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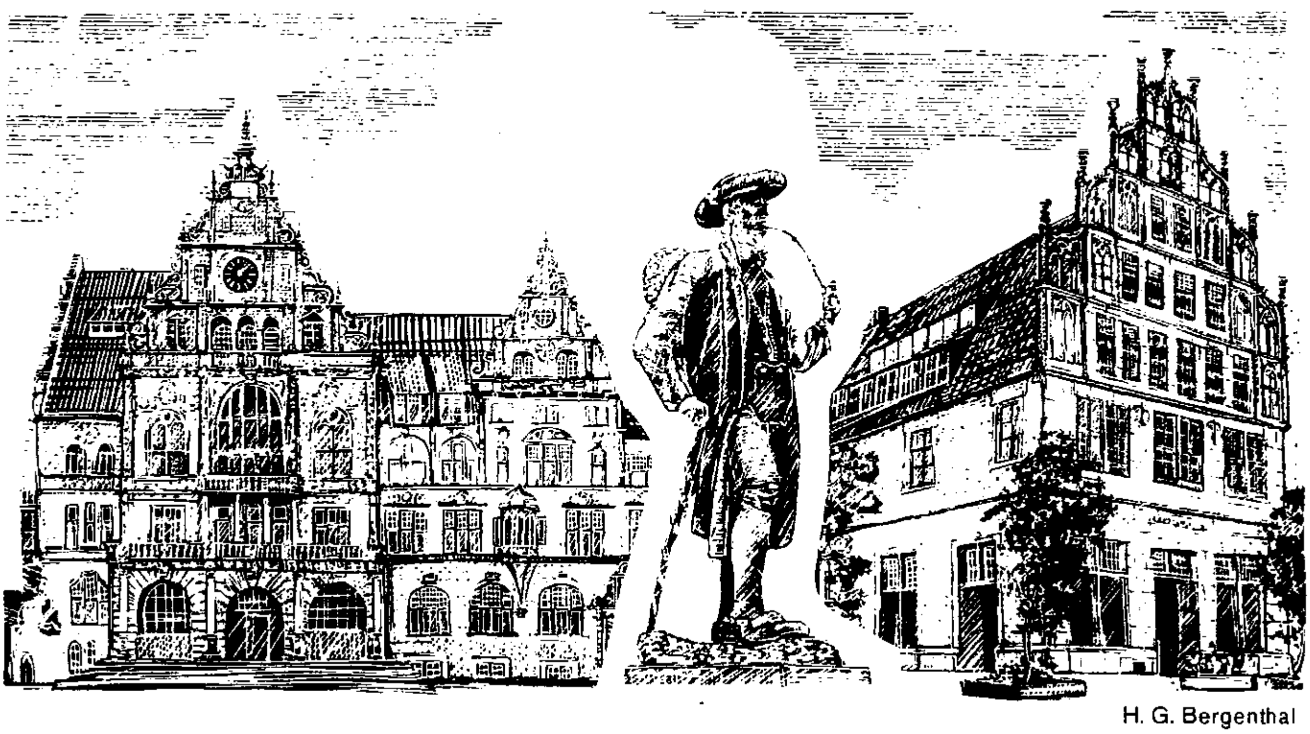

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A NOTE ON "AN IMPOSSIBILITY RESULT CONCERNING N-PERSON BARGAINING GAMES"

G. Huttel and W. F. Richter

In a recent paper published in this journal A. E. ROTH [1979] demonstrated that the axiomatization of E. KALAI and M. SMORODINSKY [1975] of solutions for two person bargaining games cannot be generalized to more than two persons. This result should not pass without due qualification. We are able to provide a straightforward generalization of the KALAI-SMORODINSKY axiomatization if only the system of feasible bargaining games is appropriately *restricted*. The difference between the general case and the two person case thus seems to be less "fundamental" (ROTH [1979, p. 132]) than ROTH's counterexample suggests.

1. A POSSIBILITY RESULT

We employ ROTH's [1979] definitions and notation. A n-person bargaining game $(S,d) \in B$ is then given whenever $S \subseteq \mathbb{R}^n$ is compact and convex and if some $x \in S$ exists with $x > d \in S$. In contrast with this general definition we shall restrict attention to some subset $B_0 \subseteq B$. However, we first need some further notational conventions.

Let $\bar{x} = \bar{x}(S,d)$ denote the ideal point. Put $A = A^{S,d} : \mathbb{R}^n \to \mathbb{R}^n$, $A_i(x) = (x_i - d_i) / (\bar{x}_i - d_i)$ for $i \in N$. Then A denotes the affine transformation of utilities normalizing threat- and ideal point: $A(d) = 0$ and $A(\bar{x}) = e = (1,\ldots,1)$. Note that $(S,d) \in B$ implies $\bar{x} > d$.

Let (S,d) be an element of B_0 by definition iff

$$d \leq y \leq x \in S \quad \text{implies} \quad y \in S \tag{1}$$

and

whenever $x \in S$ can be written as
$x = \lambda\bar{x} + (1-\lambda)d$ for some $\lambda \in [0,1]$
and there exists $y \in S$ such that (2)
$y \geq x$, $y \neq x$ then there also exists
$z \in S$ such that $z > x$.

(2) is less restrictive than to consider only such (S,d) for which the sets of strong and weak PARETO-optimal points coincide. Note that the application of any affine transformation of utilites $\bar{A} : \mathbb{R}^n \to \mathbb{R}^n$ with $\bar{A}_i(x) = \alpha_i x_i + \beta_i$; $\alpha_i, \beta_i \in \mathbb{R}$, and $\alpha_i > 0$ does not lead out of B_0 .

Now we define the solution $G : B_0 \to \mathbb{R}^n$ by

$$G(S,d) : = \arg(\max_{x \in S} \min_{i \in N} A_i(x)) \quad . \tag{3}$$

Hence $G(S,d) \in S$ maximizes $\min_{i \in N} A_i(x)$. Let $g : = G(S,d)$ for fixed $(S,d) \in B_0$. We claim:

$$\begin{aligned} &\text{There is some } \gamma \in [0,1] \text{ s. t.} \\ &A_i(g) = \gamma \quad \text{for all} \quad i \in N \ . \end{aligned} \tag{4}$$

This is the more conventional definition of KALAI-SMORODINSKY's monotone solution. ((4) might also be written as $g = \gamma\bar{x} + (1 - \gamma)d$.)

Suppose that in contradiction to (4)

$0 \leq A_j(g) = \min_{i \in N} A_k(g) < A_i(g) \leq 1$.

Then define $y : = A_j(g)e$. Note first that $y \in S$ by (1). Secondly, y can be written as

$y = A_j(g)e + (1 - A_j(g))0$.

So we may apply property (2) which yields some $x \in S$ with $x > y = A_j(g)e$, contradicting the definition of g .

Note that by (4) it is clear that there is a unique maximizer of $\min_{i \in N} A_i(x)$. Thus $G(S,d)$ is a well defined mapping. Let us now state the

THEOREM: $G: B_0 \to \mathbb{R}^n$ satisfies PARETO-optimality, symmetry, independence of affine transformations of utilities, and restricted monotonicity. Furthermore G is uniquely determined by these properties (1,2,3,5; see ROTH [1979], p. 130 - 131).

PROOF: Let us only show PARETO-optimality and restricted monotonicity for G .

Assume that there exists some $x \in S$ with $x \geqq g$, $x \neq g$. (4) implies that the assumptions of (2) are fulfilled so that there is some $z \in S$ with $z > g$, contradicting (3).

To show the restricted monotonicity let us assume (T,d) , $(S,d) \in B_0$, $T \subseteq S$, $\bar{x} = \bar{x}(T,d) = \bar{x}(S,d)$. Then

$$\max_{x \in T} \min_{i \in N} A_i(x) \leq \max_{x \in S} \min_{i \in N} A_i(x)$$

By (4) we obtain

$A_i(G(T,d)) \leqq A_i(G(S,d))$ for all $i \in N$

which in turn implies

$G_i(T,d) \leqq G_i(S,d)$.

To show that G is uniquely determined we essentially reproduce KALAI-SMORODINSKY's argument [1975, p. 517].

Let $f : B_0 \to \mathbb{R}^n$ meet all noted axioms.

Fix $(S,d) \in B_0$ where $\bar{x}(S,d) = e$, $d = 0$, w.l.o.g.

We have to show $f(S,0) = G(S,0)$.

By (4) $G(S,0) = \gamma e$ with $\gamma \in [0,1]$.

Define $T \subset S$ to be the convex hull of $0, e^1=(1,0,\ldots,0),\ldots,e^n,\gamma e$. Clearly, $(T,0) \in B_0$. The symmetry of T implies $f(T,0) = \gamma e = G(S,0)$.

The restricted monotonicity requires $f(S,0) \geq f(T,0) = G(S,0)$, but as $G(S,0)$ is PARETO-optimal in S we obtain $f(S,0) = G(S,0)$, which was to be shown.

2. A HISTORICAL NOTE

ROTH quotes several papers where the solution G has been discussed before. We would like to add to this list some prominent nineteenth century's economists like F. Y. EDGEWORTH [1897] and A. J. COHEN-STUART [1889]. The solution G is no invention of Game Theory but of Public Finance. In the latter field this solution is better known as *equal proportional sacrifice.*

To see this relationship we have to say a word on the problem of normatively justifying progressive taxation. Consider n tax payers with incomes $y_i > 0$ before, $y_i - t_i$ after taxes. $i \in N$ derives utility from money income according to some utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$ with neoclassical properties $(u_i' > 0\ ,\ u_i'' \leq 0)$. Denote $y = (y_1,\ldots,y_n)$, $t = (t_1,\ldots,t_n)$, $u = (u_1,\ldots,u_n)$. The set fo all feasible tax distributions is assumed to be

$$\mathit{t} := \{t \in \mathbb{R}^n \mid \sum_{i=1}^{n} t_i = \gamma\ ,\ 0 \leq t \leq y\}\ .$$

The tax revenue γ is exogenously fixed.

The classical question is : Which properties characterize a just and equitable distribution of taxes $t^* \in \mathit{t}$? The classics more specifically tried hard to justify and designate some tax function $t^* : \{(u,y)\} \to \mathit{t}$ holding (at least generically)

$$1 > \partial t_i^* / \partial y_i > t_i^* / y_i \quad \text{for all} \quad i \in N \qquad (5)$$

independently of the utility functions under consideration. The right hand inequality of (5) defines progressivity of t_i^* with respect to y_i in a local sense.

It is enough to say that neither the classics nor their followers solved the problem. They confined themselves to a discussion of several specific choice functions ("sacrifice concepts") among which

$$(u_i(y_i) - u_i(y_i - t_i^*)) / u_i(y_i) = \text{const.} \qquad (4')$$

is known to be the *equal proportional sacrifice*.

Setting $d := u(0)$,

$S := \{x \in \mathbb{R}^n \mid \exists\, t \in \mathit{t} : d \leq x \leq (u_i(y_i - t_i))_{i \in N}\}$

and assuming $u(0) = 0$ (by transformation invariance) we realize that (4') is equivalent to the application of solution G to $(S,0)$. By the way, the tax problem yields a nice convincing interpretation of the ideal-point $\bar{x}(S,0)$ as the utility of income before taxes: $\bar{x}_i = u_i(y_i)$.

Obviously, $(S,0) \in B_0$: Convexity of S follows by concavity of $u_i(\cdot)$. S is compact as t shares this property and as the $u_i(\cdot)$ are continuous. Finally $(S,0)$ holds (2) because of $u_i' > 0$. Consequently *all* bargaining games derived in this way from the task of distributing tax shares are in B_0.

It might be of general interest that the Dutch COHEN-STUART [1889] managed to construct some "nice" utility function such that the progressivity condition (5) is violated under solution G ((4')). The determination and axiomatic characterization of choice functions t^* holding (5) is relegated to two forthcoming papers (RICHTER [1980 a,b]).

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