Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

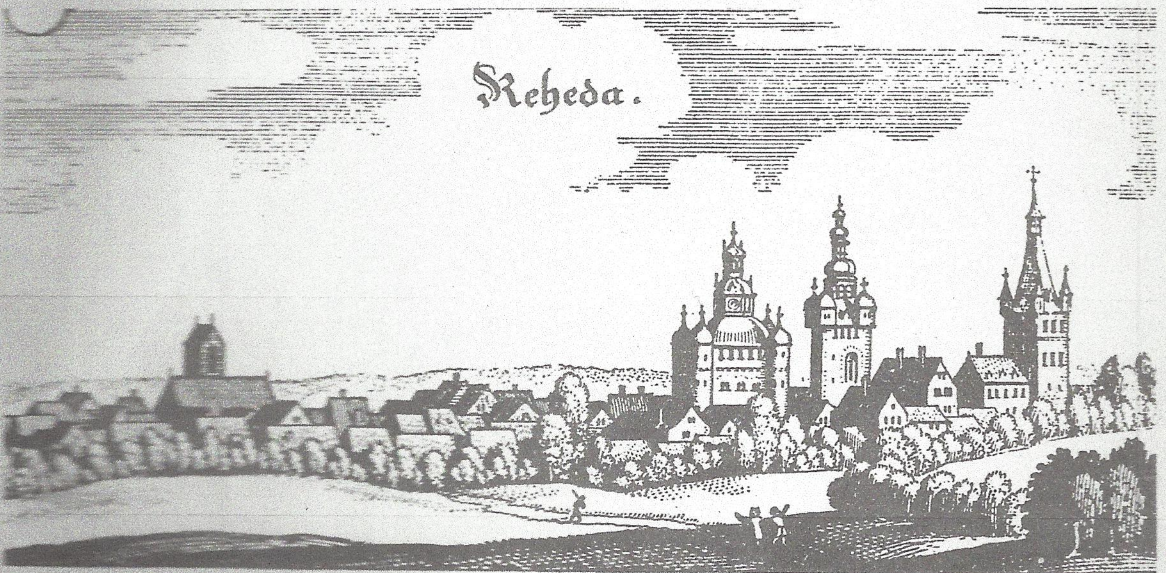
Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 11

EQUALLY DISTRIBUTED CORRESPONDENCES

Sergiu Hart and Elon Kohlberg

July 1973



Anno 1600 nach Merian

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Schloß Rheda
484 Rheda
Bundesrepublik Deutschland
Federal Republic of Germany

EQUALLY DISTRIBUTED CORRESPONDENCES

by

Sergiu Hart[*] and Elon Kohlberg[**]

July 1973

---

[*] Sergiu Hart, Department of Mathematics, Tel-Aviv University, Israel.

[**] Elon Kohlberg, Department of Economics, Harvard University, Mass., U.S.A.

This research was carried out while both authors were participating in the Workshop on Mathematical Economics at the Institute for Mathematical Economics, University of Bielefeld, Rheda, W. Germany, June-July 1973. The research of the first author is supported by the National Council for Research and Development in Israel.

1. Introduction

The study of measurable and integrable selections from a correspondence (set-valued function) has been of interest for some time (e.g., see [A], [H], etc.).

Here we take up the study of distributions of such selections. The motivation for doing this comes from Mathematical Economics (see [K, section 7]; the reader interested in the economic applications is referred to [H-H-K]).

The first question we consider is the following: Does the distribution of a correspondence determine the distributions of its selections? If the underlying space contains atoms, then of course we cannot expect an affirmative answer (e.g., consider the correspondence whose image is always the set $\{1,2\}$, once on a space consisting of one atom, and once on a space consisting of two atoms). But what if the underlying space is atomless?

Consider the following example (which is essentially a reformulation of an example due to G. Debreu [K, section 7]). Let $\phi_1(t) = \{t,-t\}$ for all $t \in [0,1]$ and

$$\phi_2(t) = \begin{cases} \{2t,-2t\}, & \text{for } t \in [0,1/2], \\ \{2t-1,1-2t\}, & \text{for } t \in (1/2,1]. \end{cases}$$

Clearly, ϕ_1 and ϕ_2 must be equally distributed by any reasonable definition of this concept. However, the distributions of their measurable selections are not the same: let

$$f(t) = \begin{cases} 2t & , \quad \text{for } t \in [0,1/2] , \\ 1-2t & , \quad \text{for } t \in [1/2,1] , \end{cases}$$

then f is a measurable selection from ϕ_2, but there is no measurable selection from ϕ_1 with the same distribution.

The above example, in giving a negative answer to our question, raises serious difficulties as to possible applications of the concept of "distributions of selections". However, most of the difficulties can be eliminated provided the following is true: if ϕ_1 and ϕ_2 are equally distributed correspondences, then the <u>closures</u> of the distributions of their selections are equal. Theorem 1 is the precise formulation of this statement.

In Theorem 3 we show that when the measurable selections are restricted to have a constant integral, the closure of their distributions still depends only on the distribution of the correspondence (this restriction is of importance in Mathematical Economics; indeed, our Theorem 3 is a well-known conjecture of R.J. Aumann [K, section 7]).

2. Preliminaries

We denote by $\mathbb{R}^\ell$ the ℓ-dimensional Euclidian space, and by $\mathfrak{B}(\mathbb{R}^\ell)$ the σ-algebra of its Borel subsets.

Let $(A,\mathfrak{A},\nu)$ be a measure space; in the following, all measure spaces will be assumed to be complete (i.e., $\mathfrak{A}$ contains all subsets of null sets - see [H,D.I]). Null sets will be systematically ignored.

Let $f:(A,\mathfrak{A},\nu) \to \mathbb{R}^\ell$ be a measurable function. The distribution D_f of f is defined as $\nu \circ f^{-1}$ (i.e., the induced measure on $\mathfrak{B}(\mathbb{R}^\ell)$). The sequence* $\{f_n\}$ converges to f in distribution if the sequence $\{D_{f_n}\}$ converges weakly to D_f, where weak convergence of measures is defined as usual by $\mu_n \overset{W}{\to} \mu$ if $\int h d\mu_n \to \int h d\mu$ for all real, continuous and bounded functions h. The topology of weak convergence can be metrized, e.g. by the Prohorov metric ρ, defined as

$$\rho(\mu_1,\mu_2) = \inf\{\varepsilon > 0 \,|\, \mu_1(B) \leq \mu_2(B_\varepsilon)+\varepsilon \text{ and } \mu_2(B) \leq \mu_1(B_\varepsilon)+\varepsilon \text{ for all Borel subsets } B\},$$

---

* note that the f_n's can be defined on different spaces, but their range must always be in the same space.

where $B_\varepsilon = \{x|d(x,B) < \varepsilon\}$.

Let* $\phi: (A,\mathcal{A},\nu) \to \mathbb{R}^\ell$ be a correspondence, i.e. $\phi(a)$ is a non-empty subset of $\mathbb{R}^\ell$ for all $a \in A$. The graph of ϕ is the set

$$G_\phi = \{(a,x)|x \in \phi(a)\}.$$

ϕ has measurable graph (or "is Borel-measurable", as in [A]) if $G_\phi \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^\ell)$. ϕ is integrably bounded if there exists an integrable function $h: (A,\mathcal{A},\nu) \to \mathbb{R}^\ell$ such that $-h(a) \leq x \leq h(a)$ for all $x \in \phi(a)$ and for all $a \in A$. ϕ is closed-valued if $\phi(a)$ is a closed set (in $\mathbb{R}^\ell$) for every $a \in A$.

The set of all integrable selections from ϕ (i.e., all integrable functions f such that $f(a) \in \phi(a)$ for all $a \in A$) is denoted $\mathcal{L}_\phi$, and $\int\phi = \{\int f | f \in \mathcal{L}_\phi\}$. For every B, let

$$\phi^{-1}(B) = \{a \in A|\phi(a) \cap B \neq \phi\}$$

(the [weak] inverse of ϕ).

---

* we write $\phi: A \to \mathbb{R}^\ell$ also for correspondences, but then we mean $\phi(a) \subset \mathbb{R}^\ell$ for all $a \in A$; no confusion will arise, since correspondences are always denoted by the Greek letters ϕ or ψ.

All the definitions up to now are standard (see, e.g. [B] and [H]). At this point we must make precise the notion of "equally distributed correspondences". We therefore define the <u>distribution</u> D_ϕ <u>of the correspondence</u> ϕ having a measurable graph, by $D_\phi = \nu \circ \phi^{-1}$. This definition is meaningful since by the projection theorem [H, D.I(11)], if ϕ has a measurable graph then $\phi^{-1}(B)$ is measurable for all $B \in \mathcal{B}(\mathbb{R}^\ell)$. Note that D_ϕ is not necessarily additive; if, for all $a \in A$, $\phi(a)$ consists of just one point, then this definition coincides with the usual one for functions. We say that ϕ_1 <u>and</u> ϕ_2 <u>are equally distributed</u> if $D_{\phi_1} = D_{\phi_2}$.

The following notations will be used: $\setminus$ for set-theoretic substraction, K^c for the complement of K, and $\int f$ for $\int_A f(a)d\nu(a)$. The space $([0,1],\mathcal{L},\lambda)$ is the unit interval with the Lebesgue measure λ. For a set $\mathcal{F}$ of functions with the same range, $D(\mathcal{F})$ will denote the set of their distributions, i.e., $D(\mathcal{F}) = \{D_f \mid f \in \mathcal{F}\}$, and $\bar{D}(\mathcal{F})$ will be its closure with respect to the weak convergence of measures.

3. Statement of the Results

In the following, $(A_i, \mathcal{A}_i, \nu_i)$ (for $i = 1,2$) will always be non-atomic complete probability measure spaces, and $\phi_i\colon (A_i, \mathcal{A}_i, \nu_i) \to \mathbb{R}^\ell$ two correspondences with measurable graphs.

Theorem 1. Let ϕ_1 and ϕ_2 be integrably bounded. If ϕ_1 and ϕ_2 are equally distributed, then $\overline{D}(\mathcal{L}_{\phi_1}) = \overline{D}(\mathcal{L}_{\phi_2})$.

The following theorem is an immediate consequence of Theorem 1.

Theorem 2. Let ϕ_1 and ϕ_2 be integrably bounded and closed-valued. If ϕ_1 and ϕ_2 are equally distributed, then $\int\phi_1 = \int\phi_2$.

Theorem 3. Let ϕ_1 and ϕ_2 be integrably bounded and closed-valued, and let $x \in \mathbb{R}^\ell$. If ϕ_1 and ϕ_2 are equally distributed, then

$$\overline{D}(\{f \in \mathcal{L}_{\phi_1} \mid \int f = x\}) = \overline{D}(\{g \in \mathcal{L}_{\phi_2} \mid \int g = x\}) .$$

Clearly, if $\phi_i = \psi \circ h_i$, where the functions h_1 and h_2 have the same distribution, then ϕ_1 and ϕ_2 are equally distributed (this is, indeed, the case in Mathematical Economics). The following decomposition theorem asserts that this is the general structure of equally distributed correspondences.

<u>Theorem 4.</u> Let ϕ_1 and ϕ_2 be closed-valued. Then ϕ_1 and ϕ_2 are equally distributed if and only if there exist $h_i: (A_i, \mathcal{A}_i, \nu_i) \to ([0,1], \mathcal{L}, \lambda)$ with the same distribution and $\psi: ([0,1], \mathcal{L}, \lambda) \to \mathbb{R}^\ell$, such that $\phi_i = \psi \circ h_i$ for $i = 1,2$.

4. Proof of the Results

In the proof of Theorem 1 we rely on the following lemma:

Lemma A. Let $(A, \mathcal{A}, \nu)$ be a non-atomic measure space. Let $\{S_i\}_{i=0}^m \subset \mathcal{A}$ and $\{\alpha_i\}_{i=0}^m$, $\alpha_i \geq 0$ be such that

(1) $\nu(\bigcup_{i \in I} S_i) \geq \sum_{i \in I} \alpha_i$, for all $I \subset \{0, 1, \ldots, m\}$,

and

(2) $\nu(\bigcup_{i=0}^m S_i) = \sum_{i=0}^m \alpha_i$.

Then there exist disjoint sets $\{T_i\}_{i=0}^m$ such that $T_i \subset S_i$ and $\nu(T_i) = \alpha_i$ for all $i = 0, 1, \ldots, m$.

The proof of this lemma may be carried out in complete analogy with the proof of a well known result in Combinatorics (see [H-V]). Since it is quite short, we repeat it here.

Proof.

We use induction on m. For $m = 0$ it is trivial.

Let $m > 0$. We distinguish two cases.

Case (i). For some $I \subsetneq \{0,1,\ldots,m\}$, there is equality in (1). Then it is easily verified that both $\{S_i,\alpha_i\}_{i\varepsilon I}$ and $\{S_i,\alpha_i\}_{i\notin I}$ satisfy (1) and (2), and we may apply the induction hypothesis separately to each one of them.

Case (ii). For all $I \subsetneq \{0,1,\ldots,m\}$, there is strict inequality in (1). In particular, $\nu(S_o) > \alpha_o$ and $\nu(S_o \setminus \bigcup_{i\neq o} S_i) < \alpha_o$. Since ν is non-atomic, we may find an S_o' such that

$$S_o \setminus \bigcup_{i\neq o} S_i \subset S_o' \subset S_o$$

and the replacement of S_o by S_o' will preserve all inequalities in (1), but at least one of them will be an equality. Clearly, (2) is still valid, and we may proceed as in case (i).

Q.E.D.

Proof of Theorem 1

Let $f \,\varepsilon\, \mathcal{L}_{\phi_1}$ and let $\varepsilon > 0$. We must find $g \,\varepsilon\, \mathcal{L}_{\phi_2}$ such that $\rho(D_f,D_g) < \varepsilon$.

Since f is integrable, there is a compact set K in $\mathbb{R}^\ell$ such that $\nu_1(f^{-1}(K^c)) < \varepsilon$. Let $K_o = K^c$ and let $K_1, K_2,\ldots,K_m$ be a partition of K into disjoint Borel sets of diameter less than ε.

Define $\alpha_i = \nu_1(f^{-1}(K_i))$ and $S_i = \phi_2^{-1}(K_i)$ for $i = 0,1,\ldots,m$.

Since ϕ_1 and ϕ_2 are equally distributed it follows that for every $I \subset \{0,1,\ldots,m\}$

$$\nu_1(\phi_1^{-1}(\bigcup_{i\varepsilon I} K_i)) = \nu_2(\phi_2^{-1}(\bigcup_{i\varepsilon I} K_i)) = \nu_2(\bigcup_{i\varepsilon I} S_i).$$

The K_i's are disjoint, hence

$$\nu_1(\phi_1^{-1}(\bigcup_{i\varepsilon I} K_i)) \geq \nu_1(f^{-1}(\bigcup_{i\varepsilon I} K_i)) = \sum_{i\varepsilon I} \nu_1(f^{-1}(K_i)) = \sum_{i\varepsilon I} \alpha_i .$$

We now have

(1) $\nu_2(\bigcup_{i\varepsilon I} S_i) \geq \sum_{i\varepsilon I} \alpha_i$, for all $I \subset \{0,1,\ldots,m\}$, and

(2) $\nu_2(\bigcup_{i=o}^{m} S_i) = 1 = \sum_{i=o}^{m} \alpha_i$,

therefore we may apply Lemma A to get a partition $\{T_i\}_{i=o}^{m}$ of A_2 such that $T_i \subset S_i = \phi_2^{-1}(K_i)$ and $\nu_2(T_i) = \alpha_i$.

Define $g|_{T_i}$ to be an integrable selection of the correspondence $\phi_2 \cap K_i$; since ϕ_2 is integrably bounded, this selection is possible

[H, D.II.4, Theorem 2]. Then $g \in \mathcal{L}_{\phi_2}$ and it is easy to verify that $\rho(D_f, D_g) < \varepsilon$.

Q.E.D.

Proof of Theorem 2.

Let $x \in \int\phi_1$ and let $f \in \mathcal{L}_{\phi_1}$ be such that $\int f = x$; by Theorem 1 there is a sequence $\{g_n\}$ converging in distribution to f, $g_n \in \mathcal{L}_{\phi_2}$. Since ϕ_2 is integrably bounded, $\int g_n \to \int f = x$ [B, Theorem 5.4]. Therefore x is in the closure of $\int\phi_2$. But $\int\phi_2$ is compact [A, Theorem 4], and the proof is completed.

Q.E.D.

Proof of Theorem 3

Let $C = \int\phi_1 = \int\phi_2$ (by Theorem 2). C is convex and compact [A, Theorems 1 and 4].

We proceed by induction on the dimension of C in $\mathbb{R}^\ell$. If $\dim(C) = 0$, then the theorem follows at once from Theorem 1.

Next, suppose $\dim(C) = n$.

Let $x \in C$ and let $f \in \mathcal{L}_{\phi_1}$ be such that $x = \int f$. We must find a $g \in \mathcal{L}_\phi$ such that $x = \int g$ and $\rho(D_f, D_g) < \varepsilon$.

<u>Case (i):</u> $x \in$ rel-int C. Let $r > 0$ be the distance of x from the boundary of C. Applying theorem 1 and the integrable boundedness of ϕ_2, we can find $g' \in \mathcal{L}_{\phi_2}$ such that $\rho(D_f, D_{g'}) < \varepsilon/2$ and

$$|x - \int g'| < r \cdot \varepsilon/2.$$

Let y be the intersection of the boundary of C with the half line from $\int g'$ to x, and let $g'' \in \mathcal{L}_{\phi_2}$ be such that $\int g'' = y$. Clearly, $x = \alpha y + (1-\alpha)\int g'$, where $\alpha < \varepsilon/2$. Applying Lyapunov's theorem [L], we obtain a set $S \subset A_2$ satisfying $\nu_2(S) = \alpha$, $\int_S g' = \alpha \int g'$ and $\int_S g'' = \alpha \int g''$. Define g by

$$g(a) = \begin{cases} g'(a), & a \notin S, \\ g''(a), & a \in S, \end{cases}$$

then clearly $\int g = x$ and $\rho(D_f, D_g) < \varepsilon$.

<u>Case (ii):</u> $x \notin$ rel-int C. Let q define a supporting hyperplane such that $q \cdot x = \max q \cdot C$.

Denote

$$\hat{\phi}_i(a) = \{y \in \phi_i(a) | q \cdot y = \max q \cdot \phi_i(a)\}, \quad \text{for} \quad a \in A_i, \quad i = 1,2.$$

By [H, D.II.3, Proposition 3], $\hat{\phi}$ have measurable graphs. Also, $\hat{\phi}_i$ are integrably bounded and closed-valued, and $f \in \mathcal{L}_{\hat{\phi}_1}$ (by [H, D.II.4, Proposition 6]).

We now claim that $\hat{\phi}_1$ and $\hat{\phi}_2$ are equally distributed. To see this, apply* theorem 4 to get decompositions $\phi_i = \psi \circ h_i$; then $\hat{\phi}_i = \hat{\psi} \circ h_i$, where

$$\hat{\psi}(t) = \{y \in \psi(t) | q \cdot y = \max q \cdot \psi(t)\}, \quad \text{for} \quad t \in [0,1].$$

But $\dim (\int \hat{\phi}_1) < n$, hence by the induction hypothesis there is a $g \in \mathcal{L}_{\hat{\phi}_2} \subset \mathcal{L}_{\phi_2}$ such that $x = \int g$ and $\rho(D_f, D_g) < \varepsilon$.

Q.E.D.

In the proof of Theorem 4 we will use the following lemma:

---

* the proof of theorem 4 does not depend on theorem 3.

<u>Lemma B.</u> Let $\{S_i\}_{i \in I}$ be a finite collection of sets in a measure space $(A, \mathcal{A}, \nu)$. For every $J \subset I$, the measure $\nu(\bigcup_{i \in J} S_i)$ is given.

Then, for every $J \subset I$, the measure

$$\nu(\bigcap_{i \in J} S_i \setminus \bigcup_{i \notin J} S_i)$$

is uniquely determined.

<u>Proof.</u>

It is easily verified that the following system of linear equations is non-singular:

$$\sum_{K \subset J} \nu(\bigcap_{i \in K} S_i \setminus \bigcup_{i \notin K} S_i) = \nu(\bigcup_{i \in J} S_i) \;, \quad \text{all} \quad J \subset I.$$

Q.E.D.

<u>Proof of Theorem 4.</u>

Let $\phi : (A, \mathcal{A}, \nu) \to \mathbb{R}^\ell$ be a closed-valued correspondence on a non-atomic probability measure space. We will show a method of constructing a mapping $h : (A, \mathcal{A}, \nu) \to ([0,1], \mathcal{L}, \lambda)$ and a correspondance $\psi : ([0,1], \mathcal{L}, \lambda) \to \mathbb{R}^\ell$, such that D_h and ψ depend <u>only</u> on the distribution of ϕ, and $\phi = \psi \circ h$. By applying this same method of construction to both ϕ_1 and ϕ_2, one obtains decompositions $\phi_i = \psi_i \circ h_i$ such that D_{h_i} and ψ_i depend only on D_{ϕ_i} (for $i = 1,2$). Since ϕ_1 and ϕ_2 are equally distributed, it follows that ψ_1 and ψ_2 must coincide, and that h_1 and h_2 have the same distribution; this will complete the proof.

Let ϕ be as above. Let $\mathcal{K} = \{K_i\}_{i\varepsilon I}$ be any finite collection of sets in $\mathbb{R}^\ell$. Define $\mathcal{C} = \mathcal{C}(\mathcal{K}) = \{C_J\}_{J\subset I}$ by setting

$$C_J = \bigcap_{i\varepsilon J} \phi^{-1}(K_i) \setminus \bigcup_{i\notin J} \phi^{-1}(K_i).$$

The measures of all sets $\bigcup_{i\varepsilon J} \phi^{-1}(K_i) = \phi^{-1}(\bigcup_{i\varepsilon J} K_i)$ are determined by D_ϕ. By Lemma B, the measures of all the C_J's are also determined by D_ϕ.

Let $\mathcal{K}$ be a finite partition of $\mathbb{R}^\ell$, then $\mathcal{C} = \mathcal{C}(\mathcal{K})$ is a finite partition of A. It is then possible to construction a partition $\mathcal{T} = \{T_J\}_{J\subset I}$ of $[0,1]$, where T_J is an interval closed from the left, such that $\lambda(T_J) = \nu(C_J)$.

Define the correspondence $\psi = \psi(\mathcal{K})$ on the unit interval by

$$\psi|_{T_J} = \bigcup_{i\varepsilon J} K_i ,$$

and let $h = h(\mathcal{K})$ be a measure-preserving function from A to $[0,1]$ such that C_J is mapped onto T_J for all $J \subset I$.

For every n, divide the set $E_n = \{x \in \mathbb{R}^\ell \mid |x^i| \leq n, i = 1,\ldots,\ell\}$ into $(2n\cdot 2^n)^\ell$ disjoint cubes of edge 2^{-n}. Let $\mathcal{K}_n$ be the partition of R^ℓ consisting of all those cubes and the complement of E_n. Let $\mathcal{C}_n = \mathcal{C}(\mathcal{K}_n)$, $\mathcal{T}_n = \mathcal{T}(\mathcal{K}_n)$, $\psi_n = \psi(\mathcal{K}_n)$ and $h_n = h(\mathcal{K}_n)$, and define $\phi_n = \psi_n \circ h_n$.

Clearly $\mathcal{K}_{n+1}$ is finer than $\mathcal{K}_n$; therefore $\mathcal{C}_{n+1}$ is finer than $\mathcal{C}_n$, and $\mathcal{T}_{n+1}$ may be chosen such that it will be finer than $\mathcal{T}_n$.

Since ϕ is closed-valued and since the diameter of the cubes in $\mathcal{K}_n$ converges to zero, it may be verified that $\phi_n(a) \searrow \phi(a)$ for all $a \in A$.

Next, we will show that the $\{h_n\}$ can be chosen to be a pointwise converging sequence.

Let $a \in A$, then for every n there is $C_n = C_n(a) \in \mathcal{C}_n$ such that $a \in \bigcap_{n=1}^{\infty} C_n$. We distinguish two cases:

Case (i). $\nu(\bigcap_{n=1}^{\infty} C_n) > 0$. Since ϕ_n is constant on C_n, it follows that ϕ is constant on $\bigcap_{n=1}^{\infty} C_n$. In this case, we redefine $\{h_n\}$ on $\bigcap_{n=1}^{\infty} C_n$ by setting $h_n \equiv h_1$ there (clearly, each such change does not affect the measure preserving property, and there are at most countably many such changes).

Case (ii). $\nu(\bigcap_{n=1}^{\infty} C_n) = 0$. Let $T_n = h_n(C_n)$, then $\lambda(\bigcap_{n=1}^{\infty} T_n) = 0$ and therefore $h_n(a) \in T_n$ is a converging sequence (recall that $\{T_n\}$ is a decreasing sequence of intervals).

In both cases, $\{h_n(a)\}$ is now a converging sequence; let $h(a)$ be its limit (for all $a \in A$).

For every $t \in [0,1]$, $\psi_n(t)$ is a decreasing sequence; let $\psi(t) = \lim_{n\to\infty} \psi_n(t)$.

It remains to prove that $\psi \circ h = \phi$ ν-almost everywhere. If, for all n large enough, $h(a)$ and $h_n(a)$ are in the same interval of $\mathcal{T}_n$, then (since ψ_n is constant on this interval)

$$\psi \circ h(a) = \lim_{n\to\infty} \psi_n(h(a)) = \lim_{n\to\infty} \psi_n(h_n(a)) = \lim_{n\to\infty} \phi_n(a) = \phi(a).$$

If not, then $t = h(a)$ must be the right end-point of the intervals $h_n(C_n(a))$ for all n large enough and a must belong to case (ii). But then $h^{-1}(t)$ has ν-measure zero, and the number of end-points t is countable, so $\psi \circ h \neq \phi$ on a null set.

Q.E.D.

Remark. Exactly the same proof shows that Theorem 4 is true for any collection of equally distributed correspondences.

ACKNOWLEDGEMENT

The authors wish to express their deep gratitude to Professor Werner Hildenbrand for introducing them to these problems and for many enlightning conversations.

R E F E R E N C E S

[A] Aumann, R.J. - Integrals of Set-Valued Functions. J. Math. Anal and Appl. 1965(12), 1-12.

[B] Billingsley, P. - Convergence of Probability Measures. J. Wiley & Sons, New York 1968.

[H-V] Halmos, P.R. and H.E. Vaughan - The Marriage Problem. Am. J. of Math. 1950 (72), 214-215.

[H-H-K] Hart, S., W. Hildenbrand and E. Kohlberg - Upper-Hemicontinuity in Distribution of the Walras Correspondence. Forthcoming.

[H] Hildenbrand, W. - Core and Equilibria in a Large Economy. Forthcoming.

[K] Kannai, Y. - Continuity Properties of the Core of a Market. Econometrica 1970(38), 791-815.

[L] Lyapunov, A. - Sur les Fonctions-Vecteurs Complement Additivies.
Bull. Acad. Sci. USSR, Ser. Math. 1940(4), 465-478.