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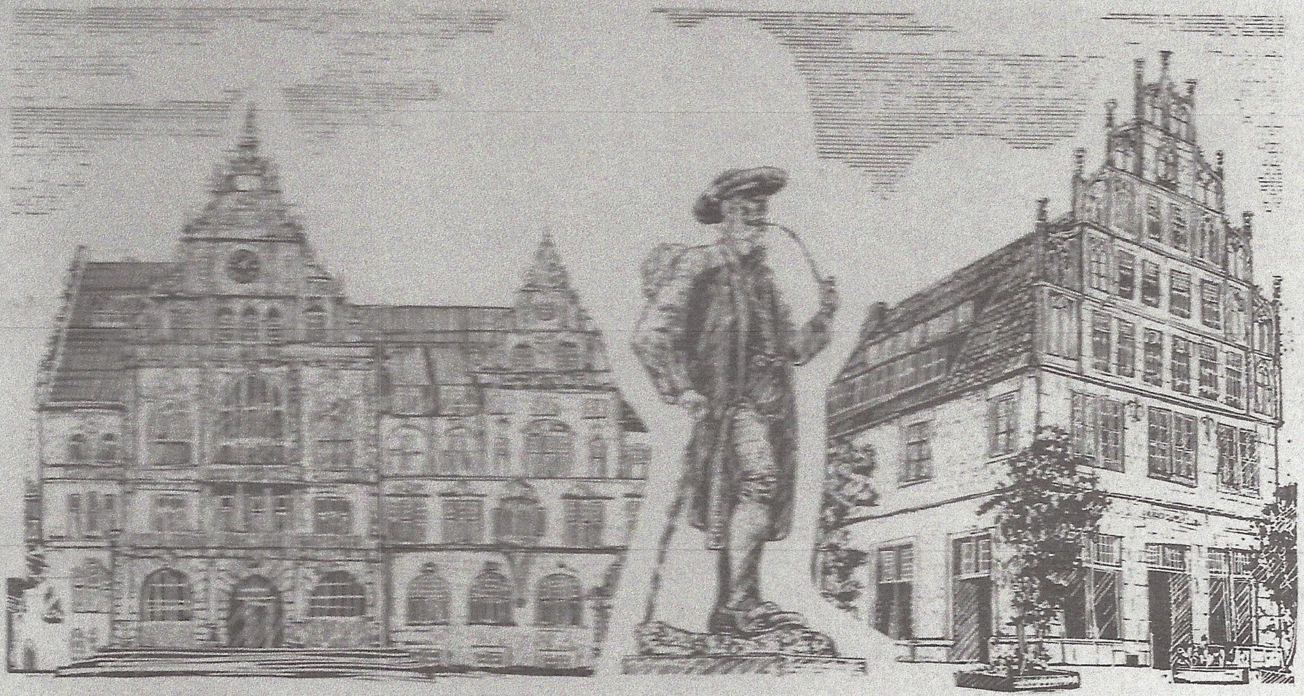

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ON A PROBLEM OF OPTIMAL SEARCH

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Abstract

A class of one-dimensional search problems is considered. The formulation results in a functional-minimization equation of the dynamic programming type. In a special case the optimal solution for both the objective function and the search procedure is found.

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1. General Outline of Search Problems

Suppose that an object moves around within a finite number of regions (cells) at each time instant according to known probabilistic laws.

A searcher adopting a search rule checks sequentially one region at a time, until the object is found. At his disposal is an 'effort budget' which tells him when to stop searching, i.e. when the costs of searching exceed the prospective gains of searching. Various optimality conditions for search rules may be invoked: they may vary between those that maximize the probability of detecting the target with a given effort and those that minimize the expected search costs needed to find the target. Alternatively, one may consider minimizing the expected number of periods to find the object or, if appropriate, minimizing total expected losses until the object is found.

Such optimization problems are of eminent interest in operations research, see J.M. Dobbie (1968), or S.M. Pollock (1971), most of the models studied so far have been Markovian-type search models.

Also in economic decision problems one is interested in comparing various decision rules with regard to the costs involved by implementing such rules. Here rather than facing the impossibility of choosing an optimal strategy 'rules of thumb' may constitute satisfactory strategies constituting a reaction to the complexity of decision rules (see H.A. Simon (1972) and C. Futia (1975)).

Search and complexity appear to be intimately related - as we encounter choice problems invoking a combinatorial structure (such as chess). If we are in a very large market the optimal search consists of finding the minimal price offered by one agent for one unit of commodity, however, with every search costs are involved and finding the minimal price after many searches need not be optimal since search costs may be prohibitive. Here the proper optimal search procedure would consist of minimizing the expected cost of search for a minimal price (among all alternative searches), or, alternatively, of maximizing the probability of finding the agent with a given minimal price and within a fixed number of searches (constituting the cost budget). Both optimization procedures are also genuine for statistical search problems (see de Groot (1970)). The standard search problem, treated statistically, is as follows: Suppose that an object is hidden in one of r possible locations ($r \geq 2$) and let p_i be the prior probability that the object is in location i. The statistician must find the object but he can search in one location at a time. Hence he must devise a sequential search procedure which specifies at each stage that a certain one of the r locations is to be searched.

An important subclass of such problems, restricted to one-dimensional search will be considered in this paper. It involves a functional minimization technique of the dynamic programming type.

2. The Problem and the Optimization Equation

We consider a storage unit of an information system consisting of N cells, with information stored in tabular form. That is, the record $r(i)$ stored in cell i is in the form of a pair $[x_i, f(x_i)]$, the file being arranged in ascending order of the argument x_i. An example of such an arrangement is a dictionary.

Given a particular argument x, we find $f(x)$ by searching for the cell containing $[x, f(x)]$. The search proceeds by comparing x against the arguments in a sequence of cells $i_1, i_2, \ldots$ This sequence is to be chosen so as to minimize the average number of comparisons required for locating the correct cell, in other words, for a given search cost associated to each comparison we want to minimize the expected cost of search over the sequence under consideration.

We begin with the following assumptions.

(i) In a comparison of x against x_i, only three possible outcomes exist, namely,

$$x > x_i, \qquad x < x_i, \qquad x = x_i.$$

(ii) Let X be an integer-valued random variable denoting the location of x. We assume that the prior probabilities $p_k = \text{Prob}[X = k]$ are given, with

$$(1) \qquad \sum_{k=1}^{N} p_k = 1$$

(iii) Let S be the set of integers 1 through N, and let S' be a non-empty subset of S. We assume that the posterior probability distribution of X is unchanged except for renormalization; i.e.

$$(2) \qquad \text{Prob}[X = k \mid X \in S'] = \frac{p_k}{P(S')}, \quad k \in S'$$
$$= 0, \quad k \notin S'$$

where $P(S') = \sum_{i \in S'} p_i$.

Let $T[(p_k), N]$ formally denote the minimum average number of comparisons per successful search, given N cells and prior distribution (p_k). It is clear that the search procedure starts with the selection of a cell for the first comparison. Suppose cell n is selected and x is compared with x_n. The following situation then results:

(a) With probability p_n, $x = x_n$ and the search terminates.

(b) With probability $P_{n-1} = \sum_{i=1}^{n-1} p_i$, $x < x_n$ and x must be contained in the first n-1 cells. If we renumber the first n-1 cells backwards starting with cell n-1, the new distribution becomes

$$(3) \quad p_k' = \frac{p_{n-k}}{P_{n-1}}, \quad k = 1,\ldots,n-1.$$

(c) With probability $1-P_n = \sum_{i=n+1}^{n} p_i$, $x > x_n$. Upon renumbering the last N-n cells, we find the new distribution to be

$$(4) \quad p_k'' = \frac{p_{n+k}}{1-P_n}, \quad k = 1,\ldots,N-n.$$

It is clear that whichever cell is optimal for the first choice, succeeding choices must remain optimal for the overall sequence to be optimal. Therefore, $T[\cdot,N]$ must satisfy the following functional equation:

$$(5) \quad T\left[(p_k),N\right] = \min_{1\leq n\leq N} \left\{1 + P_{n-1}\, T\left[\left(\frac{p_{n-k}}{P_{n-1}}\right), n-1\right] + (1-P_{n-1})\, T\left[\left(\frac{p_{n+k}}{1-P_n}\right), N-n\right]\right\}.$$

Equation (5) is in the formalism of dynamic programming, Bellman (1957), yielding as solutions the objective $T[(p_k),N]$ and the optimal policy $n^*[(p_k),N]$. As initial conditions we set $P_o = 0$, $T(\cdot,0) = 0$, and $T(\cdot,1) = 0$. (Note that this last condition implies that if there is only one cell no comparison is necessary. This is a consequence of (1))

3. Optimal Solutions for Uniform Distribution

If $p_k = \frac{1}{N}$, $k=1,2\dots,N$, explicit solution of (5) can be found. In this case, it is clear that $T(.,.)$ and $n^*(.,.)$ are functions of N only. With a slight change in notation we can rewrite (5) as

$$(6) \quad T(N) = 1 + \min_{1 \le n \le N} \left\{ \frac{n-1}{N} T(n-1) + \left(1 - \frac{n}{N}\right) T(N-n) \right\}.$$

The solution $T(N)$ of (6) is given by

$$(7) \begin{cases} (2^{k+1}+2m-1)T(2^{k+1}+2m-1) = 2^{k+1}(k-\frac{1}{2}) + 2mk+3m+1, & k=0,1,2,\dots \\ & m=0,1,\dots,2^k \\ (2^{k+1}+2m)T(2^{k+1}+2m) = 2^{k+1}(k-\frac{1}{2})+(2m+1)k+3(m+1), & m=0,1,\dots 2^k- \end{cases}$$

The policy $n^*(N)$ which yields the minimum is not unique. In fact, the multiplicity of solutions can be quite large. The complete set of solutions is

$$(8) \begin{cases} n^*(2^{k+1}+2m) = 2^k+j, & j = 0,1,\dots,2m+1, \; m < 2^{k-1} \\ & j = 2m-2^k+1,\dots,2^k, \; m \geq 2^{k-1} \\ n^*(2^{k+1}+2m-1) = 2^k+2j, & j = 0,1,\dots,m, \; m \leq 2^{k-1} \\ & j = m-2^{k-1},\dots,2^{k-1}, \; m > 2^{k-1} \end{cases}$$

For example, consider $N = 2^4+9 = 25$.

Then $n^*(N) = 10, 12, 14, 16$.

The policy solution is interesting and somewhat surprising. Intuitively, one would expect that the optimal solution $n^*(N)$ should be such as to divide the remaining $N-1$ cells into nearly equal subsets, i.e., $N-n^* \cong n^*-1$. Thus, the large multiplicity of solution is not expected. Furthermore, in some cases the midpoint is in fact not a solution. For example, for $N = 25$, the point $n = 13$ divides the remaining 22 cells equally, but is not among the solutions.

4. Proof of Optimality

In this section we shall prove that the solutions of (6) are indeed given by (7) and (8). The proof proceeds in three stages. First, it is shown that the right-hand side of (6) is minimized by a specific choice of policy $n^*(N)$. Next, $T(N)$ will be derived. Finally, the multiplicity of the policy solution is found.

A. If we let $f(N) = NT(N)$, (6) is simplified and can be rewritten as

$$(9)\quad f(N) = N + \min_{1 \le n \le N} \left\{ f(n-1) + f(N-n) \right\}$$

We begin by proving the following theorem:

Theorem 1: Under the conditions $f(0) = f(1) = 0$, the minimization in (9) is achieved with $n = n^*(N)$, where for all positive integers m,

$$(10)\quad n^*(4m-2) = n^*(4m-1) = n^*(4m) = n^*(4m+1) = 2m$$

Proof: It is seen that Theorem 1 is equivalent to the following set of equations with m ranging over all positive integers:

$$(11a)\quad f(4m-2) = 4m-2 + f(2m-2) + f(2m-1)$$

$$(11b)\quad f(4m-1) = 4m-1 + f(2m-1) + f(2m-1)$$

$$(11c)\quad f(4m) = 4m + f(2m-1) + f(2m)$$

$$(11d)\quad f(4m+1) = 4m+1 + f(2m-1) + f(2m+1)$$

We proceed by induction. First, by enumerating all possibilities, we find that (11) is true for m=1. Next, we assume (11) to be true for $m = 1, \dots, k$, and prove the following lemma:

Lemma: Equation (11) being valid for $m = 1, 2, \dots, k$, implies

$$(12)\quad f(n+1) - f(n) \geqslant f(n-1) - f(n-2), \quad n = 2, \dots, 4K$$

$$(13)\quad f(n+1) > f(n), \quad n = 1, 2, \dots, 4K$$

$$(14)\quad f(2n) - f(2n-1) > f(2n+1) - f(2n), \quad n = 1, 2, \dots, 2K$$

<u>Proof</u>: If (11) is true for m=1,...,K, then

$$(15)\quad [f(4m+1) - f(4m)] - [f(4m-1) - f(4m-2)]$$
$$= [f(2m+1) + f(2m-2)] = [f(2m) + f(2m-1)], \quad m = 1,2,\ldots,K.$$

Now under the same assumption, [compare (11c) and (9)],

$$(16)\quad f(2m) + f(2m-1) = \min_{1\leq n\leq 4m} \{f(n-1) + f(4m-n)\} \quad 1\leq m\leq K$$

Therefore, it follows that

$$(17)\quad f(2m+1) + f(2m-2) \geq f(2m) + f(2m-1), \quad 1\leq m\leq K$$

and

$$(18)\quad f(4m+1) - f(4m) \geq f(4m-1) - f(4m-2), \quad 1\leq m\leq K$$

Similarly, we find that

$$(19)\quad f(4m-1) - f(4m-2) = f(4m-3) - f(4m-4)$$
$$(20)\quad f(4m) - f(4m-1) \geq f(4m-2) - f(4m-3)$$
$$(21)\quad f(4m-2) - f(4m-3) = f(4m-4) - f(4m-5), \quad m\leq K$$

Relationships (18)-(21) imply (12), and together with the fast that $f(2)-f(1)>0$ and $f(3)-f(2)>0$ imply (13).

Now, if (11) is valid for m=1,...,K, then

$$f(2m) - f(2m-1) > f(2m+1) - f(2m)$$

implies

$$f(4m-2) - f(4m-3) > f(4m-1) - f(4m-2)$$

and

$$f(4m) - f(4m-1) > f(4m+1) - f(4m),$$

für m=1,2,...,K. Therefore, (14) is implied by $f(2) - f(1) > f(3) - f(2)$. This latter is easily verified.

Now, we proceed with the main part of the proof for Theorem 1. First we write $f(4K+2)$ as

$$(22) \quad f(4K+2) = 4K+2+\min_{2\leq n\leq 4K+1}\left\{f(n-1) + f(4K+2-n)\right\}$$

$$= 4K+2+\min\left\{\min_{1\leq n\leq K}\left[f(2n-1) + f(4K+2-2n)\right],\right.$$

$$\left.\min_{1\leq n\leq k}\left[f(2n) + f(4K+1-2n)\right]\right\}$$

By (12) of the lemma, (22) is reduced to

$$f(4K+2) = 4K+2 + \min\left\{\left[f(2K+2) + f(2K-1)\right], \left[f(2K) + f(2K+1)\right]\right\}$$

$$= 4K+2 + f(2K) + f(2K+1),$$

where the last step follows from (12). We note that we have extended (11a) to $m=K+1$, and (12) to $n=4K+1$.

Similarly, by the use of (12) and (13), $f(4K+3)$ can be written

$$(23) \quad f(4K+3) = 4K+3 + \min\left\{2f(2K+1), f(2K) + f(2K+2)\right\}$$

It follows from (14) that

$$f(2K+2) - f(2K+1) \geqslant f(2K+3) - f(2K+2)$$

and it follows from (12) that

$$f(2K+3) - f(2K+2) > f(2K+1) - f(2K).$$

Therefore,

$$f(2K+2) + f(2K) > 2f(2K+1)$$

and from (23),

$$(24) \quad f(4K+3) = 4K+3 + 2f(2K+1).$$

Following a procedure nearly identical to the above, we can show that

$$(25) \quad f(4K+4) = 4K+4+f(2K+1) + f(2K+2)$$

and

$$(26) \quad f(4K+5) = 4K+5 + f(2K+1)+f(2K+3).$$

By induction, Theorem 1 follows.

B. The functional form of f(N) is given by the following theorem:

<u>Theorem 2</u>. Equation (9) is satisfied if and only if

$$(27a)\quad f(2^{k+1}+2m-1) = 2^{k+1}(k-\tfrac{1}{2}) + 2mk + 3m+1,\quad k = 0,1,\ldots \quad m = 0,1,\ldots,2^{k},$$

$$(27b)\quad f(2^{k+1}+2m) = 2^{k+1}(k-\tfrac{1}{2}) +(2m+1)k + 3m+3, k = 0,1,\ldots \quad m = 0,1,\ldots,2^{k-1}$$

<u>Proof:</u> The "only if" part follows simply from the fact that no two functions can both be the minimum without being equal. To prove (27), we again use induction. That is, we verify (27) for k = 0 and assume it to be valid for k = 0,1,...,K-1. If it follows thereby that (27) is valid for k=K, then (27) must be true for all k. The detailed proof involves substitution of (27) in (11) and elementary manipulation, and will be omitted here.

C. Theorem 1 is strengthened by the following result:

$$(28)\quad f\left[n^*(N) - 1\right] + f\left[N-n^*(N)\right] = \min_{1\leq n\leq N} \left\{f(n) + f(N-n)\right\}$$

if and only if

$$(29a)\quad n^*(2^{k+1}+2m) = 2^{k}+j,\quad j = 0,1,\ldots,2m+1,\ 0\leq m < 2^{k-1}$$
$$j = 2m-2^{k}+1,\ldots,2^{k},\ 2^{k-1}\leq m < 2^{k}-1$$

$$(29b)\quad n^*(2^{k+1}+2m-1) = 2^{k}+2j,\quad j = 0,1,\ldots,m,\ 0\leq m\leq 2^{k-1}$$
$$j = m-2^{k-1},\ldots,2^{k-1},\ 2^{k-1}\leq m\leq 2^{k}$$

<u>Proof:</u> The "if" part is proved by substituting (29) and (27) in (28) and verify. In the process it is also shown that for $2^{k+1}\leq N\leq 2^{k+2}-1$, the only solution in the range $2^{k}\leq n^*\leq 2^{k+1}$ are those given by (29). Thus, it remains only to show that no value of n* greater than 2^{k+1} or less than 2^{k} is a solution.

Consider $N=2^{k+1}+2m$, $0 \leq m < 2^{k-1}$. Since we know that $n^* = 2^k$ is a solution, we need only to show that (similar results follow for $n^* > 2^{k+1}$ by symmetry)

$$(30) \quad f(2^k-1) + f(2^k+2m) < f(2^k-2) + f(2^k+2m+1)$$
$$\leq f(2^k-3) + f(2^k+2m+2) \leq \ldots$$

The first of inequalities in (30)is easily verified using (27). The remaining inequalities follow from (12). For $2^{k-1} \leq m \leq 2^k$, we use $n^* = 2^{k+1}$, and from (27) and (12) show that

$$(31) \quad f(2^{k+1}-1) + f(2m) < f(2^{k+1}) + f(2m-1) \leq f(2k^{+1}+1) + f(2m-2)$$
$$\leq \ldots$$

For $N = 2^{k+1}+2m-1$, the proof follows nearly identical lines and will not be given here.

<u>Remark:</u>

A somewhat related one-dimensional, continuous search problem has been treated by A. Beck (1964) with different tools.

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