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Wolfram F. Richter

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H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

A NORMATIVE JUSTIFICATION OF PROGRESSIVE TAXATION: HOW TO COMPROMISE ON NASH AND KALAI-SMORODINSKY

Wolfram F. Richter

Institute of Mathematical Economics
University of Bielefeld
Bielefeld
F.R.G.

The repartitioning of a fixed tax revenue among finitely many taxpayers formally defines a bargaining problem. Nash's and Kalai-Smorodinsky's solution concepts call for the determination of equitable distributions but, in general, violate fundamental principles of just taxation. We therefore axiomatize a solution concept for general bargaining problems which was recently proposed by the author (1980). This solution concept deserves interest since upon application to taxation a marginal tax rate is demanded that independently of the utility functions under consideration is less than one but greater than the average tax rate. We thus provide the first known sacrifice theoretic justification of non-excessively progressive taxation.

INTRODUCTION

The determination of an equitable distribution of tax shares can and will be viewed as a matter for bargaining theory. This view raises hopes that normative principles of taxation can be justified on game-theoretic grounds.

Game theory provides various solution concepts which call for the application to allocating taxes. Of the many attempts let me only mention the illuminating approach of Aumann and Kurz (1977). If a solution concept finds strong normative support in game theory one might expect that the resulting distribution of tax shares displays all essential characteristics of fair taxation. However, most work carried out in this spirit falls short of such expectations.

In this paper we shall focus on equitable repartitions of a fixed amount of tax revenue. We shall consider two prominent bargaining solutions, namely Nash's one (1950) and Kalai-Smorodinsky's (1975) monotone one. As a matter of fact both solution concepts perform rather unsatisfactorily in this application. (Let us identify concepts with persons.) *Nash* generically requires a marginal tax rate of one and thus an equalization of incomes after tax. On the other hand *Kalai-Smorodinsky* amounts to the infliction of an *equal proportional sacrifice* which is the technical term traditionally used in public finance. It has been known since the nineteenth century that the application of this sacrifice concept can lead to regressive tax schedules if specific non-pathodological utility functions are considered. Cf. Cohen-Stuart (1889).

Thus neither *Nash* nor *Kalai-Smorodinsky* meet principles that seem to be fundamental to an equitable and just distribution of tax shares: The marginal tax rate ought to be greater than the average one (which defines progressivity) but less than one. Both conditions seem to find overwhelming support by empirical facts.

We shall therefore axiomatize a new solution concept for general bargaining problems that in application accords to the just mentioned basic principles of fair taxation. However, the axiomatization is not of the same logic quality as the ones of Nash and Kalai-Smorodinsky. Our axiomatization is placed on a meta-level. The proposed concept will be justified as a compromise on Nash and Kalai-Smorodinsky to be made in some *meta-bargaining conflict*. This axiomatization provides the first known rigorous sacrifice theoretic justification of non-excessively progressive taxation.

THE MODEL

Let $M = \{1,..,m\}$ denote the set of taxpayers. $i \in M$ has income $y_i > 0$ from which tax t_i is payed. Let $y = (y_1,...,y_m)$ denote the vector of incomes and $t = (t_1,..,t_m)$ the one of tax shares. Financing the public budget requires a tax revenue $g < \Sigma y_i$. The expenditure side of the budget is assumed to be exogenously fixed. Hence it is only the distribution of the fiscal burden and not its absolute magnitude that matters in the sequel. Let

$$X(y) := \{t \in \mathbb{R}^m \mid \Sigma t_i = g,\ 0 \leq t \leq y\}$$

denote the set of *feasible tax distributions*. Positive subsidies to income are thus excluded. On the other hand people cannot be taxed by more than they earn income.

Individuals derive utility from their income after tax, $y_i - t_i$, according to $u_i : \mathbb{R} \to \mathbb{R}$. We assume $u_i' > 0$, $u_i'' \leq 0$. Let $\underline{\underline{U}}$ denote the set of all such twice differentiable utility functions. Write $u = (u_1,..,u_m) \in \underline{\underline{U}}^m$.

Some notation: $t>0$ iff $t \geq 0$ and $t \neq 0$. $t \in \mathbb{R}^m_+$ iff $t \geq 0$. $t>>0$ iff $t_i > 0$ for all $i \in M$; $t \in \mathbb{R}^m_{++}$ iff $t>>0$.

We are interested in choice functions $t^* : \underline{\underline{U}}^m \times \mathbb{R}^m_{++} \to \mathbb{R}^m$, $t^*(u,y) \in X(y)$ that are *invariant with respect to affine transformations of utilities.*

The tax problem parametrized by (u,y) is mapped into the utility space. Put

$$V(u,y) := \{v \in \mathbb{R}^m \mid \exists\, t \in X(y) : v_i = u_i(y_i - t_i)\},$$

$$a(u,y) := u(0), \quad b(u,y) := (u_i(y_i))_{i \in M}.$$

The following properties are straight forward: $V(u,y)$ is non-empty, compact and convex from above. The latter will say that for all $v^0, v^1 \in V(u,y)$, $\alpha \in (0,1)$ some $v \in V(u,y)$ exists with $v \geq \alpha v^1 + (1-\alpha) v^0$. Furthermore there exists some $v \in V(u,y)$ s.t. $a(u,y) << v$. For all $v \in V(u,y)$ we obtain $a(u,y) \leq v \leq b(u,y)$. The latter property justifies to call b *bliss-point* and a *anti-bliss-point*. I hesitate to call a "threat-point" as it would reflect some institutional assumption which makes little sense in our context. Taxpayers do not threaten by the utility payoff of zero income. However, there is an institutional coercion implicit to the model to compromise on some $t^* \in X(y)$ or some $v^*(u,y) = (u_i(v_i - t_i^*))_{i \in M}$, respectively.

We assume that the mapping $v^* : \underline{\underline{U}}^m \times \mathbb{R}^m_{++} \to \mathbb{R}^m$ factorizes with respect to (V,a,b), i.e. $v^* = f(V,a,b)$ - as functions of (u,y). Hence (V,a,b) is the only piece of information getting relevant for the determination of an equitable tax distribution.

W.l.o.g. we may assume $a(u,y) = u(0) = 0$ as $t^*(u,y)$ is required to be invariant under affine transformations of utilites. Put $(V,b) := (V,0,b)$, $f(V,b) := f(V,0,b)$. Furthermore let $\underline{\underline{P}}^{tax}$ denote the set of all $(V,b) = (V,b)(u,y)$.

Instead of considering $t^* : \underline{\underline{U}}^m \times \mathbb{R}^m_{++} \to \mathbb{R}^m$ we focus on point-valued mappings
$f : \underline{\underline{P}} \to \mathbb{R}^m$, $f(V,b) \in V$ where $\underline{\underline{P}} \supseteq \underline{\underline{P}}^{tax}$.
In particular we assume

$f(AV,Ab) = Af(V,b)$ for all $(V,b) \in \underline{\underline{P}}$ and all linear transformations (1)

$A : (Av)_i = \alpha_i v_i$, $\alpha_i > 0$. There are two prominent axiomatizations of choice functions when (1) is to hold:

NASH (1950): Let $n(V,b) \in V$ denote the solution of $\max_{v \in V} \prod_{i \in M} v_i$;

KALAI-SMORODINSKY (1975),(Huttel et al.(1980)): let $k(V,b) \in V$ denote the solution of $\max_{v \in V} \min_{i \in M} {}^{v_i}/_{b_i}$ $\left(\Longleftrightarrow \min_{v \in V} \max_{i \in M} \frac{b_i - v_i}{b_i}\right)$.

Nash's rule tells us to maximize $\prod u_i(y_i - t_i)$ for $t \in X(y)$. Let t^n be optimal. We only consider interior solutions: $0 \ll t^n \ll y$. The first-order condition then is $u_i'(y_i - t_i^n)/u_i(y_i - t_i^n) = c(y)$ $(i \in M)$ which suggests - actually by passing to the limit $|M| \to \infty$ - to look at T_i being defined by

$$\frac{u_i'(Y - T_i(Y))}{u_i(Y - T_i(Y))} = c(y) , \quad T_i(y_i) = t_i^n(u,y) \quad \text{for all } Y \text{ out of a neighbourhood}$$

$\mathcal{O}(y_i)$. The marginal tax rate MTR_i under Nash is now *defined* by $MTR_i := T_i'(y_i)$ which obviously equals 1 in case that $u_i'' < 0$.

Consider alternatively the minimization of $\max \frac{u_i(y_i) - u_i(y_i - t_i)}{u_i(y_i)}$ for $t \in X(y)$.
For an optimal interior solution t^k we obtain $u_i(y_i) - u_i(y_i - t_i^k) / u_i(y_i) = c(y)$ $(i \in M)$ which again - by passing to the limit - suggests to look at T_i being defined by

$$\frac{u_i(Y) - u_i(Y - T_i(Y))}{u_i(Y)} = c(y) , \quad Y \in \mathcal{O}(y_i) , \quad T_i(y_i) = t_i^k(u,y) . \tag{2}$$

Again, define $MTR_i := T_i'(y_i)$. Condition (2) is known as *equal proportional sacrifice* from public finance. As Cohen-Stuart (1889) demonstrated non-pathodological utility functions u_i exist such that (2) implies regressivity of T_i at y_i:
$T_i'(y_i) \leq T_i(y_i)/y_i$.

The performance of $n(\cdot,\cdot)$ and $k(\cdot,\cdot)$ when applied to $(V,b) \in \underline{\underline{P}}^{tax}$ considerably relativizes the worth of their axiomatizations. Having the outcomes in mind one might doubt that a constituent assembly would ever unanimously agree upon one of the two characterizing sets of axioms.

Hence there are good reasons to consider the next solution concept:

PROPORTIONAL UTILITARIANISM

Let $p(V,b) \in V$ denote the solution of $\min_{v \in V} \sum_{i \in M} b_i/v_i$. (3)

Note that in the general case of $a \neq 0$ (3) has to be replaced by $\max_{v \in V} \sum_{i \in M} (v_i - b_i)/v_i - a_i)$. The broken linear transformation $(x-b_i)/(x-a_i)$ normalizes the bliss-point to zero and the anti-bliss-point to $-\infty$. The utilitarian summation rule is then applied. This may eplain the name *proportional utilitarianism.*

Note that the functional form $(\sum b_i/v_i)^{-1} =: \varphi(v)$ defines a CES-production function. Thus we know that φ^{-1} is strictly convex in $\mathbb{R}^m_{++}$ and that $p(\cdot,\cdot)$ is a well-defined point-valued mapping on $\underline{\underline{P}}^{tax}$.

Let t^p denote an optimal interior solution of $\min_{t \in X(y)} \sum u_i(y_i)/u_i(y_i - t_i)$.
Then t^p is equivalently characterized by

$$u_i(y_i)\, u_i'(y_i - t_i^p)\, [u_i(y_i - t_i^p)]^{-2} = c(y) \qquad (i \in M) .$$

For notational convenience write $U = u_i$ and

$$\frac{U(Y)\, U'(Y - T_U(Y))}{[U(Y - T_U(Y))]^2} = c(y)$$ where T_U is defined in a neighbourhood $\mathcal{O}(y_i)$

holding $T_U(y_i) = t_i^p(u,y)$. The following result is mathematically trivial:

Theorem (Richter (1980)): For all $U \in \underline{\underline{U}}$, $Y \in \mathcal{O}(y_i)$ we obtain

$1 > T_U'(Y) > T_U(Y)/Y$ where the right-hand inequality is subject to the condition: $0 < T_U(Y)/Y \leq 1/2$.

Proportional utilitarianism thus generates the wanted result. All we need is the rather mild condition on the average tax rate - namely $T(Y)/Y \leq 1/2$. The important point is that this condition has to be met simultaneously by all $U \in \underline{\underline{U}}$. Note that $U(Y) = \sqrt[q]{Y}$ implies $T_U(Y) = Y - a \sqrt[q+1]{Y}$. Such tax functions were introduced into the literature by Edgeworth (1919). They are characterized by a constant residual progression. Cf. Ricter et al. (1980) for a detailed discussion.

Unfortunatelly no convincing axiomatization of (3) is known which were logically comparable to Nash and Kalai-Smorodinsky. We therefore propose another route.

A META-BARGAINING PROBLEM

Consider some constituent assembly with two members, only. Let them be called John and Ehud. These two members have to agree on some $f : \underline{\underline{P}} \to \mathbb{R}^2$, $\underline{\underline{P}} \supseteq \underline{\underline{P}}^{tax}$ to be written into the constitution and to be applied to all future conflicts $(V,b) \in \underline{\underline{P}}$. Assume that John proposes $f=n$ and Ehud $f=k$. These individual preferences on aggregation procedures could be rationalized. E.g. imagine that John has some prior expectation with respect to the future distribution of income which favours Ehud. Such a prior expectation might make John propose $n(\cdot,\cdot)$. However, such a rationalization is not directly needed in the sequel. The supposed preferences are just taken for granted. How could or would John and Ehud compromise on this *meta-bargaining problem*? This is the question guiding our reasoning.

Let (V,b) be in $\underline{\underline{P}}^{PU}$ by definition iff

a) V is non-empty, compact and convex from above;

b) $0 \leq v \leq b$ for all and $0 \ll v$ for some $v \in V$;

c) For all $i,j \in M = \{1,2\}$ and all $v \in V$ such that $v_i > 0$ there is some $\bar{v} \in V$ holding $\bar{v}_i \leq v_i$ and $\bar{v}_j > v_j$.

Condition c) ensures that whenever $i \in M$ is not at his subsistence level there is a potential reallocation in favour of j and possibly at the cost of i. Obviously, $\underline{\underline{P}}^{tax} \subseteq \underline{\underline{P}}^{PU}$. Note that $(AV,Ab) \in P^{PU}$ if $(V,b) \in \underline{\underline{P}}^{PU}$ and if $(Av)_i = \alpha_i v_i$, $\alpha_i > 0$.

Axioms: For all $(V,b) \in \underline{\underline{P}}^{PU}$:

A0) $0 \ll f(V,b) \in V$;

A1) *Pareto efficiency*: $f(V,b) > v$ implies $v \notin V$;

A2) *Nash's independence of irrelevant alternatives relative to bliss-point* b: (V,b), $(V',b) \in \underline{\underline{P}}^{PU}$, $V \subseteq V'$, $f(V',b) \in V$ implies $f(V',b) = f(V,b)$;

A3) $f(V,b) = f(V,\beta b)$ for all $\beta \in \mathbb{R}$, $\beta \leq 1$;

A4) *transformation invariance* (1).

A0) is a technical requirement. A1-4) are all met by Nash as well as Kalai-Smorodinsky. Thus John and Ehud should all accept by *unanimity* in the meta-bargaining problem. Clearly, there is no set of axioms on which both would unanimously agree and which at the same time would single out a unique aggregation procedure. We shall therefore propose an additional axiom, below, which is justified by *symmetry* considerations on the meta-level. However, let us first note some elementary facts.

Remark 1: Under A0,1) there exists some $\lambda = \lambda^V \in \mathbb{R}^2_{++}$ s.t.

$$f(V,b) = \max \lambda \cdot V \quad (:= \max_{v \in V} \lambda \cdot v) .$$

The proof is standard. Property c) of $\underline{\underline{P}}^{PU}$ ensures strict positivity of λ.

Remark 2: $k_1/b_1 = k_2/b_2$ where $k = k(V,b)$.

Proof: Suppose $0 \leq k_1/b_1 < k_2/b_2$. By property c) some $\bar{v} \in V$ exists holding $\bar{v}_1 > k_1$, $\bar{v}_2 \leq k_2$. For $\alpha \in (0,1)$ being close to 1

$$\min_{i \in M} (\alpha k_i + (1-\alpha)\bar{v}_i)/b_i > k_1/b_1 = \min_{i \in M} k_i/b_i .$$

As V is convex from above some $\bar{\bar{v}} \in V$ exists with $\bar{\bar{v}} \geq \alpha k + (1-\alpha)\bar{v}$. The existence of such $\bar{\bar{v}}$ contradicts the definition of $k(V,b)$.

Corollary: $k(V,b) \gg 0$.

Remark 3: a) $n(V,b) \gg 0$.

b) Let $\bar{v} \gg 0$. Then $\bar{v} = n(V,b)$ iff $\lambda \cdot \bar{v} = \max \lambda \cdot V$ where $\lambda_i := 1/\bar{v}_i$.

Proof: Standard.

The following axiom demands a uniform distribution of utility payoffs for situations which are characterized by symmetry. John and Ehud, respectively, propose outcomes the ratios of which are inverse to each other. These outcomes are infact comparable as they are supported by the same vector of social weights.

A5) *Compromise*: Let $\lambda_i := 1/n_i(V,b)$ $(i=1,2)$. If $\lambda \cdot n(V,b) = \max \lambda \cdot V = \lambda \cdot k(V,b)$ and if $\dfrac{n_1(V,b)}{n_2(V,b)} = \dfrac{k_2(V,b)}{k_1(V,b)}$

then $f_1(V,b) = f_2(V,b)$.

Theorem: $p : \underline{\underline{P}}^{PU} \to \mathbb{R}^2$ satisfies A0-5) and is uniquely determined by A0-5).

Proof: The proof of $p(\cdot,\cdot)$ holding A0-4) may be skipped. The proof of A5) is given in three steps. Write n,k for $n(V,b)$ and $k(V,b)$. As $k,n \gg 0$ $\alpha := k_1/(n_2+k_1) \in (0,1)$. $\bar{p}_i := \alpha n_i + (1-\alpha)k_i$.

Step 1) $\bar{p}_1 = \bar{p}_2$

which follows from the assumption $k_1 n_1 = n_2 k_2$ after inserting the value of α.

As V is convex from above there is some $v \in V$ such that $v \geq \bar{p}$. Actually, the latter inequality must be an equality as we otherwise obtain a contradiction to $\lambda \gg 0$ and $\lambda \cdot (\alpha n + (1-\alpha)k) = \max \lambda \cdot V$. Hence

step 2) $\bar{p} \in V$. We finally show (step 3) that $\bar{p} = p(V,b)$ which considering step 1 proves A5).

Proof of $\bar{p} = p(V,b)$: $k_i/b_i = \text{const}$, $\lambda_i n_i = \text{const}$, and $n_i k_i = \text{const}$ imply

$$\frac{b_i}{\lambda_i} = \frac{b_i}{k_i}\frac{k_i n_i}{\lambda_i n_i} = \text{const}$$. By step 1 $b_i/\bar{p}_i^2 = \text{const}\ \lambda_i$. $\bar{p}$ therefore is

solution to $\min\{\Sigma b_i/v_i \mid v \in \mathbb{R}^2 : \lambda \cdot v \leq \lambda \cdot n = \lambda \cdot k\}$. By step 2 $\bar{p} \in V \subseteq \{v \in \mathbb{R}^2 \mid \lambda \cdot v \leq \lambda \cdot n\}$. Hence $\bar{p}$ solves the minimization of $\Sigma b_i/v_i$ with respect to the restricted domaine V.

We now show that $p(\cdot,\cdot)$ is uniquely determined. Let $f : \underline{\underline{P}}^{PU} \to \mathbb{R}^2$ satisfy A0-5). We fix (V,b) and claim $f(V,b) = p(V,b)$. By remark 1 some $\lambda \in \mathbb{R}^2_{++}$ exists with $\lambda \cdot f(V,b) = \max \lambda \cdot V$. Put $\alpha_i := \sqrt{\lambda_i/b_i} > 0$, $A := \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$,

$V' := AV$, $b' := Ab$. Then
$b'\cdot f(V',b') = (Ab)\cdot f(AV,Ab) = (A^{-1}\lambda)\cdot Af(V,b) = \lambda\cdot f(V,b) = \max \lambda\cdot V = \max (\lambda A^{-1})\cdot(AV) = \max b'\cdot V'$. Define

$V'' := \{ v\in \mathbb{R}^2_+ \mid b'\cdot v \leq b'\cdot f(V',b')\} \supseteq V'$, $b'' := \beta b'$ where $\beta \geq 1$ is chosen sufficiently large to guarantee $v \leq b''$ for all $v\in V''$ .Then

$k := k(V'',b'') = \text{const } b''$, $n := n(V'',b'') = \text{const } (\ldots,(b''_i)^{-1},\ldots)$ and $k_1/k_2 = b''_1/b''_2 = n_2/n_1$. A5) then implies $f_1(V'',b'') = f_2(V'',b'')$. As $p(\cdot,\cdot)$ also fulfills A5) $f(V'',b'') = \text{const } p(V'',b'')$. Because of Pareto efficiency $f(V'',b'') = p(V'',b'')$. The assertion then follows by considering the sequence of equalities:

$Af(V,b) = f(V',b') = f(V',b'') = f(V'',b'') = p(V'',b'') = \ldots = Ap(V,b)$.

q.e.d.

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REFERENCES

[1] Aumann, R.J. and Kurz, M., Power and taxes, Econometrica 45 (1977) 1137-1161.

[2] Cohen-Stuart, A.J., Bijdrage tot de theorie der progressieve inkomstenbelasting, Den Haag, 1889, Engl. translation of extracts in: Musgrave, R.A. and Peacock, A.T. (eds.), Classics in the theory of public finance (Macmillan, London, 1967).

[3] Edgeworth, F.Y., Graduation of taxes, Economic Journal (1919), in: Papers relating to political economy vol.II (Burt Franklin, New York, 1925, reprinted 1970).

[4] Huttel, G. and Richter, W.F., A note on "an impossibility result concerning n-person bargaining games", Disc. Paper, IMW, University of Bielefeld, F.R.G. (1980).

[5] Kalai, E. and Smorodinsky, M., Other solutions to Nash's bargaining problem, Econometrica 43 (1975) 513-518.

[6] Nash, J.F., The bargaining problem, Econometrica 18 (1950) 155-162.

[7] Richter, W.F., Taxation according to ability to pay, Disc. Paper, IMW, University of Bielefeld, F.R.G. (1980).

[8] Richter, W.F. and Hampe, J.F., On income tax functions, Disc. Paper, IMW, University of Bielefeld, F.R.G. (1980).

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