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On Location Conflicts and their Fair Solution Concepts

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Abstract

In this paper we consider some special geometrical problems of optimization that can be seen as a generalization of the theory of the center of circum-circle. In particular this theory is essential for the study of the location structures of location games. We classify the location structures by the behaviour of the concepts of fair solution given by Rawls (1) and Kolm (2). We check these concepts on continuity, single-valuedness and equality between each other, and we give some relations between the concerned spaces of "regular" location structures. A study on methods is given by examination of the space of location structures of triangle conflicts.

§ 1. A Special geometrical optimization, ist construction of solution and an interpretation into locational conflicts.

(1.1) The convex hull of a subject D of an $\mathbb{R}^m$ is denoted by C_D. Given a pair (A,B) of finite subsets of an $\mathbb{R}^m$ with $\emptyset \neq B \subset A$. $\| \ \|$ is the euclidian norm and d_e the corresponding distance function $d_e(x) = \|e - x\|$ for e.

(1.2) (Primal version of the optimization problem)
For such a pair (A,B) we consider the following maximizations on C_A:

(R-M) $\max\limits_{x \in C_A} \min\limits_{b \in B} d_b(x)$ and

(K-M) $\text{L-max}_{x \in C_A} \ \alpha((d_b(x))_{b \in B})$ with

L-max symbolizing lexicographic maximization and α being the permutation on the coordinates given by $\alpha: y \to y'$, $y'_i \leq y'_j$ for $i \leq j$.

(1.3) (Dual version)
Concerning the corresponding minimization problems we are going to offer just a few remarks. Their study ist something analogous to that of 1.2. So we simply give their definition at first:

(R-m) $\min\limits_{x \in C_A} \max\limits_{b \in B} d_b(x)$ and

(K-m) $\text{L-min}_{x \in C_A} \ \alpha^*((d_b(x))_{b \in B})$ with

$\alpha^*: y \to y'$, $y'_j \leq y'_i$ for $i \leq j$.

(1.4) We define the corrsponding level for the maximization problems:

(R-M) $l_R(A,B) = l_R : C_A \to \mathbb{R}^+ : x \to \min\limits_{b \in B} d_b(x)$ and

(K-M) $l_K(A,B) = l_K : C_A \to \mathbb{R}^\infty : x \to i \ \alpha((d_b(x))_{b \in B})$

with i being the embedding filling up the coordinates not yet used by repetition of the value of the last coordinate. The mapping that maps (A,B) on the set of solutions of (R-M) or short

$R: (A,B) \to M_{C_A}\ l_R(A,B)$ is called

RAWLS-correspondence. We define the KOLM-correspondence in the same way:

$K: (A,B) \to M_{C_A}\ l_K(A,B).$

The dual concepts are called l_R^*, l_K^*, R^*, K^*.

(1.5) We will demonstrate LEMMA 1.5:

R and K are well-defined and finite-valued.

(1.6) (Proof of Lemma 1.5)
We remark that (R-M) and (K-M), but also (R-m) and (K-m), are generalizations of the center of circumcircle.
We define for a given (A,B) and $b \in B$ a "sphere of command"

$D_b := \{x \in C_A;\ d_b(x) = l_R(x)\}.$

The boundary of D_b consists of fragment of (finitely many) middle-hyperplanes of pairs from B and eventually a fragment of the border of C_A. D_b is polytope and maxima do exist. It is clear, the solution points for 1.2 can only lie on the edges of the D_b's ($b \in B$). The set of these edges is finite. We summarize:

Be $E: (A,B) \to \{e \text{ edge of a } D_b,\ b \in B\}$,

so we have $\emptyset \neq E. \supset R. \supset K.$, because R and K each are generated by further optimization. Because of that fact and by the finiteness of E, K (and R) is finite and nonempty, too.

(1.7) (The dual version)

Let us look at the dual concepts for a while. Obviously for all $x \in C_A \setminus C_B$ there exists a $y \in C_B$ with $l_R^*(x) > l_R^*(y)$.
So $K^*(A,B) \subset R^*(A,B) \subset C_B$; i. e.;
the solutions only depend on the second component.
Every B can be seen as the extreme points ext(B) of C_B and the rest. We have $R^*(A,B) \subset R^*(\text{ext}(B), \text{ext}(B))$, because nonextreme points get a better i. e. smaller distance in any way. The diameter of B equals the distance between a suitable pair of extreme points, denoted by dext(B).
All other points of B are elements of the corresponding ball K(B). To get all possible ext(B), i. e. all strictly convex polytopes given by their extreme points, we can start with polytopes of two points and give one more point within K(B) with every step. This induction is the technique to show:

LEMMA 1.7: K* and R* are single-valued and equal.
They are equal to a center of circumcircle of dim C_B-1 points, among them dext(B). If there is equality of maximum and minimum solution, then B = ext(A).

(1.8) Imagine a group deciding on alternatives $C_{A'}$ lying between the extreme positions A' and having equal-scaled ordinal preferences, that can be represented by utility functions d_b, $b \in B' \subset C_{A'}$ - i. e. the individuals have standpoints b (their poorest alternatives) inside the extremes, then we can define for $A = A' \cup B'$ a game without sidepayments V(A,v) with v being a nontrivial superadditive game:

$$V(A,v)(B) = \begin{cases} CH_B d_B C_A & \text{for } v(B) = 1 \\ CH_B 0 & \text{for } v(B) = 0 \end{cases}$$

with CH_B the comprehensive hull taken in B-coordinates ($B \subset A$).
Substituting damage functions for utility functions the threat-point 0 becomes a bliss-point. In this interpretation corresponding to 1.3 and 1.7 V(A,v) is called dual game.

(1.9) Such models are especially occuring in locational conflicts. Be the parameter m in 1.1 not greater than 2. Under a given structure of souvereignity v is to decide on a location of say a garbage collection (primal problem) or of a hospital (dual problem). In this interpretation 1.2 and 1.3 give the formal definition of fairness within B: where B would build the object of planning, having enough power to carry its point and being fair with each other (perhaps being forced to be it).

(1.10) Imagine the decision on an industrial area between municipalities given the objective of maximal reachability and minimal environmental burden as simultaniously solving 1.2 and 1.4 (see 1.7 Lemma).

(1.11) Interpreting B as facilities of supply, we can also see 1.2 as search for the worst-supplied point in C_A, when we can scale the quality of supply by negative distance. Maximization by 1_R takes in consideration only one unit of supply, that by 1_K takes into account all of them.

(1.12) For an interpretation of m-dimensional dual conflicts as conflicts on public goods we refer to ZECKHAUSER/WEINSTEIN (6), GUESNERIE (7) and RICHTER (3).

(1.13) The concepts of fairness gain in game theory, because conventional concepts are difficult to apply, insufficiently analysing or difficult to justify in the given situation. We give two examples and refer for further studies to RICHTER (3) and ROSENMOLLER (4,5).

(1.14) The first example is on <u>NASH's</u> value. This value-concept requires invariance under affine transformations of utility and convexity of V(B); in our class of games thereis neither convexity nor affine invariance. Asking for ZEUTHEN-NASH-principle and the maximators of $\prod_{i \in B} d_i$ we must realize that the latter are boundary elements, when we look at triangles. For a proof look at the CASSINI-curves around two points. The CASSINI's give lines of constant distance-products.

(1.15) Mostly the <u>Core</u> is empty. For an example take

$$v(B) = \begin{cases} 1 & \text{for } 2 \circ \lambda B > \lambda A \\ 0 & \text{otherwise} \end{cases}$$, the majority game on

a triangle A. If A ist equilateral and

$$v(B) = \begin{cases} 1 & a \in B \text{ and } b \notin B \\ 0 & a \notin B \text{ or } b \in B \end{cases}$$, then the Core equals {b}.

§2. The spaces of the location structures of locational conflicts.

(2.1) Interpreted as games $V(A,v)$, locational conflicts are to study on their souvereignity structure v, on their locational structures (A,B), $\emptyset \neq B \subset A$, and on the relation between both. In this paper we only deal with location structures w. r. t. fairness, or more independent on the special interpretation: we classify pairs (A,B) according to R and K.

(2.2) (The topological space considered)
We define $\Pi^m := \{(A,B);\ \emptyset \neq B \subset A \subset \mathbb{R}^m\}$ and $\Pi = \varinjlim(\Pi^m, i_m)$ with the natural embedding $i_m: \Pi^m \to \Pi^{m+1}$, filling up the new coordinate by zero. Π is to have the Hausdorff-topology (generated by : $d((A_1,B_1),(A_2,B_2)) =$

$$\max_{A_1} \min_{A_2} \|a_1 - a_2\| + \max_{A_2} \min_{A_1} \|a_1 - a_2\| + \max_{B_1} \min_{B_2} \|b_1 - b_2\| +$$

$$+ \max_{B_2} \min_{B_1} \|b_1 - b_2\|,$$

$\|\ \|$ taken in any $\mathbb{R}^m$, where both (A_i,B_i) are lying in its Π^m); this formalizes the intuition of lying close together.

(2.3) (Equivalences)
Distance functions d_b are invariant under any i_m and euclidian movements. K and R commutate with centered strechings in any $\mathbb{R}^m$ and with permutations on A. We define the following equivalence on Π:

$(A_1,B_1) \sim (A_2,B_2)$ if they can be transformed into each other by embeddings, similarities and permutations.
It is enough to study R and K on a "~-normalized" subspace Π_N of Π with $\Pi/\sim = \Pi_N/_\sim$. Π_N is a representation system of Π and ~, if it is homeomorphic to $\Pi/_\sim$.

(2.4) (Subspaces)
Let $\Pi^{(n)} := \{(A,B) \in \Pi, \aleph A \leq n\}$ and $\Pi^{(n)}/_\sim$ be the corresponding subspace of $\Pi/_\sim$. In § 3 we will give a representation system for $\Pi^{(3)}$ and describe R and K on it. In § 4 we learn something about regular subspaces and their density, but for a study of largeness of the spaces of irregular R or K the global viewpoint of Π does not work. But a later paper will show that it does work on $\Pi^{(n)}/_\sim$ because of its special structure. As a first step to this view we prove the quasi-compactness of $\Pi^{(n)}/_\sim$ in § 5.

(2.5) We remember that $\underrightarrow{\lim}$ preserves all properties that are preserved by the sum as well as by the quotient (see CECH (8)).

(2.6) $\Pi/_\sim$ is not pseudosemimetrizable, since its points are not closed: the equivalence-class 0 of the "conflictless conflict" (0,0), $0 \in \dot{\mathbb{R}}^0$ is element of the closure of any other point. We can converge to it by strechings.
But there is the following <u>LEMMA 2.6:</u> $\Pi/_\sim \setminus \{0\}$ is metrizable.

(2.7) (Sketch of a proof of Lemma 2.6)
Let us regard only the first components $\pi_1(\Pi/_{\sim}\setminus\{0\})$,

$$\pi_1 : (A,B) \to A.$$

$$\mathfrak{d}(\mathfrak{A},\mathfrak{B}) = \inf \{d(A,B);\ A\in\mathfrak{A},\ B\in\mathfrak{B},\ \emptyset A = 1\}$$

with $\emptyset$ denoting the diameter, fulfills the conditions.
For a proof look at $\pi K_\varepsilon^{d}(A)$ and $\pi^{\leftarrow} K_\varepsilon^{\mathfrak{d}}(A)$ with π the canonical factorization and the K_ε the ε-balls in question.

§ 3 Location Structures for Triangle Conflicts

(3.1) The following theorem gives a representation system of $\pi_1\, \Pi^{(3)}$ and $\sim$.

THEOREM 3.1: (1) $\pi_1\, \Pi^{(3)}/_{\sim} \setminus \{0\}$ is homeomorphic to the disc. And 0 is element of every neighbourhood of any other element.

(2) A representation system is given by $\mathfrak{T} \cup \{0\}$,

$\mathfrak{T} := \{\{(0,0), \tau, (1,0)\} \in \pi_1\, \Pi^{(3)};\ \|\tau\| \leq 1,\ \pi_1\tau \geq \frac{1}{2}, \pi_2\tau \geq 0\}$

(3) $\mathfrak{T}$ is homeomorphic to

$\mathfrak{T}' := \alpha(\{\{0,1,z\};\ 0,1,z \in \mathbb{C},\ \mathrm{re}\ z \geq \frac{1}{2},\ \mathrm{im}\ z \geq 0, |z-1| \geq 1\})$

with α denoting the one-point compactification.

Remark: $\pi_1(\Pi^{(3)} \setminus \Pi^{(2)})$ can be obtained by factorizing the affine group of the euclidian plane by similarities. The affine group is transitive and effective on $\pi_1(\Pi^{(3)} \setminus \Pi^{(2)})$. We get PSL_2/SO_2.

(3.2) (Proof of Theorem 3.1)

$\mathfrak{T}$ and $\mathfrak{T}'$ are homeomorphic to the disc. For the required property of 0 see 2.6. So we only have to prove (2) and (3). We obtain $\mathfrak{T}$ and $\mathfrak{T}'$ by normalizations. Let $A=\{a_1,a_2,a_3\}$. We proceed by the following steps:

(a) denote the edges of the shortest side by a_1 and a_2 (permutations on A)

(b) substitute 0 for a_1 by translations

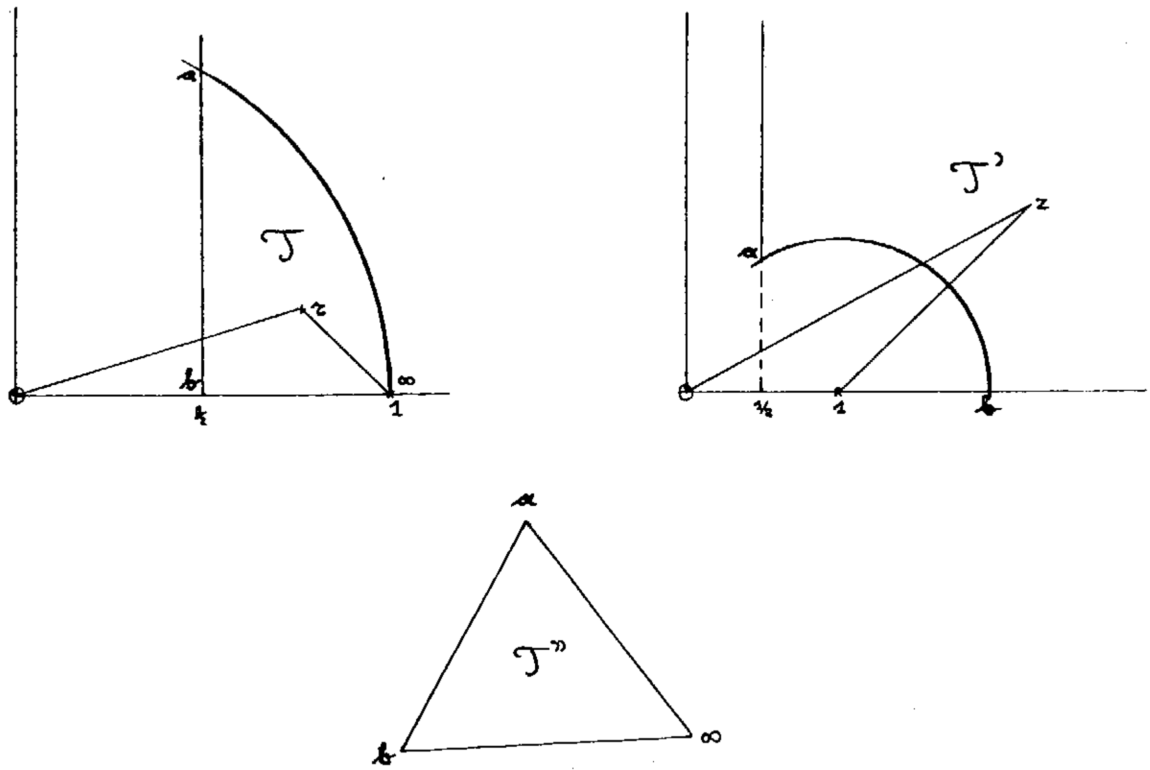
(c) stretch the largest side (the diameter) to attain unit length

(d) substitute $(1,0)=:1$ for a_2 by rotations.

By (a) - (d) we obtain $\mathfrak{T}$. We may obtain $\mathfrak{T}'$ by substituting

"(c') ... shortest ..." for "(c) ... largest ...".
As a simplification we also represent $\pi, \Pi^{(3)}/_{\sim} \setminus \{0\}$ by a triangle $\mathfrak{T}''$.

Consider the following figures:



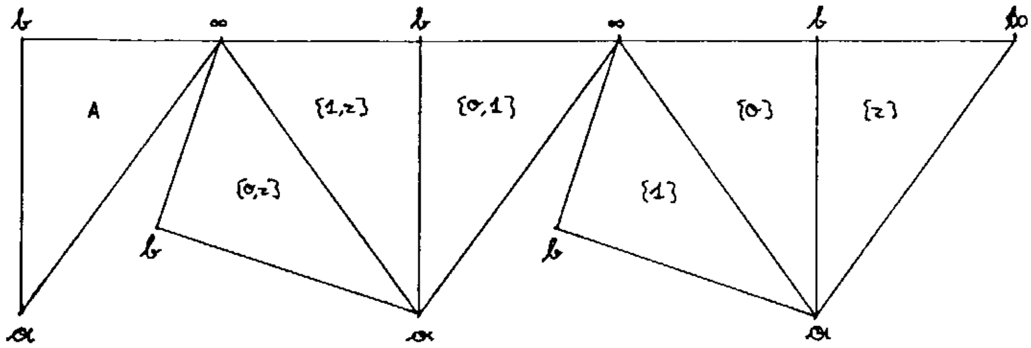
The point "$\mathfrak{a}$" represents the equilateral triangle, "∞" the two-point triangle, and "$\mathfrak{b}$" the three-point symmetric one-dimensional triangle. The topology is evident. $\mathfrak{T}'$ is not a representation system, since $\infty \notin \pi_1 \Pi^{(3)}$. We further remark that the boundary consists in symmetric and less-dimensional "conflicts".
Unless otherwise stated we shall use the representation

$\pi_1 \, \Pi^{(3)}/_{\sim} \smallsetminus \{0\}$ by $\mathfrak{T}'$.

(3.3) <u>COR. 3.3:</u> The space $\Pi_N^{(3)} \smallsetminus \{0\}$ is a two-dimensional connected compact CW-complex.

(3.4) (Proof of Cor. 3.3)

It is seen by the following figure that $\Pi_N^{(3)} \smallsetminus \{0\}$ is an intersecting sum of seven leaves homeomorphic to $\mathfrak{T}'$, indexed by the subcoalitions. Proof is evident by consideration of symmetry and embedding the smaller power sets into that of {0,1,z}.



(3.5) <u>LEMMA:</u> K and R are equal and continuous functions on $\text{II}^{(3)}$ except the following irregularities:

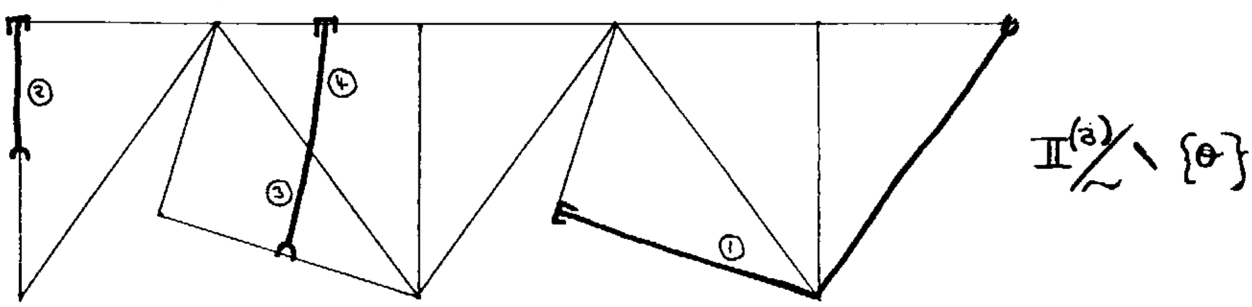
(1) a halfopen one-diemsional intervall located on the boundary of the leaves{0}, {1}, {z}: isoceles triangles with vertex 0 resp. 1, z.

(2) a halfopen one-dimensional intervall located on the boundary of the leaf A: isoceles obtuse-angled triangles.

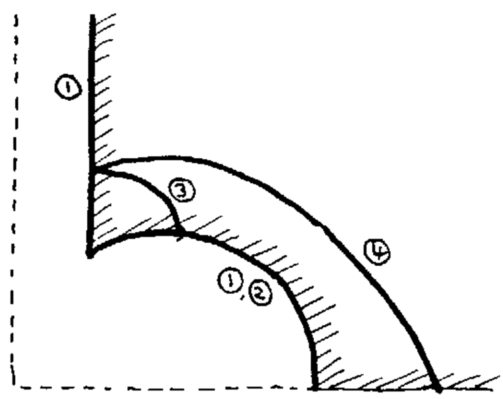
(3), (4) a halfopen one-dimensional intervall on the leaves {0,z} and {1,z}: triangles that admit a decomposition into two isoceles triangles.

The corresponding irregularities exhibit the following features: For (1) and (2) R and K are still equal, but they are no more functions and not lower hemicontinuous (lhc). For (3) and (4) we get R≠K, R is not a function, R and K are not lhc, K ist not upper hemicontinuous (uhc).

The content of the Lemma is suggested by the following figure:



... and projected by π_1 :



(3.6) (Proof of Lemma 3.5)

The statements of Lemma 3.5 are proved by verifying the results given in the following table.

A = {0,1,z}

B ⊂ A

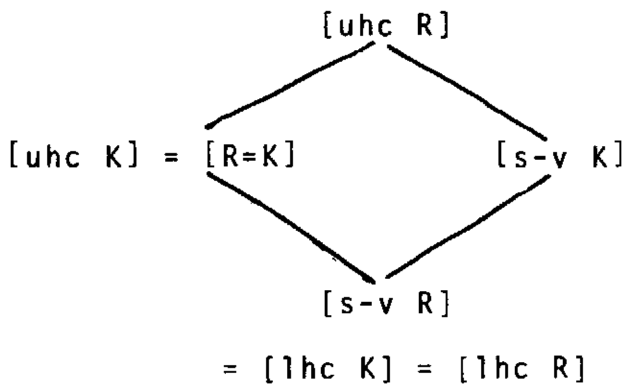
M: the center

m,n,l: the cut-point of the middle-line of {1,z} resp. {0,z} resp. {0,1} with the side of {0,z} resp. {1,z} resp. {0,z}.

B	R(A,B)	K(A,B)	Irreg.
{0}	{z,1} for equilateral, {z} otherwise		(1)
{1}	{z,0} for isoceles with vertex 1 , {z} otherwise		(1)
{z}	{0,1} for isoceles with vertex z, {0} otherwise		(1)
{0,1}	{z}		/
{1,z}	{0} for |m-1| <1 {0,m} |m-1| =1 {m} |m-1| >1	{0} for |m-1|≤1 {m} |m-1|>1	(4)
{0,z}	{1} for |n|<1 {1,n} |n|=1 {n} |n|>1	{1} for |n|≤1 {n} |n|>1	(3)
A	{M} for not obtuse-angled {m,l} obtuse-angled {m} otherwise		(2)

(3.7) We consider the subspaces [$\mathfrak{p}$] of $\Pi^{(3)}$ consisting the elements with property $\mathfrak{p}$.

<u>Cor. 3.7:</u> The following diagramm represents the strict inclusions between the subspaces in question. Moreover, the diagramm is a semilattice - it is closed under intersection; i. e.;



s-v means single-valuedness.

The proof is evident by 3.6.

(3.8) (Remark)

All subspaces of 3.7 are dense. Their complements are less-dimensional. All the subspaces are connected.

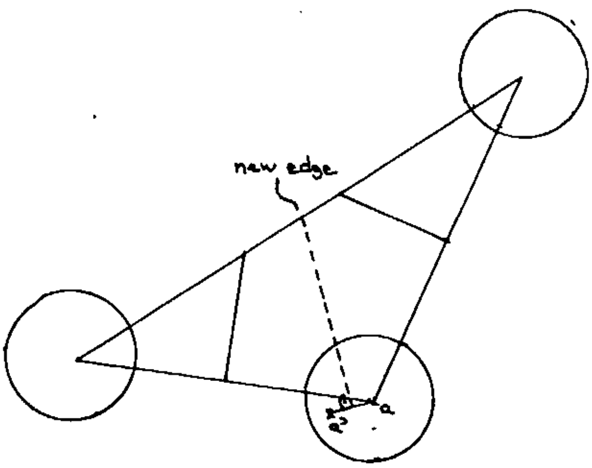
(3.9) (Remark)

R^* and K^* are single-valued, continuous and equal. $R^*(A,A)$ is equal to {M} for not-obtuse-angled triangles, and to $\frac{1}{2}(0+z)$ for obtuse-angled ones.

§ 4 Classifications of Location Structures

(4.1) THEOREM 4.1: The correspondence E (defined in 1.6) is (1) lhc and (2) not uhc, but it is (3) continuous on $\Pi^{(n)} \setminus \Pi^{(n-1)}$. (4) The correspondence R is uhc.

(4.2) (Elements of a proof of Theorem 4.1)
Let (A',B') be an element of an ε-neighbourhood of (A,B). One can imagine that neighbourhood as union of euclidian balls $K_\varepsilon a$ around the $a \in A$. For all $b \in B$ let $\hat{b}=\hat{b}((A',B'))$ be the set $\{b' \in B', b' \in K_\varepsilon b\}$ of b-corresponding b''s.
If we look at the construction of the edges of D_b and $D_{b'}$ by middlehyperplanes and boundary sets. we see that for all edges e of D_b there exists an edge e' of $D_{b'}$ with $e' \in K_\varepsilon e$ (compare OSTMANN (10) 2.5 (d)): i. e "the edges of D_b are locally conserved - maybe some edges fuse". This property is equivalent to (1). We further realize that newly appearing edges are due to an increasing number of points. This property is equivalent to (2) and (3). An illustration is given by the following figure.



A=B

A'=B'

A∪{a'}=A'

The new edge can be generated far from all old edges by a proper choice of the direction of a' w. r. t. a. These "far new edges" are necessarily suboptimal; i. e. have lower R-levels. Since R can be derived from E by maximization of the R-level, we obtain the proof of (4): The upper hemicontinuity of R (compare HILDENBRAND (9), p. 29).

(4.3) We need some <u>auxiliary definitions</u>.
Let $L(x) = \{i \in S;\ d_i x = \mu d_S\}$ the subcoalition of the least favoured, and $\lambda(x) = L(x)$ its number. L generates a correspondence between R and K:
$R^L\colon (A,B) \rightarrow \{x \in R(A,B);\ \lambda(x) = \min\ \{\lambda(y);\ y \in R(A,B)\}\}$;
i. e. K minimizes λ on R.

(4.4) <u>LEMMA 4.4</u>: [s-v R] is dense in Π.

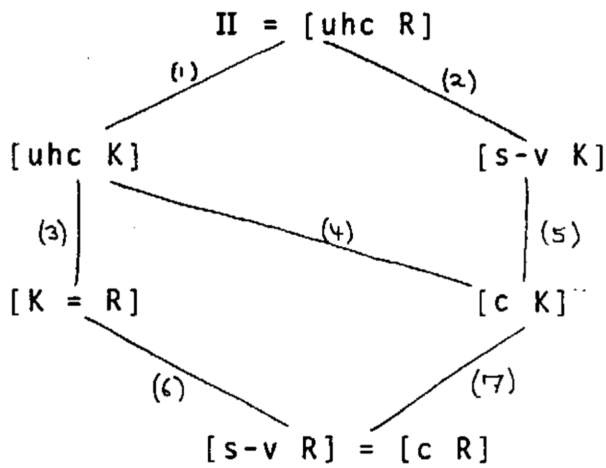
(4.5) (Proof of Lemma 4.4)
We prove this lemma in three steps:
(1) If $R(A,B) \cap C_A \neq \emptyset \neq R(A,B) \cap C_B$ (if R has both boundary and inner solutions), then there is a (A',B') with $R(A',B') \subset C_B$ in every ε-neighbourhood of (A,B).
(2) If $R(A,B) \subset C_A$, then there is an element of [s-v R] in every ε-neighbourhood of (A,B).
(3) The same as (2) for the condition $R(A,B) \subset C_B$.

First step: $(A',B') = (A^{\delta},B^{\delta})$, where δ moves inward all the boundary hyperplanes of the polytope A, that have no common point with the subpolytope generated by B. By small moves we get $R(A',B') \subset C_B$, since the edges of E, that are not in C_B, are no longer optimal.
Second step: to get an (A',B') with $\aleph R(A',B') = 1$, we introduce a ney player j in a neighbourhood of a player of L(R(A,B)), who can discriminate the elements of R(A,B). $A' = A \cup \{j\}$ and $B' = B \cup \{j\}$.
Last step: When $R(A,B) \cap C_A = \emptyset$ and $R(A,B) \subset C_B$, then $R(A,B) \subset \overset{o}{C}_B$, i. e. the inner points of C_B, otherwise the element not in the inner would not be maximal.
Take some $x \in R(A,B) \subset \overset{o}{C}_B$ and its L(x). We enlarge the R-level of x by stretching L(x) with center x. So $R(A,B) \setminus \{x\}$ is no longer optimal.

4.6 LEMMA 4.6 on finite correspondences: If Ψ is a finite-valued correspondece and [s-v Ψ] is dense in the domain of Ψ, then from "Ψ lhc in x" follows "Ψ is s-v in x", i. e. [lhc Ψ] $\subset$ [s-v Ψ].

4.7 (Proof of Lemma 4.6)
Since Ψx is finite, we can separate the $y \in \Psi x$ by ε-balls around the y's. Then from "Ψ lhc in x" follows "for every ε and every $y \in \Psi x$ is to exist a neighbourhood V_{δ} of x in which the Ψ-values are within the conserved ε-balls".
But in this neighbourhood V_{δ} the correspondence Ψ is no more single-valued. That is a contradiction to the density of [s-v Ψ].

4.8 THEOREM 4.8: All subspaces of Π directly generated by the properties s-v, lhc, uhc, R = K and logic conjunction (= set-th. intersection) are dense. The inclusions and intersections (not unions!) are given by the following diagramm:



4.9 (Proof of the theorem 4.8)
Since [uhc R] = Π (1), (2) and (3) are trivially satisfied. Since K ⊂ R (6) is evident and [s-v R] ⊂ [s-v K] too. (4) follows from definition. Since [s-v R] and the larger space [s-v K] are dense (see Lemma 4.4) follows by Lemma 4.6: [lhc K] ⊂ [s-v K] and [lhc K] = [c K] = [c s-v K]. That proves (5) and the intersection of ((4),(5)). For [s-v R] Lemma 4.6 further yields:
[lhc R] = [uhc R] ∩ [s-v R] = [s-v R] = [c R] = [c s-v R].
But [c s-v R] is a subspace of [c s-v K]. That proves (7) and the intersection of ((6),(7)) - and completes the proof, since [s-v R] is dense (Lemma 4.4).

4.10 THEOREM 4.10: [R = K] = [uhc K].

4.11 (Proof of the theorem 4.10)

It is only to be shown that uhc K implies R = K. Let us assume R(A,B) ≠ K(A,B) and $x \in K$, $y \in R \setminus K$. We look at L(y) and L(x) as defined in 4.3 and at the corresponding sheres S(y) and S(x) with radius L_R around y resp. x.

Case one: There exists a $b \in L(y) \cap L(x)$. Then there is a direction on S(y), where b can be moved to get $d(.,x) < L_R$ (and conserve $d(.,y) = L_R$); i. e. in every neighbourhood of (A,B) "x" vanishes from K and substitutes from y. But that fact distroys upperhemicontinuity.

Case two: $L(y) \cap L(x) = \emptyset$. Then there are stretchings of L(y) centered in y to give an elevated R-level to y and let jump K from x to y. This completes the proof.

4.12 COR. 4.12: [c K] = [s-v R].

4.13 (Proof of Cor. 4.12)

With 4.9 we got [s-v R] = [c R] = [c s-v R] ⊂ [c K]. It is sufficient to show c K ⇒ s-v R. But we know c K ⇒ uhc K ⇒ R=K and c K ∧ R = K ⇒ c R. In 4.9 we demonstrated c R ⇔ s-v R. This completes the proof.

4.14 COR. 4.13: The diagramm of 4.8 is simplified to that of 3.6. All of the four remaining spaces are different ones.

§ 5 Compactness of factorized spaces with number restricted

5.1 LEMMA 5.1: The mappings $\aleph$: (A,B) → $\aleph$A counting the number and dim: (A,B) → dim C_A counting the dimension of the convex hull of A, are continuous w. r. t. the topology of final fragments on $\mathbb{N}$ (i. e. the topology of sequence convergence). (Proof is simple, and so is the following (compare OSTMANN (10)).

5.2 COR. 5.2: (1) $\Pi^{(n)}$ and $\Pi^{(m)}$ are closed.

(2) $(\Pi^{(n)} \setminus \Pi^{(n-1)}) \cap (\Pi^{(n-1)} \setminus \Pi^{(n-2)})$ is dense in $\Pi^{(n)}$.

(3) $\Pi^{(n)} \setminus \Pi^{(n-1)}$ is open in $\Pi^{(n)}$.

(4) Π, $\Pi/_\sim$, $\Pi^{(n)}$, $\Pi^{(n)}/_\sim$ are of countable base.

5.3 (Elements of the proof of Cor. 5.2)

(1) is evident by 5.1.

(2): If (A,B) $\in \Pi^n$, so in every neighbourhood of A lies an element with maximal number and dimension.

(3): Since $\Pi^{(n-1)}$ is closed.

(4): $\Pi \setminus \{0\}$ resp. $\Pi/_\sim \setminus \{0\}$ are metrizable by 2.2 and 2.6 - so they have locally countabel bases. But they have a countable base, too, by counting {rational coordinates} x {dimensions}.

5.4 THEOREM 5.4: $\Pi^{(n)}/_\sim \setminus \{0\}$ is compact. $\Pi^{(n)}/_\sim$ is quasicompact.

5.5 (Proof of the Theorem 5.4)

$\Pi^{(2)}$ is quasicompact and $\Pi^{(2)} \setminus \{0\}$ is compact. We will demonstrate: from the quasicompactness of $\Pi^{(n)}$ follows quaiscompactness of $\Pi^{(n+1)}$; "compactness" works in the same way.

Take a $\sim$-normalized subspace Π_N of Π by 2.3 and "(c'') diameter is not greater than one". Because of 5.3.(4) it is enough to show that $(A_\nu)_{\nu \in \mathbb{N}}$, $A_\nu \in \Pi_n^{(n+1)} \setminus \{0\}$ has a cluster point in $\Pi^{(n+1)}$ ("countably quasicompact", see SCHUBERT (11) p. 63).
If $(A_\nu)_{\mathbb{N}}$ has an infinite familiy in $\Pi^{(n)}$, all becomes true. So let $M \subset N$ with $\aleph M = \aleph_0$ and for every $\nu \in M$: $\aleph A_\nu = n+1$. We give an arbitrary index to every element of the A_ν like $(a_\nu^1 \ldots a_\nu^{n+1})$. The families $(a_\nu^i)_{\nu \in M}$ have cluster points, say a^i, because of the boundedness caused by normalization. We get: $A = \{a^i\}_{i=1\ldots n+1}$ is clusterpoint of $(A_\nu)_M$. The number of A is not greater than n+1. May be it is one. That point is cluster point of every infinite family (2.6). So $\Pi^{(n)}$ is quasicompact. 0 is open and $\Pi^{(n)} \setminus \{0\}$ closed and that is why it is quasicompact, too. In $\Pi^{(n)} \setminus \{0\}$ we have Hausdorff separation and therefore compactness.

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