Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 14

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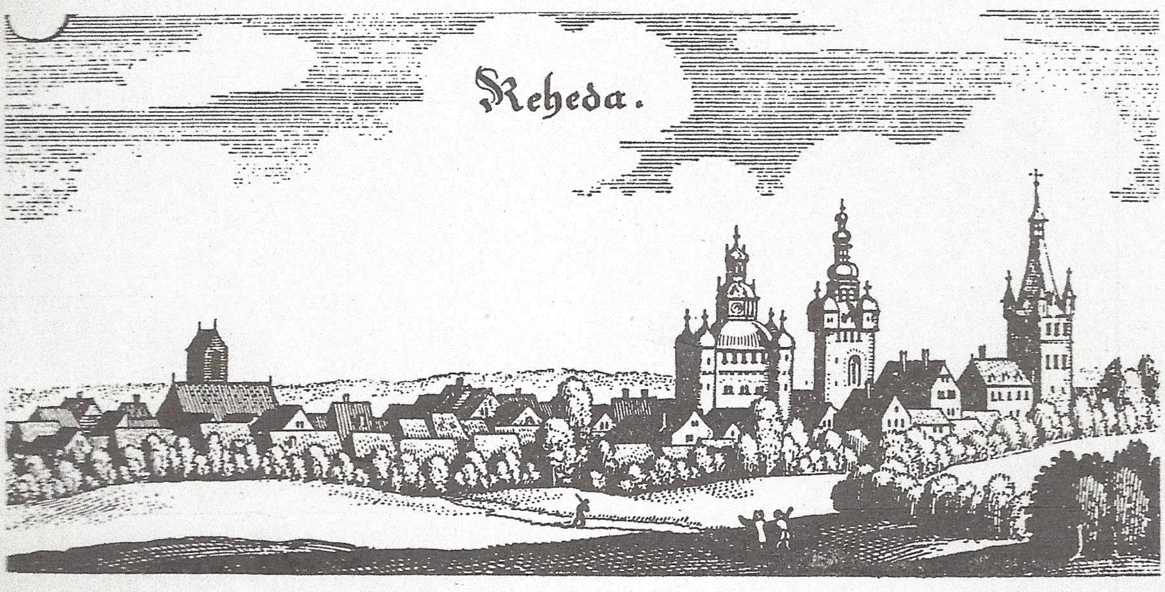
Substitution Along The Time Axis
(preliminary version)

February 1974



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I

The empirical validity of orthodox economics rests very much on the principle of substitution. No single paradigm, such as the one of neoclassical economics, can claim to have models of the real world, which are more than approximations. There are always forces and phenomena in the real world which have no counterpart in the models of the world as they are designed by any given school of thought. It is therefore not enough for such a school to say that the phenomena central to their theory exist. These phenomena must be important, the forces analysed by the theory must be strong in reality to make the theory interesting and relevant.

As an example I take the famous beaver deer story of Adam Smith and of every economist ever since. If we can ignore other factors of production the relative price of beaver and deer depends on the average amount of labour time necessary to hunt the one and the other. Why? The proof is indirect. If the relative price were different hunters would only hunt what yields a higher income per working day. There would therefore be excess supply of one animal and excess demand for the other, a situation which cannot prevail and which will be corrected by the market forces until the price ratio is established which corresponds to the labour values. The implicit assumption is a particular principle of substitution; hunters easily substitute the activity of beaver hunting for the activity of deer hunting and the other way round.

What would happen in a society, in which it happens only rarely, if ever, that a person crosses the line between the two professions of deer hunters and beaver hunters, in which moreover the son follows the father in his profession? Relative prices may be determined by demand, or relative prices are fixed by custom and consumers buy beavers as long as there is supply; otherwise they buy deer, their behaviour being determined by the moral and/or social obligation to make sure that the more prestigious class or caste of beaver hunters earn their living. Or relative prices as well as quantities bought are determined by custom and both professions have the obligation to work until the demand for their product at the customary price can be satisfied.

To which extent can neoclassical economics explain the wage structure in modern western society? What is the element of custom, tradition, equity as seen by those concerned or of the bargaining skill of particular groups in an appropriate explanation of this wage structure. What is the relative importance of marginal productivity or of the elasticity of supply of certain skills in the explanation of their relative wage? It is fairly obvious that the more orthodox theories of wage determination are more important in a situation where the elasticities of supply and demand are high, whereas they are not as important, if the supply of skill is rather inelastic and if any given skill cannot easily be substituted by other skills and cannot easily substitute them in the production process. Marginal rates of substitution are not defined between two inputs, if one cannot at all be substituted by the other. By definition marginal productivity theory does not apply. Should we trust a marginalist explanation of relative input prices, if their partial elasticity of substitution is positive, but very, very small?

As a last example I should like to refer to the well known formula of Kaldor for the distribution of income between a capitalist and a working class. Kaldor assumes that both classes have a saving behavior such that they save a constant fraction of their income, but the two savings propensities are different. Then there exists a one to one relationship between the share national income going to the working class and the proportion of net national product used for investment. While I think that Kaldor's formula is much too simple to explain the distribution of income, it is on the other hand a relevant aspect of income distribution in a market economy, granted that we really can distinguish groups with high savings propensities from groups with low savings propensities. Given a certain constancy of behaviour over time, the high saving group can easily be identified with the group of people owning most of the capital in the economy. What does then Kaldor's formula mean? A large difference in the savings propensity of capitalist and workers means a high elasticity of supply of capital with respect to the rate of return on capital: the higher the rate of profits, the higher is the income share of capitalists and the higher is the aggregate of supply of savings or capital. The difference to what we usually call high elasticities is only that it here results from interpersonal rather than intrapersonal effects.

But Kaldor's formula is not interesting, if the class difference in savings behaviour is small. Then redistribution of income between classes has only a small substitution effect between present and future consumption. Other mechanisms to solve the problem to decide between present and future consumption will dominate. In other words, the aggregate savings ratio cannot be explained by the distribution between classes and therefore income distribution cannot well be explained by reference to the savings propensities.

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What I want to say it quite obvious: Orthodox economic theory tries to explain prices, income distribution and many other things by the working of the market mechanism, i.e. by the equilibrium resulting from the market interaction of decisions taken by individuals free to choose within the limits of certain economic constraints. Such an explanation can only be interesting if the individuals concerned really have relevant choices to make, i.e. if their "noneconomic" constraints do not already explain their behaviour almost completely. This is what is meant by the principle of substitution: if prices and thus budget constraints change, an individual which in other respects is the same, will change his decisions even if the old decision could still be taken. The quantitative importance of the principle of substitution is essential for the quality of the approximation of reality favored by the neoclassical school. It should be noted that a broad interpretation of this principle of substitution is necessary: interpersonal differences are to a degree a substitute for intrapersonal substitution. Also endogenous changes in behaviour (tastes) or available technology should be interpreted as the working of the principle of substitution. One important aspect of substitution is that it takes time. This was already pointed out by Marshall. But this means that orthodox economics is - if at all - more revealing for the long run than for the short. It also means that a certain constancy or predictability of the things given exogenously ("the Datenkranz", German economists would say) is a prerequisite for the theory to be interesting. Other schools of thought may of course need this prerequisite also.

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II

It should be interesting to use mathematical methods to make precise what a high tendendy to substitute means. This is not a trivial exercise. Hicks introduced the concept of the elasticity of substitution forty years ago. It is a measure for the possibility to substitute one input for another and has been extensively used in theoretical and empirical work. One particular application of this concept has been in the discussion of substitution between capital and labour. This is also one which is of interest in the context of capital theory. But then it is not sufficient for capital theory. After all we could dispose of most capital theorists, if we all could agree to consider the Ramsey-Solow one sector model an appropriate approximation of reality for all those purposes capital theorists think to be important.

Unless we are satisfied with the Hicksian elasticity of substitution our task of formalising the ideas discussed above is not an easy one. To make clear why, let me start with a discussion of substitution of inputs. A partial Hicksian elasticity of substitution for two inputs (whatever its precise definition) may be used as an indicator for the potential to substitute between these two inputs. Hence we can expect to "explain" the price ratio of the two inputs by their marginal rate of substitution, if their elasticity of substitution is sufficiently large, i.e. if their marginal rate of substitution is sufficiently stable within a reasonable range. Thus, as the partial elasticity of substitution approaches infinity we can be more and more certain that the marginalist theory in this instance will be correct. But, given a very high elasticity of substitution, have we explained very much by the marginalist principle? Two commodities with a very high elasticity of

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substitution might be considered to be almost identical commodities, and thus in the limit, as the elasticity of substitution approaches infinity we obtain the result that an identical commodity is traded at a uniform price, something which is an axiom of orthodox theory rather than a result. This axiom already is assumed to hold, when the Hicksian elasticity of substitution is defined. Marginalist theory only obtains explanatory power if it relates the prices of different inputs to their marginal rates of substitution. But if we measure "difference" by the inverse of the elasticity of substitution or some such measure and if we believe in the validity of the marginalist theory in direct proportion to the elasticity of substitution we may reduce its explanatory power to almost nothing.

The obvious way out of this dilemma is the search for a measure of similarity or distance between commodities which does not depend or at least does not completely depend on their substitutability relations. Everbody knows that land and labour are different things. Hence a marginalist theory of distribution of revenue between workers and land owners is an interesting proposition. In principle it could be tested and our hypothesis would be that its chance of success depends on the degree of substitutability between land and labour.

Can we say something similar for the problems we are concerned with in capital theory? If, according to Marx, the capitalist class is the class of people owning the means of production we could ask for a measure of substitutability between labour and means of production. But there are obvious difficulties with such an approach. If, say, the wage rate rises and firms tend to switch to more "capital-intensive" methods of production this often means that they also replace their old equipment by some other equipment which "embodies" the new methods of production. Apart from the changes in composition of the stock of means of production there is the more fundamental problem that, different from land, these means of production are themselves produced. From the point of view of the firm this fact can be ignored. It cannot be ignored from the point of view of the economy. The "paradox" of capital theory - the possibility of cases where across steady states the capital stock in constant prices rises with a rising interest rate - crucially rests of the fact that means of production are themselves produced. This paradox would imply an elasticity of substitution between labour and means of production which can become negative.

Orthodox economics has not been disturbed too much by the capital theoretic paradox. There were two answers to defend the crucial position of orthodoxy. One is the Walrasian one: anyone who knows General Equilibrium theory knows that comparative statics of General Equilibrium Systems would easily allow "paradoxes" of this kind. Indeed there is a close analogy between the observation that demand functions aggregated over persons need not fulfill the weak axiom of revealed preference, even if each individual demand function does, and the capital theoretic paradox which involves an aggregation over commodities. These paradoxes do not touch on the basic marginalist propositions which are valid in a general equilibrium situation.

The second answer is what may be called the Solow theorem: with or without paradoxes in capital theory it remains true that in equilibrium the rate of interest is equal to the marginal social rate of return on investment, i.e. on postponing consumption. It is this particular intertemporal aspect of orthodox capital theory which I want to use for my approach to a definition of substitutability between commodities which are intrinsically distinct.

The time index not only allows us to give an objective and evidently economically relevant criterion by which to distinguish commodities. It also provides a natural metric for the degree of difference between two commodities of different time index. Using the Solow theorem on the social rate of return I shall therefore in this paper adopt the Austrian or Neo Austrian black-box approach to production. Its merits and weaknesses are related to the strategy of only treating primary inputs and final (consumption good) outputs explicitely and emphasising the time structure of these flows of inputs and outputs. Here we have a chance to avoid the confusion arising from the fact that inputs are themselves produced.

III

The point-input - point-output model is so to speak the Austrian analog of the Ramsey-Solow simplification in describing growth and accumulation processes. I therefore shall use it as a starting point here. Even if we distinguish commodities by their time index it is true that in the point-input - point-output model a given final output (consumption good) is produced with a single primary input. Substitution here takes the form of a change in the period of production, i.e. a complete replacement of one input by another. Even a very slight change of relative input prices will cause a complete replacement of the input so far used. In a sense therefore the partial elasticity of substitution of any two different inputs is infinite. Indeed, a linear production structure prevails in the sense that a given final output can be efficiently produced by a certain quantity of input T_1 or a certain quantity of input T_2 or any convex combination of these two inputs. There is no complementarity between the two inputs.

We could leave it at that and talk about the model as one with partial elasticities of substitution of infinity. But I think this would be misleading. The interesting question is how much the period of production changes, if the rate of interest is changed by one percentage point. There are again two ways, how to approach this question. One of them is inspired by the traditional attempt to treat capital in a way analogous to primary factors of production. It certainly would have been taken by Böhm-Bawerk and it also has been taken by me when I first proposed an intertemporal elasticity of substitution (Steady State Capital Theory, 1971, p. 56-57). It centers around the question of how to explain the share of profits in national income, etc. If, following Böhm-Bawerk capital intensity of production should be measured by the period of production (which here is an unambiguous concept), T, i.e. if the period of production replaces the capital output ratio in this respect, then the intertemporal elasticity of substitution should be measured by

$$\sigma = -\frac{r}{T}\frac{dT}{dr}$$

where r is the rate of interest.

Today I prefer a different approach to the intertemporal substitution. If one is prepared to take the Austrian, black-box view of production, a concept like net national product or national income should no longer be of interest. For it comprises elements of final production (consumption) with elements of processes going on within the black-box (net investment) which in our mode of analysis are only of secondary interest. Also what is interesting is not so much the distribution of national income itself but rather the distribution of consumption. If we take this view it would be wrong to ignore the potential of intertemporal substitution when the rate of interest is zero or close to zero. But σ takes on the value zero, whenever r is zero, even though $\frac{dT}{dr}$ may be quite large. I therefore now prefer what I would call the coefficient of substitution, s, defined to be

$$s = \frac{d}{dr}\left(\frac{1}{T}\right)$$

I chose the derivative of $\frac{1}{T}$ rather than T with respect to r in order to get an expression which is dimensionless. It has the following interpretation. For an economy with period of production T it may be interesting to know the present value of money (or consumption goods) accruing at time T from now.If the present value of a consumption good of period T is reduced by one percent, this means that the annual rate of interest is raised by $\frac{1}{T}$ percentage points (a rise of the rate of interest from 100r percent to 100r + $\frac{1}{T}$ percent). Then $\frac{1}{T}$ changes by the amount $\frac{1}{100}$ $\frac{1}{T}$ s which means by s percent. Then T has changed by -s percent.

This second approach to intertemporal substitution is of course not unrelated to problems of distribution, since, after all one of the main motives for an analysis of the principle of substitution is the question whether the neoclassical theory of distribution is adequate. But it appears to be more appropriate to define the substitution parameter not with a direct reference to the distribution of national income.

This definition of the coefficient of substitution is more in the spirit of our more general remarks concerning a precise definition of an economy's potential to substitute. As we had seen there one has to know something about the similarity or dissimilarity of two commodities before one can interpret their common partial elasticity of substitution. Indeed, as we see in the point-input - point-output model, substitution may take the form of a complete replacement of one commodity by another. Other examples, where this happens, are of course at hand. Whenever indivisibilities are involved the question often is not how much to buy, but rather which quality. Where a quality can be measured by a one dimensional positive parameter we formally have the same situation as in the point-input - point-output model. Let us denote by t the quality parameter and let p(t) be the price of the commodity with quality parameter t. The consumer buys the consumer durable with a certain specific parameter value T. His choice of T depends on the price function p(t). Let us assume that small changes in the price function p(t) induce only small changes in T, the chosen parameter value. This then means that dT, the small change in T, can be explained completely by the price changes in a neighborhood of T (as long as all price changes are small). Now, let r(t) be defined by

$r(t) = \frac{dp}{dt}\frac{1}{p}$. Consider a small change $\delta p(t)$ of the function p(t).

We then obtain $\delta r(t) = \dfrac{p\frac{d}{dr}\left[\delta p(t)\right] - \frac{d}{dt}\left[p\right]\delta p(t)}{p^2}$.

We may now relate the ensuing change δT to $\delta r(t)$ and define as coefficient of substitution $s = \frac{\delta \frac{1}{T}}{\delta r(T)}$. If indeed only the immediate neighborhood of T is relevant, then s is unambiguously defined for any given price function p(t).

A different way to apply the coefficient of substitution is to ask what the factor price implications of a change in the time distribution of consumption are. In the point-input - point-output model a one percent reduction of consumption over the (short) initial period Δ implies a rise in the period of production $\frac{1}{100}\Delta$. The percentage change is $\frac{\Delta}{T}$. Accordingly $\frac{1}{T}$ changes by $-\frac{\Delta}{T}$ percent or by the amount $-\frac{1}{100}\frac{\Delta}{T}\frac{1}{T} = -\frac{1}{100}\frac{\Delta}{T^2}$. The implied change in the rate of interest is $-\frac{1}{100}\frac{\Delta}{T^2}\frac{1}{s}$. It is well known that a one percentage point change in the rate of interest implies a T percent change of the wage rate in the opposite direction. Thus here the wage rate will change by $\frac{\Delta}{T^2}\frac{1}{s}$ T percent $= \frac{\Delta}{T}\frac{1}{s}$ percent.

This is an example for the use of s in relating factor price changes to quantity changes. Factor prices are relatively stable or robust whenever s is large. This robustness of prices is what the neoclassical economist should look for, if, as he believes, other effects than those which he considers are of little importance for the formation of prices.

IV

I now proceed to generalize the analysis of section III. In the present paper I shall mainly be concerned with what perhaps might be called the Quasi-Robinson-Crusoe model, which has been widely used in the modern literature, in particular in the theory of optimal growth. The classic reference is Malinvauds 1953 Econometrica Paper, although his model was more general than what we shall discuss here. By the Quasi-Robinson-Crusoe property I mean that the assumption of a social welfare function precludes the discussion of interpersonal conflicts of interest. Certain things which are of interest to us here become much easier to handle in the Quasi-Robinson-Crusoe model.

I shall assume here that there exists only one consumption good and only one kind of labour. There exists a social welfare or utility function U ordering the relevant set of consumption - labour flows extending from the present to infinity. There exists a convex subset S of feasible consumption-labour flows. To simplify matters even more we shall assume that to each efficient flow

$$\{C_o, L_o, C_1, L_1, \ldots\}$$

in S there exists a set of positive shadow prices

$$\{p_o, w_o, p_1, w_1, \ldots\}$$

unique up to a positive and arbitrary factor of proportionality so that for each

$$\{C_o', L_o', C_1', L_1', \ldots\}$$

in S we have

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$$\sum_{t=o}^{\infty} p_t (C_t - C'_t) - \sum_{t=o}^{\infty} w_t (L_t - L'_t) \geq 0$$

We shall not concern ourselves with problems of divergence of infinite sums, and in cases of doubt we shall assume all convergence properties necessary for our analysis.

In a similar way the utility function is assumed to provide a set of positive shadow prices $\{q_o, v_o, q_1, v_1, \dots\}$ - unique up to an arbitrary factor of proportionality - for each non-negativ flow $\{C_o, L_o, C_1, L_1, \dots\}$ such that for every other nonnegative flow $\{C'_o, L'_o, C'_1, L'_1, \dots\}$ we either have

$$U\{C_o, L_o, \dots\} \geq U\{C'_o, L'_o, \dots\}$$

or

$$\sum_{t=o}^{\infty} q_t (C_t - C'_t) - \sum_{t=o}^{\infty} v_t (L_t - L'_t) < o$$

or both.

An equilibrium situation in this economy is characterised by the coincidence of the production shadow prices and the utility shadow prices, as is well known.

I now introduce the average period of production and the average period of waiting or consumption. To make things slightly more elegant, I shall assume constant returns to scale. Let $K_I, K_{II}, K_{III}, \dots$ be the vector of initial capital goods available in the economy. Let $w_I, w_{II}, w_{III}, \dots$ be their shadow prices. Then for each efficient path $\{C_o, L_o, \dots\}$ in S constant returns to scale imply

$$\sum_{t=o}^{\infty} p_t C_t - \sum_{t=o}^{\infty} w_t L_t - \sum_{z=I}^{\infty} w_z K_z = o$$

if we use the appropriate shadow prices.

By the average period of production, T, we now mean

$$T = \frac{\sum p_t C_t t}{\sum p_t C_t} - \frac{\sum w_t L_t t}{\sum w_t L_t + \sum w_z k_z}$$

Correspondigly the average waiting period is

$$\hat{T} = \frac{\sum q_t C_t t}{\sum q_t C_t} - \frac{\sum v_t L_t t}{\sum q_t C_t}$$

The period of production is the "center of gravity" of the value flow of outputs on the time axis minus the center of gravity of the value flow of inputs. In a similar way the period of consumption is defined. There is one peculiarity here: initial wealth of the consumer as computed from the difference between the value of consumption and the value of labour supply (in shadow price terms) is also reckoned as part of the factor supply of the consumer (here it is society at large who is the consumer). This is why the denominator is $\sum q_t C_t$ in both terms whose difference is $\hat{T}$.

I have decided to use a concept for the period of production and the period of consumption which depends on weighting system determined by the appropriate price system. This is not in the spirit of Böhm-Bawerk, rather in the spirit of Hicks (Value and Capital). As far as I know, Hicks now does not think his concept of the period of production to be very useful. So I should perhaps say a word about the general philosophy of preference for it over the Böhm-Bawerk period of production which is "uncontaminated" by a change in the prices providing the weighting system. As it turns out there are no really interesting results to be found with the Böhm-Bawerk concept, I hope to provide some results for the Hicksian period of production.

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But there is perhaps a more basic reason to prefer th period of production with the price influenced weighting system. It is easy to see that one has no problems of extending the analysis to more than one primary input and more than one final output. Moreover the concept is independent of any intertemporally consistent concept of real output or the like. By this I mean the following. In any given period we must of course be able to distinguish variations of the quantities from variations of prices. This we need if we want to tell whether a certain change in the value of consumption (due to a change in the quantities of consumption) has raised or reduced the period of production, keeping the weighting system the same. But such an exercise does not require that we identify certain consumption goods produced in one period with certain consumption goods produced in a different period. Thus there exists no problem of measuring real growth in the context of new commodities, etc. All this would of course be necessary, if we used a period of production like Böhm-Bawerk's. In what follows we shall ignore this possibility of not having to define real growth or the real rate of interest.

V

Before we discuss the coefficient of substitution in this more general model we want to discuss a few substitution effects. First we ask about the substitution effect of an equal change of the rate of interest in every period. Let us first look at the production side. Let the price of consumption goods in period t depend on a parameter i in the following way

$$p_t(i) = p_t(o)\ e^{-it}$$

Thus a unit change of i works like a unit change of the instantaneous rate of interest. Since it is arbitrary where to put $i=o$, we do it in such a way as to make $p_t(o)$ equal to the initial situation. At the same time w_t depends also on the parameter i. Here things are a little more complicated, because a change in the (consumption) rate of interest will not only change the present value of the wage rate, but it will also change the spot wage rate itself. Otherwise the condition "revenues=costs" cannot be sustained. So we write

$$w_t(i) = h(i)\ w_t(o)\ e^{-it}$$

where $h(i)$ is given by the equation

$$\sum_t p_t(i)\ C_t(i) = \sum_t w_t(i) L_t(i)$$

Differentiation of this equation with respect to i yields,

$$\sum_t \left[p_t(i)\ \frac{dC_t(i)}{di} + C_t(i)\ \frac{dp_t(i)}{di}\right] = \sum_t \left[w_t(i)\frac{dL_t(i)}{di} + L_t(i)\frac{dw_t(i)}{di}\right]$$

Because of profit maximisation (cf. the shadow price property discussed above) we have

$$\sum_t p_t(i)\frac{de_t}{di} = \sum_t w_t(i)\frac{dL_t}{di}$$

Moreover

$$\frac{dp_t(i)}{di} = -te^{-it}\ p_t(o) = -tp_t(i)$$

$$\frac{dw_t(i)}{di} = h'(i)\ w_t\ (o)e^{-it}\ -th(i)w_t(o)\ ^{-it} =$$

$$= \frac{h'(i)}{h(i)}\ w_t(i)\ -\ t\ w_t(i)$$

It then follows

$$\sum_t C_t\ (i)\ \frac{dp_t(i)}{di} = \sum_t L_t\ (i)\ \frac{dw_t(i)}{di}$$

and thus

$$-\sum_t C_t(i)\ p_t\ (i)\ t = -\sum_t L_t\ (i)\ w_t(i)\ t + \frac{h'(i)}{h\ (i)}\ \sum_t L_t\ (i)\ w_t\ (i)$$

Upon division by $\sum C_t p_t = \sum L_t w_t$

We obtain

$$-\ T = \frac{h'(i)}{h(i)}$$

The percentage change of the spot wage rate is equal to minus T if the rate of interest rises by one percent. It should be kept in mind that the prices of initial capital goods are also treated as "wages" in our set up. A change in the rate of interest such that initial capital goods keep their value would imply an even greater fall in the wage rate.

Let us now investigate the direction of the change in consumption and labour inputs which are implied by a change in the rate of interest. From the profit maximisation condition we have

$$\sum_t p_t (o) \left[C_t (o) - C_t (i)\right] - \sum_t w_t (o) \left[L_t (o) - L_t (i)\right] \geq o$$

and

$$\sum_t p_t (i) \left[C_t (i) - C_t (o)\right] - \sum_t w_t (i) \left[L_t (i) - L_t (o)\right] \geq o$$

addition of these two equations yields

$$\sum_t \left[p_t (i) - p_t (o)\right] \left[C_t (i) - C_t (o)\right] - \sum_t \left[w_t (i) - w_t (o)\right] \left[L_t (i) - L_t (o)\right] \geq o$$

Let i approach zero and thus yield in terms of derivatives

$$\sum_t \frac{dp_t}{di} \frac{dC_t}{di} - \sum_t \frac{dw_t}{di} \frac{dL_t}{di} \geq o$$

Using again the expressions for $\frac{dp_t}{di}$ and $\frac{dw_t}{di}$ we derive

$$- \sum_t t\, p_t \frac{dC_t}{di} + \sum_t t\, w_t \frac{dL_t}{di} + T \sum_t w_t \frac{dL_t}{di} \geq o$$

This formula gives an indication about the substitution effect of a change in the rate of interest. We can now show that the quantity changes due to a rise in the rate of interest tend to reduce the period of production. Let us investigate the implications of quantity changes on the period of production. Let $\delta_s T$ be this change (s for substitution), let $\delta_p T$ be a a change in the period of production due to a change in prices.

We then have by differentiation of the expression forming T

$$\delta_s T = \frac{\left(\sum p_t C_t\right)\left(\sum p_t t dC_t\right) - \left(\sum p_t C_t t\right)\left(\sum p_t dC_t\right)}{\left(\sum p_t C_t\right)^2}$$

$$- \frac{\left(\sum w_t L_t\right)\left(\sum w_t t dL_t\right) - \left(\sum w_t L_t t\right)\left(\sum w_t dL_t\right)}{\left(\sum w_t L_t\right)^2}$$

keeping in mind that $\sum p_t dC_t - \sum w_t dL_t = o$

we obtain (T_C, T_C are the "points of gravity" of C and L)

$$\delta_s T = \frac{\sum p_t t dC_t}{\sum p_t C_t} - T_C \frac{\sum p_t dC_t}{\sum p_t C_t}$$

$$- \frac{\sum w_t t dL_t}{\sum w_t L_t} + T_L \frac{\sum w_t dL_t}{\sum w_t L_t} =$$

$$= \frac{\sum p_t t dC_t - \sum w_t t dL_t - T \sum w_t dL_t}{\sum p_t C_t}$$

Now it is easily seen that $\delta_s T$ has the opposite sign of di. The aggregate time substitution effect of a uniform rise in the rate of interest is negative.

There exist analogous theorems which are proved in a similar manner. On the consumption side we can show: an uniform rise in the rate of interest such that the utility level is not affected by the price change will induce a time substitution effect $\delta_s \hat{T}$ of positive sign, i.e. will induce a rise in the period of consumption. Then there exist the dual theorems to these two theorems. The dual theorem on the production side says the following: A change of quantities in the manner

$dC_t = (t-T)\, C_t$, $dL_t = tL_t$ induces a change $\delta_p T$ of the period of production due to price changes in the positive direction. To formulate it more heuristically: a uniform rise in the rate of growth of labour inputs and consumption good outputs implies a decline in the average of the period to period real rate of interest.

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The policy change assumed in this dual theorem, $dC_t=(t-T)C_t$ is a special case of something which I would like to call a "simple investment". By a simple investment I mean a policy change such that ΔC_t first is negative and then later becomes positive either indefinitely or for a finite period of time. Its qualitative time structure is unambiguous. It does not depend on the weighting system. Simple investments are probably particularly interesting because they are what most people have in mind when they discuss the question of whether a society accumulates too little or too much. Perhaps they should be studied more. Let me now by an argument similar to the one already used show that a simple investment has a tendency to lower the rate of interest in some average sense.

As shown above, for any two paths of accumulation and the corresponding price system we have

$$\sum_t (p_i'-p_t)\ (C_t'-C_t)\ -\sum_t (w_t'-w_t)\ (L_t'-L_t) \geq o$$

which we can write

$$\sum_t \Delta p_t\ \Delta C_t\ -\ \sum_t \Delta w_t\ \Delta L_t \geq o$$

or, for "small" differences

$$\sum_t \delta p_t\ \delta C_t\ -\ \sum_t \delta w_t\ \delta L_t \geq o$$

where profit maximisation implies for "small" differences

$$\sum p_t\ \delta C_t\ - \sum w_t\ \delta L_t\ = o$$

Let us now consider a simple investment, for which $\delta L_t = o$ and

$\delta C_t \leq o$ for $t \leq t_o$

$\delta C_t \geq o$ for $t > t_o$

The discounted value of this differential flow of consumption goods at the "old" prices p is equal to zero. On the other hand, if evaluated at the "new" prices $p + \delta p$ it is nonnegative. This means there is a tendency for a change in prices so that the positive part of the flow δC gets a relatively higher weight than the negative part. But, because δC is a simple investment, the positive part of δC comes after the negative part and hence the relative price of later goods rises which means that on average the rate of discount must have gone down. This is a very general law of diminishing returns. It says: if you double your sacrifice in consumption the later benefit of additional consumption will not be doubled.

It also applies for those cases where the paradox of capital theory prevails. Let us look at a steady state situation with a rate of interest r and, say a rate of growth zero. Assume that across steady states consumption per head rises as the rate of interest rises in the neighbourhood of r, so that we have the paradox. The economy now tries a transition to a steady state with higher consumption and thus with a higher rate of interest. It tries this transition by means of a simple investment. This means δC_t first is negative, then becomes positive and approaches a positive limit value $\delta \bar{c} > o$ as it approaches the new steady state. In the end the rate of interest will be higher than in the initial steady state. But our theorem of the interest lowering effect of simple investments shows that the transition of interest rates cannot be monotonic in time, since otherwise the condition $\sum \delta C_t \, \delta p_t \geq o$ could not be sustained.

In the case where we have a finite number of different techniques it is easy to show that a transition from the use of one technique to the use of another with a higher steady state rate of interest and more consumption by means of a simple investment will involve at least the partial use of a technique with a lower steady state rate of interest.

VI

Let us now introduce the coefficient of substitution.
It is the obvious generalisation of the coefficient of substitution as defined for the point-input - point-output model. We have derived the sign of $\delta_s T$ as the rate of interest changes uniformly.

We thus can use $s = \frac{\delta_s \frac{1}{T}}{di}$ as a measure for the size of the substitution effect and call it the intertemporal coefficient of substitution. For what purpose can we use this concept?

Our starting point was the belief that a high substitutability is a prerequisite for the empirical relevance of neoclassical economics, in particular for the empirical relevance for the proposition that market prices and "shadow prices" (or marginal rates of substitution) tend to be close together. For high substitutability is a guarantee of a certain robustness of these shadow prices. From this point of view it is useful to look at the implications of changes of accumulation policies on shadow prices.

Let us look at a stationary economy with a positive rate of interest. Now imagine a sudden small drop in the rate of interest-δi. This will imply a change in the undiscounted steady state wage rate of + $Tw\delta i$. At the same time it will mean that the period of production rises by the amount $T^2 s\delta i$ which means that per unit change of the wage rate the period of production changes by

$s \frac{T}{w}$, or per unit percentage change of the wage rate the period of production changes by s percentage units. Consider now the change δC_t in the flow of consumption. Let the period of adaptation, until the new equilibrium is reached be Δ and imagine that δC_t takes the form of a simple investment (which means we are not in a "paradoxical" situation). The relative size of the change in consumption can be computed as follows. If Δ or the rate of interest is small we can ignore compound interest effects within the period Δ and so the present value of the initial loss of consumption can be considered to be $-\Delta\delta C'$ where $\delta C' > o$ is equal to the average reduction of consumption in the period of adaptation. The present value of the additional consumption in the new steady state can (if Δ is small) be reckoned to be $\frac{\delta \bar{C}}{i}$ where $\delta \bar{C}$ is the gain in consumption per period. We then must have $\frac{\delta \bar{C}}{i} - \Delta \delta C' = o$.

The change in the period of production (since L_t does not change) due the change in consumption is (treating $\delta C'$ as if it all occurred at time t = o)

$$\delta_s T = \frac{\sum_t \frac{1}{(1+i)^t} \delta C_t \, t}{\sum_t \frac{1}{(1+i)^t} C_t} = \frac{\delta \bar{C}}{C} = \frac{\sum \frac{t}{(1+i)^t}}{\sum \frac{1}{(1+i)^t}} = \frac{\delta \bar{C}}{C} \frac{1}{i}$$

It therefore follows

$$\frac{\Delta \delta C'}{C} = \frac{1}{C} \frac{\delta \bar{C}}{i} = \delta_s T$$

or the average percentage reduction in consumption in the investment period Δ is given by

$$\frac{\delta C'}{C} = \frac{\delta s^T}{\Delta} = \frac{T^2 s \delta i}{\Delta}$$

The percentage reduction in consumption in relation to the ensuing percentage rise in the wage rate thus would be

$$\frac{\delta C'}{C} : \frac{\delta w}{w} = \frac{T}{\Delta} s$$

It is the result already derived in the ppint input-output model.The relative robustness of prices, in this case the wage rate, can be measured by the social sacrifice of consumption necessary to obtain a one percent change of the prices within a given period of time Δ.

Another application of the coefficient of substitution is the following. It is well known that the Golden Rule of Accumulation holds quite generally. That is, for a given rate of growth consumption per unit of labour, γ, is maximized across steady states at the steady state where the real rate of interest r = g.

If we consider γ to be a function of the steady state parameter r we have $\frac{d\gamma}{dr} = o$ at r = g. We now may ask how sensitve is γ with respect to a change in the rate of interest r in a neighborhood of g. In other words we are asking what the second derivative of γ with respect to ris. Our conjecture would be that this depends on the degree of substitutability.

As we now do steady state comparative statics we have to change our definition of the period of production slightly. A steady state is characterised by the equality of past and future. As opposed to our planning point of view which we have adopted so far, there is no reason only to treat the future as open and the past as fixed. We are comparing "islands" each of which had a different past as it will have a different future. But then the initial capital stock makes no sense and we have to find a different way to define the period of production. Connected with this reason for a different definition is the observation that in steady states one always has divergence problems. If the rate of interest is not smaller than the rate of growth the integrals extended into minus infinity diverge, if the rate of interest is not larger than the rate of growth the integrals extended into plus infinity diverge.

Hicks (in Capital and Time) and I myself (in Steady State Capital Theory) have worked with the concept of production processes, each of which is described by a flow of labour inputs and consumption good outputs. Given the assumption that these processes do not change their character if used in a synchronised manner with other processes of the same kind, we can imagine the steady state to be a well synchronised system of such overlapping processes all of the same kind. This is probably very much in the spirit of the Austrian school in capital theory. For these processes it is now possible to define a period of production in the same way as I have done above. Again it is possible to define the coefficient of substitution as we have done already.

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If we work with the model of the time profile of production processes then we obtain the result we are looking for. A process is characterised by the flow β_τ of consumption goods and of labour inputs α_τ where τ is project time. Constant returns to scale imply that we could multiply both flows with a scalar and still get a feasible process. We shall use this property in our derivation. In a steady state (i.e. completely synchronised) system using the process, consumption per head γ is given by (integration of total project length is indicated by $\int \ldots d\tau$)

$$\gamma = \frac{\int e^{-g\tau} \beta_\tau \, d\tau}{\int e^{-g\tau} \alpha_\tau \, d\tau}$$

where g is the rate of growth (for a derivation see Hicks, Capital and Time or my book). The budget equation for the process reads

$$\int e^{-r\tau} \beta_\tau d\tau = w \int e^{-r\tau} \alpha_\tau \, d\tau$$

Let us now look at a rate of interest "close" to g. We then can approximate

$$e^{-r\tau} \approx c^{-g\tau} \; (1 + (g - r) \; \tau)$$

Using this approximation and using the profit maximisation property of equilibrium prices we obtain the following equation for small changes $\delta\beta_\tau$ and $\delta\alpha_\tau$

$$\int e^{-g\tau} \; (1+(g-r)\tau) \; \delta\beta_\tau d\tau = w(r) \int e^{-g\tau} (1+\tau(g-r)) \; \delta a_t d\tau$$

To simplify computation we use the constant returns to scale property which makes the choice of scale for any process arbitrary. Also we can shift the flows α_τ and β_τ uniformly on the τ-axis, since the choice of project time $\tau = o$ is arbitrary. We use these two degrees of freedom so as to put

$$\int e^{-g\tau} \delta\alpha_\tau \, d\tau = o \text{ and } \int e^{-g\tau} \tau \delta\alpha_\tau d\tau = o$$

The period of production evaluated at prices $r=g$ is

$$T = \frac{\int \tau e^{-g\tau} \beta_\tau d\tau}{\int e^{-g\tau} \beta_\tau d\tau} - \frac{\int \tau e^{-g\tau} \alpha_\tau d\tau}{\int e^{-g\tau} \alpha_\tau d\tau}$$

hence, keeping in mind that only the first term changes we have

$$\delta_s T = \frac{\int e^{-g\tau} \tau \, \delta\beta_\tau d\tau}{\int e^{-g\tau} \beta_\tau \, d\tau} - T \frac{\int e^{-g\tau} \delta\beta_\tau \, d\tau}{\int e^{-g\tau} \beta_\tau d\tau}$$

The change in γ , or rather $\log \gamma$ is

$$\delta \log \gamma = \frac{\int e^{-g\tau} \delta\beta_\tau d\tau}{\int e^{-g\tau} \beta_\tau d\tau}$$

Because of the convention adopted concerning $\delta\alpha_\tau$ the profit maximisation condition derived above now reads

$$\int e^{-gt} (1+(g-r)\tau) \, \delta\beta_\tau d\tau = o$$

Dividing this through $\int e^{-g\tau} \beta_\tau \, d\tau$ yields

$$\delta \log \gamma + (g-r) \, \delta_s T + (g-r) T \, \delta \log \gamma = 0$$

or

$$\delta \log \gamma = \frac{(r-g) \delta_s T}{1+(g-r) T}$$

This may now be related to a change in the rate of interest, using the definition

$$s = \frac{\delta_s \frac{1}{T}}{\delta r}$$

$$\frac{\delta \log \gamma}{\delta r} = \frac{(g-r) T^2 s}{1+(g-r) T}$$

$$\frac{1}{T} \frac{\delta \log \gamma}{\delta r} = s \frac{(g-r) T}{1+(g-r) T}$$

A further approximation relates changes in consumption to changes in wages. As is well known in a steady state economy the value of the capital stock per unit of consumption good output is approximated by the period of production if r is close to g. Therefore the national accounting identities

$$\gamma + g v = w + r v$$

with v capital per man hour yields

$$\gamma + g T \gamma = w + r T \gamma$$

or

$$w = (1 + (g-r) T) \gamma$$

We also know that $\frac{d\log w}{dr} = - T$

and therefore we obtain the approximation

$$\frac{\delta \log \gamma}{\delta \log w} = s\frac{(r-g)\ T}{1+(g-r)T} = s\ \frac{\gamma - w}{w}$$

A one percent change in the steady state wage rate implies a percentage change in the consumption per head which is proportional to the coefficient of substitution and proportional to the percentage difference of consumption and wages.