Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

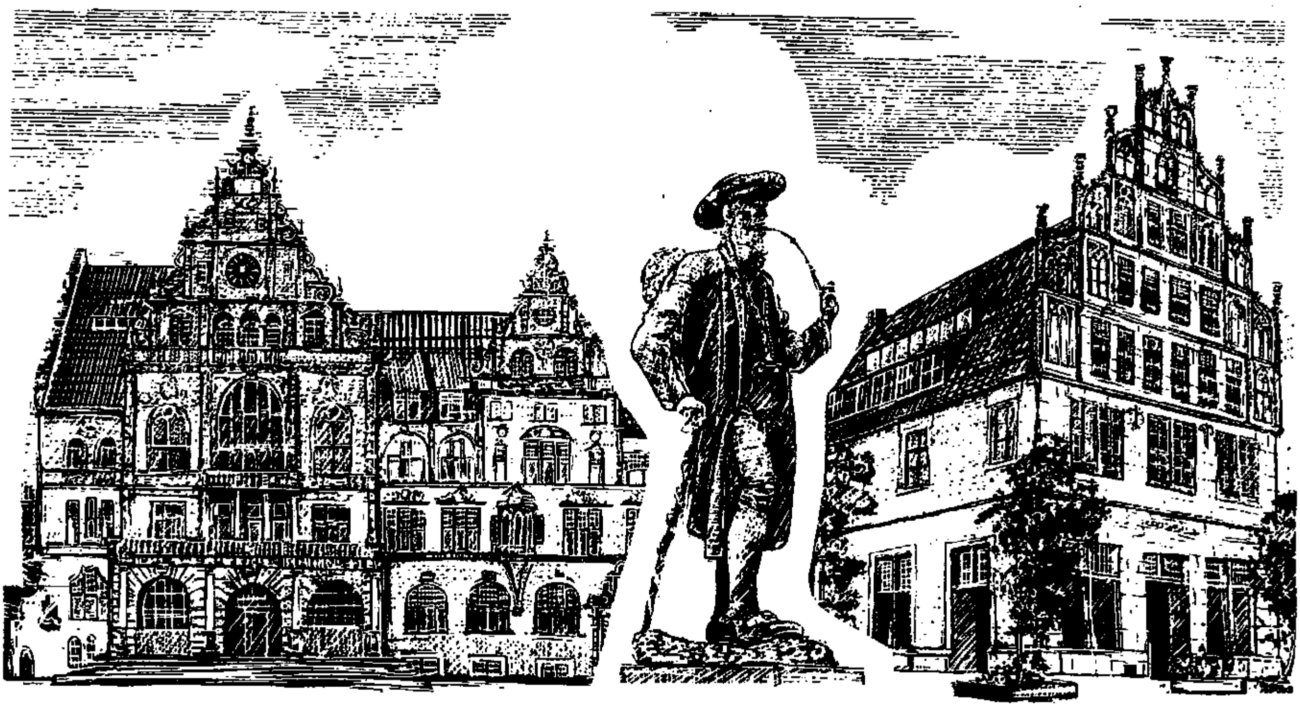
Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 48

Siegfried Brune

Computation of the Nucleolus for
Superadditive 4-Person-Games

September 1976



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This research has been done as a part of a M.S. thesis in mathematics at the University of Bielefeld and under the supervision of Prof. Dr. R. Selten (Feb. 1976).

Computation of the Nucleolus for Superadditive 4-Person-Games

1. Introduction

This paper is a supplement to the author's former paper (S. Brune 1976). It serves the purpose to provide a method to compute the nucleolus for 4-person-games by hand, without using a computer. This may be useful for experimentors who want to compare experimental results with theoretical solution concepts. Furthermore, the paper specifies certain regions in the game space, on each of which the nucleolus is a linear function. The basic ideas and theorems are given in the above mention

Essentially these ideas are the following: let $N = \{1,...,n\}$ be a set of n players. Then any maximal coalition array $B: = [B_o,B_1,...,B_q]$ - these arrays are called $\mathcal{B}$-finest by J.H. Grotte [J.H. Grotte 1972] - yields a system (P_B) of n linearly independent equations, using the equality of excesses of coalitions in the same array parts B_i. Further, we get a system of inequalities determining a game region D_B, on which the nucleolus is a linear function. For any game $v \in D_B$ the solution $N_B(v)$ of the above system (P_B) is equal to the nucleolus $N(v)$ of the game v. The system (P_B) and the collection of inequalities corresponding to a given maximal array B essentially depend on those array parts which possess more than one coalition, and on the array part B_o. These array parts we called "critical array parts". By forming classes of those maximal arrays which possess the same critical array parts we introduce an equivalence relation on the set of all maximal coalition arrays. The notion of "normalized" arrays enables us to find suitable representatives for each of the equivalence classes. Roughly speaking, a normalized array is an array whose first array parts contain more than one coalition. In section 4 we only write down the critical array parts of the normalized representatives of the equivalence classes. In order to get the whole arrays of any equivalence class we have only to add one-coalition array parts in any ordering.

By the above mentioned introduction of an equivalence relation we are able to extend the game regions D_B. The new regions $D_{[B]}$ are convex and specified in section 3. On each of them the nucleolus is a linear function. Note that in general these game regions $D_{[B]}$ are not the greatest possible ones but they are larger than E.Kohlberg's regions [E.Kohlberg 1971]. For further informations, especially about the greatest regions on which the nucleolus is a linear function, see[S.Brune 1976].

2. A search program for finding the "fitting" game region for a given 4-person-game

2.1 Description of the search program

The game regions $D_{[B]}$ are only evaluated up to permutations of the four players. Therefore, any given game has to be taken into a certain "form" by a suitable permutation of the players. Within this form some "form-invariant" permutations are allowed; in each case these special permutations will be described.

When we have taken the game v into the prescribed form v' we have to examine some "permutation-invariant" inequalities. If all inequalities are satsified, we look at the corresponding "block" (see section 3) in order to determine whether the game v' fits into one of the described game regions. Note that we have to examine all prescribed form-invariant permutations of the new game v'. If we do not find a suitable game region or if one of the above mentioned inequalities is not satisfied, then we have to proceed in the search program until we have found a fitting game region.

The described sequence of checking the inequalities of the different blocks is a subjective one. We think that this sequence is the most economical one. A simplification of the program seems only to be possible if we essentially enlarge the number of game regions to be considered. At the end of section 2.2 we give an example for the application of the search program.

We think that we have evaluated all necessary normalized maximal arrays which have to be considered for computing the nucleolus of any superadditive 4-person-game. In order to obtain a check on this we have described the critical array parts of all normalized arrays up to permutations of the four players. Thus, any maximal array to be considered must possess such more-coalition array parts which belong to one of the truncated arrays in section 4.

To compute the nucleolus for superadditive 4-person-games all normalized arrays $B = [B_o, B_1, \ldots, B_q]$ with $B_o = \emptyset$ have to be considered. Furthermore, we only need very few arrays B with $B_o \neq \emptyset$ (see the regions B6, B7, C8 and the corresponding arrays in section 4). The remaining normalized arrays of this kind are not necessary for superadditive games and will not be described in section 4.
To illuminate these cases we give the following examples.

i) $B_o = \{4\}$, $B_1 = \{124,3\}$, $B_2 = \{134,23\}$, $B_3 = \{123\}$, $B_4 = \{2\},\ldots$

If a game v is an element of the region D_B, then the first array part of the array C belonging to $(v,N(v))$ must contain the set $B_o \cup B_1 \cup B_2 \cup B_3$. Since we only consider superadditive games the third and the fourth player will get zero (note the excesses). Thus the array C can be derived from the following normalized array B':

$B'_o = \emptyset$, $B'_1 = \{124,3\}$, $B'_2 = \{134,23\}$, $B'_3 = \{123,4\}$, $B'_4 = \{2\},\ldots$

and we have $v \in D_{B'}$, too. Thus we do not need the array B.

ii) $B_o = \{4\}$, $B_1 = \{12,34\}$, $B_2 = \{134,234\}$, $B_3 = \{123\}$, $B_4 = \{2\},\ldots$

For any game $v \in D_B$ we generally have $e(\{12\},N(v)) = e(\{34\},N(v)) \geq 0$ because the fourth player will get zero. If $e(\{12\},N(v)) = 0$, then the first array part of the array C belonging to $(v,N(v))$ must contain the set $B_o \cup B_1 \cup B_2 \cup B_3$, and thus C can be derived from the following array B':

$B'_o = \emptyset$, $B'_1 = \{12,34\}$, $B'_2 = \{134,234\}$, $B'_3 = \{123,4\}$, $B'_4 = \{2\}$, ... Therefore without loss of generality we can assume $e(\{12\},N(v)) = e(\{34\},N(v)) > 0$ and thus $e(\{12\},N(v)) + e(\{34\},N(v)) > 0$. This implies $v(12)+ v(34) > v(N)$ and thus the game v is not superadditive. So we can omit the above array B.

2.2 The search program

We shall use the following notations:

1. $A := v(N)$;

   $a := v(123)$, $b: = v(124)$, $c: v(134)$,

   $d := v(234)$;

   $p := v(12)$, $q := v(13)$, $r := v(14)$,

   $s := v(23)$, $t := v(24)$, $u := v(34)$.

2. $M2 := \max \{v(12), v(13), v(14), v(23), v(24), v(34)\}$

   $M3 := \max \{v(123), v(124), v(134), v(234)\}$

   $M22 := \max \{v(12)+v(34), v(13)+v(24), v(14)+v(23)\}$

   If the term M3 takes the value of the coalition $v(ijk)$ with $i,j,k \in \{1,2,3,4\}$ then

   $\overline{M3} := \max \{\{v(123), v(124), v(134), v(234)\} \setminus \{v(ijk)\}\}$

   If the term M22 takes the value of the sum $v(ij)+v(k\ell)$ with $i,j,k,\ell \in \{1,2,3,4\}$ then

   $\overline{M22} := \max\{\{v(12)+v(34), v(13)+v(24), v(14)+v(23)\} \setminus \{v(ij)+v(k\ell)\}\}$

   $M4 := \max \{v(ijk)+v(ij\ell)+v(k\ell) : i,j,k,\ell \in \{1,2,3,4\}\}$

3. The demand "form: M3: → v(123)" means that, if necessary, the given game v has to be changed by a suitable permutation of the players such that for the new game v' the term M3 takes the value v'(123). The other notations of section 2.2.2 are to be treated analogously. If there is no suitable permutation we have to go further to the next form.

4. The demand "check" means that we have to examine whether the given game v resp. the new game v' satisfies all of the described inequalities. If only one inequality is not satisfied we have to go further to the next form.

5. The quadrupel $(ijk\ell)$ with $i,j,k,\ell \in \{1,2,3,4\}$ denotes the following permutation of the four players: $1 \to i$; $2 \to j$; $3 \to k$; $4 \to \ell$.

6. The question "$v \in B$?" means that we have to prove whether the given game v resp. the game v'(see 2.2.3) fits into one of the game regions of the block B. Notice that this has to be checked for all permutations prescribed under the question "$v \in B$?";these permutations must be applied to the given game v resp. to the new game v' according to 2.2.3.

7. The term "B2" for example denotes the second game region of the block B.

8. The capital letters beside the arrows of the flow-diagram describe those blocks which are to be examined

in the corresponding part of the diagram.

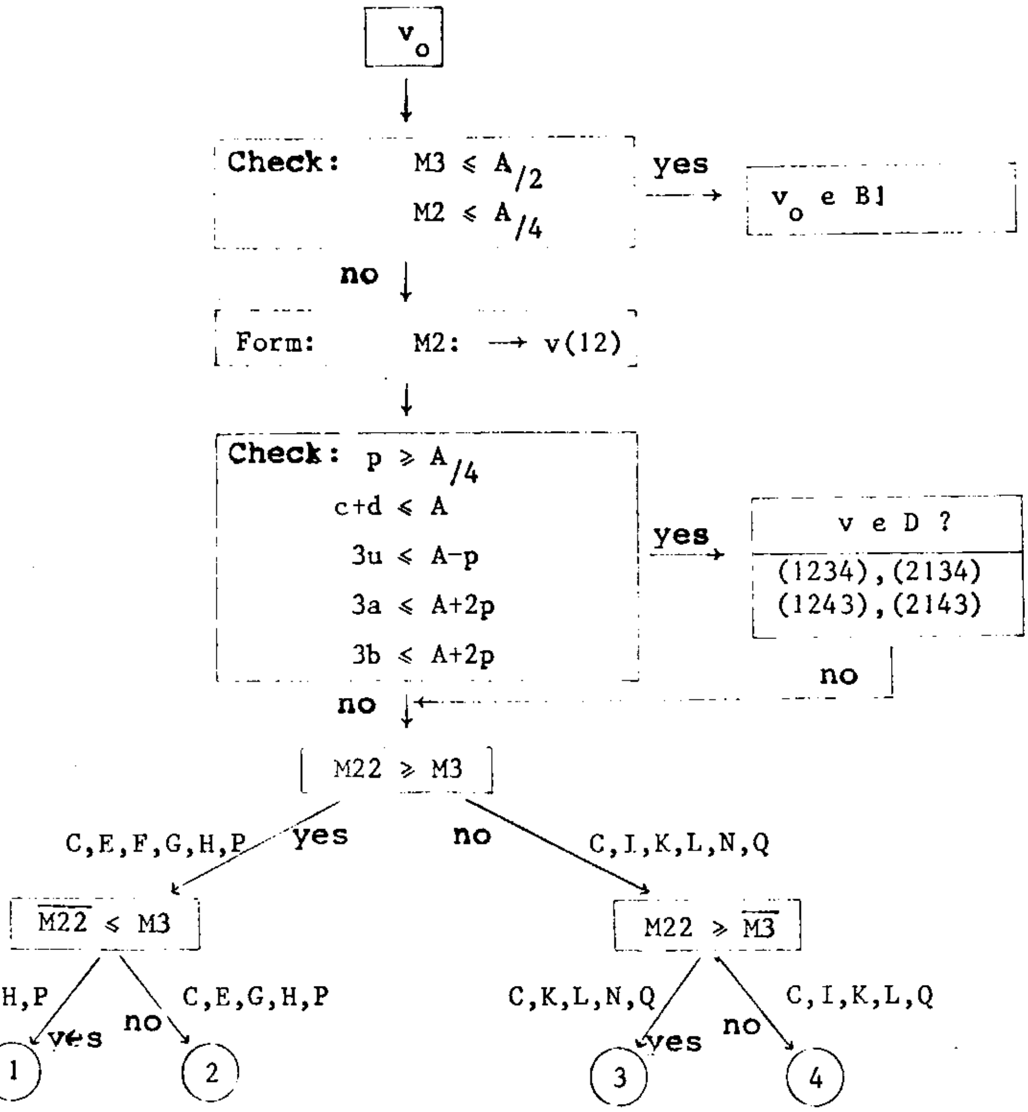
9. Connection points indicating where the search has to be continued are denoted by numbered circles.

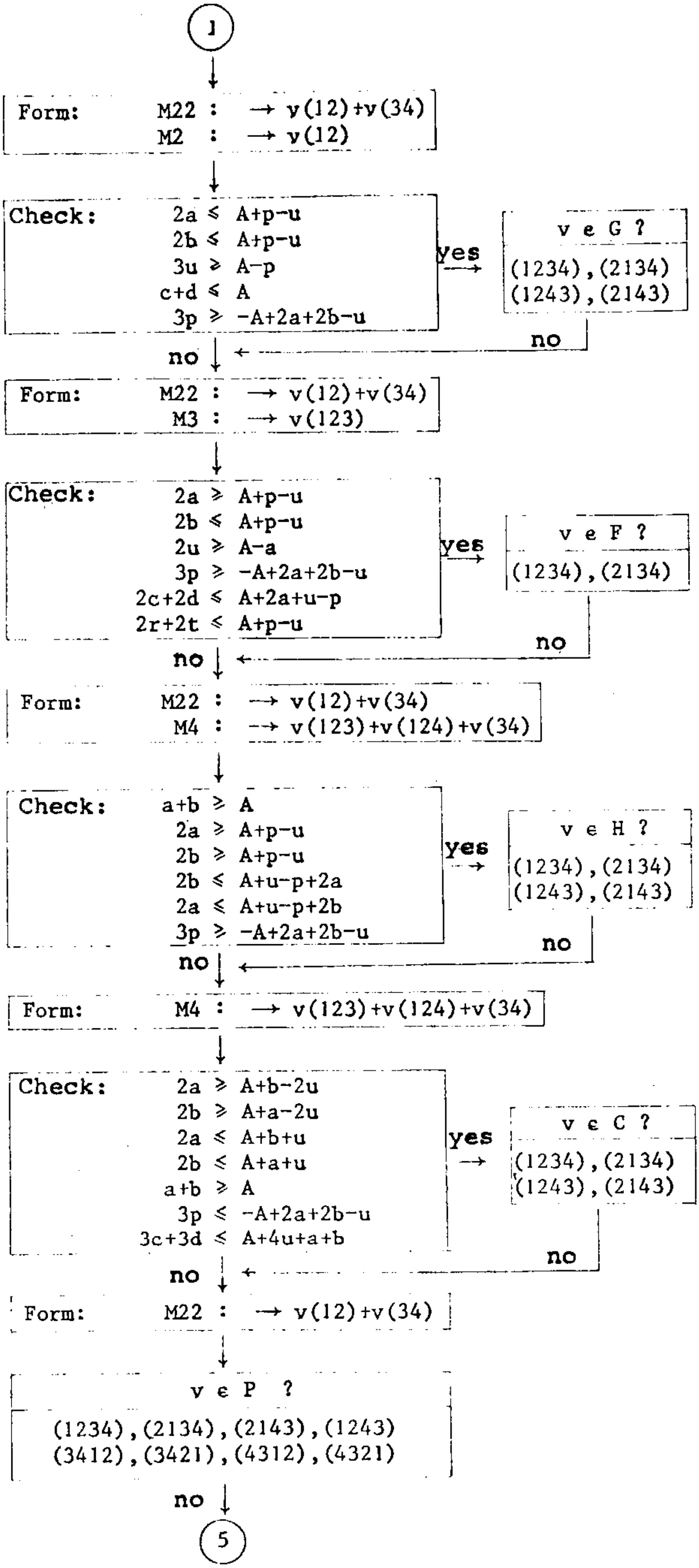
If for a given game v resp. for the changed game v', generated by some permutations of the fours players (see 2.2.3), we have found a fitting game region, then by the equations described there for x_i, $i \in \{1,2,3,4\}$, we get the nucleolus of the game v resp. of the game v'.

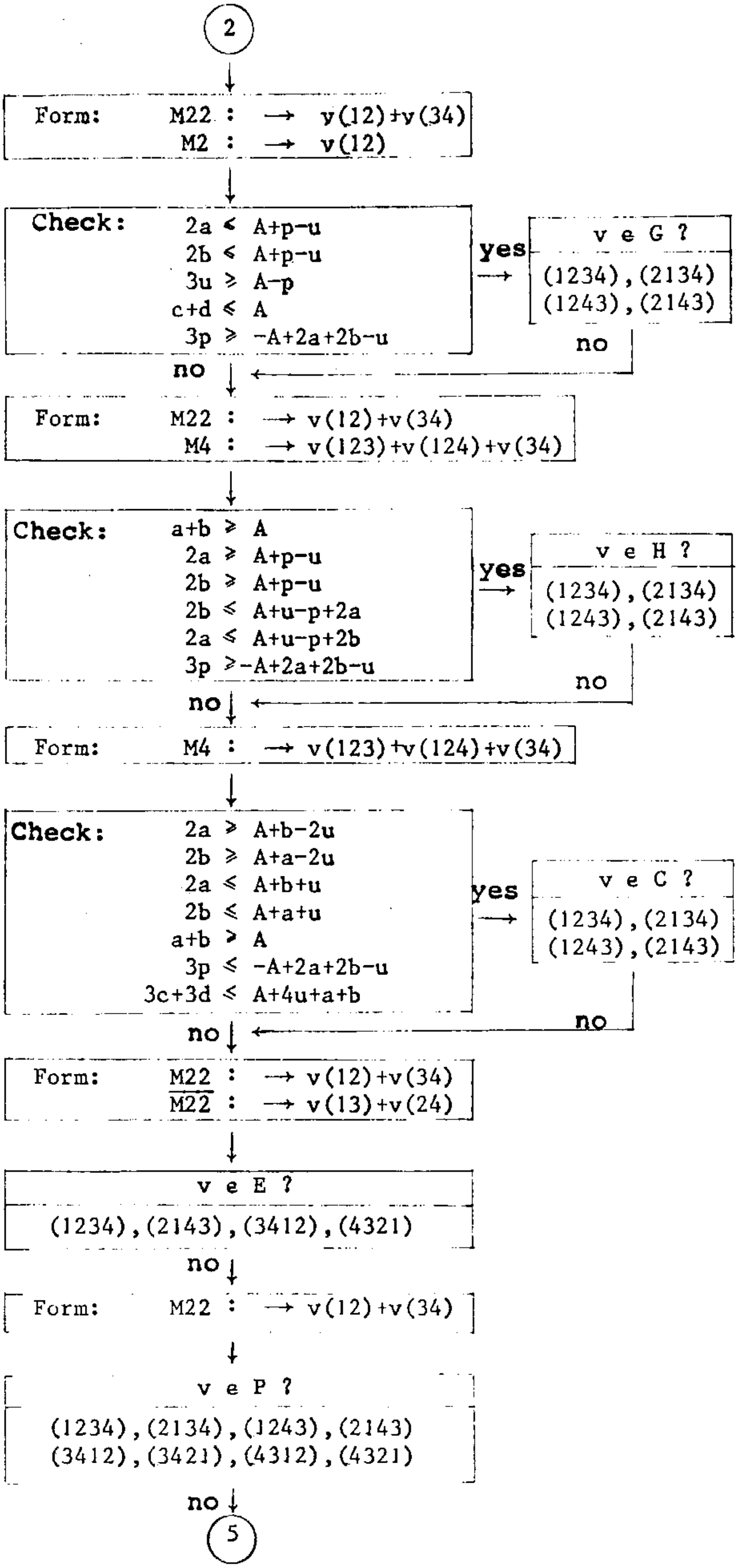
Note that games fitting into the game regions of block E are exactly those on which the Kernel and the nucleolus do not coincide. D. Bitter gave a practial necessary and sufficient criterion for these games [D.Bitter 1976]. Therefore, sometimes it may be convenient first to check this criterion and then immediately look at the game regions of block E. Another way is to compute the kernel of such a game (see [D. Bitter 1976]) and then evaluate the middle point of the kernel (note that in these cases the kernel is a line segment). We think that this will yield the nucleolus, too.

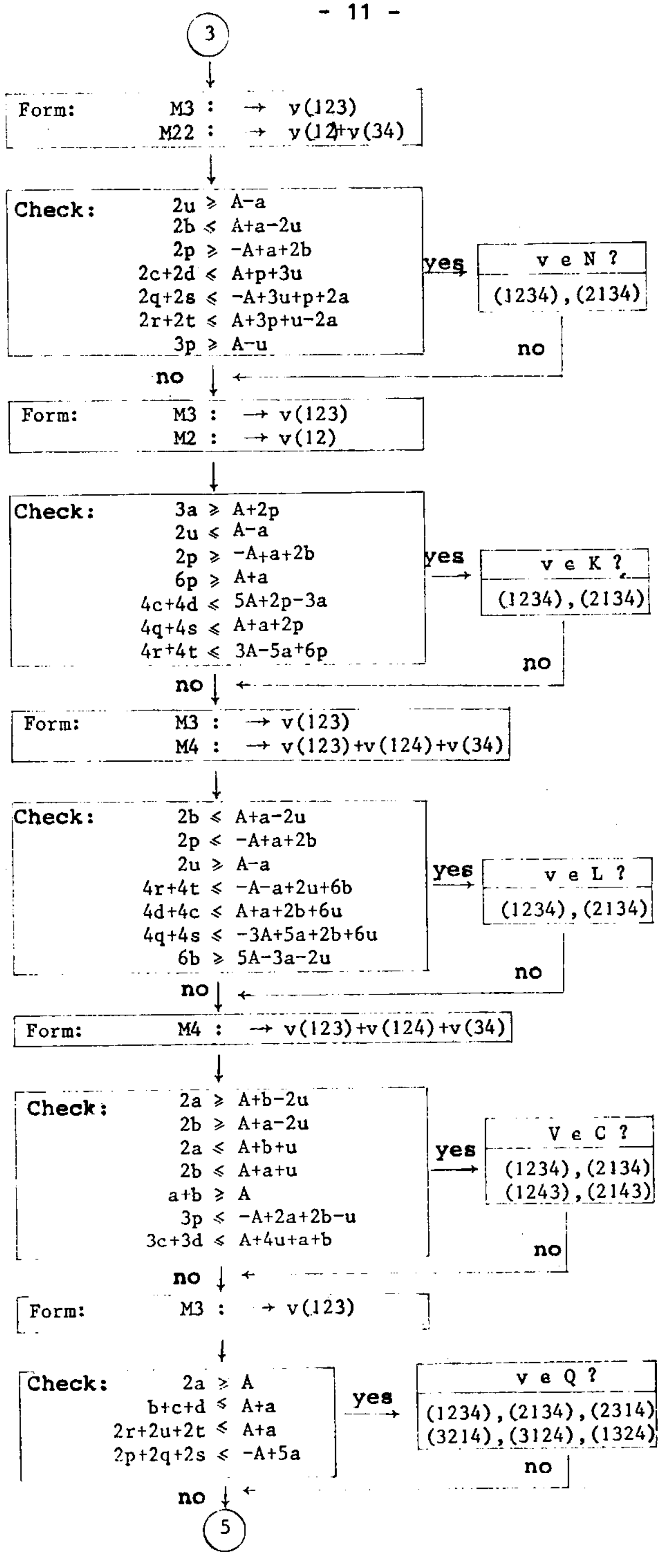
10. The search program

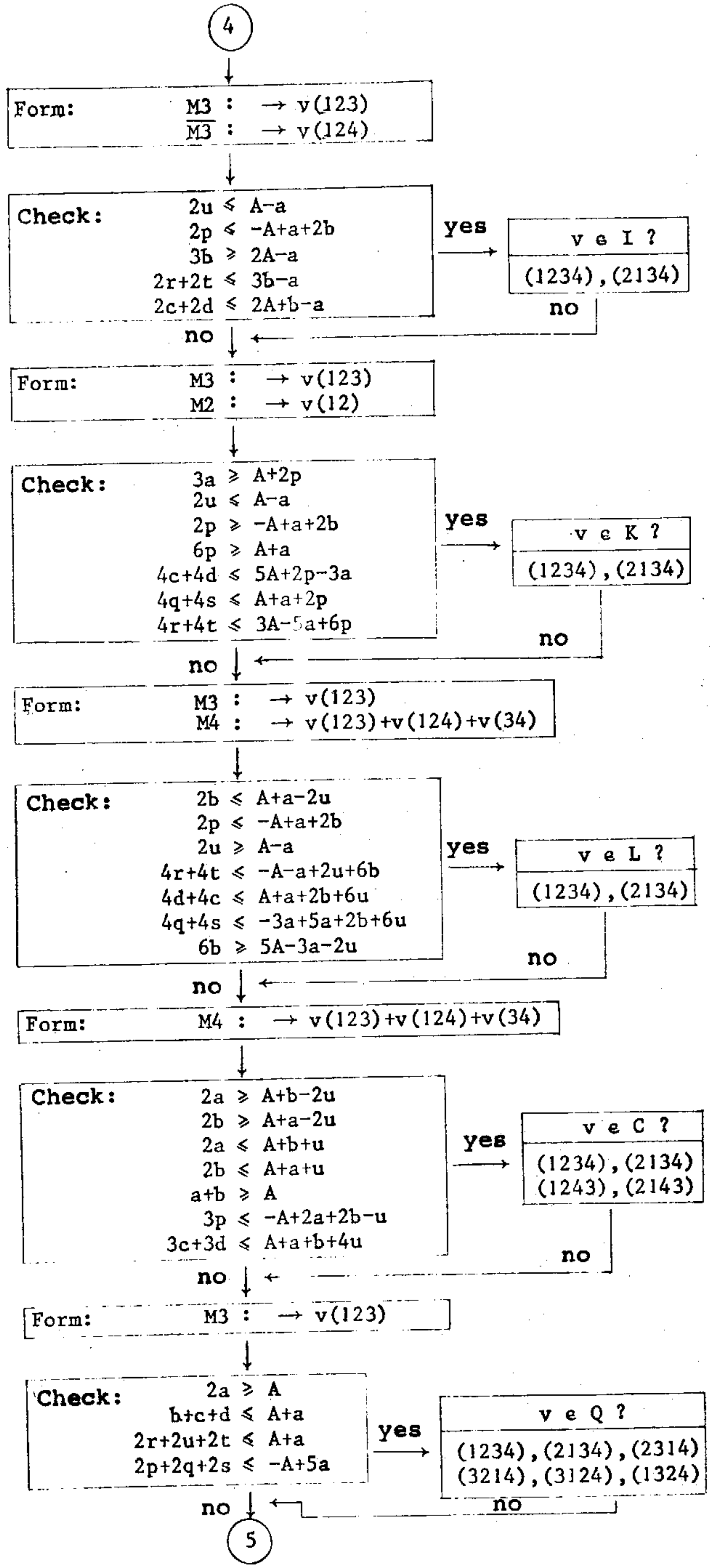
Let v_o be a superadditive 4-person game

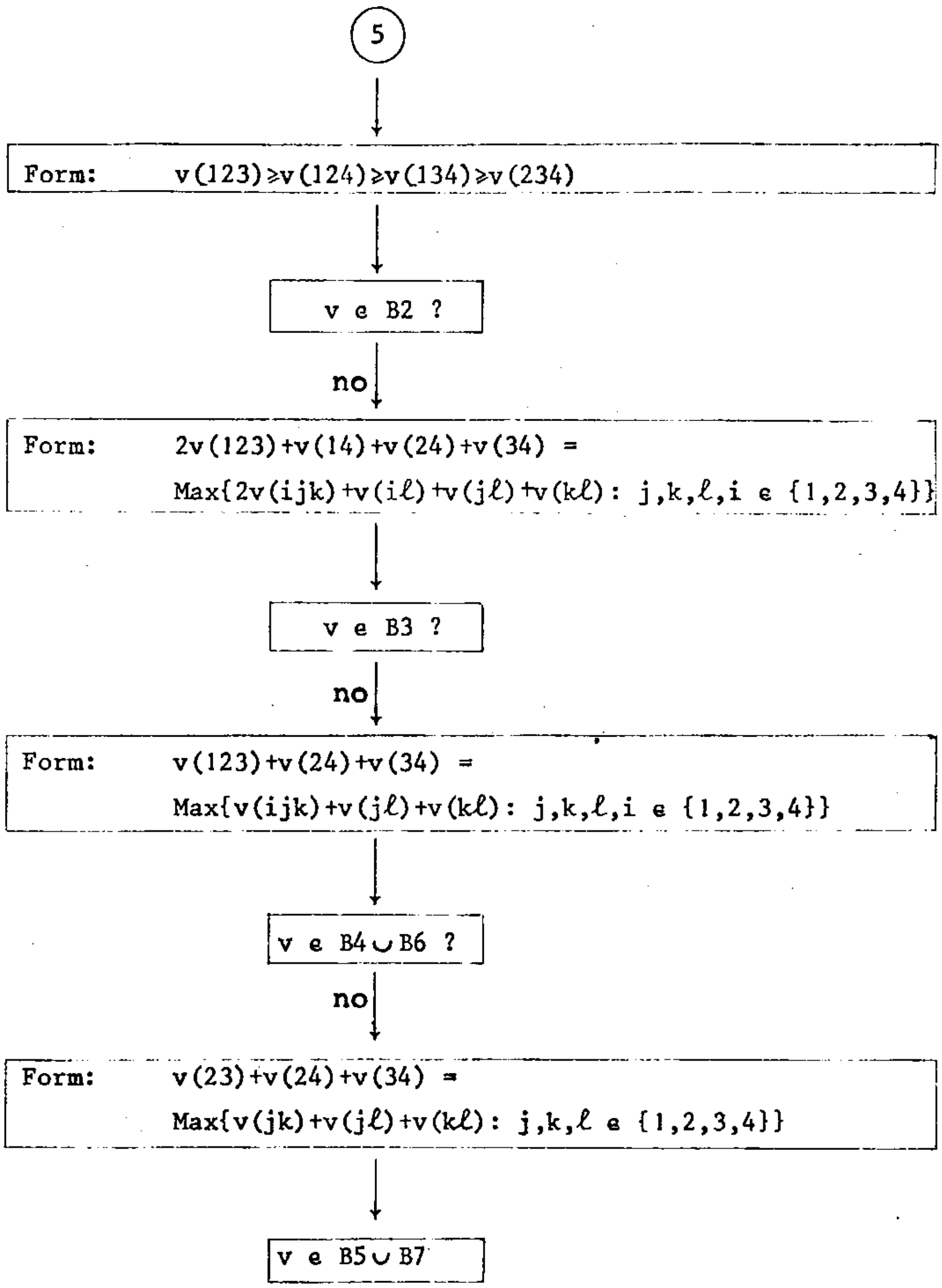



1
Form: M22 : → v(12)+v(34)
M2 : → v(12)
Check: 2a ≤ A+p-u
2b ≤ A+p-u
3u ≥ A-p
c+d ≤ A
3p ≥ -A+2a+2b-u
yes
v ∈ G ?
(1234),(2134)
(1243),(2143)
no
no
Form: M22 : → v(12)+v(34)
M3 : → v(123)
Check: 2a ≥ A+p-u
2b ≤ A+p-u
2u ≥ A-a
3p ≥ -A+2a+2b-u
2c+2d ≤ A+2a+u-p
2r+2t ≤ A+p-u
yes
v ∈ F ?
(1234),(2134)
no
no
Form: M22 : → v(12)+v(34)
M4 : → v(123)+v(124)+v(34)
Check: a+b ≥ A
2a ≥ A+p-u
2b ≥ A+p-u
2b ≤ A+u-p+2a
2a ≤ A+u-p+2b
3p ≥ -A+2a+2b-u
yes
v ∈ H ?
(1234),(2134)
(1243),(2143)
no
no
Form: M4 : → v(123)+v(124)+v(34)
Check: 2a ≥ A+b-2u
2b ≥ A+a-2u
2a ≤ A+b+u
2b ≤ A+a+u
a+b ≥ A
3p ≤ -A+2a+2b-u
3c+3d ≤ A+4u+a+b
yes
v ∈ C ?
(1234),(2134)
(1243),(2143)
no
no
Form: M22 : → v(12)+v(34)
v ∈ P ?
(1234),(2134),(2143),(1243)
(3412),(3421),(4312),(4321)
no
5


2
Form: M22 : → v(12)+v(34)
M2 : → v(12)
Check: 2a ≤ A+p−u
2b ≤ A+p−u
3u ≥ A−p
c+d ≤ A
3p ≥ −A+2a+2b−u
yes
v e G ?
(1234),(2134)
(1243),(2143)
no
no
Form: M22 : → v(12)+v(34)
M4 : → v(123)+v(124)+v(34)
Check: a+b ≥ A
2a ≥ A+p−u
2b ≥ A+p−u
2b ≤ A+u−p+2a
2a ≤ A+u−p+2b
3p ≥ −A+2a+2b−u
yes
v e H ?
(1234),(2134)
(1243),(2143)
no
no
Form: M4 : → v(123)+v(124)+v(34)
Check: 2a ≥ A+b−2u
2b ≥ A+a−2u
2a ≤ A+b+u
2b ≤ A+a+u
a+b ≥ A
3p ≤ −A+2a+2b−u
3c+3d ≤ A+4u+a+b
yes
v e C ?
(1234),(2134)
(1243),(2143)
no
no
Form: M22 : → v(12)+v(34)
$\overline{M22}$: → v(13)+v(24)
v e E ?
(1234),(2143),(3412),(4321)
no
Form: M22 : → v(12)+v(34)
v e P ?
(1234),(2134),(1243),(2143)
(3412),(3421),(4312),(4321)
no
5


3
Form: M3 : → v(123)
M22 : → v(12)+v(34)
Check: 2u ≥ A−a
2b ≤ A+a−2u
2p ≥ −A+a+2b
2c+2d ≤ A+p+3u
2q+2s ≤ −A+3u+p+2a
2r+2t ≤ A+3p+u−2a
3p ≥ A−u
yes
v ∈ N ?
(1234),(2134)
no
no
Form: M3 : → v(123)
M2 : → v(12)
Check: 3a ≥ A+2p
2u ≤ A−a
2p ≥ −A+a+2b
6p ≥ A+a
4c+4d ≤ 5A+2p−3a
4q+4s ≤ A+a+2p
4r+4t ≤ 3A−5a+6p
yes
v ∈ K ?
(1234),(2134)
no
no
Form: M3 : → v(123)
M4 : → v(123)+v(124)+v(34)
Check: 2b ≤ A+a−2u
2p ≤ −A+a+2b
2u ≥ A−a
4r+4t ≤ −A−a+2u+6b
4d+4c ≤ A+a+2b+6u
4q+4s ≤ −3A+5a+2b+6u
6b ≥ 5A−3a−2u
yes
v ∈ L ?
(1234),(2134)
no
no
Form: M4 : → v(123)+v(124)+v(34)
Check: 2a ≥ A+b−2u
2b ≥ A+a−2u
2a ≤ A+b+u
2b ≤ A+a+u
a+b ≥ A
3p ≤ −A+2a+2b−u
3c+3d ≤ A+4u+a+b
yes
V ∈ C ?
(1234),(2134)
(1243),(2143)
no
no
Form: M3 : → v(123)
Check: 2a ≥ A
b+c+d ≤ A+a
2r+2u+2t ≤ A+a
2p+2q+2s ≤ −A+5a
yes
v ∈ Q ?
(1234),(2134),(2314)
(3214),(3124),(1324)
no
no
5


4
Form: M3 : → v(123)
M̄3 : → v(124)
Check: 2u ≤ A−a
2p ≤ −A+a+2b
3b ≥ 2A−a
2r+2t ≤ 3b−a
2c+2d ≤ 2A+b−a
yes
v ∈ I ?
(1234),(2134)
no
no
Form: M3 : → v(123)
M2 : → v(12)
Check: 3a ≥ A+2p
2u ≤ A−a
2p ≥ −A+a+2b
6p ≥ A+a
4c+4d ≤ 5A+2p−3a
4q+4s ≤ A+a+2p
4r+4t ≤ 3A−5a+6p
yes
v ∈ K ?
(1234),(2134)
no
no
Form: M3 : → v(123)
M4 : → v(123)+v(124)+v(34)
Check: 2b ≤ A+a−2u
2p ≤ −A+a+2b
2u ≥ A−a
4r+4t ≤ −A−a+2u+6b
4d+4c ≤ A+a+2b+6u
4q+4s ≤ −3a+5a+2b+6u
6b ≥ 5A−3a−2u
yes
v ∈ L ?
(1234),(2134)
no
no
Form: M4 : → v(123)+v(124)+v(34)
Check: 2a ≥ A+b−2u
2b ≥ A+a−2u
2a ≤ A+b+u
2b ≤ A+a+u
a+b ≥ A
3p ≤ −A+2a+2b−u
3c+3d ≤ A+a+b+4u
yes
v ∈ C ?
(1234),(2134)
(1243),(2143)
no
no
Form: M3 : → v(123)
Check: 2a ≥ A
b+c+d ≤ A+a
2r+2u+2t ≤ A+a
2p+2q+2s ≤ −A+5a
yes
v ∈ Q ?
(1234),(2134),(2314)
(3214),(3124),(1324)
no
no
5



Because we have covered all necessary game regions $D_{[B]}$, the search program must end now.

11. Example of the application of the search program

The following game $v_o \varepsilon V_4$ may be given:

$v_o(N) = 20$; $v_o(123) = 11$, $v_o(124) = 12$, $v_o(134) = 12$,
$v_o(234) = 14$; $v_o(12) = 5$, $v_o(13) = 2$, $v_o(14) = 3$,
$v_o(23) = 10$, $v_o(24) = 2$, $v_o(34) = 12$.

The first question $M3 \leq {}^A/2$ must be answered in the negative, since $M3 = v_o(234) > {}^A/2 = 10$. Therefore the game has to be taken into the form $M2: \rightarrow v(12)$; this can be made by the permutation (3421). This yields:

$v_1(N) = 20$; $v_1(123) = 14$, $v_1(124) = 12$, $v_1(134) = 11$,
$v_1(234) = 12$; $v_1(12) = 12$, $v_1(13) = 10$, $v_1(14) = 2$,
$v_1(23) = 2$, $v_1(24) = 3$, $v_1(34) = 5$.

One of the described permutation-invariant **inequalities** is not satisfied, since we have $c+d = v_1(134) + v_1(234) > A=20$. Thus we have to proceed in the diagram. The next permutation-invariant inequalities $M22 \geq M3$, $\overline{M22} \leq M3$ and $M22 \geq \overline{M3}$ can be proved with the values of the original game v_o. Because of the inequalities $M22 = v_o(12) + v_o(34) > M3 = v_o(234)$ and $\overline{M22} = v_o(14) + v_o(23) < M3 = v_o(234)$ we reach branch (1).

The above game v_1 already possesses the first prescribed form $M22 : \rightarrow v(12) + v(34)$ and $M2 : \rightarrow v(12)$. But one of the permutation-invariant inequalities is not satisfied, since $c+d = v_1(134) + v_1(234) > A = 20$. Therefore we have to proceed in the program. Also the next form $M22 : \rightarrow v(12) + v(34)$ and $M3 : \rightarrow v(123)$ is satisfied by the game v_1. Since $2c + 2d = 2v_1(134) + 2v_1(234) >$
$> A + 2a + u - p = v_1(N) + 2v_1(123) + v_1(34) - v_1(12) = 41$, one of the described inequalities is not true and thus we

have to proceed.

An inspection of the games v_o and v_1 shows that the original game v_o satisfies the now demanded form M22: → v(12)+ v(34) and M4 : → v(123) + v(124) + v(34). Because for the game v_o all described permutation-invariant inequalities are true, we now apply the four permutations, described under the question "v ε H?", on the game v_o which satisfies the prescribed form. For all these four games we have to examine whether one of them fits into one of the game regions of block H. For this purpose it is convenient first to check the largest values of the four games. Then we shall recognize that none of the four games fits into a game region of block H.

The now following form M4 : → v(123) + v(124) + v(34) is also satisfied by the original game v_o. But the inequality $3p \leq -A+2a+2b-u$ is not true and thus we have to go further.

Obviously, the above game v_1 satisfies the next form M22 : → v(12)+ v(34). On this game we now have to apply the eight permutations described under the question "v ε P?". We see that none of the eight games fits into one of the game regions of block P and thus we proceed to the connection point (5) .

The first prescribed form v(123) ≥ v(124) ≥ v(134) ≥ v(234) can be reached by applying the permutation (4321) on the game v_o. The new game v_3 does not fit into the game region B2.

The permutation (1243), applied on the original game v_o, yields the next prescribed form because, obviously, we have

$2v_o(124) + v_o(13) + v_o(23) + v_o(34) =$

$\max \{2v_o(ijk) + v_o(i\ell) + v_o(j\ell) + v_o(k\ell) : i,j,k,\ell \varepsilon \{1,2,3,4\}\}.$

Again, this new game v_4 does not fit into the game region B3. The game v_4 is the following:

$v_4(N) = 20$; $v_4(123) = 12$, $v_4(124) = 11$, $v_4(134) = 12$,

$v_4(234) = 14$; $v_4(12) = 5$, $v_4(13) = 3$, $v_4(14) = 2$,

$v_4(23) = 2$, $v_4(24) = 10$, $v_4(34) = 12$.

This game satisfies the now prescribed form

$v(123) + v(24) + v(34) = \max \{v(ijk)+v(j\ell)+v(k\ell):i,j,k,\ell \varepsilon \{1,2,3,4\}\}$

Furthermore, the game v_4 is an element of the game region B4. The equalities described there for x_i, $i \varepsilon \{1,2,3,4\}$, yield the point $N_{B4}(v_4) := (x_1,x_2,x_3,x_4)$ with $x_1 = 1.5$, $x_2 = 5$, $x_3 = 7$, $x_4 = 6.5$.

The coalition array C belonging to $(v_4, N_{B4}(v4))$ is the following:

$C_o = \emptyset$, $C_1 = \{123, 24, 34, 1, 12\}$,

$C_2 = \{124\}$, $C_3 = \{134\}$, $C_4 = \{234\}$, $C_5 = \{2\}$, $C_6 = \{13\}$,

$C_7 = \{14\}$, $C_8 = \{4\}$, $C_9 = \{3\}$, $C_{10} = \{23\}$.

Obviously, C is balanced and therefore $N_{B4}(v_4)$ is the nucleolus of the game v_4. Again applying the permutation (1243) we get the original game v_o and its nucleolus $N(v_o) = (1.5; 5; 6.5; 7)$.

Note that we choose this example such that the search would be extremely long. For many games this procedure may be a shorter one.

3. The game regions $D_{[B]}$.

Here we describe all game regions necessary for the computation of the nucleolus for superadditive 4-person-games. The corresponding normalized coalition arrays are given in section 4. [1]

Block B

B1
$$x_1 = A/4$$
$$x_2 = A/4$$
$$x_3 = A/4$$
$$x_4 = A/4$$

$a,b,c,d \leq A/2;\ p,q,r,s,t,u \leq A/4$

B2
$$x_1 = (A+a+c+b-3d)/4$$
$$x_2 = (A+a+b+d-3c)/4$$
$$x_3 = (A+a+c+d-3b)/4$$
$$x_4 = (A+b+c+d-3a)/4$$

$a \geq A+d-b-c$
$a \geq A+c-b-d$
$a \geq A+b-c-d$
$d \geq A+a-b-c$
$4p \leq -A+3a+3b-c-d$
$4q \leq -A+3a+3c-b-d$
$4r \leq -A+3b+3c-a-d$
$4s \leq -A+3a+3d-b-c$
$4t \leq -A+3b+3d-a-c$
$4u \leq -A+3c+3d-a-b$
$3d \leq A+a+c+b$
$3c \leq A+a+b+d$
$3b \leq A+a+c+d$
$3a \leq A+b+c+d$

B3
$$x_1 = (A+a+3r-2t-2u)/5$$
$$x_2 = (A+a+3t-2u-2r)/5$$
$$x_3 = (A+a+3u-2r-2t)/5$$
$$x_4 = (2A-3a+r+t+u)/5$$

$4r \geq 2A-3a+t+u$
$4t \geq 2A-3a+r+u$
$4u \geq 2A-3a+r+t$
$2u \geq A+a-2r-2t$
$5b \leq A+a+3r+3t-2u$
$5c \leq A+a+3r+3u-2t$
$5d \leq A+a+3t+3u-2r$
$5p \leq -A+4a+2r+2t-3u$
$5q \leq -A+4a+2r+2u-3t$
$5s \leq -A+4a+2t+2u-3r$
$2u \leq A+a+3r-2t$
$2r \leq A+a+3t-2u$
$2t \leq A+a+3u-2r$
$3a \leq 2A+r+t+u$

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[1] Note that all superadditive 4-person-games are assumed to be 0-normalized.

B4

$$x_1 = (2A-u-t-a)/4$$
$$x_2 = (t-u+a)/2$$
$$x_3 = (a-t+u)/2$$
$$x_4 = (2A+u+t-3a)/4$$

$$u \leq 2A-a-t$$
$$3t \geq 2A+u-3a$$
$$3u \geq 2A+t-3a$$
$$u \geq a-t$$
$$4b \leq 2A+3t-a-u$$
$$4c \leq 2A+3u-a-t$$
$$2d \leq a+u+t$$
$$2p \leq a+t-u$$
$$2q \leq a+u-t$$
$$4s \leq -2A+5a+u+t$$
$$4r \leq 2A+u+t-3a$$

B5

$$x_1 = (2A-s-u-t)/5$$
$$x_2 = (A+2t+2s-3u)/5$$
$$x_3 = (A+2s+2u-3t)/5$$
$$x_4 = (A+2t+2u-3s)/5$$

$$s \leq 2A-u-t$$
$$3t \geq A+2s-3u$$
$$3t \geq A+2u-3s$$
$$3u \geq A+2t-3s$$
$$5a \leq 2A+4s-t-u$$
$$5b \leq 2A+4t-s-u$$
$$5c \leq 2A+4u-s-t$$
$$5d \leq A+2s+2u+2t$$
$$5p \leq A+2t+2s-3u$$
$$5q \leq A+2u+2s-3t$$
$$5r \leq A+2u+2t-3s$$

B6

$$x_1 = 0$$
$$x_2 = (A+a+t-2u)/3$$
$$x_3 = (A+a+u-2t)/3$$
$$x_4 = (A+t+u-2a)/3$$

$$u > 2A-a-t$$

$$2u \leq A+a+t$$
$$2t \leq A+a+u$$
$$2a \leq A+t+u$$
$$d \leq A$$
$$b \leq t$$
$$c \leq u$$
$$3p \leq -A+2a+2t-u$$
$$3q \leq -A+2a+2u-t$$
$$3r \leq -A+2t+2u-a$$
$$s \leq a$$

B7

$$x_1 = 0$$
$$x_2 = (A+t+s-2u)/3$$
$$x_3 = (A+u+s-2t)/3$$
$$x_4 = (A+u+t-2s)/3$$

$$s > 2A-t-u$$

$$2u \leq A+t+s$$
$$2t \leq A+u+s$$
$$2s \leq A+u+t$$
$$d \leq A$$
$$a \leq s$$
$$b \leq t$$
$$c \leq u$$
$$3q \leq -A+2s+2u-t$$
$$3r \leq -A+2t+2u-s$$
$$3p \leq -A+2t+2s-u$$

Block C

C1
$$x_1 = (A+4b+3q-2a-3t-2u)/6$$
$$x_2 = (A+4a+3t-2b-2u-3q)/6$$
$$x_3 = (A+a+u-2b)/3$$
$$x_4 = (A+u+b-2a)/3$$

$$3q \leqslant -A+2a+2b+2u-3t$$

$$2u \geqslant A+a-2b$$
$$2u \geqslant A+b-2a$$
$$3q \geqslant A+a+u-2b$$
$$3t \geqslant A+b+u-2a$$
$$3c \leqslant A+3q+u+b-2a$$
$$3d \leqslant A+3t+u+a-2b$$
$$3p \leqslant -A+2a+2b-u$$
$$r \leqslant b+q-a$$
$$s \leqslant a+t-b$$
$$3t \leqslant A+4b+3q-2a-2u$$
$$3q \leqslant A+4a+3t-2b-2u$$
$$2b \leqslant A+a+u$$
$$2a \leqslant A+b+u$$

C2
$$x_1 = (-A+2b+2a+3c-4u)/6$$
$$x_2 = (A-c)/2$$
$$x_3 = (A+a+u-2b)/3$$
$$x_4 = (A+b+u-2a)/3$$

$$3c \leqslant -A+2a+2b+2u$$

$$A \geqslant c$$
$$2u \geqslant A+b-2a$$
$$2u \geqslant A+a-2b$$
$$2u \leqslant -2A+a+b+3c$$
$$3d \leqslant 2A+2u-a-b$$
$$3p \leqslant -A+2a+2b-u$$
$$3q \leqslant -A+3c+2a-b-u$$
$$3r \leqslant -A+3c+2b-a-u$$
$$3s \leqslant A+a+u-2b$$
$$3t \leqslant A+b+u-2a$$
$$2b \leqslant A+a+u$$
$$2a \leqslant A+b+u$$

C3
$$x_1 = x_2 = (A+a+b-2u)/6$$

$$x_3 = (A+a+u-2b)/3$$
$$x_4 = (A+b+u-2a)/3$$

$$a \geqslant A-b$$

$$2u \geqslant A+b-2a$$
$$2u \geqslant A+a-2b$$
$$2u \leqslant A+a+b$$
$$3c \leqslant 2A+2u-a-b$$
$$3d \leqslant 2A+2u-a-b$$
$$3q \leqslant A+a+u-2b$$
$$3s \leqslant A+a+u-2b$$
$$3t \leqslant A+b+u-2a$$
$$3r \leqslant A+b+u-2a$$
$$3p \leqslant -A+2a+2b-u$$
$$2a \leqslant A+b+u$$
$$2b \leqslant A+a+u$$

C4
$$x_1 = (A+3r+a+b-3t-2u)/6$$
$$x_2 = (A+a+b+3t-2u-3r)/6$$
$$x_3 = (A+a+u-2b)/3$$
$$x_4 = (A+b+u-2a)/3$$

$$3r \leqslant -A+5b+2u-a-3t$$

$$2u \geqslant A+b-2a$$
$$2u \geqslant A+a-2b$$
$$3r \geqslant A+u+b-2a$$
$$3t \geqslant A+u+b-2a$$
$$3c \leqslant A+3r+a+u-2b$$
$$3d \leqslant A+3t+a+u-2b$$
$$3p \leqslant -A+2a+2b-u$$
$$q \leqslant r+a-b$$
$$s \leqslant t+a-b$$
$$3t \geqslant -A+3r+2u-a-b$$
$$3r \geqslant -A+3t+2u-a-b$$
$$2b \leqslant A+a+u$$
$$2a \leqslant A+b+u$$

C5
$$x_1 = (2A+2b-a-u-3t)/6$$
$$x_2 = (a+t-u)/2$$
$$x_3 = (A+a+u-2b)/3$$
$$x_4 = (A+b+u-2a)/3$$

$3t \leqslant -2A+a+4b+u$

$3t \leqslant 2A-a-u+2b$
$3t \geqslant A+b+u-2a$
$2u \geqslant A+b-2a$
$2u \geqslant A+a-2b$
$3c \leqslant 2A+2u-a-b$
$3d \leqslant A+3t+a+u-2b$
$3p \leqslant -A+2b+2a-u$
$3q \leqslant A+a+u-2b$
$3r \leqslant A+b+u-2a$
$s \leqslant t+a-b$
$2b \leqslant A+a+u$
$2a \leqslant A+b+u$

C6
$$x_1 = (A+a+b+3c-3d-2u)/6$$
$$x_2 = (A+a+b+3d-3c-2u)/6$$
$$x_3 = (A+a+u-2b)/3$$
$$x_4 = (A+u+b-2a)/3$$

$3c \leqslant A+4u+a+b-3d$

$3c \geqslant 2A+2u-a-b$
$3d \geqslant 2A+2u-a-b$
$2u \geqslant A+b-2a$
$2u \geqslant A+a-2b$
$3p \leqslant -A+2a+2b-u$
$3q \leqslant -A+2a+3c-b-u$
$3r \leqslant -A+2b+3c-a-u$
$3s \leqslant -A+2a+3d-b-u$
$3t \leqslant -A+2b+3d-a-u$
$3d \leqslant A+a+b-2u+3c$
$3c \leqslant A+a+b+3d-2u$
$2b \leqslant A+a+u$
$2a \leqslant A+b+u$

C7
$$x_1 = (c+b-u-t)/2$$
$$x_2 = (2A+2a-b-u+3t-3c)/6$$
$$x_3 = (A+a+u-2b)/3$$
$$x_4 = (A+b+u-2a)/3$$

$c \leqslant u+b-t$

$3c \geqslant 2A-a-b+2u$
$3t \geqslant A+b+u-2a$
$2u \geqslant A+b-2a$
$2u \geqslant A+a-2b$
$3d \leqslant A+a+u-2b+3t$
$3p \leqslant -A-u+2a+2b$
$3q \leqslant -A-u-b+2a+3c$
$3r \leqslant -A-u-a+2b+3c$
$s \leqslant t+a-b$
$c \geqslant u+t-b$
$3c \leqslant 2A+2a-b-u+3t$
$2b \leqslant A+a+u$
$2a \leqslant A+b+u$

C8
$$x_1 = 0$$
$$x_2 = (A+a+b-2u)/3$$
$$x_3 = (A+a+u-2b)/3$$
$$x_4 = (A+b+u-2a)/3$$

$t \leqslant b$

$3t \geqslant 2A+2b-a-u$
$2u \leqslant A+a+b$
$2a \leqslant A+b+u$
$2b \leqslant A+a+u$
$2u \geqslant A+a-2b$
$2u \geqslant A+b-2a$
$d \leqslant A$
$c \leqslant t+u-b$
$3p \leqslant -A+2a+2b-u$
$3q \leqslant -A+2a+2u+3t-4b$
$3r \leqslant -A+2u+3t-a-b$
$s \leqslant a-b+t$

Block D

D1 $x_1 = x_2 = (A+2p)/6$
$x_3 = x_4 = (A-p)/3$

$p \geq A/4$

$A \geq p$
$3a \leq A+2p$
$3b \leq A+2p$
$3c \leq 2A-2p$
$3d \leq 2A-2p$
$3s \leq A-p$
$3t \leq A-p$
$3r \leq A-p$
$3q \leq A-p$
$3u \leq A-p$

D2 $x_1 = (A+2p+3c-3d)/6$
$x_2 = (A+3d+2p-3c)/6$
$x_3 = x_4 = (A-p)/3$

$c \leq A-d$

$A \geq p$
$3c \geq 2A-2p$
$3d \geq 2A-2p$
$3a \leq A+2p$
$3b \leq A+2p$
$3q \leq -A+3c+p$
$3r \leq -A+3c+p$
$3t \leq -A+3d+p$
$3s \leq -A+3d+p$
$3u \leq A-p$

D3 $x_1 = (2A+p+3r-3d)/6$
$x_2 = (d+p-r)/2$
$x_3 = x_4 = (A-p)/3$

$3r \leq 2A+p-3d$

$A \geq p$
$3r \geq A-p$
$3d \geq 2A-2p$
$3a \leq A+2p$
$3b \leq A+2p$
$3c \leq A+3r-p$
$q \leq r$
$3s \leq -A+3d+p$
$3t \leq -A+3d+p$
$3u \leq A-p$

D4 $x_1 = (A+3r+2p-3t)/6$
$x_2 = (A+3t+2p-3r)/6$
$x_3 = x_4 = (A-p)/3$

$3r \leq A+2p-3t$

$A \geq p$
$3r \geq A-p$
$3t \geq A-p$
$3a \leq A+2p$
$3b \leq A+2p$
$3c \leq A+3r-p$
$3d \leq A+3t-p$
$q \leq r$
$s \leq t$
$3u \leq A-p$

D5 $x_1 = (p+r)/2$
$x_2 = 2A+p-3r)/6$
$x_3 = x_4 = (A-p)/3$

$p \geq r$

$A \geq p$
$3r \geq A-p$
$3a \leq A+2p$
$3b \leq A+2p$
$3c \leq A+3r-p$
$3d \leq 2A-2p$
$q \leq r$
$3s \leq A-p$
$3t \leq A-p$
$3u \leq A-p$

D6 $x_1 = (A+3r+2p-3s)/6$
$x_2 = (A+3s+2p-3r)/6$
$x_3 = x_4 = (A-p)/3$

$3r \leq A+2p-3s$

$A \geq p$
$3r \geq A-p$
$3s \geq A-p$
$3a \leq A+2p$
$3b \leq A+2p$
$3c \leq A+3r-p$
$3d \leq A+3s-p$
$q \leq r$
$t \leq s$
$3u \leq A-p$

D7 $x_1 = (-A+3c+4p)/6$
$x_2 = (A-c)/2$
$x_3 = x_4 = (A-p)/3$

$3c \leq A+2p$

$A \geq p$
$3c \geq 2A-2p$
$3a \leq A+2p$
$3b \leq A+2p$
$3d \leq 2A-2p$
$3q \leq -A+3c+p$
$3r \leq -A+3c+p$
$3s \leq A-p$
$3t \leq A-p$
$3u \leq A-p$

Block E

E1

$$x_1 = (A+p+q+r-s-t-u)/4$$
$$x_2 = (A+p+s+t-q-r-u)/4$$
$$x_3 = (A+q+s+u-p-r-t)/4$$
$$x_4 = (A+r+t+u-p-q-s)/4$$

$$p+u \geqslant q+t$$
$$q+t \geqslant r+s$$

$$3r \geqslant A+t+u-p-q-s$$
$$3s \geqslant A+q+u-p-r-t$$
$$3s \geqslant A+p+t-q-r-u$$
$$3r \geqslant A+p+q-s-t-u$$
$$4a \leqslant A+3s+p+q+r-t-u$$
$$4b \leqslant A+3r+p+s+t-q-u$$
$$4c \leqslant A+3r+q+s+u-p-t$$
$$4d \leqslant A+3s+r+t+u-p-q$$
$$A+p+q+r \geqslant s+t+u$$
$$A+s+t+p \geqslant q+r+u$$
$$A+q+s+u \geqslant p+r+t$$
$$A+r+t+u \geqslant p+q+s$$

E2

$$x_1 = x_3 = (A+q-t)/4$$
$$x_2 = (A+2p+t-2u-q)/4$$
$$x_4 = (A+2u+t-2p-q)/4$$

$$p+u \geqslant q+t$$
$$3q \geqslant A-t$$

$$t \leqslant A+q$$
$$q \leqslant u+t-p$$
$$q \leqslant p+t-u$$
$$2a \leqslant A+p-u$$
$$2b \leqslant A+t-q$$
$$2c \leqslant A+u-p$$
$$2d \leqslant A+t-q$$
$$4r \leqslant A+2u+t-2p-q$$
$$4s \leqslant A+2p+t-2u-q$$

E3

$$x_1 = (A+2q+p-2t-u)/4$$
$$x_2 = (A+2t+p-2q-u)/4$$
$$x_3 = x_4 = (A+u-p)/4$$

$$p+u \geqslant q+t$$
$$2q \geqslant A+p-u-2t$$

$$p \leqslant A+u$$
$$q \geqslant u+t-p$$
$$q \leqslant p+t-u$$
$$2a \leqslant A+p-u$$
$$2b \leqslant A+p-u$$
$$2c \leqslant A+q-t$$
$$2d \leqslant A+t-q$$
$$4r \leqslant A+2q+p-2t-u$$
$$4s \leqslant A+2t+p-2q-u$$

E4

$$x_1 = (A+2b+2q+p-2a-2t-u)/4$$
$$x_2 = (A+2a+2t+p-2b-2q-u)/4$$
$$x_3 = (A+2a+u-2b-p)/4$$
$$x_4 = (A+2b+u-2a-p)/4$$

$$p+u \geqslant t+q$$
$$2a \leqslant A+2q+2t+p-2b-u$$

$$2b \geqslant A+t-q$$
$$2a \geqslant A+q-t$$
$$2a \geqslant A+p-u$$
$$2b \geqslant A+p-u$$
$$2c \leqslant 2b+u+q-p-t$$
$$2d \leqslant 2a+u+t-p-q$$
$$4r \leqslant -A+6b+2q+u-2a-2t-p$$
$$4s \leqslant -A+6a+2t+u-2b-2q-p$$
$$2b \geqslant -A+2a+2t+u-2q-p$$
$$2a \geqslant -A+2b+2q+u-2t-p$$
$$2a \geqslant -A+2b+p-u$$
$$2b \geqslant -A+2a+p-u$$

E5
$$x_1 = (A+2c+2p+q-2a-2u-t)/4$$
$$x_2 = (A+2a+t-2c-q)/4$$
$$x_3 = (A+2a+2u+q-2c-2p-t)/4$$
$$x_4 = (A+2c+t-2a-q)/4$$

$$p+u \geqslant q+t$$
$$2a < A+3q+t-2c$$

$$2c \geqslant A+u-p$$
$$2a \geqslant A+q-t$$
$$2a \geqslant A+p-u$$
$$2c \geqslant A+q-t$$
$$2b \leqslant 2c+p+t-q-u$$
$$2d \leqslant 2a+t+u-p-q$$
$$4r \leqslant -A+6c+2p+t-2a-2u-q$$
$$4s \leqslant -A+6a+2u+t-2c-2p-q$$
$$2c \geqslant -A+2a+2u+t-2p-q$$
$$2a \geqslant -A+2c+q-t$$
$$2a \geqslant -A+2c+2p+t-2u-q$$
$$2c \geqslant -A+2a+q-t$$

E6
$$x_1 = (2A+p+q-t-u-2a)/4$$
$$x_2 = (2a+p+t-q-u)/4$$
$$x_3 = (2a+q+u-p-t)/4$$
$$x_4 = (2A+u+t-p-q-2a)/4$$

$$p+u \geqslant q+t$$
$$2a \leqslant p+3q+t-u$$

$$2a \leqslant 2A-u-t+p+q$$
$$2a \geqslant A+q-t$$
$$2a \geqslant A+p-u$$
$$q \leqslant u+t-p$$
$$2b \leqslant A+t-q$$
$$2c \leqslant A+u-p$$
$$2d \leqslant 2a+u+t-p-q$$
$$4r \leqslant 2A+p+q-2a-u-t$$
$$4s \leqslant -2A+6a+u+t-p-q$$

E7
$$x_1 = (2A+p+q-2s-t-u)/6$$
$$x_2 = (A+2s+2p+t-q-2u)/6$$
$$x_3 = (A+2s+2q+u-p-2t)/6$$
$$x_4 = (A+u+t-p-q-s)/3$$

$$p+u \geqslant q+t$$
$$2t \geqslant A+2s+u-4q-p$$

$$2s \leqslant 2A+p+q-u-t$$
$$4s \geqslant A+2q+u-2t-p$$
$$4s \geqslant A+2p+t-2u-q$$
$$u \geqslant p+q-t$$
$$6a \leqslant 2A+4s+p+q-t-u$$
$$2b \leqslant A+t-q$$
$$2c \leqslant A+u-p$$
$$3d \leqslant A+2s+u+t-p-q$$

E8
$$x_1 = (A+p+q+r-a-t-u)/3$$
$$x_2 = (A+2a+p+2t-2q-2r-u)/6$$
$$x_3 = (A+2a+q+2u-2p-2r-t)/6$$
$$x_4 = (2A+2r+t+u-2a-p-q)/6$$

$$p+u \geqslant q+t$$
$$4q \geqslant -A+4a+2r+u-2t-p$$

$$4r \geqslant 2A+t+u-2a-p-q$$
$$2a \geqslant A+q-t$$
$$2a \geqslant A+p-u$$
$$2r \geqslant A+p+q-a-t-u$$
$$6b \leqslant A+2a+2t+4r+p-2q-u$$
$$6c \leqslant A+2a+2u+4r+q-2p-t$$
$$2d \leqslant 2a+t+u-p-q$$
$$3s \leqslant -A+4a+t+u-p-q-r$$
$$u \leqslant A+p+q+r-a-t$$
$$2r \leqslant A+2a+p+2t-2q-u$$
$$2r \leqslant A+2a+q+2u-2p-t$$
$$2a \leqslant 2A+2r+t+u-p-q$$

E9
$$x_1 = (A+p+q-a-t-u)/2$$
$$x_2 = (a+t-q)/2$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$$p+u \geq q+t$$
$$a \leq q+t$$

$$A \geq a$$
$$u \leq p+q-t$$
$$2a \geq A+q-t$$
$$2a \geq A+p-u$$
$$2b \leq A+p-u$$
$$2c \leq A+q-t$$
$$2d \leq 2a+t+u-p-q$$
$$2r \leq A+p+q-a-t-u$$
$$2s \leq -A+3a+t+u-p-q$$

Block F

F1
$$x_1 = x_2 = (A+p-u)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$$a \leq p+u$$
$$2a \geq A+u-p$$

$$A \geq a$$
$$2a \geq A+p-u$$
$$2b \leq A+p-u$$
$$2c \leq A+u-p$$
$$2d \leq A+u-p$$
$$2q \leq a+u-p$$
$$2s \leq a+u-p$$
$$2r \leq A-a$$
$$2t \leq A-a$$

F2
$$x_1 = (A+p+q-a-t-u)/2$$
$$x_2 = (a+t-q)/2$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$$a \leq p+u$$
$$a \geq q+t$$

$$A \geq a$$
$$2a \geq A+p-u$$
$$2q \geq a+u-p$$
$$2t \geq A-a$$
$$2b \leq A+p-u$$
$$2c \leq A+2q-a$$
$$2d \leq a+u-p+2t$$
$$2r \leq A+2q+p-2a-u$$
$$2s \leq -A+2a+2t+u-p$$

F3
$$x_1 = (c+p-u)/2$$
$$x_2 = (A-c)/2$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$$a \leq p+u$$
$$a \geq c$$

$$A \geq a$$
$$2c \geq A+u-p$$
$$2a \geq A+p-u$$
$$2b \leq A+p-u$$
$$2d \leq A+u-p$$
$$2q \leq -A+2c+a$$
$$2r \leq 2c+p-a-u$$
$$2s \leq a+u-p$$
$$2t \leq A-a$$

F4
$$x_1 = (A+2c+p-2d-u)/4$$
$$x_2 = (A+2d+p-2c-u)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$$a \leq p+u$$
$$2c \leq A+2a-2d-p+u$$

$$A \geq a$$
$$2a \geq A+p-u$$
$$2c \geq A+u-p$$
$$2d \geq A+u-p$$
$$2b \leq A+p-u$$
$$2q \leq -A+a+2c$$
$$2r \leq 2c+p-a-u$$
$$2s \leq -A+a+2d$$
$$2t \leq 2d+p-a-u$$

F5
$$x_1 = (A+2c+2p-a-2t-2u)/4$$
$$x_2 = (A+a+2t-2c)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$$a \leq p+u$$
$$2c \leq A+a-2t$$

$$A \geq a$$
$$2a \geq A+p-u$$
$$2c \geq A+u-p$$
$$2t \geq A-a$$
$$2b \leq A+p-u$$
$$2d \leq a+u-p+2t$$
$$2q \leq -A+2c+a$$
$$2r \leq 2c+p-a-u$$
$$2s \leq -A+2a+2t+u-p$$

F6
$$x_1 = (A+2q+p-2s-u)/4$$
$$x_2 = (A+2s+p-2q-u)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$$a \leq p+u$$
$$2q \leq -A+4a+u-2s-p$$

$$A \geq a$$
$$2a \geq A+p-u$$
$$2q \geq a+u-p$$
$$2s \geq a+u-p$$
$$2b \leq A+p-u$$
$$2c \leq A-a+2q$$
$$2d \leq A-a+2s$$
$$2r \leq A+2q+p-2a-u$$
$$2t \leq A+2s+p-2a-u$$

F7
$$x_1 = (a+2r+p-u)/4$$
$$x_2 = (2A+p-a-u-2r)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \leqslant p+u$
$2r \leqslant a+p-u$

$A \geqslant a$
$2a \geqslant A+p-u$
$2r \geqslant A-a$
$2b \leqslant A+p-u$
$2c \leqslant 2r+a+u-p$
$2d \leqslant A+u-p$
$2q \leqslant -A+2r+2a+u-p$
$2s \leqslant a+u-p$
$2t \leqslant A-a$

F8
$$x_1 = (A+2r+p-2t-u)/4$$
$$x_2 = (A+2t+p-2r-u)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \leqslant p+u$
$2r \leqslant A+p-u-2t$

$A \geqslant a$
$2a \geqslant A+p-u$
$2r \geqslant A-a$
$2t \geqslant A-a$
$2b \leqslant A+p-u$
$2c \leqslant 2r+a+u-p$
$2d \leqslant 2t+a+u-p$
$2q \leqslant -A+2a+2r+u-p$
$2s \leqslant -A+2a+2t+u-p$

F9
$$x_1 = (A+2q+2p-2u-a)/4$$
$$x_2 = (A+a-2q)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \leqslant p+u$
$3a \geqslant A+2p$

$A \geqslant a$
$2a \geqslant A+p-u$
$2q \geqslant a-p+u$
$2b \leqslant A+p-u$
$2c \leqslant A-a+2q$
$2d \leqslant A+u-p$
$2r \leqslant A+2q+p-2a-u$
$2s \leqslant a+u-p$
$2t \leqslant A-a$

Block G

G1
$$x_1 = (A+2c+p-2d-u)/4$$
$$x_2 = (A+2d+p-2c-u)/4$$
$$x_3 = x_4 = (A+u-p)/4$$

$3u \geqslant A-p$
$c \leqslant A-d$

$A \geqslant p-u$
$2c \geqslant A+u-p$
$2d \geqslant A+u-p$
$2a \leqslant A+p-u$
$2b \leqslant A+p-u$
$4q \leqslant -A+p-u+4c$
$4r \leqslant -A+p-u+4c$
$4s \leqslant -A+4d+p-u$
$4t \leqslant -A+4d+p-u$

G2
$$x_1 = x_2 = (A+p-u)/4$$
$$x_3 = x_4 = (A+u-p)/4$$

$3u \geqslant A-p$
$p \geqslant u$

$A \geqslant p-u$
$2a \leqslant A+p-u$
$2b \leqslant A+p-u$
$2c \leqslant A+u-p$
$2d \leqslant A+u-p$
$4q \leqslant A+u-p$
$4r \leqslant A+u-p$
$4s \leqslant A+u-p$
$4t \leqslant A+u-p$

G3
$$x_1 = (A+2r+p-2t-u)/4$$
$$x_2 = (A+2t+p-2r-u)/4$$
$$x_3 = x_4 = (A+u-p)/4$$

$3u \geqslant A-p$
$2r \leqslant A+p-2t-u$

$A \geqslant p-u$
$4r \geqslant A+u-p$
$4t \geqslant A+u-p$
$2a \leqslant A+p-u$
$2b \leqslant A+p-u$
$4c \leqslant A+4r+u-p$
$4d \leqslant A+4t+u-p$
$q \leqslant r$
$s \leqslant t$

G4
$$x_1 = (A+4c+3p-4t-3u)/8$$
$$x_2 = (3A+4t+p-4c-u)/8$$
$$x_3 = x_4 = (A+u-p)/4$$

$3u \geqslant A-p$
$4t \leqslant 3A+p-u-4c$

$A \geqslant p-u$
$2c \geqslant A+u-p$
$4t \geqslant A+u-p$
$2a \leqslant A+p-u$
$2b \leqslant A+p-u$
$4d \leqslant A+4t+u-p$
$4q \leqslant -A+4c+p-u$
$4r \leqslant -A+4c+p-u$
$s \leqslant t$

G5
$$x_1 = (A+2q+p-2t-u)/4$$
$$x_2 = (A+2t+p-2q-u)/4$$
$$x_3 = x_4 = (A+u-p)/4$$

$3u \geqslant A-p$
$2q \leqslant A+p-u-2t$

$A \geqslant p-u$
$4q \geqslant A+u-p$
$4t \geqslant A+u-p$
$2a \leqslant A+p-u$
$2b \leqslant A+p-u$
$4c \leqslant A+4q+u-p$
$4d \leqslant A+4t+u-p$
$r \leqslant q$
$s \leqslant t$

G6
$$x_1 = (3A+p-u-4t)/8$$
$$x_2 = (A+4t+3p-3u)/8$$
$$x_3 = x_4 = (A+u-p)/4$$

$3u \geqslant A-p$
$4t \leqslant A+3p-3u$

$A \geqslant p-u$
$4t \geqslant A+u-p$
$2a \leqslant A+p-u$
$2b \leqslant A+p-u$
$2c \leqslant A+u-p$
$4d \leqslant 4t+A+u-p$
$4q \leqslant A+u-p$
$4r \leqslant A+u-p$
$s \leqslant t$

Block H

H1

$$x_1 = (A+2c+p-2d-u)/4$$
$$x_2 = (A+2d+p-2c-u)/4$$
$$x_3 = (A+2a+u-2b-p)/4$$
$$x_4 = (A+2b+u-2a-p)/4$$

$2a \leqslant A+3p+u-2b$
$c \leqslant a+b+u-d-p$

$2c \geqslant A+u-p$
$2d \geqslant A+u-p$
$2a,2b \geqslant A+p-u$
$4q \leqslant -A+2a+p+4c-2b-u$
$4r \leqslant -A+2b+p+4c-2a-u$
$4s \leqslant -A+2a+p+4d-2b-u$
$4t \leqslant -A+2b+p+4d-2a-u$
$2d \leqslant A+2c+p-u$
$2c \leqslant A+2d+p-u$
$2a \leqslant A+2b+u-p$
$2b \leqslant A+2a+u-p$

H2

$$x_1 = (A+2a+3p+4r-3u-2b)/8$$
$$x_2 = (3A+2b+p-4r-2a-u)/8$$
$$x_3 = (A+2a+u-2b-p)/4$$
$$x_4 = (A+2b+u-2a-p)/4$$

$2a \leqslant A+3p+u-2b$
$4r \leqslant -3A+6b+2a+u-p$

$2b \leqslant -A+2a+4r+p-u$
$2a \geqslant A+p-u$
$2b \geqslant A+p-u$
$4c \leqslant A+4r+2a+u-2b-p$
$2d \leqslant A+u-p$
$q \leqslant a+r-b$
$4s \leqslant A+2a+u-2b-p$
$4t \leqslant A+2b+u-2a-p$
$2a \leqslant 3A+2b+p-4r-u$
$2a \leqslant A+2b+u-p$
$2b \leqslant A+2a+u-p$

H3

$$x_1 = (A+2r+p-2t-u)/4$$
$$x_2 = (A+2t+p-2r-u)/4$$
$$x_3 = (A+2a+u-2b-p)/4$$
$$x_4 = (A+2b+u-2a-p)/4$$

$2a \leqslant A+3p+u-2b$
$2t \leqslant -A+4b+u-2r-p$

$4r \geqslant A+2b+u-2a-p$
$4t \geqslant A+2b+u-2a-p$
$2a \geqslant A+p-u$
$2b \geqslant A+p-u$
$4c \leqslant A+4r+2a+u-2b-p$
$4d \leqslant A+4t+2a+u-2b-p$
$q \leqslant a+r-b$
$s \leqslant a+t-b$
$2t \leqslant A+2r+p-u$
$2r \leqslant A+2t+p-u$
$2b \leqslant A+2a+u-p$
$2a \leqslant A+2b+u-p$

H4

$$x_1 = (A+2b+2q-2a-2t+p-u)/4$$
$$x_2 = (A+2a+2t-2b-2q+p-u)/4$$
$$x_3 = (A+2a-2b+u-p)/4$$
$$x_4 = (A+2b-2a+u-p)/4$$

$2a \leqslant A+3p+u-2b$
$2a \geqslant A+2t+2b-2q-u+p$

$4q \geqslant A+2a-2b+u-p$
$4t \geqslant A+2b-2a+u-p$
$2a \geqslant A+p-u$
$2b \geqslant A+p-u$
$4c \leqslant 4q+A-2a+2b+u-p$
$4d \leqslant 4t+A+2a-2b+u-p$
$q > a+r-b$
$s \leqslant a+t-b$
$2b \geqslant -A+2a+2t-2q+u-p$
$2a \geqslant -A+2b+2q-2t+u-p$
$2a \leqslant A-p+u+2b$
$2b \leqslant A+2a+u-p$

H5

$x_1 = (c+p-u)/2$
$x_2 = (A-c)/2$
$x_3 = (A+2a+u-2b-p)/4$
$x_4 = (A+2b+u-2a-p)/4$

$2a \leqslant A+3p+u-2b$
$2a \geqslant A-2b+2c+p-u$

$A \geqslant c$
$2a \geqslant A+p-u$
$2b \geqslant A+p-u$
$2c \geqslant A-p+u$
$2d \leqslant A+u-p$
$4q \leqslant -A+4c+2a+p-u-2b$
$4r \leqslant -A+4c+2b+p-u-2a$
$4s \leqslant A+2a+u-p-2b$
$4t \leqslant A+2b+u-p-2a$
$2a \leqslant A+u-p+2b$
$2b \leqslant A+u-p+2a$

H6

$x_1 = x_2 = (A+p-u)/4$
$x_3 = (A+2a+u-p-2b)/4$
$x_4 = (A+2b+u-p-2a)/4$

$2a \leqslant A+3p+u-2b$
$a \geqslant A-b$

$A \geqslant u-p$
$2a \geqslant A+p-u$
$2b \geqslant A+p-u$
$2c \leqslant A+u-p$
$2d \leqslant A+u-p$
$4r \leqslant A+2b+u-2a-p$
$4t \leqslant A+2b+u-2a-p$
$4q \leqslant A+2a+u-2b-p$
$4s \leqslant A+2a+u-2b-p$
$2b \leqslant A+u-p+2a$
$2a \leqslant A+u-p+2b$

Block I

I1

$x_1 = x_2 = (a+b)/4$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$3b \geqslant 2A-a$

$A \geqslant a$
$2c \leqslant 2A-a-b$
$2d \leqslant 2A-a-b$
$2p \leqslant -A+a+2b$
$2q \leqslant A-b$
$2s \leqslant A-b$
$2t \leqslant A-a$
$2r \leqslant A-a$
$2u \leqslant A-a$

I2

$x_1 = (-A+a+b+c)/2$
$x_2 = (A-c)/2$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$b \geqslant c$

$A \geqslant a$
$2c \geqslant 2A-a-b$
$2d \leqslant 2A-a-b$
$2p \leqslant -A+2b+a$
$2q \leqslant -A+2c+a$
$2r \leqslant -A+2c+b$
$2s \leqslant A-b$
$2t \leqslant A-a$
$2u \leqslant A-a$

I3

$x_1 = (a+b+2r-2t)/4$
$x_2 = (a+b+2t-2r)/4$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$2r \leqslant 3b-a-2t$

$A \geqslant a$
$2t \geqslant A-a$
$2c \leqslant A+2r-b$
$2d \leqslant A+2t-b$
$2p \leqslant -A+2b+a$
$2q \leqslant 2r+a-b$
$2s \leqslant 2t+a-b$
$2u \leqslant A-a$
$2r \geqslant A-a$

I4

$x_1 = (a+b+2c-2d)/4$
$x_2 = (a+b+2d-2c)/4$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$2c \leqslant 2A+b-2d-a$

$A \geqslant a$
$2c \geqslant 2A-a-b$
$2d \geqslant 2A-a-b$
$2p \leqslant -A+2b+a$
$2q \leqslant -A+2c+a$
$2r \leqslant -A+2c+b$
$2s \leqslant -A+2d+a$
$2t \leqslant -A+2d+b$
$2u \leqslant A-a$

I5

$x_1 = (-A+a+2b+2c-2t)/4$
$x_2 = (A+a-2c+2t)/4$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$2t \leqslant A-a+2b-2c$

$A \geqslant a$
$2t \geqslant A-a$
$2c \geqslant 2A-a-b$
$2d \leqslant A-b+2t$
$2p \leqslant -A+a+2b$
$2q \leqslant -A+a+2c$
$2r \leqslant -A+b+2c$
$2s \leqslant a-b+2t$
$2u \leqslant A-a$

I6

$x_1 = (-A+2a+2c+b-2s)/4$
$x_2 = (A+2s+b-2c)/4$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$2s \leqslant A+b-2c$

$A \geqslant a$
$2c \geqslant 2A-a-b$
$2s \geqslant A-b$
$2d \leqslant A-a+2s$
$2p \leqslant -A+a+2b$
$2q \leqslant -A+2c+a$
$2r \leqslant -A+2c+b$
$2t \leqslant 2s+b-a$
$2u \leqslant A-a$

I7

$x_1 = (-A+2a+b+2r)/4$
$x_2 = (A+b-2r)/4$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$3b \geqslant A+2r$

$A \geqslant a$
$2r \geqslant A-a$
$2c \leqslant A+2r-b$
$2d \leqslant 2A-a-b$
$2p \leqslant -A+2b+a$
$2q \leqslant 2r+a-b$
$2s \leqslant A-b$
$2u \leqslant A-a$
$2t \leqslant A-a$

I8

$x_1 = (-A+2b+a+2q)/4$
$x_2 = (A+a-2q)/4$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$2q \geqslant A-a-2b$

$A \geqslant a$
$2q \geqslant A-a$
$2c \leqslant A+2q-a$
$2d \leqslant 2A-a-b$
$2p \leqslant -A+2b+a$
$2r \leqslant 2q+b-a$
$2s \leqslant A-b$
$2t \leqslant A-a$
$2u \leqslant A-a$

I9

$x_1 = (b+q-t)/2$
$x_2 = b+t-q)/2$
$x_3 = (A-b)/2$
$x_4 = (A-a)/2$

$a \geqslant b$
$b \geqslant q+t$

$A \geqslant a$
$2q \geqslant A-b$
$2t \geqslant A-a$
$2c \leqslant A+2q-a$
$2d \leqslant A+2t-b$
$2p \leqslant -A+a+2b$
$2r \leqslant b-a+2q$
$2s \leqslant a-b+2t$
$2u \leqslant A-a$

<u>Block K</u>

K1

$x_1 = (3A+2p-a-4t)/8$
$x_2 = (-A+3a+2p+4t)/8$
$x_3 = (A+a-2p)/4$
$x_4 = (A-a)/2$

$3a \geqslant A+2p$
$3a \leqslant A+6p-4t$

$A \geqslant a$
$2t \geqslant A-a$
$2b \leqslant A+2p-a$
$4c \leqslant 3A-a-2p$
$4d \leqslant A+a-2p+4t$
$4q \leqslant A+a-2p$
$2r \leqslant A-a$
$4s \leqslant -A+3a+4t-2p$
$2u \leqslant A-a$

K2

$x_1 = (p+q)/2$
$x_2 = (A+a-2q)/4$
$x_3 = (A+a-2p)/4$
$x_4 = (A-a)/2$

$3a \geqslant A+2p$
$q \leqslant p$

$A \geqslant a$
$4q \geqslant A+a-2p$
$2b \leqslant A+2p-a$
$2c \leqslant A+2q-a$
$4d \leqslant 3A-2p-a$
$4r \leqslant A+4q-3a+2p$
$4s \leqslant A+a-2p$
$2t \leqslant A-a$
$2u \leqslant A-a$

K3

$x_1 = x_2 = (A+a+2p)/8$
$x_3 = (A+a-2p)/4$
$x_4 = (A-a)/2$

$3a \geqslant A+2p$
$6p \geqslant A+a$

$A \geqslant a$
$2b \leqslant A+2p-a$
$4c \leqslant 3A-2p-a$
$4d \leqslant 3A-2p-a$
$4q \leqslant A+a-2p$
$2r \leqslant A-a$
$2t \leqslant A-a$
$4s \leqslant A+a-2p$
$2u \leqslant A-a$

K4

$x_1 = (A+4c+a+2p-4d)/8$
$x_2 = (A+4d+a+2p-4c)/8$
$x_3 = (A+a-2p)/4$
$x_4 = (A-a)/2$

$3a \geqslant A+2p$
$4c \leqslant 5A+2p-4d-3a$

$A \geqslant a$
$2p \geqslant 3A-a-4c$
$2p \geqslant 3A-a-4d$
$2b \leqslant A+2p-a$
$2q \leqslant -A+2c+a$
$4r \leqslant -A+4c+2p-a$
$2s \leqslant -A+2d+a$
$4t \leqslant -A+4d+2p-a$
$2u \leqslant A-a$

K5

$x_1 = (A+a+2p+4q-4s)/8$
$x_2 = (A+a+2p+4s-4q)/8$
$x_3 = (A+a-2p)/4$
$x_4 = (A-a)/2$

$3a \geqslant A+2p$
$4q \leqslant A+a+2p-4s$

$A \geqslant a$
$4q \geqslant A+a-2p$
$4s \geqslant A+a-2p$
$2b \leqslant A+2p-a$
$2c \leqslant A+2q-a$
$2d \leqslant A+2s-a$
$4r \leqslant A+4q+2p-3a$
$4t \leqslant A+4s+2p-3a$
$2u \leqslant A-a$

K6
$$x_1 = (-A+3a+4c+2p-4s)/8$$
$$x_2 = (3A+4s+2p-a-4c)/8$$
$$x_3 = (A+a-2p)/4$$
$$x_4 = (A-a)/2$$

$$3a \geqslant A+2p$$
$$4s \leqslant 3A+2p-a-4c$$

$$A \geqslant a$$
$$4c \geqslant 3A-a-2p$$
$$4s \geqslant A+a-2p$$
$$2b \leqslant A+2p-a$$
$$2d \leqslant A+2s-a$$
$$2q \leqslant -A+2c+a$$
$$4r \leqslant -A+4c+2p-a$$
$$4t \leqslant A+2p+4s-3a$$
$$2u \leqslant A-a$$

K7
$$x_1 = (A+a+2p+4r-4t)/8$$
$$x_2 = (A+a+2p+4t-4r)/8$$
$$x_3 = (A+a-2p)/4$$
$$x_4 = (A-a)/2$$

$$3a \geqslant A+2p$$
$$4r \leqslant 3A-5a+6p-4t$$

$$A \geqslant a$$
$$2t \geqslant A-a$$
$$2r \geqslant A-a$$
$$2b \leqslant A+2p-a$$
$$4c \leqslant A+a-2p+4r$$
$$4d \leqslant A+a-2p+4t$$
$$4q \leqslant -A+3a-2p+4r$$
$$4s \leqslant -A+3a-2p+4t$$
$$2u \leqslant A-a$$

K8
$$x_1 = (c+p-t)/2$$
$$x_2 = (A+a+2t-2c)/4$$
$$x_3 = (A+a-2p)/4$$
$$x_4 = (A-a)/2$$

$$3a \geqslant A+2p$$
$$t \leqslant A+p-a-c$$

$$A \geqslant a$$
$$4c \geqslant 3A-a-2p$$
$$2t \geqslant A-a$$
$$2b \leqslant A+2p-a$$
$$4d \leqslant A+a+4t-2p$$
$$2q \leqslant -A+2c+a$$
$$4r \leqslant -A+4c+2p-a$$
$$4s \leqslant -A+3a+4t-2p$$
$$2u \leqslant A-a$$

K9
$$x_1 = (-A+a+2p+2c)/4$$
$$x_2 = (A-c)/2$$
$$x_3 = (A+a-2p)/4$$
$$x_4 = (A-a)/2$$

$$3a \geqslant A+2p$$
$$2p \geqslant -A+a+2c$$

$$A \geqslant a$$
$$2p \geqslant 3A-a-4c$$
$$2b \leqslant A+2p-a$$
$$4d \leqslant 3A-a-2p$$
$$2q \leqslant -A+a+2c$$
$$4r \leqslant -A-a+2p+4c$$
$$4s \leqslant A+a-2p$$
$$2t \leqslant A-a$$
$$2u \leqslant A-a$$

K10
$$x_1 = (A-a+2p+2q-2t)/4$$
$$x_2 = (a+t-q)/2$$
$$x_3 = (A+a-2p)/4$$
$$x_4 = (A-a)/2$$

$$3a \geqslant A+2p$$
$$2q \leqslant A-a+2p-2t$$

$$A \geqslant a$$
$$4q \geqslant A+a-2p$$
$$2t \geqslant A-a$$
$$2b \leqslant A+2p-a$$
$$2c \leqslant A+2q-a$$
$$4d \leqslant A+a+4t-2p$$
$$4r \leqslant A-3a+2p+4q$$
$$4s \leqslant -A+3a-2p+4t$$
$$2u \leqslant A-a$$

Block L

L1
$x_1 = (b+c-u-t)/2$
$x_2 = (A+a+2t-2c)/4$
$x_3 = (A+a+2u-2b)/4$
$x_4 = (A-a)/2$

$2u \leqslant A+a-2b$
$u \geqslant c+t-b$

$A \geqslant a$
$2u \geqslant A-a$
$2t \geqslant A-a$
$4c \geqslant 3A-a+2u-2b$
$4d \leqslant A+a+2u-2b+4t$
$2p \leqslant -A+a+2b$
$2q \leqslant -A+2c+a$
$4r \leqslant -A-a-2u+2b+4c$
$4s \leqslant -A+3a+2u-2b+4t$

L2
$x_1 = (-A+3a+2b-2u-4s+4c)/8$
$x_2 = (3A-a+2b-2u+4s-4c)/8$
$x_3 = (A+a+2u-2b)/4$
$x_4 = (A-a)/2$

$2u \leqslant A+a-2b$
$4c \leqslant -A+3a+2b+6u-4s$

$A \geqslant a$
$2u \geqslant A-a$
$4s \geqslant A+a+2u-2b$
$4c \geqslant 3A-a-2b+2u$
$2d \leqslant A-a+2s$
$2p \leqslant -A+a+2b$
$2q \leqslant -A+a+2c$
$4r \leqslant -A-a-2u+2b+4c$
$4t \leqslant A-3a-2u+2b+4s$

L3
$x_1 = (-A+3a+2b+4r-2u)/8$
$x_2 = (3A+2b-a-2u-4r)/8$
$x_3 = (A+a+2u-2b)/4$
$x_4 = (A-a)/2$

$2u \leqslant A+a-2b$
$4r \leqslant -3A+a+6b+2u$

$A \geqslant a$
$2u \geqslant A-a$
$2r \geqslant A-a$
$4c \leqslant A+a+2u+4r-2b$
$4d \leqslant 3A+2u-a-2b$
$2p \leqslant -A+2b+a$
$4q \leqslant -A+3a+4r+2u-2b$
$4s \leqslant A+a+2u-2b$
$2t \leqslant A-a$

L4
$x_1 = (A+a+2b-2u+4r-4t)/8$
$x_2 = (A+a+2b-2u+4t-4r)/8$
$x_3 = (A+a+2u-2b)/4$
$x_4 = (A-a)/2$

$2u \leqslant A+a-2b$
$4r \leqslant -A-a+2u+6b-4t$

$A \geqslant a$
$2u \geqslant A-a$
$2r \geqslant A-a$
$2t \geqslant A-a$
$4c \leqslant A+a+2u-2b+4r$
$4d \leqslant A+a+2u-2b+4t$
$2p \leqslant -A+a+2b$
$4q \leqslant -A+3a+2u-2b+4r$
$4s \leqslant -A+3a+2u-2b+4t$

L5
$x_1 = (A+a+2b+4c-2u-4d)/8$
$x_2 = (A+a+2b+4d-2u-4c)/8$
$x_3 = (A+a+2u-2b)/4$
$x_4 = (A-a)/2$

$2u \leqslant A+a-2b$
$4d \leqslant A+2b+6u+a-4c$

$A \geqslant a$
$2u \leqslant -3A+a+2b+4c$
$2u \leqslant -3A+a+2b+4d$
$2u > A-a$
$2p \leqslant -A+2b+a$
$2q \leqslant -A+2c+a$
$4r \leqslant -A+4c+2b-2u-a$
$2s \leqslant -A+a+2d$
$4t \leqslant -A-a+2b-2u+4d$

L6
$x_1 = (-A+a+2b+2c-2u)/4$
$x_2 = (A-c)/2$
$x_3 = (A+a+2u-2b)/4$
$x_4 = (A-a)/2$

$2u \leqslant A+a-2b$
$2c \leqslant -A+a+2b+2u$

$A \geqslant a$
$2u \geqslant A-a$
$4c \geqslant 3A-a-2b+2u$
$4d \leqslant 3A-a-2b+2u$
$2p \leqslant -A+a+2b$
$2q \leqslant -A+a+2c$
$4r \leqslant -A-a-2u+2b+4c$
$4s \leqslant A+a+2u-2b$
$2t \leqslant A-a$

L7
$$x_1 = (b+q-u)/2$$
$$x_2 = (A+a-2q)/4$$
$$x_3 = (A+a+2u-2b)/4$$
$$x_4 = (A-a)/2$$

$2u \leqslant A+a-2b$
$b \geqslant A+q-a-u$

$A \geqslant a$
$2u \geqslant A-a$
$4q \geqslant A+a+2u-2b$
$2c \leqslant A-a+2q$
$4d \leqslant 3A-a-2b+2u$
$2p \leqslant -A+2b+a$
$4r \leqslant A+2b+4q-3a-2u$
$4s \leqslant A+a+2u-2b$
$2t \leqslant A-a$

L8
$$x_1 = (A-a-2u+2b+2q-2t)/4$$
$$x_2 = (a+t-q)/2$$
$$x_3 = (A+a+2u-2b)/4$$
$$x_4 = (A-a)/2$$

$2u \leqslant A+a-2b$
$2t \leqslant -A+a+2u+2b-2q$

$A \geqslant a$
$2u \geqslant A-a$
$2t \geqslant A-a$
$4q \geqslant A+a+2u-2b$
$2c \leqslant A-a+2q$
$4d \leqslant A+a+2u-2b+4t$
$4r \leqslant A-3a-2u+2b+4q$
$4s \leqslant -A+3a+2u-2b+4t$
$2p \leqslant -A+a+2b$

L9
$$x_1 = (A+a+2b-2u+4q-4s)/8$$
$$x_2 = (A+a+2b-2u+4s-4q)/8$$
$$x_3 = (A+a+2u-2b)/4$$
$$x_4 = (A-a)/2$$

$2u \leqslant A+a-2b$
$4q \leqslant -3A+5a+2b+6u-4s$

$A \geqslant a$
$2u \geqslant A-a$
$4s \geqslant A+a+2u-2b$
$4q \geqslant A+a+2u-2b$
$2c \leqslant A-a+2q$
$2d \leqslant A-a+2s$
$2p \leqslant -A+a+2b$
$4r \leqslant A-3a+2b-2u+4q$
$4t \leqslant A-3a+2b-2u+4s$

L10
$$x_1 = x_2 = (A+a+2b-2u)/8$$
$$x_3 = (A+a+2u-2b)/4$$
$$x_4 = (A-a)/2$$

$2u \leqslant A+a-2b$
$6b \geqslant 5A-3a-2u$

$A \geqslant a$
$2u \geqslant A-a$
$4c \leqslant 3A-a+2u-2b$
$4d \leqslant 3A-a+2u-2b$
$2p \leqslant -A+a+2b$
$4q \leqslant A+a+2u-2b$
$2r \leqslant A-a$
$4s \leqslant A+a+2u-2b$
$2t \leqslant A-a$

Block N

N1
$$x_1 = (A+2p+2q-a-2u)/4$$
$$x_2 = (A+a-2q)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geqslant p+u$
$2p \geqslant A-a-2u+2q$

$A \geqslant a$
$2u \geqslant A-a$
$2q \geqslant a+u-p$
$2b \leqslant A-a+2p$
$2c \leqslant A-a+2q$
$2d \leqslant A+u-p$
$2r \leqslant A-2a-u+p+2q$
$2s \leqslant a+u-p$
$2t \leqslant A-a$

N2
$$x_1 = (A-a+2p+2c-2u-2t)/4$$
$$x_2 = (A+a+2t-2c)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geqslant p+u$
$2c \leqslant A-a+2u+2p-2t$

$A \geqslant a$
$2u \geqslant A-a$
$2c \geqslant A+u-p$
$2t \geqslant A-a$
$2b \leqslant A-a+2p$
$2d \leqslant a+u-p+2t$
$2q \leqslant -A+a+2c$
$2r \leqslant -a-u+p+2c$
$2s \leqslant -A+2a+u-p+2t$

N3

$$x_1 = (c+p-u)/2$$
$$x_2 = (A-c)/2$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geq p+u$
$c \leq p+u$

$A \geq a$
$2u \geq A-a$
$2c \geq A+u-p$
$2b \leq A-a+2p$
$2d \leq A+u-p$
$2q \leq -A+a+2c$
$2r \leq -a-u+p+2c$
$2s \leq a+u-p$
$2t < A-a$

N4

$$x_1 = (A+p-u+2c-2d)/4$$
$$x_2 = (A+p-u+2d-2c)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geq p+u$
$2c \leq A+p+3u-2d$

$2u \geq A-a; \; A \geq a$
$2d \geq A+u-p$
$2c \geq A+u-p$
$2b \leq A-a+2p$
$2q \leq -A+a+2c$
$2r \leq -a-u+p+2c$
$2s \leq -A+a+2d$
$2t \leq -a-u+p+2d$

N5

$$x_1 = (A-a+p+q-u-t)/2$$
$$x_2 = (a+t-q)/2$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geq p+u$
$p \geq q+t-u$

$A \geq a$
$2u \geq A-a$
$2q \geq a+u-p$
$2t \geq A-a$
$2b \leq A-a+2p$
$2c \leq A-a+2q$
$2d \leq a+u-p+2t$
$2r \leq A-2a+p-u+2q$
$2s \leq -A+2a+u-p+2t$

N6

$$x_1 = (A+p-u+2q-2s)/4$$
$$x_2 = (A+p-u+2s-2q)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geq p+u$
$2q \leq -A+3u+p+2a-2s$

$A \geq a$
$2u \geq A-a$
$2s \geq a+u-p$
$2q \geq a+u-p$
$2b \leq A-a+2p$
$2c \leq A-a+2q$
$2d \leq A-a+2s$
$2r \leq A+p-u-2a+2q$
$2t \leq A+p-u-2a+2s$

N7

$$x_1 = (a+p-u+2r)/4$$
$$x_2 = (2A+p-u-a-2r)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geq p+u$
$2r \leq -a+u+3p$

$A \geq a$
$2u \geq A-a$
$2r \geq A-a$
$2b \leq A-a+2p$
$2c \leq a+u-p+2r$
$2d \leq A+u-p$
$2q \leq -A+2a+u-p+2r$
$2s \leq a+u-p$
$2t \leq A-a$

N8

$$x_1 = (A+p-u+2r-2t)/4$$
$$x_2 = (A+p-u+2t-2r)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geq p+u$
$2r \leq A+3p+u-2a-2t$

$A \geq a$
$2u \geq A-a$
$2t \geq A-a$
$2r \geq A-a$
$2b \leq A-a+2p$
$2c \leq a+u-p+2r$
$2d \leq a+u-p+2t$
$2q \leq -A+2a+u-p+2r$
$2s \leq -A+2a+u-p+2t$

N9

$$x_1 = x_2 = (A+p-u)/4$$
$$x_3 = (a+u-p)/2$$
$$x_4 = (A-a)/2$$

$a \geq p+u$
$3p \geq A-u$

$A \geq a$
$2u \geq A-a$
$2b \leq A-a+2p$
$2c \leq A+u-p$
$2d \leq A+u-p$
$2q \leq a+u-p$
$2r \leq A-a$
$2s \leq a+u-p$
$2t \leq A-a$

Block P

P1

$$x_1 = (A+p+2q-2s-u)/4$$
$$x_2 = (A+p+2s-2q-u)/4$$
$$x_3 = (A+2s+2q+3u-3p)/8$$
$$x_4 = (3A+u-p-2s-2q)/8$$

$$5u \geqslant A-3p+2s+2q$$

$$2s \leqslant 3A+u-p-2q$$
$$3u \leqslant -A+6q+3p-2s$$
$$3u \leqslant -A+6s+3p-2q$$
$$p \leqslant -A+u+2s+2q$$
$$4a \leqslant A+p-u+2s+2q$$
$$2b \leqslant A+p-u$$
$$8c \leqslant 3A+u-p+6q-2s$$
$$8d \leqslant 3A+u-p+6s-2q$$
$$4r \leqslant A+p-u+2q-2s$$
$$4t \leqslant A+p-u+2s-2q$$

P2

$$x_1 = (2c+p-s-u)/3$$
$$x_2 = (3A+p-u+2s-4c)/6$$
$$x_3 = (c+s+u-p)/3$$
$$x_4 = (3A+u-p-2c-2s)/6$$

$$2u \geqslant c+s-p$$

$$2c \leqslant 3A+u-p-2s$$
$$2c \geqslant A+u-p$$
$$2s \geqslant c+u-p$$
$$4c \geqslant 3A+p-u-4s$$
$$3a \leqslant 2c+2s+p-u$$
$$3b \leqslant A+p-u$$
$$6d \leqslant 3A+u-p+4s-2c$$
$$6q \leqslant -3A+p-u+8c+2s$$
$$3r \leqslant 2c+p-u-s$$
$$6t \leqslant 3A+p-u+2s-4c$$

P3

$$x_1 = (A+3p+4c-3u-2s-2t)/8$$
$$x_2 = (3A+2s+2t+p-u-4c)/8$$
$$x_3 = (A+u+2s-2t-p)/4$$
$$x_4 = (A+u+2t-2s-p)/4$$

$$4c \leqslant A+5u+3p-2s-2t$$

$$6s \geqslant 3A-4c+u+2t+p$$
$$6t \geqslant 3A-4c+u+2s+p$$
$$2c \geqslant A+u-p$$
$$2t \geqslant A-2s-p+u$$
$$4d \leqslant A+2s+2t+u-p$$
$$8a \leqslant A+6s+3p+4c-3u-2t$$
$$8b \leqslant A+6t+3p+4c-3u-2s$$
$$4q \leqslant -A+4c+2s+p-u-2t$$
$$4r \leqslant -A+4c+2t+p-u-2s$$
$$4c \leqslant 3A-u+p+2s+2t$$
$$4c > -A+3u+2s+2t-3p$$
$$2t > -A+2s+p-u$$
$$2s > -A+2t+p-u$$

P4

$$x_1 = (A+3p+2q-3u)/6$$
$$x_2 = x_4 = (A-q)/3$$
$$x_3 = (A+3u+2q-3p)/6$$

$$3p \geqslant A+2q-3u$$

$$A \geqslant q$$
$$3u \leqslant -A+4q+3p$$
$$3p \leqslant -A+4q+3u$$
$$3a \leqslant A+2q$$
$$3c \leqslant A+2q$$
$$2b \leqslant A+p-u$$
$$2d \leqslant A+u-p$$
$$6r \leqslant A+3p+2q-3u$$
$$6s \leqslant A+3u+2q-3p$$
$$3t \leqslant A-q$$

P5 $x_1 = (A+4c+3p-2a-3u-2t)/6$
$x_2 = (A+a+t-2c)/3$
$x_3 = (A+4a+3u-2c-2t-3p)/6$
$x_4 = (2A+2t+2c-4a)/6$

$3p \geqslant -A+2a+2c+2t-3u$

$2c \geqslant A+u-p$
$2a \geqslant A+p-u$
$2t \geqslant A+c-2a$
$2t \geqslant A+a-2c$
$6b \leqslant A+4c+4t+3p-2a-3u$
$6d \leqslant A+4a+4t+3u-2c-3p$
$3q \leqslant -A+2a+2c-t$
$6r \leqslant -A+8c-4a-3u+3p+2t$
$6s \leqslant -A+8a-4c-3p+3u+2t$
$3u \leqslant A+4c+3p-2a-2t$
$3p \leqslant A+4a+3u-2c-2t$
$2c \leqslant A+a+t$
$2a \leqslant A+c+t$

<u>Block Q</u>

Q1 $x_1 = (A+a)/6$
$x_2 = x_3 = x_1$
$x_4 = (A-a)/2$

$2a \geqslant A$

$A \geqslant a$
$3b \leqslant 2A-a$
$3c \leqslant 2A-a$
$3d \leqslant 2A-a$
$6p \leqslant A+a$
$6q \leqslant A+a$
$6s \leqslant A+a$
$2r \leqslant A-a$
$2t \leqslant A-a$
$2u \leqslant A-a$

Q2 $x_1 = (a+2r-t)/3$
$x_2 = (a+2t-r)/3$
$x_3 = (3A-a-2r-2t)/6$
$x_4 = (A-a)/2$

$a \geqslant t+r$

$A \geqslant a$
$2t \geqslant A-a$
$2r \geqslant A-a$
$3b \leqslant a+2r+2t$
$6c \leqslant 3A+4r-a-2t$
$6d \leqslant 3A+4t-a-2r$
$6p \leqslant -3A+5a+4r+4t$
$3q \leqslant a+2r-t$
$3s \leqslant a+2t-r$
$2u \leqslant A-a$

Q3 $x_1 = (a+b+q-2s)/3$
$x_2 = (a+b+s-2q)/3$
$x_3 = 3A+2q+2s-a-4b)/6$
$x_4 = (A-a)/2$

$q \leqslant 2a-b-s$

$A \geqslant a$
$2q \geqslant a+b-2s$
$4s \geqslant 3A+2q-a-4b$
$4q \geqslant 3A+2s-a-4b$
$2c \leqslant A-a+2q$
$2d \leqslant A-a+2s$
$2p \leqslant -A+a+2b$
$3r \leqslant 2b+2q-a-s$
$3t \leqslant 2b+2s-a-q$
$6u \leqslant 3A+4s+4q-5a-2b$

Q4
$$x_1 = (A+a+4r-2t-2u)/6$$
$$x_2 = (A+a+4t-2r-2u)/6$$
$$x_3 = (A+a+4u-2r-2t)/6$$
$$x_4 = (A-a)/2$$

$$2u \leqslant A+a-2r-2t$$

$$A \geqslant a$$
$$2r \geqslant A-a$$
$$2t \geqslant A-a$$
$$2u \geqslant A-a$$
$$6b \leqslant A+a+4r+4t-2u$$
$$6c \leqslant A+a+4r+4u-2t$$
$$6d \leqslant A+a+4t+4u-2r$$
$$3p \leqslant -A+2a+2t+2r-u$$
$$3q \leqslant -A+2a+2u+2r-t$$
$$3s \leqslant -A+2a+2u+2t-r$$

Q5
$$x_1 = (A+a+2b+2c-4d)/6$$
$$x_2 = (A+a+2d+2b-4c)/6$$
$$x_3 = (A+a+2d+2c-4b)/6$$
$$x_4 = (A-a)/2$$

$$b \leqslant A+a-c-d$$

$$A \geqslant a$$
$$b \leqslant -2A+a+2c+2d$$
$$c \leqslant -2A+a+2b+2d$$
$$d \leqslant -2A+a+2b+2c$$
$$2b > A-a+2p$$
$$2c > A-a+2q$$
$$6r \leqslant -A+4b+4c-a-2d$$
$$2s \leqslant -A+a+2d$$
$$6t \leqslant -A+4b+4d-a-2c$$
$$6u \leqslant -A+4c+4d-a-2b$$

Q6
$$x_1 = (A+a+2p+2q-4s)/6$$
$$x_2 = (A+a+2p+2s-4q)/6$$
$$x_3 = (A+a+2q+2s-4p)/6$$
$$x_4 = (A-a)/2$$

$$2p \leqslant -A+5a-2q-2s$$

$$A \geqslant a$$
$$4q \geqslant A+a+2p-4s$$
$$4s \geqslant A+a+2q-4p$$
$$4q \geqslant A+a+2s-4p$$
$$2b \leqslant A-a+2p$$
$$2c \leqslant A-a+2q$$
$$2d \leqslant A-a+2s$$
$$3r \leqslant A+2p+2q-2a-s$$
$$3t \leqslant A+2p+2s-2a-q$$
$$3u \leqslant A+2q+2s-2a-p$$

Q7
$$x_1 = (-A+3a+4r)/6$$
$$x_2 = x_3 = (A-r)/3$$
$$x_4 = (A-a)/2$$

$$3a \geqslant A+2r$$

$$A \geqslant a$$
$$2r \geqslant A-a$$
$$3c \leqslant A+2r$$
$$3b \leqslant A+2r$$
$$6d \leqslant 5A-3a-2r$$
$$6q \leqslant -A+3a+4r$$
$$6p \leqslant -A+3a+4r$$
$$2u \leqslant A-a$$
$$2t \leqslant A-a$$
$$3s \leqslant A-r$$

Q8
$$x_1 = (-A+2b+2c+3a-4s)/6$$
$$x_2 = (A+b+s-2c)/3$$
$$x_3 = (A+c+s-2b)/3$$
$$x_4 = (A-a)/2$$

$$3a \geqslant -A+2c+2b+2s$$

$$A \geqslant a$$
$$4c \geqslant 5A-3a-4b+2s$$
$$2s \geqslant A+b-2c$$
$$2s \geqslant A+c-2b$$
$$2d \leqslant A-a+3s$$
$$3r \leqslant -A+2c+2b-s$$
$$2q \leqslant -A+2c+a$$
$$2p \leqslant -A+2b+a$$
$$6u \leqslant A+4c+4s-2b-3a$$
$$6t \leqslant A+4b+4s-2c-3a$$

4. Normalized coalition arrays

Here we describe all minimal balanced sets for the case N = {1,2,3,4} and all normalized arrays up to permutations of the players. We have only considered those arrays which are necessary for the computation of the nucleolus for superadditive games (103 arrays). Note that for each array we only have written down the corresponding critical array parts in the sequence B_o, B_1, B_2,... (see section 1).

4.1 Minimal balanced sets

{12,34}; {123,4}; {12,3,4}; {123,124,34}; {1,2,3,4};
{12,13,23,4}; {123,14,24,3}; {123,124,134,234}; {123,14,24,34} .

4.2 Normalized coalition arrays

Block B

B1 ∅
{1,2,3,4}

B2 ∅
{123,124,134,234}

B3 ∅
{123,14,24,34}

B4 ∅
{123,24,34,1}

B5 ∅
{23,24,34,1}

B6 {1}
{123,24,34}

B7 {1}
{23,24,34}

Block C

C1 ∅
{123,124,34}
{13,24}

C2 ∅
{123,124,34}
{134,2}

C3 ∅
{123,124,34}
{1,2}

C4 ∅
{123,124,34}
{14,24}

C5 ∅
{123,124,34}
{24,1}

C6 ∅
{123,124,34}
{134,234}

C7 ∅
{123,124,34}
{134,24}

C8 {1}
{123,124,34}
{24}

Block D

D1 Ø
{12,3,4}
{1,2}

D2 Ø
{12,3,4}
{134,234}

D3 Ø
{12,3,4}
{234,14}

D4 Ø
{12,3,4}
{14,24}

D5 Ø
{12,3,4}
{14,2}

D6 Ø
{12,3,4}
{14,23}

D7 Ø
{12,3,4}
{134,2}

Block E

E1 Ø
{12,34}
{13,24}
{14,23}

E2 Ø
{12,34}
{13,24}
{1,3}

E3 Ø
{12,34}
{13,24}
{3,4}

E4 Ø
{12,34}
{13,24}
{123,124}

E5 Ø
{12,34}
{13,24}
{123,134}

E6 Ø
{12,34}
{13,24}
{123,1}

E7 Ø
{12,34}
{13,24}
{23,1}

E8 Ø
{12,34}
{13,24}
{123,14}

E9 Ø
{12,34}
{13,24}
{123,4}

Block F

F1 Ø
{12,34}
{123,4}
{1,2}

F2 Ø
{12,34}
{123,4}
{13,24}

F3 Ø
{12,34}
{123,4}
{134,2}

F4 Ø
{12,34}
{123,4}
{134,234}

F5 Ø
{12,34}
{123,4}
{134,24}

F6 Ø
{12,34}
{123,4}
{13,23}

F7 Ø
{12,34}
{123,4}
{14,2}

F8 Ø
{12,34}
{123,4}
{14,24}

F9 Ø
{12,34}
{123,4}
{13,2}

Block G

G1 Ø
{12,34}
{3,4}
{134,234}

G2 Ø
{12,34}
{3,4}
{1,2}

G3 Ø
{12,34}
{3,4}
{14,24}

G4 Ø
{12,34}
{3,4}
{134,24}

G5 Ø
{12,34}
{3,4}
{13,24}

G6 Ø
{12,34}
{3,4}
{1,24}

Block H

H1	H2	H3	H4	H5
∅	∅	∅	∅	∅
{12,34}	{12,34}	{12,34}	{12,34}	{12,34}
{123,124}	{123,124}	{123,124}	{123,124}	{123,124}
{134,234}	{14,2}	{14,24}	{13,24}	{134,2}

H6
∅
{12,34}
{123,124}
{1,2}

Block I

I1	I2	I3	I4	I5
∅	∅	∅	∅	∅
{123,4}	{123,4}	{123,4}	{123,4}	{123,4}
{124,3}	{124,3}	{124,3}	{124,3}	{124,3}
{1,2}	{134,2}	{14,24}	{134,234}	{134,24}

I6	I7	I8	I9
∅	∅	∅	∅
{123,4}	{123,4}	{123,4}	{123,4}
{124,3}	{124,3}	{124,3}	{124,3}
{134,23}	{14,2}	{13,2}	{13,24}

Block K

K1	K2	K3	K4	K5
∅	∅	∅	∅	∅
{123,4}	{123,4}	{123,4}	{123,4}	{123,4}
{12,3}	{12,3}	{12,3}	{12,3}	{12,3}
{24,1}	{13,2}	{1,2}	{134,234}	{13,23}

K6	K7	K8	K9	K10
∅	∅	∅	∅	∅
{123,4}	{123,4}	{123,4}	{123,4}	{123,4}
{12,3}	{12,3}	{12,3}	{12,3}	{12,3}
{134,23}	{14,24}	{134,24}	{134,2}	{13,24}

Block L

L1	L2	L3	L4	L5
∅	∅	∅	∅	∅
{123,4}	{123,4}	{123,4}	{123,4}	{123,4}
{124,34}	{124,34}	{124,34}	{124,34}	{124,34}
{134,24}	{134,23}	{14,2}	{14,24}	{134,234}

L6	L7	L8	L9	L10
∅	∅	∅	∅	∅
{123,4}	{123,4}	{123,4}	{123,4}	{123,4}
{124,34}	{124,34}	{124,34}	{124,34}	{124,34}
{134,2}	{13,2}	{13,24}	{13,23}	{1,2}

Block N

N1	N2	N3	N4	N5
∅	∅	∅	∅	∅
{123,4}	{123,4}	{123,4}	{123,4}	{123,4}
{12,34}	{12,34}	{12,34}	{12,34}	{12,34}
{13,2}	{134,24}	{134,2}	{134,234}	{13,24}

N6	N7	N8	N9
Ø	Ø	Ø	Ø
{123,4}	{123,4}	{123,4}	{123,4}
{12,34}	{12,34}	{12,34}	{12,34}
{13,23}	{14,2}	{14,24}	{1,2}

Block P

P1	P2	P3	P4	P5
Ø	Ø	Ø	Ø	Ø
{12,34}	{12,34}	{12,34}	{12,34}	{12,34}
{13,23,4}	{134,23,4}	{134,23,24}	{13,2,4}	{123,134,24}

Block Q

Q1	Q2	Q3	Q4	Q5
Ø	Ø	Ø	Ø	Ø
{123,4}	{123,4}	{123,4}	{123,4}	{123,4}
{1,2,3}	{14,24,3}	{124,13,23}	{14,24,34}	{124,134,234}

Q6	Q7	Q8
Ø	Ø	Ø
{123,4}	{123,4}	{123,4}
{12,13,23}	{14,2,3}	{124,134,23}

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