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- A Numerical Example -

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by

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A generalized Nash solution for bargaining situations under incomplete information has been developped by John C. Harsanyi and the author [1]. In the following this generalized Nash solution will be applied to a one-parameter class of numerical examples.

As we know from everyday experience, two person bargaining is a dynamic process. Usually one observes a sequence of offers and counteroffers which either converges to an agreement or ends in conflict.What is the purpose of this dynamic process? In a bargaining situation with complete information it is hard to understand why there should be a bargaining process. Here both bargainers have full knowledge about all relevant aspects of the situation which determine the outcome of the bargaining process. Both should be able to anticipate the final agreement and to reach it at once. There should be no need for a sequence of mutual concessions.

The theory of bargaining under incomplete information provides a rational explanation for the dynamics of bargaining. Here each of both bargainers knows some aspects of the situation not known to the other. In particular a bargainer may not know whether the other is in a strong bargaining position or not. As we shall see, in the example of this paper, the bargaining process serves the purpose to resolve such uncertainties.

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1) John C. Harsanyi and Reinhard Selten, A Generalized Nash Solution for Bargaining Games with Incomplete Information, Management Science, Vol. 18, No. 5, January, Part 2,1972, p-80 - p-106. - In the following this paper will be referred to us [HS]- The author is very grateful for many discussions with John C. Harsanyi who strongly influenced the content of this paper

In the course of the bargaining process a player in a strong bargaining position may get the opportunity to prove his strength and a player in a weak bargaining position may be forced to reveal his weakness. In order to show his strength a player will act in a stubborn way. Thereby he takes a risk of conflict which is too high for a player in a weak bargaining position. The dynamics of the bargaining process appears to be a vehicle for the credible exchange of information. Creditibility is supplied by the willingness to take a risk of conflict.

1. A simple bilateral monopoly situation

The incomplete information bargaining situation considered in this paper has the following economic interpretation: The government is willing to give a contract to two firms, called 1 and 2. In order to get the contract the two firms must agree about the division of the amount of 100, which is the gross profit from this contract. One firm alone cannot get the contract. In order to decide whether an agreement on the division of the gross profit is profitable for firm i, it is necessary to know the opportunity costs for firm i, which depend on the degree of capacity utilization. We assume that the degree of capacity utilization is either low, in which case the opportunity costs are 0, or high, in which case the opportunity costs are a, where a is a constant with $0 \leq a \leq 50$. Table 1 shows the dependence of the opportunity costs on the degree of capacity utilization. The number in the upper left corner of a field refers to firm 1 and the number in the lower right corner refers to

		firm 2	
		H	L
firm 1	H	a a	a 0
	L	0 a	0 0

Table 1: Opportunity costs (conflict point)

firm 2. The symbols H and L stand for "high" and "low".

If for firm i the degree of capacity utilization is high, then we say that firm i is of type H; in the same way we speak of type L if the degree of capacity utilization is low. We assume that the four type combinations (H,H), (H,L,), (L,H) and (L,L) have equal probabilities (table 2). The information given by tables 1 and 2 is common to both firms.

		firm 2	
		H	L
firm 1	H	$\frac{1}{4}$	$\frac{1}{4}$
	L	$\frac{1}{4}$	$\frac{1}{4}$

Table 2: Probability matrix

Each of both firms has additional information because it knows its own type but this information remains incomplete since the type of the other firm is unknown. We assume that the firms are unable to prove that they are of a certain type. Therefore the incompletemess of information cannot be removed by communication. Since a firm of type L would improve its bargaining position by a false statement about its type, if such lies were believed, a firm of type H cannot effectively communicate the fact that it is of type H.

An agreement between the two firms is a division of the amount of 100; let z be the share of firm 1; then 100 - z is the share of firm 2. In the theory of [HS] an agreement is given

		firm 2	
		H	L
firm 1	H	z 100-z	z 100-z
	L	z 100-z	z 100-z

Table 3: Bimatrix representation of an agreement

by a bimatrix $u = (u_{ikm})$ where u_{ikm} is the payoff of player i if player 1 is of type k and player 2 is of type m. The bimatrix corresponding to an agreement between the two firms is given by table 3. In order to adapt to the example to the notation of the theory, we shall refer to the types H and L as types 1 and 2, respectively, whenever this is more convenient.

Formally, the conflict point (table 1) is also an agreement in the sense that it is an element of the agreement set U of [HS].

Since the theory of [HS] has been developed for finite agreement sets only, we assume that there is a smallest unit of money whose value is ε; the share z must be an integer multiple of ε with $0 \leq z \leq 100$; furthermore $1/\varepsilon$ is a (possibly very large) positive integer.

In order to be able to apply the theory to the example we assume that the utilities of both firms are linear in money. The firms are the players in an incomplete information bargaining situation $S = (U,c,r)$, where the agreement set U contains the conflict point c and the agreement bimatrices of table 3. The conflict point c is the opportunity cost bimatrix of table 1 and the probability matrix r is given by table 2.

Sometimes it is convenient to regard the bargaining situation of the example as a game played by the four types or "subplayers" as we shall also call them. The subplayers will be numbered from 1 to 4. The numbers 1,...,4 refer to types H and L of player 1 and types H and L of player 2 in that order.

2. The bargaining model

The theory of [HS] is based on a bargaining model which describes the bargaining process as a sequence of discrete stages $t = 1,\dots,T$. The model transforms a bargaining situation into a bargaining game. It will be useful to summarize the model by the following five rules:

Rule 1: At each stage each of both players makes an offer. An offer is an agreement $u \varepsilon U$. The offers are made simultaneously without knowledge of the offer made by the opponent at the same stage. Only after both offers have been made each offer is made known to the other player.

Rule 2: An offer which has not been made before by the same player is called a new offer. Bargaining results in conflict if at some stage t both players fail to make a new offer. If this happens bargaining ends and the players receive their conflict payoffs.

Rule 3: An offer which at some stage t has been made by both players - possibly at an earlier stage by one of them - is called an accepted offer. If a situation is reached where there is exactly one accepted offer, agreement is reached at that offer. If this happens, bargaining ends and the players receive the payoffs specified by the agreement.

Rule 4: If a situation is reached where there are two accepted offers - this happens if both players make offers which have been made by the other player in previous stages - then a random decision selects one of both players with equal probabilities; the selected player then chooses one of both accepted offers which thereby becomes the agreement reached by the bargaining process. Bargaining ends and the players receive the payoffs specified by the agreement.

Rule 5: If at stage t a situation is reached where there is at least one new offer and no accepted offer, then bargaining proceeds to the next stage t+1.

The bargaining game: The application of rules 1 - 5 to a bargaining situation yields a bargaining game. The bargaining game can be regarded as a finite game in extensive form played by the subplayers 1,...,4. The strategies in this game are called bargaining strategies. A bargaining strategy prescribes an offer $u \varepsilon U$ to every possible situation in which a decision may be required from a subplayer. (In the case of **rule 4** the choice is restricted to the two accepted offers.)

3. The solution concept

The generalized Nash solution developed in [HS] is based on the idea that the bargaining game is played in a noncooperative way. Generally only equilibrium points are regarded as legitimate candidates for a solution of a noncooperative game. The view which is taken by the theory of [HS] is in one sense narrower and in another sense wider than that. On the one hand, only such equilibrium points are considered as reasonable which have an additional property, called strictness, and on the other hand, not only strict equilibrium points but also probability mixtures of strict equilibrium points are regarded as possible rational ways of playing the noncooperative bargaining game.

A strict equilibrium point is characterized by the property that the payoffs of the players remain unchanged if one player,say player j, uses an alternative best reply to the equilibrium strategies of the other players, whereas they continue to play their equilibrium strategies. It has been argued in [HS] that only strict equilibrium points lead to stable payoff configurations. The example will show in which sense non-strict equilibrium points may be very unstable.

The idea that probability mixtures of strict equilibrium points represent reasonable ways of behavior, too, is based on the assumption that the players can make their behavior dependent on commonly observed random events which take place before the beginning of the game [2]. The random events serve as **signals** which tell the players which of several strict equilibrium points will be played in the game.

If one takes this point of view one might wish to have a theory which prescribes a unique probability mixture of strict equilibrium points to every bargaining game. The theory proposed in [HS] is less detailed than that. It uniquely determines a payoff vector for the subplayers in the bargaining game which belongs to at least one probability mixture of strict equilibrium points. This payoff vector is called the generalized Nash solution. A probability mixture of strict equilibrium points which yields the generalized Nash solution as its payoff vector will be referred to as a representation of the solution.

The finiteness of the agreement set U facilitates the development of the theory but it also complicates its application. Since it does not seem to be very interesting to explore the question how the generalized Nash solution for the example depends on ε we shall not look for the exact solution but for an approximate one which comes arbitrarily close to the exact solution as ε approaches zero. We shall exhibit an approximate solution which can be represented by a probability mixture of at most two strict equilibrium points in pure strategies. In order to have a short name for this representation of the approximate solution we shall call it the main representation. The next section describes the equilibrium points of the bargaining game which occur in the main representation.

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[2] As has been pointed out by R.J. Aumann, an even wider range of coordinated ways of behavior should be regarded as reasonable. We shall not explore these possibilities here. See R.J. Aumann, Subjectivity and Correlation in Randomized Strategies, Research Program in Game Theory and Mathematical Economic Research Memorandum No. 84, Jerusalem/Israel, January 1973.

4. A distinguishing and a non-distinguishing equilibrium point of the bargaining game

For the whole range $0 \leq a < 50$ of the opportunity cost parameter a the bargaining game has one special strict equilibrium point in pure strategies which we call non-distinguishing since here both types behave in the same way. For $25 + \varepsilon < a \leq 50$ we also consider another strict pure strategy equilibrium point which we call distinguishing since here the behavior depends on the type of the player.

The non-distinguishing and the distinguishing equilibrium points are very different from each other but in some less important situations they prescribe the same behavior. Both equilibrium points require all four subplayers to follow the "general" recommendations (G1) and (G2) stated below. Later the general recommendations will be completed by "specific" recommendations which are different for both equilibrium points.

the recommendations first describe the situation they apply to and then the required behavior. Unfortunately, it is necessary to cover many situations which can never arise if the equilibrium strategies are played. For the purpose of constructing a representation of the approximate solution, if does not matter very much in many of theses cases, which recommendations are given. Nevertheless, it is desirable to represent the approximate solution by equilibrium points which specify reasonable choices in all possible situations even if this complicates the exposition.

Sometimes we shall say that a player "demands d" or "chooses d as his demand" , where de is a number between 0 and 100. These words mean that an agreement is offered which corresponds to $z = d$ if we speak about player 1 and to $z = 100-d$ if we speak about player 2. (See table 3.)

Whereever the recommendations refer to a stage k we shall assume $k > 1$. Separate recommendations are given for stage 1.

The lowest payoffs in the offers which a player has made in stage 1,...,k-1 is called his "lowest previous demand" at stage k. Consider the highest payoff which a player receives in the offers made by the other player in the stages 1,...,k-1. It will be convenient to call this amount the player's "conceded payoff" at stage k.

The general recommendations are as follows:

(G1) At a stage k your lowest previous demand is lower than your conceded payoff. In this case choose your conceded payoff at stage k as your demand at stage k.

(G2) You have to choose between two accepted agreements. In this case choose that one which is more favorable to you if there is one; if both are equally favorable to you$^{3)}$ select the conflict point.

The specific recommendations do not apply wherever the general recommendations are relevant. This will not be explicitly mentioned in the text of the specific recommendations. The non-distinguishing equilibrium point is described by the following recommendations (N1) to (N3).

(N1) At stage 1 demand 50.

(N2) Your conceded payoff at stage k is not greater than 50. In this case at stage k demand 50.

(N3) Your conceded payoff at stage k is greater than 50. In this case choose your conceded payoff at stage k as your demand at stage k.

The description of the distinguishing equilibrium point is more complicated. Separate recommendations must be given to both types. A type H subplayer obeys the recommendations (H1), (H2)

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3) This case cannot arise unless one of the two accepted agreements is the conflict point. Since formally the conflict point is a possible agreement, this case is not excluded by the rules of the bargaining game.

and (H3) stated below. Unless he himself has deviated from his equilibrium strategy in the past, he always demands 75-ε. Why he demands exactly 75-ε will be partly explained in section 6.

(H1) At stage 1 demand 75-ε.

(H2) At stages 1,...,k-1 your demand was always 75-ε. In this case at stage k demand 75-ε.

(H3) At at least one of the stages 1,...,k-1 your demand was different from 75-ε. In this case select your demand at stage k as prescribed by recommendations (N2) and (N3).

A type L subplayer in the distinguished equilibrium point obeys the specific recommendations (L1) to (L4) stated below. For the purpose of having a suggestive way of expressing the content of (L3) and (L4) we introduce the following definition. "The other player's expected offer at stage k" is that offer which will be made by the other player at stage k if he is a type H subplayer who obeys the general recommendations (G1) and (G2) and the specific recommendations (H1), (H2) and (H3).

Recommendations (L3) and (L4) have the interpretation that in these situations the type L subplayer acts on the assumption that the other player is a type H subplayer.

(L1) At stage 1 demand 75-ε.

(L2) At stage 1 your demand was 75-ε and the other player's demand was 75-ε. In this case at stage 2 demand 50.

(L3) At a stage k where (L2) does not apply your conceded payoff is not greater than your payoff in the other player's expected offer. In this case choose the other player's expected offer at stage k as your offer at stage k.

(L4) At a stage k where (L2) does not apply your conceded payoff is greater than your payoff in the other player's expected offer. In this case choose your conceded payoff at stage k as your demand at stage k.

Equilibrium plays: If both players follow the specific recommendation (N1), both of them demand 50 at stage 1 and agreement is reached immediately. The non-distinguishing equilibrium point yields an equilibrium payoff of 50 for each of the four subplayers.

Suppose that the strategies prescribed by the distinguishing equilibrium point are played. If this happens the demands of the subplayers are those shown in table 4. A type H subplayer demands

	stage 1	stage 2	stage 3
type H	$75 - \varepsilon$	$75 - \varepsilon$	$75 - \varepsilon$
type L	$75 - \varepsilon$	50	$25 + \varepsilon$

Table 4: Equilibrium demands of the distinguishing equilibrium point

$75 - \varepsilon$ at stage 1,2 and 3 because of (H1) and (H2). A type L subplayer demands $75 - \varepsilon$ at stage 1 and 50 at stage 2 according to (L1) and (L2). At stage 3 a type L subplayer applies (L3). He chooses the other player's expected offer as his offer and demands $25 + \varepsilon$. After at most three stages the play is over.

If two type H subplayers meet, conflict results at stage 2. If a type H subplayer has a type L opponent, then at stage 3 an agreement is reached where the type H subplayer receives $75 - \varepsilon$ and the type L subplayer receives $25 + \varepsilon$. In the case of two type L subplayers an agreement is reached at stage 2 where both receive 50. The equilibrium payoffs are shown in the bimatrix of table 5. Since for each of both types both types of the other player have the conditional probability 1/2, the

	H	L	expected equilibrium payoff
H	a a	$75-\varepsilon$ $25+\varepsilon$	$37\frac{1}{2} - \frac{\varepsilon}{2} + \frac{a}{2}$
L	$25+\varepsilon$ $75-\varepsilon$	50 50	$37\frac{1}{2} + \frac{\varepsilon}{2}$

Table 5: Payoffs at the distinguishing equilibrium point

types have the expected payoffs indicated at the right of the bimatrix.

5. Strictness of the non-distinguishing and the distinguishing equilibrium points

Up to now we did not yet prove that the equilibrium points described in the last section are in fact equilibrium points of the bargaining game. In the following we shall do this and,moreover, we shall show that both equilibrium points are strict.

The non-distinguishing equilibrium point: We first look at the non-distinguishing equilibrium point. Obviously no deviation yields a payoff of more than 50 if the other player plays his equilibrium strategy. This shows that the non-distinguishing equilibrium point is an equilibrium point. In order to prove that it is strict, we must look at the alternative best replies. Clearly, an alternative best reply must lead to an agreement where both players receive a payoff of 50 if the other player follows his equilibrium strategy. An alternative best reply leaves the other player's payoff unchanged. This shows that the non-distinguishing equilibrium point is strict.

For the limiting case $a = 50$ the behavior prescribed by the non-distinguishing equilibrium point still has the properties of an equilibrium point but this equilibrium point fails to be strict. Strategies which lead to conflict are alternative best replies for a type H subplayer. Therefore the limiting case $a = 50$ is excluded from the range $0 \leq a < 50$ where we consider the non-distinguishing equilibrium point.

The distinguishing equilibrium point, situation of a type H subplayer: Let us now turn our attention to the distinguishing equilibrium point which is defined for $25 + \varepsilon < a \leq 50$. Consider the situation of a type H subplayer. Suppose that the other player behaves as prescribed by the distinguishing equilibrium point. If the type H subplayer plays his equilibrium strategy, he receives 75 - ε,if the other player is of type L, and he receives a, if the other player is of type H. As we shall see no deviation can improve his payoff in at least one of these cases.

Suppose that the other player is of type H. Because of (H1) and (H2) the other player's demand will always be 75 - ε. A deviation will either lead to conflict or to an agreement where the deviator receives 25 + ε. Since we have 25 + $\varepsilon \angle a$, this does not improve his payoff.

Suppose that the other player is of type L. The deviation is without consequence if it does not occur before stage 3. If the deviator's demand was 75-ε at stages 1 and 2 then this offer will be accepted by a type L opponent at stage 3. Assume that the first deviation from the demand 75-ε occurs at stage 1 or stage 2. After this has happened the type L opponent will have to apply (L3). According to (N2) he will expect a demand of 50 and act accordingly. This means that the deviator must either face conflict or accept an agreement which gives him 50 or less. (He has the choice to demand even less than 50,if he is foolish enough to do so.)

A deviation cannot improve the payoff of a type H subplayer. An alternative best reply must have the property that it leads to conflict if the other player is of type H and to the same agreement as the equilibrium strategy if the other player is of type L. This shows that as far as the equilibrium strategy of a type H subplayer is concerned, the distinguishing equilibrium point has the properties of a strict equilibrium point.

<u>The distinguishing equilibrium point, situation of a type L subplayer</u>: Consider the situation of a type L subplayer. Suppose that the other player behaves as prescribed by the distinguishing equilibrium point. If the type L subplayer

plays his equilibrium strategy he receives 25 + ε if the other player is of type H. He receives 50 if the other player is of type L.

Assume that the other player is a type H subplayer. No deviation of the type L subplayer yields more than 25 + ε. if at least once he asks for less, he will not even get that much. This follows by (G1) and (G2). If he never asks for less, then the other player's demand will always be 75-ε. This follows by (H1) and (H2). Clearly, in this case the type L subplayer cannot get more than 25 + ε. Moreover he cannot get this payoff unless the same agreement is reached with a type H subplayer as by the equilibrium strategy.

In the following it will be useful to distinguish between two kinds of deviation strategies of a type L subplayer. Consider a deviation strategy where a type L subplayer demands 75 - ε at stage 1 and where he also demands 75 - ε at stage 2 if both players' demands at stage 1 were 75 - ε. If he does this he behaves as if he were a type H subplayer. We may say that he imitates type H. Therefore we call such strategies "imitation strategies". Other strategies are called "non-imitation strategies".

We shall first look at the non-imitation strategies. Here it will be shown that a non-imitation strategy cannot yield a higher payoff than 50 if the other player is a type L subplayer who uses his equilibrium strategy. Moreover, in order to yield 50 the non-imitation strategy must lead to the same agreement as the equilibrium strategy. In view of our result for the case that the other player is of type H, this is sufficient in order to prove that as far as the non-imitation strategies are concerned the distinguishing equilibrium point has the properties of a strict equilibrium point.

<u>Non-imitation strategies:</u> We can distinguish two classes of non-imitation strategies. Class 1 contains all those non-imitation strategies where at stage 1 the demand is different

from 75 - ε. Class 2 contains the remaining non-imitation strategies.

In the following it will be assumed that the other player is a type L subplayer who plays his equilibrium strategy. We have to show that under these circumstances the deviator cannot get more than 50 and that he cannot get 50 unless his non-imitation strategy always leads to the agreement where both he and the other type L subplayer receive 50.
We shall first do this for the non-imitation strategies in class 1.

Non-imitation strategies, class 1: It will be convenient to distinguish 4 subclasses of class 1.

Subclass 1.1 : the demand at stage 1 is smaller than 25+ε.

Subclass 1.2 : the demand at stage 1 is 25+ε.

Subclass 1.3 : the demand at stage 1 is greater than 25+ε and smaller than 50.

Subclass 1.4 : the demand at stage 1 is not smaller than 50.

In the case of subclass 1.1, at stage 2 the other player finds himself in a situation where (G1) applies. He accepts the deviator's offer from stage 1. The deviator's expected payoff is less than 25 + ε.

In the case of subclass 1.2 agreement is reached at stage 1. The deviator receives 25 + ε.

In the case of subclass 1.3, at stage 2 the other player finds himself in a situation where (L4) applies. He expects a demand of 50 but his conceded payoff is greater than 50. He demands his conceded payoff. At stage 3 there will be one or two accepted agreements. The deviator's expected payoff is less than 50.

In the case of subclass 1.4, at stage 2 the other player finds himself in a situation where (L3) applies. He expects a demand of 50 and his conceded payoff is not greater than 50. His demand at stage 2 is 50. Moreover at any later stage k which may occur he will always find himself in a situation where (G1), (G2), (L3) or (L4) apply. Whereever (L3) or (L4) apply, he will expect a demand of 50. The deviator cannot get more than 50 and he cannot get 50 unless the agreement is reached where both receive 50.

The exploration of the 4 subclasses has shown that as far as the non-imitation strategies of class 1 are concerned, the distinguishing equilibrium point has the properties of a strict equilibrium point.

Non-imitation strategies, class 2: Here we shall distinguish 3 subclasses according to the demand of the deviator at stage 2 if the demands of both players were 75 - ε at stage 1. For the sake of shortness we shall refer to this demand as the demand at stage 2.

Subclass 2.1 : the demand at stage 2 is smaller than 50.

Subclass 2.2 : the demand at stage 2 is 50.

Subclass 2.3 : the demand at stage 2 is greater than 50.

In the case of subclass 1.1 the other player applies (G1) at stage 2. In the case of subclass 2.2 agreement is reached at stage 2. In both cases the deviator does not get more than 50. If he gets 50 then the other player gets 50, too.

In the case of subclass 2.3 the other player applies (L3) at stage 3. He expects a demand of 50 and his own demand at stage 3 is 50. Moreover at any later stage k which may occur he will always find himself in a situation where (G1), (G2), (L3) or (L4) apply. His own demand will be never less than 50. The deviator cannot get more than 50 and if he gets 50, the other player receives 50, too.

The exploration of the non-imitation strategies is now completed. As far as the non-imitation strategies are concerned, the distinguishing equilibrium point has the properties of a strict equilibrium point.

Imitation strategies: In the following we shall show that the expected payoff of a type L subplayer is smaller than his expected equilibrium payoff at the distinguishing equilibrium point if he uses an imitation strategy whereas the other player behaves as prescribed by the distinguishing equilibrium point. This remains to be shown in order to prove that the distinguishing equilibrium point is a strict equilibrium point of the bargaining game.

Suppose that the other player is a type H subplayer who follows his equilibrium strategy. In this case the use of an imitation strategy leads to conflict at stage 2. The deviator's payoff is 0.

Among the imitation strategies there is one where the type L subplayer always behaves as if he were a type H subplayer who obeys the prescriptions of the distinguishing equilibrium point. This imitation strategy will be called the "bluff strategy". Unlike the bluff strategy other imitation strategies may fail to imitate the type H subplayer's behavior in many situations which may arise at stage 2 or later stages.

Now assume that the other player is a type L subplayer who follows his equilibrium strategy. In this case the use of the bluff strategy will produce the following result. The deviator demands $75 - \varepsilon$ at stages 1,2 and 3. At stage 1 the other player demands $75 - \varepsilon$, at stage 2 he demands 50 and at stage 3 he demands $25 + \varepsilon$. At stage 3 an agreement is reached where the deviator receives $75 - \varepsilon$ and the other player receives $25 + \varepsilon$.

Up to stage 2 other imitation strategies will lead to the same demands as the bluff strategy but there may be a difference at stage 3. A difference will not be important as long as the deviator's demand at stage 3 is neither $25 + \varepsilon$ nor 50. Only if this happens two accepted agreements are reached at stage 3. Clearly, this situation is less favorable to the deviator than that produced by the bluff strategy. Consequently, we can concentrate our attention on the bluff strategy.

If the other player obeys the prescriptions of the distinguishing equilibrium point, a type L subplayer who uses the bluff strategy will receive a payoff of 0 if the other player is of type H,and a payoff of $75 - \varepsilon$ if the other player is of type L. Since both types of the other player are equally probable this yields an expected payoff of $37\frac{1}{2} - \frac{\varepsilon}{2}$ for the deviating type L subplayer. This expected payoff is smaller than the expected payoff $37\frac{1}{2} + \frac{\varepsilon}{2}$ at the distinguishing equilibrium point. Consequently, the distinguishing equilibrium point is a strict equilibrium point.

6. Why exactly $75 - \varepsilon$?

At first glance it is hard to understand why the demand prescribed by (H1), (H2) and (L1) should be exactly $75 - \varepsilon$. In order to give a partial explanation for this, which shows why this demand is not greater than $75 - \varepsilon$, let us look at a modified set of recommendations where the demand $75 - \varepsilon$ is replaced by some demand b with $50 < b < 100$ where b is a multiple of ε. The modified recommendations (H1), (H2) and (L1) together with the other recommendations for the distinguishing equilibrium point describe a strategy combination for the bargaining game which we shall call the "b-modification" of the distinguishing equilibrium point.

As we shall see a b-modification with $b > 75 - \varepsilon$ fails to be a strict equilibrium point. For $b = 75$ the strictness property is lost and for $b > 75$ the b-modification fails to be an equilibrium point. For $b < 75$ a b-modification is a strict equilibrium point if a is sufficiently large. $100 - b < a$ is a sufficient condition.

Suppose that we have $b < 75$ and $100 - b < a$. In the same way as this has been done in section 5 for the distinguishing equilibrium point, we can prove that the b-modification is a strict equilibrium point. It is sufficient to replace $75 - \varepsilon$ by b and $25 + \varepsilon$ by $100 - b$. This shall not be done here in detail.

In order to show that for $b > 75 - \varepsilon$ the b-modification fails to be an equilibrium point for the whole range $0 < a \leq 50$ of the opportunity cost parameter, it is sufficient to look at those strategies of a type L subplayer which correspond to the bluff strategy. In order to have a convenient name we shall speak of a **b-modification bluff strategy**. A type L subplayer uses a b-modification bluff strategy if he always behaves in the same way as if he were a type H subplayer who obeys the prescriptions of the b-modification.

If all four subplayers behave as prescribed by the b-modification, then the payoffs are as described in the first two rows of table 6. If one type L subplayer uses his b-modification bluff strategy whereas the other three subplayers behave as prescribed by the b-modification, then the payoffs in the third row of table 6 result. for $b > 75$ the expected payoff b/2 is greater then 75-(b/2). The b-modification bluff strategy yields a higher expected payoff than the type L strategy prescribed by the b-modification. Consequently for $b > 75$ the b-modification fails to be an equilibrium point.

	H	L	Expected payoff
H	a a	b 100-b	$\frac{a+b}{2}$
L	100-b b	50 50	$75 - \frac{b}{2}$
L-bluff	O a	b 100-b	$\frac{b}{2}$

Table 6 : b-modification payoffs

With the help of arguments used in section 5 it can be seen easily that for b = 75 the b-modification is an equilibrium point. The strictness property, however, is lost. The b-modification bluff strategy is an alternative best reply which drastically changes the expected payoffs of the subplayers on the other side. It is very important for them whether the equilibrium strategy or the b-modification bluff strategy is used by the type L player.

It seems to be desirable to have a theory which yields $b = 75 - \varepsilon$ rather than b = 75. Intuitively it is clear that the b-modification with b = 75 is a much less stable equilibrium point than the distinguishing equilibrium point. This is the reason why John C. Harsanyi and the author felt that they should require strictness in the theory of [HS].

It is now understandable that the demand prescribed by (H1), H2) and (L1) is not greater than $75 - \varepsilon$.

7. Some remarks on the structure of the distinguishing equilibrium point.

If the distinguishing equilibrium is played conflict results if the type combination is (H,H) and different agreements are reached for the three other type combinations (H,L), (L,H) and (L,L). (Here the letters indicate the types of player 1 and 2 in that order). It is interesting to see how this dependence on the type combination is achieved by the dynamics of the bargaining process.

At stage 1 both types make the same demand $75 - \varepsilon$. Therefore at the beginning of stage 2 a player does not know more about the other player's type than before the start of the game. After both players have made their opening demands of $75 - \varepsilon$, a type H player must risk conflict by the repetition of this demand. If he does not repeat his demand a type L player on the other side will insist on a payoff of 50. If he repeats his demand he demonstrates the strength of his bargaining position. A type L player does not repeat his opening demand. His demand of 50 reveals the weakness of his bargaining position. If both play their equilibrium strategies, both players know the other player's type after the completion of stage 2. The dynamics of the bargaining process are such that this information is effectively transmitted.

Some of the features of the distinguishing equilibrium point are arbitrary and without much importance from the point of view of the theory of [HS]. (The same is true for the the non-distinguishing equilibrium point.) In order to make this clear, we introduce the following definition. Two equilibrium points of the bargaining game are called "result equivalent" if for every type combination both equilibrium points produce the same probability distribution over the possible end results (the agreements in U and conflict). In particular, an equilibrium point is result equivalent to the distinguishing equilibrium point if (H,H) leads to conflict and (H,L), (L,H), (L,L) lead to the agreements (75-ε, 25+ε), (25+ε, 75-ε), (50,50), resp. (here the first number is player 1's payoff and the second one is player 2's payoff.)

It can be seen easily that the bargaining game has many equilibrium points which are result equivalent to the distinguishing equilibrium point. Nevertheless, some of the details of the behavior prescribed by the distinguishing equilibrium point are less arbitrary than one might think at first glance. In order to show this we shall prove a proposition about the demands in stage 1.

<u>Proposition on first stage demand equality:</u> Consider an equilibrium point of the bargaining game where the equilibrium strategies of the type H subplayers prescribe demands of 75 - ε at stage 1. If an equilibrium point of

this kind is result equivalent to the distinguishing equilibrium point then the equilibrium strategies of the type L subplayers prescribe demands of 75 - ε at stage 1.

Proof of the proposition: Suppose that there is an equilibrium point of the bargaining game which is result equivalent to the distinguishing equilibrium point and has the property that at stage 1 the type H subplayers demand 75 - ε, whereas at least one type L subplayer, say the type L subplayer of player 1, demands something else. Assume that all subplayers with the exception of player 2's type L subplayer obey the prescriptions of this hypothetical equilibrium point. As we shall see player 2's type L subplayer can improve his payoff by adopting the following mode of behavior: at stage 1 de demands 75 - ε; if he observes that the other player's demand is 75 - ε at stage 1, he demands 25 + ε at stage 1; if he observes that the other player's demand at stage 1 is different from 75 - ε, then at stage 2 and all later stages which may occur, he behaves as if he were a type H subplayer who obeys the prescriptions of the hypothetical equilibrium point. In this way he receives 25 + ε if the other player is of type H and 75 - ε if the other player is of type L. The hypothetical equilibrium strategy yields 25 + ε if the other player is of type H and 50 if the other player is of type L. This is a contradiction which shows that the proposition is true.

Interpretation: the fact that at stage 1 a type L subplayer behaves in the same way as a type H subplayer is not an arbitrary feature of the distinguishing equilibrium point. At stage 1 both types begin with the same high demand. A type L player does not prematurely reveal the weakness of his bargaining position. Thereby he deters the type L subplayer of the other player from posing as a type H subplayer. The type H subplayer of the other player is forced to demonstrate his strength by taking the risk of conflict.

8. The main representation

The nature of the solution concept applied in this paper has been explained in section 3. The approximate solution is a payoff vector for the four subplayers. The main representation is a special probability mixture of equilibrium points which yields this payoff vector. It is the purpose of this section to describe the main representation and its dependence on the opportunity cost parameter. Proofs are deferred to later sections.

The main representation is a probability mixture of the non-distinguishing equilibrium point and the distinguishing equilibrium point. The probabilities depend on the opportunity cost parameter. Let p(a) be the probability of the distinguishing equilibrium point; the probability of the non-distinguishing equilibrium point is 1-p(a).

Since we are not interested in the influence of the parameter ε we do not compute the exact expected payoffs at the main representation but the limit of these payoffs for $\varepsilon \to 0$. These payoffs will be called the "limit payoffs". The limit payoffs for the distinguishing equilibrium point are $37\frac{1}{2} + \frac{a}{2}$ for type H and $37\frac{1}{2}$ for type L. (See table 5). In order to compute the limit payoffs for the main representation one multiplies these payoffs by p(a) and then adds 50(1-p(a)), the term which comes from the non-distinguishing equilibrium point. Table 7 shows how p(a) and the limit payoffs depend on the opportunity cost parameter. The limit payoffs are also shown in figure 1.

Only in the small intervall between $33\frac{1}{2}$ and $37\frac{1}{2}$ the main representation is a proper probability mixture where both equilibrium points have positive probabilities. Everywhere else the main representation prescribes just one of both equilibrium points.

opportunity cost parameter	probability of the distinguished equilibrium point	limit payoffs	
a	p(a)	type H	type L
$0 \leq a \leq 33\frac{1}{3}$	0	50	50
$33\frac{1}{3} < a < 37\frac{1}{2}$	$\frac{3a-100}{a-25}$	$\frac{3}{2}a$	$\frac{25a}{2a-50}$
$37\frac{1}{2} \leq a \leq 50$	1	$37\frac{1}{2} + \frac{a}{2}$	$37\frac{1}{2}$

Table 7: the main representation

For $0 \leq a \leq 33\frac{1}{3}$ only the non-distinguishing equilibrium point appears in the main representation. The generalized Nash solustion does not yield different payoffs for both types unless the opportunity cost parameter is sufficiently high.
For $37\frac{1}{2} \leq a \leq 50$ only the distinguishing equilibrium point appears in the main representation. As one would expect, p(a) is a non decreasing function of the opportunity cost parameter.

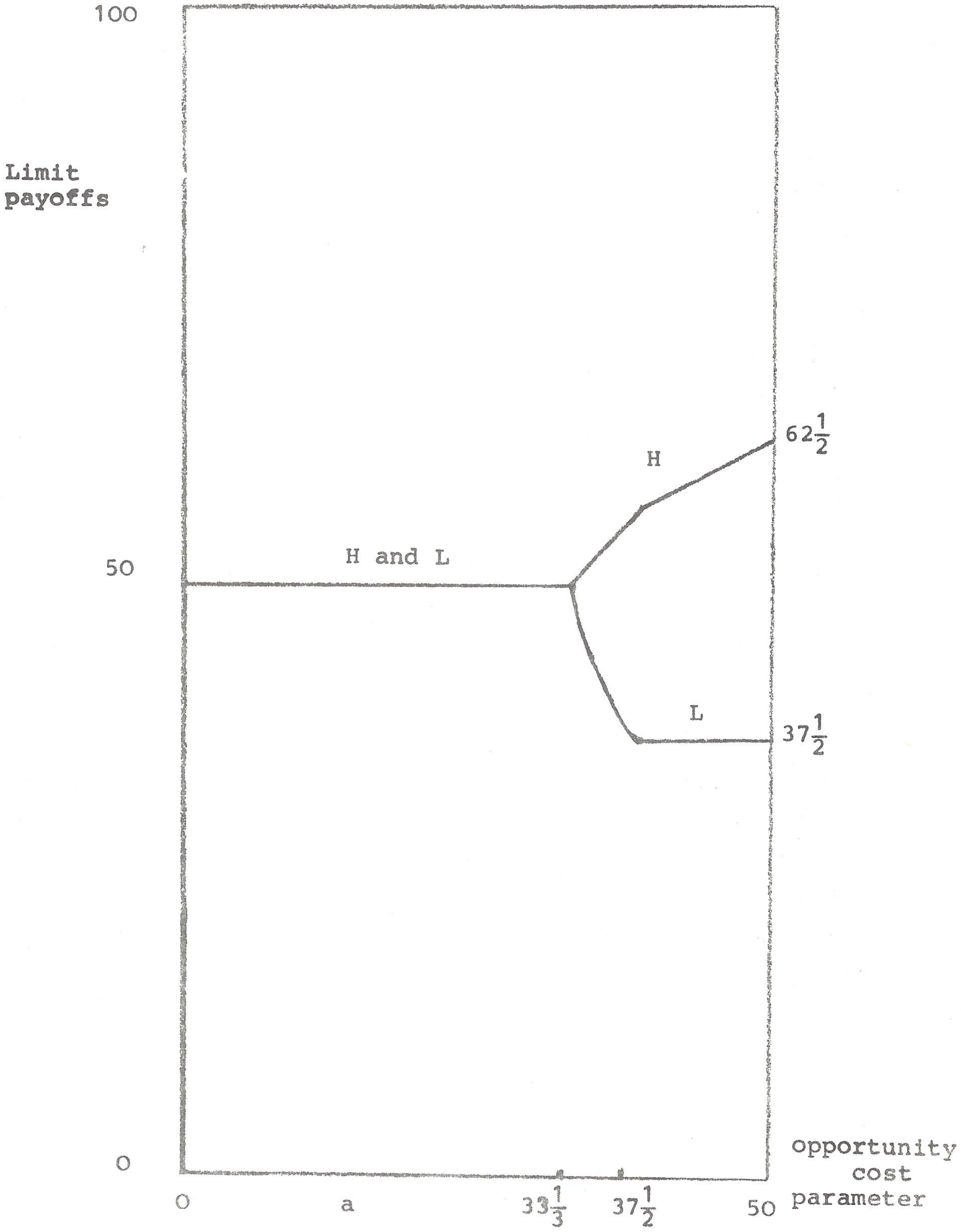

Figure 1: limit payoffs of the types

9. Preparation of the proof that the main representation represents an approximate generalized Nash solution

The remaining sections of the paper will show that the main representation described in section 8 is in fact the representation of an approximate generalized Nash solution.

In the following we shall use the notation and the terminology of [HS]. As has been explained at the end of section 1, the subplayers will be numbered from 1 to 4.

The generalized Nash solution $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_4)$ is a vector of expected payoffs for each of the four subplayers. Sometimes, in order to make the dependence on ε visible, we shall use the notation $\tilde{x}^{\varepsilon}= (\tilde{x}_1^{\varepsilon},\ldots,\tilde{x}_4^{\varepsilon})$ for the generalized Nash solution.

Let $x^{\varepsilon} = (x_1^{\varepsilon},\ldots,x_4^{\varepsilon})$ be the payoff vector which belongs to the main representation described in section 8. (Here the ε-terms are not neglected.) Let $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_4)$ be the limit payoff vector with the limit payoffs from table 7. For $\varepsilon \to 0$ the payoff vector x^{ε} converges to $\bar{x}$.

Suppose that $\hat{x}^{\varepsilon}= (\hat{x}_1^{\varepsilon},\ldots,\hat{x}_4)$ is an expected payoff vector which in the same way as x^{ε} has a representation by a probability mixture of strict equilibrium points of the bargaining game. ($\hat{x}^{\varepsilon}$is a function of ε and a.) A payoff vector $\hat{x}^{\varepsilon}$ of this kind is called an "approximate solution" of the bargaining situation if for $\varepsilon \to 0$ everywhere in the intervall $0 \leq a \leq 50$ the payoff vectors $\hat{x}^{\varepsilon}$ and $\tilde{x}^{\varepsilon}$ converge to the same limit. In this precise sense we shall prove that x^{ε} is an approximate solution.

The generalized Nash solution $\tilde{x}$ is an element of the equilibrium set X. The equilibrium set X is the convex hull of the set of all strict equilibrium payoff vectors. Since we do not have a complete overview over all strict equilibrium points of the bargaining game, we cannot maximize the generalzed Nash product over X. The task of finding the limit of the generalized Nash solution for ε 0 will be attacked in another way. We shall construct a "substitute equilibrium set" X' which contains X as a subset. The generalized Nash product assumes its maximum over X' at the limit payoff vector $\bar{x}$ for the main representation. With the help of this fact it is possible to show that x^{ε} is an approximate solution.

In this section we shall prove that $\tilde{x}$ must satisfy an important condition (equation (21)) which can be used for the construction of the substitute equilibrium set. This condition can be derived from the noncooperative equilibrium properties of $\tilde{x}$ and from axion 2 (player symmetry). One consequence of this condition is that we must have $\tilde{x}_1 \geq \tilde{x}_2$ and $\tilde{x}_3 \geq \tilde{x}_4$ as one would expect intuitively. It is interesting that this can be shown with the help of axiom 2 alone without making use of the other axioms.

Since our example is symmetric with regard to the players, it follows from axiom 2 (player symmetry) that the generalized Nash solution satisfies:

(1) $$\tilde{x}_1 = \tilde{x}_3$$

and

(2) $$\tilde{x}_2 = \tilde{x}_4$$

since $\tilde{x}$ is an element of X, it can be represented as a mixture of a finite number of strict equilibrium payoff vectors y^j:

(3) $$\tilde{x} = \sum_{j=1}^{J} \alpha_j y^j \qquad \text{with} \quad \sum_{j=1}^{J} \alpha_j = 1$$

and $0 < \alpha_j \leq 1$ for $j = 1,\ldots J$. To each of the y^j we can find a strict equilibrium point

$$t^j = (t_1^j, t_2^j, t_3^j, t_4^j) \tag{4}$$

Because of the symmetry of the bargaining game we can construct a new strict equilibrium point t^{J+j} by interchanging the roles of player 1 and 2 in t^J. Let y^{J+j} be the conditional payoff vector of t^{J+j}. Obviously y^{J+j} is derived from y^j by interchanging the players. Therefore because of (1) and (2) we have

$$\tilde{x} = \sum_{j=1}^{J} \alpha_j y^{J+j} \tag{5}$$

It is now clear that the conditional payoff vector $\tilde{x}$ can be realized by a probability mixture of strict equilibrium points which is completely symmetric with regard to the players; the equilibrium points $t^1, \dots, t^{2J}$ must be used with probabilities $\alpha_j/2$ where α_j equals to α_{J-j} for $j = J+1, \dots, 2J$. From now on we shall refer to that mixture as the "symmetric mixture".

Let W_{km} be the probability that the conflict point c is reached in the symmetric mixture. (The conflict point can be reached as the result of a conflict offer sequence or as an agreement.) Le $\overline{W}_{km}$ be the complementary probability $1-W_{km}$. Now consider the case that an agreement other than c is reached in the symmetric mixture; let z_{km} be the conditional expectation of player 1's payoff in such an agreement if player 1 is of type k and player 2 is of type m. Since agreements other than c have the form of table 3, the conditional expectation of player 2's payoff corresponding to z_{km} is $100-z_{km}$. We use the symbol $\overline{z}_{km}$ for $100-z_{km}$.

Since player 1 and player 2 behave exactly symmetrically in the symmetric mixture we must have

$$W_{12} = W_{21} \tag{6}$$

and

$$z_{km} = \bar{z}_{mk} \qquad \text{for } k = 1,2 \text{ and } m = 1,2 \tag{7}$$

For k = m equation (7) yields

$$z_{11} = z_{22} = 50 \tag{8}$$

It can now be seen easily that the components of $\tilde{x}$ can be written as follows:[4)]

$$\tilde{x}_1 = \tilde{x}_3 = \frac{1}{2}(W_{11}+W_{12})a + \frac{1}{2}W_{11}\cdot 50 + \frac{1}{2}W_{12}z_{12} \tag{9}$$

$$\tilde{x} = \tilde{x}_4 = \frac{1}{2}\bar{W}_{12}\bar{z}_{12} + \frac{1}{2}\bar{W}_{22}\cdot 50 \tag{10}$$

The fact that the symmetric mixture is a mixture of equilibrium points puts constraints on the parameters W_{km} and z_{km} in (9) and (10). In order to derive such constraints we look at certain specific deviations from the behavior prescribed by the symmetric mixture.

Consider the following deviation of subplayer 1: while all other subplayers stick to the equilibrium behavior, subplayer 1 "imitates" subplayer 2, i.e. instead of t_1^j he uses t_2^j in all equilibrium points t^j occurring in the symmetric mixture. Let x_1' be the first component of the conditional payoff vector which belongs to the mixture of strategy combinations resulting from this deviation. In the new mixture the probabilities of conflict between subplayer 1 and subplayers 3 and 4 are W_{21} and W_{22} because subplayer 1 now behaves in the same way as subplayer 2. Nevertheless, x_1' is different from $\tilde{x}_2$ since the conflict payoff of player 1 is not 0. The deviation payoff $\tilde{x}_1$ can be written as follows:

[4)] For $\bar{W}_{12} = 0$ a conditional expectation z_{12} does not exist; but in that case an arbitrary value can be given to z_{12} without any influcence on the validity of (8) and (9).

(11) $$x_1' = \frac{1}{2}(W_{12} + W_{22})a + \frac{1}{2}\bar{W}_{12}\bar{z}_{12} + \frac{1}{2}\bar{W}_{22}\cdot 50$$

We now consider the following deviation of subplayer 2: while all other subplayers stick to the behavior prescribed by the symmetric mixture, subplayer 2 imitates subplayer 1, i.e. instead of t_2^j he uses t_1^j in every t^j. The following conditional payoff x_2'' of subplayer 2 results from this deviation:

(12) $$x_2'' = \frac{1}{2}\bar{W}_{11}\cdot 50 + \frac{1}{2}\bar{W}_{12}z_{12}$$

Since the symmetric mixture is a mixture of equilibrium points, we must have

(13) $$x_1 \leq \tilde{x}_1$$

and

(14) $$x_2'' \leq \tilde{x}_2$$

Because of (13) and (14) the following is true:

(15) $$x_1 - \tilde{x}_2 \leq \tilde{x}_1 - \tilde{x}_2 \leq \tilde{x}_1 - x_2''$$

Equations (9) - (11) yield

(16) $$x_1' - \tilde{x}_2 = \frac{1}{2}(W_{12} + W_{22})a$$

and

(17) $$\tilde{x}_1 - x_2'' = \frac{1}{2}(W_{11} + W_{12})a$$

Therefore (15) is equivalent to (18):

(18) $$\frac{1}{2}(W_{12} + W_{22})a \leq \tilde{x}_1 - \tilde{x}_2 \leq \frac{1}{2}(W_{11} + W_{12})a$$

The quantities

(19) $$W_1 = \frac{1}{2}(W_{11} + W_{12})$$

and

(20) $$W_2 = \frac{1}{2}(W_{12} + W_{22})$$

have an obvious interpretation: W_i is the probability that a player of type i reaches conflict (i.e. the conflict point or agreement at c.) Condition (18) can now be written as follows:

(21) $$W_2 a \leq \tilde{x}_1 - \tilde{x}_2 \leq W_1 a$$

Since W_2 cannot be negative it follows from (21) that we must have $\tilde{x}_1 \geq \tilde{x}_2$. Because of axiom 1 (profitability) we must have $\tilde{x}_1 > a$. Therefore it follows from (9) that $W_i < 1$ is true. This together with (21) yields:

(22) $$W_2 \leq W_1 < 1$$

Condition (21) gives upper and lower borders for the difference between $\tilde{x}_1$ and $\tilde{x}_2$. In the beginning of the next section we shall derive an equation for the sum of $\tilde{x}_1$ and $\tilde{x}_2$.

10. Maximizing the generalized Nash product over a substitute equilibrium set

Adding up equations (9) and (10) we receive

(23) $$\tilde{x}_1 + \tilde{x}_2 = W_1 a + \frac{1}{2}(\overline{W}_{11} + \overline{W}_{22})50 + \frac{1}{2}\overline{W}_{12} \cdot 100$$

because of

(24) $$\frac{1}{2}(\overline{W}_{11} + \overline{W}_{22})50 + \frac{1}{2}\overline{W}_{12} \cdot 100 = (2-W_1-W_2)50$$

this is equivalent to

(25) $$\tilde{x}_1 + \tilde{x}_2 = 100 - (50-a)W_1 - 50W_2$$

With (21) and (25) we have derived two conditions for $\tilde{x}$ which involve only W_1 and W_2 and neither z_{12} nor any of the probabilities W_{mk}. For the application of these conditions to our problem of maximizing the generalized Nash product it is important that W_1 and W_2 cannot be chosen independently. Because of (19) and (20) we must have

(26) $$W_1 - W_2 = \frac{1}{2}(W_{11} - W_{12}) \leq \frac{1}{2}$$

(22) and (26) yield

(27) $$0 \leq W_1 - W_2 \leq \frac{1}{2}$$

Let X' be the set of all conditional payoff vectors $x = (x_1,...,x_4)$ with $x_1 = x_3$ and $x_2 = x_4$, such that the conditions (27),

(28) $$W_2 a \leq x_1 - x_2 \leq W_1 a$$

and

(29) $$x_1 + x_2 = 100 - (50-a)W_1 - 50W_2$$

are satisfied for at least one pair of probabilities W_1 and W_2 with $W_1 < 1$. Because of (21) and (25) the generalized Nash solution $\tilde{x}$ must be an element of X'. Instead of maximizing the generalzed Nash product over the equilibrium set X we shall approach our task by maximizing over X'. Therefore X' might be thought of as a substitute for the equilibrium set X.

Since in our example all the marginal probabilities are equal to 1/2 and since in X' we always have $x_1 = x_3$ and $x_2 = x_4$, the generalized Nash product over X' is equal to

(30) $$P = (x_1-a)x_2$$

Define

(31) $$D = \frac{x_1-x_2}{2}$$

and

(32) $$E = \frac{x_1+x_2}{2}$$

Equation (30) is equivalent to

(33) $$P = (E+D-a)(E-D)$$

With the help of (31) and (32) the conditions (28) and (29) can be written as follows:

(34) $$W_2 \cdot \frac{a}{2} \leq D \leq W_1 \cdot \frac{a}{2}$$

(35) $$E = 50 - (25-\frac{a}{2})W_1 - 25W_2$$

Taking the partial derivative of (33) with respect to D we receive

(36) $$\frac{\partial P}{\partial D} = -2D + a$$

Because of (34) and $W_1 < 1$ we must have

(37) $$D < \frac{a}{2}$$

(36) and (37) yield

(38) $$\frac{\partial P}{\partial D} > 0$$

Consequently, for fixed W_1 and W_2 we must make D as big as possible in order to maximize P. Because of (34) this means that D must be chosen in the following way:

(39) $$D = W_1' \frac{a}{2}$$

The task of maximizing P over X' has now been reduced to the task of maximizing P as a function of W_1 and W_2, subject to the side condition (27). Because of (35) and (39) we have

(40) $$E + D - a = 50 - a + (a-25)W_1 - 25W_2$$

and

(41) $$E - D = 50 - 25W_1 - 25W_2$$

It is clear from (33), (40 and (41) that for fixed W_1 the probability W_2 must be chosen as small as possible. Because of (27) this means

(42) $$W_2 = 0 \qquad \text{for } W_1 \leq \frac{1}{2}$$

and

(43) $$W_2 = W_1 - \frac{1}{2} \qquad \text{for } W_1 \geq \frac{1}{2}$$

In view of equations (40) to (43) we can now maximize P as a function of W_1: for $W_1 \leq \frac{1}{2}$ we have

(44) $$P = (50 - a + (a-25)W_1)(50 - 25W_1)$$

and for $W_1 \geq \frac{1}{2}$ we receive

(45) $$P = (62\frac{1}{2} - a + (a-50)W_1)(62\frac{1}{2} - 5oW_1)$$

In order to show that we can exclude the possibility that the maximum is assumed in the interval $\frac{1}{2} \leq W_1 < 1$ we take derivative of (45):

(46) $$\frac{dP}{dW_1} = 12\frac{1}{2}\ (9a-500+(400-8a)W_1)$$

Because of $a < 50$ the expression 400-8a is non-negative. Therefore we have

(47) $$\frac{dP}{dW_1} \leq 12\frac{1}{2}(a-100) < 0 \qquad \text{for } W_1 \geq \frac{1}{2}$$

Consequently, the function (45) assumes its maximum over the interval $\frac{1}{2} \leq W_1 < 1$ at $W_1 = \frac{1}{2}$. This means that the maximum of P over $0 \leq W_1 < 1$ must be assumed in the interval $0 \leq W_1 \leq \frac{1}{2}$. Taking the derivative of (44) we receive

(48) $$\frac{dP}{dW_1} = 25(3a-100-2(a-25)W_1) \quad \text{for } W_1 \leq \frac{1}{2}$$

For $0 < a \leq 33\frac{1}{3}$ the following is true:

(49) $$3a - 100 - 2(a-25)W_1 \leq W_1(A-50) \leq 0$$

Therefore within this interval dP/dW_1 is always negative. Consequently, the maximum of P is assumed at

(50) $$W_1 = 0 \qquad \text{for } 0 < a \leq 33\frac{1}{3}$$

From (48) it can be seen easily that the maximum is assumed at

(51) $$W_1 = \frac{3a-100}{2a-50} \qquad \text{for } 33\frac{1}{3} \leq a \leq 37\frac{1}{2}$$

For $37\frac{1}{2} \leq a \leq 50$ and $0 \leq W_1 \leq \frac{1}{2}$ we have

(52) $$3a-100-2(a-25)W_1 \geq 0$$

Therefore the maximum is assumed at

(53) $$W_1 = \frac{1}{2} \qquad \text{for } 37\frac{1}{2} \leq a \leq 50$$

From (42), (50), (51) and (53) we can compute D and E by (39) and (35); from there the components of the vector which maximizes the generalized Nash product over X' can be determined with the help of (30) and (31). The result is nothing else but the limit payoff vector $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_4)$ for the main representation. The subplayers 1 and 3 receive the type H limit payoff and the subplayers 2 and 4 receive the type L limit payoff from table 7.

11. Completion of the proof that the main representation represent an approximate generalized Nash solution

In order to prove that x^ε, the payoff vector of the main representation, is an approximate solution we have to show that for $\varepsilon \to 0$ the exact solution $\tilde{x}^\varepsilon$ and x^ε converge to the same limit:

$$(54) \qquad \lim_{\varepsilon\to 0} x^\varepsilon = \lim_{\varepsilon\to 0} \tilde{x}^\varepsilon$$

This is the definition of an approximate solution. $\bar{x}$ is the limit of x^ε:

$$(55) \qquad \lim_{\varepsilon\to 0} x^\varepsilon = \bar{x}$$

It remains to be shown that we have

$$(56) \qquad \lim_{\varepsilon\to 0} \tilde{x}^\varepsilon = \bar{x}$$

For this purpose we introduce the following definition. Let X" be the closed hull of the substitute equilibrium set. Since the generalized Nash product is a continuous function of x, the maximum of the generalized Nash product over X" is assumed at $\bar{x}$, too.

The substitute equilibrium set X' is defined by the linear inequalities (27), (28), (29) and $W_1 \angle 1$ together with $x_1 = x_3$ and $x_2 = x_4$. Since these inequalities are linear in x_1, x_2, W_1 and W_2 it can be seen easily that X' is convex. Therefore X'' is convex, too. Since for $x_1 > a$ and $x_2 > 0$ the logarithm of the generalized Nash product P is a strictly concave function of x_1 and x_2, the maximum of P over X'' cannot be assumed anywhere else but at $\bar{x}$.

For every value of ε let X^ε be the equilibrium set of the bargaining situation. Unlike X^ε the substitute equilibrium set X' does not depend on ε. The set X^ε is a subset of X'. Every $\tilde{x}^\varepsilon$ is an element of X'.

Let P^ε be the generalized Nash product of x^ε and let $\tilde{P}^\varepsilon$ be the generalized Nash product of $\tilde{x}^\varepsilon$. Moreover, let $\bar{P}$ be the generalized Nash product of $\bar{x}$. Since $\tilde{P}^\varepsilon$ is the maximum of the generalized Nash product over X^ε and since X^ε is a subset of X' we must have

$$(57) \qquad P^\varepsilon \leqq \tilde{P}^\varepsilon \leqq P$$

(57) together with (55) yields

$$(58) \qquad \lim_{\varepsilon \to 0} \tilde{P}^\varepsilon = \bar{P}$$

Suppose (56) does not hold. Then in view of the boundedness of X' it must be possible to find a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of values of ε such that the sequence of the corresponding generalized Nash solutions $\tilde{x}^\varepsilon$ converges to some limit $\tilde{x}$ which is different from $\bar{x}$. Nevertheless, because of the continuity of the generalized Nash product it follows by (58) that $\bar{P}$ is the generalized Nash product of $\tilde{x}$. On the other hand, $\tilde{x}$ belongs to X'' and the generalized Nash product assumes its maximum over X'' at $\bar{x}$ only. We must have $\tilde{x} = \bar{x}$. This controdiction shows that (56) is true and that x^ε is an approximate solution.

12. Some remarks on the relationship between the approximate and the exact solution

It is interesting to note that the maximization of the generalized Nash product over X' does not only uniquely determine the components of $\bar{x}$ but also the probabilities W_{km} and the conditional expectations z_{km} connected to $\bar{x}$. Because of $W_2 = 0$ we have

(59) $\quad W_{12} = 0$

and

(60) $\quad W_{22} = 0$

(59) together with (50, (51) and (53) yields

(61) $\quad W_{11} = 0 \qquad$ for $0 \leq a \leq 33\frac{1}{3}$

(62) $\quad W_{11} = \frac{3a-100}{a-25} \qquad$ for $33\frac{1}{2} \leq a \leq 37\frac{1}{2}$

(63) $\quad W_{11} = 1 \qquad$ for $37\frac{1}{2} \leq a \leq 50$

The conditional expectations z_{11} and z_{22} are determined by (8). With the help of (59) and (60) the conditional expectation z_{12} can be computed from the components of $\bar{x}$.
We receive

(64) $\quad z_{12} = 50 \qquad$ for $0 \leq a \leq 33\frac{1}{3}$

(65) $\quad z_{12} = 50 + 25\ \frac{3a-100}{a-25} \qquad$ for $33\frac{1}{3} \leq a \leq 50$

(66)) $\quad z_{12} = 75 \qquad$ for $37\frac{1}{2} \leq a \leq 50$

W_{11} is nothing else than the probability p(a) from table 7. The results (59) to (66) show that the main representation has some important features other than the limit payoffs which are uniquely determined by the generalized Nash solution. In particular it follows that for $\varepsilon \rightarrow 0$ the conflict probabilities connected to any representation of $\tilde{x}^{\varepsilon}$ must converge to those given in (59) to (63).

One may ask the obvious question whether there is any difference between the approximate solution and the exact one. There is no such difference for $0 \leqq a \leqq 33\frac{1}{3}$ since here x^{ε} is equal to $\bar{x}$ and $\bar{x}$ belongs to every X^{ε}. Contrary to this for $33\frac{1}{3} \leqq a \leqq 50$, the approximate solution is different from the exact one. In the following we shall sketch a proof for this assertion.

It is possible to construct a mixed strict equilibrium point of the bargaining game which can be substituted for the distinguishing equilibrium point in the main representation; thereby one receives an expected payoff vector with a higher generalized Nash product. In the following this equilibrium point will be called the"mixed distinguishing equilibrium point".

We shall not describe the mixed distinguishing equilibrium point in detail. Instead of this, only those plays will be described which may result if all four subplayers play their equilibrium strategies. Along the lines of sections 4 and 5 one can find recommendations which prescribe a strict equilibrium point with these equilibrium plays.

The equilibrium behavior is as follows:

Stage 1: Both types demand 75.

Stage 2: A type H subplayer demand 75, a type L subplayer demands 50.

Stage 3: Both players select one of the demands d with $d > 75$. Each of these demands is chosen with the same probability.

Stage 4: (a) If in stage 3 both players have made the same demand then the type H subplayer demands $75-\varepsilon$ and the type L subplayer demands $25+\varepsilon$.

(b) If in stage 3 different demands have been made by the players then the type H subplayer demands 75 and the type L subplayer demands 25.

If an equilibrium point of this kind is played instead of the distinguishing equilibrium point in a modified main representation then an expected payoff vector $\hat{x}^{\varepsilon}$ results which is a convex linear combination of x^{ε} and $\bar{x}$. Because of the concavity properties of the generalized Nash product $\hat{x}^{\varepsilon}$ yields a higher generalized Nash product than x^{ε}.

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