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On Income Tax Functions

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ABSTRACT:

The paper sheds some light on distributive properties of income tax functions. For this purpose the ARROW-PRATT notion of risk (or inequality) aversion is adapted to income taxation. JAKOBSSON's (1976) theorems concerning the residual progression of tax schedules are generalized. The question is then raised how progressive income tax must be to offset the dispersing effect of saving on the distribution of residual income. The answer will relate the elasticity of saving, the residual progression and the proportional inequality aversion. The paper is completed by a short discussion of historically effective German income tax formulas.

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1. INTRODUCTION

This paper analyses distributive properties of income tax functions. We only consider tax schedules that are given in functional form $T(\cdot)$ such that $T(Y)$ denotes tax liability and $R(Y) = Y - T(Y)$ residual income for arbitrary income $Y \geq 0$.

The discussion will centre around three magnitudes, namely the *residual progression* (of T at Y)

$$\rho(Y,T) \;:=\; YR'(Y)/R(Y)$$

the *proportional inequality aversion* (of T at Y)

$$\alpha(Y,T) \;:=\; -YR''(Y)/R'(Y) \;=\; YT''(Y)/(1-T'(Y))$$

and

$$\pi(Y,T) \;:=\; \frac{d}{dY}(R(Y)/R'(Y)) \;=\; 1 + \alpha(Y,T)/\rho(Y,T).$$

(Primes indicate derivates.) It is the purpose of this paper to make explicit the specific type of distributional information carried by $\rho(\cdot,T)$, $\alpha(\cdot,T)$ and $\pi(\cdot,T)$. Thereby we are able to heavily rely on earlier work - in particular JAKOBSSON (1976) in the case of ρ and PRATT (1964) in the case of α .

We shall restate some of their results for the sake of a unified presentation.

Section 3 deals with $\alpha(\cdot,T)$. It seems as if tax schedules have never before been subject to a discussion in terms of their inequality aversion. We shall argue that α measures the "degree of tax progression" as seen by the individual taxpayer. This follows by mere reinterpretation of the work of PRATT (1964) and ATKINSON (1970). α determines the distribution of residual income as drawn by the taxpaying unit, say, through time.

On the other hand ρ can be said to measure the "degree of tax progression" as seen by society. ρ determines the (personal) distribution of residual income across society. This observation is primarily due to JAKOBSSON (1976; see also KAKAWANI, 1976). However, JAKOBSSON's statements lack the (formal) precision which might be desirable and which is considered to be necessary for our purposes. Hence in section 4 we shall reproduce JAKOBSSON's main results with some formal corrections and slight extensions.

Section 5 collects arguments for EDGEWORTH tax functions that are characterized by a *constant* residual progression.

Section 6 introduces saving. We there deal with the question "how progressive taxation must be" to maintain the achieved level of personal income inequality if we allow for a rising propensity to save out of residual income. The answer will involve the elasticity of saving and function π - thus indirectly both measures of tax progression ρ *and* α . To avoid potential confusion let us stress that this approach to the determination of some "minimal degree of progression" has little to do with the optimal tax approach in the MIRRLEES (1971) tradition which focuses on *work incentives*.

The proof of the main theorem 11 is given in section 7. Section 8 is devoted to a short discussion of historically effective German tax formulas in terms of ρ , α and π .

2. EDGEWORTH TAX FUNCTIONS

Throughout this paper we assume that T is sufficiently times continuously differentiable to justify the proofs. (Three times will be sufficient but not necessary.) Furthermore $T(0)=0$, $T'(Y)<1$ or equivalently $R(0)=0$ and $R'(Y)>0$ for all $Y\geq 0$.

Theorem 1: Let a,b be constants, $a\neq 0$. The following conditions are equivalent:

a) $R(Y) = bY^a$ for all $Y > 0$;

b) $\rho(Y,T) = a$ for all $Y > 0$;

c) $\alpha(Y,T) = 1-a$ for all $Y > 0$;

d) $\pi(Y,T) = 1/a$ for all $Y > 0$.

The proof is trivial. By this theorem the case of $R(Y) = bY^a$ clearly deserves special interest. Let us call functions T, where $T(Y) = Y - bY^a$, EDGEWORTH *tax functions* as he will have been the first to discuss them explicitly (1919, p. 249).

3. Inequality Aversion

Let $N = \{1,..,n\}$, $n \geq 2$ stand for some household whose members $i \in N$ are all liable to pay tax from their income y_i. The income profile is $y = (y_1,..,y_n) \geq 0, \neq 0$ and the income *per capita* $\bar{y} = \Sigma y_i/n$. The interpretation of N as some household is not the only one which makes good sense. One could equally think of y as a flow of income, where y_i is the income of some fixed individual at time $i=1,..,n$. Probabilistic interpretations also present themselves.

We assume that tax laws generally prescribe to tax $y_1,..,y_n$ separately, one by one. Residual income then amounts to $\Sigma R(y_i)/n$ on the average. There might be legal provisions that allow to average out incomes *before* tax. In that case residual income *per capita* amounts to $R(\bar{y})$. German tax laws, e.g., provide such a "privilege of splitting" for married couples. The marriage certificate gives the right to increase residual income *per capita* from $R(y^e) := \Sigma R(y_i)/n$ to $R(\bar{y}) = R(\Sigma y_i/n)$. y^e is the income before tax which, if equally distributed among N yields the same residual income *per capita* as y without privilege of splitting. If, on the other hand, y stands for a flow of dated incomes $R(\bar{y}) - R(y^e)$ measures the loss of residual income due to fluctuations through time. Hence it does not matter whether any legal provisions of averaging out $y_1,\ldots,y_n$ do in fact exist. More favourable economic circumstances might as well render possible a steady flow of income. Yet, let us stick to the idea of household, and of the privilege of splitting. Thus let us call

$AGS(y,T) := \bar{y} - y^e$ the *absolute gain from splitting* and

$RGS(y,T) := (\bar{y}-y^e)/\bar{y}$ the *relative gain from splitting*.

These definitions have their counter-parts in PRATT (1964) and ATKINSON (1970). Confer these publications for additional interpretative comments.

Fix $\bar{y} > 0$ and consider only such $y \in \mathbb{R}^n$ holding $\Sigma y_i/n = \bar{y}$. Let $\sigma^2(y) = \frac{1}{n}\Sigma(y_i-y)^2$ denote the variance of income. $\mathcal{O}(\sigma^2(y))$ is a term which tends to zero for $\sigma^2(y) \to 0$.

<u>Theorem 2 (PRATT, 1964):</u>

$$AGS(y,T) = -\tfrac{1}{2}\sigma^2(y)R''(\bar{y})/R'(\bar{y}) + \mathcal{O}(\sigma^2(y)) \quad (1)$$

$$RGS(y,T) = \tfrac{1}{2}\sigma^2(\bar{y}^{-1}y)\alpha(\bar{y},T) + \mathcal{O}(\sigma^2(\bar{y}^{-1}y)) \quad (2)$$

Thus AGS(y,T) - and RGS *mutatis mutandis* - falls into two factors if the variance of income is "sufficiently small". The first factor, $\frac{1}{2}\sigma^2(y)$, is independent of $T(\cdot)$. The second factor, namely $-R''(\bar{y})/R'(\bar{y})$, is only depending on the average income $\bar{y}$ and not on y itself.

The degree of approximation is probably best illustrated by an example. The computation of RGS is based on the polynomial tax formula of §32a, German income tax, effective 1979/80: $n=2$, $y_1 = 21600$ and $y_2 = 26400$ yield RSP = 2.1709 E-3.

Computing the right-hand side of (2) yields

$$\frac{1}{2} \times 0.433901 \times \frac{1}{100} = 2.1695 \text{ E-3}.$$

The relative error is less than 1‰.

It is not being suggested to use (1) and (2) for computing AGS and RGS . Both values can more directly be computed for arbitrary (!) σ^2 by recurring to the official tax schedules which list tax liabilities $T(Y_i)$ for increasing equidistant $Y_i \geq 0$. The error incurred by rounding off is in fact negligible for all practical purposes.

Theorem 2 becomes relevant when the personal tax load of two alternative tax functions T_1 and T_2 stands for comparison. (2) tells us that $\alpha(\cdot,T)$ is a *cardinal* measure of $RGS(\cdot,T)$ - at least when income variances are small. This measure is still an *ordinal* one if σ^2 is allowed to take arbitrary values:

Theorem 3 (PRATT, 1964):

The following two conditions are equivalent in either the weak form (in which case the bracketed material has to be deleted) or in the strong form (indicated in brackets).

a) $RGS(y,T_1) \leq [<] RGS(y,T_2)$
for all $y \in \mathbb{R}^n_+$ [that are non-equally distributed, i.e. $y \notin \mathbb{R}(1,..,1)$].

b) $\alpha(Y,T_1) \leq [<] \alpha(Y,T_2)$
for all Y out of a dense[1)] subset of $\mathbb{R}_+$.

The theorem remains true if, *mutatis mutandis*, attention is restricted to some fixed income interval. Furthermore in condition a) RGS may be replaced by AGS.

For a linear tax function T_1, i.e. $T_1(Y) = a+bY$, we obtain a nice corollary. Then $AGS(y,T_1) = 0$. Note furthermore that

$$T''(Y) > 0 \quad \text{iff} \quad -R''(Y)/R'(Y) > 0$$

as $R' > 0$ by assumption.

Theorem 4: The assertion

$0 < AGS(y,T)$ for all $y \in \mathbb{R}^n_+$, $\notin \mathbb{R}(1,..,1)$

is equivalent to

$0 < T''(Y)$ on a dense subset of $\mathbb{R}_+$.[2)]

The literature does not unanimously agree on the "correct" definition of progressivity. Although the condition $\frac{d}{dY}(\frac{T(Y)}{Y}) > 0$ is generally accepted $T''(Y) > 0$ is often offered as a possible alternative. See, for instance, SADKA (1976). SEIDL et al. (1970) even favour to call T progressive at Y if $\frac{d}{dY}(\frac{T(Y)}{Y}) > 0$ *and* $T''(Y) > 0$. BÖS et al. (1977) take a similar viewpoint.

The foregoing discussion strongly suggests to call T (at Y)

inequality averse[3) if $T''(Y) > 0$ and

progressive if $\frac{d}{dY}(\frac{T(Y)}{Y}) > 0$.

Developing T at zero according to TAYLOR

$$0 = T(0) = T(Y) - YT'(Y) + \frac{1}{2}Y^2T''(Z)$$

(for some $Z \in [0,Y]$) yields the

Remark: If T is inequality averse for all $Y \geq 0$ then T is also progressive for all $Y > 0$.[4)

The reversal is clearly false as linear tax functions show. A stronger counter-example is $T(Y) = Y(1-e^{-Y})$ which is progressive for all $Y \geq 0$ however, *inequality preferring*, i.e. $T''(Y) < 0$, for all $Y > 2$.

4. Residual Progression

In this section $y = (y_1,\ldots,y_n)$ denotes the income profile of the whole tax paying population. (In case that households are allowed to split in the sense of section 3 the members

appear in y by their average income before tax.) Assume throughout that y is increasingly ordered:
$0 < y_1 \leq \ldots \leq y_n$.

Let $x,z \in \mathbb{R}^n_+$; $x_1 \leq \ldots \leq x_n$; $z_1 \leq \ldots \leq z_n$. We say that x is [strictly] LORENZ dominated by z and write $x \leq_L z$ [$x <_L z$], respectively if

$$\sum^k x_i / \sum^n x_i \leq [<] \sum^k z_i / \sum^n z_i \quad \text{for all } k=1,..,n-1 .$$

Sometimes we write $z \geq_L x$ if $x \leq_L z$, also $x =_L z$ if both $x \leq_L z$ and $x \geq_L z$.

In theorems 5 and 6 $x(y)$, $x^j(y)$ denote vectors of residual incomes. E.g.:

$$x^j_i(y) := y_i - T_j(y_i) \quad (i \in N,\ j=1,2).$$

Theorem 5 (JAKOBSSON, 1976):

The following two conditions are equivalent in weak or strong form:

a) $x^2(y) \leq_L x^1(y)$ [$x^2(y) <_L x^1(y)$, respectively] for all y [$\notin \mathbb{R}(1,..,1)$] ;

b) $\rho(Y,T_2) \geq [>] \rho(Y,T_1)$ for all Y out of a dense subset of $\mathbb{R}_+$.

The theorem strengthens JAKOBSSON's proposition 1. Careful inspection of his proof yields our assertion.

Let us choose T_2 to be proportional, i.e. $T_2(Y)=bY$. Then $x^2(y) =_L y$ and $\rho(Y,T_2) = 1$. Note furthermore that $\rho(Y,T_1) < 1$ iff T_1 is progressive at Y.

Thus we obtain the following corollary (strong form):

Theorem 6 (JAKOBSSON, 1976):

> $y <_L x(y)$ for all $y \in \mathbb{R}(1,..,1)$ iff T is progressive on a dense set in $\mathbb{R}_+$.

Note the structural parallel between theorems 3,4 and 5,6.

5. ARGUMENTS FOR EDGEWORTH TAX FUNCTIONS

α and ρ, respectively, induce quasi-orderings (reflexive and transitive) on the set of all tax functions:

$$T_2 \succsim(\alpha)\, T_1 \quad :\Leftrightarrow \quad \alpha(\cdot,T_2) \geq \alpha(\cdot,T_1) \ ,$$

$$T_2 \succsim(\rho)\, T_1 \quad :\Leftrightarrow \quad \rho(\cdot,T_2) \leq \rho(\cdot,T_1) \ .$$

By theorems 3 and 5 these quasi-orderings allow some straight forward economic interpretation. They are both equal, $\succsim(\alpha) = \succsim(\rho)$, if they are restricted to EDGEWORTH tax functions. This follows from theorem 1:

$$1-a_2 = \alpha(\cdot,T_2) \geq \alpha(\cdot,T_1) = 1-a_1 \quad \text{iff}$$

$$a_2 = \rho(\cdot,T_2) \leq \rho(\cdot,T_1) = a_1 \quad ,$$

where $R_i(Y) = b_i Y^{a_i}$. In this restricted case $\succsim(\alpha) = \succsim(\rho)$ is even complete, thus defining an ordering.

The distributive effects of parameter variations are easy to assess in the case of EDGEWORTH tax functions. However, simplification cannot be an ultimate goal in normative economics.

The structural simplification of the EDGEWORTH situation could be regarded as a "loss of degrees of freedom" for policy making. Politicians may sometimes wish to disconnect variations of α and ρ . The latest German tax reform may serve as an example *ex post*. (We are not claiming that the effect was *planned* as such.) Confer charts 2 and 3, below. Comparing T81 and T79 we may conclude that in the income range [28000, 42000] progressivity has been *increased* in terms of α but *decreased* in terms of ρ .

Unfortunately, little is known about the structural relationship of $\succsim(\alpha)$ and $\succsim(\rho)$ for non-EDGEWORTH tax functions. Let us consider EDGEWORTH tax functions in more detail. Which normative reasons could make one to choose them?

Constant proportional inequality aversion

An important implication of theorem 3 is:

Theorem 7 (PRATT, 1964):

$\alpha(Y,T)$ is decreasing [increasing or constant] in Y if and only if for all y

$RGS(\lambda y,T)$ is decreasing [increasing or constant] in $\lambda \in \mathbb{R}, > 0$.

The requirement that $RGS(\lambda y,T)$ should be independent of the income *level* measured by λ is appealing, though probably not compelling. Even if $\alpha(\cdot,T) = const$ were normatively imperative small variations of the values taken by $\alpha(Y,T)$ should not be overstated in practice. As has been argued before the uncompromising realization of $\alpha(\cdot,T) = const$ could unduly interfere with competing political goals.

Constant residual progression

The following theorem parallels theorem 7. It is proved upon application of theorem 5 to $T_1(Y) = T(Y)$ and $T_2(Y) = \frac{1}{\lambda}T(\lambda Y)$ for arbitrary Y and λ.

Theorem 8:

$\rho(Y,T)$ is decreasing [increasing or constant] in Y if and only if for all y

$x(\lambda y)$ is increasing [decreasing or constant] in λ with respect to the LORENZ quasi-ordering $\leq_L$.

The version concerning a constant residual progression was first proved by JAKOBSSON (1976). However, his proof does not rely on theorem 5. His argument makes use of CHAUCHY's functional equation. JAKOBSSON suggests to interpret λ as a measure of "inflation". An EDGEWORTH tax function is then called for if the redistributive reffect of inflation is to be neutralized. Note that the same argument could formally be advanced in connection with theorem 7. However, it would be less convincing as the distributive effect there only refers to isolated households. The normative strength of a constancy requirement seems to be more convincing in the case of theorem 8.

Sacrifice approach

Nineteenth century's economists vainly tried to justify progressive taxation by inflicting an *equal sacrifice* on all taxpayers. An equal absolute, proportional or marginal sacrifice does not yield the desired result. There is, however, a non-classical sacrifice concept suggested by RICHTER (1980) which implies "moderate progression" for *all* utility functions $U: \mathbb{R}_+ \to \mathbb{R}$ holding $U(0)=0$, $U > 0$, $U'' \leq 0$. This sacrifice concept is given by

$$\frac{U(Y)U'(Y-T(Y))}{[U(Y-T(Y))]^2} = \text{const.} \qquad (3)$$

A normative, game-theoretic justification for condition (3) was presented by RICHTER and SELTEN (1980). For any fixed U (3) implicitly defines some tax function $T = T_U$ on some open interval $\mho = \mho_U$ in $\mathbb{R}_+$. Irrespective of the considered U we obtain

<u>Theorem 9 (RICHTER, 1980):</u>

$1 > T'(Y) > T(Y)/Y$ for all $Y \in \mho$ where the latter inequality is subject to the condition $0 < T(Y)/Y \leq 1/2$.

Inserting iso-elastic utility functions $U(Y) = Y^{\varepsilon}/\varepsilon$ into (3) yields the EDGEWORTH tax function

$$R_{\varepsilon}(Y) = Y - T_{\varepsilon}(Y) = bY^{\frac{\varepsilon}{1+\varepsilon}} .$$

The range of the residual progression $\varepsilon/(1+\varepsilon)$ deserves special notice. The axiomatization of (3) makes use of $U'' \leq 0$ and $U(0) = 0$. Hence ε is restricted to the interval (o,1] and consequently $\varepsilon/(1+\varepsilon)$ to $(0,\frac{1}{2}]$. We thus only obtain a sacrifice theoretic justification of EDGEWORTH tax function if the residual progression belongs to the interval (0,.5] . The upper boundary, .5, is particularly noteworthy as it remains *below* empirical suggestions.

By using data from the Amercian Federal Income Tax of 1917 EDGEWORTH derived concrete proposals as to the choice of a and b in $R(Y) = bY^a$. Thus he suggested .967, .952, and

.946 for a (p.250-252).

GENSER presents seven alternative specifications of $R(Y) = bY^a$. Six of them are obtained by fitting best - according to various criteria - to the Austrian Income Tax of 1975. In all these cases a exceeds .828. There is a single proposal which comes .5 a bit nearer by suggesting $a = .6554$. However, that case does not pretend to fit the effective Austrian Income Tax. Whereas the effective average tax rate is 14.2% for some yearly income $\tilde{Y}$ of 50000 S the parameter value $a = .6554$ is obtained by requiring $T(\tilde{Y})/\tilde{Y} = .05$, i.e., by deliberate deviation of the *status-quo*.

Finally, R.PAULY (1979; see also PAULY et al., 1979) worked out several tax proposals for the F.R.G. referring to 1979. For certain income intervals in the range from 20000 DM to 60000 DM yearly income he bases his proposals on $R(Y) = bY^a$ with $a \geq .75$.

All these proposals are well substantiated by empirical facts (Confer also chart 2. The residual progression $\rho(Y,T)$ of various depicted German tax formulas never reached the range $(0,.5]$.) They are, however, not supported by the normative approach (3).

The dichotomy between *policy* and *theory* has some noteworthy parallel in the theory of the measurement of income inequality.

Let y stand for the vector of personal incomes. ATKINSON (1970) suggested to measure income inequality by means of

$$I(y,A) \quad := \quad 1 - \frac{nA^{-1}(\frac{1}{n}\Sigma A(y_i))}{\Sigma y_i} .$$

ATKINSON does not say much about the interpretation of the function A. (See also the critique by SEN, 1973.) If we chose A to be the residual income function R $I(y,R)$ would formally equate $RGS(y,T)$ serving here as a measure of residual income inequality in the sense of ATKINSON. If T were the tax schedule effectuated by some planner or political party then $\alpha(\cdot,T)$ would *reveal* their inequality aversion and hence their ideas about distributive justice. According to ATKINSON there are good reasons to choose iso-elastic functions

$$A_a(Y) \quad = \quad Y^a/a \qquad (a \in \mathbb{R})$$

for A. (An axiomatization of the corresponding class of inequality measures $\{I(\cdot,A_a) \mid a \leq 1\}$ was given by KOLM, 1975, and BÜRK et al., 1978.)

The interesting point is that whenever ATKINSON applies his measure $I(\cdot, A_a)$ to make international comparisons of income inequality he selects values of a which never exceed 1/2. (E.g. $a = 1/2, 0, -1/2$, etc. Cf. p. 259, 1970, or p. 48, 1975.) These values of a could not be justified if A_a were chosen to approximate the empirical residual income functions discussed above.

6. THE EFFECT OF SAVING ON THE DISTRIBUTION OF RESIDUAL INCOME

Theorems 3 and 5 tell us the distributive effects of reducing the residual progression or increasing the inequality aversion. They tell us nothing about the "optimal degree of progression". We shall now turn to this question.

The sacrifice approach of section 5 gave some hint however vague and controversal with view to tax policy. More clear-cut answers might be expected from the income theoretic approach to taxation. However, connecting ones arguments with the LORENZ curve may become a slippery slope. Accepting once that *strict* LORENZ domination is a definite welfare improvement - for which since the work of ROTHSCHILD and STIGLITZ (1973) much can be said - there is no getting round that residual incomes should best be equalized. On the other hand it can seriously be doubted that many people are in fact prepared to bear such extremal consequences. Equality will not be considered a prerequisite to true distributive equity.

In this section saving is introduced as an element of dynamic change. Without taxation a rising propensity to save tends to increase income inequality through time. We shall pose the question how progressive income tax must be to at least offset this distributive effect of saving. The answer will relate the elasticity of saving, the residual progression, and the proportional inequality aversion.

Let y - as before - denote the vector of personal incomes before tax - non-equally distributed and increasingly ordered. y_i comprises all taxable earnings, including interests on capital. We assume that there is an equilibrium in the sense that without saving the same income profile y would develop in all succeeding periods. By that we implicitly assume a constant population. $x^0=x(y)$ with $x_i^0 = R(y_i) = y_i-T(y_i)$ is the residual income vector of periode zero.

We then take regard to the fact that individuals save out of their residual income R. (It is convenient to use the same symbol R for functions and values of these functions. No confusion should arise.) Let the saving function be given by

$S:\mathbb{R}_+ \to \mathbb{R}$. Cross section studies suggest

$$\frac{d}{dR}\left(\frac{S(R)}{R}\right) > 0 . \qquad (4)$$

In the short run $S(R) < 0$ is possible, even realistic. From a long run point of view $S(R)<0$ does not make much sense as zero income is subsistence level. On the other hand DUESENBERRY (1949) convincingly argues that in the long run saving does not only depend on R itself but also on the *distribution* of residual income.

As we would like to stick to the specification $S(R)$ we just disregard individuals whose savings are negative. Hence we *assume* $S(R) > 0$ for all $R \geq R(y_1) > 0$. Then (4) is equivalent to

Assumption: $RS'(R)/S(R) > 1$ for all $R \geq R(y_1)$. (4')

Individual i is saving $S(R(y_i))$ in periode zero. Hence his income before tax in periode one is $y_i + rS(R(y_i))$ where r stands for the fixed market rate of interest. The residual income vector of periode one is thus given by x^1 where

$$x_i^1 = R(y_i + rS(R(y_i))) = R(y_i + rS(x_i^0)) .$$

We are going to compare x^1 and x^0 with respect to LORENZ domination.

The following two theorems are strict corollaries of theorem 12. They deal with special tax functions thus preparing for the general case.

Theorem 10 (linear taxation):

$c \geq 0$, $b > 0$. $R(Y) = c + bY$ implies
$x^0 >_L x^1$.

Hence linear taxation is not able to maintain the level of residual income inequality achieved in periode zero. Saving according to (4') makes residual incomes disperse. Legal tax schedules generally become linear from some high income upwards. "The rich then tend to become even richer by the mere fact of saving".

Theorem 11 (EDGEWORTHian taxation): Let $R(Y) = bY^a$ and

$RS'(R)/S(R) \geq [\leq] 1/a$ for all
$R \in [R(y_1), R(y_n)]$. Then $x^1 \leq_L x^0$
$[x^1 \geq_L x^0$, respectively] .

(A technical remark: $S(R) \leq R$ requires that $RS'(R)/S(R)$ converges to 1 for $R \to \infty$. The range of R thus has to be restricted if $RS'(R)/S(R) \geq 1/a > 1$ is to hold.)

According to theorem 10 it is sufficient to choose
$RS'(R)/S(R) \leq 1/a$ (for all R) if we intend to ensure
$x^1 \geq_L x^0$. However, this turns out to be a severe condition.

We learned above that the empirical studies of EDGEWORTH, GENSER, and PAULY suggest values for $1/a$ which (with one noted exception) all remain below $4/3 = 1.33$. Later we shall take a closer look at the German saving function of the mid-seventies. The data give some evidence that the critical value 1.33 is exceeded by $RS'(R)/S(R)$ at least for wide income ranges. The conclusion then is that - ignoring all other economic forces of change except saving - an EDGEWORTH tax function with $a = .75$ is not sufficiently progressive to maintain the achieved level of income inequality.

We now turn to the general case. Theorem 11 holds true in four alternative versions indicated by the symbol Θ which has to be replaced by one of the relations $\leq, \geq, <$ or $>$. Accordingly Θ_L stands for $\leq_L, \geq_L$, etc. We employ the shortened notation

$$RS'(R)/S(R)|_Y = R(Y)S'(R(Y))/S(R(Y))$$

here and at similar places.

Theorem 12 (General case):

For all $Y \in [y_1, y_n]$ and all $Z \in [Y, Y+rS(R(Y))]$

$$RS'(R)/S(R)|_Y \quad \Theta \quad \pi(Z,T) \ . \qquad (5)$$

Then $x^o \ \Theta_L \ x^1$.

$R(Y) = c + bY$ implies $\pi(Y,T) = 1$. Hence theorem 10 follows from theorem 12 by assumption (4'). In case of $R(Y) = bY^a$ $\pi(Y,T) = 1/a$. (See theorem 1.) That proves theorem 11. The proof of theorem 12 is relegated to section 7.

Chart 4 depicts $\pi(\cdot,T)$ for various tax formulas that with the one exception of T136 have all been effective in Germany.

Let us now give a rough empirical estimate of $RS'(R)/S(R)$. Data about savings are difficult to obtain. In what follows we completely rely on the official survey on incomes and consumption carried out for Germany in 1973. This survey shows data about the empirical saving function denoted $\hat{S}$. See table, below. The point-elasticity $R\hat{S}'(R)/\hat{S}(R)$ will be approximated by arc-elasticities. Writing

$$R_{-1}|_{y_i} := R(y_{i-1}) \ , \quad R_{+1}|_{y_i} := R(y_{i+1})$$

we define

$$\eta_-(R) \quad := \quad \frac{\hat{S}(R)-\hat{S}(R_{-1})}{R - R_{-1}} \, \frac{R}{\hat{S}(R)} \quad ,$$

$$\eta_+(R) \quad := \quad \frac{\hat{S}(R_{+1})-\hat{S}(R)}{R_{+1} - R} \, \frac{R}{\hat{S}(R)} \quad .$$

Note that if $\hat{S}$ were convex we should obtain

$$\eta_-(R) \leq R\hat{S}'(R)/\hat{S}(R) \leq \eta_+(R) \ .$$

The data, however, show that $\hat{S}$ is not convex ($\hat{S}''\geq 0$) everywhere.

<u>The elasticity of saving</u> (F.R.G., 1973)

(1) Household net income; monthly; from - to (DM)	(2) mean (DM) R	(3) savings (DM) $\hat{S}(R)$	(4) $\eta_-(R)$	(5) $\eta_+(R)$	(6) $\frac{(4)+(5)}{2}$
below 600	510	9	-	7.524	-
600 - 800	751	41	2.432	2.714	2.573
800 - 1000	967	73	1.962	2.019	1.991
1000 - 1200	1177	105	1.708	1.997	1.853
1200 - 1500	1452	154	1.680	1.418	1.549
1500 - 1800	1791	205	1.314	2.033	1.674
1800 - 2500	2298	323	1.656	2.068	1.862
2500 - 5000	3430	652	1.529	2.816	2.173
5000 - 15000	6993	2559	1.463	-	-

<u>Source</u>: Own computations on the basis of: Einkommens- und Verbrauchsstichprobe 1973, Fachserie 15, Heft 4, Statistisches Bundesamt Wiesbaden, p.18 .

$\hat{S}(R_i)$ will clearly depend on the grouping of incomes. For that reason the $\hat{S}(R_i)$-entries for the extremal incomes 510 and 6993 are particularly unreliable. The values $\eta_+(510) = 7.524$ and $\eta_-(6993) = 1.463$ should therefore best be ignored.

Despite of various legitimate objections the values above should not be all too far from the "true" elasticity of saving function $RS'(R)/S(R)$. Let us whence relate the data to $\pi(\cdot,T)$ where we choose T75 for T. See chart 4. T75 is the tax function that was effective in Germany from 1975 to 1977. It was passed in 1974, i.e. only one year after the savings data were collected. The tax function of the year 1973, namely T65, had been effective since 1965. T75 will best reflect the targets of income policy prevailing in 1973.

Note that

$$\max_Y \pi(Y, T75) \cong \pi(29000, T75) = 1.391 ,$$

a value which exceeds those of the table only once in the case of 1.314! Hence there are good reasons to infer that θ = > is the version of theorem 12 characterizing West-Germany during the mid-seventies. T75 was not sufficiently progressive to offset the dispersing effect of saving - at least for wide income ranges.

7. Proof of Theorem 12

Careful inspection and completion of a lemma proved by JAKOBSSON (1976) yields:

Lemma: Let $0 < x_1^o \leq .. \leq x_n^o$, $0 < x_1^1 \leq .. \leq x_n^1$ and

$$x_i^1 / x_{i-1}^1 \leq x_i^o / x_{i-1}^o \quad \text{for all } i = 1, .., n$$

[and < for at least one i].

Then $x^1 \geq_L x^o$ [$x^1 >_L x^0$, respectively].

Proof of theorem 12: (5) <=>

$$\frac{S'(R)R}{S(R)}\Big|_Y \ \Theta \ \frac{d}{dY}\left(\frac{R}{R'}\right)\Big|_Z$$

$$=> 1 + rS'(R)R'\Big|_Y \ \Theta \ \frac{R'(Y)}{R(Y)}\left[\frac{R(Y)}{R'(Y)} + rS(R(Y))\frac{d}{dY}\left(\frac{R}{R'}\right)\Big|_{\bar{Z}}\right]$$

$$= \frac{R'(Y)}{R(Y)}\frac{R}{R'}\Big|_{Y+rS(R(Y))}$$

where $\bar{Z}$ is appropriately chosen to justify the latter equality according to TAYLOR. Hence

$$\frac{d}{dY}\log R(Y + rS(R(Y)))$$

$$= \frac{R'}{R}\Big|_{Y+rS(R(Y))} [1 + rS'(R)R']\Big|_Y$$

$$\Theta \ \frac{R'(Y)}{R(Y)} = \frac{d}{dY}\log R(Y) \ .$$

If $y_{j-1} < y_j$ integration of the extreme left-hand and right-hand terms yields

$$\log\frac{R(y_j + rS(R(y_j)))}{R(y_{j-1} + rS(R(y_{j-1})))} \ \Theta \ \log\frac{R(y_j)}{R(y_{j-1})}$$

i.e. $x_j^1/x_{j-1}^1 \ \Theta \ x_j^o/x_{j-1}^o$. Here, equality would trivially hold in case of $y_{j-1} = y_j$. Theorem 12 follows by the above lemma.

8. CHARTS

The following charts refer to historically effective tax functions of West-Germany - with one exception: T136 is a proposal for 1981 made by KARL-BRÄUER-INSTITUT des Bundes der Steuerzahler e.V. (Tarifreform vorrangig, Wiesbaden, Feb. 1980). T65 became effective in 1965, T75 in 1975 and so forth. The list is not complete. There was some T78 which, however, differed little from T75. The computations are based on the mathematical formulas of §32a, German Income Tax Law. (Cf. Bundesgesetzblatt I: 1964, p.894; 1974, p.2195; 1979, p.757; 1980, p.1382.) The argument Y of $T(\cdot)$ thus stands for *taxable income*. As such it may differ from *income before tax* by various personal allowances which here are all left out of account. Legal provisions concerning the rounding off of Y and $T(Y)$ have also been neglected.

T65 is piecewise defined as a polynomial of degree 3 and less (in Y). T75, T79, T81 are piecewise representable by polynomials of degree 4 and less. T136 is not polynomial.
For $Y \in [4200, 104200]$ it is of the form

$$T(Y) = a_0(Z(Y))^{1.36} \quad \text{where} \quad Z(Y) = (Y-4200)/1000.$$

Charts 1 to 4 show T, $\alpha(\cdot,T)$, $\rho(\cdot,T)$ and $\pi(\cdot,T)$.

The curves are computed for values of Y that are multiples of 1000.

The tax functions are grouped into two sets: {T65, T75, T81} and {T79, T136, T81}. The first set is to illustrate long-run changes. The latter one is to shed some light on the latest amendment to tax laws.

Chart 3 tells us that progressivity - as measured by the residual progression - has been increased over time for middle and high incomes. There also is a recognizable horizontal shift which partially offsets the general increase of nominal incomes due to inflation and growth. T136 somewhat drops out of the general line. The progressivity burden is shifted to medium-high incomes.

Chart 2 seems to present a rather unsystematic picture. Such a first impression is, however, not correct. The "teeth" in the curves must be thought away. They are due to the habit of defining tax functions piecewise. The inequality aversion generally receives no attention by policy makers. Thus people will have made no effort "to smooth out" $\alpha(\cdot,T)$ as they will certainly have done with $\rho(\cdot,T)$.

Let us ignore low incomes as there incomes may considerably differ from taxable incomes. Thus ignoring low incomes we observe a decreasing proportional inequality aversion for all tax functions except T136. If we interpret RGS as the relative *gain* from splitting in case of married couples a decreasing $\alpha(\cdot,T)$ is at least more in line with the ability-to-pay principle than an increasing $\alpha(\cdot,T)$. If, however, RGS is to measure the *loss* of residual income due to fluctuations through time the argument could be reversed. In any case the sharp kink of T136 at Y = 104200 will allow no convincing justification.

In fact, T136 tells us an interesting lesson to learn. Chart 1 could make one believe that there is no essential difference between T136 on one side and T79, T81 on the other. However, charts 2, 3, and 4 reveal severe distributive differences.

The similarity of charts 2 and 4 might be striking. The similarity was stressed by the choice of appropriate scales. The effect is yet merely accidental. It can easily be rationalized by the equality of

$$\pi(\cdot,T) = 1 + \alpha(\cdot,T)/\rho(\cdot,T) .$$

For middle and high incomes $\rho(Y,T)$ is almost constant in Y for all considered tax functions. For such incomes $\pi(\cdot,T)$ approximately is an affine transformation of $\alpha(\cdot,T)$. In the lower income range the fluctuations of $\rho(\cdot,T)$ are swollowed by a vanishing $\alpha(\cdot,T)$.

Again, T136 drops out of the general line in chart 4. A rising $\pi(\cdot,T)$ for growing incomes is not favourable with view to theorem 11. The elasticity of saving necessarily has to decrease to one - at least asymptotically. Hence, if we liked $\pi(\cdot,T)$ to dominate $RS'(R)/S(R)$ in the sense of theorem 11 a decreasing $\pi(\cdot,T)$ would seem to be advocated.

We considerably benefited from hints given by D.Bös, R.Pethig, and E.Schlicht. Thanks to all of them.

FOOTNOTES

1) $\mathfrak{O}$ is *dense* in $\mathbb{R}_+$ if every $Y \in \mathbb{R}_+ \setminus \mathfrak{O}$ is a limit point of $\mathfrak{O}$.
2) T is known to be strictly convex iff $T'' > 0$ on a dense subset. (Cf. KATZNER, p. 189.)
3) PRATT would add "strictly".
4) Relying on footnote 2) one could prove progressivity of T for all $Y > 0$ by only assuming $T'' > 0$ on a *dense* subset of $\mathbb{R}_+$.

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Chart 1: Selected income tax functions

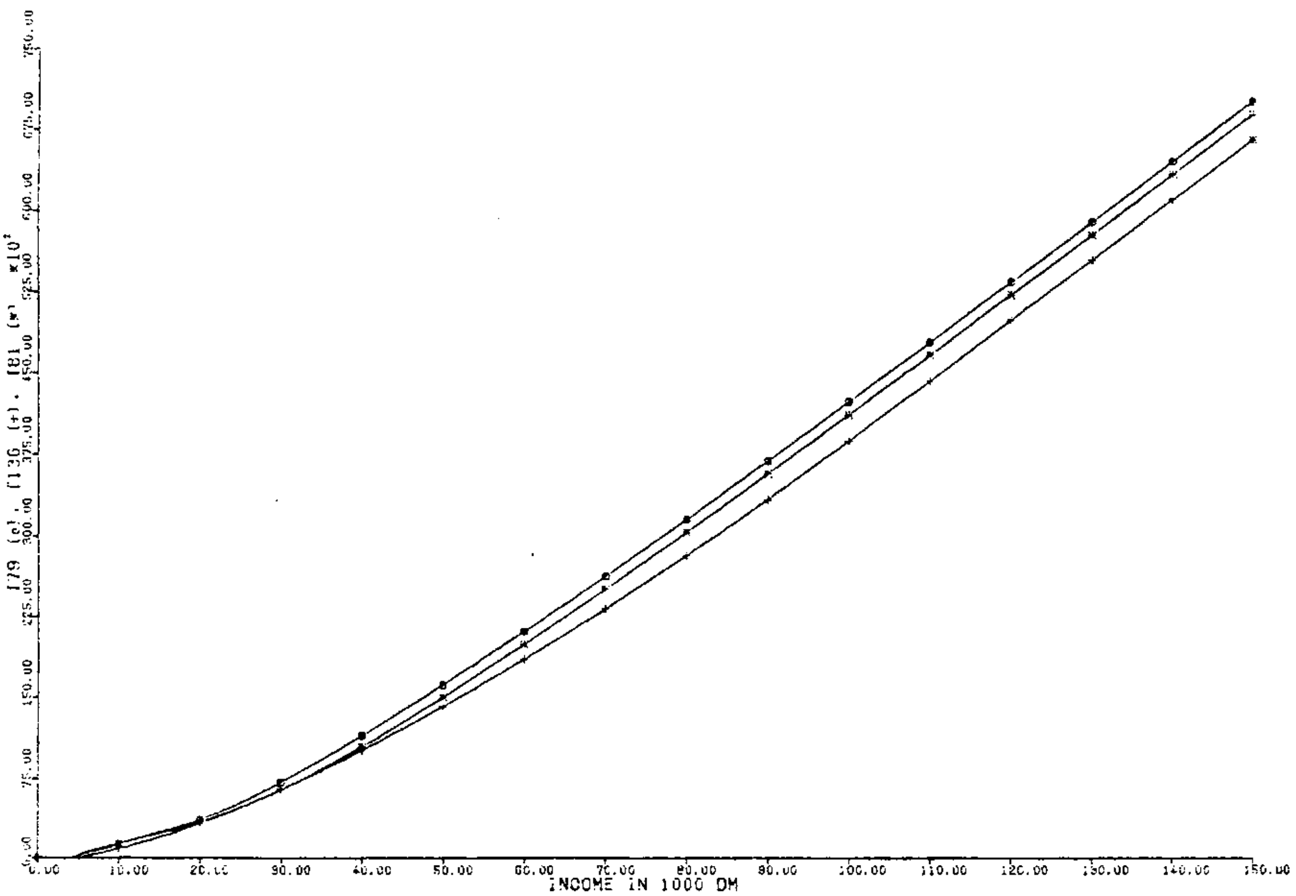
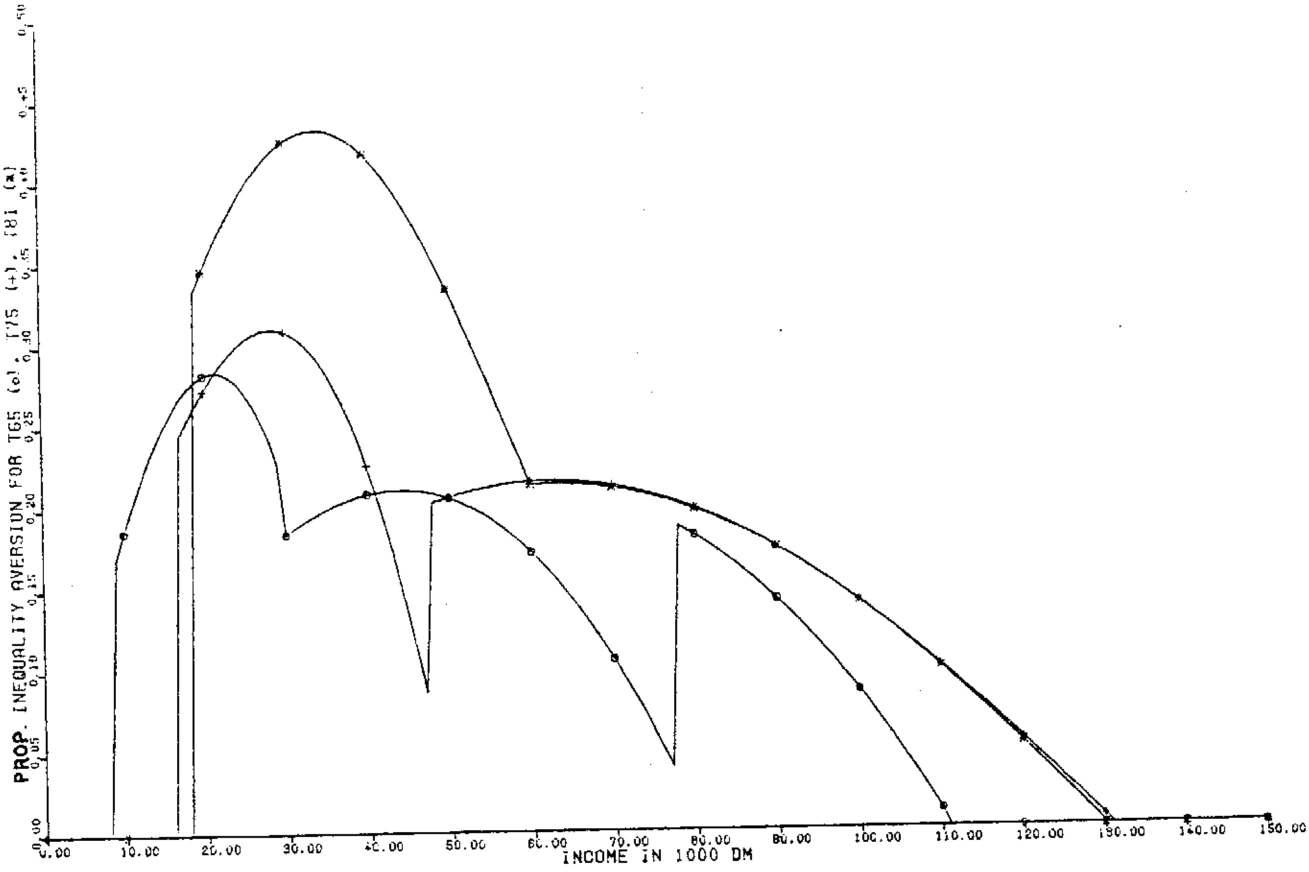
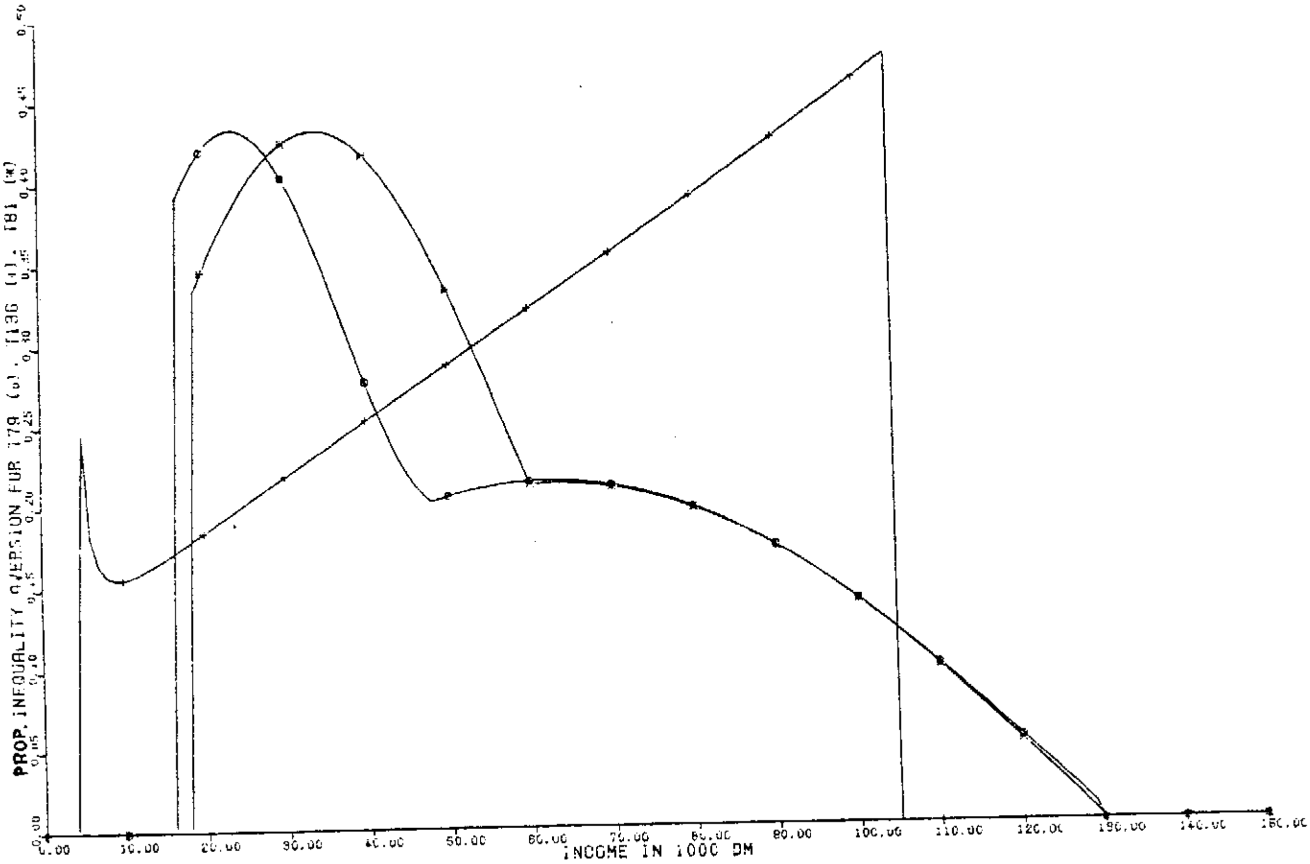
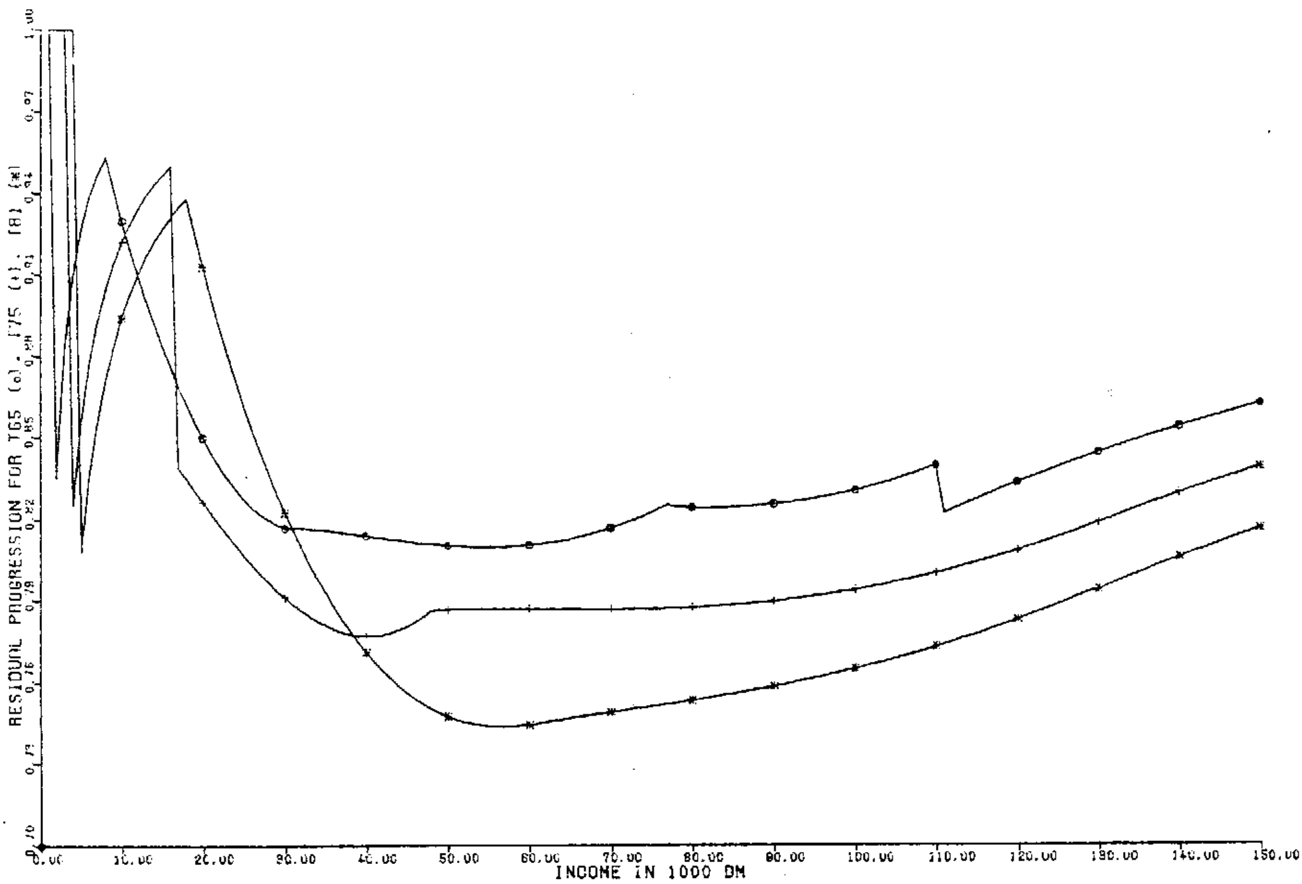


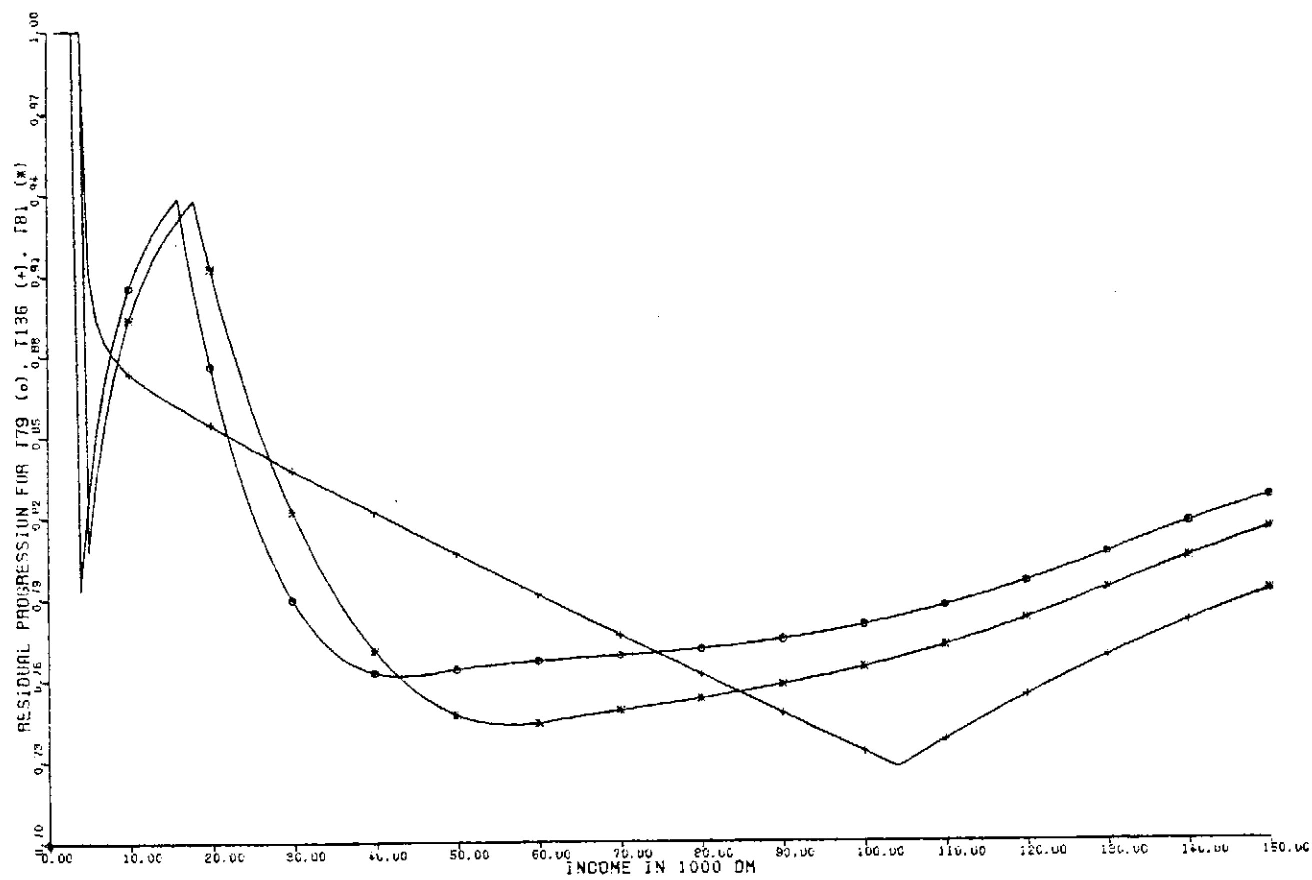
Chart 2: Proportional inequality aversion



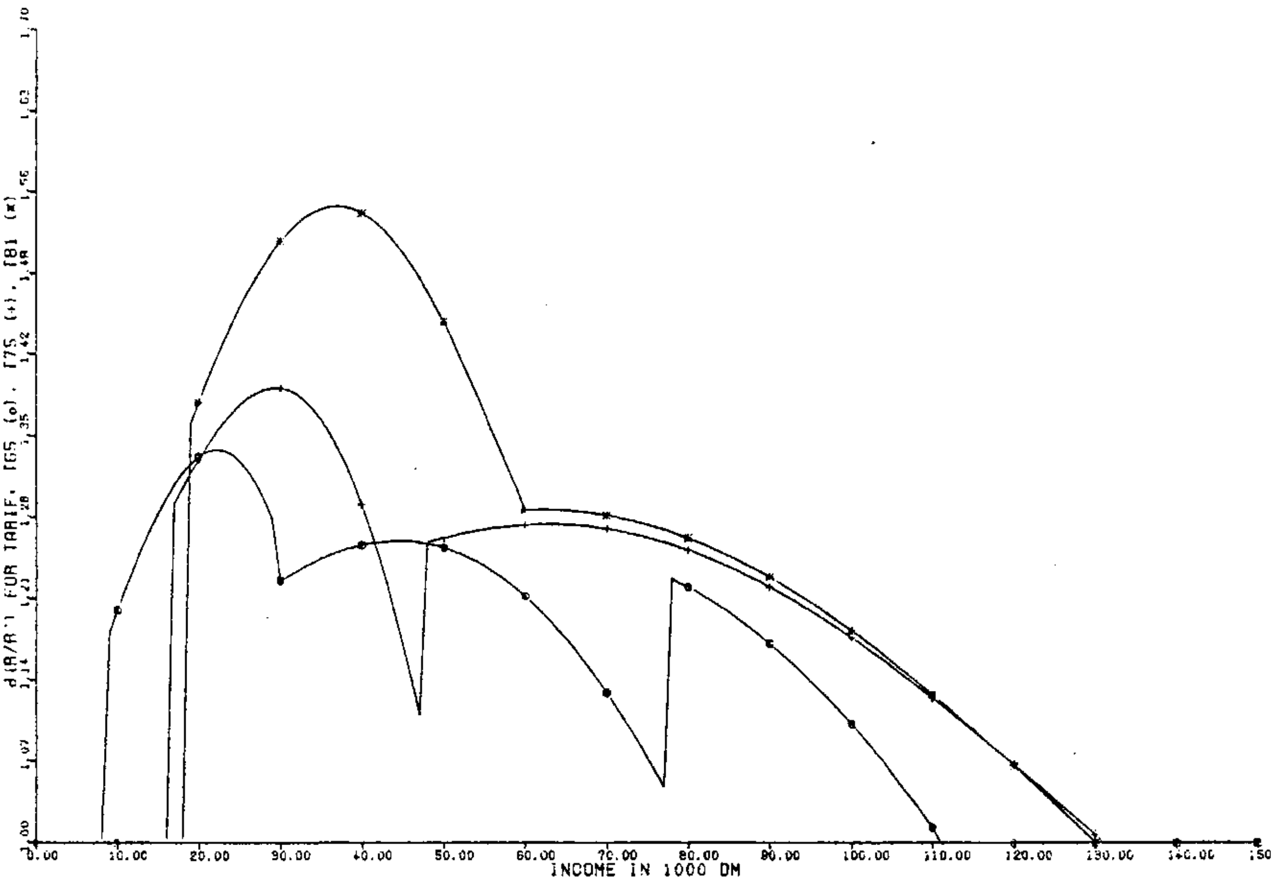


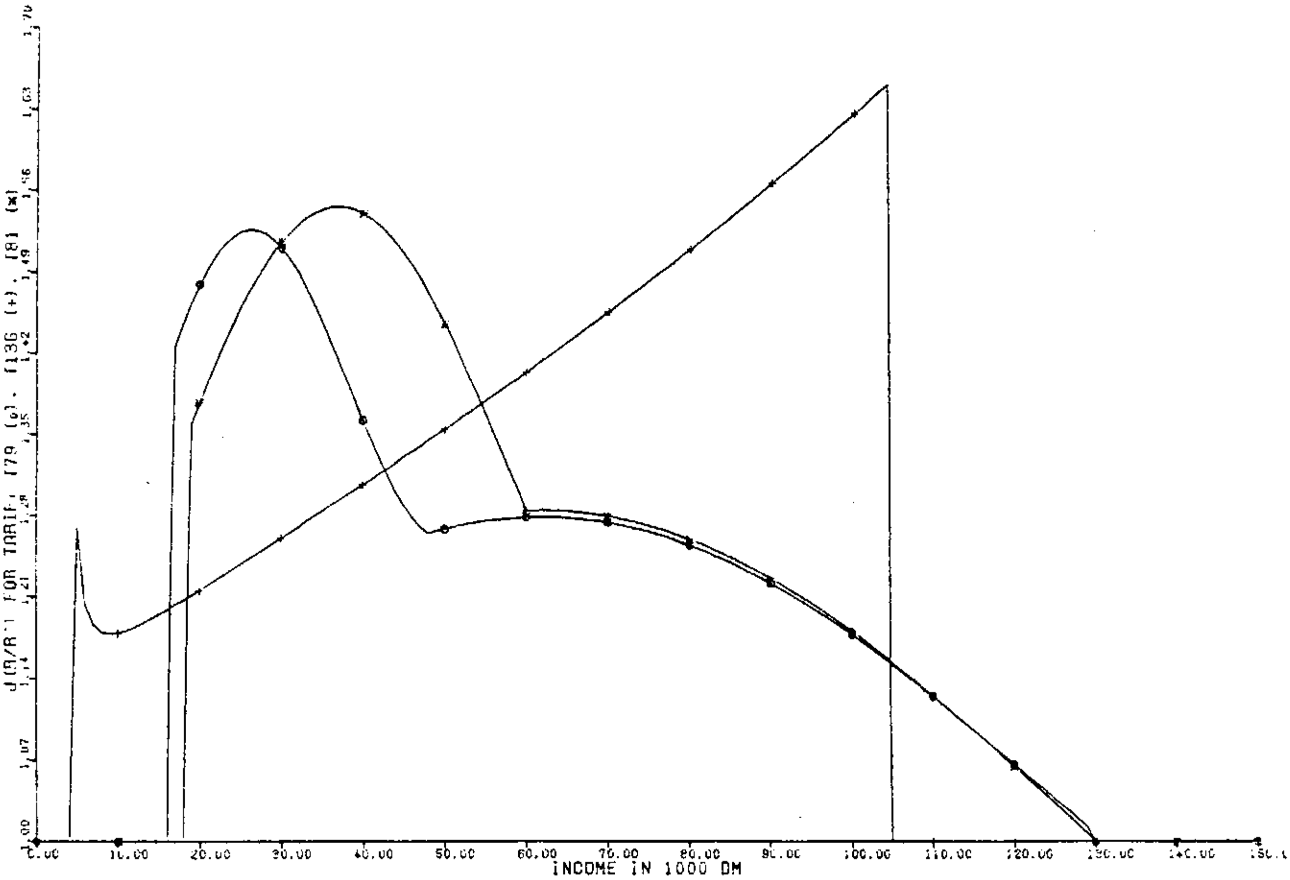
<u>Chart</u> 3: Residual progression





<u>Chart</u> 4: $\pi(\cdot,T) = \frac{d}{dY}(R/R')$





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