactory. The very nature of monetary affairs rests on disequilibrium states and period-by-period feedback. Furthermore, the monetary and credit mechanisms depend delicately upon bankruptcy conditions, information conditions and other specific rules of the game such as the precise conditions for the issue of new money.

Joint work has been done with L. Shapley on the basic model which can be considered as a one-period game in strategic form.

Various extensions of this work are available in a series of Cowles Foundation Discussion Papers entitled "A Theory of Money and Financial Institution" Parts 1 thru 12, 15, 17, 18 and several other papers.

The Cost Assignment of the Cooperative Water Resource Development

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We formulated a water resource development performed by both the construction of dams and the diversion of the water use from agriculture to city service as a five-person game in the characteristic function form played by three city water service authorities and two agricultural associations.

We suggested the nucleolus which is an acceptable and justifiable method of the assignment of the cost and benefit among the players, not only from the view-point of game theory, but also of the principle of justice set by John Rawls in his Theory of Justice.

The characteristic function was defined as the decrease of the cost of development by the cooperation. In our game, the nucleolus was given by the following equation.

$$p_i^* = V(N) - V(N - \{i\}) + \frac{1}{n}\{\Sigma_{i\epsilon N} V(N - \{i\}) - (n - 1)V(N)\}$$

$$(i = 1, \ldots, 5)$$

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EQUILIBRIUM SYSTEMS

We shall use the word "system" to describe an organic system whose main features are the following ones:

Static features:

1./ The system consists of well distinguishable parts /organs/.
2./ The state of each organ depends on the state of the other ones.
3./ The organs can exist only in well-defined states.
4./ There are criteria for the organs and therefore for the system, with the help of which they can make their choice as to changing their state.

Dynamic features:

1./ The set of possible states of the organs is changing with time.
2./ The change of the state of the organs - and therefore that of the system - is taking place in time.
3./ The number of the organs can change with time.

It is useful to distinguish two kinds of systems:

a/ Autonomic system, where the states of the organs depend only on those of the others.
b/ Non-autonomic system, where the state of the organs depends on factors being outside the system, too.

For us an organic system will be an economic system, although its features - at least in the beginning - will be simplified to the main features described proviously.

We prove first of all existence theorems for state-sets of equilibrium in the most simple case when only one organ changes its state at a time and in the second part when more organs do it.

BARGAINING SOLUTIONS AND STATIONARY SETS FOR n-PERSON GAMES

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The cooperative negotiation process in a side-payment game can be modelled as a multi-stage game, in which each stage consists of an objection raised by some coalition to a proposal under consideration, and a response made to that objection by the remaining players. A bargaining solution to this game is an equilibrium collection of objection and response strategies for the coalitions, from which no coalition is motivated to deviate. Associated with each bargaining solution is a set of stationary proposals, to which no objections will be raised. Each stationary proposal corresponds to a stable agreement between the players, for which every threat to the agreement is balanced by a response which dissuades the threatening coalition from action.

After presentation of the formal bargaining model in extensive form, application is made to several well-known classes of games. All stationary sets are determined for three-person games. For several types of games with stable cores, it is shown that their cores are also stationary sets. Two games, pathological in their behavior with respect to the classical von Neumann-Morgenstern theory, are shown to be amenable to this approach. Finally, it is noted that the bargaining theory is only partially successful in treating voting games, and suggestions are made concerning possible extensions of the theory to cover such games.