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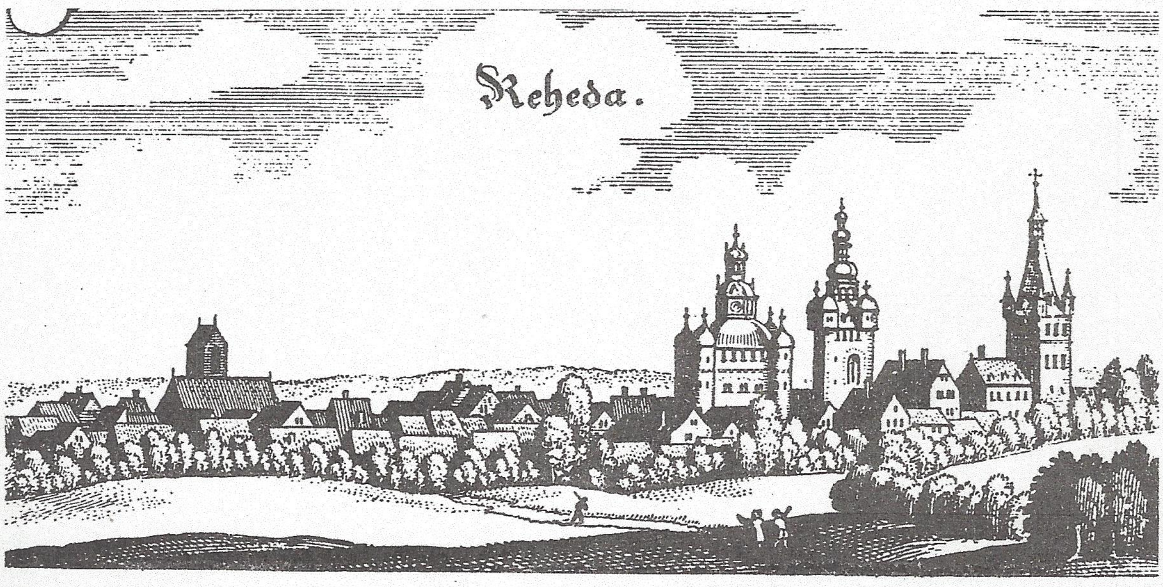

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A SIMPLE MODEL OF IMPERFECT COMPETITION, WHERE 4 ARE FEW AND 6 ARE MANY

It is a widely held belief that in imperfect markets the tendency, to cooperate depends on the number of competitors. E.H.Chamberlin's distinction between the small group and the large group is based on this assumption [1]. Cooperative forms of behaviour like joint profit maximization are assumed to be typical for markets with a small number of competitors and non-cooperative equilibria are expected, if the number of suppliers is sufficiently large.

The theory presented in this paper investigates the connection between the number of competitors and the tendency to cooperate within the context of a simple model. The proposition that few suppliers will maximize their joint profits whereas many suppliers are likely to behave non-cooperatively does not appear as an assumption but as a conclusion of the theory.

The investigation is based on the symmetric Cournot model with linear cost and linear demand, supplemented by specific institutional assumptions about the possibilities of cooperation. Cooperative forms of behavior are modelled as moves in a non-cooperative game. Game-theoretic reasoning is employed in order to find a unique solution for this game.

The distinction between the small group and the large group remains unsatisfactory as long as "small" and "large" are only vaguely defined. Where does the small group end and where does the large group begin? For the simple model of this paper a definite answer can be given to this question: 5 is the dividing line between few and many.

The formal description of the possiblities of cooperation is an important part of the model. It is assumed that the firms are free to form enforcible quota cartels, but before this can be done, each firm must decide whether it wants to participate in cartel bargaining or not. These decisions must be made without knowledge of the corresponding decisions of the other firms. Those firms who have decided to participate may then form a quota cartel. A quota is an upper bound for the supply of a firm. A quota cartel agreement is a system of quotas for all cartel members. The model assumes that each firm, which participates in cartel bargaining, proposes exactly one cartel agreement[1)] and that a quota system for a group of firms becomes binding, if all members of the group have proposed that system.

Before the supply decision is made, the outcome of the bargaining is made known to all firms in the market. If an agreement has been reached, the cartel members cannot exceed their quotas.

This is an extremely simplified picture of cartel bargaining but hopefully at least some of the relevant features of real imperfect markets are captured. Note that nobody can be forced to come to the bargaining table. Cartels may or may not include all firms in the market. Once an agreement has been reached, it cannot be broken. This means that enforcement problems are excluded from the analysis. The only kind of agreement which is allowed, is a system of quotas.

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1) One may think of this as a final proposal which is formally made after extensive informal discussions. The idea that at the end of the bargaining process the bargainers make simultaneous final proposals is maybe more realistic than it appears at first glance. Stevens' book on collective bargaining[13] conveys the impression that agreemtns are often reached by virtually simultaneous last moment concessions after a period of apparent stagnation of the bargaining process.

Within the framework of these institutional assumptions it is advantageous to form a cartel, but if the number of competitors is sufficiently large, it may be even more advantageous to stay out of a cartel formed by others. The fact that the position of an outsider becomes relatively more attractive as the number of competitors is increased, is the basic intuitive reason for the results of this paper.

The task of finding a unique solution for the model presented in this paper cannot be attacked without putting it into a wider framework. It is necessary to develop a solution concept for a class of games, which contains the model as a special case. Only in this way the desirable properties of the proposed solution of the model can be properly described.

Sections 2,3 and 4 contain some game-theoretic results which may be of interest beyond the main purpose of this paper.

1. THE MODEL

The complete model takes the form of a non-cooperative n-person game in extensive form, where the players are n firms numbered from 1,...,n. For the limited purpose of this paper it seems to be adequate to avoid a formal definition of a game in extensive form[2)], but some remarks must be made about the sense in which the words "extensive form" will be useed.

1.1 EXTENSIVE FORMS In this paper a slight generalization of the usual testbook definition of a game in extensive form is used. It is necessary to permit infinitely many choices at some or all information sets of the personal players (this excludes the random player). The set of all choices at an information set of a personal player may be a set, which it topologically equivalent to the union of a finite number of convex subsets of some euclidean space. Apart from that the properties of a finite gametree are retained as much as possible. The set of all

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2) See [5] or [6]. It will be assumed that the reader is familiar with the concept of a game in extensive form and with other basic concepts of game theory

choices at an information set of the random player is finite. Only such games are permitted, which have a finite upper bound for the length of the play. Another slight deviation from the usual definition concerns the payoff. The payoff of a player is a real number or $-\infty$.

The games considered in this paper will always be games with perfect recall, where each player always knows all his previous choices[3]. Therefore it is convenient to exclude all games which do not have this property from the definition of an extensive form. For the purpose of this paper a game in extensive form will be always a possibly infinite game with perfect recall which has the properties mentioned above. Sometimes a game in extensive form will simply be called an "extensive form" or a "game", where no confusion can arise.

It would be quite tedious to describe the model with the help of the terminology of extensive form games. Instead of this a set of rules shall be formulated, which contains all the information needed for the construction of an extensive form. Apart from inessential details like the order, in which simultaneous decisions are represented in the game tree, the extensive form representation of the model is fully determined by this description in an obvious way. Therefore it will be sufficient to relate only some of the features of the model to the formal structure of the extensive form. This will be done after the description of the rules is complete.

<u>1.2 STRUCTURE OF THE MODEL.</u> Wherever this is convenient firm i is called player i. The set N=(1,...,n) of the n first positive integers is interpreted as the set of all players. The subsets of N are called <u>coalitions</u>.

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3) The formal definition of games with perfect recall can be found in [5]. For infinite games with perfect recall see [1].

It is convenient to look at the game as a sequence of three successive stages: 1) the participation decision stage, where the firms decide, whether they want to participate in the cartel bargaining or not; 2) the cartel bargaining stage, where the proposals are made, which may or may not lead to cartel agreements; 3) the supply decision stage, where each firm selects a supply quantity.

At each stage the players know the outcome of the previous stages but they do not know the decisions of the other players at the same stage or at later stages.

The firms are motivated by their gross profits derived from the cost and demand relationship of the Cournot model. It is assumed, that the firms want to maximize expected gross profits in the sense of probability theory, subject to the constraint that the probability of negative gross profits is zero. This is not unreasonable if one imagines a situation, where non-negative gross profits are necessary for survival.

1.3 COST AND DEMAND. The same homogenous good is supplied by all firms. The supply of firm i is denoted by x_i. The quantity x_i is a non-negative real number. $x=(x_i,\dots,x_n)$ is the supply vector. It is assumed that there is no capacity limit. The cost function is the same for each firm:

$$(1) \quad K_i = F + cx_i \quad ; \quad x_i \geq 0 \quad ; \quad i = 1,\dots, n$$

F and c are positive parameters. Total supply

$$(2) \quad X = \sum_{i=1}^{n} x_i$$

determines the price p

$$(3) \quad p = \begin{cases} \beta - \alpha X & \text{for } 0 \leq X \leq \frac{\beta}{\alpha} \\ 0 & \text{for } X > \frac{\beta}{\alpha} \end{cases}$$

Here we assume $\alpha > 0$ and $\beta > c$.

It is always possible to choose the units of measurement for money and for the commodity in such a way that the parameters α and β take the following values

$$(4) \qquad \alpha = -1$$

$$(5) \qquad \beta = 1 + c$$

Therefore we shall always assume that (4) and (5) hold. This simplifies our formulas without entailing any loss of generality. Because of (4) and (5) a simple relationship between the total supply X and the profit margin

$$(6) \qquad g = p - c$$

is obtained:

$$(7) \qquad g = \begin{cases} 1 - X & \text{for} \quad 0 \leq X \leq 1 + c \\ -c & \text{for} \quad X > 1 + c \end{cases}$$

Define

$$(8) \qquad P_i = x_i g \qquad \text{for} \quad i = 1,\dots,N.$$

The variable P_i is the <u>gross profit</u> of firm i; it is the profit without consideration of fixed costs. One may imagine that the fixed costs are "prepaid" and that the availability of liquid funds depends on the gross profit.

The assumption about the motivation of the firms can be expressed by a von-Neumann-Morgenstern utility function:

$$(9) \qquad u_i = \begin{cases} P_i & \text{for} \quad P_i \geq 0 \\ -\infty & \text{for} \quad P_i < 0 \end{cases} \qquad i = 1,\dots,n$$

u_i is player i's utility. Note that u_i does not depend on the parameter c.[4)]

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4) If (9) did not have certain mathematical advantages, it would be preferable to work with the simpler assumption $u_i = P_i$. The main advantage of (9) is the possibility to prove lemma 6 in section 4.

1.4 THE PARTICIPATION DECISION STAGE. Formally the participation decision is modelled as the selection of a zero-one variable z_i. Each player i may either select $z_i=0$, which means that he does not want to participate or $z_i=1$, which means that he wants to participate. The decision is made simultaneously by all players; each player must choose his z_i without knowing the participation decisions of the other players. The result of the participation decisions is a participation decision vector $z = (z_1,\dots,z_n)$. Those players i who have selected $z_i = 1$ are called participators; the other players are called non-participators. The set of all participators, or in other words, the set of all i with $z_i=1$ is denoted by Z. At the end of the participation decision stage, the vector $z = (z_1,\dots,z_n)$ is made known to all players. In the cartel bargaining stage and the supply decision stage the players can base their decisions on the knowledge of Z.

1.5 THE CARTEL BARGAINING STAGE. In the cartel bargaining stage each participator $i \in Z$ must propose a quota system for a coalition C which contains himself as a member.

$$(10) \quad Y_i = (y_{ij})_{j\in C} \quad ; \; i \in C \subseteq Z \; ; \qquad y_{ij} \geq 0$$

Y_i is called the proposal of participator i. The notation $(y_{ij})_{j\in C}$ indicates that Y_i contains a quota y_{ij} for each participator $j \in C$. A non-participator does not make a proposal and no quotas can be proposed for non-participators. The quotas y_{ij} can be arbitrary non-negative real numbers or ∞. Within the restriction $i \in C \subseteq Z$ a participator i is free to propose a quota system for any coalition C he wants. The special case where i is the only member of C is not excluded; such proposals correspond to unilateral commitments[5].

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[5] The result of the analysis would not change, if unilateral commitments were excluded. The reader will have no difficulty to see that this is true.

The participators must make their proposals simultaneously; each participator knows the participation decision vector $z = (z_1, \ldots, z_n)$, when he makes his proposal Y_i, but he does not know the proposals of the other participators.

A quota system Y_C for a coalition $C \subseteq Z$ becomes a <u>binding agreement</u>, if and only if the following is true:

(11) $Y_C = (y_j)_{j \varepsilon C} = Y_i$ for all $i \varepsilon C$.

This means that all members of C propose the same quotas for C. Unanimity of the members is required for a cartel agreement.

The <u>system</u> <u>of</u> <u>proposals</u>

(12) $Y = (Y_i)_{i \varepsilon Z}$

determines which binding agreements are reached. In (12) the same notational convention is used as in (10) and (11): the expression $i \varepsilon Z$ indicates that Y contains exactly one proposal for each participator $i \varepsilon Z$.

If Y_C is a binding agreement, then the quotas y_i assigned by Y_C to the participators $i \varepsilon C$ are called "<u>binding</u> <u>quotas</u>". Since it is convenient to define a "<u>binding</u> <u>quota</u> <u>vector</u>" $y = (y_1, \ldots, y_n)$ which contains a binding quota y_i for every player $i \varepsilon N$, the "binding quota" $y_i = \infty$ is assigned to those players i, who are not in coalitions for which binding agreements have been reached.

At the end of the cartel bargaining stage the system of proposals Y is made known to all players. The system of proposals uniquely determines the binding quota vector $y = (y_1, \ldots, y_n)$. Note that the system of proposals Y contains a complete description of the course of the game up to the end of the cartel bargaining stage, since the knowledge of Y implies the knowledge of Z.

1.6 THE SUPPLY DECISION STAGE. In the supply decision stage each player i selects a supply quantity x_i subject to the restriction

(13) $0 \leq x_i \leq y_i \qquad i = 1,\ldots,n$

The players must make their decisions simultaneously; each player knows Z, Y and y, when he selects his quantity x_i, but he does not know the supply decisions of the other players.

At the end of the supply decision stage, each player i receives u_i as his payoff. u_i is computed according to (2), (7), (8) and (9).

1.7 SOME FEATURES OF THE EXTENSIVE FORM REPRESENTATION OF THE MODEL
In spite of the fact that a detailed formal description of the extensive form representation of the model is not needed, it may be useful to point out some of its features. Let us denote the extensive form representation of the model by Γ^1. (The symbol Γ will be used for extensive forms). The representation of the decisions in the game tree of Γ^1 follows the order of the stages and simultaneous decisions are represented in the order given by the numbering of the players, the lower numbers coming first. This arbitrary convention about simultaneous decisions is needed, since the tree structure of the extensive form requires a successive representation of simultaneous choices.

In the information partition, the participation stage is represented by n information sets, one for each player; the decision situations of a player i at the beginning of the cartel bargaining stage correspond to 2^{n-1} information sets, one for each Z with $i \in Z$; the supply decision stage is represented by infinitely many information sets: each player has one information set for each proposal system Y. A play of the game corresponds to a triple (z,Y,x), where $z = (z_1,\ldots,z_n)$ is the participation decision vector, $Y = (Y_i)_{i\in Z}$ is the proposal system and $x = (x_1,\ldots,x_n)$ is the vector of supplies.

It will be important for the game theoretic analysis of the extensive form representation Γ^1, that the game Γ^1 has subgames. Obviously after the participation decisions have been made and the set of participators Z is known to all players, the rest of the game corresponds to a subgame; this subgame is denoted by Γ^1_Z. There are 2^n subgames of this kind. We call these subgames <u>cartel bargaining subgames.</u> The cartel bargaining subgames do not have the participation decision stage, but they still have the other two stages. After a system of proposals Y has been made another kind of subgame arises, which is denoted by Γ^1_Y. In these subgames only supply decisions are made; they are called <u>supply decision subgames.</u> There are infinitely many supply decision subgames, one for each Y. Obviously for $Y = (Y_i)_{i \varepsilon Z}$, the supply decision subgame Γ^1_Y is a subgame of the cartel bargaining subgame Γ^1_Z.

A subgame, which contains at least one information set and which is not the whole game itself is called a <u>proper</u> subgame. (The information set may be an information set of the random player.) A game in extensive form is called <u>indecomposable</u>, if it does not have any proper subgames; otherwise the game is called <u>decomposable.</u> Obviously the supply decision subgames Γ^1_Y are indecomposable and the cartel bargaining subgames Γ^1_Z are decomposable.

2. PERFECT EQUILIBRIUM SETS.

Any normative theory which gives a complete answer to the question how the players should behave in a specific non-cooperative game must take the form of an equilibrium point. Theories which prescribe non-equilibrium behavior are self-destructing prohecies, since at least one player is motivated to deviate, if he expects that the others act according to the theory. Therefore, if one wants to find a rational solution for a non-cooperative game, one must look for equilibrium points.

For games in extensive form it is important to make a distinction between perfect and imperfect equilibrium points. The concept of a perfect equilibrium point will be introduced in subsection 2.3. There the reasons for the exclusion of imperfect equilibrium points will be explained.

The solution concept proposed in this paper does not prescribe perfect equilibrium points but perfect equilibrium sets. A perfect equilibrium set may be described as a class of perfect equilibrium points, which are essentially equivalent as far as the payoff interests of the players are concerned. A solution concept which prescribes perfect equilibrium sets does not give a complete answer to the question how the players should behave in the game, but the answer is virtually complete in the sense that only unimportant details are left open. Such details may be filled in by non-strategic prominence considerations.[6)]

Some basic game theoretic definitions and notations are introduced in 2.1 and 2.2.

2.1 BEHAVIOR STRATEGIES. The way in which the words "extensive form" are understood in this paper has been explained in subsection 1.1. The games considered here are always with perfect recall. H.W.Kuhn has proved a theorem about finite games with perfect recall

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6) see [9]

which shows that nothing is lost if one restricts one's attention to equilibrium points in behavior strategies.[7)] R.J. Aumann has generalized this theorem to games in extensive form, where a continuum of choices may be available at some or all information sets.[8)] In view of these results the game-theoretic analysis will be in terms of behavior strategies.

Let $\mathcal{U}_i$ be the set of all information sets U of player i in an n-person game in extensive form Γ.
A <u>behavior</u> <u>strategy</u> q_i is a system of probability distributions q_U over the choices at U, containing one distribution q for every $U \in \mathcal{U}_i$. This is expressed by the following notation:

$$(14) \qquad q_i = \left\{ q_U \right\}_{U \in \mathcal{U}_i}$$

A <u>finite</u> <u>behavior</u> <u>strategy</u> is a behavior strategy which has the property that the distributions q_U assign positive probabilities to a finite number of choices at U and zero probabilities to all other choices. Such distributions are called <u>finite</u> <u>distributions</u>.
For the purposes of this paper it will be sufficient to consider finite behavior strategies only. Therefore from now on, a <u>strategy</u> will be always a finite behavior strategy. Note that the pure strategies are included in this definition as special cases, since a <u>pure</u> <u>strategy</u> π_i can be regarded as a behavior strategy whose distributions q_U assigns 1 to one of the choices at U and zero to all others.

The set of all strategies q_i of player i in an n-person game in extensive form is denoted by Q_i. A <u>strategy</u> <u>combination</u> $q = (q_1, \ldots, q_n)$ for Γ is a vector with n components whose i-th component is a strategy $q_i \in Q_i$. The set of all pure strategies π_i of player i is denoted by Π_i. A <u>pure</u> <u>strategy</u> <u>combination</u> for Γ is a strategy combination $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi_i \in \Pi_i$. For every given strategy combination

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7) see [5] p.213

8) see [1] p.639

$q = (q_1, \ldots, q_n)$ a payoff vector $H(q) = (H_1(q), \ldots, H_n(q))$ is determined in the usual way.

The symbol Γ with various indices **attached** to it will be used for games in extensive form. The same index will be used for the game and its information sets, strategies, strategy combinations etc. In this way, notations introduced for a general game will be carried over to specific games in extensive form.

2.2 EQUILIBRIUM POINTS. It is convenient to introduce the following notation. If in a strategy combination $q = (q_1, \ldots q_n)$ the i-th component is replaced by a strategy r_i then a new strategy combination results which is denoted by q/r_i. Consider a strategy combination $s = (s_1, \ldots s_n)$ for Γ. A strategy r_i for player i with

$$(15) \qquad H_i(s/r_i) = \max_{q_i \in Q_i} H_i(s/q_i)$$

is called a best reply to the strategy combination s. An equilibrium point (in finite behavior strategies) for a game in extensive form Γ is a strategy combination $s = (s_1, \ldots, s_n)$ with the following property:

$$(16) \qquad H_i(s) = \max_{q_i \in Q_i} H_i(s/q_i)$$

An equilibrium point can be described as a strategy combination whose components are best replies to this combination.

2.3 PERFECT EQUILIBRIUM POINTS. It has been argued elsewhere [9] that one requirement which should be satisfied by an equilibrium point selected as the solution of a non-cooperative game is a property called perfectness. In order to describe this property some further definitions are needed.

Consider an n-person game Γ in extensive form. Let Γ' be a subgame of Γ and let $q = (q_1, \ldots, q_n)$ be a strategy combination for Γ. The system of probability distributions

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9) See [10] or [11]

assigned by q_i to information sets of player i in Γ' is a strategy q_i' for Γ'; this strategy q_i' is called <u>induced</u> by q_i on Γ' and the strategy combination $q' = (q_1', \ldots, q_n')$ is called <u>induced</u> by q on Γ'.

<u>A perfect equilibrium point</u> $s = (s_1, \ldots, s_n)$ for an n-person game in extensive form Γ is an equilibrium point (in finite behavior strategies) which induces an equilibrium point on every subgame of Γ. An equilibrium point which is not perfect is called <u>imperfect</u>.

An imperfect equilibrium point may prescribe absurd modes of behavior in a subgame which cannot be reached because of the behavior prescribed in earlier parts of the game; if the subgame were reached by mistake, some players would be motivated to deviate from the prescribed behavior. It is natural to require that the behavior prescribed by the solution should be in equilibrium in every subgame, regardless of whether the subgame is reached or not. Any reasonable solution concept for non-cooperative games in extensive form should have the property that it prescribes perfect equilibrium points.

<u>2.4 TRUNCATIONS.</u> A set M of subgames of a given extensive form game Γ is called a <u>multisubgame</u> of Γ, if no subgame in M is a subgame of another subgame in M. A <u>proper multisubgame</u> of Γ is a multisubgame which contains only proper subgames of Γ.

Let $s = (s_1, \ldots, s_n)$ be a strategy combination for Γ. For every proper multisubgame M of Γ we construct a new game in the following way: Every subgame $\Gamma' \in M$ is replaced by the payoff vector $H'(s')$ which in Γ' belongs to the strategy combination $s' = (s_1', \ldots, s_n')$ induced by s on Γ'. This means that every $\Gamma' \in M$ is taken away; thereby the starting point of Γ' becomes an endpoint of the new game; the payoff vector at this endpoint is the equilibrium payoff vector $H'(s')$. The new game is denoted by $T(\Gamma, M, s)$. The games $T(\Gamma, M, s)$ are called s-<u>truncations</u>.

If q_i is a strategy for Γ, then the strategy induced by q_i on $T(\Gamma,M,s)$ is defined in the same way as the strategy induced on a subgame; the induced strategy assigns the same probability distribution to an information set as q_i does. A strategy combination $\bar{q}$ for $\bar{\Gamma} = T(\Gamma,M,s)$ is called induced by a strategy combination q for Γ, if each of the components of $\bar{q}$ is induced by the corresponding component of q.

LEMMA 1. Let M be a proper multisubgame of a game Γ and let s be a strategy combination for Γ. Then $\bar{H}(\bar{s}) = H(s)$ holds for the payoff vector $\bar{H}(\bar{s})$ belonging to the strategy combination $\bar{s}$ induced by s on $\bar{\Gamma} = T(\Gamma,M,s)$.

PROOF[10]. Consider an endpoint z of Γ. Let $\bar{z}(z)$ be that endpoint of $\bar{\Gamma}$ which is on the play to z. The strategy combination s generates a probability distribution over the set of all endpoints of $\bar{\Gamma}$. The payoff vector $H(s)$ is the expected value of the payoff vectors at the endpoints with respect to this distribution. The payoff vector $H'(s')$ which belongs to the combination s' induced by s on a subgame Γ' of Γ beginning at one of the endpoints $\bar{z}$ of $\bar{\Gamma}$ is the conditional expectation of the payoff vector at z under the condition that an endpoint z of Γ with $\bar{z} = \bar{z}(z)$ is reached. This together with the definition of $\bar{\Gamma}$ and its payoff function $\bar{H}$ shows that the lemma is true.

LEMMA 2. Let M be a proper multisubgame of a game Γ and let s be a perfect equilibrium point for Γ. Then the strategy combination $\bar{s}$ induced by s on $\bar{\Gamma} = T(\Gamma,M,s)$ is a perfect equilibrium point of $\bar{\Gamma}$.

PROOF. Assume that $\bar{s}$ is not a perfect equilibrium point. Then there must be a subgame $\bar{\Gamma}'$ of $\bar{\Gamma}$ such that in this subgame at least one of the players, say player j, has a strategy $\bar{r}'_j$ for $\bar{\Gamma}'$ such that in $\bar{\Gamma}'$ his payoff $\bar{H}'_j(\bar{s}'/\bar{r}'_j)$ is greater than his payoff $\bar{H}_j(\bar{s}')$ at the combination $\bar{s}'$ induced by $\bar{s}$ on $\bar{\Gamma}'$. The subgame $\bar{\Gamma}'$ is the s'-truncation $T(\Gamma',M',s')$ of some subgame Γ' of Γ, where s' is the equilibrium point induced by s on Γ' and M' is the set of subgames of Γ' which are in M.

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10) Only a sketch of a proof is given here, since a detailed proof would require a formal definition of the extensive form. A detailed proof would be analogous to the proof of Kuhn's theorem 2. See [5] p,206.

Let r'_j be that strategy for Γ' which agrees with $\bar{r}'_j$ for the information sets in $\bar{\Gamma}'$ and agrees with player j's equilibrium strategy s'_j from s' everywhere else. It follows from $\bar{H}'_j(\bar{s}/\bar{r}'_j) > \bar{H}_j(\bar{s}')$ that because of lemma 1 for this strategy r'_j we must have $H'_j(s'/r'_j) > H'_j(s')$ for player j's payoff in Γ'. This cannot be true, since s' must be an equilibrium point.

2.5 BRICKS. Let s be a strategy combination for a game Γ. The indecomposable subgames of Γ and of the s-truncation of Γ are called s-bricks of Γ. (This includes improper subgames like indecomposable truncations or the game Γ itself if Γ is indecomposable. Obviously only the payoffs of the s-bricks depend on the strategy combination s. If Γ is a game in extensive form, then the game tree of Γ together with all the elements of the description of the extensive form apart from the payoff function (information sets, choices, probabilities of random choices etc.) is called the payoffless game of Γ. A payoffless brick of Γ is the payoffless game of an s-brick of Γ.

With respect to s-bricks and payoffless bricks, induced strategies and strategy combinations are defined in the same way as for subgames and truncations.

Obviously the payoffless bricks of an extensive form Γ generate a partition of the set of all information sets of Γ. Every information set of Γ is in one and only one payoffless brick of Γ. A strategy combination q for Γ is fully determined by the strategy combinations induced by q on the payoffless bricks of Γ.

Two strategy combinations r and s for Γ are called brick equivalent if every r-brick coincides with the corresponding s-bricks. A set S of strategy combinations for Γ is called brick-producing if two strategy combinations $r \epsilon S$ and $s \epsilon S$ are always brick equivalent. Obviously every s in a brick producing set S generates the same system of s-bricks.

2.6 THE DECOMPOSITION RANK OF A GAME. A maximal proper subgame of a game Γ in extensive form is a proper subgame Γ' of Γ which is not a proper subgame of another proper subgame of Γ.

The decomposition rank of a game Γ in extensive form is defined recursively by the following two properties: (a) indecomposable games have decomposition rank 1 and (b) for m=2,3,... a game Γ has the decomposition rank m if every maximal proper subgame of Γ has a decomposition rank of at most m-1 and if the decomposition rank of at least one maximal proper subgame of Γ is m-1.

Obviously this definition assigns a finite decomposition rank to every game in extensive form in the sense of this paper, since the play length is bounded from above.

2.7 A DECOMPOSITION PROPERTY OF PERFECT EQUILIBRIUM POINTS.

In this subsection a theorem is proved which shows that perfect equilibrium points have an important property which may be called a "decomposition property" since it relates the perfect equilibrium point to the equilibrium points induced on the bricks of the game.

Let M be the set of all maximal proper subgames of a decomposable game Γ. The s-truncation $\bar{\Gamma} = T(\Gamma,M,s)$ with respect to this multisubgame is called the indecomposable s-truncation of Γ. The notation $T(\Gamma,s)$ is used for the indecomposable s-truncation.

THEOREM 1. A strategy combination s for a game Γ in extensive form is a perfect equilibrium point of Γ, if and only if an equilibrium point is induced by s on every s-brick of Γ.

PROOF. It follows from the definition of a perfect equilibrium point and from lemma 2, that a perfect equilibrium point s induces equilibrium points on the s-bricks. Therefore we only have to show that s is a perfect equilibrium point if equilibrium points are induced on the s-bricks. In order to prove this, induction on the decomposition rank is used.

The assertion is trivially true for decomposition rank 1. Assume that it is true for decomposition ranks 1,...,m. Let s be a strategy combination for a game Γ with decomposition rank m+1, such that s induces equilibrium points on every s-brick of Γ. Since the assertion is true for 1,...,m, the strategy combination s induces a perfect equilibrium point on every maximal subgame of Γ.

Assume that s is not a perfect equilibrium point of Γ. If s were an equilibrium point, then s would be a perfect equilibrium point, since perfect equilibrium points are induced on every maximal subgame. Therefore s is not an equilibrium point. There must be a player j with a strategy r_j for Γ, such that $H_j(s/r_j) > H_j(s)$ holds for his payoff in Γ.

Consider the indecomposable s-truncation $\bar{\Gamma} = T(\Gamma,s)$. This game $\bar{\Gamma}$ is an s-brick of Γ. Let $\bar{s}$ be the strategy combination induced by s on $\bar{\Gamma}$ and let $\bar{r}_j$ be the strategy induced by r_j on $\bar{\Gamma}$.

At every endpoint of the game $\bar{\Gamma}' = T(\bar{\Gamma},s/r_j)$ the payoff of player j is at most as high as his payoff at the same endpoint in $\bar{\Gamma}$. This follows from the fact that equilibrium points are induced by s on the maximal proper subgames of Γ. Therefore $\bar{H}_j(\bar{s}/\bar{r}_j) > \bar{H}_j(\bar{s})$ must hold for player j's payoff in Γ since otherwise $H_j(s/r_j) > H_j(s)$ cannot be true. This contradicts the assumption that an equilibrium point is induced by s on the s-brick.

The following correlary is an immediate consequence of the theorem and the fact that the strategy combinations s' induced by s on a subgame Γ' of Γ or one of its s-truncations generate s'-bricks of Γ' which coincide with the corresponding s-bricks of Γ.

<u>CORRELARY</u>[11)] Let $\bar{\Gamma}=T(\Gamma,M,s)$ be an s-truncation of a game Γ in extensive form. Then the strategy combination s is a perfect equilibrium point for Γ if and only if the following two conditions are satisfied: 1) the strategy combination $\bar{s}$ induced by s on $\bar{\Gamma}$ is a perfect equilibrium point for $\bar{\Gamma}$; 2). For every $\Gamma' \varepsilon M$ the strategy combination s' induced by s on Γ' is a perfect equilibrium point for Γ'.

<u>2.8 PERFECT EQUILIBRIUM SETS.</u> Two equilibrium points r and s for a game Γ are called <u>payoff equivalent</u> if we have $H(r) = H(s)$ for the payoff vectors of r and s. An <u>equilibrium set</u> S for Γ is a non-empty class of payoff equivalent equilibrium points, s for Γ, which is not a proper subset of another class of this kind. Obviously every equilibrium point s for Γ belongs to one and only one equilibrium set for Γ. This equilibrium set is called the <u>equilibrium set of s.</u>

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11) This correlary of theorem 1 is similar to Kuhn's theorem 3. See [5],p.208.

Two perfect equilibrium points r and s for Γ are called <u>subgame payoff equivalent</u>, if for every subgame Γ' (including the improper subgame Γ) the equilibrium points r' and s' induced by r and s on Γ' are payoff equivalent. A <u>perfect equilibrium set</u> S for Γ is a non-empty class of subgame payoff equivalent perfect equilibrium points s for Γ, which is not a proper subset of another class of this kind. Obviously every perfect equilibrium points s for Γ belongs to one and only perfect equilibrium set for Γ. This perfect equilibrium set is called the <u>perfect equilibrium set of s</u>.

A set of strategy combinations R' is induced by a set R, if every element $r' \in R'$ is induced by some $r \in R$. The definition of an induced set of strategies is analogous.

<u>LEMMA 3.</u> A perfect equilibrium set S for a game Γ in extensive form induces a perfect equilibrium set S' on every subgame Γ' of Γ.

<u>PROOF.</u> Obviously the set S' induced by S on Γ' is a set of subgame payoff equivalent perfect equilibrium points. Let r' be a perfect equilibrium point for Γ' which is subgame payoff equivalent to the perfect equilibrium points $s' \in S'$. Any $s \in S$ can be changed by replacing the behavior prescribed by s on Γ' by the behavior prescribed by r'. The result is a strategy combination q for Γ. Let M be the multisubgame containing Γ' as its only element. Obviously we have $\bar{\Gamma} = T(\Gamma, M, q) = T(\Gamma, M, s)$. It follows by lemma 2 and by the correlary of theorem 1 that q is a perfect equilibrium point for Γ.

It remains to be shown that q is subgame payoff equivalent to the elements of S. If this is true r' must belong to S'. Let Γ'' be a subgame of Γ and let q" and s" be the strategy combinations induced on Γ'' by q and s, respectively. If Γ'' is a subgame of Γ'' or if Γ' is not a proper subgame of Γ'', then $H''(q'') = H''(s'')$ follows immediately from the fact that q agrees with s on $\bar{\Gamma}$ and with r' on Γ'. Let Γ' be a proper subgame of Γ'' and let S" be induced by S on Γ''; then $\bar{\Gamma}'' = T(\Gamma'',M,s'')$ is a subgame of $\bar{\Gamma} = T(\Gamma,M,s)$. Hence by lemma 1 we have $\bar{H}''(\bar{s}'') = H''(s'') = H''(q'')$ for the strategy combination $\bar{s}''$ induced by both s and q on Γ''. This proves the lemma.

Let S be a perfect equilibrium set for Γ. Obviously for $r \varepsilon S$ and $s \varepsilon S$ we always have $T(\Gamma,M,s) = T(\Gamma,M,r)$. Therefore the s-truncation $T(T,M,s)$ with $s \varepsilon S$ is denoted by $T(\Gamma,M,S)$. The games $T(\Gamma,M,S)$ are called S-truncations. Since for $s \varepsilon S$ the s-bricks are indecomposable subgames of S-truncations, every perfect equilibrium set is a brick-producing set in the sense of 2.5. If S is a brick-producing set, then the s-bricks with $s \varepsilon S$ are also called S-bricks and $T(\Gamma,s)$ is denoted by $T(\Gamma,S)$. The game $T(\Gamma,S)$ is the indecomposable S-truncation of Γ.

LEMMA 4. A perfect equilibrium set S for a game Γ induces a perfect equilibrium set S on every S-truncation $\bar{\Gamma} = T(\Gamma,N,S)$.

PROOF. It follows from lemma 2 that the elements of $\bar{S}$ are perfect equilibrium points. It remains to be shown that a) any two equilibrium points $\bar{r}$ and $\bar{s}$ with $\bar{r} \varepsilon \bar{S}$ are subgame payoff equivalent and b) if a perfect equilibrium point $\bar{q}$ for $\bar{\Gamma}$ is subgame payoff equivalent to the elements of $\bar{S}$, then $\bar{q}$ is an element of $\bar{S}$.

We first prove a). The perfect equilibrium points $\bar{r}$ and $\bar{s}$ are induced by some $r \varepsilon S$ and some $s \varepsilon S$, resp. Let r and s be such strategy combinations. Let $\bar{\Gamma}'$ be a subgame of $\bar{\Gamma}$ and let $\bar{r}'$ and $\bar{s}'$ be the strategy combinations induced by $\bar{r}$ and $\bar{s}$, resp. on $\bar{\Gamma}'$. We must show $\bar{H}'(\bar{r}') = \bar{H}'(\bar{s}')$. This is obviously true if $\bar{\Gamma}'$ is a subgame of Γ. If $\bar{\Gamma}'$ is not a subgame of Γ, then a subgame of Γ' exists, such that $\bar{\Gamma}'$ is an S'-truncation of Γ', where S' is the set which is induced by S on Γ'. Let r' and s' be the strategy combinations induced on Γ' by r and s, resp. We must have $\bar{H}'(\bar{r}') = H'(r')$ and $\bar{H}'(\bar{s}') = H'(s')$ because of lemma 1 and $H'(r') = H'(s')$ since r and s are subgame payoff equivalent. This shows that $\bar{r}$ and $\bar{s}$ are subgame payoff equivalent.

Consider a perfect equilibrium point $\bar{q}$ for $\bar{\Gamma}$ which is subgame payoff equivalent to the elements of $\bar{S}$. We have to show that $\bar{q}$ belongs to $\bar{S}$. Let q be a strategy combination for Γ which agrees with $\bar{q}$ on $\bar{\Gamma}$ and agrees with some $s \in S$ everywhere else. It follows from the correlary of theorem 1 that q is a perfect equilibrium point for Γ.

Assume that q does not belong to S. Then there must be a subgame Γ' of Γ where the payoff vector $H'(q')$ belonging to the strategy combination induced by q on Γ' does not agree with the payoff vector $H'(s')$ belonging to the strategy combination induced by s on Γ'. Obviously this subgame Γ' cannot be in M. Therefore some s-truncation $\bar{\Gamma}'=T(\Gamma',M',s)$ of Γ' must be a proper subgame of $\bar{\Gamma}$. Because of lemma 1 the payoff vector $\bar{H}'(\bar{q}')$ belonging to the strategy combination $\bar{q}'$ induced by $\bar{q}'$ on $\bar{\Gamma}'$ is the same as the payoff vector $H'(s')$. This contradiction shows that q belongs S. Therefore $\bar{q}$ belongs to $\bar{S}$. This proves the lemma.

<u>LEMMA 5.</u> A perfect equilibrium set S for a game Γ induces an equilibrium set S' on every S-brick Γ' of Γ.

<u>PROOF.</u> Since S-bricks are indecomposable subgames of S-truncations the assertion follows from lemma 3 and lemma 4.

<u>2.9 A DECOMPOSITION PROPERTY OF PERFECT EQUILIBRIUM SETS.</u> In the following it is shown that similar results as in 2.7 can be obtained for perfect equilibrium sets.

<u>THEOREM 2.</u> Let S be a perfect equilibrium set for a game Γ in extensive form. Then a strategy combination s for Γ is an element of S, if and only if for every S-brick Γ' of Γ the strategy combination s' induced by s on Γ' is an element of the equilibrium set S' induced by S on Γ'.

PROOF. The only-if part of the theorem follows from the definition of an induced set of strategy combinations. The if-part remains to be shown. This is done by induction on the decomposition rank of Γ. The assertion is trivially true for decomposition rank 1. Assume that it is true for decomposition rank 1,...,m.

Consider a strategy combination s which induces a strategy combination $s \in S$ on every S-brick Γ' of Γ. It follows from the induction hypothesis that for every proper subgame Γ'' of Γ the strategy combination s'' induced by s on Γ'' is in the perfect equilibrium set S'' induced by S on Γ''. There is no difference between an S-brick of Γ'' and the coresponding S-brick of Γ.

Let $\bar{S}$ be the equilibrium set induced on the indecomposable S-truncation $\bar{\Gamma} = T(\Gamma, S)$. The strategy combination $\bar{s}$ induced by s on the S-brick Γ belongs to $\bar{S}$. Since perfect equilibrium points s'' are induced on the maximal proper subgames Γ'' of Γ, the S-brick $\bar{\Gamma}$ is also an s-brick. Moreover every other S-brick is also an s-brick. It follows by theorem 1 that s is a perfect equilibrium point. We must have $H(s) = \bar{H}(\bar{s})$ because of lemma 1. This shows that s belongs to S.

CORRELARY. Let S be a perfect equilibrium set for a game Γ in extensive form and let $\bar{\Gamma} = T(\Gamma, M, S)$ be an S-truncation of Γ. Then a strategy combination s for Γ is an element of S, if and only if the following two conditions are satisfied: 1) The strategy combination $\bar{s}$ induced by s on $\bar{\Gamma}$ is in the perfect equilibrium set $\bar{S}$ induced by S on $\bar{\Gamma}$ and 2) For every $\Gamma' \in M$, the strategy combination s' induced by s on Γ' is in the perfect equilibrium set S' induced by S on Γ'.

PROOF. The $\bar{S}$-bricks and S'-bricks coincide with the corresponding S-bricks. Therefore for $s \in S$ the induced strategy combination $\bar{s}$ and s' are in $\bar{\Gamma}$ and Γ' resp. On the other hand, if s satisfies 1) and 2), then the strategy combinations induced by s on the S-bricks are in the equilibrium sets induced by S. This shows that the correlary follows from the theorem.

THEOREM 3. Let S be a brick-producing set of strategy combinations for a game Γ in extensive form. Then S is a perfect equilibrium set, if and only if the following two conditions are satisfied. 1) For every S-brick Γ', the set S' induced by S on Γ' is an equilibrium set for Γ'. 2) If a strategy combination s for Γ has the property that for every S-brick Γ' the strategy combination s' induced by s on Γ' is in the set S' induced by S on Γ', then s is in S.

PROOF. If 1) and 2) are satisfied, then it follows from theorem 1 that the elements s ε S are perfect equilibrium points. Take any fixed r ε S and let R be the perfect equilibrium set of r. Obviously there is no difference between corresponding r-bricks, R-bricks and S-bricks. It follows from lemma 5 that an equilibrium set R' is induced by R on every r-brick Γ'. Since every equilibrium point is in a uniquely determined equilibrium set, R' must agree with the set S' induced by S on Γ'. It follows by theorem 2, that R and S are identical sets.

If S is a perfect equilibrium set, then lemma 5 has the consequence that 1) is satisfied and it follows by theorem 2 that 2) is satisfied, too.

2.10 INTERPRETATION. The notion of a perfect equilibrium set is a natural modification of the notion of a perfect equilibrium point. Since all the perfect equilibrium points s in a given perfect equilibrium set are subgame payoff equivalent, one can take the point of view, that the differences between them are unimportant.

Theorem 1 shows that a perfect equilibrium point s is fully determined by the equilibrium points induced on the s-bricks. Theorem 3 shows that a perfect equilibrium set S is fully determined by the equilibrium sets S' induced on the S-bricks. In order to describe S it is sufficient to describe these equilibrium sets S'.

3. THE SOLUTION CONCEPT

The game-theoretic concepts developed here serve the limited purpose of constructing a theory which is just general enough to provide a solid basis for the analysis of the game Γ^1 described in section 1. The solution concept of this paper is not applicable outside a certain class of games with special properties. No attempt is made to attack the difficult task of selecting a unique solution for every non-cooperative game.[12)]

For the class of games where it is defined, the solution concept proposed here is the only one of its kind, which has four desirable properties. Two of these properties concern the relationship of the solution of a game to the solutions of its subgames and truncations. The third property is a symmetry property. The fourth property is based on the idea that the players have a tendency to act in their common interest if this is compatible with the other three properties.

3.1 SOLUTION FUNCTIONS. A solution function for a class K of games in extensive form is defined as a function which assigns a perfect equilibrium set $L(\Gamma)$ to every game Γ in the class K. The equilibrium set $L(\Gamma)$ is called the L-solution or simply the solution of Γ, where it is clear which solution function L is considered. The payoff vector belonging to $L(\Gamma)$ is called the L-value of Γ. The L-value of Γ is denoted by $V(\Gamma,L) = (V_1(\Gamma,L),\ldots, V_n(\Gamma,L))$.

It may happen that the solution $L(\Gamma)$ is a perfect equilibrium set which contains exactly one perfect equilibrium point. In this case the single perfect equilibrium point in $L(\Gamma)$ will also be called the solution of Γ, where the danger of misunderstandings cannot arise.

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12) The author is collaborating with John C.Harsanyi on the elaboration of a theory of this kind. Some of the ideas presented here go back to this common work which is not yet complete. See [4]

3.2 SUBGAME CONSISTENCY. A class K of games is called subgames complete, if for $\Gamma \in K$ every subgame of Γ is also in K. A solution function L for a class K of games is called subgame consistent, if for every $\Gamma \in K$ the L-solution $L(\Gamma')$ of Γ' is induced by $L(\Gamma)$ on every proper subgame Γ' of Γ with $\Gamma' \in K$.

Note that subgame consistency is not implied by the definition of a perfect equilibrium set. If $L(\Gamma)$ is a perfect equilibrium set then it must induce some perfect equilibrium set on a subgame Γ' of Γ, but it does not follow, that for $\Gamma' \in K$ this perfect equilibrium set is the L-solution of Γ.

Subgame consistency means that the behavior in a subgame depends on this subgame only. This is reasonable, since as far as the strategic situation of the players is concerned, those parts of the game, which are outside the subgame, become irrelevant once the subgame has been reached.

3.3 TRUNCATION CONSISTENCY. Let L be a solution function for a subgame complete class K. For any multisubgame M of a game $\Gamma \in K$, the $L(\Gamma)$-truncation $\bar{\Gamma} = T(\Gamma, M, L(\Gamma))$ can be formed. For the sake of shortness, this game $\bar{\Gamma}$ is denoted by $T(\Gamma, M, L)$. The games $T(\Gamma, M, L)$ are called L-truncations of Γ. The indecomposable L-truncations are called L-bricks. For the indecomposable $L(\Gamma)$-truncation $T(\Gamma, L(\Gamma))$ the notation $T(\Gamma, L)$ is used. $T(\Gamma, L)$ is the indecomposable L-truncation of Γ.

A class K of games in extensive form is called L-complete, if the solution function L is defined on K and if K is a subgame complete class with the additional property that for $\Gamma \in K$ every L-truncation of Γ is in K.

A solution function L for a class K of games in extensive form is, called <u>truncation consistent</u>, if for every $\Gamma \varepsilon K$ the L-solution $L(\Gamma)$ induces the L-solution $L(\bar{\Gamma})$ on every L-truncation $\bar{\Gamma} = T(\Gamma,M,L)$ with $\bar{\Gamma} \varepsilon K$.

It is intuitively clear that a reasonable subgame consistent solution function L should also be truncation consistent. If $L(\Gamma')$ is the behavior expected in the subgames $\Gamma' \varepsilon M$, then the strategic situation in $\bar{\Gamma} = T(\Gamma,M,L)$ is essentially the same as in that part of Γ which corresponds to $\bar{\Gamma}$.

<u>3.4 CONSISTENT EXTENSIONS.</u> Consider a solution function L_1 for a class K_1 of indecomposable games. In the following for any such L an extension to a wider class K will be constructed. It will be shown that the extended solution function L is the only subgame consistent and truncation consistent solution function for K such that L coincides with L_1 on K_1.

Let L be a solution function for a class K of games in extensive form. L is called a <u>consistent extension</u> of a solution function L_1 for a class K_1 of indecomposable games, if the following conditions (A) and (B) are satisfied:

<u>(A) REGION.</u> The set of all indecomposable games in K is the set K_1. For $m = 2,3,\ldots$ the set K_m of all games $\Gamma \varepsilon K$ with decomposition rank m is equal to the set of all games Γ in extensive form, such that the maximal proper subgames of Γ are in the sets $K_1,\ldots, K_{m-1}$ and the indecomposable L-truncation $T(\Gamma,L)$ is in K_1.

<u>(B) SOLUTION.</u> For every $\Gamma \varepsilon K_1$ we have $L(\Gamma) = L_1(\Gamma)$. If Γ is a decomposable game $\Gamma \varepsilon K$, then $L(\Gamma)$ induces $L(\Gamma')$ on every maximal proper subgame Γ' of Γ and $L(T(\Gamma,L))$ on the indecomposable L-truncation $T(\Gamma,L)$ of Γ.

Later it will be shown that (A) and (B) imply subgame consistency and truncation consistence. This justifies the name "consistent extension".

THEOREM 4. Every solution function L_1 for a class of indecomposable games K_1 has a uniquely determined consistent extension.

PROOF. (A) and (B) provide a recursive definition of L and K. If the classes $K_1,\dots,K_{m-1}$ are known and L is known for games in these classes, then K_m is given by (A). It remains to be shown that for every $\Gamma \in K_m$ a unique perfect equilibrium set $L(\Gamma)$ is determined by condition (B). This can be seen by induction on M. The assertion is trivially true for $\Gamma \in K_1$. If the assertion is true for games in $K_1,\dots,K_{m-1}$, then it follows by the correlary of theorem 2, that for $\Gamma \in K_m$ the set $L(\Gamma)$ is a perfect equilibrium set for Γ.

THEOREM 5. The consistent extension L of a solution function L_1 for a class K_1 of indecomposable games has an L-complete region K. The consistent extension L is subgame consistent and truncation consistent. For every $\Gamma \in K$ the L_1-solution $L_1(\hat{\Gamma})$ is induced by $L(\Gamma)$ on every L-brick $\hat{\Gamma}$ of Γ.

PROOF. Let $\bar{K}_m$ be the union of the sets $K_1,\dots,K_m$. Let L_m be that solution function for $\bar{K}_m$, which agrees with L on $\bar{K}_m$. The theorem holds, if for $m = 1,2,3,\dots$ the class $\bar{K}_m$ is L_m - complete and L_m is subgame consistent and truncation consistent. For $m = 1$ this is trivially true. Assume that the assertion holds for $\bar{K}_m$. It follows from (A) that $\bar{K}_{m+1}$ is L_{m+1} - complete. Since L_m is subgame consistent and L_m agrees with L_m for the proper subgames of games in K_{m+1}, the solution function L_{m+1} is subgame consistent because of (B).

The truncation consistency of L_{m+1} can be seen as follows. Consider an L_{m+1}-truncation $\Gamma'' = T(\Gamma,M,L_{m+1})$ of a game $\Gamma \in K_{m+1}$. It has to be shown, that $L_{m+1}(\Gamma)$ induces $L_{m+1}(\Gamma'')$ on Γ''. The maximal proper subgames of Γ'' are L_m-truncations of maximal proper subgames of Γ. The maximal proper subgames of Γ are in K_m. Since L_m is truncation consistent, $L_{m+1}(\Gamma)$ induces $L_m(\Gamma')$ on every maximal proper subgame Γ of Γ''. The indecomposable L_m-truncation of Γ is the same game as the indecomposable L_m-truncation of Γ''. It follows from (B) that $L_{m+1}(\Gamma)$ induces $L_m(T(\Gamma'',L_m))$ on $T(\Gamma'',L_m)$. This shows that $L_{m+1}(\Gamma'')$ and $L_{m+1}(\Gamma)$ induce the same perfect equilibrium sets on the maximal proper subgames Γ' of Γ'' and on $T(\Gamma'',L_m)$. According to lemma 4 a perfect equilibrium set is induced by $L_{m+1}(\Gamma)$ on Γ''. It follows by the correlary of theorem 2 that this perfect equilibrium set must be equal to $L_{m+1}(\Gamma'')$.
It is a simple consequence of the truncation consistency and the subgame consistency of L, that $L_1(\hat{\Gamma})$ is induced by $L(\Gamma)$ on every L-brick $\hat{\Gamma}$ of Γ.

THEOREM 6. The consistent extension L of a solution function L_1 for a class K_1 of indecomposable games in extensive form is the only subgame consistent and truncation consistent solution function L, which agrees with L_1 on K_1 and has the additional property that L together with its region K satisfies condition (A).

PROOF. A subgame consistent and truncation consistent solution function whose region has property (A) must have the property (B). Therefore theorem 6 is a direct consequence of theorems 4 and 5.

3.5 SIMULTANEITY GAMES. The construction of a consistent extension is a way of reducing the task of solving the decomposable games in K to the simpler task of solving the indecomposable games in K_1. For the purpose of finding a solution for the game Γ^1 of section 1, the class K_1 must be large enough to generate a class K containing Γ^1. In the following a class of very simple indecomposable games will be specified. The class K_1 underlying the solution function applied to Γ^1 will be a subclass of this class of "simultaneity games".

A simultaneity game is an n-person game in extensive form, where each of the players 1,...,n has at most one information set and where each of these information sets intersects every play of the game. A simultaneity game can be interpreted as a game, where those players, who have information sets, make simultaneous decisions without getting information about any random choices which might occur before the decisions are made.

3.6 NORMAL FORMS. Since every player has at most one information set there is no difference between behavior strategies and ordinary mixed strategies in simultaneity games. Therefore a simultaneity game is adequately described by its normal form [13].

Let Γ be an n-person game in extensive form, he normal form of Γ is the pair $G = (\Pi, H)$, where $\Pi = (\Pi_1, \ldots, \Pi_n)$ is the strategy set vector, whose i-th component is the set Π_i of all pure strategies π_i of player i in Γ and where H is the payoff function which assigns the corresponding payoff vector $H(\pi) = (H_1(\pi), \ldots, H_n(\pi))$ to every pure strategy combination $\pi = (\pi_1, \ldots, \pi_n)$ for Γ. A normal form (without reference to an extensive form) is a structure $G = (\Pi, H)$ with the same

[13] This is not true for extensive forms in general. The normal form does not preserve the distinction between perfect and imperfect equilibrium points. In simultaneity games all equilibrium points are perfect and every normal form is isomorphic to the normal form of some simultaneity game.

properties where the π_i may be arbitrary mathematical objects.

A <u>finite mixed strategy</u> of player i is a probability distribution over Π_i, which assigns positive probabilities to a finite number of pure strategies $\pi_i \in \Pi_i$ and zero probabilities to the other pure strategies of player i. Since only finite behavior strategies are considered here, in this paper a mixed strategy will be always a finite mixed strategy.

Two n-person normal forms $G = (\Pi,H)$ and $G' = (\Pi',H')$ are called <u>isomorphic</u>, if for each player i there is a one-to-one mapping f_i from the set Π_i of his pure strategies in G onto the set Π_i' of his pure strategies in G', such that the same payoff vector is assigned to corresponding pure strategy combination in both normal forms. A system of one-to-one mappings $f = (f_1,\ldots, f_n)$ of this kind is called an <u>isomorphism</u> from G to G'.

An isomorphism $f = (f_1,\ldots,f_n)$ from G to G' can be extended to the mixed strategies. For every mixed strategy q_i for G let $f_i(q_i)$ be that mixed strategy q_i for G which assigns the same probability to a pure strategy $f_i(\pi_i)$ as q_i assigns to π_i. In this way every mixed strategy combination $q = (q_1,\ldots,q_n)$ for G corresponds to a mixed strategy combination $q' = (f_1(q_1),\ldots, f_n(q_n))$ for G'.

<u>3.7 SYMMETRIES.</u> Consider a normal form G' which results from a normal form G by a renumbering of the players. In this case an isomorphism from G to G' is called a <u>symmetry</u> of G. A symmetry of G may be described as an automorphism of G, i.e. a mapping of G onto itself which preserves the structure of G.

A <u>symmetry preserving equilibrium point</u> s for a game Γ is an equilibrium point which is invariant under all symmetries of the normal form of Γ. A <u>symmetry preserving equilibrium set</u> S for a game Γ is an equilibrium set, which is invariant under all symmetries of the normal form of Γ. This means that with respect to every symmetry every $r \in S$ corresponds to some $s \in S$. Note that an equilibrium point s in a symmetry preserving equilibrium set S need not be symmetry preserving. Only the set S as a whole is invariant under the symmetries of the normal form of the game.

A perfect equilibrium point s for a game is called <u>locally symmetry preserving</u>, if a symmetry preserving equilibrium point s' is induced by s on every s-brick Γ"of Γ . A perfect equilibrium set S for a game Γ is called <u>locally symmetry preserving</u>, if a symmetry preserving equilibrium set S' is induced by S on every S-brick Γ'of Γ . Note that the elements of a locally symmetry preserving perfect equilibrium set need not be locally symmetry preserving.

The name "local" is used in these definitions since the symmetries of the normal form of an s-brick or S-brick may not be present in other parts of the game. The following two theorems show, that local symmetry preservation is in harmony with the decomposition properties of perfect equilibrium points or sets.

<u>THEOREM 7.</u> A perfect equilibrium point s for a game Γ is locally symmetry preserving if and only if a locally symmetry preserving perfect equilibrium point is induced by s on every subgame and every s-truncation of Γ .

<u>THEOREM 8.</u> A perfect equilibrium set S for a game Γ is locally symmetry preserving if and only if a locally symmetry preserving perfect equilibrium set is induced by S on every subgame and every S-truncation of Γ .

<u>PROOF OF THEOREMS 7 AND 8.</u> Since the s-bricks and S-bricks are indecomposable subgames of s-truncations and S-truncations resp., the if-parts of both theorems follow directly from the definition of "locally symmetry preserving". The equilibrium point s'induced by s on a subgame or an s-truncation generates s' -bricks which coincide with the corresponding s-bricks. This together with lemmata 1 and 2 shows, that theorem 7 holds. With the help of lemmata 3 and 4 an analogous argument can be made in order to complete the proof of theorem 8.

3.8 SYMMETRICAL SOLUTION FUNCTIONS. A solution function L for a class K of games is called symmetrical, if it assigns locally symmetry preserving perfect equilibrium set $L(\Gamma)$ to every game $\Gamma \varepsilon K$.

If one player corresponds to another under a symmetry of an L-brick Γ' of a game $\Gamma \varepsilon K$, then the strategic situation of both players in Γ' is essentially the same. It is reasonable to expect, that rational players who are in the same strategic situation behave in the same way. Therefore it is natural to require that a solution function should be symmetrical.

If Γ is an indecomposable game, then a locally symmetry preserving perfect equilibrium set of Γ is nothing else than a symmetry preserving equilibrium set of Γ. Therefore a solution function L_1 for a class K_1 of indecomposable games is symmetrical, if and only if it assigns a symmetry preserving equilibrium set $L(\Gamma)$ to every game $\Gamma \varepsilon K_1$.

THEOREM 9. The consistent extension L of a solution function L_1 for a class K_1 of indecomposable games is symmetrical if and only if L_1 is symmetrical.

PROOF. It follows directly from the definition of a symmetrical solution function that L cannot be symmetrical unless L_1 is symmetrical. If L_1 is symmetrical, then by theorem 5 for every $\Gamma \varepsilon K$ the equilibrium set $L_1(\hat{\Gamma})$ is induced by $L(\Gamma)$ on every L-brick $\hat{\Gamma}$ of Γ. This shows that L is symmetrical, if L_1 is symmetrical.

3.9 PAYOFF OPTIMALITY. A player in a game Γ in extensive form is called inessential, if in the normal form of Γ the payoffs of the other players do not depend on the strategy of player i. This is the case, if for every strategy combination π for Γ we have $H_j(\pi) = H_j(\pi/\pi_i')$ for every $\pi_i' \varepsilon \Pi_i$ and every player j with $j \neq i$. The players who are not inessential are called essential. Obviously in a simultaneity game a player without an information set is inessential.

If S is an equilibrium set or a perfect equilibrium set for a game Γ, then the payoff vector H(s) for the equilibrium points s ε S is denoted by $H(S) = (H_1(S),...,H_n(S))$. The payoff vector H(S) is called the equilibrium payoff vector at S.

Let R and S be two equilibrium sets or two perfect equilibrium sets for a game Γ. The set S is called weakly payoff superior to R if for every essential player i in Γ we have $H_i(S) \geq H_i(R)$, if in addition to this we have $H_i(S) > H_i(R)$ for at least one essential player i, then S is called strongly payoff superior to R. A perfect equilibrium set S for Γ is called weakly subgame payoff superior to another perfect equilibrium set R for Γ, if for every subgame Γ' of Γ (including Γ) the perfect equilibrium set S' induced by S on Γ' is weakly payoff superior to the perfect equilibrium set R' induced by R on Γ'. A perfect equilibrium set S for Γ is called strongly subgame payoff superior to another perfect equilibrium set R for Γ, if S is weakly subgame superior to R and if in addition to this for at least one subgame Γ' of Γ the perfect equilibrium set S' induced on Γ' by S is strongly payoff superior to the perfect equilibrium set R' induced by R on Γ'.

Let K be a class of n-person games in extensive form and let Λ be a set of solution functions for K. The solution function $\bar{L}$ ε Λ is called payoff optimal in Λ if for every L ε Λ and Γ ε K the L-solution L(Γ) is not strongly subgame payoff superior to the $\bar{L}$-solution $\bar{L}(\Gamma)$.

The solution concept of this paper is based on the idea that it is natural to select a payoff optimal solution function from a class of subgame consistent and truncation consistent symmetrical solution functions. If a perfect equilibrium set S for Γ is strongly subgame payoff superior to another perfect equilibrium set R, then it is in the common interest of the essential players in some subgames and not against the common interest of the essential players in the other subgames to coordinate their expectations at S rather than R. The concept of payoff optimality is similar to the familiar notion of Pareto-optimality. The analogy becomes clear if one takes the point of view that player i in one subgame and player i in another subgame have different interests and therefore should be treated as if they were different persons.

Definitions which do not take into account the possibility that the interests of the same player diverge in different parts of the game, cannot do justice to the structure of extensive form games. Therefore it is necessary to look at the payoffs in all possible subgames. In this respect the definition of a payoff optimal solution function is in the same spirit as the definition of a perfect equilibrium point.

3.10 DISTINGUISHED EQUILIBRIUM SETS. A distinguished equilibrium set for an indecomposable game Γ is a symmetry preserving equilibrium set S for Γ with the following additional property: if R is a symmetry preserving equilibrium set for Γ, which is different from S, then S is strongly payoff superior to R.

Obviously an indecomposable game can have at most one distinguished equilibrium set and not every indecomposable game has a distinguished equilibrium set. An indecomposable game which has a distinguished equilibrium set is called <u>distinguished</u>.

Later the class of all distinguished simultaneity games will be of special importance. It is natural to regard the distinguished equilibrium set of a distinguished simultaneity game as the solution of this game. It is in the common interest of the essential players to coordinate their expectations to an equilibrium point in this set.

In this paper the same intuitive argument is not applied to indecomposable games in general. It is not clear, whether for indecomposable games with complicated information structures the symmetries of the normal form say something meaningful about the extensive form in all possible cases. Only within the class of simultaneity games it is justified to rely on definitions based on the normal form.

3.11 <u>THE DISTINGUISHED SOLUTION FUNCTION.</u> Let $\tilde{K}_1$ be the set of all distinguished simultaneity games and let $\tilde{L}_1$ be that solution function for $\tilde{K}_1$ which assigns the distinguished equilibrium set of Γ to every $\Gamma \varepsilon \tilde{K}_1$. The *distinguished solution function* is the consistent extension $\tilde{L}$ of this solution function $\tilde{L}_1$.

The distinguished solution function is the solution concept of this paper. The following theorem summarizes the desirable properties of this solution concept.

THEOREM 10. Let $\tilde{K}$ be the region of the distinguished solution function $\tilde{L}$. The set Λ of all subgame consistent and truncation consistent symmetrical solution functions L for $\tilde{K}$ contains one and only one solution function which is payoff optimal in Λ. This is the distinguished solution function $\tilde{L}$.

PROOF. It follows from theorems 5 and 9 that $\tilde{L}$ is in Λ. It is a consequence of the definition of a distinguished equilibrium set that a solution function L, which is payoff optimal in Λ, must assign the distinguished equilibrium set to every distinguished simultaneity game in $\tilde{K}$. It follows by theorem 6 that a solution function L cannot be payoff optimal in Λ, if it

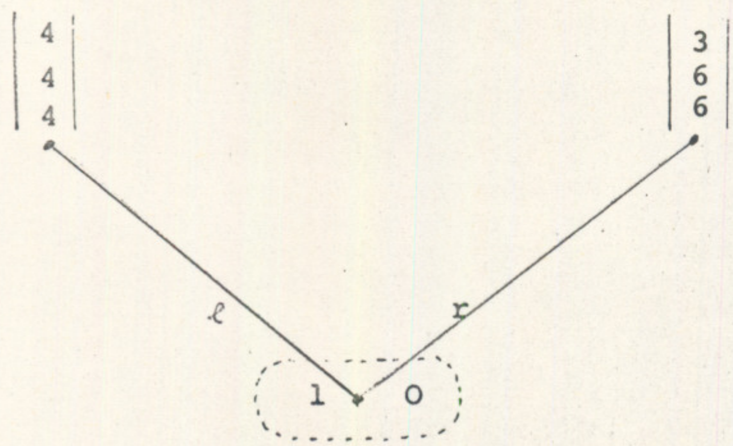

Figure 2: The indecomposable $\tilde{L}$-truncation $\bar{\Gamma} = T(\Gamma, \tilde{L})$ of the game Γ represented in figure 1.

The indecomposable $\tilde{L}$-truncation $\bar{\Gamma} = T(\Gamma,\tilde{L})$ of Γ is represented in figure 2. In $\bar{\Gamma}$ player 1 is the only essential player. Obviously $\bar{\Gamma}$ has a distinguished equilibrium set, whose only equilibrium point prescribes the left choice. Consequently Γ is in the region $\tilde{K}$ of $\tilde{L}$. The L-solution $\tilde{L}(\Gamma)$ prescribes the left choice at every information set.

The game Γ has another perfect equilibrium point which prescribes the right choice r at every information set. This equilibrium point is the only element of a perfect equilibrium set R. Obviously R is locally symmetry preserving. The $\tilde{L}$-value of Γ is $v(\Gamma,\tilde{L}) = (4,4,4)$. The equilibrium payoff vector at R is $H(R) = (5,5,5)$.

This shows that another locally symmetry preserving perfect equilibrium set can be strongly payoff superior to the $\tilde{L}$-solution of a game in $\tilde{K}$. At first glance one may think that in view of such cases it is questionable, whether $\tilde{L}$ is a reasonable solution function. With the help of the example of figure 1, it can be easily understood, why this is not a valid counterargument against the distinguished solution function. At the beginning of the game Γ of figure 1 all players prefer R to $\tilde{L}(\Gamma)$, but player 1 knows that after the subgame Γ'' will have been reached players 2 and 3 must be expected to coordinate their expectations at $\tilde{L}(\Gamma'')$, since this in their common interest. The fact that R is strongly payoff superior to $\tilde{L}(\Gamma)$ in the whole game will then be a matter of the past.

Already at the end of section 3.9 it has been pointed out, that the interests of the same player may diverge in different parts of the game and that therefore the efficiency idea behind the definition of the concept of payoff optimality must be applied to all payoffs of all subgames rather than to the payoffs of the whole game only. The numerical example of figure 1 illustrates this point.

4. THE SOLUTION OF THE MODEL

In the following the solution concept developed in sections 3 and 4 will be applied to the extensive form Γ^1 of the model described in section 1. The upper index 1 in the symbol Γ^1 has been used in order to distinguish this game from other games. Since only games related to this game will appear in the remainder of the paper, we drop the upper index 1 and use the symbol Γ without any index in order to denote the extensive form of the model described in section 1. Accordingly the notation Γ_Z will be used for the supply decision subgames and the cartel bargaining subgames will be denoted by Γ_Y. Another notational simplification concerns the distinguished solution function $\tilde{L}$. Here we shall use the symbol L instead of $\tilde{L}$, since no other solution function appears in the remainder of the paper. The distinguished solution of a game will simply be called <u>the solution</u> of this game.

The computation of the solution of the extensive form Γ of the model will follow a "cutting back procedure", which works its way backwards from the end of the game to its beginning by solving indecomposable subgames and forming truncations. First the supply decision subgames Γ_Y will be solved. Then truncated cartel bargaining subgames $\bar{\Gamma}_Z$ are formed as L-truncations of the cartel bargaining subgames. After these games have been solved the indecomposable L-truncation $\bar{\Gamma}$ of Γ can be formed and solved. The games whose solutions are found in this way are the L-bricks of Γ. Finally the solution of Γ can be put together from the solutions of the L-bricks of Γ.

The path to the solution of Γ is not the shortest possible one. The detours have the purpose to exhibit some interesting properties of the model and its solution.

4.1 LEMMATA ON THE SUPPLY DECISION SUBGAME. Obviously the supply decision subgames Γ_Y are simultaneity games. A strategy q_i for Γ_Y is a finite probability distribution over the interval $0 \leq x_i \leq y_i$. The following lemma will show, that only the pure strategies are important.

LEMMA 6. Let $s = (s_1,\ldots,s_n)$ be an equilibrium point for a supply decision subgame Γ_Y; then s is a pure strategy combination.

PROOF. In order to prove the lemma, it is sufficient to show that for every strategy combination $q=(q_1,\ldots,q_n)$ each player i has exactly one best reply r_i which is a pure strategy. Let us distinguish two cases. In case 1 the supply $x_i = 0$ is the only pure strategy which guarantees a non-negative gross profit P_i, no matter which of the pure strategies occuring in the mixed strategies q_j of the other players are realized. In case 2 player i can choose a supply $x_i > o$ which guarantees a non-negative gross profit P_i, no matter which of the pure strategies occuring in the mixed strategies q_j of the others are realized. It follows from (9) that in case 1 the supply $x_i = 0$ is the only best reply of player i.

Now consider case 2. Let $\bar{x}_j$ be the greatest supply x_j such that q_j assigns a positive probability to x_j.

Define

$$(17) \qquad \bar{X}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \bar{x}_j$$

Obviously we must have $\bar{X}_i < 1$. In order to be sure to receive a non-negative gross profit, player i must select a supply x_i in the closed intervall $0 \leq x_i \leq \hat{x}_i$, where $\hat{x}_i$ is the minimum of y_i and $1-\bar{X}_i$. It follows from $g=1 - X$, that in this intervall the expected gross profit of player i is a strictly concave quadratic function. Consequently player i has exactly one best reply, which is a supply $\tilde{x}_i$ with $0 \leq \tilde{x}_i \leq \hat{x}_i$.

REMARK. If the players had the utility function $u_i=P_i$ instead of (9), a similar argument would not go through, since over the whole range $x_i \geq 0$, the variable P_i is not a concave function of x_i.

Lemma 1 shows that we can restrict our attention to pure strategy combinations. In the following a pure strategy combination for Γ_Y is identified with the corresponding supply decision vector $x=(x_1,\ldots,x_n)$.

In lemma 2 a function $\varphi_i(X_i)$ is introduced, which is called the reaction function of player i. This function is indeed the familiar reaction function from the Cournot oligopoly theory. In lemma 8, equation (21) we shall define a related function $\eta_i(X)$, which is called the fitting-in-function [14]. The fact that this function depends on the total supply X, rather than on X_i, makes it a useful instrument for the analysis of the Cournot model.

LEMMA 7. Let $(x_1,\ldots,x_n)$ be a pure strategy combination $(x_1,\ldots,x_n)$ for a supply decision subgame Γ_y define

$$X_i = \sum_{\substack{j=1 \\ j \neq i}} x_j \tag{18}$$

Then

$$\varphi_i(X_i) = \max \left[0, \min \left[\frac{1-X_i}{2}, y_i\right]\right] \tag{19}$$

is player i's best reply to $(x_1,\ldots,x_n)$

[14] The concept of a fitting-in function has beeen introduced for a wide class of oligopoly models in [12]. The German name is "Einpassungsfunktion".

PROOF. Consider first the case $X_i \geq 1$. In this case $x_i = 0$ is the only supply which gives players i a non-negative gross profit and $\varphi_i(X_i) = 0$ is the best reply to $x = (x_1,\ldots,x_n)$.

In the case $X_i < 1$ player i's gross profit is negative outside the intervall $0 \leq x_i \leq 1 - X_i$. Within this intervall the function $x_i(1-X_i-x_i)$ assumes its maximum at $x_i = (1 - X_i)/2$. This shows that for $X_i < 1$ the best reply to $(x_1,\ldots,x_n)$ is given by (19).

LEMMA 8. Let Γ_y be a supply decision subgame with the binding quota vector $y = (y_1,\ldots,y_n)$. Define

(20) $$\eta_i(X) = \max\left[0, \min\left[1-X, y_i\right]\right]$$

for $i = 1,\ldots,n$ (the function $\eta_i(X)$ is called player i's fitting-in function). For every $X \geq 0$ and for $i=1,\ldots,n$ the function $\eta_i(X)$ satisfies the condition

(21) $$\eta_i(X) = \varphi_i(X-\eta_i(X))$$

and for every fixed $X \geq 0$ the only solution of the equation

(22) $$x_i = \varphi_i(X-x_i)$$

is $x_i = \eta_i(X)$.

PROOF. $\varphi_i(X_i)$ is monotonically non-increasing. Therefore $\varphi_i(X-x_i)-x_i$ is monotonically decreasing in x_i. Consequently for every $X \geq 0$ there is at most one x_i satisfying (22). It remains to be shown that (21) is true, (19) yields

(23) $$\varphi_i(X-\eta_i(X)) > \max\left[0, \min\left[\frac{1-X+\eta_i(X)}{2}, y_i\right]\right]$$

In order to prove (21) we distinguish the following three cases (24),(25) and (26)

(24) $\quad 1 - X \leq 0$

(25) $\quad 0 < 1 - X < y_i$

(26) $\quad y_i \leq 1 - X$

In case (24) we have $\eta_i(X) = 0$. If we insert this on the right side of (23), we see that because of (24) condition (21) is satisfied. Now consider case (25). In this case $\eta_i(X)$ is equal to 1-X. It is clear from (23) and (25) that (21) holds in this case too. In case (26) we have $\eta_i(X)$ is equal to y_i. Inequality (26) implies

(27) $$y_i \leq \frac{1-X+y_i}{2}$$

This shows that (21) is satisfied.

4.2 THE SOLUTION OF THE SUPPLY DECISION SUBGAME

In the following the results of the last section will be used in order to find the solutions of the supply decision subgames. For this purpose we introduce the total fitting-in function $\eta(X)$:

(28) $$\eta(X) = \sum_{i=1}^{n} \eta_i(X) = \sum_{i=1}^{n} \max\left[0, \min\left[1-X, y_i\right]\right]$$

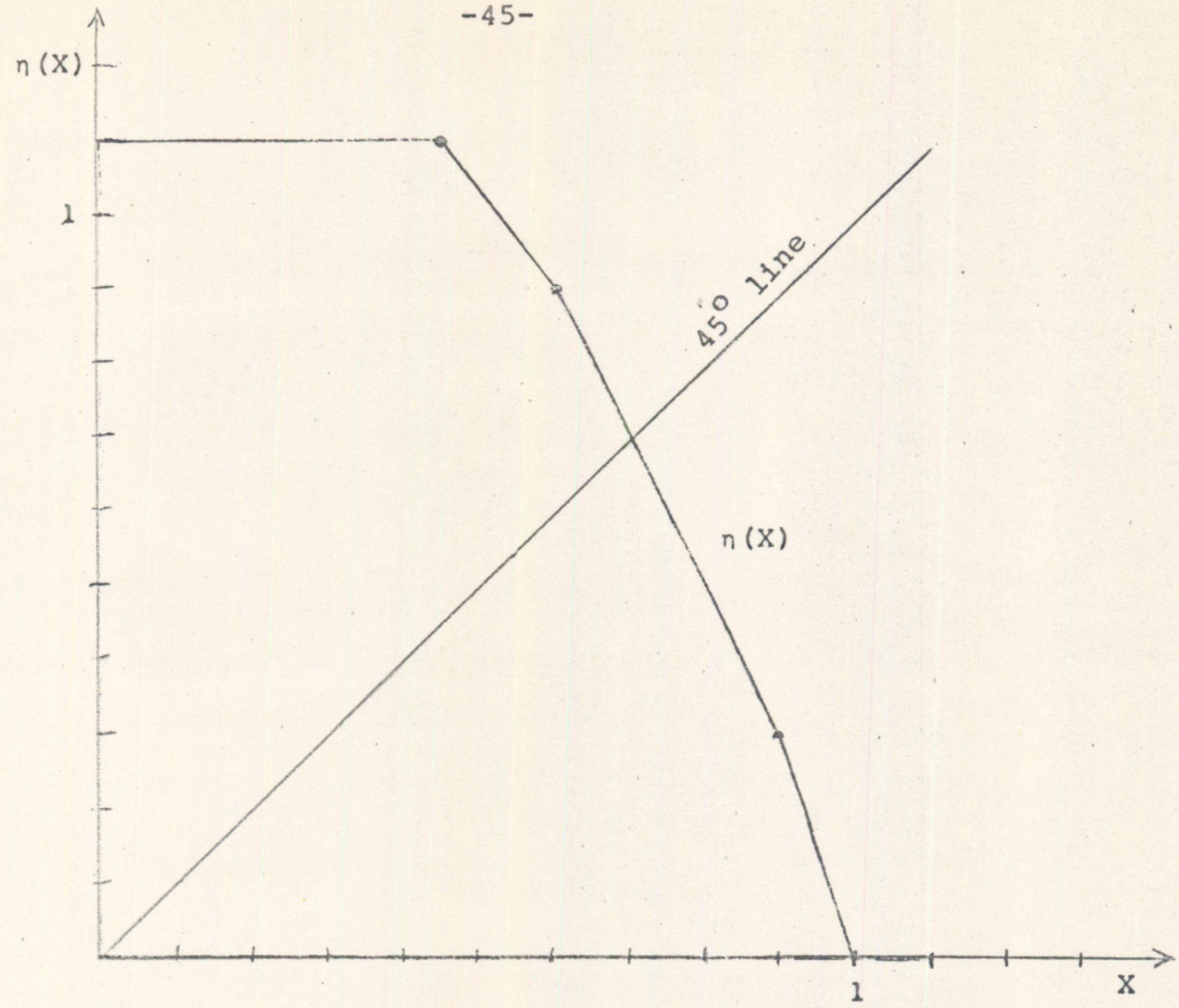

Figure 3: The fitting-in diagram for n=3 and $y_1 = .6$, $y_2 = .4$, $y_3 = .1$. The intersection of η(X) with the 45^o-line is at X= .7 . The equilibrium point is at $x_1 = .3$, $x_2 = .3$, $x_3 = .1$.

Consider the pure strategy equilibrium point $(x_1,\dots,x_n)$ of Γ_Y and let X be the total supply belonging to $x=(x_1,\dots,x_n)$. It follows from the definition of an equilibrium point and from lemma 7 that (22) must hold for i=1,...,n. In view of lemma 8 this means that we must have $x_i = \eta_i(X)$. Hence we also must have

(29) $X = \eta(X)$

Moreover, it is clear that any solution X of (29) together with equations (21) generates an equilibrium point $(x_1,\dots,x_n)$. A convenient graphical representation of the solutions of (29) can be given with the help of a diagram which shows $\eta(X)$ and the 45°-line. This diagram will be called the <u>fitting-in diagram</u>. An example is given in figure 3. In the fitting-in diagram the solutions of (29) are represented by the intersections of $\eta(X)$ with the 45°-line. Since $\eta(X)$ is a continuous non-increasing function with $\eta(0) \geq 0$ and $\eta(1) = 0$, it is clear that $\eta(X)$ has exactly one intersection with the 45°-line and Γ_Y has exactly one equilibrium point, whose total supply X satisfies the inequality

(30) $0 \leq X < 1$

The results which just have been derived, are summarized by the following theorem.

<u>THEOREM 11.</u> Let Γ_Y be a supply decision subgame. Then Γ_Y has a unique equilibrium point. This equilibrium point is an equilibrium point $(x_1,\dots,x_n)$ in pure strategies. The total supply X belonging to $(x_1,\dots,x_n)$ is the unique solution of the equation $X=\eta(X)$ and satisfies the inequality $0 \leq X < 1$. Moreover we have $x_i = \eta_i(X)$ for i = 1,...,n. (Here η and η_i are defined as in (28) and (20) resp.).

REMARK. Since Γ_Y has only one equilibrium point, the solution $L(\Gamma_Y)$ is the equilibrium set with this equilibrium point as its single element. Obviously Γ_Y is a distinguished simultaneity game.

4.3 PROPERTIES OF THE SUPPLY DECISION EQUILIBRIUM.

In this section the determination of the solution of the cartel bargaining subgames Γ_Z will be prepared by the derivation of some results on the equilibrium points of the supply decision subgames. We first look at the special case of a supply decision subgame Γ_Y with a binding quota vector $(y_1,...,y_n)$ with $y_i = \infty$ for $i = 1,...,n$. We call this case the unrestricted case. The unrestricted case is an important limiting case. If no cartel agreements were possible then the equilibrium point of the unrestricted case would be the non-cooperative solution of the model.

LEMMA 9. Let Γ_Y be a supply decision subgame with a binding quota vector $y = (y_1,...,y_n)$ with $y_i = \infty$. Then the components of the equilibrium point $(x_1,...,x_n)$ for Γ_Y are given by

$$(31) \quad x_i = \frac{1}{n+1} \text{ for } i = 1,...,n$$

and player i's profit P_i at $(x_1,...,x_n)$ is given by

$$(32) \quad P_i = \frac{1}{(n+1)^2} \text{ for } i = 1,...,n.$$

PROOF. Because of (28), (29) and (30) we have

(33) $X = n(1-X)$

(34) $X = \frac{n}{n+1}$

(31) is a consequence of (34) and (20). Equation (32) follows by (7) and (8).

LEMMA 10. Let Γ_Y be a supply decision subgame of a given cartel bargaining subgame Γ_Z. Let $(x_1,\ldots,x_n)$ be the equilibrium point of Γ_Y and let k be the number of non-participators (the number of players in N-Z). Define

(35) $X_Z = \sum_{i\varepsilon Z} x_i$

Then the following is true:

(36) $x_i = \frac{1}{k+1}(1-X_Z)$ for $i \varepsilon N-Z$

(37) $X_Z \leq \frac{n-k}{n+1}$

PROOF. Since no quotas are fixed for non-participators we have

(38) $y_i = \infty$ for $i \varepsilon N-Z$

This together with (20) and (30) yields

(39) $x_i = 1 - X$ for $i \varepsilon N-Z$

Define

(40) $X_{N-Z} = \sum_{i\varepsilon N-Z} x_i$

(39) yields

(41) $X_{N-Z} = k(1-X_Z-X_{N-Z})$

(42) $X_{N-Z} = \frac{k}{k+1}(1-X_Z)$

(39) shows that the equilibrium supply x_i is the same for all $i \varepsilon N-Z$. This together with (42) proves (36). Because of (42) we have

(43) $1 - X = 1 - X_Z - X_{N-Z}$

(44) $1 - X = \frac{1}{k+1}(1-X_Z)$

The inequality

(45) $x_i \leq 1 - X$ for $i \varepsilon Z$

is a consequence of (20) and (30). This together with (44) yields

(46) $X_Z \leq \frac{k}{1+k}(1-X_Z)$

(46) is equivalent to (37).

REMARK. Note that because of (31) in the unrestricted case X_Z is equal to the upper bound on the right side of (37).

LEMMA 11. Under the assumptions of lemma 10 let P_i be player i's gross profit at the equilibrium point $(x_1,\ldots,x_n)$ of Γ_Y.

Define

(47) $P_Z = \sum_{i \varepsilon Z} P_i$

Then the following is true:

(48) $P_Z = \frac{1}{k+1} X_Z(1-X_Z) \leq \frac{1}{4(k+1)}$

PROOF. Because of (30) we can write

(49) $P_Z = X_Z(1-X)$

This together with (44) yields

(50) $P_Z = \frac{1}{k+1} X_Z(1-X_Z)$

The right side of (50) assumes its maximum at $X_Z=1/2$. This proves (48).

4.4 THE SOLUTIONS OF THE TRUNCATED CARTEL BARGAINING SUBGAMES.

Let Γ_Z be a cartel bargaining subgame. The indecomposable L-truncation $\bar{\Gamma}_Z=T(\Gamma,L)$ of Γ_Z is called the truncated cartel bargaining subgame for Z. In this section it will be shown that $\bar{\Gamma}_Z$ has a distinguished equilibrium set.

Consider an equilibrium point $\bar{s}_Z$ of a truncated cartel bargaining subgame $\bar{\Gamma}_Z$, such that the equilibrium payoffs at $\bar{s}_Z$ are the gross profits (32) obtained in the unrestricted case of a supply decision subgame. Formally an equilibrium point of this kind may very well involve cartel agreements as we shall see in lemma 12, but such cartel agreements have no economic significance and therefore will be called inessential. No cartel bargaining is necessary in order to achieve the payoffs (32).

The solution of $\bar{\Gamma}_Z$ depends on the number k of players in N-Z. As we shall see, for $k \geq (n-1)/2$ the equilibrium payoffs connected to the equilibrium points in $L(\bar{\Gamma}_Z)$ are the gross profits (32). In this case only inessential cartel agreements result from the equilibrium points in $L(\bar{\Gamma}_Z)$. For $k < (n-1)/2$ the situation is different. Here the equilibrium payoffs at $L(\bar{\Gamma}_Z)$ are greater than those of the unrestricted case of a supply decision subgame.

Generally the solution $L(\bar{\Gamma}_Z)$ of a truncated cartel bargaining subgame contains many equilibrium points. There are two reasons for this: different proposal systems may lead to the same quota vector and different quota vectors may lead to the same equilibrium payoffs in the supply decision subgame.

For our purposes, it is not necessary to describe $L(\bar{\Gamma}_Z)$ in detail. It is sufficient to exhibit one equilibrium point in $L(\bar{\Gamma}_Z)$ and to describe $L(\bar{\Gamma}_Z)$ as that equilibrium set, which contains this equilibrium point.

<u>LEMMA 12.</u> Let $\bar{\Gamma}_Z$ be a truncated cartel bargaining subgame. Then the following system of proposals Y is an equilibrium point in pure strategies for $\bar{\Gamma}_Z$:

(51) $Y = (Y_i)_{i\varepsilon Z}$ where for every $i\varepsilon Z$

$Y_i = (y_{ij})_{j\varepsilon Z}$ with $y_{ij} = \infty$

The binding quota vector $(y_1,\dots,y_n)$ generated by this equilibrium point has the property $y_i=\infty$ for $i=1,\dots,n$.

<u>PROOF.</u> Formally an agreement results from Y, but this agreement is an inessential one, since the binding quota vector has the property $y_i= \infty$ for $i = 1,\dots,n$.
We must show that no deviation of a player $j\varepsilon Z$ can improve his gross profit. The only deviation which can change the binding quota vector is a deviation to a proposal for the one-person coalition $\{j\}$ containing j as its only element. Let y_j' be the quota which player i proposes for himself. The new binding quota vector has y_j' as its j-th component and $y_i= \infty$ for all $i \neq j$.

The proposal system (51) has the result that all players get the gross profit from (32). It is clear from the proof of lemma 4 that the new binding quota cannot lead to a different result unless we have

(52) $x_j = \min[1-X,y_j'] = y_j'$

Because of

(53) $x_i = 1-X$ for $i \neq j$

we must have

(54) $X = y_j' + (n-1)(1-X)$

(55) $X = \frac{n-1}{n} + \frac{y_j'}{n}$

(56) $1-X = \frac{1-y_j'}{n}$

This together with (52) yields

$$(57) \qquad y_j' \leq \frac{1-y_j'}{n}$$

$$(58) \qquad y_j' \leq \frac{1}{n+1}$$

Because of (7),(8) and (56) player j's gross profit P_j' after the deviation can be written as follows:

$$(59) \qquad P_j' = \frac{1}{n} y_j' (1-y_j')$$

In the interval $0 \leq y_j' \leq 1/(n+1)$ the profit P_j' is an increasing function of y_j'. Therefore we must have

$$(60) \qquad P_j' = \frac{1}{n} \cdot \frac{1}{n+1} \left(1-\frac{1}{n+1}\right) = \frac{1}{(n+1)^2}$$

This shows that the deviation to y_i' does not improve player j's gross profit above its equilibrium value from (32). Consequently (51) is an equilibrium point of $\bar{\Gamma}_Z$.

<u>THEOREM 12.</u> Let $\bar{\Gamma}_Z$ be a truncated cartel bargaining subgame where the number k of non-participators satisfies the inequality

$$(61) \qquad k \geq \frac{n-1}{2}$$

Then $\bar{\Gamma}_Z$ is a distinguished simultaneity game and the distinguished equilibrium set $\bar{S}_Z$ contains the pure strategy equilibrium point (51) from lemma 12. The equilibrium payoffs at $\bar{S}_Z$ are the gross profits (32) from lemma 9.

<u>PROOF.</u> The symmetries of $\bar{\Gamma}_Z$ correspond to those permutations of N which leave Z and N-Z unchanged. Since at the equilibrium point (51) the players in Z have equal payoffs and the players in N-Z have equal payoffs, the equilibrium set $\bar{S}_Z$ of this equilibrium point is symmetry preserving.

We have to show that $\bar{\Gamma}_Z$ is a distinguished simultaneity game and that $\bar{S}_Z$ is the distinguished equilibrium set of $\bar{\Gamma}_Z$. Since the players in N-Z are inessential and since every symmetry preserving equilibrium set must give the same payoff to all players in Z, it is sufficient to show, that the joint gross profit P_Z of the players in Z at (51) cannot be surpassed by the joint gross profit of the players in Z at any other equilibrium point of $\bar{\Gamma}_Z$.

For any supply decision subgame of Γ_Z the joint equilibrium supply X_Z of the players in Z is bounded by the right side of (37). If the lower bound for k from (61) is inserted on the right side of (37) we get

$$(62) \qquad X_Z \leq \frac{1}{2}$$

It can be seen from (50), that in the interval $0 \leq X_Z \leq 1/2$ the joint gross profit P_Z of the players in Z is a monotonically increasing function of X_Z. Therefore P_Z cannot be greater than the profit at the upper bound of X_Z in (37) which is assumed at the supplies specified in (31). This shows that the equilibrium set S_Z is the distinguished equilibrium point of $\bar{\Gamma}_Z$ and that the equilibrium payoffs at $\bar{S}_Z$ are the gross profits (32). Obviously $\bar{\Gamma}^1_Z$ is a distinguished simultaneity game.

<u>LEMMA 13.</u> Let $\bar{\Gamma}_Z$ be a truncated cartel bargaining subgame, where the number k of non-participators satisfies the inequality

$$(63) \qquad k \leq \frac{n-1}{2}$$

Then the following system of proposals Y is an equilibrium point in pure strategies for $\bar{\Gamma}_Z$.

$$(64) \qquad Y = (Y_i)_{i\varepsilon Z} \quad \text{where for every } i \varepsilon Z$$

$$Y_i = (y_{ij})_{i\varepsilon Z} \text{ with } y_{ij} = \frac{1}{2(n-k)} \text{ for all } j \varepsilon Z$$

The equilibrium payoffs at this equilibrium point are the following gross profits:

(65) $$P_i = \frac{1}{4(n-k)(k+1)} \quad \text{for} \quad i \in Z$$

(66) $$P_i = \frac{1}{4(k+1)^2} \quad \text{for} \quad i \in N-Z$$

PROOF. Let Γ_y be the supply decision subgame resulting from (64) and let $(x_1,\ldots,x_n)$ with the total supply X be the equilibrium point of Γ_y. Obviously the binding quota vector $(y_1,\ldots,y_n)$ of Γ_y is as follows:

(67) $$y_i = \frac{1}{2(n-k)} \quad \text{for} \quad i \in Z$$

(68) $$y_i = \infty \quad \text{for} \quad i \in N-Z$$

Because of (28), (29) and (30) the total supply X satisfies the condition

(69) $$X = k(1-X) + (n-k) \min \left[1-X, \frac{1}{2(n-k)}\right]$$

In the following it will be shown that we must have

(70) $$\min \left[1-X, \frac{1}{2(n-k)}\right] = \frac{1}{2(n-k)}$$

If (70) were wrong, (69) would assume the form

(71) $$X = n(1-X)$$

This yields

(72) $$X = \frac{n}{n+1}$$

Consequently (70) cannot be wrong unless the following is true

(73) $$\frac{1}{n+1} < \frac{1}{2(n-k)}$$

It is an immediate consequence of (63) that we must have

$$(74) \qquad 2(n-k) \geq 2(n-\frac{n-1}{2}) = n+1$$

This contradicts (73). Therefore (70) is correct. By theorem 11 we have $x_i = \eta_i(X)$. With the help of (20) this yields

$$(75) \qquad x_i = \frac{1}{2(n-k)} \quad \text{for} \quad i \in Z$$

The equilibrium supplies for $i \in N-Z$, can be computed from (75) and (36). We receive

$$(76) \qquad x_i = \frac{1}{2(k+1)} \quad \text{for} \quad i \in N-Z$$

The profit margin g at $(x_1,\ldots,x_n)$ is given by

$$(77) \qquad g = \frac{1}{2(k+1)}$$

It follows that the gross profits at $(x_1,\ldots,x_n)$ are the gross profits P_i in (65) and (66).

It remains to be shown that the proposal system (64) is an equilibrium point of $\bar{\Gamma}_z$. It is not necessary to look at the inessential players in N-Z. Consider a player $j \in Z$. Player j has two kinds of deviations. Some deviations have the result that the new binding quota vector gives a quota of ∞ to every player including player j. As we can see from lemma 9, if this happens player j's payoff after the deviation is equal to $1/(n+1)^2$. Later we shall show that (63) implies

$$(78) \qquad \frac{1}{4(n-k)(k+1)} \geq \frac{1}{(n+1)^2}$$

This inequality togehher with (65) has the consequence that a deviation of the kind considered above is unprofitable. The only other possibility of a deviation of a player $j \in Z$ is a deviation to a proposal for the one-person coalition {j} which would result in some binding quota y_j for player j and binding quotas $y_i=\infty$ for all other players i.

From the fact that the proposal system (51) from lemma (12) is an equilibrium point, where according to lemma 9 every player receives $1/(n+1)^2$ as his equilibrium payoff, we can conclude that such deviations are not more profitable than those which yield binding quota vectors $y_i = \infty$ for all players i.

In order to prove that (63) implies (78), we observe that the partial derivative of $4(n-k)(k+1)$ with respect to k is $4(n-1-2k)$. Obviously this is positive, if k satisfies $0 \leq k < (n-1)/2$. Therefore in the interval $0 \leq k \leq (n-1)/2$ the gross profit P_i in (65) is a monotonically decreasing function of k. At $k=(n-1)/2$ the gross profit P_i assumes the value $1/(n+1)^2$. This shows that (78) holds for $k < (n-1)/2$.

REMARK. In the course of the proof of lemma 13, it has been shown that for $k \leq (n-1)/2$ the gross profit (65) of a participator is bounded by (78). The lower bound $1/(n+1)^2$ is the supply decision equilibrium payoff of the unrestricted case. If k is equal to $(n-1)/2$ then (64) is an equilibrium point in the equilibrium set $\bar{S}_z$ from theorem 12. In this case the cartel agreement resulting from (64) is inessential. Note that both for $i \in Z$, and $i \in N-Z$ the equilibrium payoffs become smaller if the number k of non-participators is increased within the interval $0 \leq k \leq (n-1)/2$.

THEOREM 13. Let $\bar{\Gamma}_z$ be a truncated cartel bargaining subgame, where the number k of non-participators satisfies the inequality

$$(79) \qquad k < \frac{n-1}{2}$$

Then $\bar{\Gamma}_z$ is a distinguished simultaneity game and the distinguished equilibrium set $\bar{S}_z$ of $\bar{\Gamma}_z$ contains the pure strategy equilibrium point (64). The equilibrium payoffs at $\bar{S}_z$ are the gross profits (65) and (66) from lemma 13.

PROOF. Let $\bar{S}_Z$ be the equilibrium set of the equilibrium point (64). In the same way as in the proof of theorem 12 we can see that $\bar{S}_Z$ is symmetry preserving.

In order to show that Γ_Z is a distinguished simulteneity game and $\bar{S}_Z$ is the distinguished equilibrium set of $\bar{\Gamma}_Z$ it is sufficient to show that the joint equilibrium payoff of the players in Z cannot be surpassed by the joint gross profit of the players in Z at any other equilibrium point of $\bar{\Gamma}_Z$. It can be seen from (65) that the joint equilibrium payoff of the players in Z is equal to the upper bound in (48). This upper bound cannot be surpassed by the joint equilibrium gross profit P_Z of the players in Z in any supply decision subgame of Γ_Z. This completes the proof.

REMARK. Generally $\bar{S}_Z$ contains many equilibrium points. This can be seen easily for the trivial case n=1, and k=0 where any binding quota $y_1 \geq 1/2$ is compatible with the monopolist's optimal supply $x_1 = 1/2$. For $n > 1$, it is also possible that $\bar{S}_Z$ contains more than one equilibrium point. In order to see this, one may look at the case n=4, k=0. There one can find equilibrium points which achieve the binding quota vector of (64) by two 2-person agreements. Since this is an unimportant detail, no proof is given here.

4.5 THE PARTICIPATION DECISION BRICK. Let $\bar{\Gamma}$ be the indecomposable L-truncation $\bar{\Gamma} = T(\Gamma, L)$ of the extensive form of the model. $\bar{\Gamma}$ will be called the participation decision brick. In $\bar{\Gamma}$ each player i has two strategies: he may choose $z_i = 0$ or $z_i = 1$. The payoff function of the participation decision brick is described in theorem 14. Up to n=10, the numerical values of the payoffs are tabulated in table 1.

THEOREM 14. Let $z = (z_1, \ldots, z_n)$ be a pure strategy combination for the participation decision brick $\bar{\Gamma}$ and let Z be the set of all players i with $z_i = 1$ (the set of all participators). Let k be the number of players in N-Z. Then player i's payoff $\bar{H}_i(z)$ in $\bar{\Gamma}$ is as follows:

Number of players	Number of non-partici-pators	Payoff of a participator	Payoff of a non-partici-pator
n = 1	k = 0	.25000	-
	k = 1	-	.25000
n = 2	k = 0	.12500	-
	k ≥ 1	.01111	.11111
n = 3	k = 0	.08333	-
	k ≥ 1	.06250	.06250
n = 4	k = 0	.06250	-
	k = 1	.04167	.06250
	k ≥ 2	.04000	.04000
n = 5	k = 0	.05000	-
	k = 1	.03125	.06250
	k ≥ 2	.02778	.02778
n = 6	k = 0	.04167	-
	k = 1	.02500	.06250
	k = 2	.02083	.02778
	k ≥ 3	.02041	.02041
n = 7	k = 0	.03571	-
	k = 1	.02083	.06250
	k = 2	.01667	.02778
	k ≥ 3	.01562	.01562
n = 8	k = 0	.03125	-
	k = 1	.01786	.06250
	k = 2	.01389	.02778
	k = 3	.01250	.01562
	k ≥ 4	.01235	.01235
n = 9	k = 0	.02778	-
	k = 1	.01562	.06250
	k = 2	.01190	.02778
	k = 3	.01042	.01562
	k ≥ 4	.01000	.01000
n = 10	k = 0	.02500	-
	k = 1	.01389	.06250
	k = 2	.01042	.02778
	k = 3	.00893	.01562
	k = 4	.00833	.01000
	k ≥ 5	.00826	.00826

Table 1: Payoffs for the participation decision brick up to n = 10

$$(80) \quad \bar{H}_i(z)=\begin{cases} \dfrac{1}{(n+1)^2} & \text{for } i = 1, \ldots, n, \text{ if } k \geq \dfrac{n-1}{2} \\[2ex] \dfrac{1}{4(n-k)(k+1)} & \text{for } i \in Z, \text{ if } k < \dfrac{n-1}{2} \\[2ex] \dfrac{1}{(k+1)^2} & \text{for } i \in N-Z, \quad \text{if } k < \dfrac{n-1}{2} \end{cases}$$

PROOF. (80) is an immediate consequence of theorems 12 and 13.

4.6 PROPERTIES OF THE PAYOFF OF THE PARTICIPATION DECISION BRICK.

In this section several useful properties of the payoff function $\bar{H}_i$ of $\bar{\Gamma}$ shall be derived.

LEMMA 14. Let $z = (z_1,\ldots,z_n)$ be a pure strategy combination for the participation decision brick $\bar{\Gamma}$ and let i be one of the players. Let m be the number of non-participators in $N-\{i\}$. Define

$$(81) \qquad A(n,m)=\begin{cases} \dfrac{1}{4(m+2)^2} & \text{for } m < \dfrac{n-3}{2} \\[2ex] \dfrac{1}{(n+1)^2} & \text{for } m \geq \dfrac{n-3}{2} \end{cases}$$

$$(82) \qquad B(n,m) = \begin{cases} \dfrac{1}{4(n-m)(m+1)} & \text{for } m < \dfrac{n-1}{2} \\[2ex] \dfrac{1}{(n+1)^2} & \text{for } m \geq \dfrac{n-1}{2} \end{cases}$$

Then we have

(83) $\bar{H}_i(z) = A(n,m)$ for $z_i = 0$

(84) $\bar{H}_i(z) = B(n,m)$ for $z_i = 1$

<u>PROOF.</u> The lemma is an immediate consequence of theorem 14. In the case of $z_i = 0$ we have $k = m+1$ and in the case of $z_i = 1$ we have $k = m$.

<u>LEMMA 15.</u> Let m and n be integers with $0 \leq m \leq n$. Define

(85) $D(n,m) = A(n,m) - B(n,m)$

We have

$$(86)\quad D(n,m) = \begin{cases} \dfrac{1}{4(m+2)^2} - \dfrac{1}{4(n-m)(m+1)} & \text{for } m \leq \dfrac{n-4}{2} \\[2ex] \dfrac{1}{(n+1)^2} - \dfrac{1}{4(n-m)(m+1)} & \text{for } \dfrac{n-3}{2} \leq m \leq \dfrac{n-2}{2} \\[2ex] 0 & \text{for } m \geq \dfrac{n-1}{2} \end{cases}$$

and

(87) $D(4,0) = 0$

(88) $D(n,m) > 0$ for $n \geq 5$ and $m \leq \frac{n-4}{2}$

(89) $D(n,m) < 0$ for $\frac{n-3}{2} \leq m \leq \frac{n-2}{2}$

(90) $D(n,m) = 0$ for $m \geq \frac{n-1}{2}$

<u>PROOF</u>. (86) is an immediate consequence of lemma 14. The equation $D(4,0) = 0$ follows by (86) . Now assume $n \geq 5$ and $m \leq (n-4)/2$. Under this condition (88) is equivalent to

(91) $(n-m)(m+1) - (m+2)^2 > 0.$

Because of $n \geq 5$ this inequality holds for $m = 0$. Since $m \leq (n-4)/2$ implies $n \geq 2m + 4$ we receive an upper bound for the left side of (91) if we substitute $m + 4$ for $n-m$. Thus for $m > 0$ inequality (91) follows by (92).

$$(92) \quad (m + 4)(m+1) - (m+2)^2 = m > 0$$

In order to show that (89) is true, we have to examine whether

$$(93) \quad 4(n-m)(m+1) - (n+1)^2 < 0$$

holds for $m=(n-3)/2$ and for $m=(n-2)/2$. For $m=(n-3)/2$ the expression on the left side of (93) is equal to -4 and for $m=(n-2)/2$ we receive -1. Equation (90) is implied by (86).

LEMMA 16. Let m and n be integers with $0 \leq m \leq n-1$. Them we have

$$(94) \quad A(n,m+1) - A(n,m) < 0 \quad \text{for } m < \frac{n-3}{2}$$

$$(95) \quad A(n,m+1) - A(n,m) = 0 \quad \text{for } m \geq \frac{n-3}{2}$$

$$(96) \quad B(n,m+1) - B(n,m) < 0 \quad \text{for } m < \frac{n-1}{2}$$

$$(97) \quad B(n,m+1) - B(n,m) = 0 \quad \text{for } m \geq \frac{n-1}{2}$$

PROOF. (95) and (97) are an immediate consequence of (81) and (82). Obviously (94) holds for $m < (n-5)/2$. Since both for $m=(n-5)/2$ and $m=(n-4)/2$ the expression $1/4(m+2)^2$ is greater than $1/(n+1)^2$, inequality (94) holds for these values of m too. In order to show, that (96) is true we observe that the derivation of $(n-m)(m+1)$ with respect to m is equal to $n-1-2m$. For $m < (n-1)/2$ this is positive. Therefore (96) holds for $m < (n-3)/2$. For $m=(n-3)/2$ we have

$$(98) \quad 4(n-m)(m+1) = (n+3)(n-1) < (n+1)^2$$

and for $m=(n-2)/2$ we receive

$$(99) \quad 4(n-m)(m+1) = (n+2)n < (n+1)^2$$

Therefore (96) holds for these values of m too.

LEMMA 17. The payoff function $\bar{H}$ of the participation decision brick Γ has the following property:

$$\bar{H}_i(z) \geq \frac{1}{(n+1)^2} \quad \text{for } i=1,\dots,n \tag{100}$$

and for every pure strategy combination $z = (z_1,\dots,z_n)$

PROOF. Lemma 16 shows that $A(n,m)$ and $B(n,m)$ are non-increasing functions of m. For m=n these functions are equal to $1/(n+1)^2$. The assertion follows by lemma 14.

4.7 PURE STRATEGY EQUILIBRIUM POINTS OF THE PARTICIPATION DECISION BRICK. One does not have to look at the question which are the pure strategy equilibrium points of $\bar{\Gamma}$ if one wants to find the solution of $\bar{\Gamma}$, but with respect to the interpretation of the solution it is of some interest to know the answer to this question. The pure strategy equilibrium points can be classified according to the number k of non-participators. In the case k=0 we speak of a joint profit maximization equilibrium point. Here the joint gross profit of all players is the monopoly gross profit 1/4. If k is greater than 0 but smaller than $(n-1)/2$, then we speak of a partial cartel equilibrium point. Here the behavior of the players results in a cartel bargaining subgame, whose solution requires an essential cartel agreement, which is partial, since it does not include the non-participators. In the case $k \geq (n-1)/2$ we speak of an unrestricted Cournot equilibrium point. Here every player receives the payoff $1/(n+1)^2$ which is the gross profit connected to the Cournot solution of the model without any quota restrictions.

As we shall see, for small n, up to n=4 joint profit equilibrium points are available but not for $n > 4$. This is the reason why 4 is small, but 5 is not. Partial cartel equilibria can be found for every n with $n \geq 4$. The number of non-participators must be either equal to $(n-3)/2$ or to $(n-2)/2$. This means that for every $n \geq 4$ there is only one possibility for the number k of non-participators. There are altogether $\binom{n}{k}$ partial cartel equilibrium points, where k is the uniquely determined number of non-participators. All these equilibrium points can be mapped into each other by the symmetries of the game.

THEOREM 15. Let $z=(z_1,\ldots,z_n)$ be a pure strategy combination for the participation decision brick $\bar{\Gamma}$. Then z is an equilibrium point of $\bar{\Gamma}$ if and only if n and the number non-participators k connected to z satisfy one of the following three conditions (101),(102) and (103).

(101) $k = 0$ and $n \leq 4$

(102) $0 < \frac{n-3}{2} \leq k \leq \frac{n-2}{2}$

(103) $k \geq \frac{n+1}{2}$

PROOF. In the first part of the proof we show that in all three cases z is an equilibrium point. For every player i let m_i be the number of non-participators in $N-\{i\}$. It follows by lemma 14 that in the case that player i is a participator, he has no reason to deviate, if we have $D(n,m_i) \leq 0$. On the other hand, if he is a non-participator, he has no reason to deviate, if we have $D(n,m_i) \geq 0$.

If (101) is true, then $m_i=0$ holds for $i=1,\ldots,n$. Equation (90) yields $D(1,0)=0$, inequality (89) yields $D(2,0)<0$ and $D(3,0)<0$. Finally (87) covers the case $n=4$.

Now assume that (102) is satisfied. If i is a participator, then we have $m_i = k$. Inequality (89) shows that a participator has no reason to deviate. If i is a non-participator, then $m_i=k-1$. Because of (102) we must have $n \geq 4$. For $n=4$ condition (102) yields $m_i=0$. Equation (87) shows that player i has no reason to deviate. The same is true for $n \geq 5$ in view of (88) and (102).

In the case of (103) we have $m_i \geqslant (n-1)/2$ for $i=1,\ldots,n$. This means that in view of (90) nobody has a reason to deviate.

In order to prove that $\bar{\Gamma}$ has no other pure strategy equilibrium points than those covered by (101),(102) and (103), we observe that k must satisfy one of the following two conditions (104) and (105), if the former three conditions are not satisfied by k:

(104) $k < \frac{n-3}{2}$ and $n \geq 5$

(105) $\frac{n-1}{2} \leq k \leq \frac{n}{2}$

Consider the case (104) and assume that player i is a participator. We have $m_i = k$ and (88) shows that player i has a reason to deviate. Now consider case (105) and assume that player i is a non-participator. We have $m_i = k-1$ and (89) shows that player i has a reason to deviate.

REMARK. Note that generally (103) allows us to find very many unrestricted Cournot equilibrium points. All these equilibrium points are very weak in the sense that no player can loose anything by a deviation as we can see from lemma 17. For $n = 2$ and $n = 3$ the joint profit maximization equilibrium point is strong in the sense that a deviation of a player decreases his payoff. This is not true for $n = 1$ and $n = 4$, for $n \geq 5$ the partial cartel equilibrium points are strong in the same sense. Here $n = 4$ is an exception. For $n \geqslant 5$ the strongness of the partial cartel equilibrium points is due to (88) and (89); inequality (88) does not include $n = 4$.

4.8 MIXED STRATEGY EQUILIBRIUM POINTS OF THE PARTICIPATION DECISION BRICK. We shall not try to get a complete overview over the mixed strategy equilibrium points of $\bar{\Gamma}$, but we must look at some of their properties in order to derive the solution of $\bar{\Gamma}$.

A mixed strategy combination of the participation decision brick $\bar{\Gamma}$ can be represented by a vector of probabilities

(106) $w = (w_1,\ldots,w_n)$

with

(107) $0 \le w_i \le 1$ for $i = 1,\ldots,n$

where w_i is the probability that player i selects $z_i = 1$. In the following this representation of mixed strategies and mixed strategy combinations will always be used. $\bar{H}(w)=(\bar{H}_1(w),\ldots,\bar{H}_n(w))$ is the payoff vector associated with w.

<u>LEMMA 18.</u> Let $w = (w_1,\ldots,w_n)$ be a mixed strategy equilibrium point for $\bar{\Gamma}$ with

(108) $\bar{H}_j(w) > \frac{1}{(n+1)^2}$

for some player j. Then $w_j > w_j'$ implies

(109) $\bar{H}_j(w) < \bar{H}_{j'}(w)$

<u>PROOF.</u> Let A_j be the payoff of player j which he receives if he selects $z_j=0$, while all the other players i use their mixed strategies w_i in w. Similarly let B_j be the payoff of player j, if he uses $z_j=1$ while the others use w_i. Let $A_{j'}$ and $B_{j'}$ be defined in the same way for player j'. Let W_m be the probability that exactly m of the players in $N-\{j\}-\{j'\}$ become non-participators, if these players use their mixed strategies w_i. We have:

(110) $$A_j = w_{j'} \sum_{m=0}^{n-2} W_m A(n,m) + (1-w_{j'}) \sum_{m=0}^{n-2} W_m A(n,m+1)$$

(111) $$A_j = \sum_{m=0}^{n-2} W_m \left[A(n,m+1) + w_{j'} \left(A(n,m)-A(n,m+1)\right)\right]$$

Similar equations hold for $B_j, A_{j'}$ and $B_{j'}$:

(112) $$B_j = \sum_{m=0}^{n-2} W_m \left[B(n,m+1) + w_{j'}\left(B(n,m)-B(n,m+1)\right)\right]$$

(113) $$A_{j'} = \sum_{m=0}^{n-2} W_m \left[A(n,m+1) + w_{j'} \left(A(n,m) - A(n,m+1) \right) \right]$$

(114) $$B_{j'} = \sum_{m=0}^{n-2} W_m \left[B(n,m+1) + w_{j'} \left(A(n,m) - B(n,m+1) \right) \right]$$

Since w is an equilibrium point, the following must be true:

(115) $$\bar{H}_j(w) = \max \left[A_j, B_j \right]$$

(116) $$\bar{H}_{j'}(w) = \max \left[A_{j'}, B_{j'} \right]$$

Let us distinguish the two (overlapping) cases

(117) $$\bar{H}_j(w) = A_j$$

and

(118) $$\bar{H}_j(w) = B_j$$

As we shall see in case (117) there must be at least one m with $W_m > 0$ such that $A(n,m) - A(n,m+1)$ is negative and in case (118) there must be at least one m with $W_m > 0$ such that $B(n,m) - B(n,m+1)$ is negative. Cobsider the case (117). Let m' be the smallest number with $W_{m'} > 0$. Suppose that the difference $A(n,m) - A(n,m+1)$ vanishes for $m = m'$. Then this difference also vanishes for all $m \geq m'$. This follows by (94) and (95). Moreover because $A(n,m)$ is equal to $1/(n+1)^2$, equation (111) yields $A_j = 1/(n+1)^2$. Since this is excluded by (108), the difference $A(n,m) - A(n,m+1)$ is negative for $m = m'$. In the same way it can be shown that in the case (118) the difference $B(n,m) - B(n,m+1)$ must be negative for $m = m'$.

In view of this result a comparison of (111) and (113) shows that because of $w_j > w_{j'}$ the following is true for $A_j \geqslant B_j$:

(119) $\bar{H}_j(w) = A_j < A_{j'} \leq \bar{H}_{j'}(w)$

Similarly (112) and (114) yield in the case of $B_j \geq A_j$:

(120) $\bar{H}_j(w) = B_j < B_{j'} \leq H_{j'}(w)$

LEMMA 19. Let $z=(z_1,\ldots,z_n)$ be a pure strategy equilibrium point of the participation decision brick $\bar{\Gamma}$ where the number k of non-participators satisfies $0 < k < n$(i.e. z is a partial cartel equilibrium point). Then for $z_j = 1$ and $z_j' = 0$ we have

(121) $\bar{H}_{j'}(z) > \bar{H}_j(z) > \frac{1}{(n+1)^2}$

PROOF. k satisfies (102). Therefore (96) shows that $B(n,k-1)$ is greater than $1/(n+1)^2$. The payoff $H_j(z)$ is equal to $B(n,k-1)$. It follows by the application of (120) to the special case of z, that (121) is true.

LEMMA 20. Let $\bar{S}$ be a symmetry preserving equilibrium set of the participation decision brick Γ^1 with

(122) $\bar{H}_i(\bar{S}) > \frac{1}{(n+1)^2}$ for $i = 1,\ldots,n$

Let $w = (w_1,\ldots,w_n)$ be an equilibrium point in $\bar{S}$. Then we have

(123) $w_i = w_1$ for $i = 2,\ldots,n$

PROOF. $\bar{\Gamma}$ is completely symmetric. Therefore the payoff at $\bar{S}$ is the same for every player i. If (123) were not true, then in view of (122) lemma 18 could be applied to w; this would lead to the conclusion that the payoffs of two players are not equal at w.

4.9 THE SOLUTION OF THE PARTICIPATION DECISION BRICK. With the help of the results of the last section, it is now possible to find the solution of $\bar{\Gamma}$. First a theorem will show that for

$n > 1$ the game $\bar{\Gamma}$ has exactly one equilibrium point with the properties (122) and (123). This equilibrium point turns out to be the only element in the distinguished equilibrium set of $\bar{\Gamma}$.

THEOREM 16. For $n > 1$ the participation decision brick $\bar{\Gamma}$ has exactly one equilibrium point $w=(w_1,\ldots,w_n)$ with the properties (122) and (123). Moreover the following is true for this equilibrium point:

(124) $\quad w_1 = 1 \quad \text{for} \quad 1 < n \leq 4$

(125) $\quad 0 < w_1 < 1 \quad \text{for} \quad n > 4$

PROOF. The possibility $w_1=0$ is excluded by (122), since $w_1=0$ leads to the payoff $1/(n+1)^2$ for all players. Henceforth we shall assume $w_1>0$. The pure strategy $z_i=0$ is a best reply of player i to w if and only if the following expression D is non-negative.

(126) $$D = \sum_{m=0}^{n-1} \binom{n-1}{m} w_1^{\,n-m}(1-w_1)^m D(n,m)$$

It is a consequence of the definition of $D(n,m)$ that D is nothing else than player i's payoff for $z_i=0$ minus player i's payoff for $z_i=1$, if the other players use their strategies w_i in w. The pure strategy $z_i=1$ is a best reply to w, if and only if D is non-positive. Let $\bar{m}$ be that number which satisfies the condition

(127) $$\frac{n-3}{2} \leq \bar{m} \leq \frac{n-2}{2}$$

Obviously for every n there is exactly one such number $\bar{m}$. Lemma 15 shows that $D(n,m)$ vanishes for $m>\bar{m}$. Therefore we have

(128) $$D = \sum_{m=0}^{\bar{m}} \binom{n-1}{m} w_1^{\,n-m}(1-w_1)^m D(n,m)$$

For n=2 and n=3 we have $\bar{m}$=0. Inequality (89) shows that D(2,0) and D(3,0) are negative. Therefore in these two cases D is negative for every w_1 with $0 < w_1 < 1$. The same is also true for n=4 where m assumes the value 1; here we have D(4,0) =0 by (87) and D(4,1) < 0 by (89). For any equilibrium point w with $0 < w_1 < 1$ the expression D must vanish since both $z_i=0$ and $z_i=1$ are best replies to w. Since D is negative for every w_1 with $0 < w_1 < 1$ in the cases n=2, n=3 and n=4, this shows, that in these cases the joint profit maximization equilibrium point with $w_1=1$ is the only equilibrium point with the properties (122) and (123).

In the following we shall assume n>4. Theorem 15 shows, that there is no joint profit maximization equilibrium point for n>4. Therefore we must have $0 < w_1 < 1$. Define

$$(129) \qquad h = \frac{w_1}{1-w_1}$$

If one divides D by $w_1^{\,n-\bar{m}}(1-w_1)^{\bar{m}}$, one receives

$$(130) \qquad D' = \sum_{m=0}^{\bar{m}} \binom{n-1}{m} h^{\bar{m}-n} D(n,m) = 0$$

Obviously for $0 < w_1 < 1$ the expression D' vanishes, if and only if D vanishes. The condition D=0 is not only necessary but also sufficient for a strategy combination w with (122), (123) and (125) being an equilibrium point. This shows that we in order to find these equilibrium points we have to look for the solutions of the equation:

$$(131) \qquad \sum_{m=0}^{\bar{m}} \binom{n-1}{m} h^{\bar{m}-m} D(n,m)=0$$

It remains to be shown that for $n>4$ equation (131) has exactly one positive solution h. From this h the uniquely determined value of w_1 can be computed by

(132) $$w_1 = \frac{h}{1+h}$$

It can be seen from (88), (89) and (127) that the following is true

(133) $D(n,m) > 0$ for $m < \bar{m}$ and $n > 4$

(134) $D(n,\bar{m}) < 0$ for $n > 4$

In order to make use of this fact we rewrite (131) as follows

(135) $$\sum_{m=0}^{\bar{m}-1} \binom{n-1}{m} h^{\bar{m}-m} D(n,m) = -\binom{n-1}{\bar{m}} D(n,\bar{m})$$

For $n>4$ the left side of (135) is an increasing function of h which goes to infinity as h goes to infinity. The right side of (135) is a positive constant. This means that (135) has exactly one positive solution h. The proof of the theorem has shown that the following correlary is true:

<u>CORRELARY.</u> For $n>4$ the probability w_1 belonging to the uniquely determined equilibrium point $w=(w_1,\ldots,w_n)$ of $\bar{\Gamma}$ with the properties (122) and (123) can be computed by (132) where h is the unique positive solution of (135) and $\bar{m}$ is that integer which satiafies (127).

<u>THEOREM 17.</u> The participation decision brick $\bar{\Gamma}$ is a distinguished simultaneity game. For $n>1$ the distinguished equilibrium set $\bar{S}$ of $\bar{\Gamma}$ contains exactly one equilibrium point. For $n=1,\ldots,4$ the distinguished equilibrium set $\bar{S}$ contains the joint profit maximization equilibrium point where every player always chooses to participate. For $n\geq 5$ the equilibrium point $s \in \bar{S}$ is a mixed strategy equilibrium point where each player chooses to

participate with the same probability w_1 with $0 < w_1 < 1$. This probability can be computed by (132), where h is the unique positive solution of (135).

PROOF. Obviously in the trivial case n=1 the joint profit maximization equilibrium point is in $\bar{S}$. Apart from this the theorem is an immediate consequence of lemma 20, theorem 16 and the correlary of theorem 16.

4.10 THE SOLUTION OF THE MODEL. In section 2.10 we have seen that a perfect equilibrium set is fully determined by the equilibrium sets induced on the bricks of the game. In the preceding sections the L-bricks of Γ have been constructed and their solutions have been determined (theorems 11,12,13 and 17). Since all the L-bricks are distinguished simultaneity games, the game Γ is in the region of the distinguished solution function. The solution of Γ can be characterized as follows:

THEOREM 18. The distinguished solution of Γ is the set S of all strategy combinations s for Γ with the property that the strategy combinations induced by s on the supply decision subgames Γ_y, on the truncated cartel bargaining subgames $\bar{\Gamma}_z$ and on the participation decision brick $\bar{\Gamma}$ are in the distinguished equilibrium sets of these games.

PROOF. Obviously S is a brick producing set. S satisfies the conditions 1) and 2) in theorem 3. Therefore S is a perfect equilibrium set. In view of the subgame consistency and the truncation consistency of the distinguished solution function, it is clear that S is the solution of Γ.

4.11 THE PARTICIPATION PROBABILITY AS A FUNCTION OF THE NUMBER OF PLAYERS. For $n > 1$ the solution prescribes a uniquely determined probability of choosing $z_i=1$. We call this probability w_1 the participation probability. According to theorem 16 for n=2,3,4 the participation probability is equal to 1.

For $n > 4$ the participation probability w_1 can be computed as described in the corellary of theorem 16. In the following the participation probability will be denoted by $w_1(n)$ in order to indicate its dependence on the number of players. Similarly the symbol $h(n)$ will be used for the uniquely determined positive solution of (135). Table 2 in subsection 5.1 shows the values of $w_1(n)$ for $n=2,\dots,15$. It is clear from this table that $w_1(n)$ is not monotonically decreasing. Nevertheless within the range of the table $w_1(n)$ has a tendency to decrease, since for $n=4,\dots,13$ the difference $w_1(n+2)-w_1(n)$ is always negative, even though $w_1(n)$ is greater than $w_1(n-1)$ for odd values of n with $n > 5$. In the following we shall prove that $w_1(n)$ is always below a certain upper bound which goes to zero as n goes to infinity.

THEOREM 19. For $n=5,6,\dots$ let $h(n)$ be the uniquely determined positive root of equation (135) and let $w_1(n)$ be the participation probability w_1 computed from $h=h(n)$ by (132). Define

$$(136)\qquad b(n) = -\frac{(n-\bar{m})D(n,\bar{m})}{\bar{m}D(n,\bar{m}-1)} \quad \text{for } n=5,6,7,\dots$$

where $\bar{m}$ is the integer determined by (127). We have

$$(137)\qquad b(n)=\begin{cases}\dfrac{(n+5)(n-1)}{(n-4)(n+1)^2} & \text{for } n=5,7,9,\dots\\[2ex] \dfrac{(n+4)n}{2(n-4)(n+1)^2} & \text{for } n=6,8,10,\dots\end{cases}$$

For every $n=5,6,7,\dots$ the following inequalities hold:

$$(138)\qquad h(n) \le b(n)$$

$$(139)\qquad w_1(n) \le \frac{b(n)}{1+b(n)}$$

$$(140)\qquad b(n+2) < b(n)$$

Moreover we have

(141) $\lim_{n\to\infty} w_1(n) = \lim_{n\to\infty} h(n) = \lim_{n\to\infty} b(n) = 0$

PROOF. h(n) satisfies the inequality

(142) $\binom{n-1}{m-1} D(n,\bar{m}-1)h(n) \leq -\binom{n-1}{\bar{m}} D(n,\bar{m})$

for n=5,6,... . This is a consequence of (133) and (135). Inequality (142) together with (136) shows that (138) holds for n=5,6,... . In order to prove (137), we evaluate the expression on the right side of (136) with the help of (86).

(143) $(n-\bar{m})D(n,\bar{m}) = \frac{n-\bar{m}}{(n+1)^2} - \frac{1}{4(\bar{m}+1)}$

(144) $\bar{m}D(n,\bar{m}-1) = \frac{\bar{m}}{4(\bar{m}+1)^2} - \frac{1}{4(n-\bar{m}+1)}$

For n=5,7,9,... the integer $\bar{m}$ is equal to (n-3)/2. In this case equations (143) and (144) yield

(145) $(n-\bar{m})D(n,\bar{m}) = \frac{n+3}{2(n+1)^2} - \frac{1}{2(n+1)}$

(146) $\bar{m}D(n,\bar{m}-1) = \frac{n-3}{2(n-1)^2} - \frac{1}{2(n+5)}$

(147) $(n-\bar{m})D(n,\bar{m}) = - \frac{4}{2(n+1)^2(n-1)}$

(148) $\bar{m}D(n,\bar{m}-1) = \frac{4n-16}{2(n-1)^2(n+5)}$

(149) $b(n) = \frac{(n+5)(n-1)}{(n-4)(n+1)^2}$ for n=5,7,...

Now assume n=6,8,10,... . Here $\bar{m}$ is equal to (n-2)/2 and (143) and (144) can be evaluated as follows

(150) $(n-\bar{m})D(n,\bar{m}) = \frac{n+2}{2(n+1)^2} - \frac{1}{2n}$

(151) $\bar{m}D(n,\bar{m}-1) = \frac{n-2}{2n^2} - \frac{1}{2(n+4)}$

(152) $(n-\bar{m})D(n,\bar{m}) = - \frac{1}{2n(n+1)^2}$

(153) $$\bar{m}D(n,\bar{m}) = \frac{2n-8}{2n^2(n+4)}$$

(154) $$b(n) = \frac{(n+4)n}{2(n-4)(n+1)^2} \quad \text{for } n=6,8,\ldots$$

In view of the fact that h/(1+h) is a monotonically increasing function of h, it is clear that (139) is a consequence of (138). Since both for n=5,7,... and n=6,8,... the nominator of b(n) is quadratic in n whereas the denominator is cubic in n, one can see immediately, that (141) holds. In order to prove (140) we look at the derivatives of the logarithms of the expressions on the right side of (137). In this way one can see that (140) holds for n=5,7,... if we have

(155) $$\frac{1}{n+5} + \frac{1}{n-1} - \frac{1}{n-4} - \frac{2}{n+1} < 0.$$

Since n-4 is smaller than n-1 and n+1 is smaller than n+5, the right side of (155) is negative. Similarly for n=6,8,... inequality (140) is implied by

(156) $$\frac{1}{n+4} + \frac{1}{n} - \frac{1}{n-4} - \frac{2}{n+1} < 0$$

<u>4.12 THE CARTEL PROBABILITY.</u> On the basis of the assumption that the solution of the model correctly describes the behavior of the oligopolists, it is interesting to ask the question, how often it will occur that the oligopolists use the cooperative possibilities of the cartel bargaining stage in order to collude in a significant way. As an answer to this not yet precise question we shall define a "cartel probability".

As we know from 4.4, in the cartel bargaining stage the character of the behavior prescribed by the equilibrium points in the solution of the model crucially depends on the number k of non-participators. For $k \geq (n-1)/2$ an equilibrium point in the solution may lead to cartel agreements, but these cartel agreements are inessential. The equilibrium payoffs in the cartel bargaining subgame are those, which would be obtained, if no cartels were possible.

Contrary to this for $k<(n-1)/2$, the cooperative possibilities of the cartel bargaining stage are used at the equilibrium points in the solution. The participators receive greater gross profits than they could get without cartel agreements. Moreover, since their joint gross profit is equal to the upper bound on the right side of (48), one can say that they make the best possible use of their opportunity to form cartels.

In view of what has been said, it is convenient to intorudce the following way of speaking. We say that a cartel arrangement is reached by an equilibrium point s of Γ in a cartel bargaining subgame Γ_Z, if the equilibrium point s_Z induced by s on Γ_Z has the property that for each of the participators the equilibrium payoff at s_Z is greater than the payoff $1/(1+n)^2$, which is achieved at the unrestricted Cournot equilibrium. If a cartel arrangement is reached by s in Γ_Z, then the players in Z are called insiders and the players in N-Z are called outsiders with respect to the cartel arrangement.

Obviously for all equilibrium points s in the solution a cartel arrangement is reached by s in Γ_Z, if and only if the number k of non-participators is smaller than $(n-1)/2$. The probability that k will be smaller than $(n-1)/2$ if an equilibrium point s in the solution is played is the same one for all equilibrium points in the solution. This is trivially true for n=1, where the case $k<(n-1)/2$ cannot occur; for n=2,3,... every equilibrium point in $L(\Gamma)$ prescribes the same behavior in the participation decision stage, namely the selection of $z_i=1$ with probability $w_1(n)$; the probability that k will be smaller than $(n-1)/2$ is uniquely determined by $w_1(n)$. This suggests the following definition: The cartel probability is the probability that a cartel arrangement will be reached if an equilibrium point in $L(\Gamma)$ is played. The symbol $W(n)$ will be used for this cartel probability.

As we have seen above, $W(n)$ is the probability that k will be smaller than $(n-1)/2$. Obviously we have

(157) $W(1) = 0$

(158) $W(n) = 1$ for $n = 2,3,4$.

For $n=5,6,\ldots$ the cartel probability can be computed as follows:

(159) $$W(n) = \sum_{k=0}^{\bar{m}} W(n,k)$$

where $\bar{m}$ is the uniquely determined integer satisfying (127) and where

(160) $$W(n,k) = \binom{n}{k}\left[1-w_1(n)\right]^k \left[w_1(n)\right]^{n-k}$$

is the probability that there will be exactly k non-participators if the players choose to participate with probability $w_1(n)$.

<u>4.14 THE CARTEL PROBABILITY AS A FUNCTION OF THE NUMBER OF PLAYERS.</u> Table 2 in subsection 5.1 shows the values of $W(n)$ for $2,\ldots,9$. It is clear from this table that $W(n)$ does not monotonically decrease as a function of n. A weaker statement about $W(n)$ will be proved in the following. It will be shown, that $W(n)$ is below a certain upper bound which goes to zero as n goes to infinity. With the help of this upper bound it can be seen, that $W(n)$ is very small outside the table. A further property of $D(n,m)$ is needed, in order to derive these results.

<u>LEMMA 21.</u> $D(n,m)$ has the following property

(161) $D(n,m+1) < D(n,m)$

for $n=6,7,\ldots$ and $m=0,\ldots,\bar{m}-2$ where $\bar{m}$ is the uniquely determined integer satisfying (127).

PROOF. In view of (86) for $m=0,\dots,\bar{m}-1$ we have

$$(162)\qquad D(n,m) = \frac{1}{4(m+2)^2} - \frac{1}{4(n-m)(m+1)}$$

In order to prove the lemma it is sufficient to show that the following is true:

$$(163)\qquad \frac{\partial D(n,m)}{\partial m} < 0 \quad \text{for } 0 \leq m \leq \bar{m}-1$$

(162) yields

$$(164)\qquad \frac{\partial D(n,m)}{\partial m} = -\frac{1}{2(m+2)^3} + \frac{n-2m-1}{4(n-m)^2(m+1)^2}$$

In order to find an upper bound for the right side of (164) we make use of the fact that m+2 is notgreater than 2(m+1) and that n-2m-1 is smaller than n-m:

$$(165)\qquad \frac{\partial D(n,m)}{\partial m} \leq -\frac{1}{2(m+2)^3} + \frac{1}{2(n-m)(m+1)(m+2)}$$

In view of (162) this is equivalent to

$$(166)\qquad \frac{\partial D(n,m)}{\partial m} \leq -\frac{2}{m+2} D(n,m)$$

(88) shows that $D(n,m)$ is positive for $m=1,\dots,\bar{m}-1$

LEMMA 22. For $n=5,6,\dots$ the cartel probability $W(n)$ has the following property:

$$(167)\qquad W(n) \leq \left[1+\frac{\bar{m}b(n)}{n-\bar{m}+1}\right] W(n,\bar{m})$$

where $\bar{m}$ is the integer satisfying (127) and $W(n,k)$ is defined by (160).

PROOF. As we have seen in the proof of theorem 19 expression D in (128) is equal to zero for $n=5,6,\dots$, since there $w_1(n)$ is positive and smaller than 1, which has the consequence that both $z_i=0$ and $z_i=1$ are best replies to w in $\bar{\Gamma}$. If one makes use of

$$(168)\qquad \binom{n-1}{m} = \binom{n}{m}\frac{n-m+1}{n}$$

the equation $D = 0$ can be written as follows

$$(169)\qquad \sum_{m=0}^{\bar{m}} W(n,m)\frac{n-m+1}{n}D(n,m) = 0$$

It follows by (161) and (88) that for $m=1,\dots,\bar{m}-1$ we have

$$(170) \qquad D(n,m) \geq D(n,\bar{m}-1) > 0$$

Define

$$(171) \qquad W' = W(n) - W(n,\bar{m})$$

In view of (170) equation (169) implies the following inequality

$$(172) \qquad W'\frac{n-\bar{m}+1}{n} D(n,\bar{m}-1) \leq -W(n,\bar{m})\frac{n-\bar{m}}{n} D(n,m)$$

With the help of (136) it can be seen that this is equivalent to

$$(173) \qquad W' \leq \frac{\bar{m}}{n-\bar{m}+1} b(n)W(n,\bar{m})$$

(167) is an immediate consequence of (173) and (171).

<u>LEMMA 23.</u> For $n=5,6,\dots$ the probability $W(n,\bar{m})$ has the following property:

$$(174) \qquad W(n,\bar{m}) \leq \binom{n}{\bar{m}}\frac{[b(n)]^{n-\bar{m}}}{[1+b(n)]^{n}}$$

where $\bar{m}$ is the integer satisfying (127).

<u>PROOF.</u> For the sake of shortness we shall sometimes write w_1 and b instead of $w_1(n)$ and $b(n)$ resp. Obviously we have

$$(175) \qquad \frac{b^{n-\bar{m}}}{(1+b)^{n}} = \left(\frac{b}{1+b}\right)^{n-\bar{m}} \left(1-\frac{b}{1+b}\right)^{\bar{m}}$$

Therefore it is sufficient to show that the following is true:

$$(176) \qquad w_1^{n-\bar{m}}(1-w_1)^{\bar{m}} < \frac{b^{n-\bar{m}}}{(1+b)^{n}}\left(1-\frac{b}{1+b}\right)^{\bar{m}}$$

In order to prove this we show that the derivative

$$(177)\quad \frac{\partial w_1^{n-\bar{m}}(1-w_1)^{\bar{m}}}{\partial w_1} = (n-\bar{m} - nw_1)w_1^{n-\bar{m}-1}(1-w_1)^{\bar{m}-1}$$

is non-negative in the interval $0 \leq w_1 \leq b/(1+b)$. This is true if we have

$$(178)\quad \frac{b(n)}{1+b(n)} \leq \frac{n-\bar{m}}{n}$$

Condition (178) is equivalent to

$$(179)\quad b(n) \leq \frac{n-\bar{m}}{\bar{m}}$$

With the help of (137) we can compute

$$(180)\quad b(5) = 1.111$$

$$(181)\quad b(6) = .306$$

$$(182)\quad b(7) = .375$$

(180) shows that (179) holds for n=5. Since $(n-\bar{m})/n$ is always greater than 1 and both b(6) and b(7) are already smaller than 1, it can be seen with the help of (140) that (179) is satisfied for n=5,6,7,... .

THEOREM 20. For n=5,6,... define

$$(183)\quad V(n) = \left[1+\frac{\bar{m}b(n)}{n-\bar{m}+1}\right]\binom{n}{\bar{m}}\frac{[b(n)]^{n-\bar{m}}}{[1+b(n)]^{n}}$$

where $\bar{m}$ is the integer determined by (127) and b(n) is defined as in (137). The cartel probability W(n) satisfies the following inequality

$$(184)\quad W(n) \leq V(n) \quad \text{for } n=5,6,...$$

Moreover we have

(185) $V(n+2) < V(n)$ for $n=5,6,\ldots$

and

(186) $\lim_{n\to\infty} W(n) = \lim_{n\to\infty} V(n) = 0$

PROOF. (184) follows by lemma 22 and lemma 23. Since $h/(1+h)$ is a monotonically increasing function of h it follows by (140) that we have

(187) $$\frac{b(n+2)}{1+b(n+2)} < \frac{b(n)}{1+b(n)}$$

for $n=5,6,\ldots$. In the same way as (176) has been proved in the proof of lemma 24, one can see that (177) implies an inequality analoguous to (177), where w_1 corresponds to $b(n+2)/(1+b(n+2))$ and b corresponds to b(n). If one makes use of the relationship (175) this inequality can be written as follows:

(188) $$\frac{[b(n+2)]^{n-\bar{m}}}{[1+b(n+2)]^{\bar{m}}} < \frac{[b(n)]^{n-\bar{m}}}{[1+b(n)]^{\bar{m}}}$$

This inequality will be used in order to prove (184). In order to do this we also have to use the following equation, which is a consequence of (137):

(189) $$\frac{\bar{m}b(n)}{n-\bar{m}+1} = \begin{cases} \dfrac{(n+3)(n-1)}{(n-4)(n+1)^2} & \text{for } n=5,7,\ldots \\[2ex] -\dfrac{(n+2)n}{2(n-4)(n+1)^2} & \text{for } n=6,8,\ldots \end{cases}$$

It can be seen easily that the derivatives of the logarithms of the expressions on the right side of (189) with respect to n are negative ;therefore the first factor in (183) is decreased, if n is increased by 2. This together with (188) shows that the following is true:

$$(190)\qquad V(n+2) \le \frac{\binom{n+2}{\bar{m}+1}}{\binom{n}{\bar{m}}} \frac{b(n+2)}{(1+b(n+2))^2} V(n)$$

Here it is important to notice that $\bar{m}$ is always increased by 1 if n is increased by 2. Inequality (190) is equivalent to

$$(191)\qquad V(n+2) \le \frac{(n+2)(n+1)}{(\bar{m}+1)(n-\bar{m}+1)} \frac{b(n+2)}{[1+b(n+2)]^2} V(n)$$

Since $\bar{m}+1$ is not smaller than $(n-1)/2$ and $(n-\bar{m}+1)$ is not smaller than $(n+4)/2$ we have

$$(192)\qquad \frac{(n+2)(n+1)}{(\bar{m}+1)(n-\bar{m}+1)} \le 4\frac{(n+1)(n+2)}{(n-1)(n+4)}$$

(192) is equivalent to

$$(193)\qquad \frac{(n+2)(n+1)}{(\bar{m}+1)(n-\bar{m}+1)} \le 4 + \frac{28}{(n-1)(n+4)}$$

Obviously the expression on the right side of (193) is a monotonically decreasing function of n. For n=5 this expression assumes the value 4.77778.

This shows that the following is true for n=5,6,...

$$(194)\qquad \frac{(n+2)(n+1)}{(\bar{m}+1)(n-\bar{m}+1)} \le 4.77778$$

Since the derivative

$$(195)\qquad \frac{d}{db}\left(-\frac{b}{(1+b)^2}\right) = \frac{1-b^2}{(1+b)^4}$$

is positive in the interval $0 < b < 1$, we can conclude from

$$(196)\qquad \frac{b(7)}{(1+b(7))^2} = .19835$$

and

$$(197)\qquad \frac{b(8)}{(1+b(8))^2} = .11238$$

that in view of (140) we have

(198) $$\frac{b(n+2)}{[1+b(n+2)]^2} \le .19835$$

for n=5,6,... . This together with (195) and (197) yields

(199) $$V(n+2) \le .94768\ V(n) \quad \text{for } n=5,6,...$$

(186) is an immediate consequence of (199).

REMARK. Table 2 in subsection 5.1 contains the statement that for n=10,11,... the cartel probability W(n) is smaller than .0001. for n=10,...,15 the computation of W(n) from $w_1(n)$ shows that this is true. V(15) and V(16) are both smaller than .0000001. Therefore it follows by (184) and (185) that for n=15,16,... the cartel probability is below .0000001.

5. INTERPRETATION OF THE RESULTS.

It is the purpose of this section to discuss the intuitive significance of the results obtained in section 4 and to draw some heuristic conclusions with respect to possible generalizations to more complicated models. An informal description of the solution of the model is given, mainly for the benefit of those readers who are not interested in technical details.

5.1 WHAT HAPPENS AT THE SOLUTION OF THE MODEL. Technically the solution of the model is a set of equilibrium points. Mainly in the cartel bargaining stage differences between the equilibrium points in the solution arise, but these differences are unimportant, since all the equilibrium points in the solution lead to the same equilibrium payoffs, not only in the game as whole, but also in every subgame.

In order to have an easy way of speaking about the behavior at the solution, a distinction between a cartel agreement and a cartel arrangement has been introduced in 4.12. Since the formation of cartels is costless, the solution does not exclude that economically ineffective cartel agreements are reached, where nothing can be gained by a cartel. Thus for example it may happen, that the participators agree to limit their supplies by very high quotas which do not restrict them in any significant way. In such cases we say that the cartel agreements do not constitute a cartel arrangement. We speak of a cartel arrangement, if the participators successfully use the possibilities of cartel formation in order to get a higher profit, than they would get, if cartels were not possible.

Let us first look at the trivial case $n=1$ which has the pecularity that the solution permits any behavior at the participation decision stage. This is due to the fact that here the participation decision stage is strategically irrelevant. As a participator at the cartel bargaining stage the monopolist should not fix a quota below his monopoly supply 1/2, but apart from that the solution permits anything. In the supply decision stage the monopolist supplies the quantity 1/2. His payoff is the monopoly gross profit 1/4. The monopolist never reaches a cartel arrangement, since he does not need any cartel agreements, in order to achieve his monopoly profit.

For n=2,3,... every equilibrium point in the solution prescribes the same behavior in the participation decision stage: each of the players decides to participate with the same probability $w_1(n)$. For n=2,...,15 this participation probability $w_1(n)$ is tabulated in table 2. The participation probability $w_1(n)$ goes to 0 as n goes to infinity.

In the cartel bargaining stage the behavior at the solution crucially depends on the number k of non-participants. Every equilibrium point in the solution has the property that a cartel arrangement is reached if and only if the number k of non-participators is smaller than (n-1)/2.

In the case $k \geq (n-1)/2$ it may simply happen that no cartel agreement is reached but the solution also permits the possibility that economically ineffective cartel agreements are reached. In the case $k < (n-1)/2$ where a cartel arrangement occurs, the simplest way in which this may happen is the formation of one cartel where all the participators are members and have equal quotas, such that the quotas of all participators sum up to the joint quota of 1/2. The joint quota of 1/2 maximizes the joint equilibrium payoff of the cartel in the supply decision subgame after the quota agreement. The solution also permits the possibility that the participators achieve the same quota system by splitting into several coalitions with seperate cartel agreements. At least for some n this is possible.

In the case $k < (n-1)/2$ where a cartel arrangement is reached, the non-participators are also called outsiders and the participators are also called insiders. For various n and k the equilibrium payoffs in the cartel bargaining subgame at the solution are given in table 2 under the headings "gross profit of an insider" and "gross profit of an outsider".

number of players	number of out-siders	gross profit of an in-sider	gross profit of an out-sider	gross profit of a supplier in the unrestricted Cournot equilibrium	partici-pation probabi-lity	probabi-lity of a cartel arrange-ment with k out-siders	cartel proba-bility	expected gross profit of an oligo-polist
n	k	$\frac{1}{4(n-k)(k+1)}$	$\frac{1}{4(k+1)^2}$	$\frac{1}{(n+1)^2}$	$w_1(n)$	$W(n,k)$	$W(n)$	
2	0	.1250		.1111	1.0000	1.0000	1.0000	.1250
3	0	.0833		.0625	1.0000	1.0000	1.0000	.0833
4	0	.0625		.0400	1.0000	1.0000	1.0000	.0625
5	0	.0500		.0278	.5263	.0404	.2221	.0304
	1	.0312	.0625			.1817		
6	0	.0417		.0204	.1857	.0000	.0130	.0205
	1	.0250	.0625			.0011		
	2	.0208	.0278			.0118		
7	0	.0357		.0156	.2380	.0000	.0103	.0157
	1	.0208	.0625			.0010		
	2	.0167	.0278			.0093		
8	0	.0312		.0123	.1067	.0000	.0006	.0124
	1	.0179	.0625			.0000		
	2	.0139	.0278			.0000		
	3	.0125	.0156			.0006		
9	0	.0278		.0100	.1587	.0000	.0009	.0100
	1	.0156	.0625			.0000		
	2	.0119	.0278			.0001		
	3	.0104	.0156			.0008		

For n=10,11,... the cartel probability W(n) is smaller than .0001.

number of suppliers	partici-pation probability
n	$w_1(n)$
10	.0755
11	.1203
12	.0585
13	.0971
14	.0476
15	.0822

Table 2: The solution up to n=15

For a cartel bargaining subgame with $k \geq (n-1)/2$ the equilibrium payoffs at the solution are those of the unrestricted Cournot equilibrium . For n=2,...,9 these gross profits are also tabulated in table 2.

The solution exhibits a surprising change of behavior at n=5. For n=2, n=2 and n=4 each of the oligipolists decides to participate in the cartel bargaining and the outcome of the cartel bargaining is the maximization of the joint profit of all players . For n>4 the joint profit maximization by all players fails to occur at the solution; the mixed strategy behavior in the participation decision stage only occasionally results in a cartel bargaining subgame, where all players are participators. The probability W(n,0) for this event is given in table 2 under the heading "probability of a cartel arrangement with k outsiders". Already for n=5 this probability is only .0404 and for n>5 it is always smaller than .0001.

<u>5.2 WHY 4 ARE FEW AND 6 ARE MANY.</u> The probability that a cartel arrangement is reached, if an equilibrium point in the solution is played, is called cartel probability. This cartel probability W(n) is tabulated in table 2. For n=2,3,4 the cartel probability is equal to 1. One may say that with respect to the solution of the model up to n=4 the number of oligopolists is small. For n>5 the cartel probability is approximately 1% or smaller, which means that an outside observer will only rarely observe a cartel arrangement. Economically for n>5 the solution is not very different from the behavior which could be expected, if no cartel agreements were possible. This can be seen, if one compares the equilibrium payoff at the solution for the whole game with the equilibrium profit for the unrestricted Cournot equilibrium. Both profits are tabulated in table 2 under the headings "expected gross profit of an oligipolist" and "gross profit of a supplier at the unrestricted Cournot equilibrium", For n>5 the expected gross profit of an oligopolist at the solution is only slightly greater than the gross profit of a supplier at the unrestricted Cournot equilibrium.

The case n=5 may be considered an intermediate case, since here the cartel probability of approximately 22% is still quite substantial. Note that for n=5 most of the cartel arrangements are cartel arrangements with 4 insiders and 1 outsider.

Why is n=5 the dividing line between the small group and the large group? The main reason for this can be explained with the help of a heuristic argument. Assume that n is at least 3 and suppose that player j expects that each of the other players will decide to participate and that the joint profit of all players will be maximized and split evenly, if he decides to participate too. If he does not participate, he expects the others to form a cartel with a joint quota of 1/2 in order to maximize the joint equilibrium payoff of the cartel in the supply decision stage. In the case of the joint profit maximization by all players his share of the joint gross profit of 1/4 is equal to 1/4n. If he does not participate, he becomes an outsider with respect to a cartel whose total supply is 1/2. His optimal supply will be 1/4, the price will be 1/4 and his gross profit will be 1/16. The basic fact is, that up to n=4 the joint gross profit share of 1/4n is not smaller than the outsider gross profit of 1/16, whereas for n>4 the outsider gross profit is greater than the joint gross profit share. This destroys the possibility of a joint profit maximization equilibrium for n>4.

5.3 THE STRATEGIC SITUATION IN THE PARTICIPATION DECISION STAGE. In order to understand the strategic situation in the participation decision stage, one must look at the game which has been introduced in 4.5 as the "participation decision brick". The participation decision brick results from the model, if one substitutes every cartel bargaining subgame by the payoff vector which is obtained in this subgame if the players behave in a way which is compatible with the solution.

In 4.7 the pure strategy equilibrium points of the participation decision brick have been explored. For n=1,...,4 the participation decision brick has a "joint profit maximization equilibrium point", where every player always decides to participate and a maximal joint profit for all players is reached. This pure strategy equilibrium point is not available for n>4. There the only pure strategy equilibrium point which treats the players symmetrically is the "unrestricted Cournot equilibrium point", where every player decides not to participate.

For $n \geq 4$ the participation decision brick has "partial cartel equilibrium points" where for even n exactly (n-2)/2 players and for odd n exactly (n-3)/2 players are non-participators. Here the non-participators have higher payoffs than the participators. The players are treated in an asymmetrical way. Therefore the symmetry requirement underlying the solution concept of this paper excludes the partial cartel equilibrium points as possible candidates for a solution of the participation decision brick. Apart from the lack of symmetry the partial cartel equilibrium points are quite attractive. Thus for example in the case n=5 and k=1 an insider receives .0312 and the outsider receives .0625, whereas at the solution every player receives .0304 only (see table 2.). Nevertheless it is not implausible to expect that the players will fail to coordinate their expectations at a partial cartel equilibrium point, since nobody has more reason than anybody else to be satisfied with the less profitable role of an insider.

<u>5.4 POSSIBLE GENERALIZATIONS.</u> One may ask the question how much of the analysis depends on the linearity assumptions about cost and demand. Only a detailed investigation can show what happens if these assumptions are relaxed, but it is a plausible conjecture that apart from some special cases one will always find a more or less sharp dividing line between few and many beyond which the players fail to exhibit the typical small group behavior. Whether the dividing line will be at n=5 or somewhere else, will depend on the cost and demand functions.

The model is symmetric with respect to the players. It would be desirable to develop a theory for a more general model which admits some asymmetries like different cost functions for different players. For this purpose one would need a more general solution concept.

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