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A Model of Oligopolistic Size Structure and Profitability

by Reinhard Selten

Empirical studies have explored the connection between measures of concentration and measures of profitability [2], [3],[6],[7],[12]. Standard oligopoly models do not seem to suggest an easy explanation of the observed relationships. This is expecially true for the marginal concentration effect [6],[7]. It is the purpose of this paper to introduce a new oligopoly model as an attempt to provide an integrated theoretical explanation of several empirical phenomena.

1. Size Structure

Different variables can be used in order to measure the relative size of a firm in a market, e.g. sales, revenue, capacity or assets. Profitability can be measured in various ways, too. At least as far as profitability is concerned the empirical literature conveys the impression that different definitions tend to produce similar results [3]. The special properties of the new model to be introduced later suggest a definition of relative size and profitability in terms of fixed costs.

Consider a market with n suppliers and let s_i be the relative size of firm i, where the firms are numbered in order of decreasing size. We have:

$$(1) \qquad s_1 \geq s_2 \geq \dots \geq s_n > 0$$

and

$$(2) \qquad \sum_{i=1}^{n} s_i = 1$$

Readily available statistical sources do not contain data for individual firms but only for groups of four. Therefore the four firm concentration ratio

(3) $c = s_1+s_2+s_3+s_4$

has become a popular measure of concentration. Another characteristic of the size structure is the marginal concentration ratio:

(4) $m = s_5+s_6+s_7+s_8$

Size distributions of firms in actual markets exhibit certain statistical regularities. They tend to be skewed positively. Typically one finds few large firms and many small firms. References to the empirical literature on this subject can be found in [8]. The log normal distribution and the Pareto distribution seem to fit the data reasonably well.

On the basis of these findings it becomes understandable why relatively crude measures like the four firm concentration ratio c and the marginal concentration ratio m can be used successfully in empirical studies. In view of the similarities between observed size distributions these two parameters may characterize a given distribution sufficiently well.

Theories have been proposed which explain economic size distributions as limit distributions of stochastic processes. References to the literature can be found in [8]. A possible basis of such explanations is the empirical observation that there seems to be no strong relationship between the size and the rate of growth of a firm [4]. This suggests an assumption often referred to as Gibrat's law: the growth rate of a firm is a random variable independent of size. The log normal distribution is the limit distribution of a stochastic process governed by Gibrat's law. The Pareto distribution can be obtained by modified assumptions on the nature of the stochastic process [8].

Obviously stochastic theories of this kind are essentially unrelated to price theoretical considerations. Random deviations from economic law rather than economic forces towards long run equilibrium are seen as the basic cause of size differences between firms.

It is reasonable to assume that in the long run all firms in a mature market operate under the same cost and demand conditions. Competition and the diffusion of knowledge will tend to eliminate technological advantages and superior marketing positions.

In the framework of standard oligopoly models this picture of long run equilibrium is incompatible with persistent size differences. [1] As an example we may look at the simple case of the Cournot model with linear costs and linear demand. If the cost function is the same for all firms, the equilibrium supply according to Cournot's theory is the same for all firms.

It would be premature to conclude that stochastic processes based on Gibrat's law or on modified assumptions provide the only sensible explanation of the empirical observations. An alternative view is suggested by the new oligoply model to be introduced in this paper. There the long run equilibrium solution specifies a different size for each firm. The exact size structure depends on cost and demand conditions. The typical properties of empirical size distributions can be reproduced by suitable assumptions on functional forms and parameter values.

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1) This is true under rather general assumptions for strategically aggregatable static oligopoly models in the sense of [9]. It is an immediate consequence or theorem 3,p.157 in [9] that the Cournot Nash equilibria of symmetric models are symmetric.

2. Concentration and Profitability

Empirical investigations of the relationship between concentration and profitability have used different measures of profitability such as profit margin over cost [3] or rate of return on equity [2].

In the following r_i will denote the profit rate of firm i and the symbol r will be used for the profit rate of the market as a whole. The method of measurement for the s_i and r_i will be left unspecified. In this way it will be easier to summarize the most important empirical findings. For the sake of shortness the emerging general picture will be described in a necessarily superficial way. The reader who wants more precise information must look at original sources like [3] and [12].

Of course, if a firm operates in several markets, the relative size s_i refers only to that part of the firm involved in production for the market under consideration.

We are interested in relationships between the size variables s_i, r, m on the one hand and profit variables r_i and r on the other hand. The following three "effects" have been observed in the data:

Relative size effect: The relative size s_i of a firm i is positively correlated with its profit rate r_i.

Concentration effect: The four firm concentration ratio c is positively correlated with the profit rate r of the market.

Marginal concentration effect: For the medium range $.3 \leq c \leq .7$ of the four firm concentration ratio c, the marginal concentration ratio m is negatively correlated with the profit rate r of the market.

Generally, less attention is paid to the marginal concentration effect than to the concentration effect. Nevertheless, the correlation between m and r is of considerable theoretical importance since the negative sign does not seem to conform to the idea that an increase of overall concentration always reduces competition.

Miller's first presentation of the marginal concentrations effect has been criticized on the basis that the negative correlation between r and m may not be more than a statistical artifact arising from the fact that m is bounded from above by both c and 1-c [3],[6]. Therefore, later the effect has been explored by statistical investigations restricted to intervals in the medium range of c, where this objection looses its force [7], It can be said that the evidence for the marginal concentration effect is strong enough to require the attention of the theorist.

The usual interpretation of the three effects does not rely on formal models but rather on qualitative arguments. One line of reasoning which seems to be present in the thinking of researchers in the field of industrial organization runs as follows:

The relative size effect may be partly due to increasing returns to scale but since bigger firms can be expected to operate beyond minimum optimum scale this explanation is not very plausible for the range of large relative sizes. Therefore it is more likely that the higher profit rates connected with greater shares of the market are due to advantages of market power

The concentration effect can be explained by the assumption of a causal relationship between concentration and the strength of oligopolistic interdependence of behavior. Oligopolists are believed to depend on each other via the anticipation of their competitor's reactions.

The usual interpretation of the marginal concentration effect is also connected to the idea that oligopolistic interdependence explains the excessive profitability of concentrated markets. It is believed that this kind of behavioral coordination is facilitated by the absence of more than a few firms of greater relative size. If the number of such firms is increased without any change in the four firm concentration ratio c, the marginal concentration ratio m is increased and the conditions for oligopolistic interdependence become less favorable.

The new model to be introduced in this paper is motivated by a somewhat different interpretation of the three effects.

Descriptions of special industries convey the impression that the typical statistical market is an aggregate of many submarkets. Alemson's account of the Australian chocolate industry may serve as an example [1]. There, simple products like cooking chocolate are produced by many suppliers whereas only the big firms have the necessary technical equipment in order to make more sophisticated products like chocolate confections. As a result of stronger competition the submarkets for simpler products are less profitable than the submarkets for technologically more advanced products.

This suggests the following explanation of the relative size effect: A greater relative size is associated with the capability to produce technoligically more complicated products. Fewer competitors supply the submarkets for technologically more complicated products. Therefore these submarkets tend to be more profitable. Firms of greater relative size have higher profit rates since they have access to more profitable segments of the market.

In the framework of the new model the three effects are interpreted as long run equilibrium phenomena. It is important to point out that this excludes direct causal relationships between size variables and profit varibales.

The long run equilibrium values of both types of variables are determined by the cost and demand conditions of the market. The economic environment is seen as the common cause of concentration and profitability.

The concentration effect is not unrelated to the relative size effect. A higher concentration results from a greater relative importance of submarkets for technologically more complicated products. The same causal factor favors the profitability of the market as a whole.

The explanation of the marginal concentration effect as a long run equilibrium phenomenon is more complicated. It is necessary to consider a shift of the cost and demand conditions which leaves the four firm concentration ratio c constant and increases the relative importance of submarkets for technologically more complicated products. The new model suggests that the decrease of relative size as a function of rank will be steeper after a shift of this kind. In this way a higher profit rate r will be associated with a lower marginal concentration ratio m and the same four firm concentration ratio c.

3. Preliminary Description of the Model

Basically, the model is very simple. Nevertheless, a theoretically satisfactory description requires some technical detail. In order to avoid a spurious impression of complexity we shall first introduce the model in a somewhat informal way. A more rigorous presentation will be given later.

The market is modelled as an aggregate of a continuum of submarkets. As far as demand is concerned the submarkets are independent of each other. On the production side the market as a whole is bound together by a strong complementarity of fixed costs.

Each of the submarkets is characterized by a parameter $z>0$, the technological level of the product.

The salient decision parameter of firm i is its technological level z_i. A firm in the market cannot produce any products other than those at or below its technological level. There may be potential competitors with $z_i=0$ who do not enter the market. Fixed costs are a function $F(z_i)$ of the firm's technological level. This function F is the same for all potential competitors.

The submarkets are represented by symmetric Cournot models with linear costs and linear demand. The word "symmetric" indicates that costs are the same for all suppliers. No fixed costs in addition to $F(z_i)$ but only proportional costs arise in the submarkets.

We assume that Cournot-equilibrium is reached on each of the submarkets. (A game theoretical justification is given later). Let g_k be Cournot equilibrium profits of a supplier in a symmetric Cournot market with linear demand, proportional costs, and k suppliers. It is well known and it can be verified easily that the following relationship holds independently of the demand and cost parameters:

$$(5) \qquad g_k = \frac{4}{(k+1)^2}\, g_1 \qquad \text{for } k = 1,2,\ldots.$$

Therefore we need not specify the demand and cost parameters for the submarkets. It is more convenient to make assumptions directly on $g_1(z)$, the monopoly profits on submarket z. The oligopoly profits $g_k(z)$ for k suppliers are related to $g_1(z)$ as in equation (5).

In order to show that long run equilibrium exists and determines a unique size structure it is sufficient to make qualitative assumptions on $g_1(z)$ and $F(z)$. If this were the only purpose of the model we could leave functional forms unspecified and rely on the following requirements: $g_1(z)$ and $F(z)$ are continuously differentiable functions defined on $z \geq 0$. The derivative $f(z)$ of $F(z)$ is nondecreasing and $g_1(z)$ is decreasing with $g_1(z) \to 0$ for $z \to \infty$. Moreover, we have $F(0) = 0$.

The derivative $f(z)$ of $F(z)$ can be interpreted as marginal costs of technological capability. It is not unreasonable to assume that these marginal costs are nondecreasing.

The assumption that $g_1(z)$ is a decreasing function of z can be justified by the idea that technologically more complicated products tend to be more specialized and that there is a smaller demand for more specialized products.

Since we are interested in more than existence and uniqueness we shall make the simplifying assumption that both $f(z)$ and $g_1(z)$ are linear functions.

Figure 1 exhibits the graphical representation of the model and its equilibrium solution. The technological level z is shown on the abscissa. The Cournot oligopoly profits $g_k(z)$ and the marginal costs of technological capability $f(z)$ are represented by lines in the diagram.

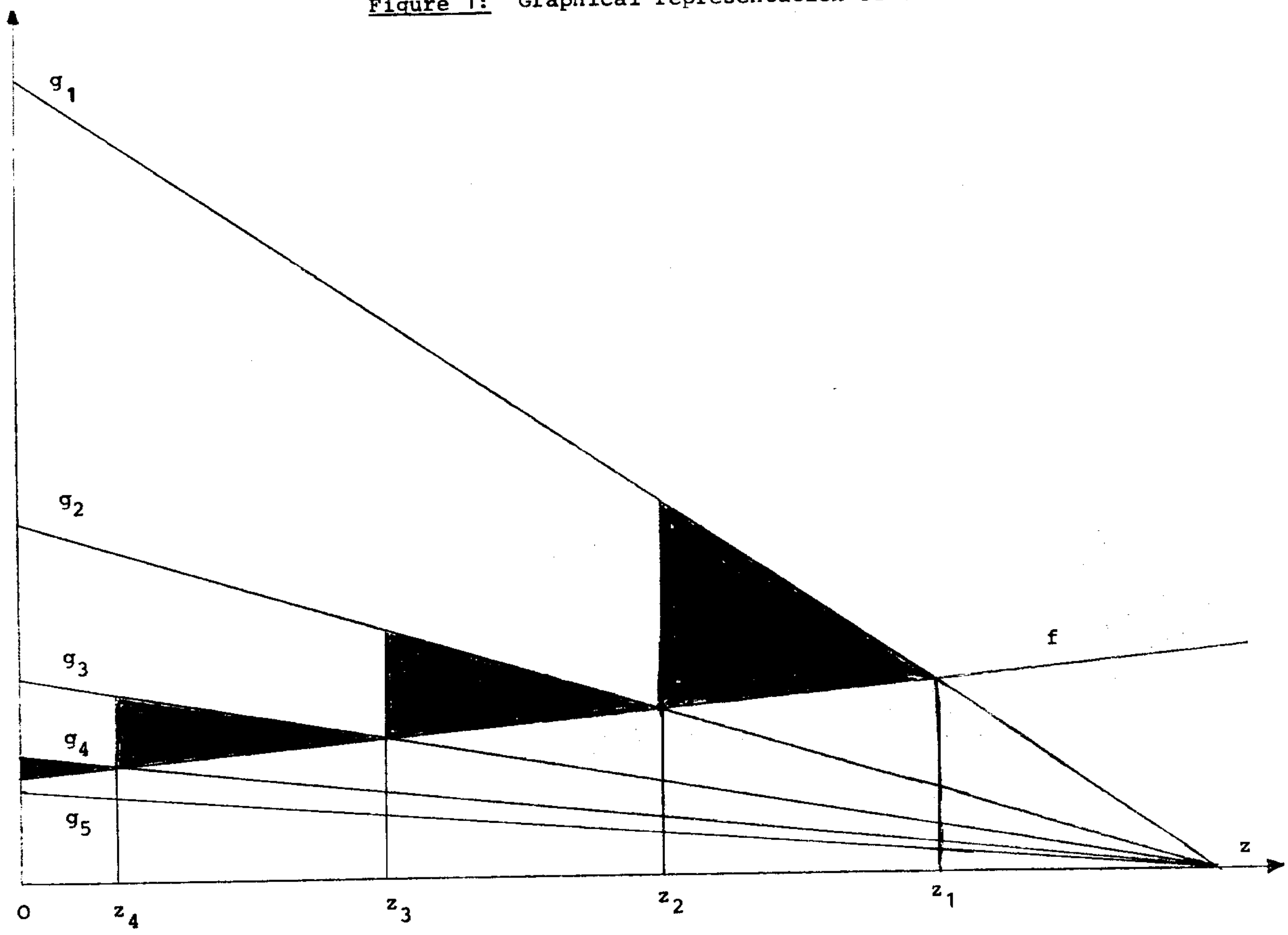
Since we consider a continuum of submarkets, each of them must be thought of as infinitely small. Strictly speaking the $g_k(z)$ are profit densities rather than profits.

Suppose that the firms are numbered in the order of decreasing technological level z_i. Then the equilibrium values of the z_i must satisfy the following marginal condition:

(6) $\quad g_i(z_i) = f(z_i) \quad$ for $i = 1,\ldots,n$

where n is the number of firms in the market. The reasons for this will become clear very soon. In the diagram the equilibrium levels z_i are found as the z-values corresponding to the intersections of the g_i-lines with the f-line.

Figure 1: Graphical representation of the model



We assume that there are more potential suppliers than firms in the market. This means that we have

(7) $z_{n+1} = 0$

In the example of figure 1 we have n = 4 since for $z \geq 0$ the line g_5 does not intersect the line f. In this way the model determines the number of firms in the market.

Since each firm i supplies all submarkets z with $0 < z \leq z_i$, the submarkets in the interval

(8) $z_{k+1} < z \leq z_k$

are supplied by exactly k firms. We refer to interval (8) as the k-th market segment. In a submarket z in the k-th market segment each of the suppliers i with $z \leq z_i$ receives $g_k(z)$. This is not yet the density of net profits but rather the density of gross profits before the deduction of fixed costs.

Total individual net profits G_k arising from the k-th market segment can be identified easily in the diagram. G_k is nothing else than the area of the shaded triangle over interval (8).

The area above (8) and below f is nothing else than the additional fixed costs which are incurred by a rise of the firm's technological level from z_{k+1} to z_k. On the other hand, the area below g_k and above (8) represents total indivual gross profits arising from the k-th market segment. The difference between the latter and the former is G_k.

It is now clear that total net profits P_i of firm i can be obtained as the sum of all G_k with k=i,...,n.

In order to see why the marginal conditions (6) must be satisfied we consider small deviations from the equilibrium condition. Suppose that firm i increases its technological level by a small amount. For the sake of concreteness assume $i>1$. For $i=1$ essentially the same argument is valid. After the increase firm i will supply some submarkets formerly belonging to the (k-1)-th market segment but now there are k suppliers on these submarkets. Additional gross profits are represented by an area below g_k whereas additional fixed costs correspond to a greater area below f.

Now consider the case that firm i decreases its technological level by a small amount. It is clear that the savings in fixed costs are less than the loss of gross profits.

A more rigorous justification of the equilibrium solution requires the explicit specification of a game structure. This will be the task of the next section.

It will be convenient to consider a slightly more general version of the model where the choice of the technological level z_i is restricted to an interval.

(9) $0 \leq z \leq \bar{z}$

where $\bar{z}$ is a positive constant. Obviously one receives the same results as in the unrestricted case if $\bar{z}$ is chosen sufficiently great.

For the sake of graphical clarity an extreme example with a four firm concentration ratio c=1 has been chosen for figure 1. Obviously the number of firms in the market depends on f(0). The smaller f(0) is, the more firms will be found in the market. A small f(0) together with a sufficiently steep slope of f will produce an equilibrium size structure with a few big firms and many small ones.

4. Game Structure of the Model

In order to clarify the game theoretical nature of the equilibrium solution it is necessary to describe the model as an extensive game. For this purpose we need more detailed assumptions on the submarkets.

4.1 Cost and demand: Let

$$(10) \qquad p(z) = \max\,[0, \alpha_z - \beta_z x(z)]$$

be the demand function of submarket z. Here x(z) is the total supply on submarket z. Let K(z) be the set of all firms i with $z \le z_i$. Thus x(z) can be written as follows:

$$(11) \qquad x(z) = \sum_{i \in K(z)} x_i(z)$$

The rate of proportional costs on submarket z is denoted by γ_z. If α_z, β_z and γ_z are positive and $\alpha_z \ge \gamma_z$ holds, the monopoly profits $g_1(z)$ on submarket z are given by the following equation:

$$(12) \qquad g_1(z) = \frac{(\alpha_z - \gamma_z)^2}{4\beta_z}$$

General assumptions on cost and demand and the number N of potential competitors are listed below. For the sake of completeness we repeat some assumptions already introduced in section 3.

(a) Fixed costs: F(z) is a continuously differentiable function of z defined on the interval (9). The derivative f(z) of F(z) is positive and nondecreasing in the whole interval. Moreover we have

$$(13) \qquad F(0) = 0$$

(b) Submarkets: α_z, β_z and γ_z are continuous functions of z in the interval (9). Moreover everywhere in (9) the parameters α_z, β_z and γ_z are positive and we have $\alpha_z \geq \gamma_z$. In addition to this $g_1(z)$ as defined by (12) is decreasing in the whole interval (9).

(c) Number of potential competitors: The number N of potential competitors is sufficiently great in the following sense:

$$(14) \qquad \frac{4}{(N+1)^2} g_1(0) < f(0)$$

4.2 Linear specification: Results on the existence and uniqueness of the equilibrium size structure can be obtained with the help of (a), (b) and (c) in 4.1. The investigation of the three effects described in section 3 requires more detailed assumptions. Therefore we introduce linear specifications for g_1 and f:

$$(15) \qquad g_1(z) = a-bz$$

$$(16) \qquad f(z) = u+vz$$

Here according to (a), (b) and (c) in 4.1 the parameters a,b and u are positive and v is non-negative.

4.3 Total net profits: With or without the linear specifications total net profits for a firm i with $z_i>0$ are as follows:

$$(17) \qquad P_i = \int_0^{z_i} x_i(z)(p(z)-\gamma_z)dz - F(z_i)$$

For firms with $z_i=0$, total net profits P_i are defined as zero.

<u>4.4 Remarks</u>: One might wish to substitute direct assumptions on α_z, β_z and γ_z for the assumptions on g_1. Suppose that quantity units are chosen in such a way that α_z is equal to a constant α in the interval (9). Further assume that γ_z is nondecreasing and β_z is increasing. If this is the case g_1 is a decreasing function of z. The linear specification (15) can be obtained by assuming that proportional costs are constant and that the saturation quantity α/β_z is a linear function of z.
The continuity assumptions on α_z, β_z and γ_z secure the existence of the integral in (17) if the $x_i(z)$ are continuous functions of z.

Requirement (c) is necessary in order to give the model the character of an open market with at least one potential competitor who remains outside at the equilibrium solution.

<u>4.5 The extensive game</u>: The players of the game Γ are N potential competitors, referred to as players 1,...,N.
The game is played in two stages:

<u>Stage 1</u>: At stage 1 each player i selects a technological level z_i in the interval (9). These choices are simultaneous and independent of each other. After the selections have been made the resulting combination of technological levels

(18) $$Z = (z_1, \ldots, z_N)$$

is made known to all players.

<u>Stage 2</u>: Let K be the set of all players i with $z_i > 0$. Only the players in K have to make choices at stage 2. Each player $i \in K$ selects a continuous and non-negative supply function defined on the interval

(19) $$0 \leq z \leq z_i$$

These choices are simultaneous and independent of each other. After the choices have been made each player i receives his total net profits as payoffs. For $i \in K$ total net profits are given by (17) and for $i \notin K$ we have $P_i = 0$.

4.6 Interpretation: The two stage structure of Γ is motivated by the idea that the technological level is a long run decision whereas the choice of supply quantities is a short run decision. It is reasonable to suppose that the long run decisions on the z_i are taken as given, when the short run decisions on the $x_i(z)$ are made. Exactly this is achieved by modelling the situation as a two stage game.

4.7 Tree structure: The game Γ may be characterized as an extensive game of finite length with infinitely many choices at the information sets. The usual game theoretical definitions of choices, information sets, strategies, etc. can be transferred without difficulty to such games [5], [10].

It is useful to look at the tree structure of Γ in some detail. At stage 1 every player i has one information set where he selects his technological level z_i. Formally the tree structure must specify an order in which these decisions are made but it is clear that the order does not matter.

Since after the decisions of stage 1, the combination of technological levels $Z = (z_1,\ldots,z_N)$ is made known to all players, the game has as many subgames as there are such combinations Z. The subgame corresponding to Z is denoted by Γ_Z.

In each of the subgames Γ_Z each player $i \in K$ has one information set where he selects his supply function $x_i(z)$. In view of the nature of the decisions to be made the subgames Γ_Z will also be called supply games.

A pure strategy π_i of player i in Γ is a pair $\pi_i = (z_i, \sigma_i)$ where σ_i is a function which assigns a non-negative continuous supply function $x_i=\sigma_i(Z)$ to every Z. Only pure strategies will be considered here.

4.8 The solution concept: The game Γ will be analyzed as a non-cooperative game. The solution concept to be used will be that of a subgame perfect pure strategy equilibrium point. It has been argued elsewhere that equilibrium points which do not satisfy the requirement of subgame perfectness cannot be regarded as rational solutions of extensive games [10]. The more refined concept of perfectness defined in [11] cannot be applied here since it has not been extended to games with infinitely many choices.

An equilibrium point is defined as a strategy combination where no player can improve his payoff by the selection of a different strategy as long as the other players stick to their strategies in the combination. An equilibrium point is subgame perfect if it induces an equilibrium point on every subgame. For the game Γ to be analysed here, this means that it is not sufficient to find an equilibrium point of the game Γ as a whole. The equilibrium solution must also specify equilibrium points for each of the subgames Γ_Z.

The subgame perfect pure strategy equilibrium point of Γ can be found as follows: We first analyse the subgames Γ_Z. As we shall see, each Γ_Z has one and only one equilibrium point in pure strategies.

After the completion of the task of solving the subgames we must solve the truncated game $\bar{\Gamma}$. This truncated game $\bar{\Gamma}$ is obtained as follows: Let P_Z be the equilibrium payoff vector of Γ_Z. In $\bar{\Gamma}$ stage 2 is deleted and the players receive payoffs according to P_Z directly after Z has been determined at stage 1.

The strategies of $\bar{\Gamma}$ are the technological levels z_i. In a somewhat informal way the game $\bar{\Gamma}$ has been analysed already in section 3. The truncated game $\bar{\Gamma}$ has many equilibrium points in pure strategies but they all agree up to the numbering of the players. If the players are renumbered in order of decreasing technological levels, the result is always the same.

The subgame perfect pure strategy equilibrium points of Γ can be obtained by combining the pure strategy equilibrium points of $\bar{\Gamma}$ with the uniquely determined pure strategy equilibrium points of the supply games Γ_Z.

4.9 Interpretation: The application of the subgame perfect equilibrium point concept implies the absence of cooperation. One may either assume that cartel laws are effectively enforced or that institutional factors like the possibilty of secret rebates prevent stable agreements.

In view of the purpose of the investigation the assumption of non-cooperative behavior is of additional significance. The usual explanations of the three effects described in section 2 involve references to collusion or oligopolistic interdependence in the sense of reaction function theory. It is important to emphasize the difference between these explanations and those proposed here. The influence of cost and demand conditions on the structure of long run equilibrium rather than collusion or oligopolistic behavior is seen as the basic force behind the empirical phenomena.

5. The Equilibrium Solution

The formal derivation of the equilibrium solution will follow the procedure outlined in 4.8. We begin with the investigation of the supply games Γ_Z.

Lemma 1: Under the assumptions (a) and (b) of 4.1 the supply game Γ_Z corresponding to $Z=(z_1,\dots,z_N)$ has one and only one pure strategy equilibrium point. The equilibrium supply quantities are as follows:

$$x_i(z) = \frac{\alpha_z - \gamma_z}{\beta_z(k(z)+1)} \qquad (20)$$

for $0<z\leq z_i$; where $k(z)$ denotes the number of players i with $z\leq z_i$.

Proof: Profit maximizing supply quantities are uniquely determined on each of the submarkets. In view of the continuity requirements on the $x_i(z)$ this has the consequence that given the supply functions of the other players each player $i\varepsilon K$ has exactly one optimal supply function. (Without the continuity requirements deviations on a set of measure zero were possible). The assertion of the lemma follows by the fact that each of the submarkets has exactly one Cournot equilibrium with supply quantities as in (20).

Ordered equilibrium points: A pure strategy equilibrium point of Γ or of the truncated game $\bar{\Gamma}$ will be called ordered if the equilibrium values of the technological levels satisfy the following inequality

$$z_1 \geq z_2 \geq \dots \geq z_N \qquad (21)$$

Since both Γ and $\bar{\Gamma}$ are completely symmetric it is clear that in both cases every pure strategy equilibrium point can be obtained from an ordered pure strategy equilibrium point by renumbering the players in some suitable way.

As we shall see $\bar{\Gamma}$ has only one ordered pure strategy equilibrium point. The structure of this equilibrium point is that of the combination described in the following lemma 2.

Lemma 2: Under the assumptions (a) and (b) of 4.1 there is one and only one combination $Z=(z_1,\dots,z_N)$ with the following properties:

(22) $z_i = \bar{z}$ for $g_i(\bar{z}) \geq f(\bar{z})$

(23) $g_i(z_i) = f_i(z_i)$ for $g_i(0) > f(0)$ and $g_i(\bar{z}) < f(\bar{z})$

(24) $z_i = 0$ for $g_i(0) \leq f(0)$

This combination Z satisfies (21).

Proof: If the conditions in (23) are satisfied equation (23) has exactly one solution since g_i is decreasing and f is non-increasing. Moreover in view of $g_{i+1} < g_i$ we have $z_i \geq z_{i+1}$.

Lemma 3: Under the assumptions (a) and (b) in 4.1 the combination Z characterized by (22), (23) and (24) is an equilibrium point of the truncated game $\bar{\Gamma}$.

Proof: Total net profits P_i at Z can be written as follows:

$$(25) \qquad P_i = \sum_{k=i}^{N} \int_{z_{k+1}}^{z_k} [g_k(z) - f(z)]dz$$

where z_{N+1} is defined as zero. Only those integrals contribute non-vanishing terms to the sum on the right hand side of (25) where we have $z_{k+1} < z_k$. This is the case where (23) applies to $i = k$ or $i = k+1$. Since g_k is decreasing it follows that the integrand is positive for all z with $z_{k+1} < z < z_k$.

Since the integrand is positive almost everywhere a player who deviates from Z by a decrease of z_i will diminuish his total net profits.

Now consider an increase of z_i. After the increase the net profit density $g_k(z) - f(z)$ for $z_{k+1} < z < z_k$ with $k < i$ will be changed to $g_{k+1}(z) - f(z)$ which is negative. Therefore a player who deviates from Z by an increase of z_i will diminuish his total net profits.

Lemma 4: Under the assumptions (a) and (b) in 4.1 the truncated game $\bar{\Gamma}$ has one and only one ordered pure strategy equilibrium point, namely the combination Z characterized by (22), (23) and (24).

Proof: It remains to show that every ordered pure strategy equilibrium point $Z^* = (z_1^*, \ldots, z_N^*)$ of $\bar{\Gamma}$ must be equal to the combination Z characterized by (22), (23) and (24). The theorem will be proved by showing that for $Z^* \neq Z$ at least one player k is not in equilibrium with respect to Z^*.

Let k be the smallest number with $z_k \neq z_k^*$. Suppose that we have $z_k^* < z_k$. Then a deviation from z_k^* to z_k will increase player k's total net profits since as far as the submarkets $z > z_k^*$ are concerned the situation is the same as at Z.

Now suppose that we have $z_k^* > z_k$. A similar argument applies here. As far as the submarkets $z > z_k$ are concerned the situation is the same as at Z. Therefore a deviation from z_k^* to z_k will increase player k's total net profits.

Theorem 1: Under the assumptions (a) and (b) of 4.1 the game Γ has one and only one ordered subgame perfect pure strategy equilibrium point. At this equilibrium point the technological levels are characterized by (22), (23) and (24), the supply functions are given by (20) and the equilibrium payoffs are the total net profits P_i in (25).

Proof: The theorem is an immediate consequence of the definition of subgame perfectness together with lemmas 1,2,3 and 4.

Equilibrium solution: For the sake of shortness the ordered subgame perfect pure strategy equilibrium point of Γ will be called the equilibrium solution.

In order to avoid unnecessary technical detail the following correlary of theorem 1 is expressed without giving a formal definition of the words "transformed by a renumbering of the players".

Correlary: Any renumbering of the players transforms the equilibrium solution into a subgame perfect pure strategy èquilibrium point of Γ and every subgame perfect pure strategy equilibrium point of Γ can be obtained in this way.

Proof: The correlary is an immediate consequence of the complete symmetry of the game Γ.

Lemma 5: Let n be the greatest positive integer with $g_n(0) > f(0)$. If there is no such n, define n=0. Let h be the greatest positive integer with $g_h(\bar{z}) \geq f(\bar{z})$. If there is no such h, define h=0. Under the assumptions (a), (b) and (c) of 4.1 the technological levels $z_1, \ldots, z_N$ at the equilibrium solution have the following properties:

(26) $z_i = \bar{z}$ for $i = 1, \ldots, h$

(27) $0 < z_i < \bar{z}$ for $i = h+1, \ldots, n$

(28) $z_i = 0$ for $i = n+1, \ldots, N$

Moreover the z_i with $i=1,\ldots,n$ do not depend on N.

Proof: It follows by the definition of n and h that (22), (23) and (24) apply to (26), (27) and (28). Assumption (c) secures $N > n$.

Remark: Lemma 5 shows that cost and demand conditions alone determine the essential features of the equilibrium solution provided that N is sufficiently great in the sense of assumption (c).

Theorem 2: Let g_1 and f be linearly specified as in (15) and (16) with positive parameters a, b, u and with $v \geq 0$. let n and m be defined as in lemma 5 and assume $N > n$. Then the technological levels $z_1, \ldots, z_N$ at the equilibrium solution are as follows:

(29) $z_k = \bar{z}$ for $k=1,\ldots,h$

(30) $$z_k = \frac{\frac{4a}{(k+1)^2} - u}{v + \frac{4b}{(k+1)^2}} \qquad \text{for } k = h+1,\ldots,n$$

(31) $$z_k = 0 \qquad \text{for } k = n+1,\ldots,N$$

Proof: The assumptions on a,b,u,v and N are such that conditions (a),(b) and (c) in 4.1 are satisfied. (30) is obtained by solving (23) with the help of (5), (15) and (16). Therefore lemma 6 is an immediate consequence of lemma 5.

Remark: Both lemma 5 and theorem 2 do not exclude the case that there is no firm on the market. For n=0 the cases (26), (27), (29) and (30) do not apply to any i. For sufficiently great $\bar{z}$ the number h is zero. Then only (27), (28),(30) and (31) are relevant for the description of the equilibrium solution.

6. Relative Size and Profit Rate

It is the purpose of this section to investigate the relative size effect as a property of the equilibrium solution. We shall do this on the basis of the assumptions of theorem 2.

Visual inspection of figure 1 suggests that the k-th market segment is more profitable in relation to additional fixed costs than the (k+1)-th market segment. As we shall see, this impression is confirmed by theoretical results. The higher profitability of low numbered market segments leads to a relative size effect.

The most natural way to define relative size in the framework of the model proposed here is based on fixed costs as a measure of size. Other definitions would require the introduction of additional assumptions. Moreover it seems to be plausible that fixed costs are closely related to equity, a variable which has been used in empirical definitions of profitability [12].

Total fixed costs F are the sum of the fixed costs F_i of the firms in the market

(32) $$F = \sum_{i=1}^{h} F_i$$

Relative size s_i is defined as follows:

(33) $$s_i = \frac{F_i}{F}$$

for $i = 1,\ldots,n$.

A measure of profitability which fits the definition of relative size is total net profits in relation to fixed costs

(34) $$r_i = \frac{P_i}{F_i}$$

for $i = 1,\ldots,n$. We shall refer to r_i as the profit rate although it might be more adequate to speak of r_i as a rate of supernormal profit. In order to see this, suppose that there is no other capital than fixed capital. F_i may be thought of as the sum of two components, the costs of maintaining the level of fixed capital and interests on fixed capital. If the capital market is perfect the interest rate will reflect the rate of normal profits which can be identified with opportunity costs of investment. Inasmuch as normal profits are included in F_i they are not included in P_i.

6.1 Segment profitability: In the following we shall look at the profitability differences between the market segments. Define

(35) $$C_k = \int_{z_{k+1}}^{z_k} f(z)dz$$

and

(36) $$G_k = \int_{z_{k+1}}^{z_k} [g_k(z) - f(z)]dz$$

Obviously C_k and G_k are those parts of fixed costs and total net profits which are attributable to the k-th market segment. We shall call

(37) $$q_k = \frac{G_k}{C_k}$$

the profit rate of the k-th market segment. It is useful to distinguish between border segments with k=1,...,h and k=n and internal segments with k=h+1,...,n-1. In the linear specification case an explicit formula can be derived for the segment profit rate of the internal segments.

Define

(38) $$w = \frac{v}{b}$$

and

(39) $$\bar{q}_k = \frac{2k+3}{(k+1)^2} \cdot \frac{1}{2 + \frac{(2k+3)w}{4+(k+1)^2 w}}$$

It will be shown that in the linear specification case the profit rate q_k for internal segments is equal to $\bar{q}_k$

Lemma 6: Under the assumptions of theorem 2 the segment profit rates at the equilibrium solution have the following properties:

(40) $q_k = q_h$ for k=1,...,h-1

(41) $q_h \geq \bar{q}_h$

(42) $q_k = \bar{q}_k$ for k=h+1,...,n-1

(43) $$q_n \leq \bar{q}_n$$

where $\bar{q}_k$ is defined by (38) and (39).

Proof: In order to prove (42) we make use of figure 2. The k-th segment is presented there in the same way as the whole market in figure 1.

Obviously G_k is the area of the shaded triangle CDE and C_k is the area of the trapezoid ABCE. In view of $g_{k+1}(z_{k+1}) = f(z_{k+1})$ the length of ED is equal to $g_k(z_{k+1}) - g_{k+1}(z_{k+1})$. Therefore we have

(44) $$G_k = \frac{1}{2}(z_k - z_{k+1})(g_k(z_{k+1}) - g_{k+1}(z_{k+1}))$$

The area of the trapezoid ABCDE is as follows:

(45) $$C_k = \frac{1}{2}(z_k - z_{k+1})\left(g_{k+1}(z_{k+1}) + g_k(z_k)\right)$$

this yields

(46) $$q_k = \frac{g_k(z_{k+1}) - g_{k+1}(z_{k+1})}{g_{k+1}(z_{k+1}) + g_k(z_k)}$$

In view of (5) we have

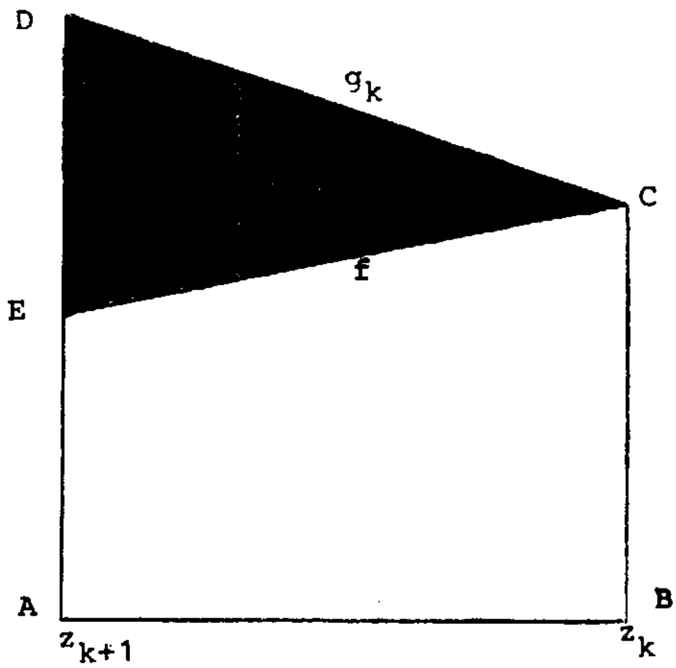
(47) $$g_k(z_{k+1}) = \left(\frac{k+2}{k+1}\right)^2 g_{k+1}(z_{k+1})$$

With the help of (47) one receives (48) by dividing numerator and denumerator of (46) by $g_{k+1}(z_{k+1})$.

(48) $$q_k = \frac{\left(\frac{k+2}{k+1}\right)^2 - 1}{1 + \frac{g_k(z_k)}{g_{k+1}(z_{k+1})}}$$

The insertion of the right hand side of (30) into (5) yields the following results

Figure 2: The k-th market segment



(49) $$g_k(z_k) = \frac{4(av+bu)}{(k+1)^2v+4b}$$

(50) $$g_{k+1}(z_{k+1}) = \frac{4(av+bu)}{(k+2)^2v+4b}$$

This together with

(51) $$\left(\frac{k+2}{k+1}\right)^2 - 1 = \frac{2k+3}{(k+2)^2}$$

has the consequence

(52) $$q_k = \frac{2k+3}{(k+1)^2} \frac{1}{1 + \frac{(k+2)^2v+4b}{(k+1)^2v+4b}}$$

Equation (52) is equivalent to (42).

It is clear that (40) holds since the players $k=1,\dots,h$ have the same technological level $z_k=\bar{z}$. Let $\bar{z}_h$ be the value obtained for $k=h$ on the right hand side of (30). Obviously we have $\bar{z}\leq\bar{z}_h$, For $\bar{z}=\bar{z}_h$ the segment profit rate q_h coincides with $\bar{q}_h$. Suppose that we have $\bar{z}<\bar{z}_h$. In order to see that (41) holds we consider the additional net profits and fixed costs of a supplier of the first h segments which would be obtained if $\bar{z}$ were increased to $\bar{z}_h$. Since f is non-decreasing and g_h is decreasing the ratio of additional profits to additional fixed costs is lower than q_h. This shows that we have $q_h\geq\bar{q}_h$.

Now consider the n-th market segment. Let $\bar{z}_{n+1}$ be the possibly negative value obtained by inserting n+1 for k on the right hand side on (30). For $\bar{z}_{n+1} = 0$ the segment profit rate q_n coincides with $\bar{q}_n$. A similar argument as in the case of q_h shows that we have $q_n<\bar{q}_n$ for $\bar{z}_{n+1}<0$. The rate of additional profits to additional fixed cost obtained by decreasing $z_{n+1}=0$ to $\bar{z}_{n+1}$ is greater than $\bar{q}_n$.

<u>Theorem 3</u>: Under the assumptions of theorem 2 the segment profit rates $q_h,\ldots,q_n$ at the equilibrium solution satisfy the following inequality:

$$(53) \qquad q_k > q_{k+1} \qquad \text{for } k = h,\ldots,n-1$$

<u>Proof</u>: In view of lemma 6 it is sufficient to show that $\bar{q}_k$ is a decreasing function of k. Equation (39) can be rewritten as follows:

$$(54) \qquad \bar{q}_k = \frac{2k+3}{(k+1)^2} \cdot \frac{4+(k+1)^2 w}{8+(k+1)^2 w+(k+2)^2 w}$$

Consider the derivative of $\log \bar{q}_k$ with respect to k

$$(55) \qquad \frac{\partial \log \bar{q}_k}{\partial k} = \frac{2}{2k+3} - \frac{2}{k+1} + \frac{2(k+1)w}{4+(k+1)^2 w} - \frac{(4k+6)w}{8+(k+1)^2 w+(k+2)^2 w}$$

An upper bound for the third term on the right hand side (55) is supplied by inequality (56)

$$(56) \qquad \frac{4(k+1)w}{8+2(k+1)^2 w} < \frac{(6k+7)w}{8+(k+1)^2 w+(k+2)^2 w}$$

One receives the right hand side of (56) by adding $(2k+3)w$ both to the numerator and the denumberator. Since $4(k+1)$ is always smaller or equal to $2(k+1)^2$ for $k \geq 1$ the fraction is increased by this operation. If we substitute the upper bound for the third term on the right hand side of (55) and then add the third and the fourth term, we receive

$$(57) \qquad \frac{\partial \log \bar{q}_k}{\partial k} < \frac{2}{2k+3} - \frac{2}{k+1} + \frac{(2k+1)w}{8+(k+1)^2 w+(k+2)^2 w}$$

The third term on the right hand side of (57) is increased if $8+2(k+1)^2$ is substituted for its denumerator. In this way it can be seen that this term is smaller than $1/(k+1)$. The first term on the right hand side (57) is smaller than $1/(k+1)$, too. This yields

$$(58) \qquad \frac{\partial \log \bar{q}_k}{\partial k} < 0$$

for $k \geq 1$. Therefore (53) holds.

<u>6.2 Relative size effect</u>: Inequality (50) in theorem 3 shows that low numbered market segments are more profitable than high numbered onws. This is the basic reason for the fact that the equilibrium solution exhibits the relative size effect expressed by theorem 4.

Theorem 4: Under the assumptions of theorem 2 profit rates and relative sizes at the equilibrium solution are related as follows: for $i = 1,\dots,n$ and $j = 1,\dots,n$ we have

(59) $r_i > r_j$ if and only if $s_i > s_j$

Proof: It is sufficient to prove

(60) $r_i > r_{i+1}$ for $i = h,\dots,n-1$

The profit rate r can be written as a weighted average of segment profit rates q_k.

(61) $$r_i = \sum_{k=i}^{n} \frac{C_k}{F_i} q_k$$

This yields

(62) $$r_i = \frac{F_{i+1}}{F_i} r_{i+1} + \frac{C_i}{F_i} q_i$$

Inequality (53) shows that for $i \geq h$ the segment profit rate q_i is greater than any of the q_k with $k>i$. It follows by (61) and (62) that (60) is true.

6.3 Remarks: The formula for $\bar{q}_k$ may still hold approximately for cases where f and g_1 are not linear if the deviations from linearity are not too severe. Of course, linear approximations for f and g_1 would have to be computed for each segment separately and instead of w in (39) one would have to use a different w_k

	bounds for $\bar{q}_k$	
k	lower bound	upper bound
1	.3846	.6250
2	.2800	.3889
3	.2195	.2812
4	.1803	.2200
5	.1529	.1806
6	.1327	.1531
7	.1172	.1328
8	.1050	.1172
9	.0950	.1050
10	.0868	.0950
11	.0799	.0868
12	.0740	.0799
13	.0689	.0740
14	.0644	.0689
15	.0606	.0645
16	.0571	.0606
17	.0540	.0571
18	.0512	.0540
19	.0488	.0512
20	.0465	.0488

Table 1: Bounds for the segment profit rate $\bar{q}_k$

for every k. It follows by (39) that $\bar{q}_k$ is a decreasing function of w. The limiting cases w=0 and $w \to \infty$ supply upper and lower bounds for $\bar{q}_k$:

$$(63) \qquad \frac{2k+3}{(k+1)^2} \cdot \frac{1}{2 + \frac{2k+3}{(k+1)^2}} < \bar{q}_k \leq \frac{2k+3}{2(k+1)^2}$$

Table 1 shows these bounds for k=1,...,20. It is interesting to note that the lower bound for $\bar{q}_k$ is smaller than the upper bound for $\bar{q}_{k+1}$ but in relation to the range of $\bar{q}_k$ the difference is very small. In view of the fact that there is little overlap between the ranges of $\bar{q}_k$ and $\bar{q}_{k+1}$ it seems to be plausible to expect that in cases of moderate deviations from linearity $\bar{q}_k$ will still be a decreasing function of k. The proof of theorem 4 does not need more than that. The relative size effect does not seem to depend in a crucial way on the linearity assumptions.

7. Concentration and Profitability in the Model

Explicit formulas for the concentration rates c and m and the profit rate r at the equilibrium solution of the linear specification case can be derived without difficulty. Unfortunately, these formulas are quite complicated. They do not seem to yield a relationship between c, m and r which can be used in order to explore the connection of r with c and m analytically.

An alternative way to investigate the question whether the equilibrium solution exhibits the concentration effect and the marginal concentration effect is the computation of numerical examples. Some first encouraging results will be presented in this section. A separate study will be devoted to a more thoroughly planned investigation of a great number of cases.

The linear specification case of the model has five parameters, a, b, u, v, and $\bar{z}$. The units of measurement for z and g_1 can be chosen in such a way that we have a=b=1. Figure 1 shows that this normalization does not influence relative sizes and profit rates. The normalized model has only three parameters, namely u,v and $\bar{z}$.

For the sake of computational conveniences the numerical examples considered here have been chosen in such a way that (30) holds not only for k=h+1,...,n but also for k=h and k=n+1; the right hand side of (30) yields 0 for k=n+1 and $\bar{z}$ for k=h. Obviously, examples with this special property can be characterized by a triple (v,n,h). The parameters v and n determine u and u and h determine $\bar{z}$.

Only three values of v have been used for the computation of the examples, namely v=0, v=0.o25 and v=0.05. It turns out that only relatively small values of v yield concentration rates c in the medium range.

Altogether 74 examples have been computed. The set of all 74 cases may best be characterized as the union of two overlapping subsamples, one containing all the 40 cases with v=0,n=8,...,12 and h=1,...,8 and the other containing all the 39 cases with v=0, v=0.025 or v=0.05 and h=1 and n=8,...,20. Obviously 5 cases are common to both subsamples.

A strong concentration effect can be seen in the numerical examples. For all 74 cases together the Spearman rank correlation coefficient between r and c is .994. The Spearman rank correlation coefficient between r and m is -.475 but this result cannot be taken as an indication of the presence of a marginal concentration effect, since the rank correlation coefficient between c and m is -.428. The correlation between r and m is mainly due to the correlation between c and m, even if the difference between the two correlation coefficients is in the right direction.

In view of the close relationship between r and c it is not surprising that a marginal concentration effect cannot be detected by rank correlation over the 74 cases. It is necessary to look at a smaller sample where the four firm concentration ratio c shows less variation. For this purpose a regression analysis has been made for those 23 cases where c is in the interval

(64) $.5 < c < .6$

This interval has been chosen, partly since it contains more cases than other intervals bounded by multiples of .1 and partly since the empirical marginal concentration effect is a phenomenon of the medium range of c. The following regression results have been obtained for the 23 cases in the interval (64).

(65) $r = .277c - .021$
with $R^2 = .87$

(66) $r = .240c - .056m + .019$
with $R^2 = .96$

Here R^2 is the measure of determination. Even if a regression on computational data does not have the usual statistical interpretation, the use of conventional statistical measures supplies some insight into the validity of (65) and (66) as approximate relationships.

It can be seen that the inclusion of the variable m improves the goodness of fit. The dependence of r on m is strong enough to lead to the rejection of the null hypothesis of random deviations from (65). (The level of significance for the F-test is 1% two-sided.)

Equation (66) shows the type of relationship between r and the concentration measures c and m which one would expect in view of empirical investigations. Both the concentration effect and the marginal concentration effect are present.

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