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ON VALUES, LOCATION CONFLICTS, AND PUBLIC GOODS

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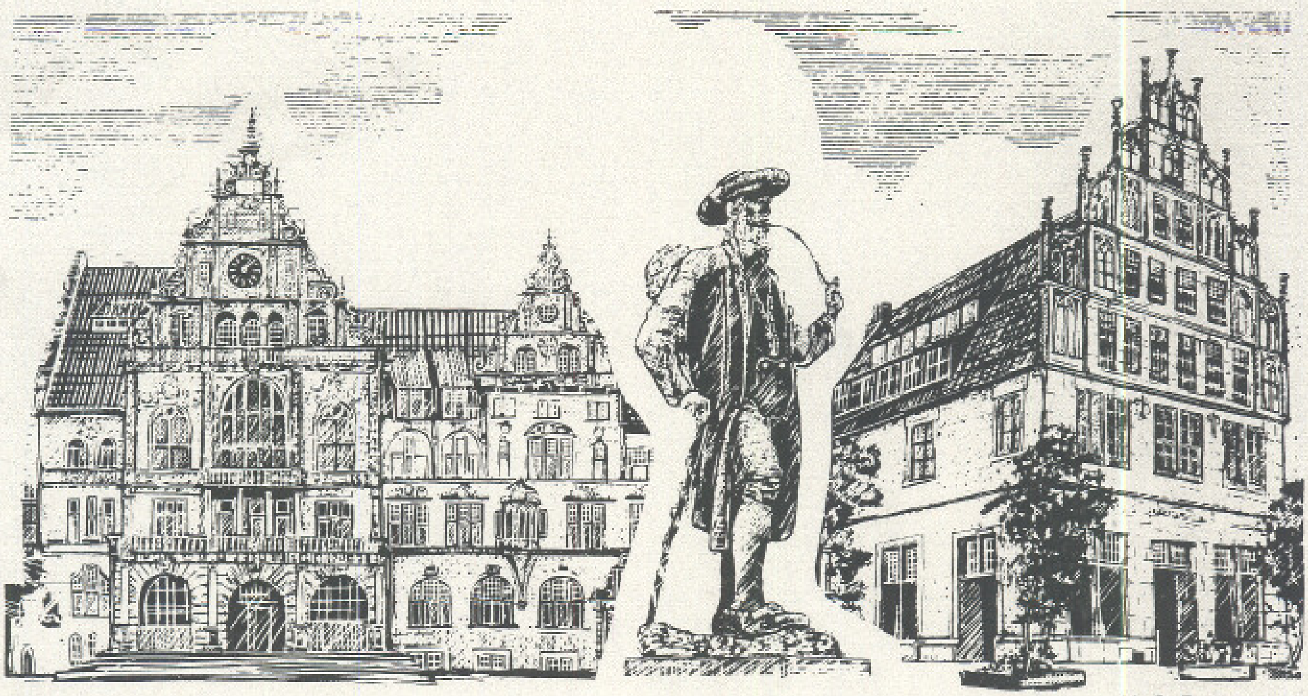

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SECTION 0

INTRODUCTION

General Remarks

The purpose of this paper is to define a concept of equilibrium for eoconomies with public goods and production. Generally speaking, this concept represents an attempt to generalize the notion of the Lindahl-equilibrium while simultaneously the idea of the "fair value", as defined within the framework of Game Theory, is introduced into the model. Also we are dealing with the question which kind of possible taxation policies are feasible in order to admit our notion of equilibrium.

As the production mechanism of our economy ist just given by an aggregate production set, the taxation mechanism may be viewed just as a method to decentralize the decision as to which bundle of public goods should be produced. In fact, there are two quantities that can be viewed as externally imposed institutional financing machanism: The value (in the sense of Cooperative Game Theory) and the taxation structure or policy. These notions will interact as follows.

As the consumer pays taxes towards the consumption of the public goods, his budget set, given prices for private goods, is restricted. Hence, if it so happens that for large bundles of public goods the marginal cost of providing these goods for the consumer exceeds his marginal utilities for the private goods, then his preference for public goods eventually will decrease. This idea is discussed e. g. in the paper of ZECKHAUSER-WEINSTEIN [23]. It will then happen that, given prices for private goods, the consumer will have "satiation points", "bliss points", with respect to the public goods. However, in general these satiation points will vary with different consumers, and hence the question occurs which public bundle actually should be produced.

The Lindahl equilibrium concept can be interpreted as to answer this question as follows: Taxes can be set up linearly and individually for each consumer such that the consumers do have a common bliss point while simultaneously maximizing their utility within their budget constraints. In our context, however, it is suggested that different policies of taxations are considered; for instance taxation might be linear and equal for each consumer. In this case we cannot expect that consumers have a common "bliss point". This is where the game theoretical value enters the picture as has been discussed in several recent papers (OSTMANN [12],[13], RICHTER [14],[15],[16], ROSENMÜLLER [18],[19]). Values, as discussed in Game Theory, can be redefined or newly defined in conflict situations that do not feature the usual game theoretical assumption of a "threat point" but rather enjoy the property of having "bliss points". Such situations typically occur in what we would like to call "location conflicts": This is a structure which features utility functions having a bliss point that is usually different for all players involved. The value is then a mapping which assigns fair solutions or "fair locations" to a certain class of bliss point problems or "location problems".

As our analysis of the underlying economy leads to a system where the consumers do have different bliss points with respect to the utility of public goods, it is suggested that a notion of the fair value as developed in [13],[16],[19] might be applied to the corresponding location problem. This then can be seen as adding the notion of "fairness" to the institutional financing mechanism. Hence the society has the possibility of choosing a certain value concept which is then applied to the location problem resulting from the utilities as defined for public goods. This yields what one could call the "fair bundle of public goods". Prices for private goods and taxes (within the constitutional framework) should then be set in a way such that every consumer maximizes utility within his budget set with respect to the private goods. Moreover, the "fair bundle" is produced and finally the result is feasible in a sense that it can be obtained by the underlying production mechanism. As it turns out in most cases (and in particular in the case of linear but equal taxation as considered here) that production is effective in the sense that it takes place at the boundary of the aggregate production set provided the production structure is of constant returns to scale. However, typically it will happen that the distribution of private goods cannot be identified to be Pareto optimal; that is, the advocated equilibrium concept is a "second best model" or, to use a phrase of [23], a "mechanism constrained Pareto optimum".

In other words, if the society chooses to have the notion of justice enter via introduction of a "value" and a "fair tax system", then it usually will have to pay for it with a non-Pareto-optimal solution. On the other hand, the Lindahl equilibrium is an example where the taxation policy admits of a Pareto optimal solution - but in the framework of our concept cannot be called a fair taxation policy.

It should be noted that our present model also bears a resemblance to certain set-ups which appear within the framework of "voting equilibria", "voting market equilibria", etc. In this context, we restrain ourselves to citing only a few relevant papers, e. g. McKELVEY-WENDELL [9], SLUTSKY [22], and DENZAU-PARKS [5]. Conceivably, a "voting equilibrium" is a "value" in the sense of Section 2 and thus, our general existence theorem may have implications with respect to the existence problem treated in [22], [5].

However, the general version of a value (in the sense of Game Theory, or as defined in Section 2) respects the shape of the players' utilities more generally compared to "voting in finitely many directions". Also, we prefer the taxation functions not to be fixed in advance, the relation, say, between the equilibrium treated in [22] and the present model is not quite obvious. At this state of affairs we prefer to postpone this question to a possible later treatment.

In what follows Sec tion 1 gives a short introduction and specifies the conditions to be imposed upon utilities and taxations. Section 2 will recall some basic features of location conflicts as defined in [14], [16]. Finally Section 3 will define the equilibrium concept and give proofs for existence.

Notations

Vectors $x \in \mathbb{R}^m \times \mathbb{R}$ will be written $x = (\overset{\circ}{x}, x_{m+1}) = (\overset{\circ}{x}, \xi)$ where $\overset{\circ}{x} \in \mathbb{R}^m_+$ and $x_{m+1} = \xi \in \mathbb{R}$. Similarly, if

$$\mathbb{P}^{m+1} := \{p \in \mathbb{R}^{m+1} \mid p \geq 0, \sum_{i=1}^{m+1} p_i = 1\}$$

denotes the "price simplex", then $p \in \mathbb{P}^{m+1}$ is written $p = (\overset{\circ}{p}, p_{m+1}) = (\overset{\circ}{p}, \rho)$.

Symbols e^i are reserved for the i'th unit vector while $e = (1,..,1)$.

$| \quad |$ is the Euclidian norm, α^+ denotes $\max(\alpha, 0)$ (for $\alpha \in \mathbb{R}$). As for vectors, x^+ is defined coordinate-wise. If $f: T \to \mathbb{R}$ is a function, then $M_{T_0} f$ denotes the maximizers of f with respect to $T_0 \subseteq T$.

$S_r(x)$ denotes the sphere with center x and radius r.

SECTION 1

ECONOMIES

General Definitions

An economy (with public goods and production) is a tupel

$\mathfrak{M} = (\Omega, X \times \mathbb{R}^l_+, U, A, b, Y)$

where the quantities are specified as follows.

$\Omega = \{1,...,n\}$ represents the set of players (traders, agents, consumers) and $X = \mathbb{R}^m_+ \times \mathbb{R}$ is the system of bundles of private goods. Private good $m+1$ appears in negative quantities and has certain aspects of a booking account; thus players may be willing to accept negative quantities of $m+1$ in exchange for goods of other types. $\mathbb{R}^l_+$ represents bundles of public goods.

Next, $U = (u^i)_{i \in \Omega}$, where $u^i : = X \times \mathbb{R}^l_+ \to \mathbb{R}$ $(i \in \Omega)$ is the family of utility functions of the players. Each u^i is assumed to be continuous, strictly monotone, and concave.

$A = (a^i)_{i \in \Omega}$, $0 < a^i \in X$ $(i \in \Omega)$ indicates the players' initial holdings of private goods, while $b \in \mathbb{R}^l_+$ is the economy's initial endowment of public goods.

Finally, $Y \subseteq \mathbb{R}^{m+1}_- \times \mathbb{R}^l_+$ is the aggregate production set. We assume that Y is closed, convex, and contains 0. Moreover, if $(x,y) \in Y$ and $x' \in \mathbb{R}^{m+1}$, $x' \leq x$, then $(x',y) \in Y$ ("free disposal"). Thus, in particular, $(x,0) \in Y$ for all $x \in \mathbb{R}^{m+1}_-$ is required. In addition,

$$\{(x,y) \in Y \mid x \geq x'\}$$

is assumed to be compact for every $x' \in \mathbb{R}^{m+1}_+$.

If $(x,y) \in \mathcal{Y}$ is such that $(x',y') \geq (x,y)$, $(x',y') \in \mathcal{Y}$ implies $x'=x$, $y'=y$, then we shall say that (x,y) is efficient.

For technical reasons, it will be necessary to have truncated versions of $\mathbb{M}$. Thus, let k be an integer such that

(1) $\sum_{i\in\Omega} a^i \leq ke$.

Define

(2) $\;{}^{o}\mathcal{X}^k := \{x = (\overset{o}{x},\xi) \in \mathbb{R}_+^{m+1} \mid x \leq ke\}$

(3) $\;{}^{k}\mathcal{X}^k := \{x = (\overset{o}{x},\xi) \in \mathbb{R}_+^{m} \times \mathbb{R} \mid \overset{o}{x} \leq ke,\ -k \leq \xi \leq k\}$

(4) $\;{}^{k}\mathcal{Z}^k := \{x = (\overset{o}{x},\xi) \in \mathbb{R}^{m} \times \mathbb{R} \mid -ke \leq x \leq ke\}$

(5) $\;\mathcal{Y}^k := \{(x,y) \in \mathcal{Y} \mid -x \in {}^{o}\mathcal{X}^{nk}\}$

(6) $\;\underline{\bar{Y}}^k := \{y \in \mathbb{R}^{l+} \mid \exists\, x \in {}^{o}\mathcal{X}^{nk} : (-x,y) \in \mathcal{Y}^k\} = \mathrm{Proj}_2 \mathcal{Y}^k$

Then

(7) $\;\mathbb{M}^k := (\Omega, {}^{k}\mathcal{X}^k \times \underline{\bar{Y}}^k, U, A, b, \mathcal{Y})$ is the

k-truncated version of $\mathbb{M}$.

Note that, by our requirements, $\mathcal{X}^k$, $\underline{\bar{Y}}^k$, and $\mathcal{Y}^k$ are compact sets. Also, u^i $(i \in \Omega)$ should be restricted to $\mathcal{X}^k \times \underline{\bar{Y}}^k$ in (7), however, we shall not introduce a further symbol.

As has been said, commodity $m+1$ plays a particular role as it is available in large negative quantities; in fact we are tempted to call it "fiat money". (In particular our results should be applicable in the case of a "transferable utility market" as introduced by SHAPLEY [20] and extensively treated e. g. in AUMANN/SHAPLEY [4], AUMANN [3], and similar papers.)

However, we want the consumer to be aversive against high debts in terms of "money" compared with positive quantities of the non-monetarian private goods. More exactly, if he pays proportionally to private goods in terms of the monetarian good, then given any proportional constant, the consumer should not be willing to buy arbitrarily large amounts of private non-monetarian goods in return for "money". This is a requirement to the utility functions as expressed by the following definition. In fact, mathematically it is necessary to insure that maximizing within someones budget constraints is a reasonable concept.

Definition 1.1: u^i is said to be admissible if, for $y \in \mathbb{R}^l_+$, $t > 0$, there exists $r > 0$ such that

(8) $$u^i(\chi, -t|\chi|, y) < u^i(0, 0, y) \text{ whenever } |\chi| \geq r.$$

$\mathfrak{M}$ is admissible if u^i is admissible for $i \in \Omega$.

Given prices for private goods, say $p \in \mathbb{P}^{m+1}$ and a bundle $y \in \mathbb{R}^l_+$, consumer i, having some wealth $w \in \mathbb{R}$ available, wants to maximize his utility within his budget constraints. Indeed, we have

Lemma 1.2: If u^i is admissible, then, for $p \in \mathbb{P}^{m+1}$, $y \in \mathbb{R}^l_+$, $w \in \mathbb{R}$

$$\max \{u^i(x,y) \mid x \in X, px \leq w\}$$

exists whenever $p > 0$.

The proof is an easy exercise:

If $|x|$ is sufficiently large and w.l.o.g. $\chi \neq 0$, then

$$px \leq w, \quad \beta\chi + \rho\xi \leq w$$

implies

$$\xi \leq \frac{w - \beta\chi}{\rho} \leq -t|\chi| \qquad \text{for a suitable } t > 0.$$

Hence

$$u^i(x,y) = u^i(\overset{\circ}{x},\xi,y)$$
$$\leq u^i(\overset{\circ}{x},-t|\overset{\circ}{x}|,y) < u^i(0,0,y) .$$

For fixed prices of private goods, the players will have utility functions defined on public goods and wealth via

<u>Definition 1.3</u>: Let u^i be admissible.
Then, for $p \in \mathbb{P}^{m+1}$, $p > 0$

$$\overset{\circ}{u}^{pi} : \mathbb{R}^l_+ \times \mathbb{R} \to \mathbb{R}$$

is defined by

(9) $$\overset{\circ}{u}^{pi}(y,w) := \max \{u^i(x,y) \mid x \in \mathcal{X},\ px \leq w\} .$$

Similarly, for $p \in \mathbb{P}^{m+1}$ and k sufficiently large,

$$\overset{kpi}{u} : \bar{\underline{Y}}^k \times [-p_{m+1}k,\infty) \to \mathbb{R}$$

is defined by

(10) $$\overset{kpi}{u}(y,w) := \max \{u^i(x,y) \mid x \in {}^k\mathcal{X}^k, px \leq w\}.$$

Of course, the existence of (10) is trivial while Lemma 1.1.2 ensures the existence of (9).

Taxation

We have so far discussed properties of the underlying economy. Let us now turn to the taxation structure. Generally speaking, taxation is used as a means to decentralize the decision of producing a certain bundle of public goods. Hence the players will be asked to pay a certain fee towards the production of public goods. This in turn will restrict their budget possibilities. We shall assume that taxation functions will provide increasing marginal costs for the players.

<u>Definition 1.4</u>: A <u>taxation function</u> is a continuous, strictly monotone, and convex mapping

$$c : \mathbb{R}^l_+ \to \mathbb{R} .$$

If c^i is a taxation function ($i \in \Omega$), then $C = (c^1,\ldots,c^n)$ is a <u>taxation scheme</u>.

Let $\mathbb{C} := \{C \mid C \text{ is a taxation scheme}\}$, then $\mathbb{C}^0 \subseteq \mathbb{C}$ is called a <u>taxation policy</u>.

By introducing the notation of a "taxation policy" as a subset of the possible taxation schemes we imagine that the society has certain notions in advance as to which tax schemes are in some sense "feasible" by customs or institutional restrictions. This is a decision which enters the picture externally. It concerns the type of taxes that are to be applied. Later on we will then discuss the welfare implications of a certain taxation policy. For instance, one might face the institutional restriction that the individual tax burden should be proportional to the output level of public goods without discriminating between individuals. This obviously defines a certain specified taxation policy. Another decision would be that the taxation policy constists of linear functions possibly differing among the players. A taxation policy is based upon reasoning which is not inherent to the economical structure. It is imposed upon the economy in order to find a means for allocation private goods and supplying a certain bundle of public goods.

<u>Example 1.5</u>

(11)
$$\mathbb{Q} := \{C = (c^1,\ldots,c^n) \mid \Omega \ni i \ \exists\ q^i \in \mathbb{R}^l_+,\ \alpha_i \in \mathbb{R} : c^i(y) = q^i y + \alpha_i\} \subseteq \mathbb{C}$$

is the <u>affine taxation policy</u>. If $\mathbb{Q}$ is adopted, then players pay towards public goods proportionally to the size of the various public commodities. However, the proportionality constants might differ among the players. Within the framework of the Lindahl equilibrium (see LINDAHL [8] or MILLERON [10] for a survey) the "Lindahl prices" constitute a particular example of an "affine taxation scheme".

Here we have $\alpha_i = 0$ while q^i represents the individual marginal rates of substitution of private for public goods. As it turns out, α_i is of no significance; we shall write "q^i" instead of "c^i" whenever $\mathbb{Q}$ is adopted.

Example 1.6

(12)
$$\mathbb{Q}^o := \{C = (c^1, \ldots c^n) \mid \exists\, q \in \mathbb{R}^l_+, \alpha \in \mathbb{R}: c^i(y) = qy + \alpha \quad (y \in \mathbb{R}^l_+ \ , \ i \in \Omega)\} \subseteq \mathbb{C} \ .$$

We shall write "q" instead of "c^i" whenever $\mathbb{Q}^o$ is adopted.

The next step apparently is common in the literature, see e. g. ZECKHAUSER-WEINSTEIN [23] or SLUTSKY [22]. Given a taxation scheme C, the consumer take private goods prices p and the provision of public goods y for granted. He then evaluates his budget constraints to be

$$w = pa^i + c^i(b) - c^i(y) \quad .$$

Thereafter he is going to maximize his utility with respect to the private goods only. This maximization procedure allows us to eliminate private goods completely from consideration; thus we are capable of deriving the utility of the public goods bundle y for a fixed taxation scheme C. Hence we have the following definition.

Definition 1.7: Let $\mathbb{M}$ be an admissible economy and C a taxation system, s. t. $c^i(b) \geq 0 \ (i \in \Omega)$. Given $p \in \mathbb{P}^{m+1}$, $p > 0$, the function
$$\hat{u}^i := \hat{u}^{pc^i i} : \mathbb{R}^l_+ \to \mathbb{R}$$
(the "derived utility function") is defined by

(12)
$$\hat{u}^i(y) = \overset{o}{u}{}^{pi}(y, pa^i + c^i(b) - c^i(y)) = \max \{u^i(x,y) \mid x \in X, px \leq pa^i + c^i(b) - c^i(y)\} .$$

Moreover, let

(13) $\hat{X}^i(p,c^i,y) := \{x \in X \mid px \leq pa^i + c^i(b) - c^i(y),\ u^i(x,y) = \hat{u}^i(y)\}$

denote the <u>maximizing private bundles</u> given prices p, taxation and a public bundle. Then $\hat{X}(\cdot,\cdot,\cdot)$ is a correspondence, which, given a suitable topology on $\mathbb{C}$, might turn out to be u.h.c at certain arguments p, c^i, y.

<u>Definition 1.8</u>: Let $\mathbb{M}$ be an economy and C a taxation system. Given $p \in \mathbb{P}^{m+1}$ and k sufficiently large, define

(14) $^kB^i := \bar{\underline{Y}}^k \cap \{y \in \mathbb{R}^l_+ \mid c^i(y) \leq pa^i + c^i(b) + p_{m+1}k\}$

and a function

$$^k\hat{u}^i(y) := {}^k\hat{u}^{pc^i i} : {}^kB^i \to \mathbb{R}$$

by

(15) $^k\hat{u}^i(y) := {}^k_{u}{}^{pi}(y, pa^i + c^i(b) - c^i(y))$

$$= \max\{u^i(x,y) \mid x \in {}^kX^k,\ px \leq pa^i + c^i(b) - c^i(y)\}.$$

Moreover let, for $y \in \bar{\underline{Y}}^k$

(16) $$^k\hat{X}^i(p,c^i,y) := \begin{cases} \{x \in {}^kX^k \mid px \leq pa^i + c^i(b) - c^i(y),\ u^i(x,y) = {}^k\hat{u}^i(y)\} & \text{for } y \in {}^kB^i \\ \{x \in {}^kX^k \mid px \leq -kp_{m+1},\ u^i(x,y) = {}^o_{u}{}^{pi}(y, -kp_{m+1})\} & \text{otherwise} \end{cases}$$

It should be noted that, for large bundles y of public goods, $c^i(y)$ might increase as to render the budget set

$$\{x \in {}^kX^k \mid px \leq pa^i + c^i(b) - c^i(y)\}$$

to be empty. However, for $y \in {}^kB^i$, clearly the private bundle (0,k) is within this budget constraints.

Once a taxation system has been chosen and prices for private goods are fixed, the consumer values public goods according to how much utility he will gain by maximizing w. r. t. the private goods given his imposed budget constraints; this is expressed by $\hat{u}^i$. Certain properties of u^i are at once carried over to the derived utility functions.

<u>Theorem 1.9</u>: $\hat{u}^{pi}$ is continuous, strictly monotone, and concave.
$\hat{u}^{pc^i i}$ is continuous and concave.
The same holds true for the k-truncated version.

The <u>proof</u> is an easy exercise. See also [23], where a proof is indicated for the case of one private and one public good as well as linear taxation. However, arbitrarily many private and public goods and a generally convex taxation are feasible.

As has been observed by ZECKHAUSER-WEINSTEIN [23] we cannot expect that the derived utility functions $\hat{u}^i$ are in addition monotone. However, the remark in their paper that the tax payers' bliss point is achieved at an internal location where the marginal cost to him of providing more of each public good just equals his valuation of a marginal unit cannot be seen as a statement or theorem.

In fact it is rather a condition concerning the utility functions although it is a very reasonable one. Of course the properties of a bliss point ought to be defined in terms of marginal utilities (partial derivatives of the utility functions), that is, by some version of ,,Gossen's Law". However, its <u>existence</u> is a requirement concerning the <u>global</u> behaviour of the utilities "far outside".

Intuitively we are to require that, given prices p and a taxation system, the marginal increase of utility of large public bundles will <u>eventually</u> become small compared to marginal costs of the sacrifices that are necessary in order to provide these bundles. As we do not want to introduce differentiability conditions at this stage of affairs, we shall choose a rather different formulation in order to ensure that the players enjoy bliss points with respect to the derived utility functions.

Since commodity $m+1$ plays a separate role, we shall just ensure that the consumer facing large bundles of public goods for which he has to pay by the taxation system as well as large bundles of non-monetarian private goods for which he has to pay proportionally will eventually have decreasing utility if his debts are to appear in his money coordinate.

More precisely, we have

Definition 1.10: Let $\mathfrak{M}$ be an economy and C a taxation system. (u^i, c^i), $(i \in \Omega)$, is said to be compatible if the following condition holds true:

For $w \in \mathbb{R}$, $s > 0$ there is $R > 0$ such that

(17) $$u^i(\overset{\circ}{x}, w - c^i(y) - s|\overset{\circ}{x}|, y) < u^i(0,0,0)$$

whenever $|(\overset{\circ}{x}, y)| \geq R$.

$\mathfrak{M}$ and C are compatible if (17) holds true for $i \in \Omega$.

It is not hard to see that, given $p \in \mathbb{P}^{m+1}$, $p > 0$ and u^i admissible, it follows that $\hat{u}^{pc^i i}$ has maximizers within some compact set and only there. For the purpose of a later existence theorem we shall, however, rephrase our definition for the case of linear taxation and prove somewhat more.

Definition 1.11: An economy $\mathfrak{M}$ is said to be compatible with nearly linear taxation if, for $i \in \Omega$, $w \in \mathbb{R}$, $s,t > 0$, there is $R \geq 0$ such that

(18) $$u^i(\overset{\circ}{x}, w - t|y| - s|\overset{\circ}{x}|, y) < u^i(0,0,0)$$

whenever $|(\overset{\circ}{x}, y)| \geq R$.

Note that (0,0,0) is chosen for convenience, the definition may be reformulated such that the existence of any $(\overset{\circ}{\bar{x}},\bar{\xi},\bar{y})$ replacing (0,0,0) is required.

<u>Theorem 1.12</u>: Let $\mathbb{M}$ be an admissible economy, compatible with nearly linear taxation, and let $\mathbb{C}^{o} \subseteq \mathbb{C}$ be a taxation policy such that there is $\eta_{o} > 0$, $r_{o} > 0$ satisfying

(19) $$c^{i}(y) \geq \eta_{o} |y| \quad (C \in \mathbb{C}^{o},\ |y| \geq r_{o})$$

(20) $$c^{i}(b) = 0 \quad .$$

Then, for any $\varepsilon_{o} > 0$, there is $R_{o} > 0$ such that, for all $p \in \mathbb{P}^{m+1}$, $p > \varepsilon_{o} e$ and for all $C \in \mathbb{C}^{o}$ it follows that

(21) $$\emptyset \neq M_{\mathbb{R}^{1}_{+}}\, \hat{u}^{pc^{i}i} \subseteq S_{R_{o}}(0) \qquad (i \in \Omega) \quad .$$

That is, if prices are bounded away from zero and taxes behave nearly linearly, then maximizers fo the derived utility functions exist and are uniformly located within some compact set.

<u>Proof</u>

Choose $\varepsilon_{o} > 0$. Write $\bar{w} := \sum_{i\in\Omega} ea^{i}$ and let R_{o} be such that

(22) $$|y| > R_{o} \quad \text{implies} \quad c^{i}(y) > \eta_{o}|y|$$

as well as

(23) $$\eta_{o}|y| > \bar{w}$$

(24) $$u^{i}(\overset{\circ}{x},\ \bar{w}-\eta_{o}|\bar{y}|-\varepsilon_{o}|\overset{\circ}{x}|,\ \bar{y}) < u^{i}(0,0,0)$$

for $|(\overset{\circ}{x},y)| \geq R_{o}$; this is clearly possible by Definition 1.11.

Now, let $|\bar{y}| > R_{o}$. We are going to show that

$$\hat{u}^{pc^{i}i}(\bar{y}) < \hat{u}^{pc^{i}i}(0)$$

for all $p > \varepsilon_{o} e$ and $C \in \mathbb{C}^{o}$.

Indeed, if $x = (\overset{\circ}{x}, \xi^{\circ})$ is such that

$$px + c^i(\bar{y}) \leq pa^i + c^i(b)$$

then, observing (20)

$$\overset{\circ}{p}\overset{\circ}{x} + \rho\xi + c^i(\bar{y}) \leq pa^i \leq ea^i \leq \bar{w} \quad ,$$

i. e., using (22)

$$\xi \leq \frac{\bar{w} - c^i(\bar{y}) - \overset{\circ}{p}\overset{\circ}{x}}{\rho}$$

$$\leq \frac{\bar{w} - \eta_o|\bar{y}| - \varepsilon_o|\overset{\circ}{x}|}{\rho} \quad .$$

Hence, as the numerator is negative by (23),

$$\xi < \bar{w} - \eta_o|\bar{y}| - \varepsilon_o|\overset{\circ}{x}|$$

and, using (24),

$$u^i(\overset{\circ}{x}, \xi, \bar{y}) \leq u^i(\overset{\circ}{x}, \bar{w} - \eta_o|y| - \varepsilon_o|x|, \bar{y})$$
$$< u^i(0,0,0) \leq \hat{u}^i(0)$$

q. e. d.

<u>Remark 1.12</u>: Let $\mathfrak{M}$ be an admissible economy and let C be a taxation system such that $\mathfrak{M}$ and C are compatible. If $p \in \mathbb{P}^{m+1}$, $p > 0$, then

$$M_{\mathbb{R}^l_+} \hat{u}^{pc^i i} \neq \emptyset \quad (i \in \Omega) .$$

The proof is trivial.

SECTION 2

LOCATION CONFLICTS

Location Conflicts and Values

We are going to use some topics of the theory of location conflicts and their values as well as of games with "bliss points" as developed in recent papers by A. OSTMANN [13],[14], W. F. RICHTER [14],[15],[16], and the author [18],[19]. However, in order to make our treatment selfcontained, we shall just informally discuss the concept of location conflicts and their fair values, the latter term referring to the "value" as defined within the framework of Game Theory.

Suppose a planning agency has to consider the problem where to locate an attractive object (for instance a park, a swimming pool, a public library) given the location of n individuals (players, communities, cities) within the plain (or n-dimensional space) that are interested in the site of our object and are capable of expressing their interest in terms of a utility function which attaches a utility to each possible location of the object. Such a utility may or may not be proportional to the negative or inverse of the distance to the object. It is not unreasonable to expect that each player involved would prefer to have the object located as close as possible to his own location, in other words we expect the utilities to have satiation points (maximizers) which frequently might coincide with the players' locations.

Let us adopt a formalization: a location conflict is a tripel $\Sigma = (\Omega, B, U)$, where the data are defined as follows: $\Omega = \{1,..,n\}$ is the set of players, $B \subseteq \mathbb{R}^l$ is a convex closed subset of $\mathbb{R}^l$ which is called the planning area, $U = (u^i)_{i \in \Omega}$, $u^i : B \to \mathbb{R}$ is a family of continuous concave utility functions for the players such that $M_B u^i \neq \emptyset$.

Given a location conflict as defined, the planning agency faces the problem of finding a "fair location" or "fair value" of Σ. "Classical location theory" when dealing with such problems is frequently adopting a naive solution concept like minimizing the sum of the distances between individual players and the object to be located. However, we would rather consider this a problem which should be tackled by methods of Game Theory as has been done for instance in [14].

Formally this would mean that we consider cooperative games without sidepayments, that is, tripels $(\Omega,\underline{\underline{P}},V)$, where again $\Omega = \{1,...,n\}$, while $\underline{\underline{P}} = \mathcal{P}(\Omega)$ is the power set of Ω (meaning the coalitions) and $V : \underline{\underline{P}} \to \mathcal{P}(\mathbb{R}^n)$ is a mapping which attaches a set of feasible utility vectors to each coalition.

The mapping V should obey certain regularity conditions. For instance, it should take values which are closed convex and comprehensive subsets of $\mathbb{R}^n$ and it should be superadditive or the like. Now, clearly a location conflict Σ gives rise to a game $(\Omega,\underline{\underline{P}},V\ )$ where V is defined for instance by means of the formula:

$$V_\Sigma(S) = \text{comprehensive hull of } \{(u^i(x))_{i\in S} \mid x \in B\}$$
$$\text{(imbedded in } \mathbb{R}^n)$$

However, the game which is induced by the mapping $\Sigma \to V_\Sigma$ will in general <u>not</u> enjoy superadditivity. In fact, the usual interpretation of the mapping V_Σ as common in Game Theory cannot be maintained because we cannot assume that coalitions will be able to place the desirable object within the limits (convex hull) of their own locations. Rather one should adopt the idea, that the planning agency just considers the merits or demerits that coalitions would obtain if planning the object was restricted to the interests of such coalitions. This topic is discussed at length in [16].

In fact, the mapping $\Sigma \to V_\Sigma$ as defined above is not the only feasible one. Quite naturally, one might restrict the discussion to the case where only the utility vectors of the grand coalition and of the single player coalitions really matter. This means that we consider the set $V_\Sigma(\Omega)$ which gives the utilities of all players obtained by varying the object in the planning area and certain sets $V(\{i\})$ which define the utility of a single player if the object is to be located in his "position", that is, in a satiation point of his utility. Such a cooperative game without sidepayments is usually called a "pure bargaining game". However, it should be noted that there is no threatpoint at hand. Rather, this game exhibits a "bliss point". This is the vector $\underline{x}(V_\Sigma) \in \mathbb{R}^n$ (utility space) which is obtained by computing the utility of each player at his satiation point simultaneously, i. e.,

$$\underline{x}_i(V_\Sigma) = u^i(x) \qquad (x \in M_B(u^i)) \quad .$$

In general, the blisspoint is not feasible ($\underline{x}(V_\Sigma) \notin V_\Sigma(\Omega)$), and the planning agency considers the problem as to which point of the feasible set of the grand coalition would be a fair value given the bliss point of the game.

There are values for games without sidepayments which may easily be defined for non-superadditive games, that is for games with bliss points. Compare for instance SHAPLEY [20], HARSANYI [6], [7] and [15], [19] . As this is not our present topic it suffices to mention that such values in principle could be carried over as to define values for location conflicts; for instance by a formula like

$$\Psi(\Sigma) := \Psi(V_\Sigma)$$

where Ψ is a value which is defined for games without sidepayments featuring blisspoints.

On the other hand it seems sometimes reasonable to discuss values for location conflicts axiomatically. As it turns out one might argue that a value like the NASH bargaining solution (NASH [11]) might not be intuitively feasible for location conflicts. An axiomatic definition of values for location conflicts has been given in RICHTER [16]. This author adopts the viewpoint that defining fair values for location conflicts is rather a problem of Wellfare Theory and not so much of Game Theory. He then gives axioms and definitions of values and introduces a certain class of values which for instance are defined for games in the "pure bargaining" form, that is where only the grand coalition and the bliss point matters. Typically such a value would be obtained by minimizing the distance of the bliss point and some point on the Pareto surface of the feasible set of the grand coalition where the word "distance" has to be interpreted in the sense of a certain p-norm. It is important to note that such a value would enjoy in addition to the axiomatically stated properties also certain continuity properties which are to be used in our later treatment. Such contiuity properties will possibly not be attached to values for games with bliss points as defined for instance in [19]. It is quite possible that they enjoy the property of being upper hemicontinuous correspondences given the appropriate topology on location conflicts. At this stage of our analysis we shall not enter the problem of continuity but rather make this a requirement. For the following discussion we shall just assume that values for location conflicts are available and that they enjoy certain properties to be defined in the next section.

Fair distribution of Public Goods

Definition 2.1: Let $\mathfrak{T} := \{\Sigma = (\Omega, B, U)\}$ denote the set of location conflicts.

Given $\mathfrak{T}^{o} \subseteq \mathfrak{T}$, a value (for $\mathfrak{T}^{o}$) is a correspondence $\Psi : \mathfrak{T}^{o} \to \mathcal{P}(\mathbb{R}^{1})$ enjoying the following properties:

1. $\Psi(\Sigma) \subseteq B$.
2. Ψ preserves blisspoints.

   (i. e., if $M_{\Sigma} := \bigcap_{i \in \Omega} M_{B} u^{i} \neq \emptyset$, then $\Psi(\Sigma) = M_{\Sigma})$.
3. Ψ is finitely determined.

   (i. e., for any family $\mathfrak{T}' \subseteq \mathfrak{T}^{o}$ of location conflicts admitting of a compact set K such that $M_{B'} u^{i'} \subseteq K$ $(\Sigma' \in \mathfrak{T}')$, there is a compact convex set $\bar{K}$ such that $\bar{K} \supseteq K$ and

$u^{i'}|_{\bar{K}} = u^{i''}|_{\bar{K}} \quad (i \in \Omega \; ; \Sigma', \Sigma'' \in \Sigma')$

implies $\Psi(\Sigma') = \Psi(\Sigma'')$.

4. $\Psi(\Sigma) \subseteq$ convex hull of $\bigcup_{i \in \Omega} M_B u^i$.

The fourth condition is not extremely appealing. Preferably it should be replaced by Pareto optimality. There are clearly situations such that both requirements are easily satisfied (e. g., if the location conflict is one-dimensional, i. e. if $l = 1$, or if the utilities of the players enjoy some symmetry, i. e., if they are norms). On the other hand, in the case of two players and two dimensions of the planning area, the intersection of the contract curve and the convex hull might just contain the satiation points of the players. By technical reasons we shall impose the above convexity condition (cf. Theorem 3.8, second step) and leave Pareto optimality for later treatments. Further requirements possible are invariance under permutations of the players, invariance under linear rescaling of the utility functions and the like. However, every value should at least choose Pareto optimal points and, if bliss points are feasible at all, then t value should choose these feasible bliss points.

Let us now recall the results of Section I. We have seen that given suitable conditions, positive prices and an appropriate taxation system, the resulting derived utility functions are concave and enjoy satation points. This means that given prices and a taxation systems the market or economy induces a location conflict. Now let us assume that some planning agency, knowing the preference structure of the individuals with respect to the public goods and some "concept of fairness" Ψ , will compute some public good to be produced which might in the ideal case of course be a "bliss point" but usually will be something which is just "fair". The value to be chosen may be opted for by society in advance. What is left to the planning agency is just to compute the fair value given the location conflict that results from the market at present prices and tax structure (There is of course the serious counter-argument to this procedure that individuals might possilby tend to reveal false preferences in order to influence the choice of the public goods.

<u>Definition 2.2</u>: Let $\mathbb{M}$ be an economy and C a taxation system s.t. $\mathbb{M}$ and C are compatible. Given $p \in \mathbb{P}^{m+1}$, $p > 0$,

(1) $\Sigma^{pC} := (\Omega, \mathbb{R}^{l}_{+}, \hat{U}^{pC})$ (where $\hat{U}^{pC} = (\hat{u}^{pc^i i})_{i \in \Omega}$)

is the <u>location conflict generated by $\mathbb{M}$</u> (via C at p).

<u>Definition 2.3</u>: Let $\mathbb{M}$ be an economy and C a taxation system. Define

$$^{k}B := \bigcap_{i \in \Omega} {}^{k}B^{j} \quad \text{(cf. Def. 1.8)}.$$

Then, given $p \in \mathbb{P}^{m+1}$

(2) $^{k}\Sigma^{pC} := (\Omega, {}^{k}B, {}^{k}\hat{U}^{pC})$ (where $^{k}\hat{U}^{pC} = ({}^{k}\hat{u}^{pc^i i})_{i \in \Omega}$)

is the <u>location conflict generated by $\mathbb{M}^k$</u>.

<u>Note</u>. If $C \in \mathbb{Q}$ (Example 1.5) is an affine taxation system, then budget restrictions of the type

$$px \leq pa^i + c^i(b) - c^i(y)$$

can be rewritten as

$$px \leq pa^i + q^i(b - y)$$

or

$$px + q^i y \leq pa^i + q^i b$$

provided $c^i(y) = q^i y + \alpha_i$. Hence, $\hat{u}^i = \hat{u}^{pc^i i}$ depends on q^i

and not on α_i . We shall use the notations $Q = (q^1, \ldots, q^n)$ (instead of C) as well as

$$\hat{u}^i = \hat{u}^{pq^i i} , \hat{U}^{pQ} , \Sigma^{pQ}$$

and the like. Similary, if $C \in \mathbb{Q}^o$ (Example 1.6), then we shall identify c^i, $c^i(y) = qy + \alpha$, and q, i. e., write

$$\hat{u}^i = \hat{u}^{pqi} , \hat{U}^{pq} , \Sigma^{pq} .$$

SECTION 3

EQUILIBRIUM

Definitions and the General Affine Taxation Case

Definition 3.1: Let $\mathbb{M}$ be an admissible economy, $\mathbb{C}^o \subseteq \mathbb{C}$ a taxation policy, and $\Psi: \mathfrak{X}^o \to P(\mathbb{R}^n)$ a value for location conflicts. Then

$(\bar{p}, \bar{C}, \bar{X}, \bar{y}) \in \mathbb{P}^{m+1} \times \mathbb{C}^o \times (\mathbb{R}^m_+ \times \mathbb{R})^n \times \mathbb{R}^1_+$

is a Ψ-$\mathbb{C}^o$-equilibrium for $\mathbb{M}$ if

0. $\bar{p} > 0$
1. $\mathbb{M}$ and $\bar{C}$ are compatible
2. $\Sigma^{\bar{p}\bar{C}} \in \mathfrak{X}^o$
3. $\bar{y} \in \Psi(\Sigma^{\bar{p}\bar{C}})$
4. $\bar{x}^i \in \hat{X}^i(\bar{p}, \bar{c}^i, \bar{y}) \quad (i \in \Omega)$
5. $(\sum_{i\in\Omega} \bar{x}^i - a^i, \bar{y} - b) \in Y$

Of course, conditions 1 - 3 are of technical nature. Hence the obvious interpretation of the definition is that, given equilibrium prices and taxes, the public bundle is considered to be "fair" within the location conflict induced. Given this bundle, every player maximizes his utility with respect to the private goods, and finally the result obtained this way is feasible and efficient with respect to the production technology.

Mathematically, it could be considered to be a blunder carrying the taxation policy $\mathbb{C}^o$ in the definition of equilibrium. However, it is our idea that a taxation policy as well as the value are institutional data and selected in advance.

Hence one is looking for a taxation scheme which allows an equilibrium within a certain prescribed family or policy of taxations.

Let us also formulate the definition in case of the truncated version.

Definition 3.2: Let $\mathbb{M}$ be an economy and $\mathbb{C}^{o} \subseteq \mathbb{C}$ a taxation policy. Let $\Psi : \Sigma^{o} \to P(\mathbb{R}^{l})$ be a value for location conflicts. Then

$$(\bar{p}, \bar{C}, \bar{X}, \bar{y}) \in \mathbb{P}^{m+l} \times \mathbb{C}^{o} \times ({}^{k}X^{k})^{n} \times \mathbb{R}^{l}_{+}$$

is a Ψ-$\mathbb{C}^{o}$-equilibrium for $\mathbb{M}^{k}$ if

2'. ${}^{k}\Sigma^{\bar{p}\bar{C}} \in \Sigma^{o}$.

3'. $\bar{y} \in \Psi({}^{k}\Sigma^{\bar{p}\bar{C}})$.

4'. $\bar{x}^{i} \in {}^{k}\hat{X}^{i}(\bar{p}, \bar{c}^{i}, \bar{y}) \quad (i \in \Omega)$.

5'. $\sum_{i\in\Omega} (\bar{x}^{i} - a^{i}, \bar{y} - b) \in Y$.

A first and obvious result is that, given affine but arbitrary taxation, the Ψ-$\mathbb{Q}$-equilibrium is a generalization of the well-known Lindahl equilibrium. As we do not want to enter the formal definition of Lindahl equilibria the reader is referred to [8], [10].

The definition of a "Lindahl equilibrium" within our framework is rather obvious. Note that, given linear taxation, compatibility of $\mathbb{M}$ and $Q = (q^{1},\ldots,q^{n})$ is easily verified if Q is positive, say, in the presence of strict concavity or utility functions that are not increasing too fast with respect to the private goods. In fact, such requirement can be imposed upon linear taxation functions in a uniform manner. (Cf. Definition 1.11 and Theorem 1.12.)

Theorem 3.3: Let $\mathcal{M}$ be an admissible economy and let $\Psi : \mathfrak{T}^{o} \to \mathcal{P}(\mathbb{R}^{n})$ be a value such that

(1) $\Sigma \in \mathfrak{T}^{o}$ whenever $M_{\Sigma} = \bigcap_{i \in \Omega} M_{B} u^{i} \neq \emptyset$.

Suppose $(\bar{p}, \bar{Q}, \bar{X}, \bar{y}) \in \mathbb{P}^{m+1} \times \mathbb{Q} \times (\mathbb{R}_{+}^{m} \times \mathbb{R})^{n} \times \mathbb{R}_{+}^{1}$ is a Lindahl-equilibrium such that $\mathcal{M}$ and $\bar{Q}$ are compatible. Then $(\bar{p}, \bar{Q}, \bar{X}, \bar{y})$ is a Ψ-$\mathbb{Q}$-equilibrium for $\mathcal{M}$.

Proof. For $i \in \Omega$, $(\bar{x}^{i}, \bar{y})$ maximizes i's utility within

(2) $\{(x,y) \mid x \in X,\ y \in \mathbb{R}_{+}^{1},\ \bar{p}x + \bar{q}^{i}y \leq \bar{p}a^{i} + \bar{q}^{i}b\}$

This shows that for x with $\bar{p}x \leq \bar{p}a^{i} + \bar{q}^{i}b - \bar{q}^{i}y$ we have

$$u^{i}(x, \bar{y}) \leq u^{i}(\bar{x}^{i}, \bar{y})$$

i. e.,

$$\bar{x}^{i} \in \hat{X}^{i}(\bar{p}, \bar{q}^{i}, \bar{y})$$

and 4. of Definition 3.1 is satisfied.

Moreover, (2) implies

$$\hat{u}^{\bar{p}\bar{q}^{i}i}(\bar{y}) = u^{i}(\bar{x}^{i}, \bar{y})$$

$$= \max \{u^{i}(x,y) \mid x \in X,\ y \in \mathbb{R}_{+}^{1},\ \bar{p}x + \bar{q}^{i}y \leq \bar{p}a^{i} + \bar{q}^{i}b)$$

$$\geq \max \{u^{i}(x,y) \mid x \in X,\ \bar{p}x + \bar{q}^{i}y \leq \bar{p}a^{i} + \bar{q}^{i}b\}$$

$$= \hat{u}^{\bar{p}\bar{q}^{i}i}(y) \qquad \text{for all } y \in \mathbb{R}_{+}^{1}$$

This means obviously $\bar{y} \in \bigcap_{i \in \Omega} M_{\mathbb{R}_{+}^{1}} \hat{u}^{\bar{p}\bar{q}^{i}i}$.

Hence, $\Sigma^{\bar{p}\bar{Q}} \in \mathfrak{T}^{o}$ and $\bar{y} \in \Psi(\Sigma^{\bar{p}\bar{Q}})$, by Definition 2.1, hence 2. and 3. of 3.1 are satisfied. Given strict monotony, 0. of 3.1 is clear while 5. is part of the definition of a Lindahl equilibrium.

q. e. d.

Equal Taxation

This section is dealing with an existence proof for equal taxation. This does not mean that equal taxation is a principle advocated generally, but it should rather imply that it is possible to prove the existence of equlibria in the presence of certain restricted taxation policies.

Clearly, we have so far not been dealing with properties of the production set Y. In fact, the conditions we are to impose result from fairly general properties of Y. This is a problem that has already been carried through by ROBERTS [17]. This author provides a list of the requirements for Y sufficient to prove that our conditions as stated below are satisfied; in paricular it is claimed that Y should feature constant returns to scale, that is, Y should be a convex cone. However, we believe that decreasing returns to scale are sufficient, that is, that convexity of Y essentially would be a sufficient condition.

In any case as we do not want to enter this discussion we shall just make the results of [17] as to be our conditions in order to establish equlibrium. This motivates the following definitions:

Definition 3.4: Y is smooth, if

1. $G : \mathbb{P}^{m+1} \times \mathbb{R}^1_+ \to P(-X \times \mathbb{R}^1_+)$

   $(\tilde{p}, \tilde{y}) \to \{(\tilde{x}, \tilde{q}) \mid -\tilde{x} \in X, \tilde{q} \in \mathbb{R}^1_+, (\tilde{x}, \tilde{y}) \in Y,$

   $\tilde{p}\tilde{x}+\tilde{q}\tilde{y} = \max\{\tilde{p}x+\tilde{q}y \mid (x, y) \in Y\}\}$

   is a nonempty, convex valued, compact valued, u.h.c. correspondence.

2. For every k sufficiently large

   $G^k : \mathbb{P}^{m+1} \times \underline{\tilde{Y}}^k \to P(-{}^{o}X^{nk} \times \mathbb{R}^1_+)$

   $(\tilde{p}, \tilde{y}) \to G(\tilde{p}, \tilde{y}) \cap (-{}^{o}X^{nk} \times \mathbb{R}^1_+)$

   is nonempty (and, of course, as well convex valued, compact valued, u.h.c.).

Theorem 3.5: (Equal taxation, truncated market)

Let $\mathbb{M}$ be an economy with smooth $\mathcal{Y}$ and $b = 0$. Let $\mathbb{Q}^{o}$ be equal taxation. Furthermore, let $\psi : \Sigma^{o} \to \mathcal{P}(\mathbb{R}^{1})$ be a value such that

$$^{k}\Sigma^{pq} \in \Sigma^{o}$$

for all $p \in \mathbb{P}^{m+1}$, $q \in \mathbb{Q}^{o}$ and some $k \in \mathbb{N}$ sufficiently large. If

$$(p, q) \to \psi(^{k}\Sigma^{pq})$$

$$\mathbb{P}^{m+1} \times \mathbb{Q}^{o} \to \mathcal{P}(\mathbb{R}^{1}_{+})$$

is a convexvalued u.h.c. correspondence, then there exists a ψ-$\mathbb{Q}^{o}$-equilibrium for $\mathbb{M}^{k}$.

Proof. 1.Step. Define

$$\overset{o}{H}{}^{k} := \frac{1}{n} \operatorname{Proj}_{2} \bigcup_{\substack{p \in \mathbb{P}^{m+1} \\ y \in \underline{\bar{Y}}^{k}}} G^{k}(p,y) ,$$

this is a compact set (since G^{k} is u.h.c.). Let H^{k} be its convex hull (as well compact). We are going to define a correspondence

$$F : \Delta^{k} \to \mathcal{P}(\Delta^{k})$$

where

$$\Delta^{k} := \mathbb{P}^{m+1} \times H^{k} \times {}^{2nk}Z^{2nk} \times \underline{\bar{Y}}^{k}$$

as follows:

Given $(p,q,x,y) \in \Delta^{k}$, define

(3) $$p' := \frac{p + x^{+}}{1 + \sum_{j} x_{j}^{+}} .$$

Next, choose

(4) $$y' \in \psi(^{k}\Sigma^{pq}) .$$

As $\mathcal{Y}$ is smooth, there is

(5) $$\tilde{x} \in -{}^{o}\mathcal{X}^{nk} , \quad q' \in \mathbb{R}^{1}_{+}$$

such that

(6) $$(\tilde{x}, nq') \in G^{k}(p,y) .$$

Next, choose

(7) $$\hat{x}^{i} \in {}^{k}\hat{X}^{i}(p,q,y)$$

and define

(8) $$x' := \sum_{i \in \Omega} \hat{x}^{i} - a^{i} - \tilde{x}$$

Now, $F(p,q,x,y)$ is defined to be the set of all (p',q',x',y') that are obtained by this procedure. Formally

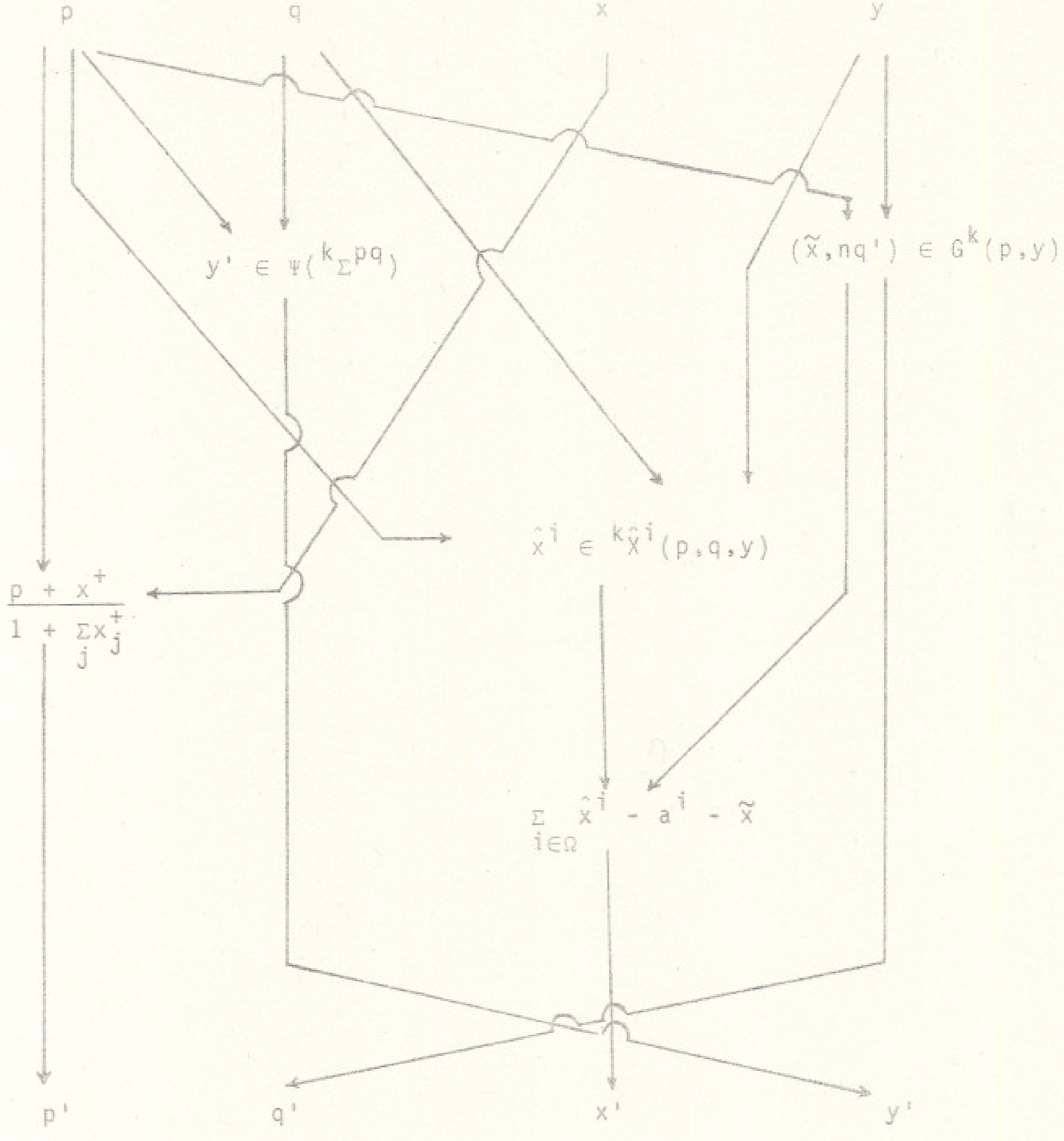
$$F(p,q,x,y) = \{(p',q',x',y') \mid$$

$$p' = \frac{p + x^{+}}{1 + \sum_{j} x_{j}^{+}} , \quad y' \in \Psi({}^{k}\Sigma^{pq}),$$

$$x' = \sum_{i \in \Omega} \hat{x}^{i} - a^{i} - \tilde{x}, \quad (\tilde{x}, nq') \in G^{k}(p,y)$$

for some $(\hat{x}^{i})_{i \in \Omega}$, $\hat{x}^{i} \in {}^{k}X^{i}(p,q,y)$, and some $\tilde{x} \in -{}^{o}\mathcal{X}^{nk}\}$

It might be useful to visualize F by means of the following diagramm.


p
q
x
y
y' ∈ Ψ(kΣpq)
(x̃,nq') ∈ Gk(p,y)
x̂i ∈ kX̂i(p,q,y)
p + x+ / 1 + Σj xj+
Σi∈Ω x̂i - ai - x̃
p'
q'
x'
y'

2. Step. Let us check that F takes values that are subsets of Δ^k.

Now, $p' \in \mathbb{P}^{m+1}$ is obvious. Next, as $y \in \bar{Y}^k$, $p \in \mathbb{P}^{m+1}$, we have

$$(\tilde{x}, nq') \in G^k(p,y)$$

i. e.

$$nq' \in \mathrm{Proj}_2\, G^k(p,y)$$

$$q' \in \frac{1}{n}\, \mathrm{Proj}_2\, G^k(p,y) \subseteq H^k .$$

Moreover, as ${}^k\Sigma^{pq} = (\Omega, {}^kB, {}^k\hat{U}^{pq})$, by Property 0. of Defition 2.1,

$$y' \in \Psi({}^k\Sigma^{pq}) \subseteq {}^kB \subseteq \bar{Y}^k$$

(cf. Definition 1.8, (14)). Therefore, it remains to show that $x' \in {}^{2nk}\mathcal{Z}^{2nk}$, i. e.

$$-2nke \leq x' \leq 2nke .$$

To this end, note that by definition of G^k, $-\tilde{x} \in {}^0\mathcal{X}^{nk}$, i. e.

$$0 \leq -\tilde{x} \leq nke .$$

Similarly $\hat{x}^i \in {}^k\mathcal{X}^k$

$$-k \leq \hat{\xi}^i \leq k$$

$$0 \leq \hat{\overset{\circ}{x}}{}^i \leq ke .$$

In view of $a^i \geq 0$ $(i \in \Omega)$, it follows that

$$x' = \sum_{i \in \Omega} \hat{x}^i - a^i - \tilde{x} \leq nke - \tilde{x}$$

$$\leq nke + nke = 2nke$$

and, as $a^i \leq ke$,

$$x' \geq \sum_{i \in \Omega} \hat{x}^i - a^i \geq -nke - nke = -2nke ,$$

completing Step 2.

3. Step. By assumption, Ψ and G^k are convexvalued and u.h.c. correspondences. It can be verified that ${}^k\hat{X}^i(\cdot,\cdot,\cdot)$ is u.h.c and convexvalued as well. From this it follows at once that

$$F : \Delta^k \to \Delta^k$$

is u.h.c. and convexvalued, and hence satisfies the conditions of Kakutani's fixed point theorem.

Let $(\bar{p},\bar{q},\bar{x},\bar{y}) \in \Delta^k$ be a fixed point. I. e., there is $\tilde{x}$, $(\hat{x}^i)_{i\in\Omega}$ s.t.

(9) $$\bar{p} = \frac{\bar{p} + \bar{x}^+}{1 + \sum_j \bar{x}_j^+}$$

(10) $$\bar{x} = \sum_{i\in\Omega} \hat{x}^i - a^i - \tilde{x}$$

(11) $$(n\bar{q},\tilde{x}) \in G^k(\bar{p},\bar{y})$$

(12) $$\hat{x}^i \in \hat{X}^i(\bar{p},\bar{q},\bar{y})$$

(13) $$\bar{y} \in \Psi({}^k\Sigma^{\bar{p}\bar{q}})$$

Let us disprove

(14) $$\bar{x}^+ \neq 0 .$$

Indeed, if (14) holds true, then from (9) :

$$\bar{p} \sum \bar{x}_j^+ = \bar{x}^+$$

and

$$0 < \bar{p}\bar{x}^+ = \bar{p}\bar{x}$$

(15) $$\overset{(10)}{=} \bar{p}(\sum_{i\in\Omega} (\bar{x}^i - a^i) - \tilde{x})$$

$$= \sum_{i\in\Omega} (\bar{p}\bar{x}^i - \bar{p}e^i) - \bar{p}\tilde{x} \ldots$$

As $b = 0$ was assumed for convenience, we have for $\hat{x}^i \in \hat{X}^i(\bar{p},\bar{q},\bar{y})$

$$\bar{p}\hat{x}^i - \bar{p}a^i \leq -\bar{q}\bar{y}$$

i. e., (15) continues

$$\ldots \leq -n\bar{q}y^- - \bar{p}\widetilde{x} \ldots$$

and since $(0,0) \in Y^k$ and $(\widetilde{x}, n\bar{q}) \in G^k(\bar{p},\bar{y})$, it follows that $n\bar{q}\bar{y} + p\widetilde{x} \geqq nq0 + p0 = 0$,

$$\ldots \leqq 0 ,$$

a contradiction.

Hence (14) is wrong, i. e.

$$\bar{x}^+ = 0$$

$$\bar{x} \leqq 0$$

meaning

(16) $$\Sigma\hat{x}^i - a^i \leq \widetilde{x} .$$

But as $(\widetilde{x},\bar{y}) \in Y^k$, this implies

(17) $$(\Sigma\hat{x}^i - a^i , \bar{y}) \in Y .$$

Now, (13), (12) and (17) imply that

$$(\bar{p}, \bar{q}, (\hat{x}^i)_{i\in\Omega} , \bar{y})$$

is a Ψ-$\mathbb{Q}^o$-equilibrium for M^k .

q. e. d.

In order to prove the general existence theorem for the non-truncated market a few additional conditions are necessary. First of all it is necessary to have some kind of strictly decreasing marginal utility imposed upon the players. In particular we shall require that the consumer has a very urgent desire to receive at least some quantity of every private good. His desire for acquiring private goods when having none should be very high compared with his desire to hold cash. This is not all together unreasonable. On the other hand we shall ask that his marginal utility with respect to the public goods is bounded. Again one could imagine that consumers in a situation where they have almost no private good would certainly prefer some private goods to public goods with great urgency.

The fact that we are going to formulate these questions in terms of partial derivatives does not mean that the theorem hinges on the existence of continuous partial derivatives as required. In fact, the experienced reader will see at once that appropriate formulations could replace the differentiability requirements in an obvious manner.

There will also be an additional requirement with respect to Y. In fact we want that taxes are bounded away from 0. This means essentially that the q-coordinates of tangents to the production set Y can be chosen as to be bounded away from 0, and this as is easily seen is only a requirement which concerns the boundary of Y. Similarly we want that taxes are ranging within some set bounded from above.

<u>Definition 3.6</u>: Let $\mathcal{M}$ be an economy. Suppose each utility function u^i $(i \in \Omega)$ admits of continuous partial derivatives $\frac{\partial u^i}{\partial x_j}$ $(x_j > 0)$, $\frac{\partial u^i}{\partial \xi}$ $(\xi \in \mathbb{R})$, $\frac{\partial u^i}{\partial y_j}$ $(y_j > 0)$, such that

(18) $\frac{\partial u^i}{\partial y_j}$ is bounded,

(19) $\frac{\partial u^i}{\partial \xi}(\mathring{x},0,y) \geqq \eta$ for some $\eta > 0$

and

(20) $x_j \frac{\partial u^i}{\partial x_j}(x_1,\ldots,x_j,\ldots,x_m,\xi,y) \uparrow \infty \quad (x_j \downarrow 0)$

(Think of $e^{\frac{-1}{x}}$, locally!)

Let $\mathcal{Y}$ have a closed efficiency set. In addition, assume that $\mathcal{Y}$ is smooth and there is some $\Delta, \delta > 0$ such that for $p \in \mathbb{P}^{m+1}$, $y \in \mathbb{R}^l_+$, there is $(x,q) \in G^k(p,y)$ with $\Delta e \geqq q \geqq \delta e$ for k sufficiently large.

Then $\underline{\mathcal{M}}$ will be called <u>smooth</u>.

We are now in a position to formulate a general existence theorem for linear and equal taxation. Of course this is running via a limit theorem of the truncated versions. It should be noted however that the general procedure applied to such limit theorems is to be changed with respect to several details. The main obstacle is that the bundle of private and public goods that the single consumer receives in the equilibrium will in general not maximize his total utility. Instead his total utility is maximized by the maximizer of the derived utility function $\hat{u}$. This causes some problems when arguing that prices in the long run will be positive.

Theorem 3.7: (Equal taxation, general version)
Let $\mathbb{M}$ be an economy which is smooth, admissible, and compatible with nearly linear taxation. Assume that $b = 0$.
Let $\mathbb{Q}^o$ be the equal taxation policy. Furthermore, let $\Psi : \mathfrak{X}^o \to P(\mathbb{R}^1)$ be a value such that

(21) $${}^k\Sigma^{pq} \in \Sigma^o$$

for all $p \in \mathbb{P}^{m+1}$, $q \in \mathbb{Q}^o$ and sufficiently large $k \in \mathbb{N}$.

(22) $$\Sigma^{pq} \in \mathfrak{X}^o$$

for all $p > 0, q > 0$.
Assume, in addition, that for all $k \in \mathbb{N}$ sufficiently large

(23) $$(p,q) \to \Psi({}^k\Sigma^{pq})$$

is a convexvalued u.h.c. correspondence.
Also,

(24) $$(p,q) \to \Psi(\Sigma^{pq})$$

is a convexvalued u.h.c. correspondence for $p,q > 0$.
Then there is a Ψ-$\mathbb{Q}^o$-equilibrium for $\mathbb{M}$.
If Y has constant returns to scale, then production is efficient.

Proof. 1. Step. Within the proof of Theorem 3.5., we may replace the correspondence G^k by $G^{k,\delta,\Delta}$ where

$$G^{k,\delta,\Delta}(\tilde{p},\tilde{y}) = G^k(\tilde{p},\tilde{y}) \cap \{(x,q) \mid \Delta e \geq q \geq \delta e\} .$$

Hence we may assume that equilibrium taxes range within some compact set and are bounded away from zero independently of $k \in \mathbb{N}$.

Now, for $k \in \mathbb{N}$ sufficiently large, Theorem 3.5 guarentees the existence of a Ψ-$\mathbb{Q}^o$-equilibrium for $\mathbb{M}^k$, say

$$(p^k, q^k, x^k, y^k) .$$

We may assume that

(25) $$p^k \underset{k}{\to} \bar{p} \in \mathbb{P}^{m+1},\ q^k \to \bar{q} \in \mathbb{R}^1_+,\ \bar{q} > 0 .$$

Also, since

(26) $$(\sum_{i \in \Omega} x^{ki} - a^i,\ y^k) \in Y$$

we have

(27) $$0 \leq \sum_{i \in \Omega} \overset{o}{x}{}^{ki} \leq \sum_{i \in \Omega} \overset{o}{a}{}^i$$

and hence, w. l. o. g.

(28) $$\overset{o}{x}{}^{ki} \underset{k}{\to} \overset{o}{x}{}^i$$

where $\overset{o}{x}{}^i \in \mathbb{R}^m_+$ and $\sum_{i \in \Omega} \overset{o}{x}{}^i \leq \sum_{i \in \Omega} a^i$.

2. Step. Let us verify that

(29) $$\bar{\rho} = \bar{p}_{m+1} > 0 .$$

To this end, assume first of all that

(30) $$p^k a^i - q^k y^k \leq 0$$

for all $i \in \Omega$ and all $k \in \mathbb{N}$ sufficiently large. By property 4. of Definition 2.1, there is $y^{ki} \in M_{kB}\hat{u}^{p^k q^k i}$; s.t. y^k is a convex combination of the y^{ki} $(i \in \Omega)$. From (30) it follows that

(31) $$p^k a^i - q^k y^{ki} \leq 0$$

for some $i \in \Omega$. Let $\overset{*}{x}^{ki} \in \hat{X}(p^k, q^k, y^{ki})$, that is

(32) $$\hat{u}^{p^k q^k i}(y^{ki}) = u^i(\overset{*}{x}^{ki}, y^{ki})$$

and

(33) $$p^k \overset{*}{x}^{ki} \leq p^k a^i - q^k y^{ki} \leq 0 .$$

Since $a^i > 0$ $(i \in \Omega)$, there is j such that y_j^{ki} is bounded away from zero. On the other hand, (33) implies that $\overset{*}{\xi}^{ki} \leq 0$. Therefore, given small $\varepsilon > 0$, the bundle

(34) $$(\tilde{x}^k, \tilde{y}^k) : = (\overset{*}{\overset{0}{x}}{}^{ki}, \overset{*}{\xi}^{ki} + \varepsilon, y^{ki} - \varepsilon\frac{\rho^k}{q_j^k} e^j)$$

(where $\rho^k : = p_{m+1}^k$) is in ${}^k\chi^k$. Obviously

$$p^k\tilde{x}^k = p^k\overset{*}{x}^{ki} + \rho^k\varepsilon \leq p^k a^i - q^k y^{ki} + \rho^k \varepsilon = p^k a^i - q^k \tilde{y}^k ,$$

and hence

$$\hat{u}^{p^k q^k i}(\tilde{y}^k) \geq u^i(\tilde{x}^k, \tilde{y}^k)$$

(35) $$= u^i(\overset{*}{x}^{ki}, y^{ki}) + \varepsilon(\frac{\partial u^i}{\partial \xi} - \frac{\rho^k}{q_j^k}\frac{\partial u^i}{\partial y_j})(\overset{*}{x}^{ki}, y^{ki}) + o^k(\varepsilon)$$

where $\frac{o^k(\varepsilon)}{\varepsilon} \to 0$ $(\varepsilon \to 0)$.

As $\overset{*}{\xi}^{ki} \leq 0$ and u^i concave, the term $\frac{\partial u^i}{\partial \xi}(\cdot,\cdot)$ in (35) is larger than η (by (19)). The term $\frac{\partial u^i}{\partial y_j}$ is bounded. If $\rho^k \to 0$ $(k \to \infty)$, the coefficient of ε will, therefore, become positive. We may then choose ε small enough (for fixed but large k) such that (35) implies

$$\hat{u}^{p^k q^k i}(\tilde{y}^k) > u^i(\overset{*}{x}^{ki}, y^{ki}) = \hat{u}^{p^k q^k i}(y^{ki}) ,$$

contradicting the maximizer property of y^{ki}.

This takes care of (30); for the rest of this step assume now

(36) $$p^k a^i - q^k y^k > 0$$

for <u>some</u> $i \in \Omega$ and (by choosing a subsequence) for all k sufficiently large.

For this player i, 0 is a feasible private bundle, i. e.,

$$p^k 0 \leq p^k a^i - q^k y^k .$$

Hence, $\xi^{ki} \to -\infty$ <u>cannot</u> happen. For, boundedness of $(y^k)_{k \in \mathbb{N}}$ (by (36)) and of $\overset{o}{x}^{ki}$ (by (27),(28)) would imply

$$u^i(x^{ki}, y^k) = u^i(\overset{o}{x}^{ki}, \xi^{ki}, y^k) \to -\infty$$

(u^i is concave and strictly monotone), i. e.

$$u^i(x^{ki}, y^k) < u^i(0, y^k) \qquad (k \text{ large}) ,$$

contradicting $x^{ki} \in \hat{X}(p^k, q^k, y^k)$.

Thus, ξ^{ki} is now bounded (by $\sum_{i \in \Omega} a^i_{m+1}$ from above).

Now, if $p^k a^i - q^k y^k \geq \delta > 0$ for some δ, then $x^\delta = (0, \frac{\delta}{\rho^k})$ would be feasible in player i's budget set. Clearly, $u^i(0, \frac{\delta}{\rho^k}, y^k) \to \infty$ if $\rho^k \to 0$, contradicting the fact that $u^i(x^{ki}, y^k)$ is bounded and x^{ki} is i's maximizer.

It remains to treat the case that

$$p^k a^i - q^k y^k := \delta^k \to 0 .$$

Now, as $(0, \frac{\delta^k}{\rho^k})$ is feasible for i, we conclude that $\frac{\delta^k}{\rho^k}$ is bounded (otherwise, like above, $u^i(0, \frac{\delta^k}{\rho^k}, y^k) \to \infty$), say

(37) $$\frac{\delta^k}{\rho^k} < C_o \quad .$$

Assuming $\rho^k \to 0$, there is j such that p_j^k is bounded away from zero; consider the bundle

$$\tilde{x}^{ki} := x^{ki} - e^{m+1} + \frac{\rho^k}{p_j^k} e^j$$

which is in i's budget set. We have by concavity

(38) $$u^i(\tilde{x}^{ki}, y^k) \geq u^i(\tilde{x}^{ki}, y^k) + (\frac{\rho^k}{p_j^k} \frac{\partial u^i}{\partial x_j} - \frac{\partial u^i}{\partial \xi})(\tilde{x}^{ki}, y^k)$$

The term $\frac{\partial u^i}{\partial \xi}$ is bounded. The j'th argument in $\frac{\partial u^i}{\partial x_j}$ is $\tilde{x}_j^{ki} = x_j^{ki} + \frac{\rho^k}{p_j^k}$.

Because of

$$p^k x^{ki} \leq \delta^k$$

we have in view of (37)

$$x_j^{ki} \leq \frac{\delta^k}{p_j^k} \leq \frac{C'\rho^k}{p_j^k}$$ (with $C' \geq C_o$, since ξ^{ki} is bounded)

and hence $\tilde{x}_j^{ki} \leq C_1 \rho^k$; $C_1 \geq \frac{C' + 1}{p_j^k}$ $(k \in \mathbb{N})$.

$$\frac{\rho^k}{p_j^k} \frac{\partial u^i}{\partial x_j} (\tilde{x}^{ki}, y^k) \geq \frac{\rho^k}{p_j^k} \frac{\partial u^i}{\partial x_j} (\ldots, \underset{\substack{\uparrow \\ j}}{C_1 \rho^k}, \ldots, y^k) \to \infty$$

by (20). This again contradicts the maximality of x^{ki} via (38).

We have thus completed step 2. and proved (29).

3. Step. As $\bar{p} > 0$ it follows clearly that $\bar{p} > 0$. For, if $\bar{p}_j = 0$, then

$$\tilde{x}^{ki} := x^{ki} + \varepsilon e^j - \varepsilon \frac{p_j^k}{p^k} e^{m+1}$$

would be a feasible bundle for i with higher utility than $\tilde{x}^{ki}$, if k is large (using $\frac{\partial u^i}{\partial x_i} > 0$).

4. Step. We are now going to invoke Theorem 1.12. Because of $\bar{p} > 0$, there is $\varepsilon_o > 0$ such that

(39) $$p^k > \varepsilon_o e$$

for large $k \in \mathbb{N}$. By Theorem 1.12., the maximizers of $\hat{u}^{p^k q^k i}$ are uniformly located within some compact set. Clearly, $\hat{u}^{p^k q^k i} \geq {}^k\hat{u}^{p^k p^k i}$; also it is not hard to see (inspect 1.12) that, for large k, the maximizing values of $\hat{u}^{p^k q^k i}$ will be attained by ${}^k\hat{u}^{p^k q^k i}$ as well; meaning that both functions have equal maximizers for large k. Hence, the maximizers of ${}^k\hat{u}^{p^k q^k i}$ will be uniformly located within some compact set. By Definition 2.1., Property 4., it follows that y^k is a bounded sequence. Hence, there is $\bar{y} \in \mathbb{R}^l_+$ s.t. $y^k \underset{k}{\to} \bar{y}$ w. l. o. g. We have established that

$$(p^k, q^k, X^k, y^k) \underset{k \in \mathbb{N}}{\to} (\bar{p}, \bar{q}, \bar{X}, \bar{y})$$

w. l. o. g., because

$$p^k x^{ki} \leq p^k a^i - q^k y^k$$

and boundedness of y^k together with $\bar{p}_{m+1} > 0$ will prevent ξ^{ki} from tending to $-\infty$, as this would contradict the budget maximizing properties of x^{ki} eventually.

5. Step. It remains to show that

$(\bar{p}, \bar{q}, \bar{X}, \bar{y})$ is a Ψ-$\mathbb{Q}^{\circ}$-equilibrium.

Conditions 0., 1., and 2. of Definition 3.1 are obviously satisfied, 5. follows from closedness of Y. As to 4., it is not hard to verify that $\hat{X}^{i}$ is u.h.c. at $\bar{p} > 0$, $\bar{q} > 0$ and $\bar{y}$. Concerning condition 3., observe that ${}^{k}\hat{u}^{p^k q^k i}$ and $\hat{u}^{p^k q^k i}$ will coincide on increasingly large sets. By Definition 2.1, as Ψ is finitely determined, this means $\Psi({}^{k}\Sigma^{p^k q^k}) = \Psi(\Sigma^{p^k q^k})$ for large k. Because Ψ is u.h.c. as $\bar{p}, \bar{q} > 0$ we conclude that $\bar{y} \in \Psi(\Sigma^{\bar{p}\bar{q}})$.

Suppose now that Y has constant returns to scale. Then $(0,0)$ is efficient, and returning to the proof (and notation) of Theorem 3.5., we find that (16) implies

$$0=\bar{p}0+n\bar{q}0=\bar{p}\tilde{x}+n\bar{q}\bar{y} \geq \bar{p}\sum_{i\in\Omega}(\bar{x}^{i}-a^{i})+n\bar{q}\bar{y} \quad .$$

This holds within the framework of $\mathbb{M}^{k}$. However, turning to the limit, we find

(40) $$0 \geq \bar{p}\sum_{i\in\Omega}(\bar{x}^{i} - a^{i}) + n\bar{q}\bar{y}$$

holds true also w. r. the equilibrium established for $\mathbb{M}$. But ">" in (39) would establish

$$\bar{p}(\bar{x}^{i} - a^{i}) + \bar{q}\bar{y} < 0$$

for at least one $i \in \Omega$ - a contradiction to the fact that u^{i} is strictly monotone.

q. e. d.

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